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NON-ZERO DEGREE MAPS TO HYPERBOLIC 3-MANIFOLDS

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Let us consider two closed, connected, orientable manifolds M, N with the same dimension n. Then, N is said to be *dominated* by M if there exists a non-zero degree map $f : M \longrightarrow N$. In this paper, we study the case where the dominated manifold N is hyperbolic. By an argument invoking the Gromov invariant, it is shown that the volume of N is bounded by a constant depending only on M; see Thurston [9, Chapter 6]. According to H.C. Wang [11], there are only finitely many hyperbolic n-manifolds with bounded volume if $n > 3$. This shows that the number of mutually non-homeomorphic, hyperbolic n-manifolds dominated by a fixed M is finite if $n \neq 3$. In the case of $n = 3$, a similar argument does not work. In fact, by Thurston's Hyperbolic Dehn Surgery Theorem [9], one can have infinitely many hyperbolic 3-manifolds with bounded volume, and hence Wang's theorem of dimension three does not hold. However, even in this case, we have the following theorem.

Theorem. *For any closed, connected, orientable* 3*-manifold* M*, the number of mutually non-homeomorphic, orientable, hyperbolic* 3*-manifolds dominated by* M *is finite.*

In [6], Reid and Wang proved the same assertion when M is hyperbolic and non-Haken by a method different from ours. We note that, by using some arguments in Boileau-Wang [2, §3], one can prove that this theorem does not hold when M is non-orientable. However, it would be impossible to apply their arguments to the orientable case; for example see Remark in §4. Moreover, our theorem is closely connected with Problem 3.100 by Y. Rong in [5], where he asked whether there are only

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finitely many irreducible 3-manifolds N admitting a degree one map $M \longrightarrow N$, that is, N is 1-*dominated* by M, for any closed orientable 3-manifold M. In [7, Corollary 4.1], Rong proved that, if this M is Seifert-fibered of infinite π_1, then M 1-dominates only finitely many Seifert fibered spaces of infinite π_1. Recently, Hayat-Legrand, Wang and Zieschang proved that any closed orientable 3-manifold 1-dominates only finitely many Seifert fibered spaces of finite π_1 by generalizing their results in [3], [4].

Our proof of Theorem is based on the argument in Thurston [10], where a certain 3-manifold M is hyperbolized with ideal 3-simplices by using a faithful, discrete representation $\rho : \pi_1(M) \longrightarrow \mathrm{Isom}^+(\mathbf{H}^3)$. Here, we "hyperbolize" our M similarly by using a non-zero degree map $f : M \longrightarrow N$ to construct a simplicial complex $\widehat{G}$ which consists of two parts, an inner part and an outer part. In [10], Thurston only needs the fact that the volume of the outer part $\widehat{\mathcal{O}}$ is small. However, in our argument, the "area" of $\widehat{\mathcal{O}}$ is needed to be small as well as the volume. To show the smallness, we consider a microchip decomposition for $\widehat{\mathcal{O}}$. Though a similar decomposition has been already used in Soma [8], we are here required to treat the decomposition more carefully. Once the smallness of the area is shown, one can deform the map f so that it takes the outer part into the "black box" of N, which restricts the variety of dominated hyperbolic 3-manifolds.

The proof of Theorem will be given in §4. In §§2-3, we will define two kinds of decompositions, inner-outer decompositions and microchip decompositions on the outer parts, for hyperbolic ideal 3-simplices and simplicial complexes. When the reader wishes to know how such decompositions are used in the proof, she/he may glance in advance at the introductory parts of §4 where outlines of the proof are presented. It seems that any arguments analogous to ours have not appeared before in the study of non-zero degree maps between 3-manifolds. The author feels that our techniques in this paper may be useful in other situations to investigate such maps.

1. Preliminaries

In this section, we will review briefly the fundamental notation and definitions needed in later sections, and refer to Thurston [9] for details on 3-dimensional hyperbolic geometry and to Beardon [1] for 2-

dimensional hyperbolic trigonometry.

For a subset A of a metric space (X,d), the δ-neighborhood of A in X is denoted by $\mathcal{N}_\delta(A;X)$, that is,

$$\mathcal{N}_\delta(A;X) = \{x \in X;\ \inf_{a\in A} d(a,x) \le \delta\}.$$

Let $\{(X_n,d_n)\}_{n=1}^\infty$ be a set of metric spaces admitting (marking) homeomorphisms $\eta_n : X \longrightarrow X_n$ for a fixed topological space X, and let $\eta_{m,n} = \eta_n \circ \eta_m^{-1} : X_m \longrightarrow X_n$. Choose a point $x \in X$, and set $x_n = \eta_n(x)$. We say that the sequence $\{(X_n,x_n)\}$ *converges geometrically* to a metric space (Y,d_Y) with base point $y_0 \in Y$ *in the right marking* if there exist sequences $\{\varepsilon_n\}$, $\{R_n\}$ with $\varepsilon_n \searrow 0$, $R_n \nearrow \infty$, and continuous (but not necessarily homeomorphic) maps $f_n : \mathcal{N}_{R_n}(x_n,X_n) \longrightarrow \mathcal{N}_{R_n}(y_0,Y)$ satisfying the following (i)-(iii):

(i) For any $x,x' \in \mathcal{N}_{R_n}(x_n,X_n)$, $|d_n(x,x') - d_Y(f_n(x),f_n(x'))| < \varepsilon_n$.

(ii) For any $y \in \mathcal{N}_{R_n}(Y,y_0)$, there exists $x \in \mathcal{N}_{R_n}(x_n,X_n)$ with $d_Y(f_n(x),y) < \varepsilon_n$.

(iii) If $m < n$, then $\eta_{m,n}(\mathcal{N}_{R_m}(x_m,X_m)) \subset \mathcal{N}_{R_n}(x_n,X_n)$ and $d_Y(f_m(x), f_n \circ \eta_{m,n}(x)) < \varepsilon_m$ for any $x \in \mathcal{N}_{R_m}(x_m,X_m)$.

Then, (Y,y_0) is called a *geometric limit* of $\{(X_n,x_n)\}$, and a continuous map satisfying the (i) and (ii) is an *ε_n-pseudo-isometric map.*

For a constant $K \ge 1$, a homeomorphism $f : (X,d_X) \longrightarrow (Y,d_Y)$ is *K-quasi-isometric* if the map satisfies

$$\frac{1}{K} d_Y(x,x') \le d_X(f(x),f(x')) \le K d_Y(x,x')$$

for any points x,x' contained in the same component of X. If both $\mathrm{diam}(X)$, $\mathrm{diam}(Y)$ are not greater than $R > 0$, then the K-quasi-isometry is a $(K-1)R$-pseudo-isometric homeomorphism.

Throughout this paper, we fix an orientation of the hyperbolic 3-space $\mathbf{H}^3$. A non-degenerate, oriented 3-simplex Δ in $\mathbf{H}^3$ is *positive* if the orientation is compatible with that of $\mathbf{H}^3$, and otherwise *negative.* If Δ is an ideal 3-simplex in $\mathbf{H}^3$ all whose vertices are contained in the sphere S^2_∞ at infinity, then Δ admits an isometric $\mathbf{Z}_2 \times \mathbf{Z}_2$-action generated by elliptic elements. Let $\{v_1,v_2,v_3,v_4\}$ be the set of vertices of the Δ, e_{ij} the edge of Δ connecting v_i with v_j, and D_i the face of Δ opposite to v_i. We suppose that the vertices are numbered so that the triad

$(v_1 - v_4, v_2 - v_4, v_3 - v_4)$ of vectors forms the frame compatible with the orientation of Δ. We direct each e_{ij} from v_i to v_j temporarily. For any even permutation (i, j, k, l) of $(1, 2, 3, 4)$, there exists a unique element $\gamma \in \mathrm{Isom}^+(\mathbf{H}^3)$ taking D_k onto D_l and fixing v_i, v_j. Then, the *edge invariant* $z(e_{ij})$ is the complex number whose modulus is the translation distance of γ with respect to the direction of e_{ij}, and whose argument is the angle of rotation of γ. Clearly, the invariant is independent of the direction of the edge, that is, $z(e_{ij}) = z(e_{ji})$. By the $\mathbf{Z}_2 \times \mathbf{Z}_2$-symmetry of Δ, mutually opposite edges of Δ have the same edge invariant. Moreover, $z(e_{23}) = z(e_{41}) = (z-1)/z$ and $z(e_{13}) = z(e_{24}) = 1/(1-z)$ if $z = z(e_{12}) = z(e_{34})$; see [9, Chapter 4] for details. Even in the case of Δ degenerated, the edge invariant is defined similarly. Then, for any edge e of the Δ, the invariant $z(e)$ takes the value in $\mathbf{R} - \{0, 1\}$.

A *Kleinian group* Γ is a discrete subgroup of $\mathrm{PSL}_2(\mathbf{C}) = \mathrm{Isom}^+(\mathbf{H}^3)$, the group of orientation-preserving isometries on $\mathbf{H}^3$. Assume that Γ is torsion free or equivalently it has no elliptic elements. Then, the quotient space $N = \mathbf{H}^3/\Gamma$ is a *hyperbolic* 3*-manifold*, and the quotient map $p : \mathbf{H}^3 \longrightarrow N$ is the locally isometric, universal covering. For an $\varepsilon > 0$, the ε*-thin part* $N_{\mathrm{thin}(\varepsilon)}$ of N is the set consisting of all points $x \in N$ such that there exists a non-contractible loop l in N with $l \ni x$ and of length $\leq \varepsilon$. The complement $N_{\mathrm{thick}(\varepsilon)} = N - \mathrm{int}N_{\mathrm{thin}(\varepsilon)}$ is called the ε*-thick part* of N. According to Margulis Lemma [9, Corollary 5.10.2], there exists an $\varepsilon_0 > 0$ independent of Γ such that, for any $\varepsilon > 0$ less than ε_0, each component of $N_{\mathrm{thick}(\varepsilon)}$ is either an embedded, tubular neighborhood of a closed geodesic (called a *Margulis tube*), or a $\mathbf{Z}$-cusp or a $\mathbf{Z} \times \mathbf{Z}$-cusp C, that is, each component of $p^{-1}(C)$ is a horoball the stabilizer of which in Γ is isomorphic to either $\mathbf{Z}$ or $\mathbf{Z} \times \mathbf{Z}$. If $\mathrm{Vol}(N) < \infty$, then N has at most finitely many Margulis tubes. If necessary replacing ε by a sufficiently smaller positive number, we may assume that $N_{\mathrm{thin}(\varepsilon)}$ has no Margulis tube components. Then, each component of $N_{\mathrm{thin}(\varepsilon)}$ is a $\mathbf{Z} \times \mathbf{Z}$-cusp.

2. Inner and outer parts of ideal simplices

In this section, we will present two kinds of δ-inner-outer decompositions for any non-degenerate, ideal 3-simplices Δ in $\mathbf{H}^3$ and any small $\delta > 0$. It is an important fact that the diameter of each component of the δ-inner part of Δ is bounded by a constant independent of Δ. We will define a δ-microchip decomposition $\mathcal{C}$ for the δ-outer part of Δ. It is crucial in the proof of the main theorem that one can choose the δ so that the total sum of the areas of the boundaries of δ-microchips in $\mathcal{C}$ is arbitrarily small.

Roughly speaking, the δ-outer part of an ideal 3-simplex is a union of small neighborhoods of its edges. It has the property that it is the union of bounded diameter 3-cells (microchips) whose total area is bounded by a linear function of δ. Furthermore, its complement, the δ-inner part, is compact. We require two kinds of decompositions to deal with cases when the δ-inner part is either connected or disconnected.

Let D be an ideal (straight) 2-simplex in $\mathbf{H}^2$ such that all vertices of D are in the circle S^1_∞ at infinity. If $0 < \delta < \operatorname{arcsinh}(1/\sqrt{3})$, then the closure T in D of the complement $D - \mathcal{N}_\delta(\partial D, D)$ is a triangle. The convex hull $D_{\mathrm{inn}(\delta)}$ in D spanned by the three vertices of T is called the *δ-inner part* of D. Note that $D_{\mathrm{inn}(\delta)}$ is a triangle with geodesic edges and containing T. The closure $D_{\mathrm{out}(\delta)}$ of the complement $D - D_{\mathrm{inn}(\delta)}$ is called the *δ-outer part.*

Let Δ be a non-degenerate, ideal 3-simplex in $\mathbf{H}^3$ such that all vertices v_1, v_2, v_3, v_4 of Δ is in S^2_∞, and D_i $(i = 1, 2, 3, 4)$ the face of Δ opposite to v_i. The edge of Δ connecting v_i with v_j is denoted by $e_{ij} = e_{ji}$. Take $\delta > 0$ such that each $D_{i,\mathrm{inn}(\delta)}$ is a triangle. For each v_i, let w_{ik} $(k \neq i)$ be the vertex of $D_{k,\mathrm{inn}(\delta)}$ adjacent to v_i, and let $T_i = T_i(\delta)$ be the totally geodesic triangle in Δ spanned by w_{ik}'s with $k \in \{1, 2, 3, 4\} - \{i\}$. Note that all T_i $(i = 1, 2, 3, 4)$ are isometric to each other.

Let u_{ikj} be the foot of the perpendicular from w_{ik} to e_{ij} in D_k. The convex hull $A_{ij} = A_{ji}$ of $w_{ik}, w_{il}, w_{jk}, w_{jl}, u_{ikj}, u_{ilj}, u_{jki}, u_{jli}$ in Δ is called a *δ-arm* of Δ for $\{k, l\} = \{1, 2, 3, 4\} - \{i, j\}$, see Figure 2.1.

FIGURE 2.1

The boundary ∂A_{ij} consists of ten totally geodesic triangles such that eight of them meeting e_{ij} non-trivially have edges of length δ and hence their areas are less than δ. Since either T' of the other two has a geodesic segment α of length $< 2\delta$ connecting some vertex of T' with the opposite edge, Area(T) is less than 4δ. This implies that Area$(\partial A_{ij}) < 16\delta$.

For any $i, j \in \{1,2,3,4\}$ with $i \neq j$, let J_{ii} (resp. J_{ij}) be the convex hull in Δ spanned by v_i and w_{ik}'s for $k \in \{1,2,3,4\} - \{i\}$ (resp. by v_i and w_{ik}, u_{ikj}'s for $k \in \{1,2,3,4\} - \{i,j\}$). We call the union $J_i = \cup_{l=i}^4 J_{il}$ is a *δ-joint* of Δ and each J_{il} is a *δ-subjoint* of J_i; see Figure 2.2.

FIGURE 2.2

The intersection $A_{ij} \cap J_i$ is the (possibly degenerated) tetrahedron spanned by $w_{ik}, w_{il}, u_{ikj}, u_{ilj}$ with $\{k,l\} = \{1,2,3,4\} - \{i,j\}$. When $j \neq i$, ∂J_{ij} consists of five totally geodesic triangles four of which have edges of length δ, and the other T'' has a geodesic segment of length $< 2\delta$ connecting some vertex of T'' with the opposite edge. Thus, the area of ∂J_{ij} is less than 8δ.

Here, we will see the behavior of $T_i(\delta)$ as $\delta \to 0$.

Lemma 1. *With the notation as above, the following* (i)-(iii) *hold:*

(i) $\sup_\delta\{\operatorname{diam}(T_i(\delta))\} < \infty$.

(ii) $\operatorname{Area}(T_i(\delta)) < 8\delta$.

(iii) *There exists a geodesic line L in $\mathbf{H}^3$ with $T_i(\delta) \subset \mathcal{N}_{2\delta}(L, \mathbf{H}^3)$ which passes through a vertex of $T_i(\delta)$ and tends toward v_i.*

Proof. We may assume that $i = 1$. Cut $D_2 \cup D_3 \cup D_4$ along e_{13} and develop it isometrically to the upper half space model $\{z \in \mathbf{C}; \operatorname{Im}(z) > 0\}$ of $\mathbf{H}^2$, as illustrated in Figure 2.3, so that the images $\widehat{w}_{12}, \widehat{w}_{13}, \widehat{w}_{14}$ of w_{12}, w_{13}, w_{14} in $\mathbf{H}^2$ are

$$\widehat{w}_{12} = \frac{1}{2} + \frac{\sqrt{-1}}{2\sinh\delta}, \quad \widehat{w}_{13} = \frac{2+u}{2} + \frac{\sqrt{-1}}{2\sinh\delta}u, \quad \widehat{w}_{14} = \frac{2+2u+v}{2} + \frac{\sqrt{-1}}{2\sinh\delta}v,$$

where $u, v > 0$ are the constants given in Figure 2.3 which depend only on Δ.

FIGURE 2.3

Note that

$$\begin{aligned}\operatorname{diam}(T_1(\delta)) &= \max_{2\le i,j\le 4}\{\operatorname{dist}_\Delta(w_{1i}, w_{1j})\}\\ &< \operatorname{dist}_\Delta(w_{12}, w_{13}) + \operatorname{dist}_\Delta(w_{13}, w_{14})\\ &< \operatorname{dist}_{\mathbf{H}^2}(\widehat{w}_{12}, \widehat{w}_{13}) + \operatorname{dist}_{\mathbf{H}^2}(\widehat{w}_{13}, \widehat{w}_{14}).\end{aligned}$$

Since

$$\operatorname{dist}_{\mathbf{H}^2}\left(\frac{\sqrt{-1}}{2\sinh\delta}, \frac{\sqrt{-1}u}{2\sinh\delta}\right) = |\log u|$$

and

$$\operatorname{dist}_{\mathbf{H}^2}\left(\frac{\sqrt{-1}u}{2\sinh\delta}, \frac{\sqrt{-1}v}{2\sinh\delta}\right) = \left|\log\frac{u}{v}\right|,$$

we have

$$\lim_{\delta\to 0}(\operatorname{dist}_{\mathbf{H}^2}(\widehat{w}_{12}, \widehat{w}_{13}) + \operatorname{dist}_{\mathbf{H}^2}(\widehat{w}_{13}, \widehat{w}_{14})) = |\log u| + \left|\log\frac{u}{v}\right|.$$

This proves (i).

We set $y_0 = \min\{1, u, v\}$. Let w_{1j} be a vertex of $T_1(\delta)$ such that the imaginary part of $\widehat{w}_{1j}$ is $\sqrt{-1}y_0/(2\sinh\delta)$, and let ρ be the geodesic ray in Δ, called a *longest ray*, emanating from w_{1j} and tending toward v_1. Then, $\operatorname{dist}_\Delta(\rho, w_{1k}) < 2\delta$ for $k \in \{2,3,4\} - \{j\}$, note that the $\widehat{\rho}$ in Figure 2.3 is the image of ρ. Thus, for the geodesic line L with $L \supset \rho$, $\mathcal{N}_{2\delta}(L, \mathbf{H}^3)$ contains $T_1(\delta)$. This shows (iii).

By (iii), $T_1(\delta)$ contains a geodesic segment of length $< 4\delta$ connecting some vertex of $T_1(\delta)$ with the opposite edge, and hence $\operatorname{Area}(T_1(\delta)) < 8\delta$, which is the consequence of (ii). q.e.d.

By Lemma 1 together with the argument in the paragraph preceding the lemma, we have

$$\lim_{\delta\to 0}\sum_{j=1}^{4}\operatorname{Area}(\partial J_{ij}(\delta)) = 0.$$

The union $\Delta_{\mathrm{out}(\delta)} = (\cup_{1\le i<j\le 4}A_{ij}) \cup (\cup_{k=1}^4 J_k)$ is called the δ-*outer part* of Δ, and the closure $\Delta_{\mathrm{inn}(\delta)}$ of the complement $\Delta - \Delta_{\mathrm{out}(\delta)}$ is the δ-*inner part*. We say that the union $\tau = T_1 \cup T_2 \cup T_3 \cup T_4$ is the *turning section* of $\Delta_{\mathrm{inn}(\delta)}$.

By Lemma 1 (i), $\mathrm{diam}(T_i(\delta))$ is bounded by a constant independent of δ. However, $\mathrm{diam}(T_i(\delta))$ diverges to the infinity if $z(e) \to 1$ in $\mathbf{C}$ for some edge e of Δ. This is inconvenient for us to analyze a geometric limit of the δ-inner parts of ideal 3-simplices. If $z(e_{ij}) = z(e_{kl})$ is sufficiently close to 1 for some $\{i, j, k, l\} = \{1, 2, 3, 4\}$, then

$$\begin{aligned} \mathrm{dist}_\Delta(u_{ijk}, u_{jik}) =& \mathrm{dist}_\Delta(u_{ijl}, u_{jil}) = \mathrm{dist}_\Delta(u_{kli}, u_{lki}) \\ =& \mathrm{dist}_\Delta(u_{klj}, u_{lkj}) \le \delta. \end{aligned} \tag{2.1}$$

We say that Δ is *δ-stretched* if it satisfies (2.1), and otherwise Δ is *δ-normal.*

Here, we will consider the case where Δ is δ-stretched, and define another δ-inner-outer decomposition for Δ. If necessary renumbering the vertices of the Δ, it may be assumed that $\mathrm{dist}_\Delta(u_{123}, u_{213}) = \mathrm{dist}_\Delta(u_{124}, u_{214}) = \mathrm{dist}_\Delta(u_{341}, u_{431}) = \mathrm{dist}_\Delta(u_{342}, u_{432}) \le \delta$; see Figure 2.4.

FIGURE 2.4

Since the loxodromic element $\gamma \in \mathrm{Isom}^+(\mathbf{H}^3)$ with $\gamma(v_1) = v_1$, $\gamma(v_2) = v_2$ and $\gamma(u_{431}) = u_{341}$ maps w_{13} to w_{14}, w_{43} to w_{34}, and u_{134} to u_{143}, we have

$$\begin{aligned} &\mathrm{dist}_\Delta(w_{ij}, w_{ik}) < \delta,\ \mathrm{dist}_\Delta(w_{jk}, w_{kj}) < \delta \text{ and} \\ &\mathrm{dist}_\Delta(u_{ijk}, u_{ikj}) < \delta \end{aligned} \tag{2.2}$$

for $(i; j, k) = (1; 3, 4), (2; 3, 4), (3; 1, 2)$ and $(4; 1, 2)$. The δ-arms A'_{12} and A'_{34} here are equal to A_{12} and A_{34} respectively. The δ-arm A'_1 is the convex hull of $w_{13}, w_{14}, w_{34}, w_{43}, u_{134}, u_{143},\ u_{341},\ u_{431}$. The boundary

$\partial A'_1$ consists of 12 totally geodesic triangles each of which has the area $< \delta$. Thus, the area of $\partial A'_1$ is less than 12δ; see Figure 2.5.

FIGURE 2.5

The δ-arms A'_2, A'_3, A'_4 are defined similarly. By the $\mathbf{Z}_2 \times \mathbf{Z}_2$-symmetry of Δ, all these A'_i are isometric to each other. We set $J'_{12} = J_{12}, J'_{21} = J_{21}, J'_{34} = J_{34}, J'_{43} = J_{43}$. The convex hull of $v_1, w_{13}, w_{14}, u_{134}, u_{143}$ is denoted by J'_{134}. The convex hulls $J'_{234}, J'_{312}, J'_{412}$ are defined similarly. Then, the unions $J'_1 = J'_{12} \cup J'_{134}, J'_2 = J'_{21} \cup J'_{234}, J'_3 = J'_{34} \cup J'_{312}, J'_4 = J'_{43} \cup J'_{412}$ are called *δ-joints* of Δ. It is easily seen that $\mathrm{Area}(\partial J'_{ij}) < 6\delta$ and $\mathrm{Area}(\partial J'_{ijk}) < 5\delta$ for all these J'_{ij} and J'_{ijk}. We need to consider the other δ-joint J'_0 which is the convex hull of $w_{12}, w_{21}, w_{34}, w_{43}, u_{123}, u_{124}, u_{213}, u_{214}, u_{341}, u_{342}, u_{431}, u_{432}$; see Figure 2.6.

FIGURE 2.6.

Since $\partial J_0'$ can be divided into 20 totally geodesic triangles each of which has an edge of length $\leq \delta$, the area of ∂J_0 is less than 20δ. The union

$$\Delta_{\mathrm{out}(\delta)'} = (\cup_{i=1}^4 A_i') \cup A_{12}' \cup A_{34}' \cup (\cup_{j=0}^4 J_j')$$

is called the δ-*outer part* of Δ, and the closure $\Delta_{\mathrm{inn}(\delta)'}$ of the complement $\Delta - \Delta_{\mathrm{out}(\delta)'}$ is the δ-*inner part* in the δ-stretched case. The intersection $\tau = \Delta_{\mathrm{inn}(\delta)'} \cap (\cup_{j=0}^4 J_j')$ is called the *turning section* of $\Delta_{\mathrm{inn}(\delta)'}$. By (2.1) and (2.2), τ consists of six geodesic segments of length $\leq \delta$. Note that $\Delta_{\mathrm{inn}(\delta)'}$ consists of two components.

A δ-*microchips* C is a compact Riemannian 3-manifold isometric to a convex polyhedron in $\mathbf{H}^3$ of diameter $< 10\delta$. For a set $\mathcal{C} = \{C_\lambda; \lambda \in \Lambda\}$ of δ-microchips, $\partial\mathcal{C}$ is the set $\{\partial C_\lambda; \lambda \in \Lambda\}$ with the total area $\mathrm{Area}(\partial\mathcal{C}) = \sum_{\lambda\in\Lambda} \mathrm{Area}(\partial C_\lambda)$. When the rule of the intersection $C_\lambda \cap C_\mu$ of any two elements $C_\lambda, C_\mu \in \mathcal{C}$ is determined, the union $\cup_{\lambda\in\Lambda} C_\lambda$ with the arcwise metric induced from those of C_λ's is denoted by $\sqcup\mathcal{C}$.

Now, we define a δ-microchip decomposition for $\Delta_{\mathrm{out}(\delta)}$ in the case where Δ is δ-normal. Let $\{P_n; n \in \mathbf{Z}\}$ be the set of totally geodesic planes in $\mathbf{H}^3$ perpendicular to e_{ij} with $\mathrm{dist}_{\mathbf{H}^3}(P_n, P_{n+1}) = 2\delta$ for any $n \in \mathbf{Z}$. These planes decompose the δ-arm A_{ij} into δ-microchips; see Figure 2.7.

FIGURE 2.7

Similarly, J_{ij}'s are decomposed into δ-microchips. For example, 2δ-equidistant, totally geodesic planes in $\mathbf{H}^3$ perpendicular to a longest ray connecting a vertex of T_1 with v_1 separate J_{11} into δ-microchips. The union $\mathcal{C}_\Delta$ of all these microchips defines a δ-microchip decomposition for $\Delta_{\mathrm{out}(\delta)}$, that is, $\sqcup\mathcal{C}_\Delta = \Delta_{\mathrm{out}(\delta)}$. We note that there may exist $C_1 \in \mathcal{C}_\Delta$ in A_{ij} and $C_2 \in \mathcal{C}_\Delta$ in J_i such that $\mathrm{int}C_1 \cap \mathrm{int}C_2 \neq \emptyset$. However, it does not cause any problem in our argument. Also in the δ-stretched case, a δ-microchip decomposition $\mathcal{C}_\Delta$ for $\Delta_{\mathrm{out}(\delta)'}$ is defined similarly.

Lemma 2. *In either case, there exists a constant $K > 1$ independent of δ and Δ such that, if $C \in \mathcal{C}_\Delta$ is contained in a δ-arm*

A (resp. a δ-subjoint J), then $\mathrm{Area}(\partial C) \leq K\mathrm{Area}(\partial C \cap \partial A)$ *(resp.* $\mathrm{Area}(\partial C) \leq K\mathrm{Area}(\partial C \cap \partial J)$*). q.e.d.*

The proof is elementary, so it will be left to the reader (cf. the proof of [8, Lemma 2]). Since $\lim_{\delta \to 0} \mathrm{Area}(\partial A) = 0$ and $\lim_{\delta \to 0} \mathrm{Area}(\partial J) = 0$ for any δ-arms A and δ-subjoints J, Lemma 2 implies the following.

Corollary 3. *Suppose that any $\varepsilon > 0$ and any non-degenerate, ideal 3-simplex Δ in $\mathbf{H}^3$ are given. Then, there exists $\delta_0 > 0$ such that, for any $0 < \delta \leq \delta_0$, there is a δ-microchip decomposition $\mathcal{C}_\Delta$ for $\Delta_{\mathrm{out}(\delta)}$ (or $\Delta_{\mathrm{out}(\delta)'}$) with the total area* $\mathrm{Area}(\partial \mathcal{C}_\Delta) < \varepsilon$.

3. Ideal simplicial complexes

In this section, we will investigate ideal simplicial complexes and extend the notation for a single 3-simplex given in the previous section to those for such complexes.

Let $\Delta_1, ..., \Delta_n$ be non-degenerate, oriented, ideal 3-simplices in $\mathbf{H}^3$ such that all vertices of Δ_i are contained in S^2_∞. Remove all edges from Δ_i and denote the resulting simplex by Δ_i°. We suppose that each face D_{ij}° of Δ_i° has the orientation induced from that of Δ_i°, that is, the combination of a positive frame of D_{ij}° and a normal vector on D_{ij}° to Δ_i° directing outward defines the orientation compatible with that of Δ_i°. Identifying faces of Δ_i°'s suitably by orientation-reversing isometries, one can construct a connected 3-manifold G°. The fundamental group $\pi_1(G^\circ)$ of G° is a free group. Since the attaching maps are orientation-reversing, G° has a unique orientation compatible with that of each Δ_i°. The boundary ∂G° is the disjoint union of all faces not identified with any other faces. We say that G° is an *ideal simplicial complex* obtained from $\Delta_1^\circ, ..., \Delta_n^\circ$.

Let $p : \widetilde{G}^\circ \longrightarrow G^\circ$ be the universal covering, and let $\mathrm{inc}_i : \Delta_i^\circ \longrightarrow \mathbf{H}^3$ be the inclusion. We will define a *developing map* $d : \widetilde{G}^\circ \longrightarrow \mathbf{H}^3$ in a usual manner. Let $\{\widetilde{\Delta}_\alpha^\circ\}$ be the set of all lifts to $\widetilde{G}^\circ$ of Δ_j°'s. Fix a base simplex $\widetilde{\Delta}_*^\circ$ in $\widetilde{G}^\circ$, and define that $d|\widetilde{\Delta}_*^\circ = \mathrm{inc}_i \circ p|\widetilde{\Delta}_*^\circ$, where $\Delta_i^\circ = p(\widetilde{\Delta}_*^\circ)$. When $\widetilde{\Delta}_*^\circ \cap \widetilde{\Delta}_\alpha^\circ \neq \emptyset$, the restriction $d|\widetilde{\Delta}_\alpha^\circ$ is defined by $f_{ki} \circ \mathrm{inc}_k \circ p|\widetilde{\Delta}_\alpha^\circ : \widetilde{\Delta}_\alpha^\circ \longrightarrow \mathbf{H}^3$, where $\Delta_k^\circ = p(\widetilde{\Delta}_\alpha^\circ)$ and $f_{ki} \in \mathrm{Isom}^+(\mathbf{H}^3)$ is the map extending the attaching map from a face of Δ_k° to that of Δ_i°. The developing map d is obtained by repeating the same process and by extending the map one by one. Note that, if adjacent simplices $\widetilde{\Delta}^\circ_\alpha$ and $\widetilde{\Delta}^\circ_\beta$ have distinct signs, then $d(\mathrm{int}\widetilde{\Delta}^\circ_\alpha) \cap d(\mathrm{int}\widetilde{\Delta}^\circ_\beta) \neq \emptyset$ in $\mathbf{H}^3$ even though $\mathrm{int}\widetilde{\Delta}^\circ_\alpha \cap \mathrm{int}\widetilde{\Delta}^\circ_\beta = \emptyset$ in $\widetilde{G}^\circ$. The developing map is illustrated in Figure 3.1 schematically.

FIGURE 3.1. The shaded triangles represent negative 3-simplices

The map d introduces a *holonomy* $\rho : \pi_1(G^\circ) \longrightarrow \mathrm{Isom}^+(\mathbf{H}^3)$ with $\rho(\gamma) \circ d = d \circ \tau_\gamma$ for all $\gamma \in \pi_1(G^\circ)$, where τ_γ is the covering transformation on $\widetilde{G}^\circ$ associated to γ. The complex G° can be extended to the union $G = \Delta_1 \cup \cdots \cup \Delta_n$ as a metric space if $\rho(\gamma)$ is either trivial or elliptic for $\gamma \in \pi_1(G^\circ)$ corresponding to the meridians of any edges in the 1-skeleton of G. We say that the G is the *complete ideal simplicial complex* (*obtained by completing* G°) if G is complete as a metric space. Some arguments in [9, Chapter 4] imply that, if $\rho(\pi_1(G^\circ))$ is discrete in $\mathrm{Isom}^+(\mathbf{H}^3)$ but G is not complete, then the metric completion of G is a union of G and finitely many geodesic loops.

For any small $\delta > 0$, the union $G_{\mathrm{inn}(\delta)} = \cup_{i=1}^n \Delta_{i,\mathrm{inn}(\delta)}$ is called the *δ-inner part* of G°, where each $\Delta_{i,\mathrm{inn}(\delta)}$ denotes the δ-inner part of Δ_i in either the δ-normal or δ-stretched case. The closure $G^\circ_{\mathrm{out}(\delta)}$ in G° of the complement $G^\circ - G_{\mathrm{inn}(\delta)}$ is the *δ-outer part* of G°. The *turning section* τ of $G_{\mathrm{inn}(\delta)}$ is the union of turning sections of all $\Delta_{i,\mathrm{inn}(\delta)}$. By Lemma 1 (i), (2.1) and (2.2), the diameter of each component of τ remains bounded as $\delta \to 0$. We suppose that ∂G° is empty, that is, each face of Δ_i°'s is necessarily attached to another face of some Δ_j°. Then, each component R of $\partial G_{\mathrm{inn}(\delta)} - \tau$ is an open annulus, and hence $\pi_1(R)$ is an infinite cyclic group generated by a single element $\gamma \in \pi_1(R)$.

Here, we consider the case where $\rho(\gamma)$ is trivial, and present a set of δ-microchips $\mathcal{C}_{R_0}$ such that $\sqcup\mathcal{C}_{R_0}$ is a solid cylinder with the side $R_0 = (\sqcup\mathcal{C}_{R_0}) \cap R$, where a 3-manifold A with $\partial A \supset B$ is called a *solid cylinder* with the *side* B if there exists a homeomorphism $h : (A, B) \longrightarrow (D^2 \times [0,1], \partial D^2 \times [0,1])$. Then, $\partial A - \mathrm{int}B$ is called the union of *top and bottom* of the solid cylinder. Renumbering 3-simplices of G° if necessary, one can assume that the first m $\Delta_1^\circ, \dots, \Delta_m^\circ$ are the 3-simplices meeting R non-trivially so that Δ_i° is adjacent to Δ_{i+1}° for $i = 1, \dots, m$ and $\Delta_{m+1}^\circ = \Delta_1^\circ$.

First, let us rearrange the position of each Δ_i in $\mathbf{H}^3$. Cut open the union $\Delta_1^\circ \cup \dots \cup \Delta_m^\circ$ along all common faces disjoint from R so that the resulting union contains R as a deformation retract. Since then $\rho(\pi_1(\Delta_1^\circ \cup \dots \cup \Delta_m^\circ)) = \rho(\pi_1(R)) = \{1\}$, the developing map d induces the continuous map $\overline{d} : \Delta_1^\circ \cup \dots \cup \Delta_m^\circ \longrightarrow \mathbf{H}^3$ with $\overline{d}(\Delta_i^\circ) = d(\widetilde{\Delta}_{\alpha(i)}^\circ)$ for some lift $\widetilde{\Delta}_{\alpha(i)}^\circ$ of Δ_i°. We regard $\overline{d}(\Delta_i^\circ)$ as Δ_i° in a new position, and hence $\overline{d}|\Delta_i^\circ$ is the identity. Let A_i be the δ-arm of Δ_i meeting R non-trivially, and set $A_i^\circ = A_i \cap \Delta_i^\circ$. If Δ_i is δ-normal, then A_i meets non-trivially only one edge e_i of Δ_i, otherwise A_i does two edges e_i', e_i'' of Δ_i. By the triviality of the holonomy, the restriction $\overline{d}|_{A_1^\circ \cup \dots \cup A_m^\circ}$ can be extended to a continuous map $\widehat{d} : A_1 \cup \dots \cup A_m \longrightarrow \mathbf{H}^3$.

If all Δ_i $(i = 1, \dots, m)$ are δ-normal, then there exists a geodesic line L in $\mathbf{H}^3$ with $\widehat{d}(e_i \cap A_i) \subset L$ for all $i \in \{1, \dots, m\}$. Let $\{P_j; j \in \mathbf{Z}\}$ be the set of totally geodesic planes in $\mathbf{H}^3$ perpendicular to L with $\mathrm{dist}_{\mathbf{H}^3}(P_j, P_{j+1}) = 2\delta$ for all $j \in \mathbf{Z}$. For $k, l \in \mathbf{Z}$ $(k < l)$, let $Q_{k,l}$ be the closure of a component of $\mathbf{H}^3 - P_k \cup P_l$ with $\partial Q_{k,l} = P_k \cup P_l$. Consider the maximum $Q_{k,l}$ such that $R_0 = \widehat{d}^{-1}(Q_{k,l}) \cap R$ is a (compact) annulus. Then, $\widehat{d}^{-1}(\cup_{j=k}^{l} P_j)$ defines the δ-microchip decomposition for the solid cylinder $\widehat{d}^{-1}(Q_{k,l}) \subset A_1 \cup \dots \cup A_m$ with the side R_0.

Now, let us consider the case where some of Δ_i's are not δ-normal. We set

$$\nu(\delta) = \frac{\mathrm{dist}_D(D_{\mathrm{inn}(\delta)}, \partial D)}{1000} \tag{3.1}$$

for a straight ideal 2-simplex D in $\mathbf{H}^2$ each vertex of which is contained in S^1_∞, where "1000" is added so as to make $\nu(\delta)$ sufficiently smaller than δ and has no other special meanings. Suppose that some of Δ_i are $\nu(\delta)/m$-stretched and all the others are δ_1-normal for some fixed $\delta_1 > \delta$, that is, all the 3-simplices belong to one of the two extreme groups. Then, there exists a geodesic line L such that $\mathcal{N}_{\nu(\delta)}(L, \mathbf{H}^3)$ contains all $A_i \cap e_i$ (or $A_i \cap e_i', A_i \cap e_i''$) for $i = 1, \dots, m$. We take a set $\{P_j; j \in \mathbf{Z}\}$ of totally geodesic planes and the set $Q_{k,l}$ as above. Note that, for the closure $\overline{R}$ of R in $\partial G_{\mathrm{inn}(\delta)}$, $\overline{R}_i = A_i \cap \overline{R}$ consists of two totally geodesic triangles T_{i1}, T_{i2}. For any $j \in \mathbf{Z}$ with $k \le j \le l$, $P_j \cap (\partial T_{i1} \cup \partial T_{i2})$ consists of three points $x^{(i)}_{j,1}, x^{(i)}_{j,2}, x^{(i)}_{j,3}$ as illustrated in Figure 3.2.

Figure 3.2

For $k \le j \le l-1$, the convex hull $C_{i,j}$ of $x^{(i)}_{j,1}, x^{(i)}_{j,2}, x^{(i)}_{j,3}, x^{(i)}_{j+1,1}, x^{(i)}_{j+1,2}, x^{(i)}_{j+1,3}, y_j, y_{j+1}$ is a δ-microchip, where y_j is the intersection point of L with P_j. We orient each $C_{i,j}$ so that the induced orientation on $\overline{R}_i \cap C_{i,j}$ is compatible with that of $\overline{R}_i \subset \overline{R}$. Let us start with mutually disjoint copies of $C_{i,j}$, still denoted by $C_{i,j}$, and consider the set $\mathcal{C}_{R_0}$ of them. We identify $C_{i,j}$ with $C_{i,j+1}$ along the totally geodesic rectangle spanned by $x^{(i)}_{j+1,1}, x^{(i)}_{j+1,2}, x^{(i)}_{j+1,3}, y_{j+1}$ for $j = k, \dots, l-1$. Let $T_{i,j}$ be the (possibly degenerated) tetrahedron in $\mathbf{H}^3$ spanned by

$x^{(i)}_{j,3} = x^{(i+1)}_{j,1}, x^{(i)}_{j+1,3} = x^{(i+1)}_{j+1,1}, y_j, y_{j+1}$. When $C_{i,j}$ and $C_{i+1,j}$ have distinct signs, we attach $C_{i,j}$ to $C_{i+1,j}$ along the two faces $F^{(1)}_{i,j}, F^{(2)}_{i,j}$ of $T_{i,j}$ contained in both $\partial C_{i,j}$ and $\partial C_{i+1,j}$ so that $C_{i,j} \cap C_{i+1,j} = F^{(1)}_{i,j} \cup F^{(2)}_{i,j}$; see Figure 3.3 (a). Otherwise, we attach $C_{i,j}$ to $C_{i+1,j}$ along $T_{i,j}$ so that $C_{i,j} \cap C_{i+1,j} = T_{i,j}$; see Figure 3.3(b).

FIGURE 3.3.

Then, the union $\sqcup \mathcal{C}_{R_0}$ is the solid cylinder with the side $R_0 = \widehat{d}^{-1}(Q_{k,l}) \cap R$.

In the δ-normal case, the solid cylinder $\sqcup \mathcal{C}_{R_0}$ is "ready-made", but in the latter case, it is "made-to-order". By our construction, we have the following lemma.

Lemma 4. *For any $\varepsilon > 0$ and $\delta_1 > 0$, there exists $\delta_0 = \delta_0(\varepsilon, m, \delta_1)$ with $0 < \delta_0 < \delta_1$ such that, for any $0 < \delta \leq \delta_0$, the set $\mathcal{C}_{R_0}$ of δ-microchips defined as above satisfies the following* (i)-(iv)*:*

(i) The union $\sqcup \mathcal{C}_{R_0}$ is a solid cylinder with the side R_0.

(ii) $\mathrm{Area}(\partial \mathcal{C}_{R_0}) < \varepsilon$.

(iii) The restriction $\overline{d}|R_0 : R_0 \longrightarrow \mathbf{H}^3$ can be extended to a continuous map $d_{R_0} : \sqcup \mathcal{C}_{R_0} \longrightarrow \mathbf{H}^3$ such that, for each $C_{i,j}$ of $\mathcal{C}_{R_0}$, $d_{R_0}|C_{i,j}$ is an isometric embedding.

(iv) Any component W of $(R - \mathrm{int}R_0) \cup Q$, called a capping disk for τ, is contained in the $(6m+5)\delta$-neighborhood of $\tau \cap W$ in W, where Q is the union of top and bottom of $\sqcup \mathcal{C}_{R_0}$.

Proof. The assertions (i) and (iii) are immediate from our construction. One can choose $\delta_0 > 0$ so that the assertion (ii) holds by the argument analogous to that for the proof of Corollary 3. So, it remains to show (iv). Let S be a component of $\overline{R} - \mathrm{int}R_0$. Since $\mathrm{dist}_{\mathbf{H}^3}(P_{k-1}, P_k) = \mathrm{dist}_{\mathbf{H}^3}(P_l, P_{l+1}) = 2\delta$ and since each edge of the triangles T_{i1}, T_{i2} in $\overline{R}_i$ connecting the two components of the annulus $\overline{R}$ meets P_j's almost orthogonally, there exists a geodesic arc α in S of length $< 3\delta$ connecting the component $\partial_+ S = \partial\overline{R} \cap S$ of ∂S with the other component $\partial_- S = \partial R_0 \cap S$. For any $x \in S$, let P_x be the totally geodesic plane in $\mathbf{H}^3$ perpendicular to L and containing $d_{R_0}(x)$. There exists an arc β in $S \cap \widehat{d}^{-1}(P_x)$ connecting x with a point in $\partial_+ R \cup \alpha$ and such that β consists of at most $2m$ geodesic segments each of which is of length $< 2\delta + 2\nu(\delta) < 3\delta$, and hence of length$(\beta) < 6m\delta$. Thus, S is contained in the $(6m+3)\delta$-neighborhood of $\partial_+ S$ in S. Let Q_0 be the component of Q with $\partial Q_0 = \partial_- S$; so $W = S \cup Q_0$ is a component of $(R - \mathrm{int}R_0) \cup Q$. Since each point of Q_0 and some point of $\partial_- S$ are connected by a geodesic segment of length $\leq \delta + \nu(\delta) < 2\delta$, S is contained in the $(6m+5)\delta$-neighborhood of $\partial_+ S = \tau \cap W$ in W. This completes the proof. q.e.d.

4. Proof of Theorem

Our main theorem is proved by reduction to absurdity. So, we may assume that there exists a closed, connected, oriented 3-manifold M dominating closed, connected, oriented hyperbolic 3-manifolds N_n ($n \in \mathbf{N}$) which are not homeomorphic to each other. Let $f_n : M \longrightarrow N_n$ be a non-zero degree map. According to Thurston [9, Chapter 6], for any $n \in \mathbf{N}$,

$$\mathrm{Vol}(N_n) = ||N_n||\mathbf{v}_3 \leq \frac{||M||\mathbf{v}_3}{|\deg(f_n)|} \leq ||M||\mathbf{v}_3,$$

where $||M||$ is the Gromov invariant of M, and $\mathbf{v}_3$ is the volume of a regular, ideal simplex in $\mathbf{H}^3$. Thus, the volumes $\mathrm{Vol}(N_n)$ are bounded. By Jϕrgensen's Theorem [9, Chapter 6], if necessary taking a subsequence of $\{N_n\}$ instead, we may assume that there exists a complete, connected, oriented, hyperbolic 3-manifold N with $\mathrm{Vol}(N) < \infty$ such that each N_n is obtained by hyperbolic Dehn surgery on N. In particular, we have sequences $\{\varepsilon_n\}$, $\{K_n\}$ with $\varepsilon_n \searrow 0$, $K_n \searrow 1$ so that there exist K_n-quasi-isometric diffeomorphisms $g_n : N_{n,\mathrm{thick}(\varepsilon_n)} \longrightarrow N_{\mathrm{thick}(\varepsilon_n)}$.

Here, we will give an outline of the proof of Theorem. It may be helpful for the reader to understand our overall strategy.

Step 1. By modifying M and f_n, we will first construct ideal simplicial complexes G_n and continuous maps $f_n' : G_n \longrightarrow N_n$ which are locally isometric on each simplex. In fact, the complex G_n is the union of ideal straight 3-simplices $\Delta_{i,n}$ obtained by straightening singular 3-simplices $f_n|\widehat{\Delta}_i : \widehat{\Delta}_i \longrightarrow N_n$ for any topological 3-simplices $\widehat{\Delta}_i$ in a fixed simplicial decomposition on M. Note that the diameter of each component of the δ-inner part of G_n is bounded, and each ideal 3-simplex in G_n is parametrized by an element of the compact set $\widehat{\mathbf{C}}$. Then, Ascoli-Arzelà's Theorem implies that there exists the "essential δ-inner part" $\mathcal{I}_n$ in $G_{n,\mathrm{inn}(\delta)}$ which has the property that, by passing to a subsequence if necessary, $\{f_n'|\mathcal{I}_n : \mathcal{I}_n \longrightarrow N_n\}$ converges to a continuous map $f' : \mathcal{I} \longrightarrow N$ which is locally isometric on the inner part of each ideal simplex in $\mathcal{I}$. We decompose G_n into the $\mathcal{I}_n$ and other two submanifolds $\mathcal{O}_n$, $\mathcal{Z}_n$ which have pairwise disjoint interiors and $\mathcal{I}_n \cap \mathcal{Z}_n = \emptyset$, and such that the topological type of $(G_n; \mathcal{I}_n, \mathcal{O}_n, \mathcal{Z}_n)$ is independent of $n \in \mathbf{N}$. The submanifold $\mathcal{Z}_n$ is the "inessential δ-inner part" which has the property that, for any $\varepsilon > 0$ and all sufficiently large n, $f_n'(\mathcal{Z}_n) \subset N_{n,\mathrm{thin}(\varepsilon)}$. The other submanifold $\mathcal{O}_n$ is the "δ-outer part" of G_n and is controlled in the following sense:

(i) $\lim_{\delta \to 0} \sup_n \{\mathrm{Vol}(\mathcal{O}_n)\} = 0$, and

(ii) There exists a δ-microchip decomposition $\mathcal{C}_n$ on $\mathcal{O}_n$ with $\lim_{\delta \to 0} \sup_n \{\mathrm{Area}(\partial \mathcal{C}_n)\} = 0$.

Step 2. Again by passing to a subsequence if necessary, one can modify G_n, f_n', $\mathcal{I}_n$, $\mathcal{O}_n$ and construct a new manifold $\widehat{G}_n$, a map $\widehat{f}_n : \widehat{G}_n \longrightarrow N_n$ and a decomposition $\widehat{\mathcal{I}}_n$, $\widehat{\mathcal{O}}_n$, $\widehat{\mathcal{Z}}_n$ $(= \mathcal{Z}_n)$ on $\widehat{G}_n$ which satisfy the same conclusions as (i), (ii) in Step 1 and moreover

(iii) $\rho_\Sigma(\pi_1(\Sigma))$ is a non-trivial parabolic group for each component Σ of $\partial\widehat{\mathcal{I}}$, where $\rho_\Sigma : \pi_1(\Sigma) \longrightarrow \mathrm{Isom}^+(\mathbf{H}^3)$ is the restriction of the "holonomy" of $\widehat{\mathcal{I}}$.

Here, let us first fix a constant $\lambda_0 > 0$ so that $\widehat{f}_n(\mathcal{L}_n^-) \cap N_{n,\mathrm{thin}(\lambda_0)} = \emptyset$, where $\mathcal{L}_n^-$ is some part added to $\widehat{\mathcal{O}}_n$ under our modification. By using the parabolicity, one can next choose $\delta > 0$ so that $\widehat{f}_n(\partial\widehat{\mathcal{I}}_n) \subset \mathrm{int} N_{n,\mathrm{thin}(\lambda_0)}$ for all $n \in \mathbf{N}$.

Step 3. We finally choose $\varepsilon > 0$ with $\varepsilon \ll \lambda_0$ so that

$$\widehat{f}_n(\widehat{\mathcal{I}}_n) \cap N_{n,\mathrm{thick}(\varepsilon)} = \emptyset.$$

By using (ii), $\widehat{f}_n$ can be modified again so that the resulting map $\widehat{\psi}_n : \widehat{G}_n \longrightarrow N_n$ satisfies that $\widehat{\psi}_n|\partial\widehat{\mathcal{I}}_n$ is a non-zero degree map onto $\partial N_{n,\mathrm{thin}(\varepsilon)}$, and $\widehat{\psi}_n(\widehat{\mathcal{I}}_n \cup \mathcal{Z}_n)$ is contained in the union of $N_{n,\mathrm{thin}(\varepsilon)}$ and a 1-complex Γ_n in $N_{n,\mathrm{thick}(\varepsilon)}$. We note that the (ii) is crucial in our argument. In fact, without the (ii), one would only show that $\widehat{\psi}_n(\widehat{\mathcal{I}}_n \cup \mathcal{Z}_n)$ would lie in the union of $N_{n,\mathrm{thin}(\varepsilon)}$ and a 2-complex in $N_{n,\mathrm{thick}(\varepsilon)}$, and hence one can not invoke Lemma 5 below. However, in our case, one can show that N_n have the same topological type for all sufficiently large $n \in \mathbf{N}$ by using Lemma 5. This contradiction completes our reduction to absurdity.

We say that a contractible 1-complex Γ is a *star of degree* n if Γ consists of n edges which have a common vertex. The following lemma suggested by David Gabai is a cleaned-up version of a certain proposition in the original manuscript.

Lemma 5. *Let W be a compact, oriented 3-manifold, $\mathcal{T} = T_1 \cup \cdots \cup T_n$ a disjoint union of tori, and Γ a star of degree n such that $\Gamma \cap T_i$ is a single end point of Γ for each $i \in \{1, \ldots, n\}$. Suppose that $\varphi : \partial W \longrightarrow \mathcal{T}$ is a continuous map such that, for each T_i, the degree d_i of $\varphi|\varphi^{-1}(T_i) : \varphi^{-1}(T_i) \longrightarrow T_i$ is non-zero. Then, there is at most one way to extend $\mathcal{T}$ to a disjoint union $\mathcal{V} = V_1 \cup \cdots \cup V_n$ of solid tori with $\partial V_i = T_i$ such that φ extends to a continuous map $\Phi : W \longrightarrow \mathcal{V} \cup \Gamma$.*

Proof. Suppose that there exists a continuous map $\Phi : W \longrightarrow \mathcal{V} \cup \Gamma$ extending φ. Consider a meridian disk D_i for V_i with $\partial D_i \cap \Gamma = \emptyset$. If necessary after modifying Φ by a proper homotopy, we may assume that Φ is transverse to $D_1 \cup \cdots \cup D_n$. Then, each $F_i = \Phi^{-1}(D_i)$ is a compact, orientable surface in W with $\partial F_i \subset \partial W$. Orient F_i so that $\varphi_*([\partial F_i]) = d_i[\partial D_i]$ in $H_1(T_i; \mathbf{Z})$. Consider another continuous map $\Phi' : W \longrightarrow \mathcal{V}' \cup \Gamma$ extending φ, where $\mathcal{V} = V_1' \cup \cdots \cup V_n'$ is a disjoint union of solid tori with $\partial V_i' = T_i$. Since $\Phi'(F_i)$ is contained in $\mathcal{V}' \cup \Gamma$, $\varphi_*([\partial F_i]) = d_i[\partial D_i] = 0$ in $H_1(\mathcal{V}' \cup \Gamma; \mathbf{Z})$. Since $d_i \neq 0$ and the homomorphism $H_1(V_i'; \mathbf{Z}) \longrightarrow H_1(\mathcal{V}' \cup \Gamma; \mathbf{Z})$ induced from the inclusion is injective, we have $[\partial D_i] = 0$ in $H_1(V_i'; \mathbf{Z})$. Hence, ∂D_i bounds a meridian disk in V_i'. This completes the proof. q.e.d.

Remark. In [2, Theorem 3.4], Boileau and Wang asserted that there exists a closed, orientable, hyperbolic 3-manifold M dominating infinitely many, mutually non-homeomorphic, hyperbolic 3-manifolds N_n. This contradicts our main theorem. However, the maps $f_n : M \longrightarrow N_n$ used in their proof seem to be of degree zero, so N_n may not be dominated by M. In fact, in their situation, each N_n is obtained by attaching a solid torus V_n to a fixed, compact 3-manifold X along the torus boundary ∂X. Moreover, there exist homeomorphisms $h_n : T^2 \times I \longrightarrow f_n^{-1}(V_n)$ with $f_n \circ h_n | \partial(T^2 \times I) = f_m \circ h_m | \partial(T^2 \times I) : \partial(T^2 \times I) \longrightarrow \partial X$ for all $n, m \in \mathbf{N}$. If $\deg(f_n) = \deg(f_n \circ h_n : T^2 \times I \longrightarrow V_n)$ were non-zero, then by Lemma 5, there would exist a homeomorphism from N_n to N_m extending the identity of X, a contradiction.

First of all, let us suppose that M admits a simplicial decomposition $\mathcal{D}$. Let $V(\mathcal{D})$ be the set of vertices of $\mathcal{D}$, and let $\widehat{\Delta}_1, \dots, \widehat{\Delta}_m$ be the 3-simplices of $\mathcal{D}$. Consider a Kleinian group Π_n with $N_n = \mathbf{H}^3/\Pi_n$, and the universal covering $p_n : \mathbf{H}^3 \longrightarrow N_n$. Choose an oriented, geodesic line $\widetilde{l}$ in $\mathbf{H}^3$ which is in general position with respect to Π_n, that is, for any mutually distinct $\gamma, \gamma' \in \Pi_n$, there exist no totally geodesic planes P containing $\gamma\widetilde{l} \cup \gamma'\widetilde{l}$, in particular $\gamma\widetilde{l} \cap \gamma'\widetilde{l} = \emptyset$. The image $l = p_n(\widetilde{l})$ is a simple, oriented geodesic in N_n. Deform f_n by homotopy, one can assume that (i) $f_n(V(\mathcal{D})) \subset l$ and (ii) for each edge e of $\mathcal{D}$, $f_n|e : e \longrightarrow N_n$ is not homotopic rel. ∂e to an arc in l. For any $i \in \{1, \dots, m\}$, straighten the singular 3-simplex $f_n|\widehat{\Delta}_i : \widehat{\Delta}_i \longrightarrow N_n$ along l and denote the resulting simplex by $\Delta_{i,n}$. Precisely, consider a lift $\widetilde{f}_n|\widehat{\Delta}_i : \widehat{\Delta}_i \longrightarrow \mathbf{H}^3$ of $f_n|\widehat{\Delta}_i$ and the components $\widetilde{l}_1, \widetilde{l}_2, \widetilde{l}_3, \widetilde{l}_4$ of $p_n^{-1}(l)$ with $\widetilde{f}_n(v_j) \in \widetilde{l}_j$ for $j = 1, 2, 3, 4$, where v_j's are the vertices of Δ_i. Each $\widetilde{l}_j$ has the orientation induced from l via p_n. Then, $\Delta_{i,n}$ is isometric to the ideal simplex in $\mathbf{H}^3$ spanned by the terminal points of $\widetilde{l}_1, \widetilde{l}_2, \widetilde{l}_3, \widetilde{l}_4$ in S^2_∞. By the assumption (ii) above, $\widetilde{l}_j \cap \widetilde{l}_k = \emptyset$ if $j \neq k$. Thus, each $\Delta_{i,n}$ is non-degenerate.

By identifying faces of $\Delta_{i,n}$ $(i = 1, \dots, m)$ suitably, we have a complete ideal simplicial complex G_n admitting a continuous map $f'_n : G_n \longrightarrow N_n$ and a (marking) homeomorphism $\eta_n : M - V(\mathcal{D}) \longrightarrow G_n$ such that (i) $\eta_n(\widehat{\Delta}_i - \widehat{\Delta}_i \cap V(\mathcal{D})) = \Delta_{i,n}$, (ii) $f'_n \circ \eta_n$ is homotopic to $f_n|(M - V(\mathcal{D}))$, and (iii) for each $\Delta_{i,n}$, $f'_n|\Delta_{i,n} : \Delta_{i,n} \longrightarrow N_n$ is a locally isometric immersion. In particular, f'_n is locally arcwise isometric, that is, for any rectifiable arc α in G_n, $\text{length}_{G_n}(\alpha) = \text{length}_{N_n}(f'_n(\alpha))$. Thus, we have the following (4.1).

(4.1) For any $x, y \in G_n$, $\mathrm{dist}_{G_n}(x,y) \geq \mathrm{dist}_{N_n}(f'_n(x), f'_n(y))$.

For the edge $e_{i,n}$ of $\Delta_{i,n}$ corresponding to a fixed edge e_i of $\widehat{\Delta}_i$, we set $z_{i,n} = z(e_{i,n})$. If necessary passing to a subsequence, we may assume that, for all $i \in \{1, \dots, m\}$, $\{z_{i,n}\}_{n=1}^{\infty}$ converges to a point $z_i \in \mathbf{C} \cup \{\infty\}$.

If $z_i \in \mathbf{R} - \{0,1\}$, then $\{\Delta_{i,n}\}_{n=1}^{\infty}$ converges geometrically to a totally geodesic, ideal rectangle R in $\mathbf{H}^3$ in the right marking. Thus, the four faces $D_{i,k;n}$ $(k = 1,2,3,4)$ of $\Delta_{i,n}$ converge geometrically to non-degenerate, ideal 2-simplices $D_{i,k}$ in R with $D_{i,1} \cup D_{i,2} = D_{i,3} \cup D_{i,4} = R$ under a suitable numbering of $D_{i,k}$'s. Let $\widetilde{f}'_{i,n} : \Delta_{i,n} \longrightarrow \mathbf{H}^3$ be a lift of the restriction $f'_n|\Delta_{i,n} : \Delta_{i,n} \longrightarrow N_n$ to $p_n : \mathbf{H}^3 \longrightarrow N_n$. By rearranging the position of $\widetilde{f}'_{i,n}(\Delta_{i,n})$ in $\mathbf{H}^3$, we may assume that $\widetilde{f}'_{i,n}(\Delta_{i,n})$ is contained in a sufficiently small neighborhood of R in $\mathbf{H}^3$. Choose $x_0 \in S^2_\infty$ so that the suspensions $\Delta^\sigma_{i,k}$ of $D_{i,k}$'s from x_0 are non-degenerate, ideal 3-simplices. For all sufficiently large $n \in \mathbf{N}$, the suspensions $\Delta^\sigma_{i,k;n}$ of $\widetilde{f}'_{i,n}(D_{i,k;n})$ from x_0 are non-degenerate. Let us start with mutually disjoint copies of $\Delta^\sigma_{i,k;n}$ $(k = 1,2,3,4)$, still denoted by $\Delta^\sigma_{i,k;n}$, and glue them along their faces which are equal to each other in $\mathbf{H}^3$. The resulting complete, simplicial complex $B_{i,n}$ is homeomorphic to a 3-ball B minus five points four of which are contained in ∂B, and the boundary $\partial B_{i,n}$ is $\partial \Delta_{i,n}$. It is easily seen that the restriction $f'_n|\partial D_{i,n} : \partial D_{i,n} \longrightarrow N_n$ can be extended to a continuous map $f'_{i,n} : B_{i,n} \longrightarrow N_n$ such that each $f'_{i,n}|\Delta^\sigma_{i,k;n} : \Delta^\sigma_{i,k;n} \longrightarrow N_n$ is a locally isometric immersion. Let us remove $\mathrm{int}\Delta_{i,n}$ from G_n and glue $B_{i,n}$ to $G_n - \mathrm{int}\Delta_{i,n}$ by the identity map of $\partial B_{i,n} = \partial \Delta_{i,n}$.

Next, we consider the case of $z_j \in \{0, 1, \infty\}$. If necessary renumbering the vertices of $\Delta_{i,n}$, we may assume that $z(e_{1,2;n}) = z(e_{3,4;n})$ converges to 1. Let C be the double of two copies of a regular ideal simplex Δ_0 in $\mathbf{H}^3$ along three faces of Δ_0. Then, C is homeomorphic to a 3-ball B minus four points three of which are contained in ∂B. Cut G_n open along the two faces $D_{j,1;n}$ and $D_{j,3;n}$ of $\Delta_{j,n}$ (but do not remove $\Delta_{j,n}$) and glue two copies C_1 and C_3 of C to the cut complex $G_n^\vee$ by an isometry $\partial C_k \longrightarrow D^+_{j,k;n} \cup D^-_{j,k;n}$ for $k = 1, 3$, where $D^+_{j,k;n}$ and $D^-_{j,k;n}$ are the 2-simplices in $\partial G_n^\vee$ corresponding to $D_{j,k;n}$. There exists a continuous map $f'_{j,k;n} : C_k \longrightarrow N_n$ extending $f'_n|D^\pm_{j,k;n} = f'_n|D_{j,k;n}$ and such that, for each ideal simplex Δ_0 in C_k, $f'_{j,k;n}|\Delta_0 : \Delta_0 \longrightarrow N_n$ is a locally isometric immersion.

We perform the same process for all $\Delta_{i,n}$ with $z_i \in \mathbf{R} \cup \{\infty\}$, and denote again the resulting complex by G_n and the corresponding decomposition for M by $\mathcal{D} = \{\widehat{\Delta}_1, \dots, \widehat{\Delta}_m\}$. The continuous map from G_n to N_n constructed from the original f'_n together with $f'_{i,n}$'s and $f'_{j,k;n}$'s is again denoted by f'_n. By these modifications, it suffices to consider the case where any z_i is contained in $(\mathbf{C} - \mathbf{R}) \cup \{0, 1, \infty\}$, and if $z_i \in \{0, 1, \infty\}$, then there exist two 3-simplices $\Delta_{k,n}$ adjacent to $\Delta_{i,n}$ each of which converges geometrically to a non-degenerate 3-simplex. Let us choose $\delta_1 > 0$ as follows.

(4.2) For any z_i contained in $\mathbf{C} - \mathbf{R}$, an ideal 3-simplex with the edge invariant z_i is δ_1-normal.

If $z_i \in \mathbf{C} - \mathbf{R}$, then it is easily seen that $\{\Delta_{i,n}\}$ converges geometrically in the right marking to an ideal simplex Δ_i with the edge invariant $z(e_i) = z_i$. Thus, for any $0 < \delta \leq \delta_1$, one can construct the ν_n-pseudo-isometric homeomorphisms $\varphi_{i,n}$ with $\nu_n \searrow 0$ realizing the geometric convergence so that $\varphi_{i,n}(\Delta_{i,\mathrm{inn}(\delta);n}) = \Delta_{i,\mathrm{inn}(\delta)}$ and $\varphi_{i,n}|\partial\Delta_{i,n} \cap \Delta_{i,\mathrm{inn}(\delta);n} : \partial\Delta_{i,n} \cap \Delta_{i,\mathrm{inn}(\delta);n} \longrightarrow \partial\Delta_i \cap \Delta_{i,\mathrm{inn}(\delta)}$ is isometric. In the case of $z_j \in \{0, 1, \infty\}$, we may assume that $z(e_{12,j;n}) = z(e_{34,j;n})$ converges to 1. For any $0 < \delta \leq \delta_1$, there exists an $n_0 \in \mathbf{N}$ such that $\Delta_{j,n}$ is $\nu(\delta)/m$-stretched if $n \geq n_0$, where $\nu(\delta)$ is the number given in (3.1). Take two base points $x^{(1)}_{j,n}$ and $x^{(3)}_{j,n}$ of $\Delta_{j,n}$ so that $x^{(1)}_{j,n}$ (resp. $x^{(3)}_{j,n}$) is contained in the component $\Delta^{(1)}_{j,\mathrm{inn}(\delta)';n}$ (resp. $\Delta^{(3)}_{j,\mathrm{inn}(\delta)';n}$) of $\Delta_{j,\mathrm{inn}(\delta)';n}$ meeting A'_{12} (resp. A'_{34}) non-trivially. Then, $\{(\Delta_{j,n}, x^{(k)}_{j,n})\}$ converges geometrically in the right marking to an ideal 2-simplex $(D^{(k)}_j, y^{(k)}_j)$ in $\mathbf{H}^2 \subset \mathbf{H}^3$ for $k = 1, 3$, where one can construct the ν_n-pseudo-isometric maps $\varphi^{(k)}_{j,n} : \mathcal{N}_{R_n}(x^{(k)}_{j,n}, \Delta_{j,n}) \longrightarrow \mathcal{N}_{R_n}(y^{(k)}_j, D^{(k)}_j)$ so that $\varphi^{(k)}_{j,n}(\Delta^{(k)}_{j,\mathrm{inn}(\delta)';n}) = D^{(k)}_{j,\mathrm{inn}(\delta)}$ and $\varphi^{(k)}_{j,n}|\partial\Delta_{j,n} \cap \Delta^{(k)}_{j,\mathrm{inn}(\delta)';n} : \partial\Delta_{j,n} \cap \Delta^{(k)}_{j,\mathrm{inn}(\delta)';n} \longrightarrow D^{(k)}_{j,\mathrm{inn}(\delta)}$ is isometric on either of the two components of $\partial\Delta_{j,n} \cap \Delta^{(k)}_{j,\mathrm{inn}(\delta)';n}$.

One can construct an ideal simplicial complex G° from Δ°_i's and $D^{(k)\circ}_j$'s so that, for any sufficiently small $\delta > 0$ and some $n_1 \in \mathbf{N}$, there exists the ν_n-pseudo-isometric map $\Phi_n : G_{n,\mathrm{inn}(\delta)} \longrightarrow G_{\mathrm{inn}(\delta)}$ extending $\varphi_{i,n}|\Delta_{i,\mathrm{inn}(\delta);n}$, $\varphi^{(k)}_{j,n}|\Delta_{j,\mathrm{inn}(\delta)';n}$ if $n \geq n_1$. Though Φ_n is in general not a homeomorphism, since each $\nu(\delta)/m$-stretched edge $\Delta_{j,n}$ has δ_1-normal neighbors, Φ_n can be deformed to a homeomorphism by a small homotopy. Thus, there exist K_n-quasi-isometric homeomorphisms $h_n : G_{\mathrm{inn}(\delta)} \longrightarrow G_{n,\mathrm{inn}(\delta)}$ for all $n \geq n_1$ with $K_n \searrow 1$ and such that $h_l \circ h_n^{-1} : G_{n,\mathrm{inn}(\delta)} \longrightarrow G_{l,\mathrm{inn}(\delta)}$ maps each $\Delta_{i,\mathrm{inn}(\delta);n}$ (resp. $\Delta_{j,\mathrm{inn}(\delta)';n}$) onto $\Delta_{i,\mathrm{inn}(\delta);l}$ (resp. $\Delta_{j,\mathrm{inn}(\delta)';l}$) and is extended to a marking-preserving homeomorphism $\eta_{n,l} : G_n \longrightarrow G_l$. By (4.2), each 3-simplex in G° is δ_1-normal, so $G_{\mathrm{inn}(\delta)}$ is a deformation retract of G°.

Let $H_1, \ldots, H_\nu$ be the components of $G_{\mathrm{inn}(\delta)}$, and $H_{\alpha,n} = h_n(H_\alpha)$ for $\alpha \in \{1, \ldots, \nu\}$. These H_α's are renumbered so that $f'_n(H_{\alpha,n})$ remains in a certain thick part of N_n for each $\alpha \in \{1, \ldots, \mu\}$, and the others leave from any thick parts. Precisely, if necessary passing to a subsequence, one can assume that the following (4.3) and (4.4) hold.

(4.3) There exists an $\varepsilon > 0$ such that $f'_n(H_{\alpha,n}) \cap N_{n,\mathrm{thin}(\varepsilon)} = \emptyset$ for all sufficiently large $n \in \mathbf{N}$ and $\alpha \in \{1, \ldots, \mu\}$.

(4.4) There exists a sequence $\{\varepsilon_n\}$ with $\varepsilon_n \searrow 0$ such that $f'_n(H_{\beta,n}) \cap N_{n,\mathrm{thin}(\varepsilon_n)} \neq \emptyset$ for all sufficiently large $n \in \mathbf{N}$ and $\beta \in \{\mu+1, \ldots, \nu\}$.

Then, we have the following commutative diagram:

$$\begin{array}{ccc} \cup_{\alpha=1}^{\mu} H_\alpha & \xrightarrow{g_n \circ f'_n \circ h_n} & N_{\mathrm{thick}(\varepsilon)} \\ h_n \big\downarrow & & \big\uparrow g_n \\ \cup_{\alpha=1}^{\mu} H_{\alpha,n} & \xrightarrow{f'_n} & N_{n,\mathrm{thick}(\varepsilon)} \end{array}$$

For simplicity, we set $\mathcal{I}_{(n)} = \cup_{\alpha=1}^{\mu} H_{\alpha,(n)}$, $\mathcal{Z}_n = \cup_{\beta=\mu+1}^{\nu} H_{\alpha,n}$ and $\mathcal{O}_n = G_{n,\mathrm{out}(\delta)}$. Note that the images $g_n \circ f'_n \circ h_n(\mathcal{I})$ are contained in the compact set $N_{\mathrm{thick}(\varepsilon)}$. Since both h_n and g_n are K_n-quasi-isometric and f'_n satisfies (4.1), by Ascoli-Arzelà's Theorem we may assume that the sequence $\{g_n \circ f'_n \circ h_n | \mathcal{I} : \mathcal{I} \longrightarrow N_{\mathrm{thick}(\varepsilon)}\}$ converges uniformly to a continuous map $f' : \mathcal{I} \longrightarrow N_{\mathrm{thick}(\varepsilon)} \subset N$. Since $K_n \searrow 1$ and f'_n is a locally isometric immersion in each ideal 3-simplex of G_n, for each simplex $\Delta_{i,\mathrm{inn}(\delta)}$ in $\mathcal{I}$, $f'|\Delta_{i,\mathrm{inn}(\delta)}$ is a locally isometric immersion. This implies that a holonomy $\rho_\alpha : \pi_1(H_\alpha) \longrightarrow \mathrm{Isom}^+(\mathbf{H}^3)$ for $\alpha \in \{1, \ldots, \mu\}$ is the composition $\rho_N \circ (f'|H_\alpha)_*$ of a holonomy $\rho_N : \pi_1(N) \longrightarrow \mathrm{Isom}^+(\mathbf{H}^3)$ of N and the induced homomorphism $(f'|H_\alpha)_* : \pi_1(H_\alpha) \longrightarrow \pi_1(N)$. In particular, the image of ρ_α is a discrete subgroup of $\mathrm{Isom}^+(\mathbf{H}^3)$ which does not contain any elliptic elements. Furthermore, the sequence $\{\rho_{\alpha,n} \circ (h_n|H_\alpha)_*\}$ converges algebraically to ρ_α in $\mathrm{Hom}(\pi_1(H_\alpha), \mathrm{Isom}^+(\mathbf{H}^3))/\mathrm{conj}$, where $\rho_{\alpha,n}$ is a holonomy of $H_{\alpha,n}$. Let τ be the turning section of $\mathcal{I}$, and set $\tau_\alpha = \tau \cap H_\alpha$. Any component R of

$\partial H_\alpha - \tau_\alpha$ is an open annulus. For a generator γ of $\pi_1(R)$, $\rho_\alpha(\gamma)$ fixes a geodesic line L in $\mathbf{H}^3$. Since $\mathrm{Im}(\rho_\alpha)$ contains no elliptic elements, ρ_α is either trivial or loxodromic, where a hyperbolic element of $\mathrm{Isom}^+(\mathbf{H}^3)$ is regarded as a loxodromic element of special type. Let τ_n be the turning section of $\mathcal{I}_n$, and let R_n be the component of $\partial\mathcal{I}_n - \tau_n$ corresponding to R.

Let us suppose that $\rho_\alpha(\gamma)$ is trivial. Take a simple, oriented loop l in R representing γ. The triviality of $\rho_\alpha(\gamma)$ and the injectivity of ρ_N imply that $f'|l$ is contractible in $N_{\mathrm{thick}(\varepsilon)}$. Thus, $f'_n \circ h_n|l$ is also contractible in $N_{n,\mathrm{thick}(\varepsilon)} \subset N_n$ for any sufficiently large $n \in \mathbf{N}$. This shows that $\rho_{\alpha,n} \circ (h_n)_*(\gamma)$ is trivial. Let R_0 and $\mathcal{C}_{R_0}$ be respectively a (compact) annulus in R_n and a set of δ-microchips given as in Lemma 4. Cut G_n open along R_0, and denote the cut complex by $G_n^\vee$. Consider the annuli R_0^+ and R_0^- in $\partial G_n^\vee$ corresponding to R_0 with $R_0^+ \subset \partial\mathcal{I}_n$ and $R_0^- \subset \partial\mathcal{O}_n$. Glue two copies E^+ and E^- of $\sqcup\mathcal{C}_{R_0}$ to $\mathcal{I}_n$ and $\mathcal{O}_n$ respectively by the identity map of R_0. Next, we identify the top and the bottom of E^+ with those of E^-. Topologically, this operation is the "0-surgery" done by attaching the solid torus $E^+ \cup E^-$ to $G_n^\vee$ along the boundary torus $R_0^+ \cup R_0^-$.

We perform the same process for all components R_n of $\partial\mathcal{I}_n - \tau_n$ with trivial holonomy, and denote the resulting manifold by $G_n^\bullet$. The continuous map $f'_n : G_n \longrightarrow N_n$ can be extended to a continuous map $f_n^\bullet : G_n^\bullet \longrightarrow N_n$ by using $p_n \circ d_{R_n} : \sqcup\mathcal{C}_{R_n} \longrightarrow N_n$, where d_{R_n} is the map given in Lemma 4. We denote the parts in $G_n^\bullet$ obtained from $\mathcal{I}_n$ (resp. $\mathcal{O}_n$) by adding the solid cylinders as above by $\mathcal{I}_n^\bullet$ (resp. $\mathcal{O}_n^\bullet$). The δ-microchip decomposition on $\mathcal{O}_n^\bullet$ is the union of the δ-microchip decompositions on all E^-'s and that on $\mathcal{O}_n$. The manifold $\mathcal{I}^\bullet$ and the continuous map $f^\bullet : \mathcal{I}^\bullet \longrightarrow N_{\mathrm{thick}(\varepsilon)}$ are defined from $\mathcal{I}$, f' and d_{R_n} similarly so that the sequence $\{g_n \circ f_n^\bullet \circ h_n^\bullet|\mathcal{I}^\bullet : \mathcal{I}^\bullet \longrightarrow N_{\mathrm{thick}(\varepsilon)}\}$ converges uniformly to $f^\bullet$, where $h_n^\bullet : \mathcal{I}^\bullet \cup \mathcal{Z} \longrightarrow \mathcal{I}_n^\bullet \cup \mathcal{Z}_n$ is a homeomorphism extending h_n. Clearly, a holonomy $\rho^\bullet_{\alpha,(n)} : \pi_1(H^\bullet_{\alpha,(n)}) \longrightarrow \mathrm{Isom}^+(\mathbf{H}^3)$ is well defined for each component $H^\bullet_{\alpha,(n)}$ of $\mathcal{I}^\bullet_{(n)}$.

We note that, if the δ-inner parts of all $\Delta_{i,n}$'s are connected, then each $\sqcup\mathcal{C}_{R_0}$ can be regarded as a submanifold of $\mathcal{O}_n$, which corresponds to the ready-made case in Lemma 4. Thus, we do not need the 0-surgery trick, and only change the post of $\sqcup\mathcal{C}_{R_0}$ from $\mathcal{O}_n$ to $\mathcal{I}_n$. However, our argument can not skip the case where some of $\Delta_{i,n}$'s have disconnected inner parts. In this case, $\sqcup\mathcal{C}_{R_0}$ is not necessarily contained in $\mathcal{O}_n$, and hence the 0-surgery trick is crucial.

For any component Σ of $\partial\mathcal{I}^\bullet$ with $\Sigma \subset \partial H_\alpha^\bullet$, we set $\tau_\Sigma = \tau \cap \Sigma$.

Then, Σ is obtained by capping off the boundary of $\tau_\Sigma \cup R_1 \cup \cdots \cup R_k$ by capping disks $W_1, \ldots, W_l$, where each R_j is a component of $\partial\mathcal{I} - \tau$ contained in Σ under the inclusion $\partial\mathcal{I} \longrightarrow \mathcal{I} \subset \mathcal{I}^\bullet$. We need to consider the following two cases.

Case 1. $\rho_\alpha^\bullet(\pi_1(\Sigma))$ contains a loxodromic element.

First, we will show that, for any component τ_j of τ_Σ, $\rho_\alpha^\bullet(\pi_1(\tau_j))$ has a fixed point in S^2_∞. Let $p : \widetilde{G}^\circ \longrightarrow G^\circ$ be the universal covering and $d : \widetilde{G}^\circ \longrightarrow \mathbf{H}^3$ the developing map. For any Δ_k° in G° with $\Delta_k^\circ \cap \tau_j \neq \emptyset$, consider the subjoint $J_{k;j}$ in Δ_k° with $\partial J_{k;j} \supset \Delta_k^\circ \cap \tau_j$ just like J_{11} in Figure 2.2. Since τ_j is connected, there exists an ideal vertex v of G° such that all $J_{k;j}$ meet a "small neighborhood" of v in G°. We set $J_j = \cup_k J_{k;j}$ and let $\widetilde{J}_j$ be a component of $p^{-1}(J_j)$. We can rearrange the position of the image $d(\widetilde{J}_j)$ in $\mathbf{H}^3$ by an isometry on $\mathbf{H}^3$ so that, for all components $\widetilde{J}_{l,k;j}$ of $p^{-1}(J_{k;j}) \cap \widetilde{J}_j$, the images $d(\widetilde{J}_{l,k;j})$ have $\infty \in S^2_\infty$ as a common ideal vertex. Then, $\rho_\alpha^\bullet(\pi_1(\tau_j))$ fixes ∞.

For each R_j, $\rho_\alpha^\bullet(\pi_1(R_j))$ is non-trivial and fixes a geodesic line in $\mathbf{H}^3$. These facts together with the discreteness of $\rho_\alpha^\bullet(\pi_1(\Sigma))$ imply that $\rho_\alpha^\bullet(\pi_1(\Sigma))$ is a cyclic subgroup of $\mathrm{Isom}^+(\mathbf{H}^3)$ generated by a loxodromic element γ. Let $q : \widetilde{N} \longrightarrow N$ be the cyclic covering associated to $f_*^\bullet(\pi_1(\Sigma)) \subset \pi_1(N)$, and let $\widetilde{f}_\Sigma^\bullet : \Sigma \longrightarrow \widetilde{N}$ be a lift of $f^\bullet|\Sigma$.

By Lemmas 1 (iii) and 4 (iv), we may choose $0 < \delta < \delta_1$ so that $\widetilde{f}_\Sigma^\bullet(\Sigma) \subset \mathcal{N}_1(c, \widetilde{N})$, where c is the geodesic core of the open solid torus $\widetilde{N}$. There exists a compact, connected, orientable 3-manifold L with $\partial L = \Sigma$ and a continuous map $\widetilde{f}_L^\bullet : L \longrightarrow \mathcal{N}_1(c, \widetilde{N})$ extending $\widetilde{f}_\Sigma^\bullet$. We set $f_L^\bullet = q \circ \widetilde{f}_L^\bullet : L \longrightarrow N$. Cut $G_n^\bullet$ open along $\Sigma_n = h_n^\bullet(\Sigma)$ and attach two copies L_n^+, L_n^- of L to the cut manifold $G_n^{\bullet\vee}$ by the homeomorphisms $h_n|\Sigma : \Sigma \longrightarrow \Sigma_n^\pm$, where $\Sigma_n^+ \subset \partial\mathcal{I}_n^\bullet$ and $\Sigma_n^- \subset \partial\mathcal{O}_n^\bullet$ are the boundary components of $\partial G_n^{\bullet\vee}$ corresponding to Σ_n. The restriction $f_n^\bullet|\Sigma_n^\pm = f_n^\bullet|\Sigma_n : \Sigma_n^\pm \longrightarrow N_n$ can be extended to a continuous map $f_{L,n}^\bullet : L_n^+ \cup L_n^- \longrightarrow N_n$ so that the sequence $\{g_n \circ f_{L,n}^\bullet|L_n^\pm\}$ converges uniformly to $f_L^\bullet$. We have not defined any microchip decomposition on L_n^-, and so $f_{L,n}$ may not be locally isometric on L_n^-.

Case 2. $\rho_\alpha^\bullet(\pi_1(\Sigma))$ is either trivial or contains a parabolic element.

Then, Σ is the union of τ_Σ and $W_1, \ldots, W_l$, and contains no open annulus components R_j. By Lemmas 1 (i) and 4 (iv), the diameter of Σ is bounded, that is, $\mathrm{diam}(\Sigma)$ depends only on δ_1 and m and is independent of δ. If $\rho_\alpha^\bullet(\pi_1(\Sigma))$ and hence $\rho_{\alpha,n}^\bullet(\pi_1(\Sigma_n))$ are trivial, then there exist handlebodies A, A_n bounded by Σ, Σ_n respectively, and continuous maps $f_A^\bullet : A \longrightarrow N$, $f_{A,n}^\bullet : A_n \longrightarrow N_n$ extending $f_{(n)}^\bullet|\Sigma_{(n)}$ such that

(4.5) $f_{A,n}^\bullet(A_n)$ is contained in the $c(\delta)$-neighborhood of a geodesic segment in N_n of bounded length with $\lim_{\delta\to 0} c(\delta) = 0$.

Cut open $G_n^\bullet$ along Σ_n and attach two copies A_n^+, A_n^- of A_n to the cut manifold $G_n^{\bullet\vee}$ as in Case 1 so that $\partial A^+ = \partial\Sigma_n^+ \subset \partial\mathcal{I}_n^\bullet$ and $\partial A_n^- = \partial\Sigma_n^- \subset \partial\mathcal{O}_n^\bullet$.

We perform the same process for any component Σ_n of $\mathcal{I}_n^\bullet$ with either loxodromic or trivial holonomy, and denote the resulting manifold by $G_n^{\bullet\bullet}$. Let $\widehat{\mathcal{I}}_{(n)}$ be the complex obtained by attaching $L_{(n)}^+$'s and $A_{(n)}^+$'s to $\mathcal{I}_{(n)}^\bullet$. The union of such $L_n^\pm$'s (resp. $A_n^\pm$'s) in $G_n^{\bullet\bullet}$ are denoted by $\mathcal{L}_n^\pm$ (resp. $\mathcal{A}_n^\pm$). The complement $G_n^{\bullet\bullet} - \widehat{\mathcal{I}}_n \cup \mathcal{Z}_n$ contains the ideal points corresponding to the vertices of $\mathcal{D}$. Excise a small end of each ideal point and attach a 3-ball B_n^-, and denote the resulting manifold by $\widehat{\mathcal{O}}_n$. The union $\widehat{G}_n = \widehat{\mathcal{I}}_n \cup \widehat{\mathcal{O}}_n \cup \mathcal{Z}_n$ is a closed, orientable 3-manifold. Extended continuous maps $\widehat{f}_n : \widehat{G}_n \longrightarrow N_n$ and $\widehat{f} : \widehat{\mathcal{I}} \longrightarrow N$ are defined naturally by using $f_{(n)}^\bullet, f_{L,(n)}^\bullet, f_{A,(n)}^\bullet$ so that it satisfies

$$\lim_{\delta\to 0}\sup_n\{\operatorname{diam}(\widehat{f}_n(B_n^-))\} = 0. \tag{4.6}$$

The union of such 3-balls B_n^-'s in $\widehat{G}_n$ is denoted by $\mathcal{B}_n^-$. In our construction, it is easily seen that there exists a homeomorphism $\widehat{h}_n : \widehat{\mathcal{I}} \cup \mathcal{Z} \longrightarrow \widehat{\mathcal{I}}_n \cup \mathcal{Z}_n$ extending $h_n^\bullet$, and a homeomorphism $\widehat{\eta}_{n,n'} : \widehat{G}_n \longrightarrow \widehat{G}_{n'}$ extending $\widehat{h}_{n'} \circ \widehat{h}_n^{-1}$ for all sufficiently large $n, n' \in \mathbf{N}$.

Now, we are ready to prove our main theorem.

Proof of Theorem. Since our modifications as above have been done in small volume parts in G_n, we have $\deg(\widehat{f}_n) = \deg(f_n) \neq 0$. The closure $\mathcal{P}_n$ of the complement $\widehat{\mathcal{O}}_n - \mathcal{L}_n^- \cup \mathcal{A}_n^- \cup \mathcal{B}_n^-$ is a union of δ-microchips C such that $\widehat{f}_n|C : C \longrightarrow N_n$ is a locally isometric immersion. We denote the set of such δ-microchips by $\mathcal{C}_n$.

First, we choose $\lambda_0 > 0$ such that $N_{\mathrm{thin}(\lambda_0)}$ consists of parabolic cusps disjoint from $\widehat{f}(\mathcal{L}^+)$. Consider the rectangle

$$R = \left\{ z \in \mathbf{C}; 0 \leq \operatorname{Re}(z) \leq 2\tanh\left(\frac{\lambda_0}{2}\right), 1 \leq \operatorname{Im}(z) \leq e \right\}$$

in the upper plane model for $\mathbf{H}^2$. The distance in $\mathbf{H}^2$ between the two horizontal sides of R are 1, and the distance between the two vertices of imaginary height 1 is λ_0. We choose λ_1, $0 < \lambda_1 < \lambda_0$ so that

$\mathrm{dist}_N(\partial N_{\mathrm{thin}(\lambda_0)} - \partial N_{\mathrm{thin}(\lambda_1)}) = 1$. By an elementary argument of hyperbolic geometry, λ_1 is the number satisfying $e \tanh(\lambda_1/2) = \tanh(\lambda_0/2)$. Note that $\lim_{n\to\infty} \mathrm{dist}_{N_n}(\partial N_{n,\mathrm{thin}(\lambda_0)} - \partial N_{n,\mathrm{thin}(\lambda_1)}) = 1$. For any $\varepsilon > \varepsilon' > 0$, we set $N_{n,\mathrm{thin}(\varepsilon;\varepsilon')} = N_{n,\mathrm{thin}(\varepsilon)} - \mathrm{int}N_{n,\mathrm{thin}(\varepsilon')}$. We know that essential annuli $(Q_n, \partial Q_n)$ of least area in $(N_{n,\mathrm{thin}(\lambda_0;\lambda_1)}, \partial N_{n,\mathrm{thin}(\lambda_0;\lambda_1)})$ satisfy $\lim_{n\to\infty} \mathrm{Area}(Q_n) = \mathrm{Area}(R)$, and hence $\mathrm{Area}(Q_n) > \mathrm{Area}(R)/2$ for all sufficiently large $n \in \mathbf{N}$.

For each component Σ of $\partial \widehat{H}_\alpha \subset \partial H_\alpha^\bullet$, $\rho_\alpha^\bullet(\pi_1(\Sigma))$ is a subgroup of $\mathrm{Isom}^+(\mathbf{H}^3)$ generated by parabolic elements. Since $\mathrm{diam}(\Sigma)$ is bounded and $\widehat{f}(\Sigma)$ converges as $\delta \searrow 0$ to the parabolic cusp of N corresponding to the fixed point of $\rho_\alpha^\bullet(\pi_1(\Sigma))$, one can choose $\delta > 0$ so that $\widehat{f}(\partial\widehat{\mathcal{I}})$ is contained in $\mathrm{int}N_{\mathrm{thin}(\lambda_1)}$. Thus, for all sufficiently large $n \in \mathbf{N}$, we have $\widehat{f}_n(\partial\widehat{\mathcal{I}}_n) \subset \mathrm{int}N_{n,\mathrm{thin}(\lambda_1)}$ and $\widehat{f}_n(\mathcal{L}_n^-) \cap N_{n,\mathrm{thin}(\lambda_0)} = \widehat{f}_n(\mathcal{L}_n^+) \cap N_{n,\mathrm{thin}(\lambda_0)} = \emptyset$; see Figure 4.1.

FIGURE 4.1. The shaded region represents $\widehat{f}_n(\widehat{\mathcal{O}}_n) \cup \mathcal{Z}_n)$

Moreover, the δ can be chosen so that

$$\mathrm{Area}(\partial\mathcal{C}_n) < \frac{\mathrm{Area}(R)}{2}.$$

By (4.3) and (4.4), for the fixed $\delta > 0$ as above, there exist $0 < \varepsilon < \lambda_1$ and $n_2 \in \mathbf{N}$ such that $\widehat{f}_n(\widehat{\mathcal{I}}_n) \cap N_{n,\mathrm{thin}(\varepsilon)} = \emptyset$ and $\widehat{f}_n(\mathcal{Z}_n) \subset \mathrm{int}N_{n,\mathrm{thin}(\varepsilon)}$

for all $n \in \mathbf{N}$ with $n \geq n_2$. For a fixed point $x_n \in N_{n,\mathrm{thick}(\lambda_0)}$, let Γ_n be a star in $N_{n,\mathrm{thick}(\varepsilon)}$ connecting x_n with the components of $\partial N_{n,\mathrm{thick}(\varepsilon)} = \partial N_{n,\mathrm{thin}(\varepsilon)}$.

Let $\mathcal{F}$ be the foliation on $N_{n,\mathrm{thin}(\lambda_0;\lambda_1)}$ consisting of equidistant surfaces from $\partial N_{n,\mathrm{thin}(\lambda_0)}$. Each leaf F of $\mathcal{F}_n$ consists of r tori. Except finitely many leaves of $\mathcal{F}$, for any $C \in \mathcal{C}_n$, $\widehat{f}_n|\partial C$ meets transversely all other leaves F, called *generic leaves.* In particular, $\Gamma_F = \widehat{f}_n^{-1}(F) \cap \sqcup(\partial\mathcal{C}_n)$ is a (possibly disconnected) 1-dimensional CW-complex such that, for a sufficiently small, saturated neighborhood $\mathcal{N}_F$ of F, $\widehat{f}_n^{-1}(\mathcal{N}_F) \cap \sqcup(\partial\mathcal{C}_n)$ is homeomorphic to $\Gamma_F \times [0,1]$, where $\sqcup(\partial\mathcal{C}_n)$ denotes the "foam" $\cup_{C\in\mathcal{C}_n}\partial C$. For all generic leaves F of $\mathcal{F}_n$, if Γ_F had a simple loop l such that $\widehat{f}_n|l$ is non-contractible in F, then

$$\begin{aligned}\mathrm{Area}(\partial\mathcal{C}_n) >& \mathrm{Area}(\sqcup(\partial\mathcal{C}_n)) > \mathrm{Area}(\sqcup(\partial\mathcal{C}_n) \cap \widehat{f}_n^{-1}(N_{n,\mathrm{thin}(\lambda_0;\lambda_1)})) \\ >& \frac{\mathrm{Area}(R)}{2}.\end{aligned}$$

This contradiction implies that there exists a generic leaf F such that the restrictions of $\widehat{f}_n$ to any components of Γ_F are contractible. Since $\widehat{f}_n^{-1}(F) \cap \mathcal{P}_n$ is a union of disks bounded by simple loops in Γ_F, the restrictions of $\widehat{f}_n$ to any components of $\widehat{f}_n^{-1}(F) \cap \mathcal{P}_n$ are contractible in F. By (4.5), we have $\lim_{\delta\to 0} \mathrm{diam}(\widehat{f}_n(A_n^-) \cap F) = 0$ for each component A_n^- of $\mathcal{A}_n^-$. By this fact together with (4.6), one can choose $\delta > 0$ so that the restrictions of $\widehat{f}_n$ to any components of $\widehat{f}_n^{-1}(F) \cap \widehat{\mathcal{O}}_n$ are contractible in F. Then, we will construct a continuous map $\psi_n : \widehat{G}_n \longrightarrow N_n$ as follows. The map ψ_n takes all components O of $\widehat{\mathcal{O}}_n - \widehat{f}_n^{-1}(\mathcal{N}_F)$ with $\widehat{f}_n(O) \subset N_{n,\mathrm{thick}(\lambda_0)} \cup \mathcal{K}_n$ to x_n, where $\mathcal{K}_n$ is the union of components of $N_{n,\mathrm{thin}(\lambda_0)} - F$ disjoint from $N_{n,\mathrm{thin}(\lambda_1)}$. Moreover, ψ_n squeezes each component of $\widehat{f}_n^{-1}(\mathcal{N}_F) \cap \widehat{\mathcal{O}}_n$ in $\mathcal{N}_F$ and then stretches it so as to connect the remaining components of $\widehat{\mathcal{O}}_n - \widehat{f}_n^{-1}(\mathcal{N}_F)$ with x_n in $\mathcal{K}_n \cup \Gamma_n$; see Figure 4.2 (a).

FIGURE 4.2

Note that, for the component U of $\widehat{G}_n - \widehat{f}_n^{-1}(\mathcal{N}_F) \cap \widehat{\mathcal{O}}_n$ meeting either $\widehat{\mathcal{I}}_n$ or $\mathcal{Z}_n$ non-trivially, $\psi_n|U$ is equal to $\widehat{f}_n|U$. Though in general ψ_n is not homotopic to $\widehat{f}_n$, the equality $\deg(\psi_n) = \deg(\widehat{f}_n)$ still holds.

Consider a small regular neighborhood $\mathcal{N}$ of $\partial\widehat{\mathcal{I}}_n$ in $\widehat{G}_n$, which is homeomorphic to $\partial\widehat{\mathcal{I}}_n \times [0,1]$. We deform $\psi_n|\mathcal{N}$ in $N_{n,\mathrm{thin}(\lambda_1)}$ by a homotopy rel. $\partial\mathcal{N}$ so that the resulting map $\psi'_{\mathcal{N}}$ meets $\partial N_{n,\mathrm{thin}(\varepsilon)}$ transversely and $\psi'_{\mathcal{N}}(\widehat{\mathcal{I}}_n \cap \mathcal{N}) \cap N_{n,\mathrm{thin}(\varepsilon)} = \psi'_{\mathcal{N}}(\partial\widehat{\mathcal{I}}_n)$. Let $\psi'_n : \widehat{G}_n \longrightarrow N_n$ be the continuous map defined by $\psi'_n|(\widehat{G}_n - \mathrm{int}\mathcal{N}) = \psi_n|(\widehat{G}_n - \mathrm{int}\mathcal{N})$ and $\psi'_n|\mathcal{N} = \psi'_{\mathcal{N}}$; see Figure 4.2 (b). Let $\widehat{\psi}_n : \widehat{G}_n \longrightarrow N_n$ be the continuous map pushing $\widehat{\mathcal{O}}_n \cap \psi_n'^{-1}(N_{n,\mathrm{thin}(\lambda_0;\varepsilon)})$ into $N_{n,\mathrm{thin}(\varepsilon)} \cup \Gamma_n$ by a deformation retract from $N_{n,\mathrm{thin}(\lambda_0)}$ to $N_{n,\mathrm{thin}(\varepsilon)}$; see Figure 4.2 (c). Thus, $\widehat{\psi}_n$ takes $(\widehat{\mathcal{I}}_n, \partial\widehat{\mathcal{I}}_n)$ to $(N_{n,\mathrm{thick}(\varepsilon)}, \partial N_{n,\mathrm{thick}(\varepsilon)})$ and $(\widehat{\mathcal{O}}_n \cup \mathcal{Z}, \partial(\widehat{\mathcal{O}}_n \cup \mathcal{Z}))$ to $(N_{n,\mathrm{thin}(\varepsilon)} \cup \Gamma_n, \partial N_{n,\mathrm{thin}(\varepsilon)})$. Since the sequence $\{g_n \circ \widehat{f}_n \circ \widehat{h}_n|\widehat{\mathcal{I}} : \widehat{\mathcal{I}} \longrightarrow N_{\mathrm{thick}(\varepsilon)}\}$ converges uniformly to $\widehat{f} : \widehat{\mathcal{I}} \longrightarrow N_{\mathrm{thick}(\varepsilon)}$, for all sufficiently large $n, n' \in \mathbf{N}$ with $n \neq n'$, $g_n \circ \widehat{f}_n \circ \widehat{h}_n|(\widehat{\mathcal{I}}, \partial\widehat{\mathcal{I}}) : (\widehat{\mathcal{I}}, \partial\widehat{\mathcal{I}}) \longrightarrow (N_{\mathrm{thick}(\varepsilon)}, N_{\mathrm{thin}(\lambda_1;\varepsilon)})$ is homotopic to $g_{n'} \circ \widehat{f}_{n'} \circ \widehat{h}_{n'}|(\widehat{\mathcal{I}}, \partial\widehat{\mathcal{I}})$ in $(N_{\mathrm{thick}(\varepsilon)}, N_{\mathrm{thin}(\lambda_1;\varepsilon)})$. Therefore, one can modify $\widehat{\psi}_n, \widehat{\psi}_{n'}$ slightly so that $g_n \circ \widehat{\psi}_n \circ \widehat{h}_n|\partial\widehat{\mathcal{I}} = g_{n'} \circ \widehat{\psi}_{n'} \circ \widehat{h}_{n'}|\partial\widehat{\mathcal{I}}$. Then, by Lemma 5, $g_{n'}^{-1} \circ g_n : N_{n,\mathrm{thick}(\varepsilon)} \longrightarrow N_{n',\mathrm{thick}(\varepsilon)}$ would be extended to a homeomorphism $N_n \longrightarrow N_{n'}$, a contradiction. This completes our reduction to absurdity and hence the proof of Theorem. q.e.d.

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