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PU(2) MONOPOLES. I: REGULARITY, UHLENBECK COMPACTNESS, AND TRANSVERSALITY

PAUL M. N. FEEHAN & THOMAS G. LENESS

1. Introduction

At seminars at Harvard and MIT, during October 1994, Edward Witten introduced the U(1) monopole equations and the Seiberg-Witten invariants to smooth four-manifold topology and conjectured their relationship with Donaldson invariants on the basis of new developments in quantum field theory [19], [98]. The conjecture, recently extended in [60], has been verified for all four-manifolds whose Donaldson and Seiberg-Witten invariants have been independently computed. Within two months of Witten's announcement, a program was outlined by V. Pidstrigach and A. Tyurin and others, which should lead to a mathematical proof of the relationship between these two invariants [68], [71], [74]. This approach is unrelated to the quantum field-theoretic arguments of [60], [98] and uses a moduli space of PU(2) monopoles to construct a cobordism between links of Seiberg-Witten moduli spaces of U(1) monopoles and the Donaldson moduli space of anti-self-dual connections, which appear as singularities in this larger stratified moduli space.

It was soon recognized, however, that despite the appeal and elegance of the PU(2) monopole program, its implementation involves substantial technical difficulties due to the contributions of moduli spaces of U(1) monopoles in the lower levels of the Uhlenbeck compactification of the moduli space of PU(2) monopoles. Many of these difficulties had

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never been resolved even in the case of Donaldson theory where similar problems arise, albeit in a rather simpler form, in attempts to prove the Kotschick-Morgan conjecture for Donaldson invariants of four-manifolds X with $b^+(X) = 1$. That conjecture asserts that the Donaldson invariants computed using metrics lying in different chambers of the positive cone of $H^2(X;\mathbb{R})/\mathbb{R}^*$ differ only by homotopy-invariant terms [45]. In the case of the Kotschick-Morgan conjecture, the heart of the problem lies in describing the links of the lower-level reducibles via gluing and then in calculating the pairings of the Donaldson cohomology classes with those links. Thus far, such links have been described and their pairings with cohomology classes computed only in certain special cases [14], [15], [16], [20], [21], [32], [58], [99]. The methods used to obtain these special cases fall very far short of the kind of general analysis needed to prove the Kotschick-Morgan conjecture. On the other hand, by assuming the Kotschick-Morgan conjecture, L. Göttsche computed the coefficients of the wall-crossing formula in [45] in terms of modular forms by exploiting the presumed homotopy invariance of the coefficients [35]. A related approach to the Witten conjecture was proposed by Pidstrigach and Tyurin [72]. Certain aspects of the PU(2) monopole program have been considered from a quantum-field theoretic viewpoint in [13], [38], [39], [50], [51], [52], [53].

In the present article and its sequels [25], [26], [27] we address the analytical problems associated with constructing the links of lower-level Seiberg-Witten moduli spaces and in establishing the analogues of the Kotschick-Morgan conjecture for PU(2) monopoles needed to compute the pairings of cohomology classes with these links. We hope to return to the actual computations and a verification of Witten's conjecture [60], [98] in a subsequent paper. In this article we describe the basic regularity, Uhlenbeck compactness, and transversality results we need for the moduli space of PU(2) monopoles, and in the sequels [26], [27] we develop the gluing theory required to construct the links of the lower-level Seiberg-Witten moduli spaces. An announcement of the main results of this article appeared in [24].

1.1. Statement of results

1.1.1. PU(2) monopoles and holonomy perturbations.

We consider Hermitian two-plane bundles E over X whose determinant line bundles $\det E$ are isomorphic to a fixed Hermitian line bundle over X endowed with a fixed C^∞, unitary connection. Let (ρ, W^+, W^-) be a spin^c structure on X, where $\rho : T^*X \to \operatorname{End} W$ is the Clifford map, and the Hermitian four-plane bundle $W = W^+ \oplus W^-$ is endowed with a C^∞ spin^c connection.

Let $k \geq 3$ be an integer and let $\mathcal{A}_E$ be the space of L^2_k connections A on the U(2) bundle E all inducing the fixed determinant connection on $\det E$. Equivalently, following [48, §2(i)], we may view $\mathcal{A}_E$ as the space of L^2_k connections A on the PU(2) = SO(3) bundle $\mathfrak{su}(E)$. We pass back and forth between these viewpoints, via the fixed connection on $\det E$, and rely on the context to make the distinction clear. Given a connection A on $\mathfrak{su}(E)$ with curvature $F_A \in L^2_{k-1}(\Lambda^2 \otimes \mathfrak{so}(\mathfrak{su}(E)))$, then $\mathrm{ad}^{-1}(F_A^+) \in L^2_{k-1}(\Lambda^+ \otimes \mathfrak{su}(E))$ is its self-dual component, viewed as a section of $\Lambda^+ \otimes \mathfrak{su}(E)$ via the isomorphism $\mathrm{ad} : \mathfrak{su}(E) \to \mathfrak{so}(\mathfrak{su}(E))$. When no confusion can arise, the isomorphism $\mathrm{ad} : \mathfrak{su}(E) \to \mathfrak{so}(\mathfrak{su}(E))$ will be implicit, and so we regard F_A as a section of $\Lambda^+ \otimes \mathfrak{su}(E)$ when A is a connection on $\mathfrak{su}(E)$. Let $D_A : L^2_k(W^+ \otimes E) \to L^2_{k-1}(W^- \otimes E)$ be the corresponding Dirac operator.

For an L^2_k section Φ of $W^+ \otimes E$, let $\Phi^* = \langle \cdot, \Phi \rangle$ be its pointwise Hermitian dual and let $(\Phi \otimes \Phi^*)_{00}$ be the component of the Hermitian endomorphism $\Phi \otimes \Phi^*$ of $W^+ \otimes E$ which lies in $\mathfrak{su}(W^+) \otimes \mathfrak{su}(E)$. The Clifford multiplication ρ defines an isomorphism $\rho : \Lambda^+ \to \mathfrak{su}(W^+)$ and thus an isomorphism $\rho = \rho \otimes \mathrm{id}_{\mathfrak{su}(E)}$ of $\Lambda^+ \otimes \mathfrak{su}(E)$ with $\mathfrak{su}(W^+) \otimes \mathfrak{su}(E)$. Then

$$
\begin{aligned}
F_A^+ - \rho^{-1}(\Phi \otimes \Phi^*)_{00} &= 0, \\
D_A \Phi &= 0,
\end{aligned} \tag{1.1}
$$

are the unperturbed PU(2) monopole equations considered in [67], [68], [71], [74], with a slightly differing trace conditions (see below), for a pair (A, Φ) consisting of a connection on $\mathfrak{su}(E)$ and a section Φ of $W^+ \otimes E$.

Donaldson's proof of the connected-sum theorem for his polynomial invariants [18, Theorem B] makes use of certain 'extended anti-self-dual equations' [18, Equation (4.24)] to which the Freed-Uhlenbeck generic metrics theorem does not apply [18, §4(v)]. These extended equations model a neighborhood of the product connection appearing in the Uhlenbeck compactification of the moduli space of anti-self-dual SU(2) connections. As the zero locus of the extended equations may not be transverse, Donaldson employs holonomy perturbations which give gauge-equivariant C^∞ maps $\mathcal{A}^*_E(X) \to \Omega^+(\mathfrak{su}(E))$ and thus perturbations of the extended anti-self-dual equations [17, §2], [18, pp. 282–287]. These perturbations are continuous with respect to Uhlenbeck limits and yield transversality not only for the top-level moduli space, but also for all lower-level moduli spaces and all intersections of the geometric representatives defining the Donaldson invariants.

In §2.5.2 and the Appendix we describe a generalization of Donaldson's idea which we use to prove transversality for the moduli space of solutions to a perturbed version of the PU(2) monopole equations (1.1). Unfortunately, in the case of of PU(2) monopoles, the analysis is considerably more intricate and the method we employ here is rather different from the one developed in [18]. We use an infinite sequence of holonomy sections defined on the infinite-dimensional configuration space of pairs; when restricted to small enough open balls in the configuration space, away from reducible connections, only finitely many of these perturbing sections are non-zero and they vanish at reducible connections.

Let $\mathcal{G}_E$ be the Hilbert Lie group of L^2_{k+1} unitary gauge transformations of E with determinant one. Let S^1_Z denote the center of $\mathrm{U}(2)$ and set ${}^\circ\mathcal{G}_E := S^1_Z \times_{\{\pm\mathrm{id}_E\}} \mathcal{G}_E$, which we may view as the group of L^2_{k+1} unitary gauge transformations of E with constant determinant. The stabilizer of a unitary connection A on E in ${}^\circ\mathcal{G}_E$ (which coincides with its stabilizer in the full group $\operatorname{Aut} E$ of unitary automorphisms of E) always contains the center $S^1_Z \subset \mathrm{U}(2)$, corresponding to the constant, central, unitary automorphisms of E. We call A *irreducible* if its stabilizer is exactly S^1_Z and *reducible* otherwise.

It is also possible, as in [71], [74], to fix a smooth representative $\omega \in \Omega^2(X, \mathbb{R})$ for $c_1(E)$ and instead consider the space of unitary connections A on E satisfying the trace condition $\operatorname{tr} F_A = -2\pi i\omega$, modulo the action of the full group $\operatorname{Aut} E$ of unitary automorphisms of E. The resulting moduli space of nonabelian monopoles is then a torus bundle over the moduli space which we define below, with fibers $H^1(X;\mathbb{R})/H^1(X;\mathbb{Z})$. These tori complicate the analysis of the links of singularities when $b^1(X) > 0$ and do not contain any additional information, so we choose to eliminate them by instead imposing the stronger fixed-determinant-connection trace condition and working with a compatible group of gauge transformations. A similar framework is used in [68], [92].

We refer to §2.5 for a detailed account of the construction of our holonomy perturbations. The large number of technical points involving regularity and uniform estimates for these perturbations (which still allow us to obtain an Uhlenbeck compactification) are discussed in the Appendix. We fix $r \geq k+1$ and define gauge-equivariant C^∞ maps (see

§2.5),

$$\begin{aligned}
&\mathcal{A}_E(X) \to L^2_{k+1}(X, \mathfrak{gl}(\Lambda^+) \otimes_{\mathbb{R}} \mathfrak{so}(\mathfrak{su}(E))), \\
&\qquad\qquad A \mapsto \vec{\tau} \cdot \vec{\mathfrak{m}}(A), \\
&\mathcal{A}_E(X) \to L^2_{k+1}(X, \mathrm{Hom}(W^+, W^-) \otimes_{\mathbb{C}} \mathfrak{sl}(E)), \\
&\qquad\qquad A \mapsto \vec{\vartheta} \cdot \vec{\mathfrak{m}}(A),
\end{aligned} \tag{1.2}$$

where $\vec{\tau} := (\tau_{j,l,\alpha})$ is a suitably convergent sequence in $C^r(X, \mathfrak{gl}(\Lambda^+))$, and $\vec{\vartheta} := (\vartheta_{j,l,\alpha})$ is a suitably convergent sequence in $C^r(X, \Lambda^1 \otimes \mathbb{C})$, while $\vec{\mathfrak{m}}(A) := (\mathfrak{m}_{j,l,\alpha}(A))$ is a sequence in $L^2_{k+1}(X, \mathfrak{su}(E))$ of holonomy sections constructed by extending the method of [17], [18], and

$$\begin{aligned}
\vec{\tau} \cdot \vec{\mathfrak{m}}(A) &:= \sum_{j,l,\alpha} \tau_{j,l,\alpha} \otimes_{\mathbb{R}} \mathrm{ad}(\mathfrak{m}_{j,l,\alpha}(A)), \\
\vec{\vartheta} \cdot \vec{\mathfrak{m}}(A) &:= \sum_{j,l,\alpha} \rho(\vartheta_{j,l,\alpha}) \otimes_{\mathbb{C}} \mathfrak{m}_{j,l,\alpha}(A).
\end{aligned}$$

We call a point (A, Φ) in the pre-configuration space of L^2_k pairs $\tilde{\mathcal{C}}_{W,E} := \mathcal{A}_E \times L^2_k(W^+ \otimes E)$ a PU(2) *monopole* if it solves

$$\begin{aligned}
F^+_A - (\mathrm{id} + \tau_0 \otimes \mathrm{id}_{\mathfrak{su}(E)} + \vec{\tau} \cdot \vec{\mathfrak{m}}(A))\rho^{-1}(\Phi \otimes \Phi^*)_{00} &= 0, \\
D_A\Phi + \rho(\vartheta_0)\Phi + \vec{\vartheta} \cdot \vec{\mathfrak{m}}(A)\Phi &= 0.
\end{aligned} \tag{1.3}$$

We let $M_{W,E}$ be the moduli space of solutions cut out of the configuration space of pairs $\mathcal{C}_{W,E} := \tilde{\mathcal{C}}_{W,E}/{}^\circ\mathcal{G}_E$ by the equations (1.3). We let $M^{*,0}_{W,E} \subset M_{W,E}$ be the subspace of pairs $[A, \Phi]$ such that A is irreducible and the section Φ is not identically zero. The sections $\vec{\tau} \cdot \vec{\mathfrak{m}}(A)$ and $\vec{\vartheta} \cdot \vec{\mathfrak{m}}(A)$ vanish at reducible connections A by construction; plainly, the terms in (1.3) involving the perturbations $\vec{\tau} \cdot \vec{\mathfrak{m}}(A)$ and $\vec{\vartheta} \cdot \vec{\mathfrak{m}}(A)$ are zero when Φ is zero.

1.1.2. Uhlenbeck compactness. The holonomy-perturbation maps in (1.2) are continuous with respect to the Uhlenbeck topology (see §4.5), just as those of [18]. Suppose $\{A_\beta\}$ is a sequence in $\mathcal{A}_E(X)$ which converges in the Uhlenbeck topology to a limit $(A, \mathbf{x})$ in $\mathcal{A}_{E_{-\ell}}(X) \times \mathrm{Sym}^\ell(X)$, where $E_{-\ell}$ is a Hermitian two-plane bundle on X with $\det(E_{-\ell}) = \det E$ and $c_2(E_{-\ell}) = c_2(E) - \ell$, and $\ell \geq 0$ is an integer. The sections $\vec{\tau} \cdot \vec{\mathfrak{m}}(A_\beta)$ and $\vec{\vartheta} \cdot \vec{\mathfrak{m}}(A_\beta)$ then converge in $L^2_{k+1}(X)$ to a section $\vec{\tau} \cdot \vec{\mathfrak{m}}(A, \mathbf{x})$ of $\mathfrak{gl}(\Lambda^+) \otimes_{\mathbb{R}} \mathfrak{so}(\mathfrak{su}(E_{-\ell}))$ and a section $\vec{\vartheta} \cdot \vec{\mathfrak{m}}(A, \mathbf{x})$ of

$\mathrm{Hom}(W^+, W^-) \otimes_{\mathbb{C}} \mathfrak{sl}(E_{-\ell})$, respectively. For each $\ell \geq 0$, the maps of (1.2) extend continuously to gauge-equivariant maps

$$
\begin{aligned}
&\mathcal{A}_{E_{-\ell}}(X) \times \mathrm{Sym}^\ell(X) \to L^2_{k+1}(X, \mathfrak{gl}(\Lambda^+) \otimes_{\mathbb{R}} \mathfrak{so}(\mathfrak{su}(E_{-\ell}))), \\
&\mathcal{A}_{E_{-\ell}}(X) \times \mathrm{Sym}^\ell(X) \to L^2_{k+1}(X, \mathrm{Hom}(W^+, W^-) \otimes_{\mathbb{C}} \mathfrak{sl}(E_{-\ell})),
\end{aligned} \tag{1.4}
$$

given by $(A, \mathbf{x}) \mapsto \vec{\tau} \cdot \vec{\mathfrak{m}}(A, \mathbf{x})$ and $(A, \mathbf{x}) \mapsto \vec{\vartheta} \cdot \vec{\mathfrak{m}}(A, \mathbf{x})$, which are C^∞ on each C^∞ stratum determined by $\mathrm{Sym}^\ell(X)$.

Our construction of the Uhlenbeck compactification for $M_{W,E}$ requires us to consider moduli spaces

$$
\mathbf{M}_{W,E_{-\ell}} \subset \mathcal{C}_{W,E_{-\ell}} \times \mathrm{Sym}^\ell(X)
$$

of triples $[A, \Phi, \mathbf{x}]$ given by the zero locus of the ${}^\circ\mathcal{G}_{E_{-\ell}}$-equivariant map

$$
\mathfrak{S} : \tilde{\mathcal{C}}_{W,E_{-\ell}} \times \mathrm{Sym}^\ell(X) \to L^2_k(\Lambda^+ \otimes \mathfrak{su}(E_{-\ell})) \oplus L^2_k(W^- \otimes E_{-\ell})
$$

defined as in (1.3) except using the perturbing sections $\vec{\tau} \cdot \vec{\mathfrak{m}}$ and $\vec{\vartheta} \cdot \vec{\mathfrak{m}}$ in (1.4) instead of those in (1.2). We call $\mathbf{M}_{W,E_{-\ell}}$ a *lower-level* moduli space if $\ell > 0$ and call $\mathbf{M}_{W,E_{-0}} = M_{W,E}$ the *top* or *highest level*.

In the more familiar case of the unperturbed PU(2) monopole equations (1.1), the spaces $\mathbf{M}_{W,E_{-\ell}}$ would simply be products $M_{W,E_{-\ell}} \times \mathrm{Sym}^\ell(X)$. In general, though, the spaces $\mathbf{M}_{W,E_{-\ell}}$ are not products when $\ell > 0$ due to the slight dependence of the section $\mathfrak{S}(A, \Phi, \mathbf{x})$ on the points $\mathbf{x} \in \mathrm{Sym}^\ell(X)$ through the perturbations $\vec{\tau} \cdot \vec{\mathfrak{m}}$ and $\vec{\vartheta} \cdot \vec{\mathfrak{m}}$. A similar phenomenon is encountered in [18, §4(iv)–(vi)] for the case of the extended anti-self-dual equations, where holonomy perturbations are also employed in order to achieve transversality.

We define $\bar{M}_{W,E}$ to be the Uhlenbeck closure of $M_{W,E}$ in the space of ideal PU(2) monopoles,

$$
\bigcup_{\ell=0}^N \mathbf{M}_{W,E_{-\ell}} \subset \bigcup_{\ell=0}^N \left(\mathcal{C}_{W,E_{-\ell}} \times \mathrm{Sym}^\ell(X) \right),
$$

for any integer $N \geq N_p$, where N_p is a sufficiently large integer to be specified below.

Theorem 1.1. *Let X be a closed, oriented, smooth four-manifold with C^∞ Riemannian metric, spin^c structure (ρ, W^+, W^-) with spin^c connection on $W = W^+ \oplus W^-$, and a Hermitian two-plane bundle E with unitary connection on* $\det E$. *Then there is a positive integer N_p,* *depending at most on the curvatures of the fixed connections on* W *and* $\det E$ *together with* $c_2(E)$*, such that for all* $N \geq N_p$ *the topological space* $\bar{M}_{W,E}$ *is compact, second countable, Hausdorff, and is given by the closure of* $M_{W,E}$ *in* $\cup_{\ell=0}^{N}\mathbf{M}_{W,E_{-\ell}}$.

Remark 1.2. The existence of an Uhlenbeck compactification for the moduli space of solutions to the unperturbed PU(2) monopole equations (1.1) was announced by Pidstrigach [71] and an argument was outlined in [74]. A similar argument for equations (1.1) was outlined by Okonek and Teleman in [68]. Theorem 1.1 yields the standard Uhlenbeck compactification for the system (1.1) and the perturbations of (1.1) described in [23], [93] — see Remark 1.4. An independent proof of Uhlenbeck compactness for (1.1) and certain perturbations of these equations is given in [92], [93].

1.1.3. Transversality. The space $\mathrm{Sym}^{\ell}(X)$ is smoothly stratified, the strata being enumerated by partitions of ℓ. If $\Sigma \subset \mathrm{Sym}^{\ell}(X)$ is a smooth stratum, we define

$$\mathbf{M}_{W,E_{-\ell}}|_{\Sigma} := \{[A,\Phi,\mathbf{x}] \in \mathbf{M}_{W,E_{-\ell}} : \mathbf{x} \in \Sigma\},$$

with $\mathbf{M}_{W,E_{-0}} := M_{W,E}$ when $\ell = 0$. We then have the following transversality result for equations (1.3), at least away from the solutions where the connection is reducible or the spinor vanishes. Let M_E^{asd} denote the moduli space of anti-self-dual connections on $\mathfrak{su}(E)$.

Theorem 1.3. *Let* X *be a closed, oriented, smooth four-manifold with* C^∞ *Riemannian metric, spin*^{c} *structure* (ρ, W^+, W^-) *with spin*^{c} *connection on* $W = W^+ \oplus W^-$*, and a Hermitian line bundle* $\det E$ *with unitary connection. Then there is a first-category subset of the space of* C^∞ *perturbation parameters such that the following holds. For each 4-tuple* $(\tau_0, \vartheta_0, \vec{\tau}, \vec{\vartheta})$ *in the complement of this first-category subset, the corresponding moduli space* $M_{W,E}^{*,0}$ *is a smooth manifold of the expected dimension,*

$$\begin{aligned}\dim M_{W,E}^{*,0} &= \dim M_E^{\mathrm{asd},*} + 2\,\mathrm{Ind}_{\mathbb{C}} D_A - 1 \\ &= -2p_1(\mathfrak{su}(E_{-\ell})) - \tfrac{3}{2}(e(X) + \sigma(X)) \\ &\quad + \tfrac{1}{2}p_1(\mathfrak{su}(E_{-\ell})) + \tfrac{1}{2}((c_1(W^+) + c_1(E))^2 - \sigma(X)) - 1.\end{aligned}$$

Moreover, for each integer $\ell \geq 0$*, and smooth stratum* $\Sigma \subset \mathrm{Sym}^{\ell}(X)$*, the moduli space* $\mathbf{M}_{W,E_{-\ell}}^{*,0}|_{\Sigma}$ *is a smooth manifold of the expected dimension,* $\dim \mathbf{M}_{W,E_{-\ell}}^{*,0}|_{\Sigma} = \dim M_{W,E_{-\ell}}^{*,0} + \dim \Sigma$.

Remark 1.4. Different approaches to the question of transversality for the PU(2) monopole equations (1.1) with generic perturbation parameters have been considered by Pidstrigach and Tyurin in [74] and Teleman in [92]. More recently, a new approach to transversality for (1.1) has been discovered independently by the first author [23] and Teleman [93]: the method uses only the perturbations (τ_0, ϑ_0) together with perturbations of the Riemannian metric g on X and compatible Clifford map ρ.

A choice of generic Riemannian metric on X ensures that the moduli space $M_E^{\mathrm{asd},*}$ of irreducible anti-self-dual connections on $\mathfrak{su}(E)$ is smooth and of the expected dimension [20], [30], although the points of $M_E^{\mathrm{asd},*}$ need not be regular points of $M_{W,E}^*$ as the linearization of (1.3) may not be surjective there. A choice of generic parameter τ_0 ensures that the moduli spaces $M_{W,E,L_1}^{\mathrm{red},0}$ of non-zero-section solutions to (1.3) which are reducible with respect to the splitting $E = L_1 \oplus (\det E) \otimes L_1^*$ are smooth and of the expected dimension [25]. Again, the points of $M_{W,E,L_1}^{\mathrm{red},0}$ need not be regular points of $M_{W,E}^0$ since the linearization of (1.3) may not be surjective there.

We note that related transversality and compactness issues have been recently considered in approaches to defining Gromov-Witten invariants for general symplectic manifolds [59], [75], [79].

1.2. Outline. We indicate how the remainder of our article is organized. In §2.2 we prove a slice result for the configuration space $\mathcal{C}_{W,E}$ (Proposition 2.8) while in §2.6 we describe the elliptic deformation complex for (1.3) and compute the expected dimension of $M_{W,E}$. We develop the regularity theory for a generalization of the PU(2) monopole equations (1.3) (obtained by allowing additional inhomogeneous terms) in §3: the main technical result there is that an L_1^2 solution to an inhomogeneous version of (1.3) and the Coulomb gauge equation is C^∞ (Corollary 3.4). By combining this with the slice result of Proposition 2.8, we then show that any L_k^2 PU(2) monopole (A, Φ) is L_{k+1}^2 gauge-equivalent to a C^∞ PU(2) monopole (Proposition 3.7).

We establish local estimates for L_1^2 solutions to the inhomogeneous version of (1.3) in §3.3 and §3.4. We use the sharp L_1^2 regularity result of Corollary 3.4 in §4 to prove the removability of point singularities for PU(2) monopoles (Theorem 4.10). In the sequels [26], [27], these regularity results and estimates are needed to prove that L_1^2 gluing solutions to (1.3) are C^∞ and to analyse the asymptotic behavior of Taubes' gluing maps and their differentials near the lower strata of the Uhlenbeck compactification.

The proof of Theorem 1.1 relies heavily on both the regularity theory of §3 and the fact that solutions (A, Φ) to (1.3) satisfy a 'universal energy bound', with constant depending only on the data in the hypotheses of Theorem 1.1. The section Φ also satisfies a universal C^0 bound: these bounds are the analogues for PU(2) monopoles of the now well-known *a priori* estimates for Seiberg-Witten monopoles [47], [62], [77], [98] and follow, in much the same way, from the maximum principle and the Bochner-Weitzenböck formula for D_A provided $k \geq 3$ (see §4.1). In §4 we prove the removability of point singularities for solutions to (1.3) (Theorem 4.10) using our *a priori* bounds and regularity results. While the PU(2) monopole equations are not conformally invariant, they are invariant under constant rescalings of the metric (in the sense of §4.2) and, as in the case of anti-self-dual connections, this scale invariance is exploited in the proof of Theorem 1.1, whose proof is completed in §4.6.

Theorem 1.3 is initially established in §5 for C^r perturbations for any fixed $3 \leq r < \infty$, in order to avail of the Sard-Smale theorem for Fredholm maps of Banach manifolds [80], while in §5.1.2 we show that generic C^∞ perturbation parameters are sufficient for transversality. (See Corollary 5.6 for the special case $\ell = 0$ and §5.1.3 for its extension to the general case $\ell \geq 0$.)

As we shall explain in §5, our proof of Theorem 1.3 ultimately hinges on the fact that if (A, Φ) is a PU(2) monopole and A is reducible on a non-empty open subset containing the support of the holonomy perturbations depending on A, then A is reducible over X. The proof of this result (Theorem 5.11) occupies §5.3; the result follows from the Agmon-Nirenberg unique continuation theorem after the system (1.3) has been transformed into an ordinary differential equation for a one-parameter family of pairs over S^3. The unique continuation property holds for both the perturbed PU(2) monopole equations (1.3), when the initial open subset of X contains the balls in X supporting holonomy perturbations, and the unperturbed equations (1.1) for any initial open subset.

1.3. Other approaches to transversality. As we noted in Remark 1.4, transversality for the PU(2) monopole equations (1.1) has also been proved very recently using only the perturbations (τ_0, ϑ_0), together with perturbations of the Riemannian metric g on X and compatible Clifford map ρ [23], [93]. The transversality proof given in [23] is more delicate than the method we employ in the present article. When using holonomy perturbations, the main technical difficulties are due to the noncompactness of the moduli space of PU(2) monopoles, and the core transversality argument itself is more straightforward, whereas in [23] the situation is entirely reversed. To place these various transversality results in a suitable context, we briefly discuss some other approaches to transversality, both for the PU(2) monopole equations and the closely related 'spin-ASD' equations of [73], which Pidstrigach and Tyurin used to define spin polynomial invariants.

Suppose (A, Φ) is a solution to either equations (1.1), equations (1.1) with perturbations (τ_0, ϑ_0), or the holonomy-perturbed equations (1.3). If A is reducible, then Φ has rank less than or equal to one (see Lemma 5.22). However, as observed by Teleman [69], [92], if Φ is rank one, then A is not necessarily reducible and he describes a simple counterexample for equations (1.1) when X is a Kähler manifold with its canonical spin^c structure.

It is not too difficult to prove that $M^{*,0}_{W,E}$ is a smooth manifold of the expected dimension away from the locus of irreducible, rank-one solutions using the perturbation parameters (τ_0, ϑ_0) alone. However, as irreducible, rank-one solutions to (1.3) could be present in $M^{*,0}_{W,E}$, it appears impossible to prove that the entire space $M^{*,0}_{W,E}$ is a smooth manifold of the expected dimension using only the parameters (τ_0, ϑ_0). A similar problem arises in the proof of transversality for the spin-ASD equations given in [73, Proposition I.3.5]; a version of these equations can be obtained from equations (1.1) by omitting the quadratic term $\rho^{-1}(\Phi \otimes \Phi^*)_{00}$. In the proof of [73, Proposition I.3.5] it is claimed that if $D_A\Phi = 0$ and Φ is rank one, then A is reducible [p. 277]: Teleman's counterexample shows that this claim is incorrect, and he points out an error in their argument [69], [92]. On the other hand, the possible presence or absence of irreducible, rank-one solutions to the PU(2) monopole equations makes no difference to the transversality argument which we describe in §5 using holonomy perturbations, as these perturbations are strong enough to yield transversality without a separate analysis of the locus of irreducible, rank-one solutions.

1.4. Applications. In [25] we discuss the singularities of the moduli space $M_{W,E}$. We introduce cohomology classes and geometric representatives on $M^*_{W,E}$, construct the links of the strata of anti-self-dual and Seiberg-Witten moduli spaces in $M_{W,E}$, and compute the Chern characters of their normal bundles in suitably defined ambient manifolds. We thus obtain a relation between Donaldson and Seiberg-

Witten invariants when the reducible solutions appear only in the top level of the Uhlenbeck compactification. In [26], [27], we develop the gluing theory for PU(2) monopoles: this is used to construct links of the lower-level Seiberg-Witten moduli spaces, to show that the pairing of the cohomology classes with this link is well-defined, and to eliminate the requirement of [25] that the reducible solutions appear only in the top level. A survey of some of the results contained in the present article and its sequels [25], [26], [27] is provided in [24].

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2. The PU(2) monopole equations

We develop a framework for gauge theory for pairs in §2.1 and §2.2. In §2.1 we define the Hilbert spaces of pairs and gauge transformations, while in §2.2 we establish the main slice result which we require for the configuration space of pairs modulo gauge transformations, paying particular attention to the structure near 'reducibles'. The relationship between the groups of SO(3) and SU(2) gauge transformations is discussed in §2.3. A few linear algebra issues which will be important for later compactness and transversality arguments are discussed in §2.4. The PU(2) monopole equations and their holonomy perturbations are introduced in §2.5. The role of the perturbations in obtaining transversality results will be explained in §5. To preserve continuity a detailed discussion of the technical points which need to be addressed when using holonomy perturbations in the present context is deferred to the Appendix. The moduli space and the elliptic deformation complex for solutions to the PU(2) monopole equations are described in §2.6.

2.1. Configuration spaces of connections and pairs. In this section we define the spaces of connections, pairs, gauge transformations, and configuration spaces which we will use throughout our work.

2.1.2. Connections on Hermitian two-plane bundles. We consider Hermitian two-plane bundles E over X whose determinant line bundles $\det E$ are isomorphic to a fixed Hermitian line bundle over X endowed with a fixed C^∞, unitary connection A_e. *The Hermitian line bundle over X and its unitary connection A_e are fixed for the remainder of this article.*

Let $k \geq 2$ be an integer and let $\mathcal{A}_E$ be the space of L^2_k connections A on the U(2) bundle E all inducing the fixed determinant connection A_e on $\det E$. Equivalently, following [48, §2(i)], we may view $\mathcal{A}_E$ as the space of L^2_k connections A on the $\mathrm{PU}(2) = \mathrm{SO}(3)$ bundle $\mathfrak{su}(E)$. We shall pass back and forth between these viewpoints, via the fixed connection on $\det E$, relying on the context to make the distinction clear. Explictly, if A is a unitary connection on E and $s \in \Omega^0(X, E)$, then $\nabla_A s \in \Omega^1(E)$. For $\zeta \in \Omega^0(X, \mathfrak{u}(E))$, then $\nabla_A \zeta \in \Omega^1(X, \mathfrak{u}(E))$ is determined by

$$(\nabla_A \zeta)s = \nabla_A(\zeta s) - \zeta(\nabla_A s),$$

so, if $A^\tau \in \Omega^1(U, \mathfrak{u}(2))$ denotes the local connection matrix defined by a choice of local frame for E over an open subset $U \subset X$, then

$$\begin{aligned}
\nabla_A s &= ds + A^\tau s, \\
\nabla_A \zeta &= d\zeta + [A^\tau, \zeta] = d\zeta + (\mathrm{ad} A^\tau)\zeta.
\end{aligned}$$

Similarly, the connection A on E induces connections on $\mathfrak{su}(E)$ and $\det E$. For example, if $\xi \in \Omega^0(X, \mathfrak{su}(E))$, then $\nabla_A \xi \in \Omega^1(X, \mathfrak{su}(E))$ is also given locally by

$$\nabla_A \xi = d\xi + [A^\tau, \xi] = d\xi + [(A^\tau)_0, \xi] = d\zeta_0 + (\mathrm{ad} A^\tau)\zeta_0,$$

while if $\lambda \in \Omega^0(X, \det E)$ then $\nabla_A \lambda \in \Omega^1(X, \det E)$ is given by

$$\nabla_A \lambda = d\lambda + (\operatorname{tr} A^\tau)\lambda.$$

The above local connection matrices are related by

$$A^\tau = (A^\tau)_0 + \tfrac{1}{2}(\operatorname{tr} A^\tau)\,\mathrm{id}_{\mathbb{C}^2} \in \Omega^1(U, \mathfrak{u}(2)),$$

where $(A^\tau)_0 = A^\tau - \frac{1}{2}(\operatorname{tr} A^\tau)\operatorname{id}_{\mathbb{C}^2} \subset \Omega^1(U, \mathfrak{su}(2))$ is the *traceless* component of A^τ, while $\operatorname{tr} A^\tau \subset \Omega^1(U, \mathfrak{u}(1)) = \Omega^1(U, i\mathbb{R})$ is the induced local connection form for $\det E$ given by the *trace* component of A^τ. Note that $(A^\tau)_0 \in \Omega^1(U, \mathfrak{su}(2))$, and that $\operatorname{ad}(A^\tau)_0 = \operatorname{ad} A^\tau \in \Omega^1(U, \mathfrak{so}(3))$ is the induced local connection matrix for the SO(3) bundle $\mathfrak{su}(E)$, where we use the standard identification $\mathfrak{so}(3) = \operatorname{ad}(\mathfrak{su}(2))$.

Conversely, given a unitary connection A_e for $\det E$ and a Riemannian connection $\hat{A}$ for $\mathfrak{su}(E)$, we obtain a unitary connection for E given in terms of local connection matrices by

$$A^\tau = \operatorname{ad}^{-1}(\hat{A}^\tau) + \tfrac{1}{2}A_e^\tau \operatorname{id}_{\mathbb{C}^2} \in \Omega^1(U, \mathfrak{u}(2)).$$

The curvatures of these connection matrices are related by

$$F_{A^\tau} = \operatorname{ad}^{-1}(F_{\hat{A}^\tau}) + \tfrac{1}{2}F_{A_e^\tau} \operatorname{id}_{\mathbb{C}^2} \in \Omega^2(U, \mathfrak{u}(2)),$$

with $F_{\hat{A}^\tau} \in \Omega^2(U, \mathfrak{so}(3))$ and $F_{A_e^\tau} \in \Omega^2(U, \mathfrak{u}(1))$. Thus, $(F_{A^\tau})_0 = \operatorname{ad}^{-1}(F_{\hat{A}^\tau})$ and $\operatorname{tr} F_{A^\tau} = F_{A_e^\tau} = F(\operatorname{tr} A^\tau)$. Therefore, globally we have $(F_A)_0 = \operatorname{ad}^{-1} F_{\hat{A}} \in \Omega^2(X, \mathfrak{su}(E))$ and $\operatorname{tr} F_A = F_{A_e} \in \Omega^2(X, \mathfrak{u}(1)) = \Omega^2(X, i\mathbb{R})$, so that

$$F_A = \operatorname{ad}^{-1}(F_{\hat{A}}) + \tfrac{1}{2}F_{A_e} \operatorname{id}_E \in \Omega^2(X, \mathfrak{u}(E)),$$

as $\mathfrak{u}(E) = \mathfrak{su}(E) \oplus (i\mathbb{R}) \operatorname{id}_E$.

When we are not explicitly discussing connections which are reducible with respect to some splitting of E, it is generally more convenient to view our connections as being defined on $\mathfrak{su}(E)$ rather than E as we can then avoid explicit mention of the otherwise unimportant choice of fixed connection on $\det E$. Of course, these viewpoints are equivalent via the choice of this fixed determinant connection. The isomorphism $\operatorname{ad} : \mathfrak{su}(E) \to \mathfrak{so}(\mathfrak{su}(E))$ will remain implicit when no confusion can arise so that, given a connection A on $\mathfrak{su}(E)$, we write its curvature as $F_A \in \Omega^2(\mathfrak{su}(E))$ and associated deformation complexes on $\Omega^\bullet(\mathfrak{su}(E))$ rather than $F_A \in \Omega^2(\mathfrak{so}(\mathfrak{su}(E)))$, with deformation complexes on $\Omega^\bullet(\mathfrak{so}(\mathfrak{su}(E)))$ (see [30, Chapter 10], for comparison).

2.1.2. Spin^c structures. The minimal, axiomatic approach to the definition of spin^c structures and the Dirac operator employed by Kronheimer-Mrowka [49] and Mrowka-Ozsváth-Yu [66] is extremely useful for our purposes, so this is the method we shall follow here.

Recall that a real-linear map $\rho_+ : T^*X \to \operatorname{Hom}(W^+, W^-)$ defines a Clifford map $\rho : \Lambda^\bullet(T^*X) \otimes \mathbb{C} \to \operatorname{End}(W)$, with $W := W^+ \oplus W^-$, if and only if [57], [77]

$$\rho_+(\alpha)^\dagger\rho_+(\alpha) = g(\alpha,\alpha)\mathrm{id}_{W^+}, \qquad \alpha \in C^\infty(T^*X), \tag{2.1}$$

where g denotes the Riemannian metric on T^*X. The real-linear map $\rho : T^*X \to \mathrm{End}(W^+ \oplus W^-)$ is obtained by defining a real-linear map

$$\rho_- : T^*X \to \mathrm{Hom}(W^-, W^+), \qquad \alpha \mapsto \rho_-(\alpha) := -\rho_+(\alpha)^\dagger,$$

and setting

$$\rho(\alpha) := \begin{pmatrix} 0 & \rho_-(\alpha) \\ \rho_+(\alpha) & 0 \end{pmatrix}. \tag{2.2}$$

This extends to a linear map $\rho : \Lambda^\bullet(T^*X) \otimes \mathbb{C} \to \mathrm{End}(W^+ \oplus W^-)$ and satisfies

$$\rho(\alpha)^\dagger = -\rho(\alpha) \quad \text{and} \quad \rho(\alpha)^\dagger\rho(\alpha) = g(\alpha,\alpha)\mathrm{id}_W, \\ \alpha \in C^\infty(T^*X). \tag{2.3}$$

A unitary connection ∇ on W and g-compatible Clifford map ρ induce a unique $\mathrm{SO}(4)$ connection ∇^g on T^*X by requiring that

$$[\nabla_\eta, \rho(\alpha)] = \rho(\nabla^g_\eta \alpha), \tag{2.4}$$

for all $\eta \in C^\infty(TX)$ and $\alpha \in \Omega^1(X, \mathbb{R})$; the sign above is opposite to that used in [66]. The unitary connection ∇ on W uniquely determines a unitary connection on $\det W^+ \simeq \det W^-$ in the standard way [43]. Conversely, the preceding data uniquely determines a unitary connection ∇ on W. The connection ∇ on W is called a *spin*^{c} connection if the connection ∇^g on T^*X is also torsion free, that is, if ∇^g is the Levi-Civita connection for the metric g.

Given a unitary connection A on an auxiliary Hermitian two-plane bundle E, we let ∇_A denote the induced unitary connection on $W \otimes E$. The corresponding Dirac operator D_A is defined by

$$D_A := \sum_{\mu=1}^{4} \rho(v^\mu)\nabla_{A,v_\mu},$$

where $\{v_\mu\}$ is a local frame for TX and $\{v^\mu\}$ is the dual frame for T^*X defined by $v^\mu(v_\mu) = \delta_{\mu\nu}$.

2.1.3. Pre-configuration spaces and automorphism groups We denote $\Lambda^\bullet := \Lambda^\bullet(T^*X)$. For any fixed L^2_k connection $A_0 \in \mathcal{A}_E$ we therefore write

$$\mathcal{A}_E = A_0 + L^2_k(\Lambda^1 \otimes \mathfrak{su}(E)),$$

for either fixed-determinant, unitary connections on E or determinant-one, orthogonal connections on $\mathfrak{su}(E)$. Our *pre-configuration space* of pairs is given by

$$\tilde{\mathcal{C}}_{W,E} := \mathcal{A}_E \times L^2_k(W^+ \otimes E),$$

and for any fixed L^2_k pair $(A_0, \Phi_0) \in \tilde{\mathcal{C}}_{W,E}$, we have

$$\tilde{\mathcal{C}}_{W,E} = (A_0, \Phi_0) + L^2_k(\Lambda^1 \otimes \mathfrak{su}(E)) \oplus L^2_k(W^+ \otimes E), \tag{2.5}$$

for the cases of either fixed-determinant, unitary connections on E or determinant-one, orthogonal connections on $\mathfrak{su}(E)$.

Given any C^∞ connection $A_0 \in \mathcal{A}_E$, our Sobolev norms are defined in the usual way: for example, if $a \in \Omega^1(\mathfrak{su}(E))$, we write

$$\|a\|_{L^p_{\ell,A_0}(X)} := \left(\sum_{j=0}^{\ell} \|\nabla^j_{A_0} a\|^p_{L^p(X)}\right)^{1/p},$$

and if $(a, \phi) \in \Omega^1(\mathfrak{su}(E)) \oplus \Omega^0(W^+ \otimes E)$, we write

$$\|(a,\phi)\|_{L^p_{\ell,A_0}(X)} := \left(\|a\|^p_{L^p_{\ell,A_0}(X)} + \|\phi\|^p_{L^p_{\ell,A_0}(X)}\right)^{1/p},$$

for any $1 \leq p \leq \infty$ and integer $\ell \geq 0$.

For convenience, we let $i\mathbb{R}_Z = (i\mathbb{R})\,\mathrm{id}_{\mathbb{C}^2} \subset \mathfrak{u}(2)$ denote the center of the Lie algebra $\mathfrak{u}(2)$, and let $S^1_Z = \exp(i\mathbb{R}_Z) \subset \mathrm{U}(2)$ denote the center $Z(\mathrm{U}(2))$ of the Lie group $\mathrm{U}(2)$ given by

$$i\mathbb{R}_Z = \left\{\begin{pmatrix} i\theta & 0 \\ 0 & i\theta \end{pmatrix} : \theta \in \mathbb{R}\right\}, \quad S^1_Z = \left\{\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{i\theta} \end{pmatrix} : \theta \in \mathbb{R}\right\}. \tag{2.6}$$

Following [48, §2], we consider the induced action for connections on $\mathfrak{su}(E)$ by the group $\mathcal{G}_E$ of determinant-one, unitary L^2_{k+1} automorphisms of E rather than the group $\mathcal{G}_{\mathfrak{su}(E)}$ of determinant-one, orthogonal L^2_{k+1} automorphisms of $\mathfrak{su}(E)$ to define quotient spaces of connections on $\mathfrak{su}(E)$. The reasons for this choice are explained further in [25]; see also §2.3. We have

$$\mathcal{G}_E := \{u \in L^2_{k+1}(\mathfrak{gl}(E)) : u^\dagger u = \mathrm{id}_E \text{ and } \det u = 1 \text{ a.e.}\},$$

and

$$^\circ\mathcal{G}_E := S^1_Z \times_{\{\pm \mathrm{id}_E\}} \mathcal{G}_E.$$

One can define the action of $^\circ\mathcal{G}_E$ on $\mathcal{A}_E$ either by push-forward or pullback, and obtain equivalent quotient spaces in either case; but the choice does affect the orientation of moduli spaces, so we specify the action here: for $u \in {}^\circ\mathcal{G}_E$ and $(A,\Phi) \in \tilde{\mathcal{C}}_{W,E}$, the action of $^\circ\mathcal{G}_E$ is defined by

$$u(A,\Phi) := (u(A), u\Phi) = (u_*A, u\Phi) = ((u^{-1})^*A, u\Phi).$$

The push-forward action, $u(A) = u_*A$, agrees with the conventions of [9], [20]. For the associated covariant derivative ∇_A on E and $u \in \operatorname{Aut} E$, we have

$$\nabla_{u(A)} = u \circ \nabla_A \circ u^{-1},$$

so that $u(A) = A - (\nabla_A u)u^{-1}$. In terms of a local connection matrix $A^\tau \in \Omega^1(U, \mathfrak{u}(2))$, this gives $u(A^\tau) = uA^\tau u^{-1} - (du)u^{-1}$, where we use u to denote both the gauge transformation and the element of $\Omega^0(U, \mathrm{U}(2))$.

In order to define quotients by the action of $^\circ\mathcal{G}_E$, we need to choose $k \geq 2$, so gauge transformations are at least continuous. The proof of the following proposition is a standard application of the Sobolev multiplication theorem; see [30, Propositions A.2, A.3, A.9].

Proposition 2.1. *Let X be a closed Riemannian four-manifold, E be a Hermitian vector bundle over X, and $k \geq 2$ be an integer. Then the following hold:*

(1) The space $^\circ\mathcal{G}_E$ is a Hilbert Lie group with Lie algebra $T_{\mathrm{id}}{}^\circ\mathcal{G}_E = L^2_{k+1}(\mathfrak{su}(E)) \oplus i\mathbb{R}_Z$.

(2) The action of $^\circ\mathcal{G}_E$ on $\tilde{\mathcal{C}}_{W,E}$ is smooth.

(3) For $(A,\Phi) \in \tilde{\mathcal{C}}_{W,E}$, the differential, at the identity $\mathrm{id}_E \in {}^\circ\mathcal{G}_E$, of the map $^\circ\mathcal{G}_E \to \tilde{\mathcal{C}}_{W,E}$ given by $u \mapsto u(A,\Phi) = (A - (d_A u)u^{-1}, u\Phi)$ is $\zeta \mapsto -d^0_{A,\Phi}\zeta := (-d_A\zeta, \zeta\Phi)$ as a map

$$L^2_{k+1}(\mathfrak{su}(E)) \oplus i\mathbb{R}_Z \to L^2_k(\Lambda^1 \otimes \mathfrak{su}(E)) \oplus L^2_k(W^+ \otimes E).$$

We denote $\mathcal{B}_E = \mathcal{A}_E/\mathcal{G}_E = \mathcal{A}_E/{}^\circ\mathcal{G}_E$. The *configuration space of pairs* is given by $\mathcal{C}_{W,E} := \tilde{\mathcal{C}}_{W,E}/{}^\circ\mathcal{G}_E$, that is,

$$\mathcal{C}_{W,E} := \left(\mathcal{A}_E \times L^2_k(W^+ \otimes E)\right)/{}^\circ\mathcal{G}_E = \mathcal{A}_E \times_{^\circ\mathcal{G}_E} L^2_k(W^+ \otimes E),$$

and is endowed with the quotient L_k^2 topology.

If π denotes the projection $\tilde{\mathcal{C}}_{W,E} \to \mathcal{C}_{W,E}$, a base for the quotient L_k^2 topology of $\mathcal{C}_{W,E}$ is given by open subsets $\pi^{-1}\pi(B_{(A,\Phi)}(\varepsilon)) \supset B_{(A,\Phi)}(\varepsilon)$, where $B_{(A,\Phi)}(\varepsilon)$ is the L_k^2-ball of radius ε and center (A,Φ) in $\tilde{\mathcal{C}}_{W,E}$ given by $\{(A_1,\Phi_1) \in \tilde{\mathcal{C}}_{W,E} : \|(A_1,\Phi_1) - (A,\Phi)\|_{L^2_{k,A}} < \varepsilon\}$.

2.1.4. Stabilizers. The space $\mathcal{C}_{W,E}$ is not a manifold — it has singularities at points $[A,\Phi]$ with non-trivial stabilizer $\mathrm{Stab}_{A,\Phi} \subset {}^\circ\mathcal{G}_E$. Recall that the stabilizer subgroup (of the group of bundle automorphisms) for a connection on a G bundle always contains the center $Z(G) \subset G$ [20, §4.2.2]. We let $S^1_Z \subset {}^\circ\mathcal{G}_E$ denote the constant, central automorphisms of E.

Definition 2.2. Suppose (A,Φ) is a point in $\tilde{\mathcal{C}}_{W,E}$. The stabilizer $\mathrm{Stab}_{A,\Phi} \subset {}^\circ\mathcal{G}_E$ of (A,Φ) in ${}^\circ\mathcal{G}_E$ is given by $\{\gamma \in {}^\circ\mathcal{G}_E : \gamma(A,\Phi) = (A,\Phi)\}$, while the stabilizers of A and Φ are denoted by Stab_A and Stab_Φ, respectively.

(1) The point (A,Φ) is a *zero-section pair* if $\Phi \equiv 0$;

(2) The point (A,Φ) is an *irreducible pair* if the connection A has minimal stabilizer $\mathrm{Stab}_A = Z(\mathrm{U}(2)) = S^1_Z \subset {}^\circ\mathcal{G}_E$ and is *reducible* otherwise, that is, if $\mathrm{Stab}_A \supsetneq S^1_Z$.

The point (A,Φ) is a *reducible, zero-section pair* if (A,Φ) is both a reducible and zero-section pair.

As usual, the stabilizer subgroup $\mathrm{Stab}_A \subset {}^\circ\mathcal{G}_E$ may be identified with a closed subgroup of $\mathrm{U}(E_{x_0}) \simeq \mathrm{U}(2)$ for any point $x_0 \in X$ by parallel translation with respect to the connection A [20, §4.2.2], [61]. The following lemma implies that the stabilizer in $\mathrm{Aut}\, E$ of a unitary connection on E coincides with its stabilizer on ${}^\circ\mathcal{G}_E$.

Lemma 2.3. *Let E be a Hermitian two-plane bundle over a connected four-manifold X, let A be a unitary connection on E, and let u be a unitary automorphism of E such that $u \in \mathrm{Stab}_A \subset \mathrm{Aut}\, E$. Then $\det u : X \to S^1$ is a constant map.*

Proof. The gauge transformation u may be viewed as an Ad-equivariant map $u : P \to \mathrm{U}(2)$, where $E = P \times_{\mathrm{U}(2)} \mathbb{C}^2$, so $\det u$ may be viewed as a map from P to S^1. Since $\det u$ is constant on the fibers of P, it descends to a map on X. Over a small enough open set $U \subset X$, we may write $u = \exp\zeta$, where $\zeta \in \Omega^0(\mathrm{ad}P)$, and $\mathrm{ad}P := P \times_{\mathrm{ad}} \mathfrak{g} \simeq \mathfrak{u}(E)$, with $G = \mathrm{U}(2)$ and $\mathfrak{g} = \mathfrak{u}(2)$. Differentiating the action of u on A

we see that $\nabla_A\zeta = 0$, so that ζ is covariantly constant over U. Thus $\det u = \exp(\operatorname{tr}\zeta)$ and $\nabla(\det u) = \operatorname{tr}(\nabla_A\zeta) = 0$, so $\det u$ is constant on U and therefore on X, since X is connected. q.e.d.

For a pair (A,Φ) on $(E, W^+\otimes E)$ or $(\mathfrak{su}(E), W^+\otimes E)$, the Lie algebras of Stab_A and $\mathrm{Stab}_{A,\Phi}$ are given by

$$\begin{aligned} H^0_A &:= \operatorname{Ker}\{d_A : L^2_{k+1}(\mathfrak{su}(E))\oplus i\mathbb{R}_Z \\ &\qquad\qquad \to L^2_k(\Lambda^1\otimes\mathfrak{su}(E))\}, \\ H^0_{A,\Phi} &:= \operatorname{Ker}\{d^0_{A,\Phi} : L^2_{k+1}(\mathfrak{su}(E))\oplus i\mathbb{R}_Z \\ &\qquad\qquad \to L^2_k(\Lambda^1\otimes\mathfrak{su}(E))\oplus L^2_k(\Lambda^1\otimes W^+\otimes E)\}. \end{aligned}$$

The first identification is standard (see [20, §4.2.2], [30, Chapter 3], or [61]) and the second follows by the same argument.

From [20, §4.2.2] and [30, Theorem 10.8] we have the following characterization of the stabilizer $\mathrm{Stab}_B \subset \mathcal{G}_V = \operatorname{Aut} V$ of a connection B on an $\mathrm{SO}(3)$ bundle V. Note that the four-manifold X below need not be simply-connected or closed and that $H^0_B = \operatorname{Ker}\{d_B : L^2_{k+1}(\mathfrak{so}(V)) \to L^2_k(\Lambda^1\otimes\mathfrak{so}(V))\}$.

Lemma 2.4. *Let B be an orthogonal connection on an* $\mathrm{SO}(3)$ *bundle V over a connected smooth four-manifold X. The following are equivalent provided B is not flat:*

(1) $\mathrm{Stab}_B \simeq \mathrm{SO}(2) \simeq S^1$*;*

(2) $H^0_B \neq 0$*;*

(2) B is reducible with respect to an orthogonal splitting $V = N\oplus\underline{\mathbb{R}}$, where N is a complex line bundle over X and $\underline{\mathbb{R}} = X\times\mathbb{R}$ (that is, B reduces to an $\mathrm{SO}(2)$ *connection);*

(4) $\mathrm{Stab}_B \neq \{\mathrm{id}_V\}$.

Finally, B is flat if $\mathrm{Stab}_B = \mathrm{SO}(3)$.

Remark 2.5. Note that the 'twisted reducible' (that is, locally reducible) connections on $\mathfrak{su}(E)$ discussed by Kronheimer and Mrowka in [48, §2(i)] are globally irreducible in the sense that they have minimal stabilizer $\{\mathrm{id}_{\mathfrak{su}(E)}\}$. The condition that B is not flat is used in the proof that (4) implies (1) [30, p. 48]: the stabilizer Stab_B is isomorphic to the centralizer of the holonomy group Hol_B so, if $\mathrm{Stab}_B \supsetneq S^1$, then $\mathrm{Hol}_B \subsetneq S^1$ and hence Hol_B is discrete. Since the Lie algebra of Hol_B vanishes, the Ambrose-Singer holonomy theorem implies that the connection B is flat. In particular, we only need X to be a connected four-manifold for these arguments to hold.

As customary, we call an SO(3) connection B on V *irreducible* if $\mathrm{Stab}_B = \{\mathrm{id}_V\}$. We say that a unitary connection A on E is *projectively flat* if the induced connection A^{ad} on $\mathfrak{su}(E)$ is flat. By modifying the proofs of Theorem 3.1 in [30] and Proposition II.8.10 [56], we obtain the following relation between (i) the stabilizers Stab_A in ${}^\circ\mathcal{G}_E$ of U(2) connections A on E or their induced SO(3) connections A^{ad} on $\mathfrak{su}(E)$ and (ii) the stabilizers $\mathrm{Stab}_{A^{\mathrm{ad}}}$ in $\mathcal{G}_{\mathfrak{su}(E)}$ of SO(3) connections A^{ad} on $\mathfrak{su}(E)$.

Lemma 2.6. *Let A be a* U(2) *connection on a Hermitian two-plane bundle E over a smooth, connected four-manifold X, and let A^{ad} be the induced* SO(3) *connection on the bundle $\mathfrak{su}(E)$. Then the following are equivalent:*

(1) $\mathrm{Stab}_{A^{\mathrm{ad}}} \simeq S^1$*;*

(2) $\mathrm{Stab}_A \simeq S^1 \times S^1_Z$*;*

(3) $H^0_A \simeq i\mathbb{R} \oplus i\mathbb{R}_Z$*;*

(4) $H^0_{A,\Phi} \supsetneq i\mathbb{R}_Z$*;*

(5) A is reducible with respect to an orthogonal splitting $E = L_1 \oplus L_2$, where L_1 and L_2 are complex line bundles over X (that is, A reduces to a $T^2 = S^1 \times S^1$ connection);

(6) $\mathrm{Stab}_A \supsetneq S^1_Z$.

Finally, the connection A on E is projectively flat (or the connection A^{ad} on $\mathfrak{su}(E)$ is flat) if $\mathrm{Stab}_A = \mathrm{U}(2)$.

Proof. (1) $\Longrightarrow$ (2): Let $\pi : \mathrm{U}(2) \to \mathrm{SO}(3) = \mathrm{U}(2)/S^1_Z$ be the projection $u \mapsto \pm(\det u)^{-1/2}u$. Then $\pi^{-1}(\mathrm{Stab}_{A^{\mathrm{ad}}}) \simeq \mathrm{Stab}_{A^{\mathrm{ad}}} \times S^1_Z$ and so $\mathrm{Stab}_{A^{\mathrm{ad}}} \simeq S^1$ implies that $\mathrm{Stab}_A \simeq S^1 \times S^1_Z$.

(2) $\Longrightarrow$ (3): Immediate from the identification of H^0_A as the Lie algebra of Stab_A.

(3) $\Longrightarrow$ (4): Trivial.

(4) $\implies$ (5): We argue as in the proof of Theorem 3.1 in [30]. Note that $\mathrm{Ker}\{d_A : L^2_{k+1}(X, \mathfrak{u}(E)) \to \Omega^1(X, \mathfrak{u}(E))\}$ is isomorphic to

$$\begin{aligned}&\mathrm{Ker}\{d_A : L^2_{k+1}(X, \mathfrak{su}(E))\} \oplus \mathrm{Ker}\{d : L^2_{k+1}(X, i\mathbb{R}_Z)\}\\ &\quad = \mathrm{Ker}\{d_{A^{\mathrm{ad}}} : L^2_{k+1}(X, \mathfrak{su}(E))\} \oplus i\mathbb{R}_Z.\end{aligned}$$

Given (4), we may choose $0 \not\equiv \zeta \in \mathrm{Ker}\{d_A : L^2_{k+1}\Omega^0(X, \mathfrak{su}(E))\}$. The pointwise traceless, skew-Hermitian endomorphism ζ of E has eigenvalues $i\lambda_1, i\lambda_2 = \pm i\lambda$ for some $0 \not\equiv \lambda \in L^2_{k+1}(X, \mathbb{R})$. On the open subset of X where $\lambda \neq 0$, let ξ_j, $j = 1, 2$, be L^2_{k+1} eigenvectors of ζ such that $\zeta(\xi_j) = i\lambda_j\xi_j$ and $\langle \xi_j, \xi_k\rangle = \delta_{jk}$. Just as in [30, p. 47], we find that λ is constant, the eigenvectors ξ_j are globally defined and $d_A\xi_j = 0$ for $j = 1, 2$. In particular, we have an orthogonal splitting $E = L_1 \oplus L_2$, where $\xi_j \in L^2_{k+1}(X, E)$ is a section of L_j and $d_A = d_{A_1} \oplus d_{A_2}$ with respect to this splitting, so this gives (5).

(5) $\implies$ (6): Given (5), the connection A reduces to $A_1 \oplus A_2$ with respect to the splitting $E = L_1 \oplus L_2$ and so has stabilizer $S^1_{L_1} \times S^1_{L_2}$, where we identify the constant maps in $\mathrm{Map}^{k+1}(X, S^1)$ with $S^1_{L_i}$ for $i = 1, 2$. Since $S^1_{L_1} \times S^1_{L_2} \simeq S^1 \times S^1_Z$, this gives (2).

(6) $\implies$ (1): Given $u \in \mathrm{Stab}_A$ and $u \notin S^1_Z$, we obtain $u_1 = \pm(\det u)^{-1/2}u \in \mathrm{Stab}_{A^{\mathrm{ad}}}$. (By Lemma 2.3, the determinant $\det u$ is a constant map.) If $u_1 = \mathrm{id}_{\mathfrak{su}(E)}$ then we would have $u_1 = \pm\mathrm{id}_E$ and $u = \pm(\det u)^{1/2}u_1 \in S^1_Z$, so $u_1 \neq \mathrm{id}_{\mathfrak{su}(E)}$. Therefore, $\mathrm{Stab}_{A^{\mathrm{ad}}} \supsetneq \{\mathrm{id}_{\mathfrak{su}(E)}\}$ and Lemma 2.4 implies that $\mathrm{Stab}_{A^{\mathrm{ad}}} \simeq S^1$.

Lastly, A is projectively flat if $\mathrm{Stab}_{A^{\mathrm{ad}}} = \mathrm{SO}(3)$ (by Lemma 2.4) and $\mathrm{Stab}_{A^{\mathrm{ad}}} = \mathrm{SO}(3) \simeq \mathrm{U}(2)/S^1_Z$ if and only if $\mathrm{Stab}_A = \mathrm{U}(2)$. q.e.d.

The equivalence of (1) and (2) above can alternatively be seen by noting that $H^0_{A^{\mathrm{ad}}} \simeq i\mathbb{R}$ if and only if $H^0_A \simeq i\mathbb{R} \oplus i\mathbb{R}_Z$, so $\mathrm{Stab}_{A^{\mathrm{ad}}} \simeq S^1$ if and only if $\mathrm{Stab}_A \simeq S^1 \times S^1_Z$.

From Lemmas 2.4 and 2.6 we see that A is an irreducible $\mathrm{U}(2)$ connection if and only if A^{ad} is an irreducible $\mathrm{SO}(3)$ connection. Note that $\mathrm{Stab}_{A,\Phi} = \mathrm{Stab}_A \cap \mathrm{Stab}_\Phi$. The stabilizer $\mathrm{Stab}_{A,\Phi}$ of a zero-section pair (A, Φ) contains S^1_Z. The stabilizer $\mathrm{Stab}_{A,\Phi}$ of a reducible pair (A, Φ) need not contain the stabilizer Stab_A since Stab_A may not fix Φ.

We write $\tilde{\mathcal{C}}^*_{W,E}$ (respectively, $\tilde{\mathcal{C}}^0_{W,E}$) for the complement of the reducible pairs (respectively, zero section pairs) in $\tilde{\mathcal{C}}_{W,E}$, and let $\tilde{\mathcal{C}}^{*,0}_{W,E} = \tilde{\mathcal{C}}^*_{W,E} \cap \tilde{\mathcal{C}}^0_{W,E}$. The quotients $\mathcal{C}^*_{W,E}$, $\mathcal{C}^0_{W,E}$, and $\mathcal{C}^{*,0}_{W,E}$ are similarly defined.

If $(A, \Phi) \in \tilde{\mathcal{C}}^{*,0}_{W,E}$, then $\mathrm{Stab}_{A,\Phi} = \{\mathrm{id}_E\}$, as the stabilizer of an irreducible connection A will be S^1_Z and if S^1_Z stabilizes the section Φ

then $e^{i\theta}\Phi = 0$ for all $\theta \in \mathbb{R}$ and so $\Phi \equiv 0 \in L^2_k(X, W^+ \otimes E)$ (note that Φ need not be continuous). Therefore, ${}^\circ\mathcal{G}_E$ acts freely on $\tilde{\mathcal{C}}^{*,0}_{W,E}$ and, as we shall see in the next section, the quotient $\mathcal{C}^{*,0}_{W,E}$ is a Banach manifold. Conversely, if (A,Φ) is a PU(2) monopole, A is reducible, and $\Phi \not\equiv 0$, then Lemma 5.22 implies that $\operatorname{Stab}_{A,\Phi} \simeq S^1$.

2.2. A slice for the action of the group of gauge transformations. Let $k \geq 2$ be an integer. The *slice* $\mathbf{S}_{A,\Phi} \subset \tilde{\mathcal{C}}_{W,E}$ through a pair (A,Φ) is given by $\mathbf{S}_{A,\Phi} := (A,\Phi) + \mathbf{K}_{A,\Phi}$, where

$$\mathbf{K}_{A,\Phi} := \operatorname{Ker} d^{0,*}_{A,\Phi} \subset L^2_k(\Lambda^1 \otimes \mathfrak{su}(E)) \oplus L^2_k(W^+ \otimes E). \tag{2.7}$$

If π is the projection from $\tilde{\mathcal{C}}_{W,E}$ onto $\mathcal{C}_{W,E} = \tilde{\mathcal{C}}_{W,E}/{}^\circ\mathcal{G}_E$, denoted by $(A,\Phi) \mapsto [A,\Phi]$, we let

$$\mathbf{B}_{A,\Phi}(\varepsilon) = \pi^{-1}B_{A,\Phi}(\varepsilon) \cap \mathbf{S}_{A,\Phi}$$

be the open L^2_k ball in $\mathbf{S}_{A,\Phi}$ with center (A,Φ) and radius ε, so that

$$\begin{aligned}\mathbf{B}_{A,\Phi}(\varepsilon) &:= \{(A_1,\Phi_1) \in \mathbf{S}_{A,\Phi} : \|(A_1,\Phi_1) - (A,\Phi)\|_{L^2_{k,A}} < \varepsilon\} \\ &= (A,\Phi) + \{(a,\phi) \in \mathbf{K}_{A,\Phi} : \|(a,\phi)\|_{L^2_{k,A}} < \varepsilon\}.\end{aligned}$$

The Hilbert Lie group ${}^\circ\mathcal{G}_E$ has Lie algebra

$$L^2_{k+1}(\mathfrak{su}(E)) \oplus i\mathbb{R}_Z \subset L^2_{k+1}(\mathfrak{su}(E))$$

and exponential map $\exp : L^2_{k+1}(\mathfrak{su}(E)) \oplus i\mathbb{R}_Z \to {}^\circ\mathcal{G}_E$ given by $\zeta \mapsto u = \exp\zeta$. Let

$$\operatorname{Stab}^\perp_{A,\Phi} := \exp((\operatorname{Ker} d^0_{A,\Phi})^\perp) \subset {}^\circ\mathcal{G}_E,$$

where

$$(\operatorname{Ker}(d^0_{A,\Phi}|_{L^2_{k+1}}))^\perp = \operatorname{Im}(d^{0,*}_{A,\Phi}|_{L^2_{k+2}}) \subset L^2_{k+1}(\mathfrak{su}(E)),$$

noting that

$$d^{0,*}_{A,\Phi} : L^2_{k+2}\Omega^1(\mathfrak{su}(E)) \to L^2_{k+1}(\mathfrak{su}(E)) \oplus L^2_k(W^+ \otimes E)$$

has closed range. Recall that $\operatorname{Stab}_{A,\Phi} \subset {}^\circ\mathcal{G}_E$ is given by

$$\{\gamma \in {}^\circ\mathcal{G}_E : \gamma(A,\Phi) = (A,\Phi)\}$$

and has Lie algebra

$$H^0_{A,\Phi} = \operatorname{Ker}(d^0_{A,\Phi}|_{L^2_{k+1}}) \subset L^2_{k+1}(\mathfrak{su}(E)) \oplus i\mathbb{R}_Z,$$

so that

$$L^2_{k+1}(\mathfrak{su}(E)) \oplus i\mathbb{R}_Z = (\operatorname{Ker}(d^0_{A,\Phi}|_{L^2_{k+1}}))^\perp \oplus H^0_{A,\Phi}. \tag{2.8}$$

The subspace $\operatorname{Stab}^\perp_{A,\Phi} \subset {}^\circ\mathcal{G}_E$ is closed and is a Banach submanifold of ${}^\circ\mathcal{G}_E$ with codimension $\dim H^0_{A,\Phi}$.

The map $d^0_{A,\Phi} : L^2_{k+1}(\mathfrak{su}(E)) \oplus i\mathbb{R}_Z \to L^2_k(\Lambda^1 \otimes \mathfrak{su}(E)) \oplus L^2_k(W^+ \otimes E)$ has closed range and so we have an L^2-orthogonal decomposition

$$T_{A,\Phi}\tilde{\mathcal{C}}_{W,E} = \operatorname{Im}(d^0_{A,\Phi}|_{L^2_{k+1}}) \oplus \mathbf{K}_{A,\Phi} \tag{2.9}$$

of the tangent space to the space of L^2_k pairs at the point (A,Φ).

The proof that the quotient space $\mathcal{C}_{W,E}$ is Hausdorff and our later proof of removable singularities for PU(2) monopoles make use of the following well-known technical result [20, Proposition 2.3.15], [30, Proposition A.5], [56, Theorem II.7.11]. Note that the space of L^2_2 unitary automorphisms of E is neither a Hilbert Lie group nor does it act smoothly on the space of L^2_2 unitary connections on E.

Lemma 2.7. *Let E be a Hermitian bundle over a Riemannian manifold X and let $k \geq 2$ be an integer. Suppose that $\{A_\alpha\}$ and $\{B_\alpha\}$ are sequences of L^2_k unitary connections on E, and that $\{u_\alpha\}$ is a sequence of unitary automorphisms of E such that $u_\alpha(A_\alpha) = B_\alpha$. Then the following hold.*

1. *(1) The sequence $\{u_\alpha\}$ is in L^2_{k+1}.*
2. *(2) If $\{A_\alpha\}$ and $\{B_\alpha\}$ converge in L^2_k to limits A_∞, B_∞, then there is a subsequence $\{\alpha'\} \subset \{\alpha\}$ such that $\{u_{\alpha'}\}$ converges in L^2_{k+1} to u_∞ and $B_\infty = u_\infty(A_\infty)$.*

The following slice result was established by Parker [70], but only for pairs which are neither zero-section nor reducible; see also [9]. It is, of course, the analogue of the usual result for the topology and manifold structure of the configuration space $\mathcal{B}_E = \mathcal{A}_E/\mathcal{G}_E$ of connections. The proof we give here is modelled on the corresponding arguments for connections given in [20, Proposition 2.3.4], [30, Theorems 3.2 & 4.4], and [56, Theorem II.10.4]. We will ultimately need a rather more involved version of this method in order to show that our gluing maps are diffeomorphisms, so we give the argument in the simpler model case below in some detail and establish some of the notation and conventions we will later require.

Proposition 2.8. *Let X be a closed, oriented, Riemannian spin^c four-manifold, let E be a Hermitian two-plane bundle over X, and let $k \geq 2$ be an integer. Then the following hold.*

(1) The space $\mathcal{C}_{W,E}$ is Hausdorff.

(2) The subspace $\mathcal{C}^{*,0}_{W,E} \subset \mathcal{C}_{W,E}$ is open and is a C^∞ Hilbert manifold with local parametrizations given by $\pi : \mathbf{B}_{A_0,\Phi_0}(\varepsilon) \to \mathcal{C}^{*,0}_{W,E}$ for sufficiently small $\varepsilon = \varepsilon(A_0, \Phi_0, k)$.

(3) The projection $\pi : \tilde{\mathcal{C}}^{*,0}_{W,E} \to \mathcal{C}^{*,0}_{W,E}$ is a C^∞ principal ${}^\circ\mathcal{G}_E$ bundle.

(4) For $(A_0, \Phi_0) \in \tilde{\mathcal{C}}_{W,E}$, the projection $\pi : \mathbf{B}_{A_0,\Phi_0}(\varepsilon)/\operatorname{Stab}_{A_0,\Phi_0} \to \mathcal{C}_{W,E}$ is a homeomorphism onto an open neighborhood of $[A_0, \Phi_0] \in \mathcal{C}_{W,E}$ and a diffeomorphism on the complement of the set of points in $\mathbf{B}_{A_0,\Phi_0}(\varepsilon)$ with non-trivial stabilizer.

Proof. The stabilizer $\operatorname{Stab}_{A_0,\Phi_0}$ (which we can identify with a Lie subgroup of U(2)) acts freely on ${}^\circ\mathcal{G}_E$ and thus on the Hilbert manifold ${}^\circ\mathcal{G}_E \times \mathbf{S}_{A_0,\Phi_0}$ by $(u, A, \Phi) \mapsto \gamma \cdot (u, A, \Phi) = (u\gamma^{-1}, \gamma(A, \Phi))$, and so the quotient ${}^\circ\mathcal{G}_E \times_{\operatorname{Stab}_{A_0,\Phi_0}} \mathbf{S}_{A_0,\Phi_0}$ is again a Hilbert manifold. We define a smooth map

$$\mathbf{\Psi} : {}^\circ\mathcal{G}_E \times_{\operatorname{Stab}_{A_0,\Phi_0}} \mathbf{S}_{A_0,\Phi_0} \to \tilde{\mathcal{C}}_{W,E}, \qquad [u, A, \Phi] \mapsto u(A, \Phi).$$

Our first task is to show that the map $\mathbf{\Psi}$ is a diffeomorphism onto its image upon restriction to a sufficiently small neighborhood ${}^\circ\mathcal{G}_E \times_{\operatorname{Stab}_{A_0,\Phi_0}} \mathbf{B}_{A_0,\Phi_0}(\varepsilon)$. Given $\delta > 0$, let $B_{\mathrm{id}}(\delta)$ be the ball

$$\{u \in {}^\circ\mathcal{G}_E : \|u - \mathrm{id}_E\|_{L^2_{k+1,A_0}} < \delta\},$$

and let $B^\perp_{\mathrm{id}}(\delta) = B_{\mathrm{id}}(\delta) \cap \operatorname{Stab}^\perp_{A_0,\Phi_0}$.

Claim 2.9. For small enough $\delta = \delta(A_0, \Phi_0, k)$, the ball $B_{\mathrm{id}}(\delta)$ is diffeomorphic to an open neighborhood in $B^\perp_{\mathrm{id}}(\delta) \times \operatorname{Stab}_{A_0,\Phi_0}$, with inverse map given by $(u_0, \gamma) \mapsto u = u_0\gamma$.

Proof. The differential of the multiplication map

$$\operatorname{Stab}^\perp_{A_0,\Phi_0} \times \operatorname{Stab}_{A_0,\Phi_0} \to {}^\circ\mathcal{G}_E, \qquad (u_0, \gamma) \mapsto u_0\gamma$$

at $(\mathrm{id}_E, \mathrm{id}_E)$ is given by

$$\operatorname{Ker}(d^0_{A_0,\Phi_0}|_{L^2_{k+1}})^\perp \oplus H^0_{A_0,\Phi_0} \to L^2_{k+1}(\mathfrak{su}(E)) \qquad (\zeta, \chi) \mapsto \zeta + \chi,$$

and is just the identity map with respect to the L^2-orthogonal decomposition (2.8) of the range. Hence, the Hilbert space implicit function theorem implies that there is a diffeomorphism from an open neighborhood of $(\mathrm{id}_E, \mathrm{id}_E)$ onto an open neighborhood of $\mathrm{id}_E \in {}^\circ\mathcal{G}_E$. For small enough δ, we may suppose that if $u \in B_{\mathrm{id}}(\delta)$, then u can be written uniquely as $u = u_0\gamma$ with $u_0 \in B_{\mathrm{id}}^\perp(\delta)$ and $\gamma \in \mathrm{Stab}_{A_0,\Phi_0}$. q.e.d.

Claim 2.10. For small enough $\varepsilon = \varepsilon(A_0, \Phi_0, k)$, the map $\mathbf{\Psi}$ is a diffeomorphism from ${}^\circ\mathcal{G}_E \times_{\mathrm{Stab}_{A_0,\Phi_0}} \mathbf{B}_{A_0,\Phi_0}(\varepsilon)$ onto its image in $\tilde{\mathcal{C}}_{W,E}$.

Proof. We first restrict the map $\mathbf{\Psi}$ to a neighborhood $B_{\mathrm{id}}(\delta) \times_{\mathrm{Stab}_{A_0,\Phi_0}} \mathbf{S}_{A_0,\Phi_0}$, which is diffeomorphic to the neighborhood $B_{\mathrm{id}}^\perp(\delta) \times \mathbf{S}_{A_0,\Phi_0}$ in $\mathrm{Stab}_{A_0,\Phi_0}^\perp \times \mathbf{S}_{A_0,\Phi_0}$ by Claim 2.9. The differential of the induced map

$$\mathbf{\Psi} : \mathrm{Stab}_{A_0,\Phi_0}^\perp \times \mathbf{S}_{A_0,\Phi_0} \to \tilde{\mathcal{C}}_{W,E}, \qquad (u, A, \Phi) \mapsto u(A, \Phi)$$

at $(\mathrm{id}_E, A_0, \Phi_0)$ is given by

$$\begin{aligned}(D\mathbf{\Psi})_{(\mathrm{id},A_0,\Phi_0)} :& T_{\mathrm{id}} \mathrm{Stab}_{A_0,\Phi_0}^\perp \oplus T_{A_0,\Phi_0}\mathbf{S}_{A_0,\Phi_0} \to T_{A_0,\Phi_0}\tilde{\mathcal{C}}_{W,E},\\ &(\zeta, a, \phi) \mapsto -d_{A_0,\Phi_0}^0\zeta + (a, \phi),\end{aligned}$$

where we recall that $T_{A_0,\Phi_0}\mathbf{S}_{A_0,\Phi_0} = \mathbf{K}_{A_0,\Phi_0}$ and

$$T_{\mathrm{id}} \mathrm{Stab}_{A_0,\Phi_0}^\perp = (\mathrm{Ker}(d_{A_0,\Phi_0}^0|_{L_{k+1}^2}))^\perp = \mathrm{Im}(d_{A_0,\Phi_0}^{0,*}|_{L_{k+2}^2}).$$

Using the L^2-orthogonal decomposition (2.9) of the range we see that the map

$$-d_{A_0,\Phi_0}^0 \oplus \mathrm{id}_E : (\mathrm{Ker}(d_{A_0,\Phi_0}^0|_{L_{k+1}^2}))^\perp \oplus \mathbf{K}_{A_0,\Phi_0} \to \mathrm{Im}(d_{A_0,\Phi_0}^0|_{L_{k+1}^2}) \oplus \mathbf{K}_{A_0,\Phi_0}$$

given by $(\zeta, b, \psi) \mapsto -d_{A_0,\Phi_0}^0\zeta + (b, \psi)$ is a Hilbert space isomorphism. So, by the Hilbert space implicit function theorem, there are positive constants $\varepsilon = \varepsilon(A_0, \Phi_0, k)$ and $\delta = \delta(A_0, \Phi_0, k)$ and an open neighborhood $\mathcal{U}_{A_0,\Phi_0} \subset \tilde{\mathcal{C}}_{W,E}$ such that the map

$$\mathbf{\Psi} : B_{\mathrm{id}}^\perp(\delta) \times \mathbf{B}_{A_0,\Phi_0}(\varepsilon) \to \mathcal{U}_{A_0,\Phi_0}, \qquad (u, A, \Phi) \mapsto u(A, \Phi)$$

gives a diffeomorphism from an open neighborhood of $(\mathrm{id}_E, A_0, \Phi_0)$ onto an open neighborhood of (A_0, Φ_0). In particular, we obtain a map $\mathcal{U}_{A_0,\Phi_0} \to \mathrm{Stab}_{A_0,\Phi_0}^\perp$, given by $(A, \Phi) \mapsto u = u_{A,\Phi}$, such that

$$\begin{aligned}\mathbf{\Psi}^{-1}(A, \Phi) = \big(u, u^{-1}(A, \Phi)\big) &\in B_{\mathrm{id}}^\perp(\delta) \times \mathbf{B}_{A_0,\Phi_0}(\varepsilon)\\ &\subset \mathrm{Stab}_{A_0,\Phi_0}^\perp \times \mathbf{B}_{A_0,\Phi_0}(\varepsilon).\end{aligned}$$

Hence, for any $(A,\Phi)\in\mathcal{U}_{A_0,\Phi_0}$ there is a unique $u\in B^{\perp}_{\mathrm{id}}(\delta)$ such that $u^{-1}(A,\Phi)-(A_0,\Phi_0)\in\mathbf{K}_{A_0,\Phi_0}$:

$$d^{0,*}_{A_0,\Phi_0}(u^{-1}(A_1,\Phi_1)-(A_0,\Phi_0))=0. \tag{2.10}$$

The neighborhood $\mathbf{B}_{A_0,\Phi_0}(\varepsilon)$ is $\mathrm{Stab}_{A_0,\Phi_0}$-invariant: if $(A,\Phi)\in\mathbf{B}_{A_0,\Phi_0}(\varepsilon)$ and $\gamma\in\mathrm{Stab}_{A_0,\Phi_0}$, then

$$\begin{aligned}\|\gamma(A_1,\Phi_1)-(A,\Phi)\|_{L^2_{k,A_0}} &= \|(A_1,\Phi_1)-\gamma^{-1}(A,\Phi)\|_{L^2_{k,\gamma^{-1}(A_0)}}\\ &= \|(A_1,\Phi_1)-(A,\Phi)\|_{L^2_{k,A_0}}<\varepsilon,\end{aligned}$$

and

$$\begin{aligned}d^{0,*}_{A_0,\Phi_0}(\gamma(A,\Phi)-(A_0,\Phi_0)) &= \gamma\left(d^{0,*}_{\gamma^{-1}(A_0,\Phi_0)}((A,\Phi)-(A_0,\Phi_0))\right)\\ &= \gamma\left(d^{0,*}_{A_0,\Phi_0}((A,\Phi)-(A_0,\Phi_0))\right)=0,\end{aligned}$$

so $\gamma(A,\Phi)\in\mathbf{B}_{A_0,\Phi_0}(\varepsilon)$.

The group ${}^\circ\mathcal{G}_E$ acts on ${}^\circ\mathcal{G}_E\times\mathbf{S}_{A_0,\Phi_0}$ by $(u,A,\Phi)\mapsto(vu,A,\Phi)$, and so gives a diffeomorphism

$$B_{\mathrm{id}}(\delta)\times\mathbf{B}_{A_0,\Phi_0}(\varepsilon)\to B_v(\delta)\times\mathbf{B}_{A_0,\Phi_0}(\varepsilon),\quad (u,A,\Phi)\mapsto(vu,A,\Phi),$$

and as this action commutes with the given action of $\mathrm{Stab}_{A_0,\Phi_0}$, it descends to a diffeomorphism

$$\begin{gathered}B_{\mathrm{id}}(\delta)\times_{\mathrm{Stab}_{A_0,\Phi_0}}\mathbf{B}_{A_0,\Phi_0}(\varepsilon)\to B_v(\delta)\times_{\mathrm{Stab}_{A_0,\Phi_0}}\mathbf{B}_{A_0,\Phi_0}(\varepsilon),\\ [u,A,\Phi]\mapsto[vu,A,\Phi],\end{gathered}$$

for each $v\in{}^\circ\mathcal{G}_E$. Consequently, the ${}^\circ\mathcal{G}_E$-equivariant map

$${}^\circ\mathcal{G}_E\times_{\mathrm{Stab}_{A_0,\Phi_0}}\mathbf{B}_{A_0,\Phi_0}(\varepsilon)\to\tilde{\mathcal{C}}_{W,E}$$

is a diffeomorphism onto its image, as desired. q.e.d.

Plainly, $[\gamma(A,\Phi)]=[A,\Phi]$ for each $\gamma\in\mathrm{Stab}_{A_0,\Phi_0}$ and $(A,\Phi)\in\mathbf{B}_{A_0,\Phi_0}(\varepsilon)$ and hence, the projection $\pi:\mathbf{B}_{A_0,\Phi_0}(\varepsilon)\to\mathcal{C}_{W,E}$ factors through $\mathbf{B}_{A_0,\Phi_0}(\varepsilon)/\mathrm{Stab}_{A_0,\Phi_0}$.

Claim 2.11. There is a positive constant $\delta=\delta(A_0,\Phi_0,k)$ with the following significance. If $(A_i,\Phi_i)\in\mathbf{B}_{A_0,\Phi_0}(\varepsilon)$ for $i=1,2$ and there is a gauge transformation $u\in B_{\mathrm{id}}(\delta)$ such that $u(A_1,\Phi_1)=(A_2,\Phi_2)$, then $u\in\mathrm{Stab}_{A_0,\Phi_0}$.

Proof. For small enough δ, Claim 2.9 implies that $u \in B_{\mathrm{id}}(\delta)$ can be written uniquely as $u = u_1\gamma$ with $u_1 \in B_{\mathrm{id}}^{\perp}(\delta)$ and $\gamma \in \mathrm{Stab}_{A_0,\Phi_0}$. Thus, $(A_2, \Phi_2) = u_1\gamma(A_1, \Phi_1) \in \mathbf{B}_{A_0,\Phi_0}(\varepsilon)$. But the neighborhood $\mathbf{B}_{A_0,\Phi_0}(\varepsilon)$ is $\mathrm{Stab}_{A_0,\Phi_0}$-invariant, so we also have $\gamma(A_1, \Phi_1) \in \mathbf{B}_{A_0,\Phi_0}(\varepsilon)$. Therefore $u_1 = \mathrm{id}_E$ by the uniqueness assertion of Claim 2.9 and so $u = \gamma \in \mathrm{Stab}_{A_0,\Phi_0}$. This completes the proof of the claim. q.e.d.

Claim 2.11 shows that $\mathbf{B}_{A_0,\Phi_0}(\varepsilon)/\operatorname{Stab}_{A_0,\Phi_0}$ injects into the quotient $\mathcal{C}_{W,E}$ modulo the assumption that the gauge transformations are close to id_E. It remains to show that if $(A_i, \Phi_i) \in \mathbf{B}_{A_0,\Phi_0}(\varepsilon)$ for $i = 1, 2$ and there is a gauge transformation $u \in {}^\circ\mathcal{G}_E$ such that $u(A_1, \Phi_1) = (A_2, \Phi_2)$, then u is necessarily close to id_E and hence that $\mathbf{B}_{A_0,\Phi_0}(\varepsilon)/\operatorname{Stab}_{A_0,\Phi_0}$ injects into the quotient $\mathcal{C}_{W,E}$.

Claim 2.12. For small enough $\varepsilon(A_0, \Phi_0)$, the projection

$$\pi : \mathbf{B}_{A_0,\Phi_0}(\varepsilon)/\operatorname{Stab}_{A_0,\Phi_0} \to \mathcal{C}_{W,E}$$

is injective.

Proof. Suppose that $(A_i, \Phi_i) \in \mathbf{B}_{A_0,\Phi_0}(\varepsilon)$ for $i = 1, 2$ and that $[A_1, \Phi_1] = [A_2, \Phi_2] \in \mathcal{C}_{W,E}$, so that $u(A_1, \Phi_1) = (A_2, \Phi_2)$ for some $u \in {}^\circ\mathcal{G}_E$.

Since $u(A_0) = A_0 - (d_{A_0}u)u^{-1}$, we see that $u \in \mathrm{Stab}_{A_0,\Phi_0}$ if and only if $d^0_{A_0,\Phi_0}u = (d_{A_0}u, -u\Phi_0) = (0, -\Phi_0)$ or, equivalently, if and only if $d^0_{A_0,\Phi_0}(u - \mathrm{id}_E) = 0$, since $\mathrm{id}_E \in \mathrm{Stab}_{A_0,\Phi_0}$. Here, we view $u \in L^2_{k+1}(\mathfrak{gl}(E))$ via the isometric embedding ${}^\circ\mathcal{G}_E \subset L^2_{k+1}(\mathfrak{gl}(E))$ and write

$$u - \mathrm{id}_E = u_0 - \gamma,$$

where $u_0 \in (\operatorname{Ker} d^0_{A_0,\Phi_0})^{\perp}$ and $\gamma \in \operatorname{Ker} d^0_{A_0,\Phi_0}$. Our first task is to estimate $\|u - \mathrm{id}_E\|_{L^2_{3,A_0}}$.

By assumption, $d^0_{A_0,\Phi_0}\gamma = (d_{A_0}\gamma, \gamma\Phi) = 0$, so $d_{A_0}\gamma = 0$ and $\gamma\Phi_0 = 0$. Since $u(A_1) \equiv A_1 - (d_{A_1}u)u^{-1} = A_2$, we have

$$A_2u = A_1u - d_{A_1}u = A_1u - d_{A_0}u - [A_1 - A_0, u],$$

and therefore, using $d_{A_0}\mathrm{id}_E = 0 = d_{A_0}\gamma$, we have

$$d_{A_0}u_0 = d_{A_0}u = u(A_1 - A_0) - (A_2 - A_0)u.$$

As $d_{A_0}^*(A_i - A_0) = (\cdot\Phi_0)^*(\Phi_i - \Phi_0)$, we obtain

$$\begin{aligned} d_{A_0}^* d_{A_0} u_0 &= - * (d_{A_0} u \wedge *(A_1 - A_0)) + u d_{A_0}^*(A_1 - A_0) \\ &\quad - (d_{A_0}^*(A_2 - A_0))u + *(*(A_2 - A_0) \wedge d_{A_0} u) \\ &= - * (d_{A_0} u_0 \wedge *(A_1 - A_0)) + u(\cdot\Phi_0)^*(\Phi_1 - \Phi_0) \\ &\quad - ((\cdot\Phi_0)^*(\Phi_2 - \Phi_0))u + *(*(A_2 - A_0) \wedge d_{A_0} u_0). \end{aligned}$$

We define the Laplacian $\Delta^0_{A_0,\Phi_0}$ by setting

$$\Delta^0_{A_0,\Phi_0} = d^{0,*}_{A_0,\Phi_0} d^0_{A_0,\Phi_0} = d^*_{A_0} d_{A_0} + (\cdot\Phi_0)^*\Phi_0.$$

Now $u_0 = u - \mathrm{id}_E + \gamma$ and $d_{A_0}\mathrm{id}_E = 0 = d_{A_0}\gamma$ and $\gamma\Phi_0 = 0$ so, using $u\Phi_1 = \Phi_2$, we obtain

$$\begin{aligned} \Delta^0_{A_0,\Phi_0} u_0 &= \Delta^0_{A_0,\Phi_0}(u - \mathrm{id}_E + \gamma) \\ &= d^*_{A_0} d_{A_0}(u - \mathrm{id}_E + \gamma) + (\cdot\Phi_0)^*(u - \mathrm{id}_E)\Phi_0 \\ &= d^*_{A_0} d_{A_0} u + (\cdot\Phi_0)^*(u\Phi_0 - u\Phi_1 + u\Phi_1 - \Phi_0) \\ &= d^*_{A_0} d_{A_0} u_0 + (\cdot\Phi_0)^* \left(u(\Phi_0 - \Phi_1) + (\Phi_2 - \Phi_0)\right). \end{aligned}$$

Our assumption that $(A_i, \Phi_i) \in \mathbf{B}_{A_0,\Phi_0}(\varepsilon)$ and the embeddings $L^2_k \subset L^2_2$ imply that for $i = 1, 2$ we have

$$\|(A_i, \Phi_i) - (A_0, \Phi_0)\|_{L^2_{2,A_0}} \le \|(A_i, \Phi_i) - (A_0, \Phi_0)\|_{L^2_{k,A_0}} < \varepsilon.$$

Since $u_0 \in (\operatorname{Ker} d^0_{A_0,\Phi_0})^\perp$, the standard elliptic estimate for the Laplacian $\Delta^0_{A_0,\Phi_0}$, the fact that $|u| = 1$ as $u \in {}^\circ\mathcal{G}_E$, the Sobolev embedding $L^2_2 \subset L^q$, $2 \le q < \infty$, and multiplication $L^2_2 \times L^2_2 \to L^2_1$, and our expression for $d^*_{A_0} d_{A_0} u = d^*_{A_0} d_{A_0} u_0$ combine to give

$$\begin{aligned} \|u_0\|_{L^2_{3,A_0}} &\le C\|\Delta^0_{A_0,\Phi_0} u_0\|_{L^2_{1,A_0}} \\ &\le C\|d^*_{A_0} d_{A_0} u_0\|_{L^2_{1,A_0}} + C\|(\cdot\Phi_0)^*(u(\Phi_0 - \Phi_1) + (\Phi_2 - \Phi_0))\|_{L^2_{1,A_0}} \\ &\le C\|d_{A_0} u_0\|_{L^2_{2,A_0}}(\|A_1 - A_0\|_{L^2_{2,A_0}} + \|A_2 - A_0\|_{L^2_{2,A_0}}) \\ &\quad + C\|u\|_{L^2_{2,A_0}}\|\Phi_0\|_{L^2_{2,A_0}}(\|\Phi_1 - \Phi_0\|_{L^2_{2,A_0}} + \|\Phi_2 - \Phi_0\|_{L^2_{2,A_0}}) \\ &\le C\|u_0\|_{L^2_{3,A_0}}\varepsilon + C\|u\|_{L^2_{2,A_0}}\|\Phi_0\|_{L^2_{2,A_0}}\varepsilon \\ &\le C\|u_0\|_{L^2_{3,A_0}}\varepsilon + C\|\Phi_0\|_{L^2_{2,A_0}}\varepsilon, \end{aligned}$$

for some constant $C = C(A_0, \Phi_0)$. For small enough ε we can rearrange the preceding inequality and use the Sobolev embedding $L^2_3 \subset C^0$ to yield

$$\|u_0\|_{L^2_{3,A_0}} \leq C\|\Phi_0\|_{L^2_{2,A_0}}\varepsilon \quad \text{and} \quad \|u_0\|_{C^0} \leq C\|\Phi_0\|_{L^2_{2,A_0}}\varepsilon.$$

Therefore, as $|u| = 1$, we have the pointwise bounds

$$1 - |u_0| \leq |\mathrm{id}_E - \gamma| \leq 1 + |u_0|,$$

and thus, for ε small, $|\mathrm{id}_E - \gamma| > 0$; the pointwise norm of $z = |\mathrm{id}_E - \gamma|$ is constant since $d_{A_0}(\mathrm{id}_E - \gamma) = 0$. Consequently, $\gamma_0 \equiv z^{-1}(\mathrm{id}_E - \gamma)$ lies in ${}^\circ\mathcal{G}_E$: clearly, $d_{A_0}\gamma_0 = 0$, so $\gamma_0 \in \mathrm{Stab}_A$. If $\Phi_0 \equiv 0$, we trivially have $\gamma_0 \in \mathrm{Stab}_{\Phi_0}$. If $\Phi_0 \not\equiv 0$, then the equalities $|\gamma_0\Phi_0| = |\Phi_0|$ (as γ_0 is a unitary gauge transformation) and $\gamma_0\Phi_0 = z^{-1}(\mathrm{id}_E - \gamma)\Phi_0 = z^{-1}\Phi_0$, so $|\gamma_0\Phi_0| = z^{-1}|\Phi_0|$ implying that $z = 1$ (it is enough to have equality of L^2 norms here as z is constant). Hence, we also have $\gamma_0 \in \mathrm{Stab}_{\Phi_0}$ and in particular, $\gamma_0 \in \mathrm{Stab}_{A_0,\Phi_0} = \mathrm{Stab}_{A_0} \cap \mathrm{Stab}_{\Phi_0}$.

We now write $\gamma_0^{-1}u = \gamma_0^{-1}u_0 + \gamma_0^{-1}(\mathrm{id}_E - \gamma) = \gamma_0^{-1}u_0 + z\,\mathrm{id}_E$, so that

$$\gamma_0^{-1}u - \mathrm{id}_E = (z-1)\mathrm{id}_E + \gamma_0^{-1}u_0.$$

Clearly, $(z-1)\mathrm{id}_E \in \operatorname{Ker} d^0_{A_0,\Phi_0}$ by the remarks of the last paragraph, while $\gamma_0^{-1}u_0 \in (\operatorname{Ker} d^0_{A_0,\Phi_0})^\perp$ since $u_0 \in (\operatorname{Ker} d^0_{A_0,\Phi_0})^\perp$ and $\gamma_0^{-1} \in \mathrm{Stab}_{A_0,\Phi_0}$. Similarly, $\gamma_0^{-1}u(A_1,\Phi_1) = \gamma_0^{-1}(A_2,\Phi_2) \in \mathbf{B}_{A_0,\Phi_0}(\varepsilon)$, as the neighborhood $\mathbf{B}_{A_0,\Phi_0}(\varepsilon)$ is $\mathrm{Stab}_{A_0,\Phi_0}$-invariant. Thus, replacing u by $\gamma_0^{-1}u$ and (A_2,Φ_2) by $(\tilde{A}_2,\tilde{\Phi}_2) \equiv \gamma_0^{-1}(A_2,\Phi_2)$ in our L^2_{3,A_0} estimate for u_0 yields

$$\|\gamma_0^{-1}u_0\|_{L^2_{3,A_0}} \leq C\|\Phi_0\|_{L^2_{2,A_0}}\varepsilon,$$

while $|z-1| \leq \|u_0\|_{C^0} \leq C\|\Phi_0\|_{L^2_{2,A_0}}\varepsilon$, so we find that

$$\|\gamma_0^{-1}u - \mathrm{id}_E\|_{L^2_{3,A_0}} \leq C\|\Phi_0\|_{L^2_{2,A_0}}\varepsilon.$$

If ε is small enough that $C\|\Phi_0\|_{L^2_{2,A_0}}\varepsilon < \delta$, where $\delta = \delta(A_0,\Phi_0)$ is the constant of Claim 2.11 with $k = 2$, then $\gamma_0^{-1}u \in \mathrm{Stab}_{A_0,\Phi_0}$ and so u lies in $\mathrm{Stab}_{A_0,\Phi_0}$, as desired. q.e.d.

Claim 2.13. The quotient space $\mathcal{C}_{W,E}$ is Hausdorff.

Proof. Let Γ be the subspace $\{\{(A,\Phi), u(A,\Phi)\} : (A,\Phi) \in \tilde{\mathcal{C}}_{W,E}$ and $u \in {}^\circ\mathcal{G}_E\}$ of $\tilde{\mathcal{C}}_{W,E} \times \tilde{\mathcal{C}}_{W,E}$. If $\{(A_\alpha,\Phi_\alpha), u_\alpha(A_\alpha,\Phi_\alpha)\}$ is a sequence in Γ

which converges in L^2_k to a point $\{(A_\infty, \Phi_\infty), (B_\infty, \Psi_\infty)\}$, then Lemma 2.7 implies that there is a subsequence $\{\alpha'\} \subset \{\alpha\}$ such that $\{u_\alpha\}$ converges in L^2_{k+1} to $u_\infty \in {}^\circ\mathcal{G}_E$ and $u_\infty(A_\infty) = B_\infty$. But then $u_{\alpha'}\Phi_{\alpha'}$ converges in L^2_k to $u_\infty\Phi_\infty$ and so $\Psi_\infty = u_\infty\Phi_\infty$. Thus, $(B_\infty, \Psi_\infty) = u_\infty(A_\infty, \Phi_\infty)$ for some $u_\infty \in {}^\circ\mathcal{G}_E$, and therefore Γ is closed. Hence the quotient $\tilde{\mathcal{C}}_{W,E}/{}^\circ\mathcal{G}_E$ is Hausdorff. q.e.d.

Claim 2.13 gives Assertion (1) of the proposition. Thus Assertions (2), (3), and (4) now follow from the preceding arguments and Claim 2.12. This completes the proof of the proposition. q.e.d.

Remark 2.14. (1) Alternatively, one can show that the quotient L^2_k topology on $\mathcal{C}_{W,E}$ is metrizable via the L^2 metric (exactly as in [20, Lemma 4.2.4]) and thus $\mathcal{C}_{W,E}$ is Hausdorff.

(2) As Mrowka pointed out to us, one can sharpen the assertions of Proposition 2.8, at least for the quotient space $\mathcal{B}_E$: one finds that charts are provided by L^4-balls in $\operatorname{Ker} d^*_{A_0}$ rather than the much smaller L^2_{k,A_0}-balls usually employed; see [22].

It is convenient to extract the following global, Coulomb gauge-fixing result (analogous to Proposition 2.3.4 in [20]) which we established in the course of proving Proposition 2.8:

Lemma 2.15. *Let X be a closed, oriented, Riemannian spin^c four-manifold and let E be a Hermitian vector bundle over X. Suppose that $k \geq 2$ is an integer and that $(A_0, \Phi_0) \in \tilde{\mathcal{C}}_{W,E}$. Then there is a positive constant $\varepsilon = \varepsilon(A_0, \Phi_0, k)$ such that for any $(A, \Phi) \in \tilde{\mathcal{C}}_{W,E}$ with*

$$\|(A, \Phi) - (A_0, \Phi_0)\|_{L^2_{k,A_0}(X)} < \varepsilon,$$

there is a gauge transformation $u \in {}^\circ\mathcal{G}_E$, unique up to an element of $\mathrm{Stab}_{A_0,\Phi_0}$, *such that*

$$d^{0,*}_{A_0,\Phi_0}(u(A, \Phi) - (A_0, \Phi_0)) = 0.$$

2.3. Connections on SO(3) bundles and groups of gauge transformations. For some local patching arguments over simply-connected open regions $Y \subset X$ in §3 and §4 it is very useful to be able to lift gauge transformations in $\mathcal{G}_{\mathfrak{su}(E)}(Y)$ to gauge transformations in $\mathcal{G}_E(Y)$. The following result tells us that this is always possible when Y is simply connected; it is an extension of Theorem IV.3.1 in [61] from SU(2) to U(2) bundles.

Proposition 2.16. *Let E be a Hermitian two-plane bundle over a connected manifold X. Then there is an exact sequence*

$$1 \to \{\pm \mathrm{id}_E\} \to \mathcal{G}_E \to \mathcal{G}_{\mathfrak{su}(E)} \to H^1(X;\mathbb{Z}/2\mathbb{Z}).$$

Proof. Let P be the U(2) principal bundle underlying E so $P^{\mathrm{ad}} = P/S^1_Z$ is the SO(3) principal bundle underlying $\mathfrak{su}(E)$. We can view elements of $\mathcal{G}_E$ as maps $u : P \to \mathrm{SU}(2)$ satisfying $u(pg) = g^{-1}u(p)g$ for $p \in P$ and $g \in \mathrm{U}(2)$, and similarly for elements of $\mathcal{G}_{\mathfrak{su}(E)}$. The map $\mathcal{G}_E \to \mathcal{G}_{\mathfrak{su}(E)}$ is then given by $u \to \mathrm{Ad}u$. Because X is connected, the kernel of this map is $\{\pm \mathrm{id}_E\}$, giving the exactness of all but the last terms in the sequence.

Let $\nu_2 \in H^1(\mathrm{SO}(3);\mathbb{Z}/2\mathbb{Z})$ be the non-trivial cohomology class given by the double cover $\mathrm{SU}(2) \to \mathrm{SO}(3)$. Then a map $u : V \to \mathrm{SO}(3)$ lifts to SU(2) if and only if $u^*\nu_2 = 0$. We would like to define the map $\alpha : \mathcal{G}_{\mathfrak{su}(E)} \to H^1(X;\mathbb{Z}/2\mathbb{Z})$ by $\alpha(u) = u^*\nu_2$. It is not immediately clear that $\alpha(u)$ actually lies in $H^1(X;\mathbb{Z}/2\mathbb{Z})$. The homotopy exact sequence

$$\pi_1(\mathrm{SO}(3)) \to \pi_1(P^{\mathrm{ad}}) \to \pi_1(X) \to 1$$

gives an exact sequence in cohomology $H^1(\cdot\,;\mathbb{Z}/2\mathbb{Z}) = \mathrm{Hom}(\pi_1(\cdot),\mathbb{Z}/2\mathbb{Z})$,

$$1 \to H^1(X;\mathbb{Z}/2\mathbb{Z}) \to H^1(P^{\mathrm{ad}};\mathbb{Z}/2\mathbb{Z}) \to H^1(\mathrm{SO}(3);\mathbb{Z}/2\mathbb{Z}).$$

Thus $u^*\nu_2$ is pulled back from a unique element of $H^1(X;\mathbb{Z}/2\mathbb{Z})$ if and only if $i_x^*u^*\nu_2 = 0$, where $i_x : \mathrm{SO}(3) \to P^{\mathrm{ad}}$ is the inclusion of a fiber.

Because $u^*\nu_2$ depends only on the homotopy class of u, the map α is constant on connected components. Let $\mathcal{G}^x_{\mathfrak{su}(E)}$ be the subgroup of elements of $\mathcal{G}_{\mathfrak{su}(E)}$ which are the identity over $x \in X$. The exact sequence of groups

$$1 \to \mathcal{G}^x_{\mathfrak{su}(E)} \to \mathcal{G}_{\mathfrak{su}(E)} \to \mathrm{SO}(3) \to 1$$

and the corresponding exact sequence of homotopy groups (π_0 to be specific) show that the inclusion $\mathcal{G}^x_{\mathfrak{su}(E)}$ induces a surjection of connected components. Now if $u \in \mathcal{G}^x_{\mathfrak{su}(E)}$ then $i_x^*u^*\nu_2 = 0$, and so $i_xu^*\nu_2 = 0$ for all $u \in \mathcal{G}_{\mathfrak{su}(E)}$. This implies that the map α takes values in $H^1(X;\mathbb{Z}/2\mathbb{Z})$.

By the defining property of ν_2, we have $\alpha(u) = 0$ if and only if there is a lift of u to SU(2). Because $\{\pm \mathrm{id}_E\}$ is central, any Ad-equivariant map $\tilde{u} : P \to \mathrm{SU}(2)$ descends to $P^{\mathrm{ad}} \to \mathrm{SU}(2)$. Thus $u \in \mathcal{G}_{\mathfrak{su}(E)}$ is induced by some $\tilde{u} \in \mathcal{G}_E$ if and only if $\alpha(u) = 0$. q.e.d.

Remark 2.17. One can also show that the map

$$\mathcal{G}_{\mathfrak{su}(E)} \to H^1(X;\mathbb{Z}/2\mathbb{Z})$$

is surjective.

2.4. Quadratic forms for coupled spinor bundles. Give $\mathfrak{gl}(E)$ (respectively, $\mathfrak{gl}(W^{\pm})$) the Hermitian fiber inner product and norm

$$\langle M,N\rangle := \tfrac{1}{2}\operatorname{tr}(M^\dagger N) \quad \text{and} \quad |M|^2 = \langle M,M\rangle,$$

for $M,N\in\Omega^0(\mathfrak{gl}(E))$ (respectively, $\mathfrak{gl}(W^{\pm})$), so that $|\mathrm{id}_E|=1$, etc. If $\Phi\in\Omega^0(W^+\otimes E)$, then $\Phi^*\in\Omega^0((W^+\otimes E)^*)=\Omega^0((W^+)^*\otimes E^*)$ is defined by $\Phi^*(\Psi):=\langle\Psi,\Phi\rangle$ for $\Psi\in\Omega^0(W^+\otimes E)$.

With respect to the induced Hermitian fiber inner products on $\mathfrak{gl}(W^+)$ and $\mathfrak{gl}(E)$, we have fiber-wise orthogonal direct sums, $\mathfrak{gl}(W^+)=\mathbb{C}\mathrm{id}_{W^+}\oplus\mathfrak{sl}(W^+)$, and similarly for $\mathfrak{gl}(E)$ and thus $\mathfrak{gl}(W^+)\otimes_{\mathbb{C}}\mathfrak{gl}(E)$. Let π_{W^+} (respectively, π_E) be the fiber-wise orthogonal projection from $\mathfrak{gl}(W^+)$ (respectively, $\mathfrak{gl}(E)$) onto the subspace $\mathfrak{sl}(W^+)$ (respectively, $\mathfrak{sl}(E)$), so that

$$\pi_{W^+}M := M-\tfrac{1}{2}\operatorname{tr}M\,\mathrm{id}_{W^+} \quad\text{and}\quad \pi_E N := N-\tfrac{1}{2}\operatorname{tr}N\,\mathrm{id}_E$$

are the projection onto the traceless components of $M\in\mathfrak{gl}(W^+)$ and $N\in\mathfrak{gl}(E)$, and

$$\pi_{W^+}^{\perp}M := \tfrac{1}{2}\operatorname{tr}M\,\mathrm{id}_{W^+} \quad\text{and}\quad \pi_E^{\perp}N := \tfrac{1}{2}\operatorname{tr}N\,\mathrm{id}_E$$

are the projections onto the trace components of $M\in\mathfrak{gl}(W^+)$ and $N\in\mathfrak{gl}(E)$. The orthogonal projection from $\mathfrak{gl}(W^+)\otimes_{\mathbb{C}}\mathfrak{gl}(E)$ onto $\mathfrak{sl}(W^+)\otimes_{\mathbb{C}}\mathfrak{sl}(E)$ is obtained by writing $\pi_{W^+}=\mathrm{id}_{W^+}-\pi_{W^+}^{\perp}$ and $\pi_E=\mathrm{id}_E-\pi_E^{\perp}$, so that

$$\begin{aligned}\pi_{W^+}\otimes\pi_E =&\mathrm{id}_{W^+}\otimes\mathrm{id}_E-\left(\pi_{W^+}\otimes\pi_E^{\perp}+\pi_{W^+}^{\perp}\otimes\pi_E\right)\\ &+\pi_{W^+}^{\perp}\otimes\pi_E^{\perp}.\end{aligned}\tag{2.11}$$

Consequently, if $\phi\in\Omega^0(W^+)$ and $\Phi\in\Omega^0(W^+\otimes E)$, we may define

$$\begin{aligned}(\phi\otimes\phi^*)_0 &:= \pi_{W^+}(\phi\otimes\phi^*),\\ (\Phi\otimes\Phi^*)_{00} &:= (\pi_{W^+}\otimes\pi_E)(\Phi\otimes\Phi^*).\end{aligned}\tag{2.12}$$

Note that

$$\mathfrak{gl}(W^+)\otimes_{\mathbb{C}}\mathfrak{gl}(E)=\mathfrak{u}(W^+)\otimes_{\mathbb{R}}\mathfrak{u}(E)+i\mathfrak{u}(W^+)\otimes_{\mathbb{R}}\mathfrak{u}(E).$$

Plainly, $(\phi\otimes\phi^*)_0$ is a section of $\mathfrak{sl}(W^+)\cap i\mathfrak{u}(W^+)=i\mathfrak{su}(W^+)$. The following identities are now well-known [47], [77, Chapter 8]:

Lemma 2.18. *For every $\phi \in \Omega^0(W^+)$ or $\Omega^0(E)$, $T \in \Omega^0(\mathfrak{sl}(W^+))$, and $\eta \in \Omega^2(X,\mathbb{C})$, the following hold. The endomorphism $(\phi\otimes\phi^*)_0$ is a section of $\Omega^0(i\mathfrak{su}(W^+))$ and satisfies:*

$$|(\phi\otimes\phi^*)_0|^2 = \tfrac{1}{4}|\phi|^4, \tag{1}$$

$$|\rho(\eta)|^2 = 2|\eta^+|^2 \quad \text{and} \quad |\rho^{-1}T|^2 = \tfrac{1}{2}|T|^2. \tag{2}$$

Similarly, $\Phi\otimes\Phi^*$ is a section of $i\mathfrak{u}(W^+\otimes_{\mathbb{C}} E) = \mathfrak{u}(W^+)\otimes_{\mathbb{R}}\mathfrak{u}(E)$ and $(\Phi\otimes\Phi^*)_{00}$ is a section of $\mathfrak{su}(W^+)\otimes_{\mathbb{R}}\mathfrak{su}(E)$. We then have the following analogue of Lemma 2.18:

Lemma 2.19. *For every $\Phi \in \Omega^0(W^+\otimes E)$ and*

$$M \in \Omega^0(\mathfrak{gl}(W^+\otimes E)),$$

the following identities hold:

$$\langle \Phi\otimes\Phi^*, M\rangle = \tfrac{1}{2}\langle M\Phi,\Phi\rangle, \tag{1}$$

$$\left(\tfrac{5}{4}-\tfrac{1}{\sqrt{2}}\right)|\Phi|^4 \le \langle(\Phi\otimes\Phi^*)_{00}\Phi,\Phi\rangle \le \tfrac{1}{\sqrt{2}}|\Phi|^4, \tag{2}$$

$$\left(\tfrac{5}{8}-\tfrac{1}{2\sqrt{2}}\right)|\Phi|^4 \le |(\Phi\otimes\Phi^*)_{00}|^2 \le \tfrac{1}{2\sqrt{2}}|\Phi|^4. \tag{3}$$

Proof. Using $\Phi = c^a\Phi^a$ (we employ the summation convention), so $M\Phi = c^aM\Phi^a$ and $\Phi^*\Phi^a = \langle\Phi^a,\Phi\rangle = \bar{c}^a$, we have

$$\begin{aligned}\langle \Phi\otimes\Phi^*, M\rangle &= \tfrac{1}{2}\operatorname{tr}((\Phi\otimes\Phi^*)^\dagger M) = \tfrac{1}{2}\langle(\Phi\otimes\Phi^*)M\Phi^a,\Phi^a\rangle \\ &= \tfrac{1}{2}\langle M\Phi^a,(\Phi\otimes\Phi^*)\Phi^a\rangle = \tfrac{1}{2}\langle M\Phi^a,\Phi(\Phi^*\Phi^a)\rangle \\ &= \tfrac{1}{2}\langle M\Phi^a,\bar{c}^a\Phi\rangle = \tfrac{1}{2}\langle c^aM\Phi^a,\Phi\rangle = \tfrac{1}{2}\langle M\Phi,\Phi\rangle,\end{aligned}$$

which gives (1). Next, we see that

$$\begin{aligned}|\Phi\otimes\Phi^*|^2 &= \langle\Phi\otimes\Phi^*,\Phi\otimes\Phi^*\rangle = \tfrac{1}{2}\langle(\Phi\otimes\Phi^*)\Phi,\Phi\rangle \\ &= \tfrac{1}{2}|\Phi|^2\langle\Phi,\Phi\rangle = \tfrac{1}{2}|\Phi|^4,\end{aligned}$$

and so the upper bound in (2) is obtained by

$$\begin{aligned}|\langle(\Phi\otimes\Phi^*)_{00}\Phi,\Phi\rangle| &\le |(\Phi\otimes\Phi^*)_{00}||\Phi|^2 \le |\Phi\otimes\Phi^*||\Phi|^2 \\ &\le (1/\sqrt{2})|\Phi|^4 < \tfrac{3}{4}|\Phi|^4,\end{aligned}$$

where we use the fact that $(\,\cdot\,)_{00} = \pi_{W^+}\otimes\pi_E$ is an orthogonal projection to obtain the second inequality.

To obtain the lower bound in (2), suppose that $\Phi = \phi^a \otimes \xi^a$ (implied summation), where $\{\phi^a\} \subset \Omega^0(W^+)$ and $\{\xi^a\} \subset \Omega^0(E)$, then $\Phi \otimes \Phi^* = (\phi^a \otimes \phi^{b,*}) \otimes (\xi^a \otimes \xi^{b,*})$ and

$$\begin{aligned}(\pi_{W^+}^\perp \otimes \pi_E^\perp)(\Phi \otimes \Phi^*) &= \pi_{W^+}^\perp(\phi^a \otimes \phi^{b,*}) \otimes \pi_E^\perp(\xi^a \otimes \xi^{b,*}) \\ &= \tfrac{1}{4} \operatorname{tr}(\phi^a \otimes \phi^{b,*}) \operatorname{tr}(\xi^a \otimes \xi^{b,*}) \operatorname{id}_{W^+ \otimes E} \\ &= \tfrac{1}{4} \langle \phi^a, \phi^b \rangle \langle \xi^a, \xi^b \rangle \operatorname{id}_{W^+ \otimes E} \\ &= \tfrac{1}{4} \langle \phi^a \otimes \xi^a, \phi^b \otimes \xi^b \rangle \operatorname{id}_{W^+ \otimes E} \\ &= \tfrac{1}{4} \langle \Phi, \Phi \rangle \operatorname{id}_{W^+ \otimes E}.\end{aligned}$$

Using the last identity together with $(\Phi \otimes \Phi^*)\Phi = \Phi\langle \Phi, \Phi \rangle = \Phi|\Phi|^2$ and (2.11) gives

$$\begin{aligned}\langle (\Phi \otimes \Phi^*)_{00}\Phi, \Phi \rangle &= \langle (\Phi \otimes \Phi^*)\Phi, \Phi \rangle \\ &\quad - \langle (\pi_{W^+}^\perp \otimes \pi_E + \pi_{W^+} \otimes \pi_E^\perp)(\Phi \otimes \Phi^*)\Phi, \Phi \rangle \\ &\quad + \langle \tfrac{1}{4}|\Phi|^2\Phi, \Phi \rangle \\ &= \tfrac{5}{4}|\Phi|^4 - \langle (\pi_{W^+}^\perp \otimes \pi_E + \pi_{W^+} \otimes \pi_E^\perp)(\Phi \otimes \Phi^*)\Phi, \Phi \rangle.\end{aligned}$$

But $\pi_{W^+}^\perp \otimes \pi_E + \pi_{W^+} \otimes \pi_E^\perp$ is just the orthogonal projection onto the middle two factors of the orthogonal decomposition of $\mathfrak{gl}(W^+) \otimes_{\mathbb{C}} \mathfrak{gl}(E)$, so

$$\begin{aligned}&\left| \langle (\pi_{W^+}^\perp \otimes \pi_E + \pi_{W^+} \otimes \pi_E^\perp)(\Phi \otimes \Phi^*)\Phi, \Phi \rangle \right| \\ &\qquad \le \left| (\pi_{W^+}^\perp \otimes \pi_E + \pi_{W^+} \otimes \pi_E^\perp)(\Phi \otimes \Phi^*) \right| |\Phi|^2 \\ &\qquad \le |\Phi \otimes \Phi^*||\Phi|^2 \le (1/\sqrt{2})|\Phi|^4 < \tfrac{3}{4}|\Phi|^4,\end{aligned}$$

and therefore

$$|\langle (\Phi \otimes \Phi^*)_{00}\Phi, \Phi \rangle| \ge \tfrac{5}{4}|\Phi|^4 - (1/\sqrt{2})|\Phi|^4 > \tfrac{1}{2}|\Phi|^4.$$

Consequently, using $(\Phi \otimes \Phi^*)_{00}^\dagger = (\Phi \otimes \Phi^*)_{00}$, we have

$$\begin{aligned}\tfrac{1}{2}\langle (\Phi \otimes \Phi^*)_{00}\Phi, \Phi \rangle &= \langle (\Phi \otimes \Phi^*)_{00}, \Phi \otimes \Phi^* \rangle = \langle (\Phi \otimes \Phi^*)_{00}, (\Phi \otimes \Phi^*)_{00} \rangle \\ &= |(\Phi \otimes \Phi^*)_{00}|^2,\end{aligned}$$

which gives (3). q.e.d.

The fact that $|(\Phi \otimes \Phi^*)_0|^2 \ge \frac{1}{4}|\Phi|^4$, for $\Phi \in \Omega^0(W^+ \otimes E)$, is used in §4 to show that the moduli space of PU(2) monopoles has an Uhlenbeck compactification. (The analogous equality, $|(\phi\otimes\phi^*)_0|^2 = \frac{1}{4}|\phi|^4$ for $\phi\in\Omega^0(W^+)$, is used in [47], [98] to show that the moduli space of Seiberg-Witten monopoles is compact.)

We adapt some terminology from the proofs of Proposition 3.4 in [30] and Lemma 4.3.25 in [20] which we will need for our proof of transversality for the PU(2) monopole moduli space in §5:

Definition 2.20. The *rank* of a section $\Phi\in\Omega^0(W^\pm\otimes E)$ *at a point* $x\in X$ is the rank of $\Phi|_x$ considered as a complex linear map $(W^\pm)^*|_x\to E|_x$ (that is, its rank as a complex two-by-two matrix). We say that Φ is *rank one* on a subset $U\subset X$ if it has rank less than or equal to one at all points in U. Similarly, for $v\in\Omega^0(\Lambda^+(T^*X)\otimes\mathfrak{su}(E))$, the *rank* of v *at a point* $x\in X$ is the rank of v considered as a real linear map from $\Lambda^+(TX)|_x\to\mathfrak{su}(E)|_x$ (that is, its rank as a real three-by-three matrix). The *rank* of v on a set $U\subset X$ is the maximum rank of $v|_x$ over all points $x\in U$. Finally, for $M\in\Omega^0(\mathfrak{su}(W^+)\otimes\mathfrak{su}(E))$, the *rank* of M *at a point* $x\in X$ is the rank of M considered as a real linear map from $\mathfrak{su}(W^+)^*|_x\to\mathfrak{su}(E)|_x$ (again, its rank as a real three-by-three matrix). The *rank* of M on a set $U\subset X$ is the maximum rank of $M|_x$ over all points $x\in U$.

For the proof of Lemma 2.21 below we shall need a simple linear algebra identity. Suppose that $\Phi=\phi\otimes\xi$, for some $\phi\in\Omega^0(W^+)$ and $\xi\in\Omega^0(E)$, and that $\Psi=\psi\otimes\zeta$, with $\phi\in\Omega^0(W^+)$ and $\xi\in\Omega^0(E)$. Then

$$\Phi\otimes\Psi^* = (\phi\otimes\xi)\otimes(\psi\otimes\zeta)^* = (\phi\otimes\psi^*)\otimes(\xi\otimes\zeta^*)\in\mathfrak{gl}(W^+)\otimes_{\mathbb{C}}\mathfrak{gl}(E).$$

Writing $B\in\mathfrak{gl}(W^+)$ (or $\mathfrak{gl}(E)$) as $B=B_h+B_s$, where $B_h=\frac{1}{2}(B+B^\dagger)$ is Hermitian and $B_s=\frac{1}{2}(B-B^\dagger)$ is skew-Hermitian, we have

$$\begin{aligned}\Phi\otimes\Psi^* &= \tfrac{1}{4}(\phi\otimes\psi^*+\psi\otimes\phi^*)\otimes(\xi\otimes\zeta^*+\zeta\otimes\xi^*)\\ &\quad+\tfrac{1}{4}(\phi\otimes\psi^*-\psi\otimes\phi^*)\otimes(\xi\otimes\zeta^*-\zeta\otimes\xi^*)\\ &\quad+\tfrac{1}{4}(\phi\otimes\psi^*+\psi\otimes\phi^*)\otimes(\xi\otimes\zeta^*-\zeta\otimes\xi^*)\\ &\quad+\tfrac{1}{4}(\phi\otimes\psi^*-\psi\otimes\phi^*)\otimes(\xi\otimes\zeta^*+\zeta\otimes\xi^*),\end{aligned}$$

and similarly for $\Psi\otimes\Phi^*$. Therefore,

$$\begin{aligned}\Phi\otimes\Psi^*+\Psi\otimes\Phi^* &= \tfrac{1}{2}(\phi\otimes\psi^*+\psi\otimes\phi^*)\otimes(\xi\otimes\zeta^*+\zeta\otimes\xi^*)\\ &\quad+\tfrac{1}{2}(\phi\otimes\psi^*-\psi\otimes\phi^*)\otimes(\xi\otimes\zeta^*-\zeta\otimes\xi^*)\\ &= -\tfrac{1}{2}i(\phi\otimes\psi^*+\psi\otimes\phi^*)\otimes i(\xi\otimes\zeta^*+\zeta\otimes\xi^*)\\ &\quad+\tfrac{1}{2}(\phi\otimes\psi^*-\psi\otimes\phi^*)\otimes(\xi\otimes\zeta^*-\zeta\otimes\xi^*).\end{aligned}\tag{2.13}$$

We shall need the following elementary observation in our proof of transversality for the moduli space of PU(2) monopoles in §5. Note that as $\rho : \Lambda^+ \to \mathfrak{su}(W^+)$ is an isomorphism, the rank of a section v of $\Lambda^+ \otimes \mathfrak{su}(E)$ is equal to the rank of the section $\rho(v)$ of $\mathfrak{su}(W^+) \otimes \mathfrak{su}(E)$.

Lemma 2.21. *If $\Phi \in C^0(W^+ \otimes E)$ and $x \in X$, then the following hold:*

(1) $\operatorname{Rank}_{\mathbb{R}}(\Phi(x) \otimes \Phi^*(x))_{00} = 1$ *if and only if* $\operatorname{Rank}_{\mathbb{C}} \Phi(x) = 1$,

(2) $\operatorname{Rank}_{\mathbb{R}}(\Phi(x) \otimes \Phi^*(x))_{00} = 3$ *if and only if* $\operatorname{Rank}_{\mathbb{C}} \Phi(x) = 2$.

Proof. For convenience we write Φ, W^+, and E for $\Phi|_x$, $W^+|_x$, and $E|_x$. If $\Phi \in \operatorname{Hom}(W^+, E)$ has complex rank one, we can write $\Phi = \phi \otimes \xi$, for $\phi \in W^+$ and $\xi \in E$. Then $(\Phi \otimes \Phi^*)_{00} = -i(\phi \otimes \phi^*)_0 \otimes i(\xi \otimes \xi^*)_0$ and so $(\Phi \otimes \Phi^*)_{00} \in \operatorname{Hom}(\mathfrak{su}(W^+), \mathfrak{su}(E))$ has real rank one.

Conversely, suppose $(\Phi \otimes \Phi^*)_{00}$ has real rank one. Let $\{\xi_1, \xi_2\}$ be an orthonormal basis for E and write $\Phi = \phi_1 \otimes \xi_1 + \phi_2 \otimes \xi_2$: if $\phi_1, \phi_2 \in E$ are linearly dependent, then Φ has complex rank one and we are done, so suppose that ϕ_1, ϕ_2 are linearly independent. Using the standard basis $\{\sigma_1, \sigma_2, \sigma_3\}$ of Pauli matrices (2.14) for $\mathfrak{su}(2)$, namely

$$\sigma_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \tag{2.14}$$

we obtain bases for the real vector spaces $\mathfrak{su}(W^+)$ and $\mathfrak{su}(E)$:

$$\begin{aligned}
\sigma_1^e &= i(\xi_1 \otimes \xi_1^* - \xi_2 \otimes \xi_2^*), \quad \sigma_2^e = \xi_1 \otimes \xi_2^* - \xi_2 \otimes \xi_1^*, \\
\sigma_3^e &= i(\xi_1 \otimes \xi_2^* + \xi_2 \otimes \xi_1^*), \\
\sigma_1^w &= i(\phi_1 \otimes \phi_1^* - \phi_2 \otimes \phi_2^*), \quad \sigma_2^w = \phi_1 \otimes \phi_2^* - \phi_2 \otimes \phi_1^*, \\
\sigma_3^w &= i(\phi_1 \otimes \phi_2^* + \phi_2 \otimes \phi_1^*).
\end{aligned}$$

Writing $\Phi = \Phi_1 + \Phi_2$, we have

$$(\Phi_a \otimes \Phi_a^*)_{00} = -i(\phi_a \otimes \phi_a^*)_0 \otimes i(\xi_a \otimes \xi_a^*)_0$$

for $a = 1, 2$ and

$$(\Phi \otimes \Phi^*)_{00} = (\Phi_1 \otimes \Phi_1^*)_{00} + (\Phi_1 \otimes \Phi_2^* + \Phi_2 \otimes \Phi_1^*)_{00} + (\Phi_2 \otimes \Phi_2^*)_{00}.$$

So, by the identity (2.13) and noting that

$$i(\xi_1 \otimes \xi_1^*)_0 = \tfrac{1}{2} i(\xi_1 \otimes \xi_1^* - \xi_2 \otimes \xi_2^*) = \tfrac{1}{2}\sigma_1^e = -i(\xi_2 \otimes \xi_2^*)_0,$$

and

$$i(\phi_1 \otimes \phi_1^*)_0 = \tfrac{1}{2}i(\phi_1 \otimes \phi_1^* - \phi_2 \otimes \phi_2^*) = \tfrac{1}{2}\sigma_1^w = -i(\phi_2 \otimes \phi_2^*)_0$$

we see that

$$\begin{aligned}
(\Phi \otimes \Phi^*)_{00} &= -i(\phi_1 \otimes \phi_1^*)_0 \otimes i(\xi_1 \otimes \xi_1^*)_0 - i(\phi_2 \otimes \phi_2^*)_0 \otimes i(\xi_2 \otimes \xi_2^*)_0 \\
&\quad - \tfrac{1}{2}i(\phi_1 \otimes \phi_2^* + \phi_2 \otimes \phi_1^*) \otimes i(\xi_1 \otimes \xi_2^* + \xi_2 \otimes \xi_1^*) \\
&\quad + \tfrac{1}{2}(\phi_1 \otimes \phi_2^* - \phi_2 \otimes \phi_1^*) \otimes (\xi_1 \otimes \xi_2^* - \xi_2 \otimes \xi_1^*) \\
&= -\tfrac{1}{4}\sigma_1^w \otimes \sigma_1^e - \tfrac{1}{4}\sigma_1^w \otimes \sigma_1^e - \tfrac{1}{2}\sigma_3^w \otimes \sigma_3^e + \tfrac{1}{2}\sigma_2^w \otimes \sigma_2^e \\
&= -\tfrac{1}{2}\sigma_1^w \otimes \sigma_1^e + \tfrac{1}{2}\sigma_2^w \otimes \sigma_2^e - \tfrac{1}{2}\sigma_3^w \otimes \sigma_3^e.
\end{aligned}$$

Thus, $(\Phi \otimes \Phi^*)_{00}$ would have real rank three, a contradiction, and so Φ must have complex rank one, which proves Assertion 1.

The preceding argument also shows that if Φ has complex rank two, then $(\Phi \otimes \Phi^*)_{00}$ has real rank three. Conversely, if $(\Phi \otimes \Phi^*)_{00}$ has real rank three, then Φ cannot have complex rank one by Assertion 1, so Φ has complex rank two. q.e.d.

2.5. The PU(2) monopole equations and holonomy perturbations. In §2.5.1 we describe the PU(2) monopole equations in their unperturbed form, following [71], [74]. In §2.5.2 we introduce the holonomy perturbations and the perturbed PU(2) monopole equations, deferring a detailed discussion of most of the technical regularity issues concerning holonomy perturbations to the Appendix. In Donaldson's application to the extended anti-self-dual equation, some important features ensure that the requisite analysis is relatively tractable: (i) reducible connections can be excluded from the compactification of the extended moduli spaces [18, p. 283], (ii) the cohomology groups for the elliptic complex of his extended equations have simple weak semi-continuity properties with respect to Uhlenbeck limits [18, Proposition 4.33], and (iii) the zero locus being perturbed is cut out of a finite-dimensional manifold [18, p. 281, Lemma 4.35, & Corollary 4.38]. For the development of Donaldson's method for PU(2) monopoles described here, none of these simplifying features hold and so the corresponding transversality argument is rather complicated. Indeed, from Proposition 7.1.32 in [20] one can see that because of the Dirac operator, the behavior of the cokernels of the linearization of the PU(2) monopole equations can be quite involved under Uhlenbeck limits. The holonomy perturbations considered by Donaldson in [18] are inhomogeneous, as he uses the perturbations to kill the cokernels $\operatorname{Coker} d_A^+$ directly. In contrast, the perturbations which we consider in (1.3) are homogeneous and we shall argue indirectly in §5 that the cokernels of the linearization vanish away from the reducible and zero-section solutions.

2.5.1. The unperturbed PU(2) monopole equations. Recall that we consider Hermitian two-plane bundles E over X whose determinant line bundles $\det E$ are isomorphic to a fixed Hermitian line bundle over X endowed with a fixed C^∞, unitary connection A_e. Let (ρ, W^+, W^-) be a spin^c structure on X, where $\rho : T^*X \to \operatorname{End} W$ is the Clifford map, and the Hermitian four-plane bundle $W = W^+ \oplus W^-$ is endowed with a C^∞ spin^c connection. Given a connection A on E with curvature $F_A \in L^2_{k-1}(\Lambda^2 \otimes \mathfrak{u}(E))$, then $(F_A^+)_0 \in L^2_{k-1}(\Lambda^+ \otimes \mathfrak{su}(E))$ denotes the traceless part of its self-dual component. Equivalently, if A is a connection on $\mathfrak{su}(E)$ with curvature $F_A \in L^2_{k-1}(\Lambda^2 \otimes \mathfrak{so}(\mathfrak{su}(E)))$, then $F_A^+ \in L^2_{k-1}(\Lambda^+ \otimes \mathfrak{su}(E))$ is its self-dual component, viewed as a section of $\Lambda^+ \otimes \mathfrak{su}(E)$ via the implicit isomorphism $\operatorname{ad} : \mathfrak{su}(E) \to \mathfrak{so}(\mathfrak{su}(E))$. Let $D_A : L^2_k(W^+ \otimes E) \to L^2_{k-1}(W^- \otimes E)$ be the corresponding Dirac operator.

For an L^2_k section Φ of $W^+ \otimes E$, let Φ^* be its pointwise Hermitian dual and let $(\Phi \otimes \Phi^*)_{00}$ be the component of the Hermitian endomorphism $\Phi \otimes \Phi^*$ of $W^+ \otimes E$ which lies in $\mathfrak{su}(W^+) \otimes \mathfrak{su}(E)$. The Clifford map ρ defines an isomorphism $\rho : \Lambda^+ \to \mathfrak{su}(W^+)$ and thus an isomorphism $\rho = \rho \otimes \mathrm{id}_{\mathfrak{su}(E)}$ of $\Lambda^+ \otimes \mathfrak{su}(E)$ with $\mathfrak{su}(W^+) \otimes \mathfrak{su}(E)$. Then

$$\begin{aligned} F_A^+ - \rho^{-1}(\Phi \otimes \Phi^*)_{00} &= 0, \\ D_A \Phi &= 0, \end{aligned} \tag{2.15}$$

are the unperturbed equations considered in [67], [68], [71], [74] (with slightly differing trace conditions) for a pair (A, Φ) consisting of a connection A on $\mathfrak{su}(E)$ and a section Φ of $W^+ \otimes E$. Equivalently, given a pair (A, Φ) with A a fixed-determinant connection on E, equations (2.15) take the same form except that F_A^+ is replaced by $(F_A^+)_0$.

2.5.2. The perturbed PU(2) monopole equations. We next introduce perturbations of the PU(2) monopole equations (2.15) which will enable us to prove the transversality result described in Theorem 1.3. The perturbations in question make use of holonomy and their construction is partly modelled on related perturbations introduced by Donaldson, Floer and Taubes [8], [17], [29], [18], [86]. All of these constructions require that a unitary connection A on a Hermitian two-plane bundle E over X be regular enough that parallel translation along a C^∞

path in X is well-defined. For example, Floer employs a configuration space of L^4_1 connections modulo L^4_2 gauge-transformations over a three-manifold, so these connections are at least continuous (as $L^4_1 \subset C^0$ in dimension three) and the standard theory of ordinary differential equations applies to define parallel translation without further qualification [29]. In dimension four, parallel translation can be defined without difficulty using L^2_k connections with $k \geq 3$. With a little more care one can see that parallel translation can be defined by L^2_2 connections, even though $L^2_2 \not\subset C^0$ [61, §3.2]. These regularity issues become rather more intractable, as one can see from the Sobolev restriction theorem, for connections which are only L^2_1 [2, Theorem V.5.4]. As explained in the Appendix — to which we refer the reader for a discussion of the more technical points — we shall ultimately restrict our attention to configuration spaces of L^2_k connections with $k \geq 3$. The perturbations will, by definition, be zero on a neighborhood of a point in X where the curvature is large and so our regularity theory for L^2_1 monopoles will apply near points where curvature has bubbled off in order to prove removability of singularities (see §4.5.2).

We follow standard convention by saying that a connection A on a G bundle E over a connected manifold Y is 'irreducible' if its stabilizer Stab_A is trivial, that is, the center of the Lie group G [20, p. 133], rather than saying (more correctly) that its holonomy group $\mathrm{Hol}_A(y_0)$ is not a proper Lie subgroup of $\mathrm{Aut}(E_{y_0}) \simeq G$, where $y_0 \in Y$ is any basepoint. However, the holonomy will be our primary concern in this section, so some care is required as the two notions do not coincide in general. Recall from [20, Lemma 4.2.8] that Stab_A is isomorphic to the centralizer of $\mathrm{Hol}_A(y_0)$ in G. If Y is simply connected, then $\mathrm{Hol}_A(y_0)$ is a connected Lie subgroup of G [44, Theorem II.4.2]. Thus, if $G = \mathrm{SU}(2)$ or $\mathrm{SO}(3)$ and Y is simply connected, then A has trivial stabilizer in G if and only if $\mathrm{Hol}_A(y_0) \simeq G$ [20, p. 133]. Indeed, if $G = \mathrm{SO}(3)$ and $\mathrm{Stab}_A = \{\mathrm{id}\}$, then $H := \mathrm{Hol}_A(y_0) = \mathrm{SO}(3)$; otherwise, we would have:

- $H = \mathrm{SO}(2)$, with centralizer $Z(H) = \mathrm{SO}(2)$ (by [11, Theorem IV.2.3(ii)]), contradicting $\mathrm{Stab}_A = \{\mathrm{id}\}$, or
- $H = \{\mathrm{id}\}$, with centralizer $Z(H) = \mathrm{SO}(3)$, again contradicting $\mathrm{Stab}_A = \{\mathrm{id}\}$.

The same argument holds for $G = \mathrm{SU}(2)$, but not for higher-dimensional Lie groups. For example, if $G = \mathrm{U}(2) = \mathrm{SU}(2) \times_{\{\pm\mathrm{id}\}} S^1$, we cannot exclude the possibility that a $\mathrm{U}(2)$ connection A with $\mathrm{Stab}_A = S^1_Z$

reduces to an SU(2) connection, that is, A has holonomy $\mathrm{Hol}_A(y_0) = \mathrm{SU}(2)$.

With the preceding comments in mind, the first ingredient in our construction of the holonomy perturbations is a local section of $\mathfrak{su}(E)$ given by the holonomy of an SO(3) connection A on $\mathfrak{su}(E)$. Let $\gamma \subset X$ be a C^∞ loop based at a point $x_0 \in X$ and let

$$h_{\gamma,x_0}(A) \in \mathrm{SO}(\mathfrak{su}(E))|_{x_0}$$

be the holonomy of the connection A around the loop γ. The exponential map $\exp : \mathfrak{so}(3) \to \mathrm{SO}(3)$ gives a diffeomorphism from a ball $2B$ around the origin in $\mathfrak{so}(3)$ to a ball around the identity in SO(3). Let $\psi : \mathbb{R} \to [0,1]$ be a C^∞ cutoff function such that $\psi(|\zeta|) = 1$ for $\zeta \in \frac{1}{2}B$, $\psi(|\zeta|) > 0$ for $\zeta \in B$, and $\psi(|\zeta|) = 0$ for $\zeta \in \mathfrak{so}(3) - B$. Then

$$\mathfrak{h}_{\gamma,x_0}(A) := \psi\left(|\exp^{-1}(h_{\gamma,x_0}(A))|\right) \cdot \mathrm{ad}^{-1}(\exp^{-1}(h_{\gamma,x_0}(A))) \tag{2.16}$$

defines a gauge-equivariant map $\mathcal{A}_E(X) \to \mathfrak{su}(E)|_{x_0}$, where $\mathrm{ad} : \mathfrak{su}(E) \to \mathfrak{so}(\mathfrak{su}(E))$ is the standard isomorphism.

Lemma 2.22. *Let $U \subset X$ be a simply connected open subset and let x_0 be a point in U. If $A|_U$ is an irreducible* SO(3) *connection on $\mathfrak{su}(E)|_U$, then there are loops $\{\gamma_l\}_{l=1}^3 \subset U$, depending on A, such that the set $\{h_{\gamma_l,x_0}(A)\}_{l=1}^3 \subset \mathrm{SO}(\mathfrak{su}(E))|_{x_0}$ lies in the open subset of $\mathrm{SO}(\mathfrak{su}(E))|_{x_0}$ given by the image under* exp *of the ball $B \subset \mathfrak{so}(\mathfrak{su}(E))|_{x_0}$ around the identity. The set $\{\mathfrak{h}_{\gamma_l,x_0}(A)\}_{l=1}^3$ is then a basis for $\mathfrak{su}(E)|_{x_0}$.*

Proof. Since $A|_U$ is an irreducible SO(3) connection over a simply connected manifold U, the holonomy group $\mathrm{Hol}_{A|U}(x_0)$ is equal to $\mathrm{SO}(\mathfrak{su}(E))|_{x_0}$ by the remarks preceding the statement of the lemma. Hence, there are three loops $\gamma_1, \gamma_2, \gamma_3$ such that the holonomies

$$h_{\gamma_l,x_0}(A) \in B \subset \mathrm{SO}(\mathfrak{su}(E))|_{x_0}$$

give a basis $\{\mathfrak{h}_{\gamma_l,x_0}(A)\}_{l=1}^3$ for $\mathfrak{su}(E)|_{x_0}$. q.e.d.

For a C^∞ connection A we may extend $\mathfrak{h}_{\gamma,x_0}(A)$ to a C^∞ section $\hat{\mathfrak{h}}_\gamma(A)$ of $\mathfrak{su}(E)$ by radial parallel translation, with respect to A over a small ball $B(x_0, 2R_0)$ and then multiplying by a C^∞ cutoff function φ on X which is positive on $B(x_0, R_0)$ and identically zero on $X - B(x_0, R_0)$. Thus if the set $\{\mathfrak{h}_{\gamma_l,x_0}(A)\}_{l=1}^3$ spans $\mathfrak{su}(E)|_{x_0}$, then the set $\{\varphi\hat{\mathfrak{h}}_{\gamma_l}(A)|_y\}_{l=1}^4$ will span $\mathfrak{su}(E)|_y$ for $y \in B(x_0, R_0)$. The constant R_0 is chosen so that $4R_0$ is smaller than the injectivity radius of (X, g).

In general, for an L^2_k connection A with $k \geq 2$, the section $\hat{\mathfrak{h}}_\gamma(A)$ will not be in L^2_{k+1} and so we use the Neumann heat operator, for fixed small $t > 0$,

$$K_t(A|_{B(x_0,2R_0)}) : L^2(B(x_0,2R_0),\mathfrak{su}(E)) \to L^2_{k+1}(B(x_0,2R_0),\mathfrak{su}(E)),$$

as discussed in Appendix A.1 to construct an L^2_{k+1} section

$$\mathfrak{h}_\gamma(A) := K_t(A|_{B(x_0,2R_0)})\hat{\mathfrak{h}}_\gamma(A) \tag{2.17}$$

of $\mathfrak{su}(E)$ over $B(x_0,2R_0)$ which converges to $\hat{\mathfrak{h}}_\gamma(A)$ in $C^0(B(x_0,R_0))$ as $t \to 0$. Therefore, for small enough $t = t(A)$, the set $\{\varphi\mathfrak{h}_{\gamma_l}(A)|_y\}_{l=1}^3$, will span $\mathfrak{su}(E)|_y$ for all $y \in B(x_0,R_0)$ just as before.

Lemma 2.23. *Let $k \geq 2$ be an integer and let A_0 be an L^2_k unitary connection on $E|_{B(x_0,2R_0)}$. Let $\gamma_1,\gamma_2,\gamma_3$ be loops in $B(x_0,2R_0)$ based at x_0 such that $\{\mathfrak{h}_{\gamma_l,x_0}(A_0)\}_{l=1}^3$ spans $\mathfrak{su}(E)|_{x_0}$. Then there is a positive constant $\varepsilon(A_0,\{\gamma_l\})$ such that if A is an L^2_k unitary connection on $E|_{B(x_0,2R_0)}$ satisfying*

$$\|A - A_0\|_{L^2_{k,A_0}(B(x_0,2R_0))} < \varepsilon,$$

then the set $\{\mathfrak{h}_{\gamma_l,x_0}(A)\}_{l=1}^3$ spans $\mathfrak{su}(E)|_{x_0}$.

Proof. According to Lemma A.3 the holonomy maps

$$h_{\gamma,x_0} : \mathcal{A}_E(X) \to \mathrm{U}(E)|_{x_0}$$

are continuous and so the maps $\mathfrak{h}_{\gamma_l,x_0} : \mathcal{A}_E(X) \to \mathfrak{su}(E)|_{x_0}$ are continuous on small open neighborhoods of $A_0 \in \mathcal{A}_E(X)$. Hence, for small enough ε, the set $\{\mathfrak{h}_{\gamma_l,x_0}(A)\}_{l=1}^3$ spans $\mathfrak{su}(E)|_{x_0}$. q.e.d.

We now specify the loops to be used in this construction, essentially following the argument used in the proof of Lemma 2.5 in [17]. Let $\{B(x_j,4R_0)\}_{j=1}^{N_b}$, be a disjoint collection of open balls, where N_b is a fixed integer to be specified later (see §4.5.2). According to the slice results of [20, p. 192], [87, Proposition 2.1] for manifolds with boundary, the quotient space $\mathcal{B}^*_E(B(x_j,2R_0))$ of irreducible L^2_k connections on $E|_{B(x_j,2R_0)}$ is a C^∞ manifold modelled on a separable Hilbert space. The quotient L^2_k topology on $\mathcal{B}^*_E(B(x_j,2R_0))$ is clearly metrizable and so Stone's Theorem implies that $\mathcal{B}^*_E(B(x_j,2R_0))$ is paracompact and thus admits partitions of unity [55, Corollary II.3.8].

For each $j = 1,\dots,N_b$ and each point $[A_0]$ in $\mathcal{B}^*_E(B(x_j,2R_0))$, we can find loops $\{\gamma_{j,l,A_0}\}_{l=1}^3$, contained in $B(x_j,2R_0)$ and based at x_j such that $\{\mathfrak{h}_{\gamma_{j,l,A_0}}(A_0)\}_{l=1}^3$, spans $\mathfrak{su}(E)|_{x_j}$. For each such point $[A_0]$, Lemma 2.23 implies that there is an L^2_{k,A_0} ball

$$B_{[A_0]}(\varepsilon_{A_0}) := \{[A] \in \mathcal{B}^*_E(B(x_j, 2R_0)) : \operatorname{dist}_{L^2_{k,A_0}}([A],[A_0]) < \varepsilon_{A_0}\}$$

such that for all $[A] \in B_{[A_0]}(\varepsilon_{A_0})$, the set $\{\mathfrak{h}_{\gamma_{j,l,A_0}}(A)\}_{l=1}^3$ spans $\mathfrak{su}(E)|_{x_j}$. These balls give an open cover of $\mathcal{B}^*_E(B(x_j, 2R_0))$, and hence there is a locally finite refinement of this open cover, $\{U_{j,\alpha}\}_{\alpha=1}^\infty$, and a positive partition $\chi_{j,\alpha}$ (see Proposition A.12) subordinate to $\{U_{j,\alpha}\}_{\alpha=1}^\infty$ in the sense that

$$\sum_\alpha \chi_{j,\alpha}[A] > 0, \qquad [A] \in \mathcal{B}^*_E(B(x_j, 2R_0)).$$

Hence, for each $U_{j,\alpha}$, we obtain loops $\{\gamma_{j,l,\alpha}\}_{l=1}^3 \subset B(x_j, 2R_0)$ such that for all $[A] \in U_{j,\alpha}$, the sections $\mathfrak{h}_{\gamma_{j,l,\alpha}}(A)$ span $\mathfrak{su}(E)|_{x_j}$.

Let β be a smooth cutoff function on $\mathbb{R}$ such that $\beta(t) = 1$ for $t \leq \frac{1}{2}$ and $\beta(t) = 0$ for $t \geq 1$, with $\beta(t) > 0$ for $t < 1$. Then the C^∞ gauge-invariant maps $\mathcal{A}_E(X) \to \mathbb{R}$, $A \mapsto \beta_j[A]$ given by

$$\beta_j[A] := \beta\left(\frac{1}{\varepsilon_0^2}\int_{B(x_j,4R_0)} \beta\left(\frac{\operatorname{dist}_g(\cdot, x_j)}{4R_0}\right)|F_A|^2\,dV\right) \tag{2.18}$$

are zero when the energy of the connection A is greater than or equal to $\frac{1}{2}\varepsilon_0^2$ over a ball $B(x_j, 2R_0)$. Here, ε_0 is the constant of Corollary 3.16. Finally, we define C^∞ cutoff functions on X by setting

$$\varphi_j(x) := \beta\left(\frac{\operatorname{dist}_g(x, x_j)}{R_0}\right), \qquad x \in X, \tag{2.19}$$

so that φ_j is positive on the ball $B(x_j, R_0)$ and zero on its complement in X.

We can now define a gauge equivariant C^∞ map

$$\mathcal{A}_E(X) \to L^2_{k+1}(X, \mathfrak{su}(E)), \qquad A \mapsto \mathfrak{m}_{j,l,\alpha}(A)$$

by setting

$$\mathfrak{m}_{j,l,\alpha}(A) := \beta_j[A]\chi_{j,\alpha}[A|_{B(x_j,2R_0)}]\varphi_j\mathfrak{h}_{\gamma_{j,l,\alpha}}(A). \tag{2.20}$$

Thus at each point $A \in \mathcal{A}_E(X)$ only a finite number of the $\mathfrak{m}_{j,l,\alpha}(A)$ are non-zero and each map $\mathfrak{m}_{j,l,\alpha}$ is C^∞ with uniformly bounded derivatives of all orders on $\mathcal{A}_E(X)$ (see Appendix A.4).

To define the perturbation of the Dirac operator in (2.15), we need elements of $\Omega^0(\mathrm{Hom}_{\mathbb{C}}(W^+, W^-))$ and these are conveniently provided by complex one-forms using the Clifford isomorphism

$$\rho : \Lambda^1_{\mathbb{C}} \to \mathrm{Hom}_{\mathbb{C}}(W^+, W^-),$$

where $\Lambda^1_{\mathbb{C}} = \Lambda^1 \otimes_{\mathbb{R}} \mathbb{C}$. These one-forms are parametrized by the following Banach space: Let $\mathbb{A}$ be the index set

$$\{(j, l, \alpha) : 1 \leq j \leq N_b, 1 \leq l \leq 3, \alpha \in \mathbb{N}\},$$

let $\delta = (\delta_\alpha)_{\alpha=1}^\infty \in \ell^1(\mathbb{R})$ be the sequence of positive weights described in Appendix A.4, let $r \geq k + 1$, and let

$$\mathcal{P}^r_\vartheta := \ell^1_\delta(\mathbb{A}, C^r(X, \Lambda^1_{\mathbb{C}})) \tag{2.21}$$

be the set of sequences $\vec{\vartheta} := (\vartheta_{j,l,\alpha})$ in $C^r(X, \Lambda^1_{\mathbb{C}})$ such that

$$\|\vec{\vartheta}\|_{\ell^1_\delta(C^r(X))} := \sum_{j,l,\alpha} \delta_\alpha^{-1} \|\vartheta_{j,l,\alpha}\|_{C^r(X)} < \infty.$$

Then $\mathcal{P}^r_\vartheta$ is a Banach space with respect to the above norm [40, §1.7].

Remark 2.24.

1. Note that we measure reducibility of connections and allow the loops $\gamma_{j,l,\alpha}$ to be contained in the *larger* balls $B(x_j, 2R_0)$ while the perturbations are supported on the *smaller* balls $B(x_j, R_0)$. Thus, if $A \in \mathcal{A}_E(X)$ is a connection such that $A|_{B(x_j,2R_0)}$ is reducible for all $j \in \{1, \ldots, N_b\}$ such that $\beta_j[A] > 0$ and $\Phi \not\equiv 0$, then $[A, \Phi]$ cannot be a point in $M^{*,0}_{W,E}$: our unique continuation result for PU(2) monopoles (see Theorem 5.11) which are reducible on an open subset of X containing all balls $\bar{B}(x_j, R_0)$ with $\beta_j[A] > 0$ would imply that A is reducible on all of X and so (A, Φ) would be a reducible PU(2) monopole. Consequently, if $[A, \Phi] \in M^{*,0}_{W,E}$ then A must be irreducible on *some* ball $B(x_j, 2R_0)$; otherwise, A would be reducible on all balls $B(x_j, 2R_0)$ with $\beta_j[A] > 0$ and so reducible on X.

2. Note that the sections $\mathfrak{m}_{j,l,\alpha}$ are defined on the entire space $\mathcal{A}_E(X)$: They are zero for connections $A \in \mathcal{A}_E(X)$ which are reducible when restricted to $B(x_j, 2R_0)$.

3. Our energy bound (4.2) for solutions (A, Φ) to the perturbed PU(2) monopole equations ensures that there is always at least one ball $B(x_j, R_0)$ where the sections $\{\mathfrak{m}_{j,l,\alpha}(A)\}_{l=1}^3$ span $\mathfrak{su}(E)|_{B(x_j,R_0)}$; see §4.5.2.

Because only a finite number of the sections $\mathfrak{m}_{j,l,\alpha}(A)$ are non-zero at each $A \in \mathcal{A}_E(X)$, the sum

$$\vec{\vartheta} \cdot \vec{\mathfrak{m}}(A) := \sum_{j,l,\alpha} \rho(\vartheta_{j,l,\alpha}) \otimes_{\mathbb{C}} \mathfrak{m}_{j,l,\alpha}(A) \tag{2.22}$$

gives a well-defined L^2_{k+1} section of

$$\mathrm{Hom}_{\mathbb{C}}(W^+, W^-) \otimes_{\mathbb{C}} \mathfrak{sl}(E) \simeq \Lambda^1_{\mathbb{C}} \otimes_{\mathbb{C}} \mathfrak{sl}(E),$$

noting that although each $\mathfrak{m}_{j,l,\alpha}(A)$ is a section of $\mathfrak{su}(E)$, we have $\mathfrak{sl}(E) = \mathfrak{su}(E) \otimes_{\mathbb{R}} \mathbb{C}$. The map

$$\mathcal{A}_E(X) \to L^2_{k+1}(X, \mathrm{Hom}_{\mathbb{C}}(W^+, W^-) \otimes_{\mathbb{C}} \mathfrak{sl}(E)), \qquad A \mapsto \vec{\vartheta} \cdot \vec{\mathfrak{m}}(A)$$

is C^∞ and gauge equivariant and so defines a C^∞ section of the vector bundle

$$\mathcal{A}^*_E(X) \times_{\mathcal{G}_E} L^2_{k+1}(X, \mathrm{Hom}_{\mathbb{C}}(W^+, W^-) \otimes_{\mathbb{C}} \mathfrak{sl}(E)) \to \mathcal{B}^*_E(X).$$

By construction (see Appendix A.4) the sum $\vec{\vartheta} \cdot \vec{\mathfrak{m}}(A)$ satisfies a C^0 estimate of the form

$$\begin{aligned} &\sup_{A \in \mathcal{A}_E(X)} \|\vec{\vartheta} \cdot \vec{\mathfrak{m}}(A)\|_{L^2_{k+1,A}(X)} \le C \|\vec{\vartheta}\|_{\ell^1_\delta(C^r(X))} \\ &\qquad\qquad \text{and} \quad \|\vec{\vartheta}\|_{\ell^1_\delta(C^r(X))} \le \varepsilon_\vartheta, \end{aligned} \tag{2.23}$$

where $C = C(g, k)$, and ε_ϑ is a positive constant which we are free to specify.

We shall also need to construct a gauge equivariant C^∞ map from $\mathcal{A}^*_E(X)$ to $L^2_{k+1}(\mathfrak{gl}(\Lambda^+) \otimes_{\mathbb{R}} \mathfrak{so}(\mathfrak{su}(E)))$. This map will define a perturbation of the quadratic form in (2.15) using the representation $\mathrm{ad} : \mathfrak{su}(E) \to \mathfrak{so}(\mathfrak{su}(E))$ and is parametrized by the Banach space

$$\mathcal{P}^r_\tau := \ell^1_\delta(\mathbb{A}, C^r(X, \mathfrak{gl}(\Lambda^+))) \tag{2.24}$$

of sequences $\vec{\tau} := (\tau_{j,l,\alpha})$ in $C^r(X, \mathfrak{gl}(\Lambda^+))$ such that

$$\|\vec{\tau}\|_{\ell^1_\delta(C^r(X))} := \sum_{j,l,\alpha} \delta_\alpha^{-1} \|\tau_{j,l,\alpha}\|_{C^r(X)} < \infty.$$

Then the sum

$$\vec{\tau}\cdot\vec{\mathfrak{m}}(A) := \sum_{j,l,\alpha} \tau_{j,l,\alpha}\otimes_{\mathbb{R}} \operatorname{ad}(\mathfrak{m}_{j,l,\alpha}(A)) \tag{2.25}$$

is pointwise finite and gives a well-defined L^2_{k+1} section of

$$\mathfrak{gl}(\Lambda^+)\otimes_{\mathbb{R}}\mathfrak{so}(\mathfrak{su}(E)).$$

The map

$$\mathcal{A}_E(X)\to L^2_{k+1}(X,\mathfrak{gl}(\Lambda^+)\otimes_{\mathbb{R}}\mathfrak{so}(\mathfrak{su}(E))),\qquad A\mapsto \vec{\tau}\cdot\vec{\mathfrak{m}}(A)$$

is C^∞ and gauge equivariant and so defines a C^∞ section of the vector bundle

$$\mathcal{A}^*_E(X)\times_{\mathcal{G}_E} L^2_{k+1}(X,\mathfrak{gl}(\Lambda^+)\otimes_{\mathbb{R}}\mathfrak{so}(\mathfrak{su}(E)))\to\mathcal{B}^*_E(X).$$

By construction (see Appendix A.4) the sum $\vec{\tau}\cdot\vec{\mathfrak{m}}(A)$ satisfies C^0 estimates

$$\begin{aligned}\sup_{A\in\mathcal{A}_E(X)}\|\vec{\tau}\cdot\vec{\mathfrak{m}}(A)\|_{L^2_{k+1,A}(X)} &\le C\|\vec{\tau}\|_{\ell^1_\delta(C^r(X))}\\ \text{and}\quad \|\vec{\tau}\|_{\ell^1_\delta(C^r(X))}&\le\varepsilon_\tau,\end{aligned}\tag{2.26}$$

where $C=C(g,k)$, and ε_τ is a positive constant which we are free to specify.

Our perturbed PU(2) monopole equations for a pair (A,Φ) on $(\mathfrak{su}(E),W^+\otimes E)$ then take the form

$$\begin{aligned}\mathfrak{S}_1(A,\Phi) :={}&F^+_A-\big(\mathrm{id}+\tau_0\otimes\mathrm{id}_{\mathfrak{su}(E)}\\ &\qquad+\vec{\tau}\cdot\vec{\mathfrak{m}}(A)\big)\,\rho^{-1}(\Phi\otimes\Phi^*)_{00}=0,\\ \mathfrak{S}_2(A,\Phi) :={}&D_A\Phi+\rho(\vartheta_0)\Phi+\vec{\vartheta}\cdot\vec{\mathfrak{m}}(A)\Phi=0,\end{aligned}\tag{2.27}$$

where $\tau_0\in C^r(X,\mathfrak{gl}(\Lambda^+))$ and $\vartheta_0\in C^r(X,\Lambda^1_{\mathbb{C}})$ are additional perturbation parameters. For brevity, we shall often denote $\vec{\tau}_A:=\vec{\tau}\cdot\vec{\mathfrak{m}}(A)$ and $\vec{\vartheta}_A:=\vec{\vartheta}\cdot\vec{\mathfrak{m}}(A)$.

Remark 2.25.

1. In [25] we consider the question of transversality of the spaces of reducible solutions or U(1) monopoles, which are identified with moduli spaces of Seiberg-Witten monopoles. In particular, we show that the moduli spaces of U(1) monopoles are cut out transversely for generic $\tau_0\in\Omega^0(\mathfrak{gl}(\Lambda^+))$.

2. If the section Φ is identically zero, then the PU(2) monopole equations reduce to those for a projectively anti-self-dual unitary connection A on E. If the connection A is reducible, then all of the sections $\mathfrak{m}_{j,l,\alpha}(A)$ are identically zero on X; the perturbations $\vec{\tau}\cdot\vec{\mathfrak{m}}(A)$ and $\vec{\vartheta}\cdot\vec{\mathfrak{m}}(A)$ in (2.27) are then zero.

3. We emphasize that the holonomy sections in the sequence $\vec{\mathfrak{m}}(A)$ are always L^2_{k+1} by construction for L^2_k connections A (with $k\geq 2$) and that the sums defining $\vec{\tau}\cdot\vec{\mathfrak{m}}(A)$ and $\vec{\vartheta}\cdot\vec{\mathfrak{m}}(A)$ are always finite for each A, although the number of terms may tend to infinity as A approaches a reducible connection.

The proof of the existence of an Uhlenbeck compactification for $M_{W,E}$ in §4 requires the perturbations $\vec{\tau}\cdot\vec{\mathfrak{m}}(A)$ and $\vec{\vartheta}\cdot\vec{\mathfrak{m}}(A)$ to satisfy the following estimates in order to obtain universal *a priori* $L^2_{1,A}$ bounds for Φ and L^2 bounds for F_A (see Lemmas 4.2 and 4.3):

$$\begin{aligned}&\sup_{A\in\mathcal{A}_E}\|\vec{\tau}\cdot\vec{\mathfrak{m}}(A)\|_{L^\infty(X)}\leq 1,\\ &\sup_{A\in\mathcal{A}_E}\|\vec{\vartheta}\cdot\vec{\mathfrak{m}}(A)\|_{L^4(X)}\leq 1.\end{aligned}\tag{2.28}$$

To obtain the more delicate universal *a priori* L^∞ bounds for Φ and F^+_A (see Lemmas 2.26 and 4.4) required by our Uhlenbeck compactness argument in §4, the perturbations τ_0, $\vec{\tau}\cdot\vec{\mathfrak{m}}(A)$, and $\vec{\vartheta}\cdot\vec{\mathfrak{m}}(A)$ must satisfy the following stronger estimates:

$$\begin{aligned}\|\tau_0\|_{L^\infty(X)}+\sup_{A\in\mathcal{A}_E}\|\vec{\tau}\cdot\vec{\mathfrak{m}}(A)\|_{L^\infty(X)}&\leq \tfrac{1}{64},\\ \|\vartheta_0\|_{L^\infty_1(X)}+\sup_{A\in\mathcal{A}_E}\|\vec{\vartheta}\cdot\vec{\mathfrak{m}}(A)\|_{L^\infty_{1,A}(X)}&\leq 1.\end{aligned}\tag{2.29}$$

We take a constant 1 on the right-hand sides of (2.28) and the second inequality in (2.29) for notational convenience only: these bounds do not need to be 'small'. There are continuous Sobolev embeddings $L^2_4(X,\mathbb{R})\subset L^\infty(X,\mathbb{R})$ and $L^2_4(X,\mathbb{R})\subset L^\infty_1(X,\mathbb{R})$ over a four-manifold X. The estimates (2.29) then follow from the estimates (2.23) and (2.26) when $k\geq 3$. *Therefore, to obtain the inequalities* (2.29)*, we require that* $k\geq 3$ in our configuration spaces of L^2_k connections $\mathcal{B}_E$ and pairs $\mathcal{C}_{W,E}$; see §A.4. The bounds in (2.29) then follow for small enough choices of ε_τ and ε_ϑ in the inequalities (2.23) and (2.26).

We shall need a slight generalization of Lemma 2.19 which applies to the perturbed quadratic form $\tau\rho^{-1}(\Phi\otimes\Phi^*)_{00}$ when $\tau\neq \mathrm{id}_{\Lambda^+}$. Without loss of generality, we may restrict our attention to automorphisms $\tau_\rho := \rho\tau\rho^{-1}$ of $\mathfrak{su}(W^+)$ such that $|\tau_\rho - \mathrm{id}_{\mathfrak{su}(W^+)}| = |\rho\tau\rho^{-1} - \mathrm{id}| < \varepsilon_\tau$ and thus

$$\langle\Phi_1, \Phi_2\rangle - \varepsilon_\tau|\Phi_1||\Phi_2| \leq \langle\tau(\Phi_1), \Phi_2\rangle \leq \langle\Phi_1, \Phi_2\rangle + \varepsilon_\tau|\Phi_1||\Phi_2|$$

for some universal positive constant $\varepsilon_\tau \ll 1$ and all

$$\Phi_1, \Phi_2 \in \Omega^0(W^+ \otimes E).$$

Hence, Lemma 2.19 yields the following bounds for the perturbed quadratic form:

Lemma 2.26. *There is a universal positive constant $\varepsilon_\tau \ll 1$ such that for all $\|\tau - \mathrm{id}_{\Lambda^+}\|_{C^0(X)} < \varepsilon_\tau$ and $\Phi \in \Omega^0(W^+ \otimes E)$ the following inequalities hold:*

$$\tfrac{1}{2}|\Phi|^4 \leq \langle\tau_\rho(\Phi\otimes\Phi^*)_{00}\Phi, \Phi\rangle \leq \tfrac{3}{4}|\Phi|^4, \tag{1}$$

$$\tfrac{1}{4}|\Phi|^4 \leq |\tau_\rho(\Phi\otimes\Phi^*)_{00}|^2 \leq \tfrac{3}{8}|\Phi|^4. \tag{2}$$

Remark 2.27. The constant $\varepsilon_\tau = 1/64$ will suffice. The constraint $\|\tau - \mathrm{id}\|_{C^0(X)} < \varepsilon_\tau$ is only used in §4.1 and consequently in §4.6, where we establish the 'universal' *a priori* bounds for PU(2) monopoles and prove the existence of an Uhlenbeck compactification, respectively.

2.6. The moduli space and the elliptic deformation complex. We define the moduli space of PU(2) monopoles and compute the index of its elliptic deformation complex.

For any integer $k \geq 2$, the PU(2) monopole equations (2.27) define a C^∞ map $\mathfrak{S} := (\mathfrak{S}_1, \mathfrak{S}_2)$ of Hilbert manifolds,

$$\mathfrak{S} : \tilde{\mathcal{C}}_{W,E} \to L^2_{k-1}(\Lambda^+ \otimes \mathfrak{su}(E)) \oplus L^2_{k-1}(W^- \otimes E), \tag{2.30}$$

given by

$$\begin{aligned}(A,\Phi) \mapsto &\begin{pmatrix}\mathfrak{S}_1(A,\Phi)\\ \mathfrak{S}_2(A,\Phi)\end{pmatrix}\\ &:= \begin{pmatrix}F_A^+ - (\mathrm{id} + \tau_0\otimes\mathrm{id}_{\mathfrak{su}(E)} + \vec{\tau}\cdot\vec{\mathfrak{m}}(A))\rho^{-1}(\Phi\otimes\Phi^*)_{00}\\ D_A\Phi + +\rho(\vartheta_0)\Phi + \vec{\vartheta}\cdot\vec{\mathfrak{m}}(A)\Phi\end{pmatrix}.\end{aligned}$$

We can then define the moduli space of PU(2) *monopoles* by setting

$$M_{W,E} := \{(A,\Phi) \in \tilde{\mathcal{C}}_{W,E} : \mathfrak{S}(A,\Phi) = 0\}/{}^\circ\mathcal{G}_E \tag{2.31}$$

and denote the moduli space of PU(2) monopoles which are neither reducible nor zero-section pairs by

$$M^{*,0}_{W,E} := M_{W,E} \cap \mathcal{C}^{*,0}_{W,E}.$$

We let $M^{\mathrm{asd}}_E := \{[A] : F^+_A = 0\} \subset \mathcal{B}_E$ be the moduli space of anti-self-dual connections on $\mathfrak{su}(E)$.

The SO(3) bundle $\mathfrak{su}(E)$ has first Pontrjagin number and second Stiefel-Whitney class given by

$$\begin{aligned} \kappa &:= -\tfrac{1}{4}p_1(\mathfrak{su}(E)) = c_2(E) - \tfrac{1}{4}c_1(E)^2 \\ w_2(\mathfrak{su}(E)) &= \; c_1(E) \pmod 2, \end{aligned} \tag{2.32}$$

and, for a connection A on $\mathfrak{su}(E)$, we have the following Chern-Weil integral identity,

$$-\tfrac{1}{4}p_1(\mathfrak{su}(E)) = \frac{1}{8\pi^2}\int_X \left(|F^-_A|^2 - |F^+_A|^2\right) dV, \tag{2.33}$$

where we view F_A as a section of $\Lambda^2 \otimes \mathfrak{su}(E)$ via the isomorphism $\mathrm{ad} : \mathfrak{su}(E) \simeq \mathfrak{so}(\mathfrak{su}(E))$ (see [20, §2.1.4]).

Since the map $\mathfrak{S}$ is ${}^\circ\mathcal{G}_E$-equivariant, it defines a section of the Hilbert vector bundle $\mathfrak{V}$ over $\mathcal{C}^{*,0}_{W,E}$ with total space

$$\mathfrak{V} := \tilde{\mathcal{C}}^{*,0}_{W,E} \times_{{}^\circ\mathcal{G}_E} \left(L^2_{k-1}(\Lambda^+ \otimes \mathfrak{su}(E)) \oplus L^2_{k-1}(W^- \otimes E)\right). \tag{2.34}$$

While the equation $\mathfrak{S}[A,\Phi] = 0$ is, of course, defined on $\mathcal{C}_{W,E}$, $\mathfrak{V}$ does not extend from $\mathcal{C}^{*,0}_{W,E}$ to a vector bundle over $\mathcal{C}_{W,E}$. The moduli space $M^{*,0}_{W,E} \subset \mathcal{C}^{*,0}_{W,E}$ is then the zero locus of the section $\mathfrak{S}$ of Hilbert vector bundle $\mathfrak{V}$ over the Hilbert manifold $\mathcal{C}^{*,0}_{W,E}$: it will be a *regular submanifold* if $\mathfrak{S}$ vanishes transversely, that is, if the differential

$$(D\mathfrak{S})_{A,\Phi} : T_{[A,\Phi]}\mathcal{C}^{*,0}_{W,E} \to L^2_{k-1}(\Lambda^+ \otimes \mathfrak{su}(E)) \oplus L^2_{k-1}(W^- \otimes E) \tag{2.35}$$

is surjective at all points $[A,\Phi]$ in $\mathfrak{S}^{-1}(0) \cap \mathcal{C}^{*,0}_{W,E}$; recall from §2 that the tangent space $T_{[A,\Phi]}\mathcal{C}^{*,0}_{W,E}$ is canonically identified with $\operatorname{Ker} d^{0,*}_{A,\Phi} \subset \tilde{\mathcal{C}}^{*,0}_{W,E}$.

Suppose (A,Φ) is a pair in $\mathcal{A}_E \times \Omega^0(W^+ \otimes E)$. Recall from Proposition 2.1 that the differential at the identity $\mathrm{id}_E \in {}^\circ\mathcal{G}_E$, of the map ${}^\circ\mathcal{G}_E \to \mathcal{A}_E \times \Omega^0(W^+ \otimes E)$ given by $u \mapsto u(A,\Phi)$ is

$$\begin{aligned} \Omega^0(\mathfrak{su}(E)) \oplus i\mathbb{R}_Z &\to \Omega^1(\mathfrak{su}(E)) \oplus \Omega^0(W^+ \otimes E), \\ \zeta &\mapsto -d^0_{A,\Phi}\zeta = (-d_A\zeta, \zeta\Phi). \end{aligned}$$

Similarly, the differential $(a,\phi)\mapsto d^1_{A,\Phi}(a,\phi):=(D\mathfrak{S})_{A,\Phi}(a,\phi)$,

$$\Omega^1(\mathfrak{su}(E))\oplus\Omega^0(W^+\otimes E)\to\Omega^+(\mathfrak{su}(E))\oplus\Omega^0(W^-\otimes E),$$

of the map $\mathfrak{S}$ at the point (A,Φ) is given by

$$\begin{aligned} &d^1_{A,\Phi}(a,\phi)\\ &:=\begin{pmatrix} d^+_A a-\vec{\tau}\cdot(\delta\vec{\mathfrak{m}})(a)(\Phi\otimes\Phi^*)_{00}-(\mathrm{id}+\tau_0\otimes\mathrm{id}_{\mathfrak{su}(E)} \\ \qquad +\vec{\tau}\cdot\vec{\mathfrak{m}}(A))\rho^{-1}(\Phi\otimes\phi^*+\phi\otimes\Phi^*)_{00} \\ D_A\phi+\rho(\vartheta_0)\phi+\vec{\vartheta}\cdot\vec{\mathfrak{m}}(A)\phi+\rho(a)\Phi+\vec{\vartheta}\cdot(\delta\vec{\mathfrak{m}})(a)\Phi\end{pmatrix},\end{aligned}\tag{2.36}$$

where $\delta\vec{\mathfrak{m}}=\delta\vec{\mathfrak{m}}/\delta A$. (In the sequels we also find it convenient to use $L_{A,\Phi}$ to denote the linearization of the map $\mathfrak{S}$ at the point (A,Φ).) The differential of the composition ${}^\circ\mathcal{G}_E\mapsto\Omega^+(\mathfrak{su}(E))\oplus\Omega^0(W^-\otimes E)$ given by

$$u\mapsto u(A,\Phi)\mapsto\mathfrak{S}(u(A,\Phi))=u(\mathfrak{S}(A,\Phi))=(u\mathfrak{S}_1(A,\Phi)u^{-1},u\mathfrak{S}_2(A,\Phi))$$

is then

$$\begin{aligned}\Omega^0(\mathfrak{su}(E))\oplus i\mathbb{R}_Z\to\ &\begin{matrix}\Omega^+(\mathfrak{su}(E))\\ \oplus\\ \Omega^0(W^-\otimes E)\end{matrix},\\ &\zeta\mapsto d^1_{A,\Phi}\circ d^0_{A,\Phi}\zeta=([\zeta,\mathfrak{S}_1(A,\Phi)],\zeta\mathfrak{S}_2(A,\Phi)).\end{aligned}$$

Therefore, $d^1_{A,\Phi}\circ d^0_{A,\Phi}=0$ if and only if $\mathfrak{S}(A,\Phi)=0$, that is, if and only if (A,Φ) is a PU(2) monopole. Consequently, the sequence

$$\Omega^0(\mathfrak{su}(E))\oplus i\mathbb{R}_Z\xrightarrow{d^0_{A,\Phi}}\begin{matrix}\Omega^1(\mathfrak{su}(E))\\ \oplus\\ \Omega^0(W^+\otimes E)\end{matrix}\xrightarrow{d^1_{A,\Phi}}\begin{matrix}\Omega^+(\mathfrak{su}(E))\\ \oplus\\ \Omega^0(W^-\otimes E)\end{matrix}\tag{2.37}$$

is a complex if and only if (A,Φ) is a PU(2) monopole. The L^2 adjoint of $-d^0_{A,\Phi}$ is given by

$$-d^{0,*}_{A,\Phi}(a,\phi)=-d^*_A a+(\cdot\Phi)^*\phi.\tag{2.38}$$

The operator

$$\mathcal{D}_{A,\Phi}:=d^{0,*}_{A,\Phi}+d^1_{A,\Phi}:\begin{matrix}\Omega^1(\mathfrak{su}(E))\\ \oplus\\ \Omega^0(W^+\otimes E)\end{matrix}\longrightarrow\begin{matrix}\Omega^0(\mathfrak{su}(E))\oplus i\mathbb{R}_Z\\ \oplus\\ \Omega^+(\mathfrak{su}(E))\\ \oplus\\ \Omega^0(W^-\otimes E)\end{matrix}\tag{2.39}$$

is elliptic (thus Fredholm) and so (2.37) is an *elliptic deformation complex* for the PU(2) monopole equations (2.27), with cohomology groups

$$H^0_{A,\Phi} := \operatorname{Ker} d^0_{A,\Phi}, \quad H^1_{A,\Phi} := \operatorname{Ker} d^1_{A,\Phi} / \operatorname{Im} d^0_{A,\Phi}, \quad H^2_{A,\Phi} := \operatorname{Coker} d^1_{A,\Phi},$$

analogous to the usual elliptic deformation complex [20, Eq. (4.2.26)] for the anti-self-dual equation, $F^+_A = 0$. (An elliptic deformation complex for stable pairs for holomorphic bundles is given by Bradlow and Daskalopoulos in [9, §2].)

Recall that $H^0_{A,\Phi} = \operatorname{Ker} d^0_{A,\Phi}$ is just the Lie algebra of the stabilizer $\operatorname{Stab}_{A,\Phi}$ of the point $[A,\Phi] \in M_{W,E}$ and $H^1_{A,\Phi}$ is the Zariski or formal tangent space. Thus, for any point $[A,\Phi] \in M^{*,0}_{W,E}$ we have $H^0_{A,\Phi} = 0$. If $H^2_{A,\Phi} = 0$, then $\operatorname{Coker}(D\mathfrak{S})_{A,\Phi} = 0$ and so $[A,\Phi]$ is a regular point of the zero locus of the section $\mathfrak{S}$ of $\mathfrak{V}$ over $\mathcal{C}^{*,0}_{W,E}$. Thus, if $H^0_{A,\Phi} = 0$ and $H^2_{A,\Phi} = 0$, then $[A,\Phi]$ is a smooth point of $M_{W,E}$ with tangent space $\operatorname{Ker} \mathcal{D}_{A,\Phi} = \operatorname{Ker}(d^{0,*}_{A,\Phi} + d^1_{A,\Phi}) = H^1_{A,\Phi}$, as we see from (2.35). Provided the zero set $\mathfrak{S}^{-1}(0)$ is regular, then $M^{*,0}_{W,E}$ will be a smooth manifold of dimension $-\operatorname{Ind} \mathcal{D}_{A,\Phi}$.

The perturbation terms in (2.27) define gauge equivariant maps

$$\tilde{\mathcal{C}}^{*,0}_{W,E} \to L^2_{k-1}(\Lambda^+ \otimes \mathfrak{su}(E)) \oplus L^2_{k-1}(W^- \otimes E),$$

given by

$$(A,\Phi) \mapsto \begin{pmatrix} \vec{\tau} \cdot \vec{\mathfrak{m}}(A)\rho^{-1}(\Phi \otimes \Phi^*)_{00} \\ \vec{\vartheta} \cdot \vec{\mathfrak{m}}(A)\Phi \end{pmatrix}.$$

For $k \geq 2$, the Sobolev multiplication theorem implies that $\vec{\vartheta} \cdot \vec{\mathfrak{m}}(A)\Phi$ is in $L^2_k(W^- \otimes E)$, while $\vec{\tau} \cdot \vec{\mathfrak{m}}(A)\rho^{-1}(\Phi \otimes \Phi^*)_{00}$ is in $L^2_k(\Lambda^+ \otimes \mathfrak{su}(E))$ when $k \geq 3$ and in $L^p_2(\Lambda^+ \otimes \mathfrak{su}(E))$, $1 \leq p < 2$ when $k = 2$. By the Rellich embedding theorem, the inclusions $L^2_k \subset L^2_{k-1}$ and $L^p_2 \subset L^2_1$, $p > 1$, are compact. In particular, it follows that the linearization of the perturbed PU(2) monopole equations (2.36) differs from the linearization of the unperturbed equations (2.15) by a compact operator [2, Theorem VI.2].

Proposition 2.28. *If the map $\mathfrak{S}$ of* (2.30) *vanishes transversely for the parameters $(\tau_0, \vartheta_0, \vec{\tau}, \vec{\vartheta})$, then $M^{*,0}_{W,E}(\tau_0, \vartheta_0, \vec{\tau}, \vec{\vartheta})$ is a smooth manifold of dimension*

$$\begin{aligned} \dim M^{*,0}_{W,E} &= -2p_1(\mathfrak{su}(E)) - 3(1 - b^1(X) + b^+(X)) \\ &\qquad + \tfrac{1}{2}p_1(\mathfrak{su}(E)) + \tfrac{1}{2}(c_1(W^+) + c_1(E))^2 - \tfrac{1}{2}\sigma(X) - 1 \\ &= \dim M^{\mathrm{asd}}_E + 2\operatorname{Ind}_{\mathbb{C}} D_A - 1, \end{aligned}$$

where $\sigma(X)$ is the signature of X.

Proof. Since $H^0_{A,\Phi}=0$ and $H^2_{A,\Phi}=0$ (by hypothesis) at any point $[A,\Phi]$ in $M^{*,0}_{W,E}$, we have $\dim M^{*,0}_{W,E} = -\operatorname{Ind}\mathcal{D}_{A,\Phi}$. By our regularity result, Proposition 3.7, any point $[A,\Phi]$ in $M^{*,0}_{W,E}$ has a smooth representative (A,Φ). Therefore, from the expressions for $d^{0,*}_{A,\Phi}$ in (2.38) and for $d^1_{A,\Phi}$ in (2.36) and the Sobolev multiplication and embedding theorems, we find that the operator $\mathcal{D}_{A,\Phi} = d^{0,*}_{A,\Phi} + d^1_{A,\Phi} : L^2_k \to L^2_{k-1}$ differs from

$$\begin{pmatrix} d_A^* + d_A^+ & 0 \\ 0 & D_A \end{pmatrix} : \begin{matrix} L^2_k(\Lambda^1\otimes\mathfrak{su}(E)) \\ \oplus \\ L^2_k(W^+\otimes E) \end{matrix} \to \begin{matrix} L^2_{k-1}(\mathfrak{su}(E))\oplus i\mathbb{R}_Z \\ \oplus \\ L^2_{k-1}(\Lambda^+\otimes\mathfrak{su}(E)) \\ \oplus \\ L^2_{k-1}(W^-\otimes E) \end{matrix}$$

by a compact operator and so has the same (real) index:

$$\operatorname{Ind}\mathcal{D}_{A,\Phi} = \operatorname{Ind}(d_A^*+d_A^+) + \operatorname{Ind}_{\mathbb{R}} D_A - 1.$$

We recall from [20, Eq. (4.2.22)] that the operator

$$d_A^*+d_A^+ : L^2_k(\Lambda^1\otimes\mathfrak{su}(E)) \to L^2_{k-1}(\mathfrak{su}(E))\oplus L^2_{k-1}(\Lambda^+\otimes\mathfrak{su}(E))$$

has index

$$\operatorname{Ind}(d_A^*+d_A^+) = -2p_1(\mathfrak{su}(E)) - 3(1-b^1(X)+b^+(X)).$$

The complex index of the Dirac operator D_A is given by

$$\begin{aligned}
\operatorname{Ind}_{\mathbb{C}} D_A &= \langle(\hat{A}(X))\operatorname{ch}(E)e^{\frac{1}{2}c_1(W^+)},[X]\rangle \\
&= \big\langle(1-\tfrac{1}{24}p_1(X))(2+c_1(E) \\
&\quad +\tfrac{1}{2}c_1(E)^2 - c_2(E))(1+\tfrac{1}{2}c_1(W^+)+\tfrac{1}{8}c_1(W^+)^2),[X]\big\rangle \\
&= \langle -\tfrac{1}{12}p_1(X) + \tfrac{1}{4}c_1(W^+)^2 \\
&\qquad + \tfrac{1}{2}c_1(E)^2 - c_2(E) + \tfrac{1}{2}c_1(E)c_1(W^+),[X]\rangle \\
&= -\tfrac{1}{4}\sigma(X) + \tfrac{1}{4}c_1(W^+)^2 + \tfrac{1}{2}c_1(E)^2 - c_2(E) + \tfrac{1}{2}c_1(E)c_1(W^+) \\
&= (\tfrac{1}{4}c_1(E)^2 - c_2(E)) + \tfrac{1}{4}((c_1(W^+)+c_1(E))^2 - \sigma(X)) \\
&= \tfrac{1}{4}p_1(\mathfrak{su}(E)) + \tfrac{1}{4}((c_1(W^+)+c_1(E))^2 - \sigma(X)).
\end{aligned}$$

We obtain our dimension formula by adding these indices and noting that $\operatorname{Ind}_{\mathbb{R}} D_A = 2\operatorname{Ind}_{\mathbb{C}} D_A$. q.e.d.

3. Regularity

Our goal in this section is to establish the basic regularity results for solutions to the PU(2) monopole equations (2.27). In §3.1 we show that global L^2_1 solutions to the first-order elliptic system comprising (2.27) and the Coulomb gauge equation (see §2) are necessarily C^∞, while in §3.2 we show that global L^2_k solutions to (2.27) are equivalent via an L^2_{k+1} determinant-one, unitary automorphism of E — provided $k \geq 2$ — to C^∞ solutions to (2.27) in Coulomb gauge relative to some C^∞ reference pair. In §3.3 and §3.4 we establish local versions of the results of §3.1 and §3.2. The regularity results and estimates of this section will be needed repeatedly throughout our work, so we state them here in sufficient generality to cover all of these applications. In the present article, we require the regularity results for our proof of transversality for the moduli space of PU(2) monopoles (see §5) and they form the cornerstone of our proofs of removable singularities and existence of an Uhlenbeck compactification (see §4). Furthermore, in sequels to this article [26], [27], the regularity results of this section are used to show that L^2_1 gluing solutions to (2.27) are necessarily C^∞ and to analyse the Uhlenbeck-boundary behavior of the gluing and obstruction maps parametrizing the moduli space ends.

In order to simultaneously address all of the intended applications, the equations we find it convenient to consider here are a quasi-linear, inhomogeneous elliptic system consisting of a generalization of the equations (2.27) and Coulomb gauge equation for a pair $(A, \Phi) \in \tilde{\mathcal{C}}_{W,E}(X)$. Specifically, we allow inhomogeneous, right-hand terms: the need for this generalization arises in our proofs of removable singularities, of regularity for gluing solutions to (2.27), and in analysing the Uhlenbeck-boundary behavior of gluing maps. Suppose that (A_0, Φ_0) is a fixed C^∞ reference pair in $\tilde{\mathcal{C}}_{W,E}(X)$: writing $(A, \Phi) = (A_0, \Phi_0) + (a, \phi)$, combining (2.27) with the Coulomb gauge equation, and allowing inhomogenous terms, we obtain an elliptic system of equations for a pair (a, ϕ) in $\Omega^1(X, \mathfrak{su}(E)) \oplus \Omega^0(X, W^+ \otimes E)$,

$$
\begin{aligned}
d^*_{A_0} a - (\cdot\Phi_0)^* \phi &= \zeta, \\
\mathfrak{S}(A_0 + a, \Phi_0 + \phi) &= (v_0, \psi_0),
\end{aligned} \tag{3.1}
$$

Considering A to be a connection on $\mathfrak{su}(E)$ and using the isomorphism $\mathrm{ad} : \mathfrak{su}(E) \to \mathfrak{so}(\mathfrak{su}(E))$ to view F_A as a section of $\Lambda^2 \otimes \mathfrak{su}(E)$, we write

(3.1) as

$$
\begin{aligned}
d_{A_0}^{*}a - (\cdot\Phi_0)^*\phi &= \zeta,\\
F_{A_0+a}^{+} - \left(\mathrm{id} + \tau_0\otimes \mathrm{id}_{\mathfrak{su}(E)} + \vec{\tau}\cdot\vec{\mathfrak{m}}(A)\right)\rho^{-1}((\Phi_0+\phi)\otimes(\Phi_0+\phi)^*)_{00} &= v_0,\\
(D_{A_0+a} + \rho(\vartheta_0) + \vec{\vartheta}\cdot\vec{\mathfrak{m}}(A))(\Phi_0+\phi) &= \psi_0,
\end{aligned}
$$

where $(v_0,\psi_0)\in\Omega^+(X,\mathfrak{su}(E))\oplus\Omega^0(X,W^-\otimes E)$. Recalling that $d^0_{A_0,\Phi_0}$ and $d^1_{A_0,\Phi_0}$ are the differential operators in the elliptic deformation complex (2.37) for the PU(2) monopole equations (2.27), the above system may be rewritten in the form

$$
\begin{aligned}
d^{0,*}_{A_0,\Phi_0}(a,\phi) &= \zeta,\\
d^1_{A_0,\Phi_0}(a,\phi) + \{(a,\phi),(a,\phi)\} &= -\mathfrak{S}(A_0,\Phi_0) + (v_0,\psi_0) =: (v,\psi),
\end{aligned}
$$

where $(v,\psi)\in\Omega^+(X,\mathfrak{su}(E))\oplus\Omega^0(X,W^-\otimes E)$, and the differentials $d^{0,*}_{A_0,\Phi_0}$ and $d^1_{A_0,\Phi_0}$ are given by (2.38) and (2.36). It will be convenient to view the quadratic term $\{(a,\phi),(a,\phi)\}$ as being defined via the following bilinear form,

$$
\begin{aligned}
&\{(a,\phi),(b,\varphi)\}\\
&\quad := \begin{pmatrix} (a\wedge b)^+ - \left(\mathrm{id} + \tau_0\otimes \mathrm{id}_{\mathfrak{su}(E)} + \vec{\tau}\cdot\vec{\mathfrak{m}}(A)\right)\rho^{-1}(\phi\otimes\varphi^*)_{00} \\ \rho(a)\varphi \end{pmatrix},
\end{aligned}
$$

where $(b,\varphi)\in\Omega^1(X,\mathfrak{su}(E))\oplus\Omega^0(X,W^+\otimes E)$; we will further abbreviate $\{(a,\phi),(a,\phi)\}$ by $q(a,\phi)$ when convenient. Our elliptic system (3.1) then takes the simple shape

$$
\mathcal{D}_{A_0,\Phi_0}(a,\phi) + \{(a,\phi),(a,\phi)\} = (\zeta,v,\psi), \tag{3.2}
$$

recalling from (2.39) that $\mathcal{D}_{A_0,\Phi_0} = d^{0,*}_{A_0,\Phi_0} + d^1_{A_0,\Phi_0}$. This is the form of the (inhomogeneous) Coulomb gauge and PU(2) monopole equations we will use for the majority of the basic regularity arguments.

Some of the regularity results and estimates of this section generalize corresponding results for the first-order anti-self-dual equation [20], [30] and, to a certain extent, those of Uhlenbeck [94] and Parker [70] for the second-order (coupled) Yang-Mills equations. As usual for a quasi-linear, first-order elliptic system with a quadratic non-linearity, over an n-dimensional manifold, the main difficulty is to prove regularity for $L^{n/2}_1$ solutions (that is, at the critical Sobolev exponent). Once the solutions are known to be in L^∞, then standard linear elliptic regularity theory can be applied [37], [64]. It is worth noting at the outset, though, that despite an extensive literature on quasi-linear elliptic systems due to Ladyzhenskaya-Ural'tseva, Morrey and others [54], [64], these classical results do not meet the usual demands of gauge theory and this explains why we develop the precise results we require here. The *constants* appearing in our estimates generally depend on the underlying Riemannian metric on X, the fixed spin^c connection on W, the fixed unitary connection on $\det E$, and the perturbations $(\tau_0, \vartheta_0, \vec{\tau}, \vec{\vartheta})$: this dependence is not explicitly noted, as these ingredients in the PU(2) monopole equations (2.27) are fixed once and for all.

3.1. Regularity for $\mathbf{L^2_1}$ solutions to the inhomogeneous Coulomb gauge and PU(2) monopole equations. We show in this subsection that an L^2_1 solution (a, ϕ) to the PU(2) monopole and Coulomb-gauge equations (3.2), with an L^2_k inhomogeneous term (with $k \geq 1$) is in L^2_{k+1}. Thus, if the inhomogeneous term is in C^∞, then (a, ϕ) is in C^∞. In passing from an L^2_1 to an L^2_2 solution we need only consider the case where $(\vec{\tau}, \vec{\vartheta}) = 0$, while no restriction is placed on the perturbation $(\vec{\tau}, \vec{\vartheta}) \in \mathcal{P}$ given an L^2_2 solution (a, ϕ).

Our regularity result contains, as special cases, Proposition 4.4.13 in [20] and Theorem 8.8 in [30], for anti-self-dual connections, and Theorem 8.11 in [77] for Seiberg-Witten monopoles. The proof we give below for PU(2) monopoles is rather different. We provide a fairly detailed argument here since regularity of L^2_2 solutions to an elliptic equation with a quadratic non-linear term does not quite follow directly from the standard theory for non-linear elliptic systems (for example, [64, Theorem 6.8.1]).

The two principal steps are, first, to get L^p_1 regularity of an L^2_1 solution (a, ϕ) when $2 < p < 4$ and (a, ϕ) is sufficiently L^4-small and, second, to apply elliptic boostrapping and the Sobolev multiplication and embedding theorems to get C^∞ regularity of an L^p_1 solution (a, ϕ) when $2 < p < \infty$. We will use these sharp regularity results and estimates repeatedly throughout our work, so we give the argument in some detail here. The main ingredient in the first step is supplied by Proposition 3.2 and uses a Fredholm alternative argument to pass from L^2_1 to the slightly stronger L^p_1 regularity [33, Theorem 5.3]. Although we will often be able to simply assume that the inhomogenous term is in C^∞, rather than just in L^p, we will later need these intermediate regularity results in our development of the gluing theory for PU(2) monopoles [26], [27] to show that L^2_1 solutions (a, ϕ) to the system (3.2) with a certain inhomogeneous term (ζ, v, ψ), where the latter is initially only known to be in L^2_1 or L^2_2, are actually in C^∞ (together, of course, with (ζ, v, ψ)).

Lemma 3.1. *Let $\mathfrak{B}_1$, $\mathfrak{B}_2$ be Banach spaces and let*

$$T \in \operatorname{Hom}(\mathfrak{B}_1, \mathfrak{B}_2)$$

have a right (left) inverse P. If $T' \in \operatorname{Hom}(\mathfrak{B}_1, \mathfrak{B}_2)$ satisfies

$$\|T' - T\| < \|P\|^{-1},$$

then T' also has a right (left) inverse.

Proof. If $P \in \operatorname{Hom}(\mathfrak{B}_2, \mathfrak{B}_1)$ is a right inverse for T, so $TP = \mathrm{id}_1$, then

$$\|(T' - T)P\| \leq \|T' - T\|\|P\| < 1,$$

and $\mathrm{id}_1 + (T' - T)P$ is an invertible element of the Banach algebra $\operatorname{End}(\mathfrak{B}_1)$. Define $P' = P(1 + (T' - T)P)^{-1}$, so $T'P' = \mathrm{id}_1$ and P' is a right inverse for T'. Similarly for left inverses. q.e.d.

The preceding elementary consequence of the usual characterization of invertible elements of a Banach algebra will be frequently invoked in this section and in particular, in the proof of the proposition below.

Proposition 3.2. *Let X be a closed, oriented, Riemannian four-manifold with metric g and spin^c structure (ρ, W), and let E be a Hermitian two-plane bundle over X. Let (A_0, Φ_0) be a C^∞ pair on the C^∞ bundles $(\mathfrak{su}(E), W^+ \otimes E)$ over X and let $2 \leq p < 4$. Then there are positive constants $\varepsilon = \varepsilon(A_0, \Phi_0, p)$ and $C = C(A_0, \Phi_0, p)$ with the following significance. Suppose that $(a, \phi) \in L^2_1(X, \Lambda^1 \otimes \mathfrak{su}(E)) \oplus L^2_1(X, W^+ \otimes E)$ is an L^2_1 solution on $(\mathfrak{su}(E), W^+ \otimes E)$ to the elliptic system* (3.2) *over X with $(\vec{\tau}, \vec{\vartheta}) = 0$, where (ζ, v, ψ) is in L^p. If $\|(a, \phi)\|_{L^4(X)} < \varepsilon$, then (a, ϕ) is in L^p_1 and*

$$\|(a, \phi)\|_{L^p_{1,A_0}(X)} \leq C \left(\|(\zeta, v, \psi)\|_{L^p(X)} + \|(a, \phi)\|_{L^2(X)}\right).$$

Proof. The operator $\mathcal{D}_{A_0,\Phi_0}$ is Fredholm (since it is elliptic with C^∞ coefficients and X is closed), and so has a finite-dimensional kernel and cokernel. In particular, $\operatorname{Ker} \mathcal{D}_{A_0,\Phi_0}|_{L^2_1} \subset C^\infty$. Let Π_1 be the L^2-orthogonal projection onto $\operatorname{Ker} \mathcal{D}_{A_0,\Phi_0}|_{L^2_1}$ and let Π_2 be the L^2-orthogonal projection onto $\operatorname{Ker} \mathcal{D}^*_{A_0,\Phi_0}|_{L^2} = (\operatorname{Im} \mathcal{D}_{A_0,\Phi_0}|_{L^2_1})^\perp$, the L^2-orthogonal complement of the image of $\operatorname{Im} \mathcal{D}_{A_0,\Phi_0}|_{L^2_1}$. The L^2-adjoint

$\mathcal{D}^*_{A_0,\Phi_0}$ is again a first-order linear elliptic operator with C^∞ coefficients and so $\operatorname{Ker}\mathcal{D}^*_{A_0,\Phi_0}|_{L^2} \subset C^\infty$. We may then rewrite our quasi-linear elliptic equation (3.2) in the form

$$\begin{aligned}\Pi_2^\perp\mathcal{D}_{A_0,\Phi_0}\Pi_1^\perp(a,\phi) &+ \Pi_2^\perp\{(a,\phi),\Pi_1^\perp(a,\phi)\}\\ &= -\Pi_2\left(\mathcal{D}_{A_0,\Phi_0}(a,\phi) + \{(a,\phi),\Pi_1^\perp(a,\phi)\}\right)\\ &\quad - \{(a,\phi),\Pi_1(a,\phi)\} + (\zeta,v,\psi) =: \Upsilon,\end{aligned}$$

where $\Pi_i^\perp = \mathrm{id} - \Pi_i$ for $i=1,2$. Thus, we need to consider the existence and uniqueness problem for solutions (b,φ) to

$$\Pi_2^\perp\mathcal{D}_{A_0,\Phi_0}(b,\varphi) + \Pi_2^\perp\{(a,\phi),(b,\varphi)\} = \Upsilon, \tag{3.3}$$

where $(b,\varphi)\in L^p_1\cap(\operatorname{Ker}\mathcal{D}_{A_0,\Phi_0})^\perp$. Plainly, the operator

$$\Pi_2^\perp\mathcal{D}_{A_0,\Phi_0} : L^p_1\cap(\operatorname{Ker}\mathcal{D}_{A_0,\Phi_0})^\perp \to L^p_1\cap(\operatorname{Ker}\mathcal{D}^*_{A_0,\Phi_0})^\perp$$

is (left and right) invertible and hence this will also be true for any operator from $L^p_1\cap(\operatorname{Ker}\mathcal{D}_{A_0,\Phi_0})^\perp$ to $L^p\cap(\operatorname{Ker}\mathcal{D}^*_{A_0,\Phi_0})^\perp$, such as $\Pi_2^\perp\mathcal{D}_{A_0,\Phi_0} + \Pi_2^\perp\{(a,\phi),\,\cdot\,\}$, which is sufficiently close in the $\operatorname{Hom}(L^p_{1,A_0},L^p)$ operator norm by Lemma 3.1. Since $\mathcal{D}_{A_0,\Phi_0}$ and $\mathcal{D}^*_{A_0,\Phi_0}$ are first-order linear elliptic operators with C^∞ coefficients and $\mathcal{D}_{A_0,\Phi_0}\Pi_1 = 0$ and $\mathcal{D}^*_{A_0,\Phi_0}\Pi_2 = 0$, standard elliptic theory implies that

$$\begin{aligned}\|\Pi_2(\xi,w,\varrho)\|_{L^r_{k,A_0}} &\le C\|(\xi,w,\varrho)\|_2, \qquad (\xi,w,\varrho)\in L^r_k,\\ \|\Pi_1(b,\varphi)\|_{L^r_{k,A_0}} &\le C\|(b,\varphi)\|_2, \qquad (b,\varphi)\in L^r_k,\end{aligned} \tag{3.4}$$

for some constant $C = C(A_0,\Phi_0,k,r)$, whenever $k\ge 1$ and $rk\ge 4/3$ or $k=0$ and $r\ge 2$.

Since (a,ϕ) is in L^2_1, the terms $\mathcal{D}_{A_0,\Phi_0}(a,\phi)$ and $\{(a,\phi),(a,\phi)\}$ and $\{(a,\phi),\Pi_1^\perp(a,\phi)\}$ are in L^2, while the term $\{(a,\phi),\Pi_1(a,\phi)\}$ is in L^2_1. The terms $\Pi_2\mathcal{D}_{A_0,\Phi_0}(a,\phi)$ and $\Pi_2\{(a,\phi),\Pi_1^\perp(a,\phi)\}$ are each in C^∞, while (ζ,v,ψ) is in L^p, and so the right-hand side Υ of (3.3) is in $L^p\cap(\operatorname{Ker}\mathcal{D}^*_{A_0,\Phi_0})^\perp$.

Let $q = 4p/(4-p)$. Then $4<q<\infty$; there is a continuous multiplication map $L^4\times L^q\to L^p$ and an embedding $L^p_1\subset L^q$. So, as $\Pi_2^\perp = \mathrm{id}-\Pi_2$,

$$\begin{aligned}\|\Pi_2^\perp\{(a,\phi),(b,\varphi)\}\|_{L^p} &\le \|\{(a,\phi),(b,\varphi)\}\|_{L^p} + C\|\{(a,\phi),(b,\varphi)\}\|_{L^2}\\ &\le C\|(a,\phi)\|_{L^4}\|(b,\varphi)\|_{L^p_{1,A_0}},\end{aligned}$$

for some positive constant $C = C(A_0, \Phi_0, p)$. By Lemma 3.1 there is a positive constant $\varepsilon = \varepsilon(A_0, \Phi_0, p)$ such that if $\|(a,\phi)\|_{L^4} < \varepsilon$, the linear operator $\Pi_2^\perp \mathcal{D}_{A_0,\Phi_0} + \Pi_2^\perp\{(a,\phi),\ \cdot\ \}$ is (left and right) invertible as a map $L_1^p \cap (\operatorname{Ker}\mathcal{D}_{A_0,\Phi_0})^\perp \to L^p \cap (\operatorname{Ker}\mathcal{D}_{A_0,\Phi_0}^*)^\perp$. Since $\Pi_1^\perp(a,\phi)$ is a solution to (3.3) when $p = 2$, it is the unique solution for $p = 2$.

Let $(a_p, \phi_p) \in L_1^p \cap (\operatorname{Ker}\mathcal{D}_{A_0,\Phi_0})^\perp$ be the L_1^p solution to (3.3). Then (a_p, ϕ_p) is also an L_1^2 solution and by the uniqueness assertion we must have that $(a_p, \phi_p) = \Pi_1^\perp(a,\phi)$. Thus, $\Pi_1^\perp(a,\phi)$ is in L_1^p and so (a,ϕ) is in L_1^p, since $\Pi_1(a,\phi)$ is in C^∞. Finally, the standard estimate for $\mathcal{D}_{A_0,\Phi_0}$, the estimate (3.4), and equation (3.2) yield

$$\begin{aligned}
\|(a,\phi)\|_{L^p_{1,A_0}} &\le C\left(\|\mathcal{D}_{A_0,\Phi_0}(a,\phi)\|_{L^p} + \|\Pi_1(a,\phi)\|_{L^2}\right) \\
&\le C\left(\|\{(a,\phi),(a,\phi)\}\|_{L^p} + \|(\zeta,v,\psi)\|_{L^p} + \|\Pi_1(a,\phi)\|_{L^2}\right),
\end{aligned}$$

and so the desired bound for (a,ϕ) follows by the Sobolev multiplication $L^4 \times L^q \to L^p$, the embedding $L_1^p \subset L^q$, and rearrangement. q.e.d.

Proposition 3.2 will have two main applications: the first is in our proof of removable singularities for PU(2) monopoles, and the second is in our proof of C^∞ regularity for L_1^2 solutions to the PU(2) monopole equation obtained by gluing [26], [27].

As is well-known from standard gauge theory, it is not possible to construct a well-defined quotient space using L_1^2 pairs modulo L_2^2 gauge transformations. We construct a quotient using L_k^2 pairs modulo L_{k+1}^2 gauge transformations, with $k \ge 2$. We first establish a regularity result for the inhomogeneous PU(2) monopole and Coulomb gauge equations, while in §3.2 we show that any PU(2) monopole in L_k^2 is L_{k+1}^2-gauge equivalent to a PU(2) monopole in C^∞.

Proposition 3.3. *Continue the notation of Proposition 3.2. Let $k \ge 1$ be an integer and let $2 < p < \infty$. Let (A_0, Φ_0) be a C^∞ pair on the C^∞ bundles $(\mathfrak{su}(E), W^+ \otimes E)$ over X. Suppose that either*

- *$(a,\phi) \in L_1^p(X, \Lambda^1 \otimes \mathfrak{su}(E)) \oplus L_1^p(X, W^+ \otimes E)$, with $(\vec{\tau}, \vec{\vartheta}) = 0$, or*
- *$(a,\phi) \in L_2^2(X, \Lambda^1 \otimes \mathfrak{su}(E)) \oplus L_2^2(X, W^+ \otimes E)$*

is a solution on $(\mathfrak{su}(E), W^+ \otimes E)$ to the elliptic system (3.2) over X, where (ζ, v, ψ) is in L_k^2. Then (a,ϕ) is in L_{k+1}^2 and there is a universal polynomial $Q_k(x,y)$, with positive real coefficients depending at most on $(A_0,\Phi_0), k$, such that $Q_k(0,0) = 0$ and

$$\|(a,\phi)\|_{L^2_{k+1,A_0}(X)} \le Q_k\left(\|(\zeta,v,\psi)\|_{L^2_{k,A_0}(X)}, \|(a,\phi)\|_{L^p_{1,A_0}(X)}\right).$$

In particular, if (ζ, v, ψ) *is in* C^∞*, then* (a, ϕ) *is in* C^∞*, and if* $(\zeta, v, \psi) = 0$*, then*

$$\|(a,\phi)\|_{L^2_{k+1,A_0}(X)} \leq C\|(a,\phi)\|_{L^p_{1,A_0}(X)}.$$

Proof. We consider the cases $k = 1$, $k = 2$, and $k \geq 3$ separately. We may further assume without loss of generality that $2 < p < 4$.

Case $(\boldsymbol{k = 1})$. Let $p_0 = p$ and $q_0 = q = 4p/(4-p)$. Then $1/p = 1/4 + 1/q$ and $4 < q < \infty$. We have a continuous Sobolev multiplication map $L^4 \times L^q \to L^p$ and an embedding $L^p_1 \subset L^q$. Since

$$\mathcal{D}_{A_0,\Phi_0}(a,\phi) = -\{(a,\phi),(a,\phi)\} + (\zeta, v, \psi), \tag{3.5}$$

elliptic regularity for $\mathcal{D}_{A_0,\Phi_0}$ implies that $(a, \phi) \in L^{p_1}_1$, where $p_1 = 2p/(4-p)$ and $q_1 = 4p_1/(4-p_1)$, so $2 < p_1 < 4$ and $4 < q_1 < \infty$. Here, we have used the Sobolev multiplication $L^4 \times L^{q_1} \to L^{p_1}$ to get $\{(a,\phi),(a,\phi)\}$ in L^{p_1} and the embedding $L^2_1 \subset L^{p_1}$ to get (ζ, v, ψ) in L^{p_1}. Let $p = 2 + \varepsilon$ and $\delta = \varepsilon/(2-\varepsilon)$, and note that

$$p_1 = \frac{2p}{4-p} = \frac{2p}{2-\varepsilon} = (1+\delta)p > 2 + \varepsilon.$$

If $p_j < 4$, we inductively define $p_{j+1} = 2p_j/(4-p_j)$ and $q_{j+1} = 4p_j/(4-p_j)$ for $j \geq 0$. Therefore, we have $p_j > 2 + \varepsilon$. Thus

$$p_{j+1} = \frac{2p_j}{4-p_j} > \frac{2p_j}{2-\varepsilon} = (1+\delta)p_j > p_j > 2 + \varepsilon,$$

and so $p_{j+1} > (1+\delta)^{j+1}p$ for $j \geq 0$.

By repeating the above regularity argument when $2 < p_{j+1} < 4$, using (3.5) at each stage, we see that $(a, \phi) \in L^{p_j}_1$ for $j \geq 0$. We continue the induction until for large enough $j \geq 0$, we find that $p' := p_{j+1} \geq 8/3$ and $q' := q_{j+1} \geq 8$, so $(a, \phi) \in L^{8/3}_1 \subset L^8$. Therefore, with $(\zeta, v, \psi) \in L^2_1 \subset L^4$ and using the Sobolev multiplication $L^8 \times L^8 \to L^4$ to get $\{(a,\phi),(a,\phi)\}$ in L^4, equation (3.5) gives $(a, \phi) \in L^4_1$. The Sobolev multiplication $L^4_1 \times L^4_1 \to L^2_1$ implies that the quadratic term $\{(a,\phi),(a,\phi)\}$ is in L^2_1 and so (3.5) yields $(a, \phi) \in L^2_2$, as required.

Case $(\boldsymbol{k = 2})$. From the case $k = 1$ we continue the induction until $p' > 8/3$ and $q' > 4$. Hence, with (ζ, v, ψ) now in $L^2_2 \subset L^{p'}_1$, equation (3.5) gives $(a, \phi) \in L^{p'}_2 \subset C^0$. The Sobolev multiplication $L^{p'}_2 \times L^2_2 \to L^2_2$ implies that the quadratic term $\{(a,\phi),(a,\phi)\}$ is then in L^2_2 and so (3.5) gives $(a, \phi) \in L^2_4$, as required.

Case $(k \geq 3)$. There is continuous Sobolev multiplication map $L^2_k \times L^2_k \to L^2_k$ and so the quadratic term $\{(a,\phi),(a,\phi)\}$ is in L^2_k, since (a,ϕ) is in L^2_k. Therefore, (3.5) gives $(a,\phi) \in L^2_{k+1}$.

This completes the proof of the proposition. q.e.d.

By combining Propositions 3.2 and 3.3 we obtain the desired regularity result for L^2_1 solutions to the inhomogeneous Coulomb gauge and PU(2) monopole equations:

Corollary 3.4. *Continue the notation of Proposition 3.2. Let (A_0,Φ_0) be a C^∞ pair on the C^∞ bundles $(\mathfrak{su}(E), W^+\otimes E)$ over X. Then there is a positive constant $\varepsilon = \varepsilon(A_0,\Phi_0)$ such that the following hold. Suppose that either*

- $(a,\phi) \in L^2_1(X,\Lambda^1\otimes\mathfrak{su}(E)) \oplus L^2_1(X,W^+\otimes E)$, *with* $(\vec\tau,\vec\vartheta)=0$, *or*
- $(a,\phi) \in L^2_2(X,\Lambda^1\otimes\mathfrak{su}(E)) \oplus L^2_2(X,W^+\otimes E)$

is a solution on $(\mathfrak{su}(E), W^+\otimes E)$ to the elliptic system (3.2) over X, where (ζ,v,ψ) is in L^2_k and $\|(a,\phi)\|_{L^4(X)} < \varepsilon$ and $k\geq 0$ is an integer. Then (a,ϕ) is in L^2_{k+1} and there is a universal polynomial $Q_k(x,y)$, with positive real coefficients depending at most on $(A_0,\Phi_0),k$, such that $Q_k(0,0)=0$ and

$$\|(a,\phi)\|_{L^2_{k+1,A_0}(X)} \leq Q_k\left(\|(\zeta,v,\psi)\|_{L^2_{k,A_0}(X)}, \|(a,\phi)\|_{L^2(X)}\right).$$

In particular, if (ζ,v,ψ) is in C^∞ then (a,ϕ) is in C^∞, and if $(\zeta,v,\psi) = 0$, then

$$\|(a,\phi)\|_{L^2_{k+1,A_0}(X)} \leq C\|(a,\phi)\|_{L^2(X)}.$$

Remark 3.5. A similar sharp regularity result for solutions to the Coulomb and anti-self-dual equations (that is, $(d^*_\Gamma + d^+_\Gamma)a + (a\wedge a)^+ = v$) on the product bundle over S^4 is given by Proposition 4.4.13 in [20]. The reader is forwarned that Corollary 3.4 does not apply directly to show that L^2_1 gluing solutions to the PU(2) monopole equation in [26] are C^∞ due to the unfavorable dependence of the constant $\varepsilon(A',\Phi',p)$ on the approximate PU(2) monopole (A',Φ') when $p>2$. However, a local version of this result, namely Corollary 3.11 below, is applicable in this situation. The point is explained further in [26].

3.2. Regularity of $\mathbf{L}^2_k$ solutions to the PU(2) monopole equations. We explain in this subsection why an L^2_k monopole (A,Φ)

on the C^∞ bundles $(\mathfrak{su}(E), W^+ \otimes E)$ over X is gauge equivalent, via an L^2_{k+1} determinant-one, unitary automorphism u of the bundle E, to a C^∞ solution $u(A,\Phi)$ on $(\mathfrak{su}(E), W^+ \otimes E)$ over X when $k \geq 2$. This regularity result contains, as special cases, Theorem 8.8 in [30], for anti-self-dual connections, and Theorem 8.11 in [77] for Seiberg-Witten monopoles.

We will need the following observation concerning the symmetry of the Coulomb gauge equation for pairs; the corresponding fact for connections is explained in [20, p. 56].

Lemma 3.6. *Let (A,Φ), (A_0,Φ_0) be L^2_k pairs on the bundles $(\mathfrak{su}(E), W^+ \otimes E)$ over X. If (A_0,Φ_0) is in Coulomb gauge relative to (A,Φ), so $d^{0,*}_{A,\Phi}((A_0,\Phi_0) - (A,\Phi)) = 0$, then (A,Φ) is in Coulomb gauge relative to (A_0,Φ_0), so $d^{0,*}_{A_0,\Phi_0}((A,\Phi) - (A_0,\Phi_0)) = 0$.*

Proof. The equation $d^{0,*}_{A,\Phi}((A_0,\Phi_0) - (A,\Phi)) = 0$ is the Euler-Lagrange equation for the functional

$$\mathcal{G}_E^{2,k+1} \ni u \mapsto \|u(A_0,\Phi_0) - (A,\Phi)\|^2_{L^2},$$

while the equation $d^{0,*}_{A_0,\Phi_0}((A,\Phi) - (A_0,\Phi_0)) = 0$ is the Euler-Lagrange equation for the functional $\mathcal{G}_E^{2,k+1} \ni v \mapsto \|v(A,\Phi) - (A_0,\Phi_0)\|^2_{L^2}$.

But for any $u \in \mathcal{G}_E^{2,k+1}$ we have

$$\|u(A_0,\Phi_0) - (A,\Phi)\|_{L^2} = \|u^{-1}(A,\Phi) - (A_0,\Phi_0)\|_{L^2},$$

and so if the functional $u \mapsto \|u(A_0,\Phi_0) - (A,\Phi)\|^2_{L^2}$ has a critical point at $u = \mathrm{id}_E$, then the functional $u^{-1} \mapsto \|u^{-1}(A,\Phi) - (A_0,\Phi_0)\|^2_{L^2}$ will also have a critical point at $u = \mathrm{id}_E$. q.e.d.

Proposition 3.7. *Let X be a closed, oriented, Riemannian four-manifold with spin^c bundle W and let E be a Hermitian two-plane bundle over X. Let $k \geq 2$ be an integer and suppose that (A,Φ) is an L^2_k solution to (2.27) on the C^∞ bundles $(\mathfrak{su}(E), W^+ \otimes E)$ over X. Then there is a L^2_{k+1} determinant-one, unitary automorphism u of the bundle E over X such that $u(A,\Phi)$ is C^∞ over X.*

Proof. It suffices, of course, to show that there is an L^2_{k+1} gauge transformation u of E over such that $u(A,\Phi)$ is in L^2_{k+1}. The C^∞ pairs (A_0,Φ_0) on $(\mathfrak{su}(E), W^+ \otimes E)$ form a dense subspace of the space of L^2_k pairs and so, given $\varepsilon = \varepsilon(A,\Phi) > 0$, there is a C^∞ pair (A_0,Φ_0) such that

$$\|(A,\Phi) - (A_0,\Phi_0)\|_{L^2_{k,A_0}} < \varepsilon.$$

For small enough ε, Proposition 2.8 gives an L^2_{k+1} determinant-one, unitary automorphism u of the bundle E such that $u^{-1}(A_0,\Phi_0)$ is in Coulomb gauge relative to (A,Φ):

$$d^{0,*}_{A,\Phi}(u^{-1}(A_0,\Phi_0)-(A,\Phi))=0.$$

Now $u(d^{0,*}_{A,\Phi}(u^{-1}(A_0,\Phi_0)-(A,\Phi)))=d^{0,*}_{u(A,\Phi)}((A_0,\Phi_0)-u(A,\Phi))$ and so

$$d^{0,*}_{u(A,\Phi)}((A_0,\Phi_0)-u(A,\Phi))=0.$$

Therefore, (A_0,Φ_0) is in Coulomb gauge relative to $u(A,\Phi)$ and Lemma 3.6 implies that $u(A,\Phi)$ is in Coulomb gauge relative to (A_0,Φ_0),

$$d^{0,*}_{A_0,\Phi_0}(u(A,\Phi)-(A_0,\Phi_0))=0.$$

Let $(a,\phi)=u(A,\Phi)-(A_0,\Phi_0)$, so that

$$(a,\phi)\in L^2_k(X,\Lambda^1\otimes\mathfrak{su}(E))\oplus L^2_k(X,W^+\otimes E);$$

the Coulomb gauge condition then takes the simpler form $d^{0,*}_{A_0,\Phi_0}(a,\phi)=0$. Since $u(A,\Phi)=(A_0,\Phi_0)+(a,\phi)$ is an L^2_k monopole, $\mathfrak{S}((A_0,\Phi_0)+(a,\phi))=0$ and so (a,ϕ) is an L^2_k solution to the quasi-linear elliptic equation

$$\mathcal{D}_{A_0,\Phi_0}(a,\phi)+\{(a,\phi),(a,\phi)\}=-\mathfrak{S}(A_0,\Phi_0),$$

with C^∞ data $-\mathfrak{S}(A_0,\Phi_0)$. The conclusion now follows from Proposition 3.3 with $\zeta=0$ and $(v,\psi)=-\mathfrak{S}(A_0,\Phi_0)$. q.e.d.

One of the convenient practical consequences of Proposition 3.7 is that we can always work, modulo global L^2_{k+1} gauge transformations, with C^∞ rather than L^2_k monopoles. For the remainder of this article, therefore, the term 'PU(2) monopole' is generally reserved for C^∞ solutions to the perturbed PU(2) monopole equations (2.27). In a similar vein, we generally reserve the terms 'gauge transformation' or 'bundle map' for gauge transformations or bundle maps which are in C^∞.

Given Proposition 3.7, we then have the following analogue of Proposition 4.2.16 in [20], the corresponding result for the moduli space of anti-self-dual connections. The proof is standard and so is left to the reader.

Corollary 3.8. *Continue the hypotheses of Proposition 3.7. Then for any $k\geq 2$ the natural inclusion of topological spaces $M^k_{W,E}\hookrightarrow M^{k+1}_{W,E}$ is a homeomorphism.*

Thus, the topology of the moduli space $M^k_{W,E}$ of L^2_k monopoles is independent of the Sobolev spaces used in its construction for $k \geq 2$ and so we simply denote the moduli space by $M_{W,E}$.

3.3. Local regularity and interior estimates for $\mathbf{L}^2_1$ solutions to the inhomogeneous Coulomb gauge and PU(2) monopole equations. In this section we specialize the results of §3.1 to the case where the reference pair is a trivial PU(2) monopole, so $(A_0, \Phi_0) = (\Gamma, 0)$ on the bundles $(\mathfrak{su}(E), W^+ \otimes E)$, over an open subset $\Omega \subset X$, where Γ is a flat connection.

We continue to assume that X is a closed, oriented four-manifold with metric g, spin^c bundle W, and Hermitian two-plane bundle E extending those on $\Omega \subset X$. We use the inhomogenous estimates and regularity results in our proof of removable singularities for PU(2) monopoles in §4.3 and in our development of the gluing theory for PU(2) monopoles in sequels [26], [27] to the present article — especially to show that a global L^2_1 gluing solution (a, ϕ) is actually C^∞. We use the homogeneous estimates and regularity results in §4 for our proof of the existence of an Uhlenbeck compactification for the moduli space of PU(2) monopoles.

We have the following local versions of Propositions 3.2 and 3.3 and Corollary 3.4:

Proposition 3.9. *Continue the notation of the preceding paragraph. Let $\Omega' \Subset \Omega$ be a precompact open subset and let $2 \leq p < 4$. Then there are positive constants $\varepsilon = \varepsilon(\Omega, p)$ and $C = C(\Omega', \Omega, p)$ with the following significance. Suppose that (a, ϕ) is an $L^2_1(\Omega)$ solution to the elliptic system* (3.2) *over Ω for $(\vec{\tau}, \vec{\vartheta}) = 0$, with $(A_0, \Phi_0) = (\Gamma, 0)$ and where (ζ, v, ψ) is in $L^p(\Omega)$. If $\|(a, \phi)\|_{L^4(\Omega)} < \varepsilon$, then (a, ϕ) is in $L^p_1(\Omega')$ and*

$$\|(a, \phi)\|_{L^p_{1,\Gamma}(\Omega')} \leq C \left(\|(\zeta, v, \psi)\|_{L^p(\Omega)} + \|(a, \phi)\|_{L^2(\Omega)}\right).$$

Proof. Choose an open subset Ω'' such that $\Omega' \Subset \Omega'' \Subset \Omega$. Let χ be C^∞ cutoff function such that $\operatorname{supp} \chi \subset \Omega''$ and $\chi = 1$ on Ω'. Let β be a cutoff function such that $\beta = 1$ on $\operatorname{supp} \chi$ and $\operatorname{supp} \beta \subset \Omega''$. Since (a, ϕ) is a solution to (3.2) with right-hand side (ζ, v, ψ) over Ω, then $\chi(a, \phi)$ solves

$$\begin{aligned}
&\mathcal{D}_{\Gamma,0}\chi(a, \phi) + \{(a, \phi), \chi(a, \phi)\} \\
&\quad = \chi\mathcal{D}_{\Gamma,0}(a, \phi) + d\chi \otimes (a, \phi) + \chi\{(a, \phi), (a, \phi)\} \\
&\quad = \chi(\zeta, v, \psi) + d\chi \otimes (a, \phi) =: (\zeta', v', \psi').
\end{aligned}$$

Thus, $\chi(a,\phi)$ is an L^2_1 solution over X to the linear elliptic system over X,

$$\mathcal{D}_{\Gamma,0}(b,\varphi) + \{\beta(a,\phi),(b,\varphi)\} = (\zeta',v',\psi'),$$

with L^2_1 coefficient $\beta(a,\phi)$ and L^2_1 right-hand side (ζ',v',ψ'). Since $\|\beta(a,\phi)\|_{L^4(X)} \le \|(a,\phi)\|_{L^4(\Omega)}$, the proof of Proposition 3.2 implies that $\chi(a,\phi)$ is in L^p_1 over X if $\varepsilon = \varepsilon(g,\Omega,p)$ is sufficiently small. Thus, (a,ϕ) is in L^p_1 over Ω' with

$$\|\chi(a,\phi)\|_{L^p_{1,\Gamma}(X)} \le C\left(\|(\zeta',v',\psi')\|_{L^p(X)} + \|\chi(a,\phi)\|_{L^2(X)}\right),$$

and therefore, for $2 \le p < 4$, we have

$$\|(a,\phi)\|_{L^p_{1,\Gamma}(\Omega')} \le C\left(\|(\zeta,v,\psi)\|_{L^p(\Omega'')} + \|(a,\phi)\|_{L^p(\Omega'')}\right).$$

The preceding bound and the Sobolev embedding $L^2_1 \subset L^p$ give the estimate

$$\|(a,\phi)\|_{L^p(\Omega'')} \le c\|(a,\phi)\|_{L^2_{1,\Gamma}(\Omega'')} \le C\left(\|(\zeta,v,\psi)\|_{L^2(\Omega)} + \|(a,\phi)\|_{L^2(\Omega)}\right).$$

Combining these inequalities then yields the required L^p_1 estimate for (a,ϕ) over Ω'. q.e.d.

Proposition 3.10. *Continue the notation of Proposition 3.9. Let $k \ge 1$ be an integer, and let $2 < p < \infty$. Suppose that either*

- *(a,ϕ) is an $L^p_1(\Omega)$, when $(\vec{\tau},\vec{\vartheta}) = 0$, or*
- *(a,ϕ) is an $L^2_2(\Omega)$*

solution to the elliptic system (3.2) over Ω with $(A_0,\Phi_0) = (\Gamma,0)$, where (ζ,v,ψ) is in $L^2_k(\Omega)$. Then (a,ϕ) is in $L^2_{k+1}(\Omega')$ and there is a universal polynomial $Q_k(x,y)$, with positive real coefficients depending at most on k, Ω', Ω, such that $Q_k(0,0) = 0$ and

$$\|(a,\phi)\|_{L^2_{k+1,\Gamma}(\Omega')} \le Q_k\left(\|(\zeta,v,\psi)\|_{L^2_{k,\Gamma}(\Omega)}, \|(a,\phi)\|_{L^p_{1,\Gamma}(\Omega)}\right).$$

If (ζ,v,ψ) is in $C^\infty(\Omega)$, then (a,ϕ) is in $C^\infty(\Omega')$ and if $(\zeta,v,\psi) = 0$, then

$$\|(a,\phi)\|_{L^2_{k+1,\Gamma}(\Omega')} \le C\|(a,\phi)\|_{L^p_{1,\Gamma}(\Omega)}.$$

Proof. Again, choose an open subset Ω'' such that $\Omega' \Subset \Omega'' \Subset \Omega$, let χ be C^∞ cutoff function such that $\operatorname{supp}\chi \subset \Omega$, and $\chi = 1$ on Ω'' and let β be a cutoff function such that $\beta = 1$ on $\operatorname{supp}\chi$ and $\operatorname{supp}\beta \subset \Omega$. Since (a,ϕ) is a solution to (3.2) with right-hand side (ζ, v, ψ) over Ω, $\chi(a,\phi)$ solves

$$\begin{aligned}
&\mathcal{D}_{\Gamma,0}\chi(a,\phi) + \{\chi(a,\phi), \chi(a,\phi)\} \\
&\quad = \chi\mathcal{D}_{\Gamma,0}(a,\phi) + d\chi \otimes (a,\phi) + \chi\{(a,\phi),(a,\phi)\} \\
&\qquad + \chi(\chi-1)\{(a,\phi),(a,\phi)\} \\
&\quad = \chi(\zeta,v,\psi) + d\chi\otimes(a,\phi) + \chi(\chi-1)\{(a,\phi),(a,\phi)\} \\
&\quad =: (\zeta', v', \psi').
\end{aligned}$$

Note that $\chi(\zeta,v,\psi)$ is in $L^2_k(X)$, while $d\chi\otimes(a,\phi)$ is in $L^p_1(X)$, and the Sobolev multiplication $L^q \times L^q \to L^{p_1}$ implies that $\chi(\chi-1)\{(a,\phi),(a,\phi)\}$ is in $L^{p_1}(X)$, where $p_1 = q/2 = 2p/(4-p) > p$. Thus, (ζ', v', ψ') is in $L^{p_1}(X)$.

The proof of the case $k = 1$ in Proposition 3.3 then implies that $\chi(a,\phi)$ is an $L^{p_1}_1(X)$, so (a,ϕ) is in $L^{p_1}_1(\Omega'')$. We now repeat this process for each of the remaining steps in the proof of Proposition 3.3, at each stage on successively smaller open subsets Ω''' such that $\Omega' \Subset \Omega''' \Subset \Omega''$, until we obtain (a,ϕ) in $L^2_{k+1}(\Omega')$ and the desired $L^2_{k+1}(\Omega')$ estimate. q.e.d.

Corollary 3.11. *Continue the notation of Proposition 3.10. Then there is a positive constant $\varepsilon = \varepsilon(\Omega)$ with the following significance. Suppose that either*

- *(a,ϕ) is an $L^2_1(\Omega)$, when $(\vec{\tau},\vec{\vartheta}) = 0$, or*
- *(a,ϕ) is an $L^2_2(\Omega)$*

solution to the elliptic system (3.2) *over Ω with $(A_0,\Phi_0) = (\Gamma, 0)$, where (ζ,v,ψ) is in $L^2_k(\Omega)$ and $\|(a,\phi)\|_{L^4(\Omega)} < \varepsilon$. Then (a,ϕ) is in $L^2_{k+1}(\Omega')$ and there is a universal polynomial $Q_k(x,y)$, with positive real coefficients, depending at most on k, Ω', Ω, such that $Q_k(0,0) = 0$ and*

$$\|(a,\phi)\|_{L^2_{k+1,\Gamma}(\Omega')} \le Q_k\left(\|(\zeta,v,\psi)\|_{L^2_{k,\Gamma}(\Omega)}, \|(a,\phi)\|_{L^2(\Omega)}\right).$$

If (ζ,v,ψ) is in $C^\infty(\Omega)$ then (a,ϕ) is in $C^\infty(\Omega')$, and if $(\zeta,v,\psi) = 0$, then

$$\|(a,\phi)\|_{L^2_{k+1,\Gamma}(\Omega')} \le C\|(a,\phi)\|_{L^2(\Omega)}.$$

Corollary 3.11 thus yields a sharp local elliptic regularity result for PU(2) monopoles (A, Φ) in L^2_1 which are given to us in Coulomb gauge relative to $(\Gamma, 0)$: this regularity result is the key ingredient in our proof (given in §4.3) of removable point singularities for PU(2) monopoles.

Proposition 3.12. *Continue the notation of Corollary 3.11. Then there is a positive constant $\varepsilon = \varepsilon(\Omega)$ and, if $k \geq 1$ is an integer, there is a positive constant $C = C(\Omega', \Omega, k)$ with the following significance. Suppose that either*

- (A, Φ) *is an* L^2_1, *when* $(\vec{\tau}, \vec{\vartheta}) = 0$, *or*
- (A, Φ) *is an* L^2_2

solution to the PU(2) *monopole equations* (2.27) *over* Ω, *which is in Coulomb gauge over* Ω *relative to* $(\Gamma, 0)$, *so* $d^*_\Gamma(A - \Gamma) = 0$, *and obeys* $\|(A-\Gamma, \Phi)\|_{L^4(\Omega)} < \varepsilon$. *Then* $(A-\Gamma, \Phi)$ *is in* $C^\infty(\Omega')$ *and for any* $k \geq 1$,

$$\|(A - \Gamma, \Phi)\|_{L^2_{k,\Gamma}(\Omega')} \leq C\|(A - \Gamma, \Phi)\|_{L^2(\Omega)}.$$

Proof. Corollary 3.11 applies to the $L^2_1(\Omega)$ pair $(a, \phi) = (A - \Gamma, \Phi)$ and yields the required regularity and estimates for $(A - \Gamma, \Phi)$ with $(\zeta, v, \psi) = 0$ in (3.2). q.e.d.

3.4. Estimates for PU(2) monopoles in a good local gauge. It remains to combine the local regularity results and estimates of §3.3, for PU(2) monopoles (A, Φ) where the connection A is assumed to be in Coulomb gauge relative to the product SO(3) connection Γ, with Uhlenbeck's local, Coulomb gauge-fixing theorem. We then obtain regularity results and estimates for PU(2) monopoles (A, Φ) with small curvature F_A, parallel to those of Theorem 2.3.8 and Proposition 4.4.10 in [20] for anti-self-dual connections.

In order to apply Corollary 3.11 we need Uhlenbeck's Coulomb gauge-fixing result [95, Theorem 2.1 & Corollary 2.2]). Let B (respectively, $\bar{B}$) be the open (respectively, closed) unit ball centered at the origin in $\mathbb{R}^4$ and let G be a compact Lie group. In order to provide universal constants we assume $\mathbb{R}^4$ has its standard metric, though the results of this subsection naturally hold for any C^∞ Riemannian metric, with comparable constants for metrics which are suitably close.

Theorem 3.13. *There are positive constants c and ε with the following significance. If $2 \leq p < 4$ is a constant and*

$$A \in L^p_1(B, \Lambda^1 \otimes \mathfrak{g}) \cap L^p_1(\partial B, \Lambda^1 \otimes \mathfrak{g})$$

is a connection matrix whose curvature satisfies $\|F_A\|_{L^p(B)} < \varepsilon$*, then there is an gauge transformation* $u \in L^p_2(B,G) \cap L^p_2(\partial B, G)$ *such that* $u(A) := uAu^{-1} - (du)u^{-1}$ *satisfies*

$$d^*u(A) = 0 \text{ on } B, \tag{1}$$

$$\frac{\partial}{\partial r} \lrcorner u(A) = 0 \text{ on } \partial B, \tag{2}$$

$$\|u(A)\|_{L^p_1(B)} \le c\|F_A\|_{L^p(B)}. \tag{3}$$

If A *is in* $L^p_k(B)$*, for* $k \geq 2$*, then* u *is in* $L^p_{k+1}(B)$*, and the gauge transformation* u *is unique up to multiplication by a constant element of* G*.*

Remark 3.14. If G is abelian, then the requirement that

$$\|F_A\|_{L^p(B)} < \varepsilon$$

can be omitted.

It is often useful to rephrase Theorem 3.13 in two other slightly different ways. Suppose A is an L^2_k connection on a C^∞ principal G bundle P over B with $k \geq 2$ and $\|F_A\|_{L^2(B)} < \varepsilon$. Then the assertions of Theorem 3.13 are equivalent to each of the following:

- There is an L^2_{k+1} trivialization $\tau : P \to B \times G$ such that
  - (i) $d^*_\Gamma(\tau(A) - \Gamma) = 0$, where Γ is the product connection on $B \times G$,
  - (ii) $\frac{\partial}{\partial r} \lrcorner (\tau(A) - \Gamma) = 0$, and
  - (iii) $\|(\tau(A) - \Gamma)\|_{L^2_1(B)} \le c\|F_A\|_{L^2(B)}$.
- There is an L^2_{k+1} flat connection Γ on P such that
  - (i) $d^*_\Gamma(A - \Gamma) = 0$,
  - (ii) $\frac{\partial}{\partial r} \lrcorner (A - \Gamma) = 0$, and
  - (iii) $\|(A - \Gamma)\|_{L^2_1(B)} \le c\|F_A\|_{L^2(B)}$, and an L^2_{k+1} trivialization $P|_B \simeq B \times G$ taking Γ to the product connection.

We can now combine Theorem 3.13 with Proposition 3.12 to give the following analogue of Theorem 2.3.8 in [20] — the interior estimate for anti-self-dual connections with L^2-small curvature.

Corollary 3.15. *Let $B \subset \mathbb{R}^4$ be the open unit ball with center at the origin with spin^c structure (ρ, W), let $U \Subset B$ be an open subset, and let Γ be the product connection on $B \times \mathrm{SO}(3)$. Then there is a positive constant ε and if $\ell \geq 1$ is an integer, there is a positive constant $C(\ell, U)$ with the following significance. Suppose that either*

- *(A, Φ) is an L^2_1, when $(\vec{\tau}, \vec{\vartheta}) = 0$, or*
- *(A, Φ) is an L^2_k, $k \geq \max\{2, \ell\}$,*

solution to the PU(2) *monopole equations* (2.27) *over B and that the curvature of the* SO(3) *connection matrix A obeys $\|F_A\|_{L^2(B)} < \varepsilon$. Then there is an L^2_{k+1} gauge transformation $u : B \to \mathrm{SU}(2)$ such that $(u(A) - \Gamma, u\Phi)$ is in $C^\infty(B)$ with $d^*(u(A) - \Gamma) = 0$ over B and*

$$\|(u(A) - \Gamma, u\Phi)\|_{L^2_{\ell,\Gamma}(U)} \leq C\|F_A\|^{1/2}_{L^2(B)}.$$

Proof. Let ε_1 be the constant in Theorem 3.13 and note that for $\varepsilon \leq \varepsilon_1$, Theorem 3.13 (taking $G = \mathrm{SO}(3)$) and the Sobolev embedding $L^2_1(B) \subset L^4(B)$ imply that there is an L^2_{k+1} gauge transformation $u : B \to \mathrm{SU}(2)$ (by lifting the SO(3) gauge transformation) such that

$$\begin{aligned} &d^*_\Gamma(u(A) - \Gamma) = 0, \\ &\|u(A) - \Gamma\|_{L^4(B)} \leq c_1\|F_A\|_{L^2(B)} < c_1\varepsilon. \end{aligned}$$

On the other hand, the quadratic equation for Φ in (2.27) and Lemma 2.26 give the L^4 and L^2 estimates,

$$\begin{aligned} &\|u\Phi\|_{L^4(B)} = \|\Phi\|_{L^4(B)} \leq 2\|F^+_A\|^{1/2}_{L^2(B)}, \\ &\|u\Phi\|_{L^2(B)} \leq 2\|F_A\|^{1/2}_{L^1(B)}. \end{aligned}$$

Let ε_2 be the constant in Proposition 3.12. Hence, if $c_1\varepsilon \leq \varepsilon_2$ and $4\sqrt{\varepsilon} \leq \varepsilon_2$, then Proposition 3.12 implies that $(u(A) - \Gamma, u\Phi)$ obeys

$$\|(u(A) - \Gamma, u\Phi)\|_{L^2_{\ell,\Gamma}(U)} \leq C\|(u(A) - \Gamma, u\Phi)\|_{L^2(B)},$$

since $d^*_\Gamma(u(A) - \Gamma) = 0$ and $\|(u(A) - \Gamma, u\Phi)\|_{L^4(B)} < \varepsilon_2$. The desired estimate follows by combining these inequalities for small enough $\varepsilon \leq 1$.

q.e.d.

Again, it is often useful to rephrase Corollary 3.15 in the two other slightly different ways. Suppose that $\ell \geq 1$ and that (A, Φ) is a PU(2) monopole in L^2_k on $(\mathfrak{su}(E), W^+ \otimes E)$ over the unit ball $B \subset \mathbb{R}^4$ with $k \geq \max\{2, \ell\}$, $\|F_A\|_{L^2(B)} < \varepsilon$ and $U \Subset B$. Then the assertions of Corollary 3.15 are equivalent to each of the following:

- There is a C^∞ trivialization $\tau : E|_B \to B \times \mathbb{C}^2$ and an L^2_{k+1} determinant-one, unitary bundle automorphism u of $E|_B$ such that, with respect to the product connection Γ on $B \times \mathfrak{su}(2)$, we have

    (i) $d^*_\Gamma(\tau u(A) - \Gamma) = 0$, and

    (ii) $\|(\tau u(A) - \Gamma, \tau u\Phi)\|_{L^2_{\ell,\Gamma}(U)} \le C\|F_A\|^{1/2}_{L^2(B)}$.

- There is an L^2_{k+1} flat connection Γ on $\mathfrak{su}(E)|_B$ such that

    (i) $d^*_\Gamma(A - \Gamma) = 0$, and

    (ii) $\|(A - \Gamma, \Phi)\|_{L^2_{\ell,\Gamma}(U)} \le c\|F_A\|^{1/2}_{L^2(B)}$, and an L^2_{k+1} trivialization $\mathfrak{su}(E)|_B \simeq B \times \mathfrak{su}(2)$ taking Γ to the product connection.

Corollary 3.15 immediately yields the following local compactness result for PU(2) monopoles analogous to the local compactness result for anti-self-dual connections in [20, Corollary 2.3.9].

Corollary 3.16. *Let $B \subset \mathbb{R}^4$ be the open unit ball and spin^c structure (ρ, W). Then there is a positive constant $\varepsilon_0(g, A_{\det W^+}, A_{\det E})$ with the following significance. Let $U \Subset B$ be an open subset and let $k \ge 2$ be an integer. If (A_α, Φ_α) is a sequence of* PU(2) *monopoles in L^2_k on $(\mathfrak{su}(E), W^+ \otimes E)$ over B such that*

$$\|F_{A_\alpha}\|_{L^2(B)} < \varepsilon_0,$$

then there is a subsequence $\{\alpha'\} \subset \{\alpha\}$, a sequence of determinant-one, unitary L^2_{k+1} automorphisms $\{u_{\alpha'}\}$ of $E|_B$ and a sequence of gauge equivalent pairs $(\tilde{A}_{\alpha'}, \tilde{\Phi}_{\alpha'}) := u_{\alpha'}(A_{\alpha'}, \Phi_{\alpha'})$ which converge in $L^2_{k,\mathrm{loc}}$ on U to a PU(2) *monopole $(\tilde{A}, \tilde{\Phi})$ over U.*

We will also need interior estimates for PU(2) monopoles in a good local gauge over more general simply-connected regions than the open balls considered in Corollary 3.15. Specifically, recall that a domain $\Omega \subset X$ is *strongly simply-connected* if it has an open covering by balls $D_1, \dots, D_m$ (not necessarily geodesic) such that for $1 \le r \le m$ the intersection $D_r \cap (D_1 \cup \cdots \cup D_{r-1})$ is connected. We recall (see [20, Proposition 2.2.3] or [43, Proposition I.2.6]):

Proposition 3.17. *If Γ is a C^∞ flat connection on a principal G bundle P over a simply-connected manifold Ω, then there is a C^∞ isomorphism $P \simeq \Omega \times G$ taking Γ to the product connection on $\Omega \times G$.*

More generally, if A is a C^∞ connection on a G bundle P over a simply-connected manifold-with-boundary $\bar{\Omega} = \Omega \cup \partial\Omega$ with L^p-small curvature (with $p > 2$), then Uhlenbeck's theorem implies that A is L^p_2-gauge equivalent to a connection which is L^p_1-close to an L^p_1 flat connection on P (see [96, Corollary 4.3] or [20, p. 163]). The following *a priori* interior estimate is a straightforward generalization of [20, Proposition 4.4.10]: the method of proof is identical to that described in [20, pp. 161–162] and uses a patching argument for gauge transformations employed by Uhlenbeck in the proof of Theorem 3.6 in [95]. The required bound for the connection in terms of its curvature is obtained by covering the given open region with balls and applying the estimate of Corollary 3.15 in place of Theorem 2.3.8 in [20].

We recall from Proposition 2.16 that any automorphism of the SO(3) bundle $\mathfrak{su}(E)|_\Omega$ over a simply-connected open subset $\Omega \subset X$ lifts to a determinant-one, unitary bundle automorphism of $E|_\Omega$. The method of [20, pp. 161–162] then yields:

Proposition 3.18. *Let X be a closed, oriented, Riemannian four-manifold with spin^c structure (ρ, W) and let $\Omega \subset X$ be a strongly simply-connected open subset. Then there is a positive constant $\varepsilon(\Omega)$ with the following significance. For $\Omega' \Subset \Omega$ a precompact open subset and an integer $\ell \geq 1$, there is a constant $C(\ell, \Omega', \Omega)$ such that the following holds. Suppose (A, Φ) is a $\mathrm{PU}(2)$ monopole in L^2_k on $(\mathfrak{su}(E), W^+ \otimes E)$ over Ω with $k \geq \max\{2, \ell\}$ such that*

$$\|F_A\|_{L^2(\Omega)} < \varepsilon.$$

Then there is an L^2_k flat connection Γ on $\mathfrak{su}(E)|_{\Omega'}$ such that

$$\|(A - \Gamma, \Phi)\|_{L^2_{\ell,\Gamma}(\Omega')} \leq C \|F_A\|^{1/2}_{L^2(\Omega)},$$

and an L^2_{k+1} trivialization $\mathfrak{su}(E)|_{\Omega'} \simeq \Omega' \times \mathfrak{su}(2)$ taking Γ to the product connection.

4. Uhlenbeck compactification for the moduli space of PU(2) monopoles

Our goal in this section is to prove the existence of an Uhlenbeck-type compactification of the moduli space of PU(2) monopoles analogous to the Uhlenbeck compactification of the moduli space of anti-self-dual connections [20]. In §4.1 we establish the Bochner-Weitzenböck formulas for the coupled Dirac operators D_A and D_A^* and the *a priori* bounds satisfied by the section Φ and the curvature F_A, if (A,Φ) is a PU(2) monopole on $(\mathfrak{su}(E), W^+\otimes E)$; these generalize the well-known *a priori* bounds for Seiberg-Witten monopoles [47], [62], [77]. In our proof of removable singularities we need to take into account the variation of spin^c structures with the Riemannian metric and the rescaling behavior of the PU(2) equations; this variation is discussed in §4.2.

In §4.3 we prove the removability of point singularities for PU(2) monopoles. T. Parker established the removability of point singularities for solutions to the second-order coupled Yang-Mills equations, namely, $d_A^*F_A = q(\Phi)$ and $D_A\Phi = 0$, for a unitary connection A on E and a section Φ of $W\otimes E$, where q is a certain quadratic form [70]. This generalizes the corresponding results of Uhlenbeck in the case $\Phi = 0$, where the above system then reduces to the second-order Yang-Mills equation [30], [94]; proofs of removable singularities for anti-self-dual connections are given in [20] (see also [96]). D. Salamon has given a proof of removable singularities for Seiberg-Witten monopoles [77, Chapter 9]. The arguments of Uhlenbeck and Parker rely on pointwise curvature and energy decay estimates; Salamon's method relies on energy decay estimates and elliptic regularity results for Seiberg-Witten monopoles. The proof we give for PU(2) monopoles is rather different and instead relies heavily on our C^∞ regularity result for Coulomb-gauge PU(2) monopoles in L^2_1 (Proposition 3.12); this is similar to the strategy used by Donaldson and Kronheimer in [20].

The technical ingredients we need for patching sequences of local gauge transformations are described in §4.4, and the Uhlenbeck closure $\bar M_{W,E}$ is defined in §4.5, by analogy with the corresponding definition for the moduli space of anti-self-dual connections in [20, §4.4]. In §4.5.2 we describe how the holonomy perturbations extend continously with respect to Uhlenbeck limits and induce holonomy perturbations on all lower-level moduli spaces of PU(2) monopoles. We then define the Uhlenbeck closure for the moduli space of perturbed PU(2) monopoles.

In §4.6 we prove Theorem 1.1, which asserts that the Uhlenbeck closure $\bar M_{W,E}$ is compact. The main analytical ingredients in the proof comprise the regularity results and estimates of §3; the *a priori* bounds of §4.1 provide a 'universal energy bound' for a PU(2) monopole (A,Φ) and this bound plays the same role here in establishing the existence of an Uhlenbeck compactification as the usual topological bound for the energy of an anti-self-dual connection in [20]. The scale invariance of the PU(2) monopole equation described in §4.2 is used here in much the same way that the conformal invariance of the anti-self-dual equation is exploited in [20].

The parameters $\tau_0, \vartheta_0, \vec{\tau}, \vec{\vartheta}$ are chosen so that the perturbation estimates (2.29) are satisfied. These bounds follow from Proposition A.13, with $k \geq 2$ for the first inequality in (2.29) and $k \geq 3$ for the second, if

$$\|\vec{\vartheta}\|_{\ell^1_\delta(C^r(X))} < \varepsilon_\vartheta \quad \text{and} \quad \|\vec{\tau}\|_{\ell^1_\delta(C^r(X))} < \varepsilon_\tau, \tag{4.1}$$

with suitable positive constants ε_τ and ε_ϑ. These constraints are needed in the proofs of the *a priori* estimates in Lemmas 4.2, 4.3, and 4.4 of §4.1 and hence in the proof of Theorem 4.20 in §4.6. *For the remainder of the article we therefore require that* $k \geq 3$.

4.1. Bochner formulas and a priori estimates for PU(2) monopoles. In this section we apply the Bochner-Weitzenböck formula for $D_A^* D_A$ together with Kato's inequality and the maximum principle to derive *a priori* estimates for solutions (A, Φ) to the PU(2) monopole equations (2.27). These estimates then lead to a vanishing result generalizing that of [98, pp. 781–782].

We first have a generalization of the usual Bochner-Weitzenböck identity for Seiberg-Witten monopoles [47], [62], [77], [89], [98].

Lemma 4.1. *Let X be an oriented, Riemannian, four-manifold with spin^c structure (ρ, W), and let E be a Hermitian bundle over X. If A denotes a* U(2) *connection on E,*

$$\nabla_A : \Omega^0(W^\pm \otimes E) \to \Omega^1(W^\pm \otimes E)$$

are the covariant derivatives defined by A, and

$$D_A : \Omega^0(W^+ \otimes E) \to \Omega^0(W^- \otimes E)$$

the Dirac operator, then

$$D_A^* D_A = \nabla_A^* \nabla_A + \tfrac{1}{4} R + \rho_+(F_A^+) + \tfrac{1}{2}\rho_+(F_{A_L}^+), \tag{1a}$$

$$D_A D_A^* = \nabla_A^* \nabla_A + \tfrac{1}{4} R + \rho_-(F_A^-) + \tfrac{1}{2}\rho_-(F_{A_L}^+), \tag{2a}$$

where R is the scalar curvature of the Riemannian metric and $A_L = A_{\det W^+}$ is the induced connection on $L = \det W^+ \simeq \det W^-$. If A denotes an SO(3) *connection on $\mathfrak{su}(E)$, then*

$$D_A^* D_A = \nabla_A^* \nabla_A + \tfrac{1}{4} R + \rho_+(F_A^+) + \tfrac{1}{2}\rho_+(F_{A_L}^+ + F_{A_e}^+), \tag{1b}$$

$$D_A D_A^* = \nabla_A^* \nabla_A + \tfrac{1}{4} R + \rho_-(F_A^-) + \tfrac{1}{2}\rho_-(F_{A_L}^- + F_{A_e}^+), \tag{2b}$$

where $A_e = A_{\det E}$ is the fixed connection on $\det E$, and the identification $\mathrm{ad} : \mathfrak{su}(E) \simeq \mathfrak{so}(\mathfrak{su}(E))$ is implicit.

Proof. We just consider the first pair of identities, as the second follow immediately from the fact that

$$F(A_E) = F(A_{\mathfrak{su}(E)}) + \tfrac{1}{2}F(A_{\det E}) \otimes \mathrm{id}_E.$$

Over any sufficiently small local coordinate neighborhood U in X we have a local spin structure and a Hermitian spin bundle S such that $W|_U = S \otimes L^{1/2}$, where $L^{1/2}$ is a Hermitian line bundle over U such that $(L^{1/2})^{\otimes 2} = L|_U$ and having an induced unitary connection $\frac{1}{2}A_L$. Applying the Bochner-Weitzenböck identity of [57, Theorem II.8.17] to the Hermitian bundle $S \otimes L^{1/2} \otimes E$ over U with unitary connection ∇_A, given by the tensor product of ∇^S, $\nabla^{L^{1/2}}_{2^{-1}A_L}$, and ∇^E_A, yields

$$D_A^2 = \nabla_A^* \nabla_A + \tfrac{1}{4}R + \rho(F_{L^{1/2}\otimes E}),$$

where $F_{L^{1/2}\otimes E}$ is the curvature of the tensor product connection $\nabla^{L^{1/2}}_{2^{-1}A_L} \otimes \mathrm{id}_E + \mathrm{id}_{L^{1/2}} \otimes \nabla^E_A$ on $L^{1/2} \otimes E$. Since

$$F_{L^{1/2}\otimes E} = F_{2^{-1}A_L} \otimes \mathrm{id}_E + \mathrm{id}_{L^{1/2}} \otimes F_A = \tfrac{1}{2}F_{A_L} \otimes \mathrm{id}_E + \mathrm{id}_{L^{1/2}} \otimes F_A,$$

we have

$$\rho(F_{L^{1/2}\otimes E}) = \tfrac{1}{2}\rho(F_{A_L}) \otimes \mathrm{id}_E + \mathrm{id}_{L^{1/2}} \otimes \rho(F_A),$$

and hence on $\Omega^0(U, W \otimes E)$,

$$D_A^2 = \nabla_A^* \nabla_A + \tfrac{1}{4}R + \tfrac{1}{2}\rho(F_{A_L}) + \rho(F_A),$$

which is plainly independent of the local splitting $W = S \otimes L^{1/2}$ and so gives an identity on $\Omega^0(X, W \otimes E)$. From the decomposition $\rho = \rho_+ \oplus \rho_-$ we see that

$$\rho^{\pm}(F_A)\Phi^{\pm} = \rho^{\pm}(F_A^{\pm})\Phi^{\pm} \quad \text{and} \quad \rho^{\pm}(F_{A_L})\Phi^{\pm} = \rho^{\pm}(F_{A_L}^{\pm})\Phi^{\pm},$$

for any $\Phi^{\pm} \in \Omega^0(X, W^{\pm} \otimes E)$, and so the result follows. q.e.d.

As in the case of the abelian monopole equations, the equations (2.27) and Lemma 4.1 combine to give *a priori* estimates for solutions (A, Φ) which play essential role in the proof of existence of the Uhlenbeck compactification.

Lemma 4.2. *Let X be a closed, oriented, Riemannian four-manifold with spin^c structure (ρ, W), and Hermitian line bundle* $\det E$ *with fixed unitary connection. Then there is a positive constant K_1, depending only on the L^2 norms of the scalar curvature R and the curvatures $F(A_{\det E})$ and $F(A_{\det W^+})$, such that the following holds. If (A, Φ) is an L^2_1 solution on $(\mathfrak{su}(E), W^+ \otimes E)$ to the* PU(2) *monopole equations* (2.27) *over X, then*

$$\|\Phi\|_{L^4(X)} \le K_1 \quad \text{and} \quad \|\nabla_A \Phi\|_{L^2(X)} \le K_1.$$

Proof. We may assume that (A, Φ) is a C^∞ pair. Then Lemma 4.1 gives

$$\begin{aligned}(D_A^* D_A \Phi, \Phi) =&(\nabla_A^* \nabla_A \Phi, \Phi) + \tfrac{1}{4}(R\Phi, \Phi) + (\rho(F_A^+)\Phi, \Phi) \\ &+ \tfrac{1}{2}(\rho(F_{A_L}^+ + F_{A_e}^+)\Phi, \Phi),\end{aligned}$$

while the first equation in (2.27) implies that

$$\rho(F_A^+) = \rho \vec{\tau}_A \rho^{-1}(\Phi \otimes \Phi^*)_{00}.$$

So, using the second equation in (2.27) and integration by parts, we obtain

$$\begin{aligned}\|\vec{\vartheta}_A \Phi\|_{L^2}^2 &= \|D_A \Phi\|_{L^2}^2 \\ &= \|\nabla_A \Phi\|_{L^2}^2 + \tfrac{1}{4}(R\Phi, \Phi) + (\rho \vec{\tau}_A \rho^{-1}(\Phi \otimes \Phi^*)_{00}\Phi, \Phi) \\ &\quad + \tfrac{1}{2}(\rho(F_{A_L}^+ + F_{A_e}^+)\Phi, \Phi).\end{aligned}$$

Consequently, Lemmas 2.18 and 2.26, Hölder's inequality, and the estimate for $\vec{\tau}_A$ in (2.29) leads to the bound

$$\begin{aligned}&\tfrac{1}{2}\|\Phi\|_{L^4}^4 + \|\nabla_A \Phi\|_{L^2}^2 \\ &\quad \le \left(\tfrac{1}{4}\|R\|_{L^2} + \tfrac{1}{2}\|F_{A_L}^+\|_{L^2} + \tfrac{1}{2}\|F_{A_e}^+\|_{L^2} + \|\vec{\vartheta}_A\|_{L^4}^2\right) \|\Phi\|_{L^4}^2.\end{aligned}$$

Thus, if $\Phi \not\equiv 0$, the above inequality and the estimate for $\vec{\vartheta}_A$ in (2.29) give the required L^4 estimate for Φ. Then the above inequality and the L^4 estimate for Φ yield the L^2 bound for $\nabla_A \Phi$. The bounds hold trivially if $\Phi \equiv 0$. q.e.d.

The preceding *a priori* L^4 estimate on the section Φ yields *a priori* L^2 bounds on the components of the curvature, F_A^+ and F_A^-, if (A, Φ) is a PU(2) monopole:

Lemma 4.3. *Continue the notation and hypotheses of Lemma 4.2. Then there is a positive constant K_2^+, depending only on the L^2 norms of the scalar curvature R, the curvatures $F(A_{\det E})$ and $F(A_{\det W^+})$, and a positive constant K_2^-, depending only on K_2^+ and $p_1(\mathfrak{su}(E))$, such that if (A,Φ) is an L_1^2 solution on $(\mathfrak{su}(E), W^+\otimes E)$ to the* PU(2) *monopole equations* (2.27), *then*

$$\|F_A^+\|_{L^2} \le K_2^+ \quad \textit{and} \quad \|F_A^-\|_{L^2} \le K_2^-.$$

Proof. Lemma 2.26 and the first PU(2) monopole equation in (2.27) imply that

$$\|F_A^+\|_{L^2} \le \tfrac{3}{4}\|\Phi\|_{L^4}^2,$$

and so the L^4 bound for Φ in Lemma 4.2 gives the first estimate. The Chern-Weil integral identity (2.33) yields

$$\int_X |F_A^-|^2 dV \le \int_X |F_A^+|^2 dV + 8\pi^2\left(c_2(E) - \tfrac{1}{4}c_1(E)^2\right),$$

and so the second estimate follows from the first. q.e.d.

Lemmas 4.2 and 4.3 imply an *a priori* bound K on the 'energy' of a PU(2) monopole (A,Φ) in terms of the scalar curvature R, the connections on $\det W^+$ and $\det E$ and $p_1(\mathfrak{su}(E))$:

$$\int_X \left(|F_A|^2 + |\Phi|^4 + |\nabla_A\Phi|^2\right) dV \le K. \tag{4.2}$$

We use Lemmas 4.2 and 4.3 to provide the 'energy bound' assumed in our proof of removable singularities (Theorem 4.10) and we use Lemma 4.3 to give a lower bound on the second Chern class of an ideal monopole appearing in the Uhlenbeck compactification of the moduli space of PU(2) monopoles (see §4.5 and the conclusion of the proof of Theorem 4.20 in §4.6).

As in the case of the U(1) monopole equations [47], [98], the Bochner-Weitzenböck identity and the maximum principle yield *a priori* C^0 estimates for Φ and F_A^+ when (A,Φ) is a PU(2) monopole.

Lemma 4.4. *Continue the notation and hypotheses of Lemma 4.2. Then there is a non-negative constant K_3, depending only on the C^0 norms of the scalar curvature R and the curvatures $F(A_{\det E})$ and $F(A_{\det W^+})$, such that the following holds. If (A,Φ) is a C^1 solution on $(\mathfrak{su}(E), W^+\otimes E)$ to the* PU(2) *monopole equations* (2.27), *then*

$$\|\Phi\|_{C^0(X)}^2 \le K_3 \quad \textit{and} \quad \|F_A^+\|_{C^0(X)} \le K_3.$$

Proof. We may assume that the pair (A,Φ) is C^∞. The analogue of equation (6.18) of [30] for *Hermitian* bundles (see [77, §8.3]) reads

$$\tfrac{1}{2}\Delta|\Phi|^2+|\nabla_A\Phi|^2=\operatorname{Re}\langle\nabla_A^*\nabla_A\Phi,\Phi\rangle,$$

where $\Delta=d^*d$ on $\Omega^0(X,\mathbb{R})$. Since Φ is continuous, $|\Phi|$ achieves its maximum at some point $x_0\in X$, so $\Delta|\Phi|^2(x_0)=-\nabla_{e_\mu}\nabla_{e_\mu}|\Phi|^2(x_0)\geq 0$ and thus at x_0:

$$\operatorname{Re}\langle\nabla_A^*\nabla_A\Phi,\Phi\rangle\geq|\nabla_A\Phi|^2.$$

Let $\{e_\mu\}$ be a local oriented, orthonormal frame for the tangent bundle TX such that $(\nabla_{e_\mu}e^\nu)_{x_0}=0$, where $\{e^\mu\}$ is the dual coframe for the cotangent bundle T^*X. Since $D_A\Phi=-\vec{\vartheta}_A\Phi$ by the second equation in (2.27), at x_0 we have

$$\begin{aligned}\langle D_A^*D_A\Phi,\Phi\rangle&=-\langle D_A^*(\vec{\vartheta}_A\Phi),\Phi\rangle\\&=-\langle\rho(e^\mu)(\nabla_{e_\mu}\vec{\vartheta}_A)\Phi,\Phi\rangle-\langle\rho(e^\mu)\vec{\vartheta}_A\nabla^A_{e_\mu}\Phi,\Phi\rangle,\end{aligned}$$

and so (using the inequality $ab\leq\frac{1}{4}a^2+b^2$) we get the following bound at x_0:

$$\begin{aligned}|\langle D_A^*D_A\Phi,\Phi\rangle|&\leq 4|\nabla_A\vec{\vartheta}_A||\Phi|^2+4|\vec{\vartheta}_A||\nabla_A\Phi||\Phi|\\&\leq 4|\nabla_A\vec{\vartheta}_A||\Phi|^2+4\left(|\vec{\vartheta}_A|^2|\Phi|^2+\tfrac{1}{4}|\nabla_A\Phi|^2\right)\\&=4\left(|\nabla_A\vec{\vartheta}_A|+|\vec{\vartheta}_A|^2\right)|\Phi|^2+|\nabla_A\Phi|^2.\end{aligned}$$

Recall that the first PU(2) monopole equation in (2.27) gives

$$\rho(F_A^+)=\rho\vec{\tau}_A\rho^{-1}(\Phi\otimes\Phi^*)_{00}.$$

The endomorphism $\rho\vec{\tau}_A\rho^{-1}(\Phi\otimes\Phi^*)_{00}$ of $W^+\otimes E$ lies in $\mathfrak{su}(W^+)\otimes\mathfrak{su}(E)$ and in particular is Hermitian, so

$$\operatorname{Re}\langle\rho\vec{\tau}_A\rho^{-1}(\Phi\otimes\Phi^*)_{00}\Phi,\Phi\rangle=\langle\rho\vec{\tau}_A\rho^{-1}(\Phi\otimes\Phi^*)_{00}\Phi,\Phi\rangle,$$

and similarly for the endomorphism $\rho(F_A^+)$. We now combine Lemmas 2.26 and 4.1 and the first equation in (2.27) to get an estimate for Φ at the point x_0:

$$\begin{aligned}\tfrac{1}{2}|\Phi|^4&\leq\langle\rho\vec{\tau}_A\rho^{-1}(\Phi\otimes\Phi^*)_{00}\Phi,\Phi\rangle\\&=\langle\rho(F_A^+)\Phi,\Phi\rangle\\&=\operatorname{Re}\langle D_A^*D_A\Phi,\Phi\rangle-\operatorname{Re}\langle\nabla_A^*\nabla_A\Phi,\Phi\rangle-\tfrac{1}{4}\langle R\Phi,\Phi\rangle\\&\quad-\tfrac{1}{2}\langle\rho(F_{A_L}^++F_{A_e}^+)\Phi,\Phi\rangle.\end{aligned}$$

We now combine the last inequality with our estimates for $\langle D_A^* D_A\Phi, \Phi\rangle$ and $\langle \nabla_A^*\nabla_A\Phi, \Phi\rangle$ to obtain the following bound for Φ at x_0:

$$\begin{aligned}\tfrac{1}{2}|\Phi|^4 &\le 4\left(|\nabla_A\vec{\vartheta}_A| + |\vec{\vartheta}_A|^2\right)|\Phi|^2 + |\nabla_A\Phi|^2 \\ &\quad - |\nabla_A\Phi|^2 - \tfrac{1}{4}R|\Phi|^2 + \tfrac{1}{2}\left(|F_{A_L}^+| + |F_{A_e}^+|\right)|\Phi|^2 \\ &\le 4\left(|\nabla_A\vec{\vartheta}_A| + |\vec{\vartheta}_A|^2\right)|\Phi|^2 - \tfrac{1}{4}R|\Phi|^2 + \tfrac{1}{2}\left(|F_{A_L}^+| + |F_{A_e}^+|\right)|\Phi|^2.\end{aligned}$$

Either $\Phi(x_0) = 0$, and so Φ is identically zero, or at the point x_0 the preceding inequality implies that

$$|\Phi|^2 \le 8\left(|\nabla_A\vec{\vartheta}_A| + |\vec{\vartheta}_A|^2\right) - \tfrac{1}{2}\inf_X R + |F_{A_L}^+| + |F_{A_e}^+|,$$

which gives the first desired estimate. Then the second desired estimate follows from Lemma 2.26, the first equation in (2.27), and the estimate for $\vec{\vartheta}_A$ in (2.29). q.e.d.

We use Lemma 4.4 in §4.6 to show that the curvature of a PU(2) monopole connection A concentrates at points with integer multiplicities given by the second Chern classes of limiting (ideal) anti-self-dual connections over S^4 (see Lemma 4.21). These C^0 estimates yield the following analogue of Witten's vanishing theorem [98, §3] for U(1) monopoles over four-manifolds with non-negative scalar curvature.

Corollary 4.5. *Continue the notation and hypotheses of Lemma 4.4 and suppose $K_3 \le 0$. If (A,Φ) is a C^1 solution on $(\mathfrak{su}(E), W^+\otimes E)$ to the* PU(2) *monopole equations* (2.27), *then $\Phi \equiv 0$ and $F_A^+ \equiv 0$.*

Remark 4.6. The proof of Lemma 4.4 shows that

$$K_3 = \max\left\{0, -\tfrac{1}{2}\inf_X R + 8\|\vec{\vartheta}_A\|_{L^\infty_{1,A}(X)} + \|F_{A_L}^+\|_{C^0(X)} + \|F_{A_e}^+\|_{C^0(X)}\right\}.$$

In particular, we see that if $X = S^4$ has its round metric of scalar curvature $R = 1$, standard spin^c structure with $c_1(W^+) = 0$ and $F_{A_L} = 0$, the Hermitian bundle E has $c_1(E) = 0$ and $F_{A_e} = 0$, and we have $\vec{\vartheta} = 0$, then $\Phi \equiv 0$ and A is an anti-self-dual SO(3) connection.

4.2. Scale invariance of the PU(2) monopole equations. In this section we describe the behavior of the PU(2) monopole under rescaling of the Riemannian metric. As is well-known, the anti-self-dual equation is conformally invariant. Although the PU(2) monopole equations are not conformally invariant they are, like the Seiberg-Witten equations, invariant under constant rescalings of the metric in a sense we describe below. The proof of the existence of an Uhlenbeck compactification (Theorem 1.1) in §4 relies on local regularity and removable singularity results for solutions to (2.27) over the unit ball B in $\mathbb{R}^4$. The requirements that the L^2 norm of the curvature F_A and the L^2_1 norm of section Φ be sufficiently small are met via a rescaling argument.

Suppose that $\lambda > 0$ is a constant and that the Riemannian metric g on T^*X is replaced by $\lambda^2 g$. Since the Clifford map ρ is compatible with g, the Clifford map

$$\rho_{\lambda^2 g} \equiv \lambda\rho_g : T^*X \to \mathrm{End}(W)$$

is compatible with $\lambda^2 g$, as for any $\alpha \in \Omega^1(X, \mathbb{R})$ it satisfies

$$\rho_{\lambda^2 g}(\alpha)^\dagger \rho_{\lambda^2 g}(\alpha) = \lambda^2 \rho_g(\alpha)^\dagger \rho_g(\alpha) = \lambda^2 g(\alpha, \alpha)\mathrm{id}_W.$$

It extends in the usual way to a linear map

$$\rho_{\lambda^2 g} : \Lambda^\bullet(T^*X) \otimes \mathbb{C} \to \mathrm{End}(W).$$

The Levi-Civita connection on T^*X for the metric g coincides with the Levi-Civita connection on T^*X for the rescaled metric $\lambda^2 g$, so the $\mathrm{SO}(4)$ connection on T^*X induced by the unitary connection on W and the Clifford map $\rho_{\lambda^2 g} : T^*X \to \mathrm{End}(W)$ is still torsion free.

Lemma 4.7. *If (A, Φ) is a solution to the $\mathrm{PU}(2)$ monopole equations (2.27) for the metric g on T^*X, then $(A, \lambda\Phi)$ is a solution to the $\mathrm{PU}(2)$ monopole equations for the rescaled metric $\lambda^2 g$ on T^*X, where λ is a positive constant.*

Proof. The projection $P^+(\lambda^2 g) = \frac{1}{2}(1 + *_{\lambda^2 g})$ from $\Lambda^2(T^*X)$ to $\Lambda^+(T^*X)$ is given by $P^+(\lambda^2 g) = P^+(g)$, while the induced map $\rho_{\lambda^2 g} : \Lambda^2(T^*X) \to \mathrm{End}(W)$ is given by $\rho_{\lambda^2 g} = \lambda^2 \rho_g$. Therefore,

$$\begin{aligned} \rho_{\lambda^2 g}(\vartheta) &= \lambda\rho_g(\vartheta), \quad \vartheta \in \Omega^1(X, \mathbb{C}), \\ D_A^{\lambda^2 g} &= \lambda D_A^g, \\ \rho_{\lambda^2 g}(P^+(\lambda^2 g)F_A) &= \lambda^2 \rho_g(P^+(g)F_A). \end{aligned}$$

Consequently, we see from (2.27) that if (A, Φ) is a solution for the metric g, then $(A, \lambda\Phi)$ is a solution for the metric $\lambda^2 g$. q.e.d.

Remark 4.8. By adapting the proof of Theorem II.5.24 in [57] we see that if g is replaced by the conformally equivalent metric $h^{-2}g$, then $D_A^{h^{-2}g} = h^{5/2}D_A^g h^{-3/2}$. Thus, while the proof of Lemma 4.7 adapts without change to show that the first equation in (2.27) is invariant under the transformation $(A,\Phi) \mapsto (A, h\Phi)$ when $g \mapsto h^{-2}g$, the second equation (when $\vartheta = 0$) is invariant under the transformation $(A,\Phi) \mapsto (A, h^{3/2}\Phi)$. It is this incompatibility which prevents the PU(2) monopole equations from being conformally invariant, although they are scale invariant in the sense described above.

4.3. Removable singularities. Given the sharp local elliptic regularity result of Proposition 3.12 for PU(2) monopoles in Coulomb gauge in L_1^2, we can now establish a removable singularities theorem for PU(2) monopoles analogous to Theorem 4.1 in [94] in the case of the Yang-Mills equations, and Theorem 8.1 in [70] in the case of the coupled Yang-Mills equations. Our method is modelled on the proof of Theorem 4.4.12 in [20] — the removable singularities result for the anti-self-dual equation — which uses a local elliptic regularity result for L_1^2 solutions to the Coulomb gauge and inhomogeneous anti-self-dual equation (namely, Proposition 4.4.13 in [20]) and which in turn has its antecedent in the proof of Theorem 4.5 in [96]. This, of course, is not the only possible approach: Uhlenbeck's original argument [94] employed a differential inequality to obtain a pointwise decay estimate for solutions near the singular point, and this was the method generalized by Parker to the case of certain coupled Yang-Mills equations; see [77, §9.2] for a proof of removable singularities for Seiberg-Witten monopoles which also uses differential inequalities.

It will be convenient to define the following annuli in X, given a point $x_0 \in X$ and a positive constant r:

$$\begin{aligned}
\Omega(x_0;r) &:= \{x \in X : \tfrac{1}{8}r < \operatorname{dist}_g(x,x_0) < r\},\\
\Omega'(x_0;r) &:= \{x \in X : \tfrac{1}{4}r < \operatorname{dist}_g(x,x_0) < \tfrac{1}{2}r\} \Subset \Omega(x_0,r).
\end{aligned}$$

If $X = \mathbb{R}^4$ and $x_0 = 0$ and $r = 1$, we denote $\Omega = \Omega(1)$ and $\Omega' = \Omega'(1)$. We will need the following special case of Proposition 3.18.

Lemma 4.9. *Let $\mathbb{R}^4$ have a C^∞ Riemannian metric g, let $\Omega \subset \mathbb{R}^4$ be an open subset with spin^c structure (ρ, W), let E be a Hermitian two-plane product bundle over Ω, and let $\Omega' \Subset \Omega$ be an open subset. Then there are positive constants C,ε such that if (A,Φ) is a* PU(2) *monopole* *in* C^∞ *on* $(\mathfrak{su}(E), W^+ \otimes E)$ *over* Ω *with*

$$\|F_A\|_{L^2(\Omega)} < \varepsilon,$$

then there is a C^∞ *flat connection* Γ *on* $\mathfrak{su}(E)|_{\Omega'}$ *such that*

$$\|A - \Gamma\|_{L^4(\Omega')} + \|\nabla_\Gamma(A - \Gamma)\|_{L^2(\Omega')} \le c\|F_A\|_{L^2(\Omega)},$$

and a C^∞ *trivialization* $\mathfrak{su}(E)|_{\Omega'} \simeq \Omega' \times \mathfrak{su}(2)$ *taking* Γ *to the product connection.*

The proof of Theorem 4.10 relies on a cutting off procedure to 'smooth out' the PU(2) monopole near the singular point, using a family of cutoff functions which we now define. Let $\chi : \mathbb{R} \to [0,1]$ be a bump function such that $\chi(t) = 1$ if $t \ge \frac{1}{2}$ and $\chi(t) = 0$ if $t \le \frac{1}{4}$. For any $r \in (0, \varrho)$, where ϱ is the injectivity radius of (X, g), define a C^∞ cutoff function on X by setting $\chi_r(\cdot) = \chi(\operatorname{dist}_g(\cdot, x_0)/r)$. Thus, we have $\chi_r = 0$ on the ball $B(x_0, \frac{1}{4}r)$, while $\chi_r = 1$ on $X - B(x_0, \frac{1}{2}r)$ and so $d\chi_r$ is supported in $\Omega'(x_0, r)$.

Theorem 4.10. *Let* $B \subset \mathbb{R}^4$ *be a geodesic ball with* C^∞ *metric* g *and center at the origin, spin*^{c} *structure* (ρ, W) *over* B*, and Hermitian two-plane bundle* E *over* $B\backslash\{0\}$*. Suppose* (A, Φ) *is a* C^∞ *solution to the* PU(2) *monopole equations* (2.27) *on* $(\mathfrak{su}(E), W^+ \otimes E)$ *with* $(\vec{\tau}, \vec{\vartheta}) = 0$ *over the punctured ball* $B \setminus \{0\}$ *and* finite energy,

$$\int_{B\backslash\{0\}} \left(|F_A|^2 + |\Phi|^4 + |\nabla_A\Phi|^2\right) dV < \infty.$$

Then there are a Hermitian two-plane bundle $\widetilde{E}$ *over* B *with* $\det \widetilde{E} = \det E$*, a* C^∞ PU(2) *monopole* $(\tilde{A}, \tilde{\Phi})$ *on* $(\mathfrak{su}(\widetilde{E}), W^+ \otimes \widetilde{E})$ *over the ball* B*, and a* C^∞*, determinant-one unitary bundle isomorphism* $u : E|_{B\backslash\{0\}} \to \widetilde{E}|_{B\backslash\{0\}}$ *such that*

$$u(A, \Phi) = (\tilde{A}, \tilde{\Phi}) \quad \text{over} \quad B \setminus \{0\}.$$

Remark 4.11. We restrict our attention to the case of $(\vec{\tau}, \vec{\vartheta}) = 0$ in the PU(2) monopole equations (2.27) since the holonomy perturbations are undefined for the L^2_1 connections which arise in the proof of Theorem 4.10. However, there is no loss of generality in making this restriction as the holonomy perturbations vanish near points where curvature has bubbled off.

Proof. We may suppose without loss of generality that the ball B has radius less than or equal to one. Since $\|F_A\|_{L^2(B)} < \infty$, then $\|F_A\|_{L^2(B_{r_0})}$ tends to zero as r_0 tends to zero. Hence, for small enough $r_0 \in (0,1]$, we may suppose that $\|F_A\|_{L^2(B_{r_0})} < \varepsilon$, where ε is the constant of Lemma 4.9. Thus, for any $r \in (0, r_0)$, Lemma 4.9 provides a C^∞ flat connection Γ'_r on $\mathfrak{su}(E)|_{\Omega'(r)}$ such that

$$\|A - \Gamma'_r\|_{L^4(\Omega'(r))} \leq c\|F_A\|_{L^2(\Omega(r))}, \tag{4.3}$$

where the constant c is *independent of* $r \in (0, r_0)$. To see that the constant c is indeed scale invariant, note that by Lemma 4.7 the pair $(A, r^{-1}\Phi)$ is a PU(2) monopole over B with respect to the rescaled metric $g_r := r^{-2}g$, so $\Omega'_g(r) = \Omega'_{g_r}(1)$ and $\Omega_g(r) = \Omega_{g_r}(1)$. We then apply Lemma 4.9 to the annuli $\Omega'_{g_r}(1) \Subset \Omega_{g_r}(1)$ and observe that the L^4 norm on one-forms and the L^2 norm on two-forms are scale invariant.

Lemma 4.9 also provides a C^∞ trivialization $\mathfrak{su}(E)|_{\Omega'(r)} \simeq \Omega'(r) \times \mathfrak{su}(2)$ taking Γ'_r to the product connection on $\Omega'(r) \times \mathfrak{su}(2)$. We can then define a smooth SO(3) bundle V_r over B by setting

$$V_r := \begin{cases} B(\tfrac{1}{2}r) \times \mathfrak{su}(2) & \text{over } B(\tfrac{1}{2}r), \\ \mathfrak{su}(E) & \text{over } B - B(\tfrac{1}{4}r), \end{cases}$$

recalling that $\Omega'(r) = B(\frac{1}{2}r) - \bar{B}(\frac{1}{4}r)$. Let Γ'_r denote the C^∞ flat connection on V_r over the ball $B(\frac{1}{2}r)$, extending Γ'_r on $\mathfrak{su}(E)|_{\Omega'(r)}$ via the product connection on $B(\frac{1}{2}r) \times \mathfrak{su}(2)$, and let E_r be the smooth U(2) bundle over B with $\det E_r = \det E$ and $\mathfrak{su}(E_r) = V_r$ over B.

Now let (A_r, Φ_r) be the C^∞ pair on B defined by

$$(A_r, \Phi_r) := \begin{cases} (\Gamma'_r + \chi_r(A - \Gamma'_r), \chi_r\Phi) & \text{over } B(\tfrac{1}{2}r), \\ (A, \Phi) & \text{over } B \setminus B(\tfrac{1}{4}r), \end{cases}$$

where we note that $\chi_r = 0$ on $B(\frac{1}{4}r)$ and $\chi_r = 1$ on $B - B(\frac{1}{2}r)$. To estimate the L^2 norm of F_{A_r}, note that over $B - B(\frac{1}{2}r)$ we have $A_r = A$ and $F_{A_r} = F_A$, while over $B(\frac{1}{2}r)$,

$$F_{A_r} = \chi_r F_A + d\chi_r \wedge (A - \Gamma'_r) + (\chi_r^2 - \chi_r)(A - \Gamma'_r) \wedge (A - \Gamma'_r).$$

Hence, by (4.3), we have

$$\begin{aligned} \|F_{A_r}\|_{L^2(\Omega'(r))} \leq & \|F_A\|_{L^2(\Omega'(r))} + \|d\chi_r\|_{L^4(\Omega'(r))}\|A - \Gamma'_r\|_{L^4(\Omega'(r))} \\ & + \|A - \Gamma'_r\|^2_{L^4(\Omega'(r))}. \end{aligned}$$

Therefore, since $\|d\chi_r\|_{L^4(\Omega'(r))} \leq c_0$, for a constant c_0 independent of $r \in (0,\infty)$, there is a positive constant c, independent of $r \in (0, r_0)$, such that

$$\|F_{A_r}\|_{L^2(B_{r_0})} \leq c\left(\|F_A\|_{L^2(B_{r_0})} + \|F_A\|^2_{L^2(B_{r_0})}\right) \leq c\|F_A\|_{L^2(B_{r_0})}. \tag{4.4}$$

Thus, $\|F_{A_r}\|_{L^2(B_{r_0})}$ tends to zero as $r_0 \to 0$, uniformly with respect to $r \in (0, r_0)$. Fix r_0 small enough so that $\|F_{A_r}\|_{L^2(B_{r_0})} < \varepsilon_1$ for all $r \in (0, r_0)$, where ε_1 is the constant of Theorem 3.13. Hence, there is a family of C^∞ flat connections Γ_r, $r \in (0, r_0)$, on the SO(3) bundles $\mathfrak{su}(E_r)$ over B_{r_0} such that

$$d^*_{\Gamma_r}(A_r - \Gamma_r) = 0 \quad \text{and} \quad \|A_r - \Gamma_r\|_{L^2_{1,\Gamma_r}(B_{r_0})} \leq c\|F_{A_r}\|_{L^2(B_{r_0})}, \tag{4.5}$$

for a positive constant c independent of $r \in (0, r_0)$, and a family of C^∞ bundle isomorphisms $\tau_r : \mathfrak{su}(E_r)|_{B_{r_0}} \simeq B_{r_0} \times \mathfrak{su}(2)$ inducing C^∞ bundle isomorphisms $\tau_r : E_r \simeq B_{r_0} \times \mathbb{C}^2$, via a choice of fixed C^∞ trivialization $\det E_r|_{B_{r_0}} = \det E|_{B_{r_0}} \simeq B_{r_0} \times \mathbb{C}$.

Since $\|\Phi\|_{L^2_{1,A}(B)} < \infty$, then $\|\Phi\|_{L^2_{1,A}(B_{r_0})}$ and $\|\Phi\|_{L^4(B_{r_0})}$ tend to zero as r_0 as tends to zero. As $\Phi_r = \chi_r\Phi$ over $B(\frac{1}{2}r)$ and using (4.3), we have

$$\begin{aligned}
\|\Phi_r\|_{L^2_{1,\Gamma_r}(B_{r_0})} &\leq \|\Phi_r\|_{L^2(B_{r_0})} + \|\nabla_{\Gamma_r}\Phi_r\|_{L^2(B_{r_0})} \\
&\leq \|\Phi_r\|_{L^2(B_{r_0})} + \|(A_r - \Gamma_r)\cdot\Phi_r\|_{L^2(B_{r_0})} \\
&\quad + \|\nabla_{A_r}\Phi_r\|_{L^2(B_{r_0})} \\
&\leq c\left(1 + \|(A_r - \Gamma_r)\|_{L^4(B_{r_0})}\right)\|\Phi_r\|_{L^4(B_{r_0})} \\
&\quad + \|\nabla_{A_r}\Phi_r\|_{L^2(B_{r_0})} \\
&\leq c\left(1 + \|F_{A_r}\|_{L^2(B_{r_0})}\right)\|\Phi\|_{L^4(B_{r_0})} + \|\nabla_{A_r}\Phi_r\|_{L^2(B_{r_0})} \\
&\leq c\|\Phi\|_{L^4(B_{r_0})} + \|\nabla_{A_r}(\chi_r\Phi)\|_{L^2(\Omega(\frac{1}{4}r,\frac{1}{2}r))} \\
&\quad + \|\nabla_A\Phi\|_{L^2(\Omega(\frac{1}{2}r,r_0))} \\
&\leq c\|\Phi\|_{L^2_{1,A}(B_{r_0})} + \|d\chi_r\cdot\Phi\|_{L^2(\Omega'(r))} \\
&\quad + \|\chi_r\nabla_{A_r}\Phi\|_{L^2(\Omega'(r))}
\end{aligned}$$

$$\begin{aligned}&\leq c\|\Phi\|_{L^2_{1,A}(B_{r_0})} + \|d\chi_r\|_{L^4(B_{r_0})}\|\Phi\|_{L^4(B_{r_0})}\\&\quad + \|\nabla_A\Phi\|_{L^2(\Omega'(r))}\\&\quad + \|(\chi_r - 1)(A - \Gamma'_r)\cdot\Phi\|_{L^2(\Omega'(r))}\\&\leq c\|\Phi\|_{L^2_{1,A}(B_{r_0})} + \|A - \Gamma'_r\|_{L^4(\Omega'(r))}\|\Phi\|_{L^4(\Omega'(r))},\end{aligned}$$

using $A_r = \Gamma'_r + \chi_r(A - \Gamma'_r) = A + (\chi_r - 1)(A - \Gamma'_r)$ over $\Omega'(r)$, and so

$$\|\Phi_r\|_{L^2_{1,\Gamma_r}(B_{r_0})} \leq c\|\Phi\|_{L^2_{1,A}(B_{r_0})}, \tag{4.6}$$

for some constant c independent of $r \in (0, r_0)$. Therefore, $\|\Phi_r\|_{L^2_{1,\Gamma_r}(B_{r_0})}$ tends to zero as $r_0 \to 0$, uniformly with respect to $r \in (0, r_0)$.

Hence, the estimates (4.4), (4.5), and (4.6) combine to give a uniform bound,

$$\|(A_r - \Gamma_r, \Phi_r)\|_{L^2_{1,\Gamma_r}(B_{r_0})} \leq c\left(\|F_A\|_{L^2(B_{r_0})} + \|\Phi\|_{L^2_{1,A}(B_{r_0})}\right), \tag{4.7}$$

for some constant c independent of $r \in (0, r_0)$. Note that

$$\begin{aligned}&L^2_{1,\Gamma_r}(B_{r_0}, \Lambda^1 \otimes \mathfrak{su}(E_r)) \oplus L^2_{1,\Gamma_r}(B_{r_0}, W^+ \otimes E_r)\\&\quad = L^2_{1,\Gamma}(B_{r_0}, \Lambda^1 \otimes \mathfrak{su}(2)) \oplus L^2_{1,\Gamma}(B_{r_0}, W^+ \otimes \mathbb{C}^2),\end{aligned}$$

via the C^∞ isomorphisms $\tau_r : E_r|_{B_{r_0}} \to B_{r_0} \times \mathbb{C}^2$, with

$$\begin{aligned}\|(A_r - \Gamma_r, \Phi_r)\|_{L^2_{1,\Gamma_r}(B_{r_0})} &= \|(\tau_r(A_r) - \Gamma, \tau_r(\Phi_r))\|_{L^2_{1,\Gamma}(B_{r_0})},\\ d^*_\Gamma(\tau_r(A_r) - \Gamma) &= 0.\end{aligned}$$

By the weak compactness of the unit ball in the Hilbert space $L^2_1(B_{r_0})$, there is a sequence $r_\alpha \to 0$ such that the pairs $(A_{r_\alpha} - \Gamma_{r_\alpha}, \Phi_{r_\alpha})$ converge weakly in $L^2_1(B_{r_0})$ to a limit $(\tilde{A}, \tilde{\Phi})$ in $L^2_1(B_{r_0})$. For brevity, we denote $(A^\tau_r, \Phi^\tau_r) := (\tau_r(A_r), \tau_r(\Phi_r))$.

Claim 4.12. Continue the above notation. Then the following hold:

(1) After passing to a subsequence, the pairs $(A^\tau_{r_\alpha}, \Phi^\tau_{r_\alpha})$ converge in C^∞ over compact subsets of $B_{r_0} \setminus \{0\}$ to $(\tilde{A}^\tau, \tilde{\Phi}^\tau)$ and so $(\tilde{A}^\tau, \tilde{\Phi}^\tau)$ is in C^∞ over $B_{r_0} \setminus \{0\}$;

(2) The pair $(\tilde{A}^\tau, \tilde{\Phi}^\tau)$ is an L^2_1 solution over B_{r_0} to the elliptic system

$$\mathfrak{S}(\tilde{A}^\tau, \tilde{\Phi}^\tau) = 0 \quad \text{and} \quad d^*_\Gamma(\tilde{A}^\tau - \Gamma) = 0.$$

Proof. For any open subset $U \Subset B_{r_0} \setminus \{0\}$, the ball $B(\frac{1}{2}r)$ containing the support of the cutoff function χ_r does not meet U when r is sufficiently small and so (A_r^τ, Φ_r^τ) is a PU(2) monopole in C^∞ over U in Coulomb gauge with respect to $(\Gamma, 0)$ with uniformly L^2-small curvature. Hence, choosing $U' \Subset B_{r_0} \setminus \{0\}$ so that $U' \Supset U$, Proposition 3.12 implies that for any integer $k \geq 1$ and r small enough that $B(\frac{1}{2}r) \cap U' = \emptyset$, we have uniform bounds

$$\begin{aligned}\|(A_r^\tau - \Gamma, \Phi_r^\tau)\|_{L^2_{k,\Gamma}(U)} &\leq C\left(\|(A_r^\tau - \Gamma, \Phi_r^\tau)\|_{L^2(U')} + \|F(A_r^\tau)\|_{L^2(U')}\right) \\ &\leq C\left(\|\Phi\|_{L^4(B)} + \|F_A\|_{L^2(B)}\right) < \infty,\end{aligned}$$

for some constant $C(g, k, U)$ independent of $r \in (0, r_0)$. Therefore, by passing to a subsequence, the pairs $(A_{r_\alpha}^\tau, \Phi_{r_\alpha}^\tau)$ converge in C^∞ over compact subsets of $B_{r_0} \setminus \{0\}$ to a C^∞ pair $(\tilde{A}^\tau, \tilde{\Phi}^\tau)$ over $B_{r_0} \setminus \{0\}$ as $r_\alpha \to 0$. But for any $U \Subset B_{r_0} \setminus \{0\}$ and small enough $r \in (0, r_0)$, we have $\mathfrak{S}(A_r^\tau, \Phi_r^\tau) = \mathfrak{S}(A_r, \Phi_r) = 0$ over U, and so

$$\mathfrak{S}(\tilde{A}^\tau, \tilde{\Phi}^\tau) = \lim_{\alpha \to \infty} \mathfrak{S}(A_{r_\alpha}^\tau, \Phi_{r_\alpha}^\tau) = 0 \quad \text{over } U.$$

Hence, $\mathfrak{S}(\tilde{A}^\tau, \tilde{\Phi}^\tau) = 0$ over $B_{r_0} \setminus \{0\}$ and so $(\tilde{A}^\tau, \tilde{\Phi}^\tau)$ is a PU(2) monopole in C^∞ over $B_{r_0} \setminus \{0\}$. This proves Assertion (1) of the claim.

Since $(\tilde{A}^\tau, \tilde{\Phi}^\tau)$ is a C^∞ monopole over $B_{r_0} \setminus \{0\}$, then $\mathfrak{S}(\tilde{A}^\tau, \tilde{\Phi}^\tau) = 0$ a.e. over B_{r_0} and so $(\tilde{A}^\tau, \tilde{\Phi}^\tau)$ is an L_1^2 monopole over B_{r_0}. Let $W_0^{2,1}(B_{r_0}) \subset L_1^2(B_{r_0})$ be the closure in $L_1^2(B_{r_0})$ of the pairs

$$C_0^\infty(B_{r_0}, \Lambda^1 \otimes \mathfrak{su}(E)) \oplus C_0^\infty(B_{r_0}, W^+ \otimes E)$$

with compact support in the open ball B_{r_0}. Then, for any $(b, \varphi) \in W_0^{2,1}(B_{r_0})$ we have

$$\begin{aligned}(b, d_\Gamma^*(\tilde{A}^\tau - \Gamma))_{L^2(B_{r_0})} &= (d_\Gamma b, \tilde{A}^\tau - \Gamma)_{L^2(B_{r_0})} \\ &= \lim_{\alpha \to \infty} (d_\Gamma b, (A_{r_\alpha}^\tau - \Gamma))_{L^2(B_{r_0})} \\ &= \lim_{\alpha \to \infty} (b, d_\Gamma^*(A_{r_\alpha}^\tau - \Gamma))_{L^2(B_{r_0})} = 0,\end{aligned}$$

and so $d_\Gamma^*(\tilde{A}^\tau - \Gamma) = 0$, as required. This completes the proof of the claim. q.e.d.

By Claim 4.12, the pair $(\tilde{A}^\tau, \tilde{\Phi}^\tau)$ is an L_1^2 monopole over B_{r_0} in Coulomb gauge relative to $(\Gamma, 0)$. From the estimate (4.7) and the

Sobolev embedding $L_1^2(B_{r_0}) \subset L^4(B_{r_0})$, we can ensure that for sufficiently small r_0,

$$\|(A_r^\tau - \Gamma, \Phi_r^\tau)\|_{L^4(B_{r_0})} < \varepsilon_2,$$

for all $r \in (0, r_0)$ and so $\|(\tilde{A}^\tau, \tilde{\Phi}^\tau)\|_{L^4(B_{r_0})} < \varepsilon_2$, where ε_2 is the constant of Proposition 3.12 and therefore, $(\tilde{A}^\tau, \tilde{\Phi}^\tau)$ is a C^∞ monopole over B_{r_0}. (As usual this means, more precisely, that there is a C^∞ monopole over B_{r_0} which coincides with $(\tilde{A}^\tau, \tilde{\Phi}^\tau)$ over B_{r_0} except over a subset of measure zero; since $(\tilde{A}^\tau, \tilde{\Phi}^\tau)$ is already C^∞ on the punctured ball $B_{r_0} \setminus \{0\}$, this C^∞ monopole is equal to $(\tilde{A}^\tau, \tilde{\Phi}^\tau)$ except possibly at the origin.)

Finally, the C^∞ bundle isomorphisms $\tau_r : \mathfrak{su}(E_r)|_{B_{r_0}} \simeq B_{r_0} \times \mathfrak{su}(2)$ may be viewed as SU(2) automorphisms acting on the SO(3) bundle $B_{r_0} \times \mathfrak{su}(2)$ by initially choosing a fixed C^∞ trivialization

$$E|_{B\setminus\{0\}} \simeq B \setminus \{0\} \times \mathbb{C}^2, \quad \text{with} \quad \mathfrak{su}(E)|_{B\setminus\{0\}} \simeq B \setminus \{0\} \times \mathfrak{su}(2).$$

Then Lemma 2.7 implies, after passing to a subsequence, that the sequence of SU(2) automorphisms τ_{r_α} converges in C^∞ over compact subsets of $B_{r_0}\setminus\{0\}$ to a C^∞ limit σ over $B_{r_0}\setminus\{0\}$. Then $(\tilde{A}^\tau, \tilde{\Phi}^\tau) = (A^\sigma, \Phi^\sigma)$ over the punctured ball $B_{r_0} \setminus \{0\}$, while $(\tilde{A}^\tau, \tilde{\Phi}^\tau)$ is smooth over B_{r_0}. Thus, the finite-energy C^∞ monopole (A, Φ) over the punctured ball $B_{r_0} \setminus \{0\}$ is equivalent via a C^∞, determinant-one, unitary bundle isomorphism to $(\tilde{A}^\tau, \tilde{\Phi}^\tau)$ over $B_{r_0} \setminus \{0\}$, where $(\tilde{A}^\tau, \tilde{\Phi}^\tau)$ is a C^∞ monopole over B_{r_0}. This completes the proof of the theorem. q.e.d.

Remark 4.13. The proof of Theorem 4.10 does not imply, of course, that the section $\tilde{\Phi}^\tau$ is zero at the center of the ball B. Even though the C^∞ sections $\Phi^\tau_{r_\alpha}$ are zero at the center, the subsequence only converges in $L_1^2(B_{r_0})$ over B_{r_0} to a limit $\tilde{\Phi}^\tau$. Similarly, while Lemma 4.4 provides a uniform C^0 bound for the sections $\Phi^\tau_{r_\alpha}$ over B_{r_0}, we would need a uniform, $C^{0,\nu}$ Hölder bound, for some $\nu \in (0,1)$, in order to extract a convergent subsequence.

4.4. Patching arguments. The standard proof of the compactness theorem for the moduli space of anti-self-dual connections employs a patching argument for gauge transformations to obtain C^∞ convergence (modulo gauge transformations) on compact subsets of $X \setminus \{x_1, \dots, x_m\}$ for a sequence of anti-self-dual connections A_α on a Hermitian bundle E over X. The gauge transformations that require patching are obtained by repeated application of Corollary 2.3.9 in [20] to geodesic balls where the L^2 norm of the curvature F_{A_α} is less than ε: since the L^2 norm of the curvature is scale invariant, these possibly small balls may be rescaled to standard size with metrics which are approximately Euclidean as in [20, Corollary 2.3.9].

Throughout this subsection, (A_α, Φ_α) will denote a sequence of C^∞ pairs (not necessarily PU(2) monopoles) on $(\mathfrak{su}(E), W^+ \otimes E)$ over Ω and u_α will denote a sequence of C^∞ determinant-one, unitary automorphisms of a Hermitian bundle E (that is, gauge transformations in $\mathcal{G}_E$ and not ${}^\circ\mathcal{G}_E$), where Ω is an oriented, Riemannian four-manifold with spin^c structure (ρ, W). Convergence will mean convergence in C^∞ on compact subsets which, as usual, can be replaced by $L^2_{k+1,\mathrm{loc}}$ convergence of L^2_k pairs (A_α, Φ_α) provided $k \geq 2$.

The following four patching results follow almost immediately from the proofs of Lemmas 4.4.5–4.4.7 and Corollary 4.4.8 in [20] (where the sequence of connections A_α is not assumed to be anti-self-dual). Their proofs are omitted and instead we refer the reader to [20] or [95] for a detailed account; patching arguments of this type are used by Uhlenbeck in her proof of Theorem 3.6 [95], where the connections (not necessarily anti-self-dual or Yang-Mills) are just assumed to be in L^p_1 and the gauge transformations are in L^p_2 with $p > 2$.

Lemma 4.14. *Suppose that (A_α, Φ_α) is a sequence of pairs on $(\mathfrak{su}(E), W^+ \otimes E)$ over a base manifold Ω (possibly non-compact), and let $\tilde\Omega \Subset \Omega$ be an interior domain. Suppose that there are gauge transformations $u_\alpha \in \mathcal{G}_E$ and $\tilde u_\alpha \in \mathcal{G}_{E|\tilde\Omega}$ such that $u_\alpha(A_\alpha, \Phi_\alpha)$ converges over Ω and $\tilde u_\alpha(A_\alpha, \Phi_\alpha)$ converges over $\tilde\Omega$. Then for any compact set $K \Subset \tilde\Omega$ there are a subsequence $\{\alpha'\} \subset \{\alpha\}$ and gauge transformations $w_{\alpha'} \in \mathcal{G}_E$ such that $w_{\alpha'} = \tilde u_{\alpha'}$ on a neighborhood of K and the pairs $w_{\alpha'}(A_{\alpha'}, \Phi_{\alpha'})$ converge over Ω.*

We have the following two extensions of this result.

Lemma 4.15. *Let Ω be exhausted by an increasing sequence of precompact open subsets $U_1 \Subset U_2 \Subset \cdots \Subset \Omega$ with $\cup_n U_n = \Omega$. Suppose (A_α, Φ_α) is a sequence of pairs on $(\mathfrak{su}(E), W^+ \otimes E)$ over Ω and that for each n there are a subsequence $\{\alpha'\} \subset \{\alpha\}$ and gauge transformations $u_{\alpha'} \in \mathcal{G}_{E|U_n}$ such that $u_{\alpha'}(A_{\alpha'}, \Phi_{\alpha'})$ converges over U_n. Then there are a subsequence $\{\alpha''\} \subset \{\alpha\}$ and gauge transformations $u_{\alpha''} \in \mathcal{G}_E$ such that $u_{\alpha''}(A_{\alpha''}, \Phi_{\alpha''})$ converges over Ω.*

Lemma 4.16. *Suppose that $\Omega = \Omega_1 \cup \Omega_2$ and that (A_α, Φ_α) is a sequence of pairs on $(\mathfrak{su}(E), W^+ \otimes E)$ over Ω. If there are sequences of* *gauge transformations $v_\alpha \in \mathcal{G}_{E|\Omega_1}$ and $w_\alpha \in \mathcal{G}_{E|\Omega_2}$ such that $v_\alpha(A_\alpha,\Phi_\alpha)$ converges over Ω_1 and $w_\alpha(A_\alpha,\Phi_\alpha)$ converges over Ω_2, then there are a subsequence $\{\alpha'\}\subset\{\alpha\}$ and gauge transformations $u_{\alpha'}\in\mathcal{G}_E$ such that the pairs $u_{\alpha'}(A_{\alpha'},\Phi_{\alpha'})$ converge over Ω.*

Lemmas 4.14, 4.15, and 4.16 combine to yield the following analogue of Corollary 4.4.8 in [20].

Corollary 4.17. *Suppose that (A_α,Φ_α) is a sequence of pairs on $(\mathfrak{su}(E), W^+\otimes E)$ over Ω such that the following holds. For each point $x\in\Omega$ there are a neighborhood D of x, a subsequence $\{\alpha'\}\subset\{\alpha\}$, and gauge transformations $v_{\alpha'}\in\mathcal{G}_{E|_D}$ such that the pairs $v_{\alpha'}(A_{\alpha'},\Phi_{\alpha'})$ converge over D. Then there are a single subsequence $\{\alpha''\}\subset\{\alpha\}$, and gauge transformations $u_{\alpha''}\in\mathcal{G}_E$ such that the pairs $u_{\alpha''}(A_{\alpha''},\Phi_{\alpha''})$ converge over Ω.*

We now assume that (A_α,Φ_α) is a sequence of PU(2) monopoles on $(\mathfrak{su}(E), W^+\otimes E)$ over Ω and obtain the required convergence from our local elliptic estimates for PU(2) monopoles and Uhlenbeck's gauge-fixing theorem. The following result is the analogue of Proposition 4.4.9 in [20], which applies to a sequence of anti-self-dual connections.

Proposition 4.18. *Let Y be an oriented four-manifold with Riemannian metric g and spin^c structure (ρ, W). Suppose that (A_α,Φ_α) is a sequence of* PU(2) *monopoles, on the bundles $(\mathfrak{su}(E), W^+\otimes E)$ over Y, with the following property. For each point $y\in Y$ there is a geodesic ball $B_g(y,r_y)$ with center y, radius r_y, and index α_y such that*

$$\|F_{A_\alpha}\|_{L^2(B_g(y,r_y))}<\varepsilon_0,\qquad \alpha\ge\alpha_y,$$

where $\varepsilon_0(g, A_{\det W^+}, A_{\det E})$ is the constant of Corollary 3.16. Then there are a subsequence $\{\alpha''\}\subset\{\alpha\}$ and a sequence of C^∞ gauge transformations $u_{\alpha''}\in\mathcal{G}_E$ such that $u_{\alpha''}(A_{\alpha''},\Phi_{\alpha''})$ converges in C^∞ on compact subsets over Y.

Proof. Fix a point $y\in Y$. If $B_g(y,r_y)$ is a geodesic ball with center y and g-radius r_y, then $B_{g_r}(y,1)$ is a geodesic ball with center y and g_r-radius one, where $g_r=r_y^{-2}g$. Thus, $(A_\alpha, r_y^{-1}\Phi_\alpha)$ is sequence of PU(2) monopoles over $B_{g_r}(y,1)$ such that

$$\|F_{A_\alpha}\|_{L^2(B_{g_r}(y,1))}<\varepsilon_0,\qquad \alpha\ge\alpha_y.$$

Corollary 3.16 implies that there are a subsequence $\{\alpha'\}\subset\{\alpha\}$ and a sequence of gauge transformations $\{u_{\alpha'}\}$ over $B_{g_r}(y,1)$ such that the sequence $u_{\alpha'}(A_{\alpha'}, r_y^{-1}\Phi_{\alpha'})$ converges over $B_{g_r}(y,\frac{1}{2})$.

Therefore, for each point $y \in Y$, we have a sequence of gauge transformations $\{u_{\alpha'}\}$ over $B_g(y, r_y)$ such that the sequence $u_{\alpha'}(A_{\alpha'}, \Phi_{\alpha'})$ converges over $B_g(y, \frac{1}{2}r_y)$. The conclusion now follows from Corollary 4.17. q.e.d.

4.5. Definition of the Uhlenbeck closure. The definition of the Uhlenbeck closure of the moduli space of solutions to the perturbed PU(2) monopole equations (2.27) is slightly more involved than that of the unperturbed PU(2) monopole equations (2.15). For this reason it is convenient to define the Uhlenbeck closure in the unperturbed case before considering the general case.

4.5.1. Definition of the Uhlenbeck closure for the moduli space of unperturbed PU(2) monopoles. Let $M_{W,E}$ (temporarily) denote the moduli space of gauge-equivalence classes of solutions (A, Φ) on $(\mathfrak{su}(E), W^+ \otimes E)$ to the unperturbed PU(2) monopole equations (2.15). We define the Uhlenbeck closure $\bar{M}_{W,E}$ of the moduli space $M_{W,E}$ by analogy with the definition of the Uhlenbeck closure of the moduli space of anti-self-dual connections [20, §4.4]. The *moduli set* $IM_{W,E}$ of unperturbed *ideal* PU(2) *monopoles* on $(\mathfrak{su}(E), W^+ \otimes E)$ is given by

$$IM_{W,E} := \bigcup_{\ell=0}^{N} M_{W,E_{-\ell}} \times \mathrm{Sym}^{\ell}(X),$$

where $N \geq N_p$, N_p is the constant defined in equation (4.15), $E_{-\ell}$ denotes a Hermitian two-plane bundle with $\det E_{-\ell} = \det E$, and $c_2(E_{-\ell}) = c_2(E) - \ell$, for $\ell \geq 0$.

Definition 4.19. Suppose that $[A_\alpha, \Phi_\alpha, \mathbf{y}_\alpha]$ is a sequence of points in $IM_{W,E}$ and that $[A_0, \Phi_0, \mathbf{x}]$ is a point in $IM_{W,E}$, where (A_α, Φ_α) and (A_0, Φ_0) are monopoles on $(\mathfrak{su}(E_\alpha), W^+ \otimes E_\alpha)$ and $(\mathfrak{su}(E_0), W^+ \otimes E_0)$ over X, respectively, with $\det E_\alpha = \det E_0 = \det E$ and $c_2(E_\alpha), c_2(E_0) \leq c_2(E)$. Then the sequence of points $[A_\alpha, \Phi_\alpha, \mathbf{y}_\alpha]$ *converges* to $[A_0, \Phi_0, \mathbf{x}]$ (or, the sequence of triples $(A_\alpha, \Phi_\alpha, \mathbf{y}_\alpha)$ *converges weakly* to $(A_0, \Phi_0, \mathbf{x})$) if the following hold:

- There is a sequence of $L^2_{k+1,\mathrm{loc}}$ determinant-one, unitary bundle isomorphisms $u_\alpha : E_\alpha|_{X\setminus\mathbf{x}} \to E_0|_{X\setminus\mathbf{x}}$ such that the sequence of PU(2) monopoles $u_\alpha(A_\alpha, \Phi_\alpha)$ converges to (A_0, Φ_0) in $L^2_{k,\mathrm{loc}}$ over $X \setminus \mathbf{x}$.
- The sequence $|F_{A_\alpha}|^2 + 8\pi^2 \sum_{y\in\mathbf{y}_\alpha} \delta(y)$ converges in the weak-* topology on measures to $|F_{A_0}|^2 + 8\pi^2 \sum_{x\in\mathbf{x}} \delta(x)$.

- We have $c_2(E) = c_2(E_0) + |\mathbf{x}|$.

Give $IM_{W,E}$ the Uhlenbeck topology specified by Definition 4.19 and let $\bar{M}_{W,E} \subset IM_{W,E}$ be the closure of $M_{W,E}$ in $IM_{W,E}$. The topological space $IM_{W,E}$ is second-countable and Hausdorff.

4.5.2. Definition of the Uhlenbeck closure for the moduli space of perturbed PU(2) monopoles. The basic idea underlying the choice of holonomy perturbations described in §2.5.2 is a generalization of an earlier construction due to S. K. Donaldson for the moduli spaces of solutions to the 'extended anti-self-dual equations', to which the Freed-Uhlenbeck generic metrics theorem does not apply [18, §IV(v), pp. 282–286]. As in the case of the moduli space of anti-self-dual connections, we shall see in §4.6 that there is an upper bound M (which is determined by g, $A_{\det W^+}$, $A_{\det E}$, and $p_1(\mathfrak{su}(E))$) on the total energy $\|F_A\|^2_{L^2(X)}$ for any solution (A, Φ) to the perturbed PU(2) monopole equations (2.27) and so an upper bound $2M/\varepsilon_0^2$ on the number of disjoint balls $B(x_j, 4R_0)$ with energy greater than or equal to $\frac{1}{2}\varepsilon_0^2$. Hence, if $N_b \geq 2M/\varepsilon_0^2 + 1$, at least one ball $B(x_j, 4R_0)$ in the collection $\{B(x_j, 4R_0)\}_{j=1}^{N_b}$ has energy less than $\frac{1}{2}\varepsilon_0^2$.

Suppose (A_α, Φ_α) is a sequence of PU(2) monopoles which converges to an ideal PU(2) monopole $(A_0, \Phi_0, \mathbf{x})$ in $\tilde{\mathcal{C}}_{W,E_{-\ell}} \times \mathrm{Sym}^\ell(X)$. If a point $x \in \mathbf{x}$ lies in a ball $B(x_j, 2R_0)$, the corresponding sections $\mathfrak{m}_{j,l,\beta}(A_\alpha)$ supported on $\bar{B}(x_j, R_0)$ converge to zero (by construction) for all l, β. Thus, the solution (A_0, Φ_0) will satisfy a version of the PU(2) monopole equations (2.27) with the perturbations supported on $\bar{B}(x_j, R_0)$ omitted. (In the situation considered by Donaldson, ideal extended anti-self-dual connections also satisfy a family of equations [18, Eq. (4.37)] which depend on the bubble points in X.) Therefore, the ideal limit $[A_0, \Phi_0, \mathbf{x}]$ is a point in the fiber $\mathbf{M}_{W,E_{-\ell}}|_{\mathbf{x}}$ over a point $\mathbf{x}$ in the base $\mathrm{Sym}^\ell(X)$. Here, $M_{W,E_{-\ell}}|_{\mathbf{x}}$ is simply the moduli space of solutions to the perturbed monopole equations (2.27) with the connection energy cutoff functions $\beta_j[A]$ of (2.18) (used in the definition of the perturbing sections $\mathfrak{m}_{j,l,\beta}(A)$ of (2.20)) replaced by cutoff functions

(4.8)
$$\beta_j[A_0, \mathbf{x}] := \beta\left(\frac{1}{\varepsilon_0^2}\int_{B(x_j,4R_0)} \beta\left(\frac{\mathrm{dist}_g(\cdot, x_j)}{4R_0}\right)\left(|F_{A_0}|^2 + 8\pi^2 \sum_{x\in\mathbf{x}} \delta(x)\right) dV\right),$$

where ε_0 is the constant of Corollary 3.16. Then $\mathbf{M}_{W,E_{-\ell}}$ is the moduli space of triples $(A_0, \Phi_0, \mathbf{x})$ solving the 'lower-level' PU(2) monopole equations

$$
\begin{aligned}
F_A^+ - \left(\mathrm{id} + \tau_0 \otimes \mathrm{id}_{\mathfrak{su}(E)} + \vec{\tau} \cdot \vec{\mathfrak{m}}(A, \mathbf{x})\right) \rho^{-1}(\Phi \otimes \Phi^*)_{00} &= 0, \\
D_A \Phi + \rho(\vartheta_0)\Phi + \vec{\vartheta} \cdot \vec{\mathfrak{m}}(A, \mathbf{x})\Phi &= 0.
\end{aligned}
\tag{4.9}
$$

In the more familiar case of the Uhlenbeck compactification of the moduli space of solutions to the unperturbed PU(2) monopole equations (2.15), the spaces $\mathbf{M}_{W,E_{-\ell}}$ above would be replaced by the products $M_{W,E_{-\ell}} \times \mathrm{Sym}^\ell(X)$.

The definition of the Uhlenbeck topology for the moduli space of solutions to the perturbed PU(2) monopole equations (2.27) is almost identical to that of the unperturbed case. The only difference is in the definition of the set of ideal solutions to (2.27). In the presence of holonomy perturbations, the Uhlenbeck closure $\bar{M}_{W,E}$ is therefore defined to be the closure of $M_{W,E}$ in

$$
IM_{W,E} := \bigcup_{\ell=0}^{N} \mathbf{M}_{W,E_{-\ell}} \subset \bigcup_{\ell=0}^{N} \mathcal{C}_{W,E_{-\ell}} \times \mathrm{Sym}^\ell(X),
$$

where $\mathbf{M}_{W,E_{-0}} := M_{W,E}$, while $N \geq N_p$ and N_p is the constant defined in equation (4.15).

4.6. Sequential compactness. In this section we apply our elliptic regularity and removable singularity results to prove our main compactness result, namely Theorem 1.1, which asserts the existence of an Uhlenbeck compactification for the moduli space of PU(2) monopoles, analogous to that given by Theorem 4.4.3 [20] in the case of the moduli space of anti-self-dual connections.

As in [20], the proof of Theorem 1.1 follows by an entirely routine argument (which we leave to the reader) from the special case below which is an analogue of similar compactness results for anti-self-dual connections; see, for example, [20, Theorem 4.4.4], [31, Theorem 3.2], [30, Chapter 8], [78, Theorem 3.1], [83, Proposition 4.4], and [84, Proposition 5.1].

Theorem 4.20. *Let X be a closed, oriented, smooth four-manifold with C^∞ Riemannian metric, spin^c structure (ρ, W) with spin^c connection, and a Hermitian two-plane bundle E with unitary connection on* $\det E$. *Then there is a positive integer N_p, depending at most on the* *curvatures of the fixed connections on W and $\det E$ together with $c_2(E)$, such that for all $N \geq N_p$, any infinite sequence in $M_{W,E}$ has a weakly convergent subsequence, with limit point in $\cup_{\ell=0}^{N} \mathbf{M}_{W,E_{-\ell}}$.*

Proof. The basic argument follows that of [20, pp. 163–165] and [83, Proposition 4.4] for the moduli space of anti-self-dual connections. Let $[A_\alpha, \Phi_\alpha]$ be a sequence of points in $M_{W,E}$ and let (A_α, Φ_α) be a corresponding sequence of PU(2) monopoles in C^∞ on $(E, W^+ \otimes E)$. By passing to a subsequence we can assume that the sequence of positive measures $\mu_\alpha := |F_{A_\alpha}|^2$ on X converges to a measure μ_∞ on X in the weak-* topology on measures, so

$$\lim_{\alpha\to\infty} \int_X |F_{A_\alpha}|^2 \, dV = \int_X \mu_\infty =: M_\infty \leq K,$$

where $K < \infty$ is the constant in our universal energy bound (4.2) for a PU(2) monopole over X. Hence there are at most M_∞/ε_0^2 distinct points in X, labelled $\{x_1, \dots, x_m\}$, which do not lie in a geodesic ball $B(x,r)$ of μ_∞-measure less than ε_0^2, where ε_0 is the constant of Proposition 4.18 and which appears in (2.18) and (4.8). Thus, for any $r > 0$, we have

$$\lim_{\alpha\to\infty} \int_{B(x_i,r)} |F_{A_\alpha}|^2 \, dV = \int_{B(x_i,r)} \mu_\infty \geq \varepsilon_0^2, \qquad i = 1, \dots, m,$$

and so we may define real numbers $\kappa_i \geq \varepsilon_0^2/8\pi^2$ by setting

$$\kappa_i := \lim_{r\to 0} \lim_{\alpha\to\infty} \frac{1}{8\pi^2} \int_{B(x_i,r)} |F_{A_\alpha}|^2 \, dV = \lim_{r\to 0} \frac{1}{8\pi^2} \int_{B(x_i,r)} \mu_\infty.$$

We may suppose, without loss of generality, that $m \geq 1$. If a point x_i lies in a ball $\bar{B}(x_j, 2R_0)$ then the holonomy perturbation sections are zero over $B(x_j, R_0)$ since $8\pi^2\kappa_i \geq \frac{1}{2}\varepsilon_0^2$. Hence, the points $\{x_1, \dots, x_m\}$ are contained in a large open subset of X where $(\vec{\tau}, \vec{\vartheta}) = 0$.

By passing to a subsequence, Proposition 4.18 supplies determinant-one, unitary gauge transformations u_α over $X \setminus \{x_1, \dots, x_m\}$ such that the sequence $u_\alpha(A_\alpha, \Phi_\alpha)$ converges over $X \setminus \{x_1, \dots, x_m\}$ to a pair (A_0, Φ_0) on $(\mathfrak{su}(E), W^+ \otimes E)|_{X \setminus \{x_1,\dots,x_m\}}$, such that the triple $(A_0, \Phi_0, \mathbf{x})$ solves the lower-level PU(2) monopole equations (4.9). Plainly,

$$\int_{X\setminus\{x_1,\dots,x_m\}} \left(|F_{A_0}|^2 + |\Phi_0|^4 + |\nabla_{A_0}\Phi_0|^2\right) \, dV \leq K < \infty.$$

By the removability of point singularities for finite-energy PU(2) monopoles (Theorem 4.10), there are a Hermitian two-plane bundle E_0 with $\det E_0 = \det E$ over X, a PU(2) monopole $(\tilde{A}_0, \tilde{\Phi}_0)$ on $(\mathfrak{su}(E_0), W^+ \otimes E_0)$, and a determinant-one, unitary bundle isomorphism u_0 from $E|_{X\setminus\{x_1,\dots,x_m\}}$ to $E_0|_{X\setminus\{x_1,\dots,x_m\}}$ such that $u_0(A_0, \Phi_0) = (\tilde{A}_0, \tilde{\Phi}_0)$ over $X \setminus \{x_1, \dots, x_m\}$.

The limiting measure μ_∞ has the form

$$\mu_\infty = |F_{A_0}|^2 + 8\pi^2 \sum_{i=1}^{m} \kappa_i \delta_{x_i},$$

where the δ_{x_i} have unit mass concentrated at the points x_i. It remains to show that the κ_i are positive integers. For this purpose we use an argument similar to that used to prove Lemma 3.8 in [31] (due to Taubes) which fits better with our later development of the gluing theory for PU(2) monopoles, though one could also use the Chern-Simons functional for this purpose as in [20, p. 164]. The proof of Theorem 3.2 in [31] is a modification of an earlier compactness result, Proposition 4.4 in [83], for connections with L^2 bounded curvature but which are only approximately anti-self-dual in a suitable sense. Theorem 4.20 follows easily from the next lemma:

Lemma 4.21. *The bundle E_0 has Chern classes $c_1(E_0) = c_1(E)$ and $c_2(E_0) = c_2(E) - \ell$, where $\ell = \sum_{i=1}^m \kappa_i$ and the constants κ_i are positive integers for $i = 1, \dots, m$.*

Proof. The equality of the first Chern classes follows from the remarks in the preceding paragraph. The proof that each κ_i is an integer requires a brief digression in order to discuss the limiting behavior of the connections A_α near the points $x_i \in X$.

Fix an index $i \in \{1, \dots, m\}$, let ϱ be the injectivity radius of (X, g), and fix a constant $\delta \in (0, \frac{1}{4}\varrho)$. Choose an orthogonal frame for $\mathfrak{su}(E_0)|_{x_i}$, use parallel translation via the connection A_0 along radial geodesics from $x_i \in X$ to trivialize $\mathfrak{su}(E_0)$ over the ball $B(x_i, \varrho)$, and let $w_{0,i} : \mathfrak{su}(E_0)|_{B(x_i,\varrho)} \to B(x_i, \varrho) \times \mathfrak{su}(2)$ be the resulting smooth bundle map. We have $\|F_{A_0}\|_{L^\infty(X)} \le C$, for some positive constant C, and so

$$\|F_{A_0}\|_{L^2(B(x_i,\delta))} \le C\delta^2.$$

Thus, we may suppose that δ is fixed small enough so that Theorem 3.13 provides an SU(2) gauge transformation $v_{0,i}$ of $B(x_i, \delta) \times \mathfrak{su}(2)$ such that

$$\|a_{0,i}\|_{L^4(B(x_i,\delta))} + \|\nabla_\Gamma a_{0,i}\|_{L^2(B(x_i,\delta))} \le c\|F_{A_0}\|_{L^2(B(x_i,2\delta))},$$

where $a_{0,i} := u_{0,i}(A_0) - \Gamma \in \Omega^1(B(x_i,\delta),\mathfrak{su}(2))$ and $u_{0,i} := v_{0,i} \circ w_{0,i}$, and Γ is the product connection on $B(x_i,\delta) \times \mathfrak{su}(2)$. The sequence of connections $u_\alpha(A_\alpha)$ converges in C^∞ on compact subsets of the punctured balls $B(x_i,\delta)\setminus\{x_i\}$ to the C^∞ connection A_0 on $\mathfrak{su}(E_0)|_{B(x_i,\delta)\setminus\{x_i\}}$; therefore, the sequence of connections $u_{0,i}u_\alpha(A_\alpha)$ converges in C^∞ on compact subsets of the punctured balls $B(x_i,\delta) \setminus \{x_i\}$ to the C^∞ connection $u_{0,i}(A_0)$ on $B(x_i,\delta) \setminus \{x_i\} \times \mathfrak{su}(2)$.

Write $u_{0,i}u_\alpha(A_\alpha) = \Gamma + a_{i,\alpha}$ over $B(x_i,\delta) \setminus \{x_i\}$, where $a_{i,\alpha} \in \Omega^1(B(x_i,\delta) \setminus \{x_i\}, \mathfrak{su}(2))$. Let $\Omega(x_i;\frac{1}{2}r,2r)$ denote the open annulus $\{x \in X : \frac{1}{2}r < \operatorname{dist}_g(x,x_i) < 2r\}$ in X. Then, for any $r \in (0,\frac{1}{2}\delta)$, there is an index $\alpha_0(r)$ such that

$$\int_{\Omega(x_i,\frac{1}{2}r,2r)} \left(|a_{i,\alpha} - a_{0,i}|^4 + |\nabla_\Gamma(a_{i,\alpha} - a_{0,i})|^2\right) dV < r^4, \qquad \alpha \geq \alpha_0.$$

Since $\|F_{A_0}\|_{L^\infty(X)} \leq C$, we have

$$\int_{B(x_i,2r)} \left(|a_{0,i}|^4 + |\nabla_\Gamma a_{0,i}|^2\right) dV \leq c \int_{B(x_i,2r)} |F_{A_0}|^2\, dV \leq Cr^4, \tag{4.10}$$

and therefore

$$\int_{\Omega(x_i,\frac{1}{2}r,2r)} \left(|a_{i,\alpha}|^4 + |\nabla_\Gamma a_{i,\alpha}|^2\right) dV \leq Cr^4, \qquad \alpha \geq \alpha_0. \tag{4.11}$$

Let $\chi : \mathbb{R} \to [0,1]$ be a bump function such that $\chi(t) = 0$ for $t \leq \frac{1}{2}$ and $\chi(t) = 1$ for $t \geq 2$. Define a cutoff function $\chi_{i,r} : X \to [0,1]$ by setting $\chi_{i,r}(x) = 1 - \chi(\operatorname{dist}_g(x,x_i)/r)$ so that $\chi_{i,r} = 1$ on $B(x_i,\frac{1}{2}r)$ and $\chi_{i,r} = 0$ on $X - B(x_i,2r)$. Fix a Riemannian metric g_i on S^4 which coincides with g on $B(x_i,\delta) = B(n,\delta)$ (after identifying the point $x_i \in X$ with the north pole $n \in S^4$) and extends g outside $B(x_i,2\delta) = B(n,2\delta)$ to a smooth metric on S^4. Define a sequence of SO(3) bundles $V_{i,r,\alpha}$ over S^4 by setting

$$V_{i,r,\alpha} := \begin{cases} \mathfrak{su}(E) & \text{over } B(n,2r), \\ S^4 \setminus \{n\} \times \mathfrak{su}(2) & \text{over } S^4 \setminus \{n\}, \end{cases}$$

where the identification of the SO(3) bundles $\mathfrak{su}(E)$ and $S^4 \setminus \{n\} \times \mathfrak{su}(2)$ over the annulus $B(n,2r) \setminus \{n\} = B(x_i,2r) \setminus \{x_i\}$ is induced from the the SO(3) bundle isomorphism

$$u_{0,i} \circ u_\alpha : \mathfrak{su}(E)|_{B(n,2r)\setminus\{n\}} \to B(n,2r) \setminus \{n\} \times \mathfrak{su}(2).$$

We cut off the sequence of connections A_α on $\mathfrak{su}(E)$ over the annulus $\Omega(x_i;\frac{1}{2}r,2r)$ and thus obtain a sequence of C^∞ connections $A_{i,r,\alpha}$ on the sequence of SO(3) bundles $V_{i,r,\alpha}$ over S^4 by setting

$$A_{i,r,\alpha} := \begin{cases} A_\alpha & \text{on } \mathfrak{su}(E)|_{B(n,\frac{1}{2}r)}, \\ \Gamma + \chi_{i,r}a_{i,\alpha} & \text{on } S^4 \setminus \{n\} \times \mathfrak{su}(2). \end{cases}$$

Recall from Lemma 4.4 that there is a constant C independent of α such that $\|F^+_{A_\alpha}\|_{L^\infty(X)} \le C$ and so

$$\|F^+_{A_\alpha}\|_{L^2(B(x_i,2r))} \le Cr^2 \quad \text{for all } \alpha. \tag{4.12}$$

Since

$$F^+_{A_{i,r,\alpha}} = \chi_{i,r}F^+_{A_\alpha} + (d\chi_{i,r}\wedge a_{i,\alpha})^+ + \chi_{i,r}(\chi_{i,r}-1)(a_{i,\alpha}\wedge a_{i,\alpha})^+,$$

the estimates (4.10), (4.11), and (4.12) imply that

$$\begin{aligned}\|F^+_{A_{i,r,\alpha}}\|_{L^2(S^4)} &\le \|F^+_{A_\alpha}\|_{L^2(B(x_i,2r))} \\ &\quad + \sqrt{2}\|d\chi_{i,r}\|_{L^4(X)}\|a_{i,\alpha}\|_{L^4(\Omega(x_i;\frac{1}{2}r,2r))} \\ &\quad + \sqrt{2}\|a_{i,\alpha}\|^2_{L^4(\Omega(x_i;\frac{1}{2}r,2r))} \\ &\le C(r+r^2), \qquad \alpha \ge \alpha_0.\end{aligned}$$

Therefore,

$$\lim_{r\to 0}\lim_{\alpha\to\infty}\|F^+_{A_{i,r,\alpha}}\|_{L^2(S^4)} = 0. \tag{4.13}$$

Similarly, as

$$F_{A_{i,r,\alpha}} = \chi_{i,r}F_{A_\alpha} + d\chi_{i,r}\wedge a_{i,\alpha} + \chi_{i,r}(\chi_{i,r}-1)a_{i,\alpha}\wedge a_{i,\alpha},$$

the estimates (4.10) and (4.11) yield

$$\begin{aligned}&\|F_{A_{i,r,\alpha}} - F_{A_\alpha}\|_{L^2(\Omega(x_i;\frac{1}{2}r,2r))} \\ &\quad \le \|\nabla_\Gamma a_{i,\alpha}\|_{L^2(\Omega(x_i;\frac{1}{2}r,2r))} + c\|a_{i,\alpha}\|_{L^4(\Omega(x_i;\frac{1}{2}r,2r))} \\ &\qquad + \|a_{i,\alpha}\|^2_{L^4(\Omega(x_i;\frac{1}{2}r,2r))} \\ &\quad < C(r+r^2), \qquad \alpha \ge \alpha_0.\end{aligned}$$

Therefore,

$$\lim_{r\to 0}\lim_{\alpha\to\infty}\|F_{A_{i,r,\alpha}} - F_{A_\alpha}\|_{L^2(\Omega(x_i;\frac{1}{2}r,2r))} = 0. \tag{4.14}$$

We can now complete the argument that the κ_i are integers:

Claim 4.22. There is an SU(2) bundle E_i over S^4 such $c_2(E_i) = \kappa_i$ for each $i = 1, \dots, m$.

Proof. Fix an index $i \in \{1, \dots, m\}$. Over S^4, the SO(3) bundles $V_{i,r,\alpha}$ lift to SU(2) bundles $E_{i,r,\alpha}$ with $V_{i,r,\alpha} = \mathfrak{su}(E_{i,r,\alpha})$ and $p_1(V_{i,r,\alpha}) = -4c_2(E_{i,r,\alpha})$. The second Chern classes of the SU(2) bundles $E_{i,r,\alpha}$ are given by

$$c_2(E_{i,r,\alpha}) = \frac{1}{8\pi^2}\int_{S^4}\left(|F^-_{A_{i,r,\alpha}}|^2 - |F^+_{A_{i,r,\alpha}}|^2\right) dV,$$

recalling that the isomorphisms $\mathrm{ad} : \mathfrak{su}(E_{i,r,\alpha}) \to \mathfrak{so}(\mathfrak{su}(E_{i,r,\alpha}))$ are implicit and that we view $F_{A_{i,r,\alpha}}$ as sections of $\Lambda^2 \otimes \mathfrak{su}(E_{i,r,\alpha})$. Therefore, by (4.14) and (4.13) and the fact that $F_{A_{i,r,\alpha}} = F_{A_\alpha}$ on $B(x_i, \frac{1}{2}r)$, we have

$$\begin{aligned}
\lim_{r\to 0}\lim_{\alpha\to\infty} c_2(E_{i,r,\alpha}) &= \lim_{r\to 0}\lim_{\alpha\to\infty}\frac{1}{8\pi^2}\int_{S^4}|F^-_{A_{i,r,\alpha}}|^2\, dV \\
&= \lim_{r\to 0}\lim_{\alpha\to\infty}\frac{1}{8\pi^2}\int_{S^4}\left(|F^-_{A_{i,r,\alpha}}|^2 + |F^+_{A_{i,r,\alpha}}|^2\right) dV \\
&= \lim_{r\to 0}\lim_{\alpha\to\infty}\frac{1}{8\pi^2}\int_{B(x_i,2r)}|F_{A_{i,r,\alpha}}|^2\, dV \\
&= \lim_{r\to 0}\lim_{\alpha\to\infty}\frac{1}{8\pi^2}\int_{B(x_i,2r)}|F_{A_\alpha}|^2\, dV = \kappa_i,
\end{aligned}$$

where the final equality follows by definition of κ_i. Thus, for small enough r and large enough α, we have $c_2(E_{i,r,\alpha}) = c_2(E_i)$ for some fixed SU(2) bundle E_i over S^4 and so $\kappa_i = c_2(E_i)$, completing the proof of the claim. q.e.d.

By Claim 4.22 the κ_i are positive integers for $i = 1, \dots, m$. We can now compute the second Chern class of the limit bundle E_0. The Chern-Weil identity (2.33) implies that, for all α,

$$-\tfrac{1}{4}p_1(\mathfrak{su}(E)) = c_2(E) - \tfrac{1}{4}c_1(E)^2 = \frac{1}{8\pi^2}\int_X\left(|F^-_{A_\alpha}|^2 - |F^+_{A_\alpha}|^2\right) dV.$$

Therefore, by (4.12) we have

$$
\begin{aligned}
c_2(E) - \tfrac{1}{4}c_1(E)^2 &= \lim_{r\to 0}\lim_{\alpha\to\infty}\frac{1}{8\pi^2}\int_X \left(|F^-_{A_\alpha}|^2 - |F^+_{A_\alpha}|^2\right) dV \\
&= \lim_{r\to 0}\lim_{\alpha\to\infty}\frac{1}{8\pi^2}\int_{X\setminus\cup_{i=1}^m B(x_i,2r)} \left(|F^-_{A_\alpha}|^2 - |F^+_{A_\alpha}|^2\right) dV \\
&\quad + \sum_{i=1}^m \lim_{r\to 0}\lim_{\alpha\to\infty}\frac{1}{8\pi^2}\int_{B(x_i,2r)} \left(|F^-_{A_\alpha}|^2 - |F^+_{A_\alpha}|^2\right) dV \\
&= \frac{1}{8\pi^2}\int_X \left(|F^-_{A_0}|^2 - |F^+_{A_0}|^2\right) dV + \sum_{i=1}^m \kappa_i \\
&= c_2(E_0) - \tfrac{1}{4}c_1(E_0)^2 + \sum_{i=1}^m \kappa_i.
\end{aligned}
$$

Now $c_1(E) = c_1(E_0)$ and thus $c_2(E_0) = c_2(E) - \sum_{i=1}^m \kappa_i$. This completes the proof of Lemma 4.21. q.e.d.

Therefore, after passing to a subsequence, the sequence of points $[A_\alpha, \Phi_\alpha]$ in $M_{W,E}$ converges to an ideal monopole $[A_0, \Phi_0, \mathbf{x}]$ in $M_{L,E_0} \times \mathrm{Sym}^\ell(X)$, for some integer $\ell \geq 0$, and Lemma 4.21 implies the Chern classes of the limit bundle E_0 are given by $c_1(E_0) = c_1(E)$ and $c_2(E_0) = c_2(E) - \ell$.

It remains to give an upper bound for the integer $\ell = c_2(E) - c_2(E_0)$. The Chern-Weil identity (2.33) implies that

$$
\begin{aligned}
\ell = c_2(E) - c_2(E_0) =& \frac{1}{8\pi^2}\int_X \left(|F^-_A|^2 - |F^+_A|^2\right) dV \\
&- \frac{1}{8\pi^2}\int_X \left(|F^-_{A_0}|^2 - |F^+_{A_0}|^2\right) dV,
\end{aligned}
$$

and thus, by Lemma 4.3, we have

$$
\ell \leq \frac{1}{8\pi^2}\int_X |F^-_A|^2\, dV + \frac{1}{8\pi^2}\int_X |F^+_{A_0}|^2\, dV \leq N_p, \tag{4.15}
$$

for some positive integer $N_p = N_p(c_1(E), c_2(E), g, F(A_{\det W}), F(A_{\det E}))$. This completes the proof of Theorem 4.20. q.e.d.

Remark 4.23. The compactness result in this section for the moduli space of PU(2) monopoles has an antecedent in [31, Theorem 3.2] (due to Taubes) in the following sense. Taubes' theorem provides a weak compactness result for connections A satisfying the 'infinite-dimensional part' of the anti-self-dual equation, namely $\Pi_{A;\mu}F_A^+ = 0$, where $\mu \notin \operatorname{Spec} d_A^+ d_A^*$, and $\Pi_{A;\mu}$ is the L^2-orthogonal projection onto the eigenvectors of $d_A^+ d_A^*$ with eigenvalue less than μ, together with the curvature bounds $\|F_A\|_{L^2(X)} + \|d_A F_A^+\|_{L^2(X)} \leq C$ for some constant C independent of A. The analogous point here is that although $\int_X |F_A|^2\, dV$ is not a topological invariant unless $F_A^+ = 0$ or $F_A^- = 0$, just as in the case of the PU(2) monopoles, it is enough for the purposes of obtaining a weak compactness result to have uniform bounds on the L^2 norm of F_A together with an L^p bound on F_A^+ for some $p > 2$.

There is one further compactness result we will need, analogous to Uhlenbeck's original compactness theorem for connections (not necessarily satisfying any elliptic equation) with L^p bounds on curvature with $p > 2$ [95, Theorem 1.5].

Proposition 4.24. *Let $p > 2$ and $K > 0$ be constants. If $[A_\alpha, \Phi_\alpha]$ is an infinite sequence in $M_{W,E}$ satisfying*

$$\|F_{A_\alpha}\|_{L^p(X)} \leq K,$$

then there is a subsequence $\{\alpha'\} \subset \{\alpha\}$ such that the sequence $[A_{\alpha'}, \Phi_{\alpha'}]$ converges to a point $[A_\infty, \Phi_\infty] \in M_{W,E}$.

Proof. Let (A_α, Φ_α) be a corresponding sequence of C^∞ pairs. Since $p > 2$, Hölder's inequality implies that for any geodesic ball $B(x,r) \subset X$ we have

$$\|F_{A_\alpha}\|_{L^2(B(x,r)} \leq c r^{2-(4/p)} \|F_{A_\alpha}\|_{L^p(B(x,r))} \leq cK r^{2-(4/p)}.$$

Hence, for small enough r, Proposition 4.18 applies and there are a subsequence $\{\alpha'\} \subset \{\alpha\}$ and a sequence of C^∞ gauge transformations $u_{\alpha'}$ such that the sequence $u_{\alpha'}(A_{\alpha'}, \Phi_{\alpha'})$ converges in C^∞ to a limit (A_∞, Φ_∞) over all of X, with no exceptional points. q.e.d.

As we shall see in §5.1.2, Proposition 4.24 allows us to work with perturbation parameters $(\tau_0, \vec{\tau}, \vec{\vartheta})$ and a metric g which are C^∞ rather than just C^r, as required by the application of the Sard-Smale theorem in our proof of transversality in §5.

5. Transversality

In this section we show that for generic perturbation parameters $(\vartheta_0, \tau_0, \vec{\tau}, \vec{\vartheta})$ the moduli space of solutions to the perturbed PU(2) monopole equations (2.27) is a smooth manifold away from the zero-section and reducible pairs.

The outline of the proof is of the now standard form introduced in [14] and [30]. In §5.1 we define a parametrized moduli space and explain why transversality for the moduli space (Corollary 5.3) follows from transversality for the parametrized moduli space (Theorem 5.2) via the Sard-Smale theorem. In §5.2 we show that the parametrized moduli space is a smooth Banach manifold (Theorem 5.2). The proof of Theorem 5.2 relies on the fact that a PU(2) monopole, which is reducible on an admissible open subset of the manifold X, is reducible on the entire manifold (Theorem 5.11) and this is proved in §5.3.

The proof of Theorem 1.3 does not apply to PU(2) monopoles which are zero-sections or which are reducible. We describe the cokernels of $D\mathfrak{S}$ evaluated at these pairs in the sequel [25] to the present article.

5.1. The parametrized moduli space. It is convenient to first consider the question of transversality for the top stratum $M^{*,0}_{W,E}$ of the Uhlenbeck compactification $\bar{M}_{W,E}$ and then consider the very slight modification required to obtain simultaneous transversality for all the lower-level moduli spaces $\mathbf{M}^{*,0}_{W,E_{-\ell}}|_\Sigma \subset \mathcal{C}^{*,0}_{W,E_{-\ell}} \times \Sigma$, for smooth strata $\Sigma \subset \mathrm{Sym}^\ell(X)$.

5.1.1. Transversality for the top-level moduli space. The condition in Proposition 2.28 that the section $\mathfrak{S}$, vanish transversely is, of course, not necessarily true for all the parameters $(\tau_0, \vartheta_0, \vec{\tau}, \vec{\vartheta})$, on which $\mathfrak{S}$ depends. As in the cases of the moduli spaces of anti-self-dual connections [14], [20], [30] and Seiberg-Witten monopoles [47], [98], we first show that the family of moduli spaces parametrized by the perturbations $(\tau_0, \vartheta_0, \vec{\tau}, \vec{\vartheta})$ is smooth and then apply the Sard-Smale theorem [80] to conclude that for generic perturbations $(\tau_0, \vartheta_0, \vec{\tau}, \vec{\vartheta})$ (that is, a subset of perturbations which is the complement of some first-category subset), the moduli space $\mathfrak{S}^{-1}(0) = M^{*,0}_{W,E}(\tau_0, \vartheta_0, \vec{\tau}, \vec{\vartheta})$ is smooth.

Set $\mathcal{P}^r_0 := C^r(X, \mathfrak{gl}(\Lambda^+)) \oplus C^r(X, \Lambda^1 \otimes \mathbb{C})$, let $\mathcal{P}^r := \mathcal{P}^r_0 \oplus \mathcal{P}^r_\tau \oplus \mathcal{P}^r_\vartheta$ denote our Banach space of C^r perturbation parameters, and define a ${}^\circ\mathcal{G}_E$-equivariant map

$$\underline{\mathfrak{S}} := (\underline{\mathfrak{S}}_1, \underline{\mathfrak{S}}_2) : \mathcal{P}^r \times \tilde{\mathcal{C}}_{W,E} \to L^2_{k-1}(\Lambda^+ \otimes \mathfrak{su}(E)) \oplus L^2_{k-1}(W^- \otimes E))$$

by setting

$$\begin{aligned}
&\underline{\mathfrak{S}}(\tau_0,\vartheta_0,\vec{\tau},\vec{\vartheta},A,\Phi)\\
(5.1)\quad &:= \begin{pmatrix}\underline{\mathfrak{S}}_1(\tau_0,\vartheta_0,\vec{\tau},\vec{\vartheta},A,\Phi)\\ \underline{\mathfrak{S}}_2(\tau_0,\vartheta_0,\vec{\tau},\vec{\vartheta},A,\Phi)\end{pmatrix}\\
&= \begin{pmatrix}F_A^+ - \left(\mathrm{id} + \tau_0\otimes \mathrm{id}_{\mathfrak{su}(E)} + \vec{\tau}\cdot\vec{\mathfrak{m}}(A)\right)\rho^{-1}(\Phi\otimes\Phi^*)_{00}\\ D_A\Phi + \rho(\vartheta_0)\Phi + \vec{\vartheta}\cdot\vec{\mathfrak{m}}(A)\Phi\end{pmatrix},
\end{aligned}$$

where (A,Φ) is a pair on $(\mathfrak{su}(E), W^+\otimes E)$ and the isomorphism ad : $\mathfrak{su}(E)\simeq\mathfrak{so}(\mathfrak{su}(E))$ is implicit, ${}^\circ\mathcal{G}_E$ acts trivially on the space of perturbations $\mathcal{P}^r$, and so $\underline{\mathfrak{S}}^{-1}(0)/{}^\circ\mathcal{G}_E$ is a subset of $\mathcal{P}^r\times\mathcal{C}_{W,E}$. We let $\mathfrak{M}_{W,E}$ denote the *parametrized moduli space* $\underline{\mathfrak{S}}^{-1}(0)/{}^\circ\mathcal{G}_E$ and let $\mathfrak{M}_{W,E}^{*,0} = \mathfrak{M}_{W,E}\cap(\mathcal{P}^r\times\mathcal{C}_{W,E}^{*,0})$.

Remark 5.1. While we assumed for convenience in §3 and §4 that the parameters $g,\tau_0,\vartheta_0,\vec{\tau},\vec{\vartheta}$ were C^∞, the only difference, if the parameters are only assumed to be C^r for some finite r, is the slight increase in bookkeeping required to keep track of the regularity of solutions to (2.27) and other associated elliptic systems.

Just as in §2.6, the ${}^\circ\mathcal{G}_E$-equivariant map $\underline{\mathfrak{S}}$ defines a section of a Banach vector bundle $\underline{\mathfrak{V}}$ over $\mathcal{P}^r\times\mathcal{C}_{W,E}^{*,0}$ with total space

$$\underline{\mathfrak{V}} := \mathcal{P}^r\times\tilde{\mathcal{C}}_{W,E}^{*,0}\times_{{}^\circ\mathcal{G}_E}\left(L^2_{k-1}(\Lambda^+\otimes\mathfrak{su}(E))\oplus L^2_{k-1}(W^-\otimes E)\right),$$

so $\mathfrak{S} := \underline{\mathfrak{S}}(\tau_0,\vartheta_0,\vec{\tau},\vec{\vartheta},\cdot)$ is a section over $\tilde{\mathcal{C}}_{W,E}^{*,0}$ of the Banach vector bundle $\mathfrak{V} := \underline{\mathfrak{V}}|_{(\tau_0,\vartheta_0,\vec{\tau},\vec{\vartheta})}$ in (2.34). In particular, the parametrized moduli space $\mathfrak{M}_{W,E}^{*,0}$ is the zero set of the section $\underline{\mathfrak{S}}$ of the vector bundle $\underline{\mathfrak{V}}$ over $\mathcal{P}^r\times\mathcal{C}_{W,E}^{*,0}$.

Theorem 5.2. *The zero set in $\mathcal{P}^r\times\mathcal{C}_{W,E}^{*,0}$ of the section $\underline{\mathfrak{S}}$ is regular and, in particular, the moduli space $\mathfrak{M}_{W,E}^{*,0}$ is a smooth Banach submanifold of $\mathcal{P}^r\times\mathcal{C}_{W,E}^{*,0}$.*

To preserve continuity, we defer the proof of Theorem 5.2 to §5.2. The differential $D\underline{\mathfrak{S}} := (D\underline{\mathfrak{S}})_{[\tau_0,\vartheta_0,\vec{\tau},\vec{\vartheta},A,\Phi]}$ of the section $\underline{\mathfrak{S}}$ at a point $[\tau_0,\vartheta_0,\vec{\tau},\vec{\vartheta},A,\Phi]$ in $\mathcal{P}^r\times\mathcal{C}_{W,E}^{*,0}$ is given by

$$(5.2)\qquad D\underline{\mathfrak{S}}(\delta\tau_0,\delta\vartheta_0,\delta\vec{\tau},\delta\vec{\vartheta},a,\phi) = \begin{pmatrix}D\underline{\mathfrak{S}}_1(\delta\tau_0,\delta\vartheta_0,\delta\vec{\tau},\delta\vec{\vartheta},a,\phi)\\ D\underline{\mathfrak{S}}_2(\delta\tau_0,\delta\vartheta_0,\delta\vec{\tau},\delta\vec{\vartheta},a,\phi)\end{pmatrix},$$

where $(a,\phi) \in \mathbf{K}_{A,\Phi} \subset L^2_k(\Lambda^1 \otimes \mathfrak{su}(E)) \oplus L^2_k(W^+ \otimes E)$ represents a vector in the tangent space $(T\mathcal{C}^{*,0}_{W,E})_{[A,\Phi]}$ and $(\delta\tau_0, \delta\vartheta_0, \delta\vec{\tau}, \delta\vec{\vartheta}) \in \mathcal{P}^r$. The differential of the first component in (5.2) is given explicitly by

$$
\begin{aligned}
D\underline{\mathfrak{S}}_1 & \left(\delta\tau_0, \delta\vartheta_0, \delta\vec{\tau}, \delta\vec{\vartheta}, a, \phi\right) \\
&= d_A^+ a - \delta\tau_0 \rho^{-1}(\Phi\otimes\Phi^*)_{00} \\
&\quad - \left(\mathrm{id} + \tau_0 \otimes \mathrm{id}_{\mathfrak{su}(E)} + \vec{\tau}\cdot\vec{\mathfrak{m}}(A)\right)\rho^{-1}(\phi\otimes\Phi^* + \Phi\otimes\phi^*)_{00} \\
&\quad - \sum_{j,l,\alpha}(\delta\tau_{j,l,\alpha}\otimes \mathrm{ad}(\mathfrak{m}_{j,l,\alpha}(A))\rho^{-1}(\Phi\otimes\Phi^*)_{00} \\
&\quad - \sum_{j,l,\alpha}\tau_{j,l,\alpha}\otimes\mathrm{ad}\left(\frac{\delta\mathfrak{m}_{j,l,\alpha}}{\delta A}a\right)\rho^{-1}(\Phi\otimes\Phi^*)_{00},
\end{aligned}
\tag{5.3}
$$

and the second component by

$$
\begin{aligned}
D\underline{\mathfrak{S}}_2 & \left(\delta\tau_0, \delta\vartheta_0, \delta\vec{\tau}, \delta\vec{\vartheta}, a, \phi\right) \\
&= D_A\phi + \vec{\vartheta}\cdot\vec{\mathfrak{m}}(A)\phi + \rho(a)\Phi \\
&\quad + \rho(\delta\vartheta_0)\Phi + \sum_{j,l,\alpha}\rho(\delta\vartheta_{j,l,\alpha})\otimes\mathfrak{m}_{j,l,\alpha}(A)\Phi \\
&\quad + \sum_{j,l,\alpha}\rho(\vartheta_{j,l,\alpha})\otimes\left(\frac{\delta\mathfrak{m}_{j,l,\alpha}}{\delta A}a\right)\Phi.
\end{aligned}
\tag{5.4}
$$

We note that from their definitions in §2.5.2 the perturbations (and their variations) are *zeroth order*, unlike the first order perturbations considered in [92].

Recall from the arguments of §2.2 that $D\underline{\mathfrak{S}}(\cdot, d^0_{A,\Phi}\zeta) = 0$ for all $\zeta \in L^2_{k+1}(\mathfrak{su}(E)) \oplus i\mathbb{R}_Z$ since $\underline{\mathfrak{S}}$ is ${}^\circ\mathcal{G}_E$-equivariant. By Proposition 3.7 we may assume, without loss of generality, that the pair (A,Φ) in $\tilde{\mathcal{C}}^{*,0}_{W,E}$ is a C^r representative for the point $[A,\Phi]$ in the zero set $\mathfrak{S}^{-1}(0) \subset \mathcal{C}^{*,0}_{W,E}$. Since the tangent space $(T\mathcal{C}^{0,*}_{W,E})_{[A,\Phi]}$ may be identified with $\mathbf{K}_{A,\Phi} := \operatorname{Ker} d^{0,*}_{A,\Phi}$ (see §2.2), we have

$$
D\underline{\mathfrak{S}}(0,0,0,0,a,\phi) = d^1_{A,\Phi}(a,\phi) = (d^{0,*}_{A,\Phi} + d^1_{A,\Phi})(a,\phi),
$$

for $(a,\phi) \in \mathbf{K}_{A,\Phi}$, so the differential $D\underline{\mathfrak{S}}|_{\{0\}\times T\mathcal{C}^{*,0}_{W,E}}$ is Fredholm, where $\{0\}\times T\mathcal{C}^{0,*}_{W,E} = T(\{\tau_0,\vartheta_0,\vec{\tau},\vec{\vartheta}\}\times\mathcal{C}^{*,0}_{W,E})$. Thus, $\underline{\mathfrak{S}}$ is a Fredholm section when restricted to the fixed parameter fibers

$$
\{\tau_0,\vartheta_0,\vec{\tau},\vec{\vartheta}\}\times\mathcal{C}^{*,0}_{W,E} \subset \mathcal{P}^r \times \mathcal{C}^{*,0}_{W,E}.
$$

The Sard-Smale theorem (in the form of Proposition 4.3.11 in [20]) then implies that there is a first-category subset of the space $\mathcal{P}^r$ such that the zero sets in $\mathcal{C}^{*,0}_{W,E}$ of the sections $\mathfrak{S} = \underline{\mathfrak{S}}(\tau_0, \vartheta_0, \vec{\tau}, \vec{\vartheta}, \cdot)$ are regular for all C^r perturbations $(\tau_0, \vartheta_0, \vec{\tau}, \vec{\vartheta})$ in the complement of this subset. Now

$$M^{*,0}_{W,E}(\tau_0, \vartheta_0, \vec{\tau}, \vec{\vartheta}) = \mathfrak{S}^{-1}(0) \cap \mathcal{C}^{*,0}_{W,E},$$

and so for generic parameters $(\tau_0, \vartheta_0, \vec{\tau}, \vec{\vartheta})$, the moduli space $M^{*,0}_{W,E}(\tau_0, \vartheta_0, \vec{\tau}, \vec{\vartheta})$ is a smooth manifold of the expected dimension. In summary, we have:

Corollary 5.3. *There is a first-category subset of the space $\mathcal{P}^r$, such that for all C^r perturbations $(\tau_0, \vartheta_0, \vec{\tau}, \vec{\vartheta})$ in the complement of this subset, the zero locus of the section $\mathfrak{S}$ is regular and so the moduli space $M^{*,0}_{W,E}(\tau_0, \vartheta_0, \vec{\tau}, \vec{\vartheta}) = \mathfrak{S}^{-1}(0) \cap \mathcal{C}^{*,0}_{W,E}$ is a smooth submanifold of $\mathcal{C}^{*,0}_{W,E}$ with the expected dimension.*

We recall that a subset S of a topological space $\mathcal{P}$ is a set of the *first category* if its complement $\mathcal{P} - S$ is a countable intersection of dense open sets or, equivalently, if S is a countable union of closed subsets of $\mathcal{P}$ with empty interior; if $\mathcal{P}$ is a complete metric space, then Baire's theorem implies that $\mathcal{P} - S$ is dense in $\mathcal{P}$ [76]. In our applications, $\mathcal{P}$ will either be a Banach or Fréchet space (with a complete metric), so $\mathcal{P} - S$ will always be dense if S is a first-category subset.

5.1.2. Reduction to the case of $\mathbf{C}^\infty$ parameters. The restriction to C^r parameters $(\tau_0, \vartheta_0, \vec{\tau}, \vec{\vartheta})$, necessary to apply the Sard-Smale theorem in §5.1.1, proves inconvenient in practice. We shall see that these restrictions can now be removed, so we need only use C^∞ parameters $(\tau_0, \vartheta_0, \vec{\tau}, \vec{\vartheta})$. Although we did not need the metric g to be generic in order for our transversality proof to work, we will nonetheless require the metric g to be generic in the sequels to the present article and so, *a priori*, g would also be restricted to a certain Banach manifold of metrics on X. An argument almost identical to the one which we describe here can be used to show that one need only consider generic C^∞ metrics in those applications.

There is an argument due to Taubes — for the moduli space of Seiberg-Witten monopoles — which reduces the case of transversality for C^∞ parameters to the case of Hölder or Sobolev parameters [77, §9.4]. (A related result for generic metrics due to Freed and Uhlenbeck appears as Proposition 3.20 in [30], although it is only stated for the moduli space of anti-self-dual SU(2) connections with second Chern class one over a simply-connected, negative definite four-manifold.)

We adapt Taubes argument here to the case of the moduli spaces of PU(2) monopoles. We define

$$
\begin{aligned}
\mathcal{P} &:= \mathcal{P}_0 \oplus \mathcal{P}_\tau \oplus \mathcal{P}_\vartheta, \\
\mathcal{P}_0 &:= \Omega^0(X, \mathfrak{gl}(\Lambda^+)) \oplus \Omega^1(X, \mathbb{C}), \\
\mathcal{P}_\tau &= \ell^1_\delta(\mathbb{A}, \Omega^0(X, \mathfrak{gl}(\Lambda^+))) := \bigcap_{r=0}^{\infty} \ell^1_\delta(\mathbb{A}, C^r(X, \mathfrak{gl}(\Lambda^+))), \\
\mathcal{P}_\vartheta &= \ell^1_\delta(\mathbb{A}, \Omega^1(X, \mathbb{C})) := \bigcap_{r=0}^{\infty} \ell^1_\delta(\mathbb{A}, C^r(X, \Lambda^1 \otimes \mathbb{C})),
\end{aligned}
$$

and let $\mathcal{P}^r$ be the Banach space

$$
\begin{aligned}
\mathcal{P}^r &:= \mathcal{P}^r_0 \oplus \mathcal{P}^r_\tau \oplus \mathcal{P}^r_\vartheta \\
&= C^r(X, \mathfrak{gl}(\Lambda^+)) \oplus C^r(X, \Lambda^1 \otimes \mathbb{C}) \\
&\qquad \oplus \ell^1_\delta(\mathbb{A}, C^r(X, \mathfrak{gl}(\Lambda^+))) \oplus \ell^1_\delta(\mathbb{A}, C^r(X, \Lambda^1 \otimes \mathbb{C})).
\end{aligned}
$$

Define metrics d_0, d_τ and d_ϑ on $\mathrm{Met}(X)$, $\mathcal{P}_0$, $\mathcal{P}_\tau$ and $\mathcal{P}_\vartheta$ by setting

$$
\begin{aligned}
d_0(\tau_{01}, \vartheta_{01}; \tau_{02}, \vartheta_{02}) &:= \sum_{r=0}^{\infty} \frac{2^{-r}(\|\tau_{01} - \tau_{02}\|_{C^r} + \|\vartheta_{01} - \vartheta_{02}\|_{C^r})}{1 + \|\tau_{01} - \tau_{02}\|_{C^r} + \|\vartheta_{01} - \vartheta_{02}\|_{C^r}}, \\
d_\tau(\vec{\tau}_1, \vec{\tau}_2) &:= \sum_{r=0}^{\infty} \frac{2^{-r}\|\vec{\tau}_1 - \vec{\tau}_2\|_{\ell^1_\delta(C^r)}}{1 + \|\vec{\tau}_1 - \vec{\tau}_2\|_{\ell^1_\delta(C^r)}}, \\
d_\vartheta(\vec{\vartheta}_1, \vec{\vartheta}_2) &:= \sum_{r=0}^{\infty} \frac{2^{-r}\|\vec{\vartheta}_1 - \vec{\vartheta}_2\|_{\ell^1_\delta(C^r)}}{1 + \|\vec{\vartheta}_1 - \vec{\vartheta}_2\|_{\ell^1_\delta(C^r)}},
\end{aligned}
$$

and observe that $\mathcal{P}_0$, $\mathcal{P}_\tau$ and $\mathcal{P}_\vartheta$ are complete metric spaces, with the above metrics inducing the C^∞ topologies. Thus, $\mathcal{P}$ is a complete metric space with respect to the product metric $d = d_0 \times d_\tau \times d_\vartheta$, which induces the C^∞ topology on $\mathcal{P}$. The C^r topology of $\mathcal{P}^r$ is induced by the product metric $d^r := d^r_0 \times d^r_\tau \times d^r_\vartheta$.

Let $\mathcal{P}_{\mathrm{reg}} \subset \mathcal{P}$ be the subspace of parameters for which the zero set of $\mathfrak{S}_{\mathbf{p}}$ is regular, and note that

$$
\mathcal{P}_{\mathrm{reg}} = \bigcap_{n \geq 1} \mathcal{P}_{n,\mathrm{reg}},
$$

where $\mathcal{P}_{n,\mathrm{reg}} \subset \mathcal{P}_{\mathrm{reg}}$ is the subspace of C^∞ parameters $\mathbf{p}$ such that the differential $D\mathfrak{S}_{\mathbf{p}}$ is surjective for all pairs (A,Φ) in $\mathfrak{S}_{\mathbf{p}}^{-1}(0) \cap \tilde{\mathcal{C}}^{*,0}_{W,E}$, satisfying

$$\|F_A\|_{L^p} \leq n \quad \text{and} \quad \nu_0(A,\Phi;\mathbf{p}) \geq 1/n, \tag{5.5}$$

for some $p > 2$, where $\nu_0(A,\Phi;\mathbf{p})$ is the least eigenvalue of the Laplacian $\Delta^0_{A,\Phi;\mathbf{p}} := d^{0,*}_{A,\Phi} d^0_{A,\Phi}$ computed with respect to the metric g. Define, in the analogous way, the subspaces $\mathcal{P}^r_{\mathrm{reg}}$ and $\mathcal{P}^r_{n,\mathrm{reg}}$ of $\mathcal{P}^r$.

The uniform upper L^p bound on F_A precludes bubbling, while the uniform lower bound on $\nu_0(A,\Phi)$ keeps (A,Φ) bounded away from the reducible or zero-section pairs. Let $\nu_2(A,\Phi;\mathbf{p})$ be the least eigenvalue of the Laplacian $\Delta^2_{A,\Phi;\mathbf{p}} := d^1_{A,\Phi} d^{1,*}_{A,\Phi}$ computed with respect to the parameters $\mathbf{p} = (\tau_0, \vartheta_0, \vec{\tau}, \vec{\vartheta})$. Then $(D\mathfrak{S}_{\mathbf{p}})_{A,\Phi}$ is surjective if and only if $\nu_2(A,\Phi;\mathbf{p}) > 0$.

Claim 5.4. The subspace $\mathcal{P}_{n,\mathrm{reg}} \subset \mathcal{P}$ is open in the C^∞ topology and $\mathcal{P}^r_{n,\mathrm{reg}} \subset \mathcal{P}^r$ is open in the C^r topology.

Proof. Let $\{\mathbf{p}_\alpha\}_{\alpha=1}^\infty \subset \mathcal{P} \setminus \mathcal{P}_{n,\mathrm{reg}}$ be a sequence of parameters and suppose that $\mathbf{p}_\alpha$ converges to $\mathbf{p} \in \mathcal{P}$ in the C^∞ topology. Then there is a sequence of solutions (A_α, Φ_α) to (2.27) in $\tilde{\mathcal{C}}^{*,0}_{W,E}$, with parameters $\mathbf{p}_\alpha$, which satisfy the bounds in (5.5) and for which $\nu_2(A_\alpha,\Phi_\alpha;\mathbf{p}_\alpha) = 0$. Proposition 4.24 and the L^p bounds in (5.5) imply that, after passing to a subsequence, there is a sequence of gauge transformations $\{u_\alpha\} \subset \mathcal{G}_E$ such that $u_\alpha(A_\alpha,\Phi_\alpha)$ converges (strongly) in L^2_k to a solution (A,Φ) in $\tilde{\mathcal{C}}^{*,0}_{W,E}$ to (2.27) which satisfies the curvature bound in (5.5). Standard perturbation theory implies that the eigenvalues $\nu_i(u_\alpha(A_\alpha,\Phi_\alpha);\mathbf{p}_\alpha)$ converge to $\nu_i(A,\Phi;\mathbf{p})$ for $i = 0, 2$ [41], so that the triple $(A,\Phi;\mathbf{p})$ satisfies the eigenvalue bound in (5.5). The eigenvalue $\nu_2(A,\Phi;\mathbf{p})$ must be zero — otherwise the eigenvalues $\nu_2(u_\alpha(A_\alpha,\Phi_\alpha);\mathbf{p}_\alpha)$ would be positive for large enough α. Hence, $\mathbf{p} \notin \mathcal{P}_{n,\mathrm{reg}}$ and so $\mathcal{P}_{n,\mathrm{reg}}$ is open. The proof that $\mathcal{P}^r_{n,\mathrm{reg}}$ is an open subset of $\mathcal{P}^r$ is identical. q.e.d.

Claim 5.5. The subspace $\mathcal{P}_{n,\mathrm{reg}} \subset \mathcal{P}$ is dense in the C^∞ topology.

Proof. By Corollary 5.3, the space $\mathcal{P}^r_{\mathrm{reg}}$ is the complement in $\mathcal{P}^r$ of a first-category subset and so is dense by Baire's theorem; clearly, $\mathcal{P}^r_{n,\mathrm{reg}}$ is also dense in $\mathcal{P}^r$, since $\mathcal{P}^r_{\mathrm{reg}} \subset \mathcal{P}^r_{n,\mathrm{reg}}$. Let $\mathbf{p} \in \mathcal{P}$ be a C^∞ parameter, and let $\{\mathbf{p}_\alpha\} \subset \mathcal{P}^r_{n,\mathrm{reg}}$ be a sequence of C^r parameters such that $d^r(\mathbf{p},\mathbf{p}_\alpha) < 2^{-\alpha-1}$, so $\mathbf{p}_\alpha$ converges in C^r to $\mathbf{p}$. Since $\mathcal{P}^r_{n,\mathrm{reg}} \subset \mathcal{P}^r$ is open by Claim 5.4 and the C^∞ parameters $\mathcal{P}$ are dense in $\mathcal{P}^r_{n,\mathrm{reg}}$, we may choose, for each α, a C^∞ parameter $\mathbf{p}'_\alpha \in \mathcal{P}^r_{n,\mathrm{reg}}$ such that $d^r(\mathbf{p}_\alpha,\mathbf{p}'_\alpha) <$

$2^{-\alpha-1}$. Since $\mathbf{p}'_\alpha$ is C^∞, then $\{\mathbf{p}'_\alpha\} \subset \mathcal{P}_{n,\text{reg}}$ and by construction we have $d^r(\mathbf{p}, \mathbf{p}'_\alpha) < 2^{-\alpha}$ and so the sequence $\{\mathbf{p}'_\alpha\}$ converges in C^r to $\mathbf{p} \in \mathcal{P}$.

Therefore, for each r, we obtain a sequence $\{\mathbf{p}'_\alpha(r)\} \subset \mathcal{P}_{n,\text{reg}}$ which converges in C^r to $\mathbf{p} \in \mathcal{P}_{\text{reg}}$. But then the diagonal sequence $\{\mathbf{p}'_\alpha(\alpha)\} \subset \mathcal{P}_{n,\text{reg}}$ converges to $\mathbf{p}$ in C^r for each r (it satisfies $d^r(\mathbf{p}, \mathbf{p}'_\alpha(\alpha)) < 2^{-\alpha}$ for all $\alpha \geq r$) and so the sequence converges in C^∞ to $\mathbf{p} \in \mathcal{P}$, as required. q.e.d.

From Claims 5.4 and 5.5 we conclude that $\mathcal{P}_{\text{reg}}$ is a countable intersection of dense, open subsets of $\mathcal{P}$ and hence is the complement of a first-category subset (in particular, the subset $\mathcal{P}_{\text{reg}} \subset \mathcal{P}$ is dense by Baire's theorem). Hence, the space $\mathcal{P}_{\text{reg}}$ of C^∞ parameters $(\tau_0, \vartheta_0, \vec{\tau}, \vec{\vartheta})$ such that the moduli spaces $M^{*,0}_{W,E}(\tau_0, \vartheta_0, \vec{\tau}, \vec{\vartheta})$ are regular is the complement of a first-category subset of $\mathcal{P}$. From this and Corollary 5.3 we conclude:

Corollary 5.6. *Let X be a closed, oriented, smooth four-manifold with C^∞ metric g. There is a first-category subset of the space $\mathcal{P}$, such that for all C^∞ perturbations $(\tau_0, \vartheta_0, \vec{\tau}, \vec{\vartheta})$ in the complement of this subset, the zero locus of the section $\mathfrak{S}$ is regular and so the moduli space $M^{*,0}_{W,E}(\tau_0, \vartheta_0, \vec{\tau}, \vec{\vartheta}) = \mathfrak{S}^{-1}(0)$ is a smooth submanifold of $\mathcal{C}^{*,0}_{W,E}$ with the expected dimension.*

Remark 5.7. The same argument shows that the standard Freed-Uhlenbeck generic metrics theorems (specifically, Corollaries 4.3.15, 4.3.18, and 4.3.19 in [20] and the refinement Lemma 2.4 in [48] for the non-simply connected case) for the moduli spaces of anti-self-dual connections on an SU(2) or SO(3) bundle over X continue to hold for the complement of a first-category subset of C^∞ metrics, rather than just C^r metrics.

5.1.3. Simultaneous transversality for the top and lower-level moduli spaces. Let Σ be a smooth stratum of $\operatorname{Sym}^\ell(X)$. Recall from §4.5.2 that a universal choice of a sufficiently large constant N_b guarantees that if $[A, \Phi, \mathbf{x}]$ is any point in $\bar{M}_{W,E}$ and A is irreducible, then A has at least one ball $B(x_j, R_0)$ which supports holonomy perturbations.

The PU(2) monopole equations cutting out the locus

$$\mathbf{M}^{*,0}_{W,E_{-\ell}}|_\Sigma \subset \mathcal{C}^{*,0}_{W,E_{-\ell}} \times \Sigma$$

from the Uhlenbeck compactification $\bar{M}_{W,E}$ are equations for triples

$(A,\Phi,\mathbf{x}) \in \tilde{\mathcal{C}}^{*,0}_{W,E_{-\ell}} \times \Sigma$. We can again define a ${}^\circ\mathcal{G}_E$-equivariant C^∞ map

$$\underline{\mathfrak{S}} : \mathcal{P}^r \times \tilde{\mathcal{C}}^{*,0}_{W,E_{-\ell}} \times \Sigma \to L^2_{k-1}(\Lambda^+ \otimes \mathfrak{su}(E)) \oplus L^2_{k-1}(W^- \otimes E)$$

by setting

$$\begin{aligned} &\underline{\mathfrak{S}}(\tau_0, \vartheta_0, \vec{\tau}, \vec{\vartheta}, A, \Phi, \mathbf{x}) \\ &:= \begin{pmatrix} F_A^+ - \left(\mathrm{id} + \tau_0 \otimes \mathrm{id}_{\mathfrak{su}(E)} + \vec{\tau} \cdot \vec{\mathfrak{m}}(A,\mathbf{x})\right) \rho^{-1}(\Phi \otimes \Phi^*)_{00} \\ D_A\Phi + \rho(\vartheta_0)\Phi + \vec{\vartheta} \cdot \vec{\mathfrak{m}}(A,\mathbf{x})\Phi \end{pmatrix}. \end{aligned}$$

The proof of Corollary 5.6 now shows that $\mathbf{M}^{*,0}_{W,E_{-\ell}}|_\Sigma \subset \mathcal{C}^{*,0}_{W,E_{-\ell}} \times \Sigma$ is a smooth submanifold of the expected dimension for generic parameters $(\tau_0, \vartheta_0, \vec{\tau}, \vec{\vartheta})$,

$$\dim \mathbf{M}^{*,0}_{W,E_{-\ell}}|_\Sigma = \dim M^{*,0}_{W,E_{-\ell}} + \dim \Sigma. \tag{5.6}$$

Furthermore, by considering regular values of the projection maps onto the second factors Σ, Sard's Theorem also shows that the fibers $M^{*,0}_{W,E_{-\ell}}|_{\mathbf{x}}$ are smooth manifolds of the expected dimension for generic points $\mathbf{x} \in \mathrm{Sym}^\ell(X)$. Indeed, the only tangent vectors in each stratum Σ, which might not appear in the image of the projection, are those arising from the radial vector on the annuli $B(x_j, 4R_0)\backslash \bar{B}(x_j, 2R_0)$. This observation shows that the projection from $\mathbf{M}^{*,0}_{W,E_{-\ell}}$ to Σ is transverse to certain submanifolds of Σ which would allow dimension-counting arguments similar to those in [25]. Issues related to dimension-counting in the presence of holonomy perturbations are also discussed by Donaldson in [18, pp. 282–287]. We can now conclude the proof of our main transversality result:

Proof of Theorem 1.3, given Corollary 5.6. For the case $\ell = 0$, the transversality assertion is given by Corollary 5.6 and the dimension formula is provided by Proposition 2.28. The case $\ell > 0$ then follows from the discussion in the preceding paragraphs. q.e.d.

5.2. Smoothness of the parametrized moduli space. We prove Theorem 5.2 by showing that the ${}^\circ\mathcal{G}_E$-equivariant map

$$\underline{\mathfrak{S}} : \mathcal{P}^r \times \tilde{\mathcal{C}}^{*,0}_{W,E} \to L^2_{k-1}(\Lambda^+ \otimes \mathfrak{su}(E)) \oplus L^2_{k-1}(W^+ \otimes E)$$

vanishes transversely, and so the parametrized moduli space $\mathfrak{M}^{*,0}_{W,E} = \underline{\mathfrak{S}}^{-1}(0)/{}^\circ\mathcal{G}_E$ is a smooth Banach manifold. The broad strategy is reminiscent of that of [30, Chapter 3] and [20, §4.3.5], where the analogous result is established for the moduli space of anti-self-dual connections parametrized by the Banach space of C^r (conformal classes of) metrics: our proof of surjectivity of the differential $D\underline{\mathfrak{S}}$ at a point $(\tau_0, \vartheta_0, \vec{\tau}, \vec{\vartheta}, A, \Phi)$ in the zero set $\underline{\mathfrak{S}}^{-1}(0)$ ultimately relies on the fact that, for $\Phi \not\equiv 0$, a monopole (A, Φ) which is reducible on an *admissible* open subset of X is necessarily reducible over all of X (Theorem 5.11). If there are sections $\mathfrak{m}_{j,l,\alpha}(A)$ which are non-zero on $B(x_j, R_0)$, we say that the connection A has *holonomy perturbations supported on* $B(x_j, R_0)$; the set $\{\mathfrak{m}_{j,l,\alpha}(A)\}_{l=1}^3$ then spans $\mathfrak{su}(E)|_{B(x_j,R_0)}$ for at least one index α. Then an admissible open set for the pair (A, Φ) is one containing the $\bar{B}(x_j, R_0)$ for all j such that $\beta_j[A] > 0$. (The supports of all the sections $\mathfrak{m}_{j,l,\alpha}(A)$ are contained in $\cup_{j=1}^{N_b} \bar{B}(x_j, R_0)$, and so any open subset of X containing $\cup_{j=1}^{N_b} \bar{B}(x_j, R_0)$ is admissible.) Recall from §4.5.2 that because of our choice of constant N_b, if $[A, \Phi]$ is any point in $M^*_{W,E}$, then at least one ball $B(x_j, R_0)$ supports holonomy perturbations for A.

Theorem 5.2 is an almost immediate consequence of

Proposition 5.8. *Suppose* $(\tau_0, \vartheta_0, \vec{\tau}, \vec{\vartheta}, A, \Phi)$ *is a point in* $\underline{\mathfrak{S}}^{-1}(0)$ *with* A *irreducible and* $\Phi \not\equiv 0$. *If* (v, ψ) *is in the cokernel of the differential* $D\underline{\mathfrak{S}}$ *at the point* $(\tau_0, \vartheta_0, \vec{\tau}, \vec{\vartheta}, A, \Phi)$, *then* $(v, \psi)|_{B(x_j,R_0)} \equiv 0$ *for each ball* $B(x_j, R_0)$ *supporting holonomy perturbations for* A.

We first observe that the elements of the cokernel of

$$D\underline{\mathfrak{S}} := D\underline{\mathfrak{S}}_{(\tau_0,\vartheta_0,\vec{\tau},\vec{\vartheta},A,\Phi)}$$

have a restricted form of the unique continuation property (sufficient for our purposes) by Aronszajn's theorem [5]:

Lemma 5.9. *If* $(v, \psi) \in \operatorname{Ker} D\underline{\mathfrak{S}}(D\underline{\mathfrak{S}})^*$ *and* $(v, \psi)|_U \equiv 0$ *on some non-empty open subset* $U \subset X$ *containing all balls* $B(x_j, R_0)$ *supporting holonomy perturbations for* A, *then* $(v, \psi) \equiv 0$ *on* X.

Proof. By hypothesis, the pair (v, ψ) solves the second-order elliptic equation

$$(D\underline{\mathfrak{S}})(D\underline{\mathfrak{S}})^*(v, \psi) = 0 \quad \text{on } X,$$

where the Laplacian $D\underline{\mathfrak{S}} = (D\underline{\mathfrak{S}}_1, D\underline{\mathfrak{S}}_2)$, given by equations (4.3) and (4.4), has C^{r-1} coefficients and (v, ψ) is at least C^{r+1}. Also, $(v, \psi) \equiv 0$ on the set of closed balls supporting holonomy perturbations,

$$\bar{B}_I(A) := \bigcup_{j \in I(A)} \bar{B}(x_j, R_0),$$

where $I(A) := \{j : 1 \le j \le N_b, \beta_j[A] > 0, \text{ and } A|_{B(x_j,2R_0)} \text{ is irreducible}\}$. Now, on the subset $X - B_{I(A)}$ where all of the holonomy perturbations and especially their derivatives with respect to A vanish (see their definition in §2.5.2), the Laplacian $(D\underline{\mathfrak{S}})(D\underline{\mathfrak{S}})^*$ is a purely differential operator. In particular, it extends to a differential operator with C^{r-1} coefficients over X, say $(D\underline{\mathfrak{S}}^0)(D\underline{\mathfrak{S}}^0)^*$, given by the linearized PU(2) monopole equations (5.3) and (5.4) with all terms involving holonomy perturbations and their derivatives set equal to zero. On the other hand, the pair (v,ψ) also solves the resulting second-order elliptic *differential* equation

$$(D\underline{\mathfrak{S}}^0)(D\underline{\mathfrak{S}}^0)^*(v,\psi) = 0 \quad \text{on } X,$$

since $(v,\psi)=0$ on $\bar{B}_I(A)$ and $D\underline{\mathfrak{S}}^0 \neq D\underline{\mathfrak{S}}$ only on $B_I(A)$, while $D\underline{\mathfrak{S}}^0 = D\underline{\mathfrak{S}}$ on $X - B_{I(A)}$.

Without loss of generality, we may scale the Dirac equation in (2.21) by $1/\sqrt{2}$. We then have

$$D\underline{\mathfrak{S}}^0(D\underline{\mathfrak{S}}^0)^* = \begin{pmatrix} d_A^+ d_A^{+,*} & 0 \\ 0 & \frac{1}{2} D_A D_A^* \end{pmatrix} + \text{First-order differential terms},$$

and so by the Bochner formulas of Lemma 4.1 and [30, Eq. (6.26)], the Laplacian $D\underline{\mathfrak{S}}^0(D\underline{\mathfrak{S}}^0)^*$ is a second order elliptic differential operator with scalar principal symbol (given by the metric $\frac{1}{2}g$ on T^*X). The desired conclusion then follows from Aronszajn's unique continuation theorem [5]. q.e.d.

Remark 5.10.

1. Aronszajn's theorem does not apply without the given restriction on the open set U in the statement of Lemma 5.9, as the Laplacian $(D\underline{\mathfrak{S}})(D\underline{\mathfrak{S}})^*$ is not a purely differential operator over all of X. One can see from equations (5.3) and (5.4) that the problem terms are those appearing in the last line of each displayed equation: the operator $\delta\mathfrak{m}_{j,\ell,\alpha}/\delta A$ acting on $a \in \Omega^1(\mathfrak{su}(E))$ is an integral operator, as is clear from the formula (A.7) for the differential of the holonomy with respect to the connection.

2. Lemma 5.9 can also be proved without using Aronszajn's theorem explicitly and instead applying the Agmon-Nirenberg unique continuation theorem (Theorem 5.25) to the equation $(D\underline{\mathfrak{S}})^*(v,\psi) = 0$, and mimicking the existing application in the proof of Theorem 5.11. Indeed, this second proof of Lemma 5.9 is virtually identical to the proof of Theorem 5.11. We leave the details to the interested reader, as the preceding use of Aronszajn's theorem appears

easier to us. We note that Aronszajn's theorem can be derived from that of Agmon-Nirenberg (see [3]).

If (v, ψ) is an L^2_{k-1} element of the cokernel of $D\underline{\mathfrak{S}}$, then elliptic regularity for the Laplacian $D\underline{\mathfrak{S}}(D\underline{\mathfrak{S}})^*$, with C^{r-1} coefficients, implies that (v, ψ) is in C^{r+1}. Our proof of Theorem 5.2 also relies on the following 'unique continuation' result for reducible monopoles:

Theorem 5.11. *If (A, Φ) is a C^r solution to the perturbed* PU(2) *monopole equations* (2.27) *with $\Phi \not\equiv 0$ over a connected, oriented, smooth four-manifold X with C^r Riemannian metric, and (A, Φ) is reducible on a non-empty open subset $U \subset X$ with $\bar{B}(x_j, R_0) \subset U$ for all j such that $\beta_j[A] > 0$, then (A, Φ) is reducible on X.*

The proof of Theorem 5.11 is lengthy, so we defer it to §5.3.

Proof of Theorem 5.2, given Proposition 5.8 and Theorem 5.11. Let $(\tau_0, \vartheta_0, \vec{\tau}, \vec{\vartheta}, A, \Phi)$ be a C^r representative for a point in $\mathfrak{M}^{*,0}_{W,E}$, so that A is irreducible and $\Phi \not\equiv 0$, and suppose (v, ψ) is in the cokernel of $D\underline{\mathfrak{S}}$. By definition of N_b in §2.5.2, the set

$$J(A) := \{j : 1 \leq j \leq N_b \text{ and } \beta_j[A] > 0\}$$

is non-empty. Since A is irreducible on X, Theorem 5.11 implies that $A|_{B(x_{j'}, 2R_0)}$ must be irreducible for some $j' \in \{1, \dots, N_b\}$ such that $\beta_{j'}[A] > 0$; otherwise, $A|_{B(x_{j'}, 2R_0)}$ would be reducible for all j such that $\beta_j[A] > 0$, and Theorem 5.11 would imply that A would be reducible over all of X, contradicting our assumption that A is irreducible. By Proposition 5.8 we see that $(v, \psi)|_{B(x_j, R_0)} \equiv 0$ for all j such that $\beta_j[A] > 0$ and $A|_{B(x_j, 2R_0)}$ is irreducible, so $(v, \psi) \equiv 0$ on X by Lemma 5.9.

q.e.d.

The proof of Proposition 5.8 occupies the remainder of this subsection. We first note that since Φ is in the kernel $D_A + \vec{\vartheta} \cdot \vec{\mathfrak{m}}(A)$, it has the unique continuation property by Aronszajn's Theorem [5]:

Lemma 5.12. *If $(D_A + \rho(\vartheta_0) + \vec{\vartheta} \cdot \vec{\mathfrak{m}}(A))\Phi = 0$ and $\Phi|_U \equiv 0$ for some non-empty open subset $U \subset X$, then $\Phi \equiv 0$.*

Proof. The perturbed Dirac operator $D_A + \rho(\vartheta_0) + \vec{\vartheta} \cdot \vec{\mathfrak{m}}(A)$ differs from D_A by a zeroth order term and so

$$\begin{aligned}(D_A + \rho(\vartheta_0) + \vec{\vartheta} \cdot \vec{\mathfrak{m}}(A))^*(D_A + \rho(\vartheta_0) + \vec{\vartheta} \cdot \vec{\mathfrak{m}}(A)) \\ = D_A^* D_A + \text{First order terms.}\end{aligned}$$

The Bochner formula of Lemma 4.1 then implies that the above Laplacian is a second order elliptic differential operator with scalar principal symbol (given by the metric g on T^*X). The conclusion now follows from Aronszajn's unique continuation theorem [5]. q.e.d.

We shall use the following linear algebra result to show that $v \in C^{r+1}(X, \Lambda^+ \otimes \mathfrak{su}(E))$ vanishes on a ball:

Lemma 5.13. *Let M, N be elements of $(\Lambda^+ \otimes \mathfrak{su}(E))|_x$. Suppose $\mathfrak{m}_1, \mathfrak{m}_2, \mathfrak{m}_3$ span $\mathfrak{su}(E)|_x$. If*

$$\langle \tau_0 M, N\rangle + \sum_{l=1}^{3} \langle (\tau_l \otimes \mathrm{ad}(\mathfrak{m}_l))M, N\rangle = 0 \tag{5.7}$$

for all $\tau_0, \tau_1, \tau_2, \tau_3 \in \mathfrak{gl}(\Lambda^+)|_x$, then either $M = 0$ or $N = 0$.

Proof. If $\langle \tau_0 M, N\rangle = 0$ for all $\tau_0 \in \mathfrak{gl}(\Lambda^+)|_x$, then by the proof of [30, Lemma 3.7], the images in $\mathfrak{su}(E)|_x$ of $M, N \in \mathrm{Hom}(\Lambda^+|_x, \mathfrak{su}(E)|_x)$ are orthogonal. (Although their lemma refers to an element of $\mathrm{Hom}(\Lambda^+|_x, \mathfrak{su}(E)|_x)$ and an element of $\mathrm{Hom}(\Lambda^-|_x, \mathfrak{su}(E)|_x)$, we can choose any isomorphism between $\Lambda^+|_x$ and $\Lambda^-|_x$ to translate the result.) We can therefore assume that M has rank one and N has rank less than or equal to two. (If M is rank two and N is rank one, we can reverse their roles by using adjoints.) If both $M \neq 0$ and $N \neq 0$ let

$$M = u \otimes m \quad \text{and} \quad N = v_1 \otimes n_1 + v_2 \otimes n_2,$$

where $u, v_1, v_2 \in \Lambda^+|_x$ and $m, n_1, n_2 \in \mathfrak{su}(E)|_x$. Since the images of M and N in $\mathfrak{su}(E)|_x$ are orthogonal, we have $\langle m, n_1\rangle = 0 = \langle m, n_2\rangle$; without loss of generality, we can assume that $\langle n_1, n_2\rangle = 0$. (If N is rank one, n_2 can be any element of $\mathfrak{su}(E)|_x$ completing m, n_1 to an orthogonal basis of $\mathfrak{su}(E)|_x$.) Under the isomorphism $\mathfrak{su}(E)|_x \simeq \mathbb{R}^3$, the adjoint representation is given by the cross-product. We can find $f_l \in \mathbb{R}$ such that $n_2 = \sum_l f_l \mathfrak{m}_l$, so

$$\sum_{l=1}^{3} \langle \mathrm{ad}(f_l \mathfrak{m}_l) m, n_1\rangle = \langle [n_2, m], n_1\rangle \neq 0, \tag{5.8}$$

$$\sum_{l=1}^{3} \langle \mathrm{ad}(f_l \mathfrak{m}_l) m, n_2\rangle = \langle [n_2, m], n_2\rangle = 0. \tag{5.9}$$

By assumption, $M \neq 0$ and $N \neq 0$, so $u \neq 0$ and either $v_1 \neq 0$ or $v_2 \neq 0$; we may suppose without loss of generality that $v_1 \neq 0$. Thus, we can find $\tau \in \mathfrak{gl}(\Lambda^+)|_x$ such that $\tau u = v_1$, and so choosing $\tau_l = f_l \tau$ for $l = 1,2,3$ and $\delta\tau_0 = 0$, we have

$$\begin{aligned}\sum_{l=1}^3 \langle (\tau_l \otimes \mathrm{ad}(\mathfrak{m}_l))\, M, N\rangle &= \sum_{l=1}^3 \langle \tau \otimes \mathrm{ad}(f_l \mathfrak{m}_l) M, N\rangle \\ &= \langle (\tau \otimes \mathrm{ad}(n_2))M, N\rangle \\ &= \langle (\tau \otimes \mathrm{ad}(n_2))(u \otimes m), v_1 \otimes n_1 + v_2 \otimes n_2\rangle \\ &= \langle \tau u \otimes [n_2, m], v_1 \otimes n_1 + v_2 \otimes n_2\rangle \\ &= \langle \tau u, v_1\rangle \langle [n_2, m], n_1\rangle + \langle \tau u, v_2\rangle\langle [n_2,m], n_2\rangle \\ &= |v_1|^2 \langle [n_2, m], n_1\rangle \neq 0 \quad \text{by (5.8) and (5.9)},\end{aligned}$$

contradicting our hypothesis in (5.7). Hence, either $M = 0$ or $N = 0$, as desired. q.e.d.

Remark 5.14. Lemma 5.13 does not hold if the rank of E is greater than two. If $a, b \in \mathfrak{su}(E)$, the above arguments would only allow one to conclude that

$$0 = \langle [\mathfrak{m}, a], b\rangle = \langle \mathfrak{m}, [a, b]\rangle$$

for all $\mathfrak{m} \in \mathfrak{su}(E)$, so $[a, b] = 0$ and a, b are simultaneously diagonalizable. The subspace of diagonal elements of $\mathfrak{su}(n)$ has dimension $n-1$, so this would not contradict the orthogonality of a, b if $n > 2$.

Lemma 5.15. *Continue the hypotheses of Proposition 5.8 and suppose $B(x_j, R_0)$ is a ball supporting holonomy perturbations for A. Then $v \equiv 0$ on $B(x_j, R_0)$.*

Proof. By hypothesis, there are holonomy sections $\mathfrak{m}_l \equiv \mathfrak{m}_{j,l,\alpha}(A)$, $l = 1,2,3$, which span $\mathfrak{su}(E)|_y$, for any point $y \in B(x_j, R_0)$. Let $\delta\tau_l := \delta\tau_{j,l,\alpha} \in \Omega^0(\mathfrak{gl}(\Lambda^+))$, $l = 1,2,3$, denote the corresponding coefficients, and let $\delta\vec\tau$ be a sequence with all other coefficients equal to zero.

By the hypothesis of Proposition 5.8, we have

$$(D\underline{\mathfrak{S}}(\delta\tau_0, \delta\vartheta_0, \delta\vec\tau, \delta\vec\vartheta, a, \phi), (v, \psi))_{L^2} = 0$$

for all $(\delta\tau_0, \delta\vartheta_0, \delta\vec\tau, \delta\vec\vartheta, a, \phi)$, and so

$$\begin{aligned} 0 &= (D\underline{\mathfrak{S}}(\delta\tau_0, 0, \delta\vec\tau, 0, 0, 0), (v, \psi))_{L^2} \\ &= (D\underline{\mathfrak{S}}_1(\delta\tau_0, 0, \delta\vec\tau, 0,0,0), v)_{L^2} \\ &= ((\delta\tau_0 \otimes \mathrm{id}_{\mathfrak{su}(E)} + \delta\vec\tau \cdot \vec{\mathfrak{m}}(A))\rho^{-1}(\Phi \otimes \Phi^*)_{00}), v)_{L^2} \\ &= (\delta\tau_0 \rho^{-1}(\Phi\otimes\Phi^*)_{00}, v)_{L^2} + \sum_{l=1}^3 (\delta\tau_l \otimes \mathrm{ad}(\mathfrak{m}_l)\rho^{-1}(\Phi\otimes\Phi^*)_{00}, v)_{L^2}.\end{aligned}$$

Taking a sequence of $\delta\tau_l$'s which approximate $\delta\tau_{l,y}\delta(\cdot,y)$, where $\delta(\cdot,y)$ is the Dirac delta distribution supported at y and $\delta\tau_{l,y}\in\mathfrak{gl}(\Lambda^+)|_y$, we obtain the pointwise identity

$$\langle\delta\tau_{0,y}\rho^{-1}(\Phi\otimes\Phi^*)_{00}|_y,v|_y\rangle+\sum_{l=1}^3\langle\delta\tau_{l,y}\otimes\operatorname{ad}(\mathfrak{m}_l)\rho^{-1}(\Phi\otimes\Phi^*)_{00}|_y,v|_y\rangle=0,$$

for all $\delta\tau_{l,y}\in\mathfrak{gl}(\Lambda^+)|_y$, $l=0,1,2,3$. Lemma 5.13 then implies that either $\rho^{-1}(\Phi\otimes\Phi^*)_{00}|_y=0$, and thus $\Phi|_y=0$ by Lemma 2.19, or else $v|_y=0$. If $v|_y\neq 0$, we see that Φ would be zero on the nonempty open subset $\{v\neq 0\}\cap B(x_j,R_0)$. But then Lemma 5.12 would imply that $\Phi\equiv 0$ on X, contradicting our assumption that $\Phi\not\equiv 0$. Thus, $v\equiv 0$ on $B(x_j,R_0)$, as desired. q.e.d.

The following similar argument shows that $\psi\equiv 0$ on the ball $B(x_j,R_0)$. Note that having only $v\equiv 0$ or $\psi\equiv 0$ on an open set does not suffice to contradict Lemma 5.9, as the non-vanishing result of Lemma 5.9 applies to the pair (v,ψ). We again begin with a linear algebra lemma:

Lemma 5.16. *Let $S^+\in(W^+\otimes E)|_x$ and $S^-\in(W^-\otimes E)|_x$. If $\mathfrak{m}_0,\dots,\mathfrak{m}_3$ span $\mathfrak{u}(E)|_x$ and*

$$\sum_{l=0}^3\langle(\vartheta_l\otimes\mathfrak{m}_l)S^+,S^-\rangle=0$$

for all $\vartheta_0,\dots,\vartheta_3\in\operatorname{Hom}_{\mathbb{C}}(W^+,W^-)|_x$, then $S^+=0$ or $S^-=0$.

Proof. Because $\{\mathfrak{m}_l\}_{l=0}^3$ spans $\mathfrak{u}(E)|_x$ and $\mathfrak{gl}(E)|_x=\mathfrak{u}(E)|_x\oplus i\mathfrak{u}(E)|_x$, we have $\mathfrak{gl}(E)|_x=\mathfrak{u}(E)|_x\otimes_{\mathbb{R}}\mathbb{C}$, and the set $\{\mathfrak{m}_l\}_{l=0}^3$ is a complex basis for $\mathfrak{gl}(E)|_x$. Thus, any element of

$$\operatorname{Hom}_{\mathbb{C}}(W^+\otimes_{\mathbb{C}}E,W^-\otimes_{\mathbb{C}}E)|_x\simeq\operatorname{Hom}_{\mathbb{C}}(W^+,W^-)|_x\otimes_{\mathbb{C}}\mathfrak{gl}(E)|_x$$

can be written as $\sum_{l=0}^3\vartheta_l\otimes\mathfrak{m}_l$, for some $\vartheta_l\in\operatorname{Hom}_{\mathbb{C}}(W^+,W^-)|_x$, $l=0,\dots,3$. Thus, if $S^+\neq 0$, the hypothesis implies that $S^-=0$ and conversely, if $S^-\neq 0$, then $S^+=0$. q.e.d.

Lemma 5.17. *Continue the hypotheses of Proposition 5.8 and suppose $B(x_j,R_0)$ is a ball supporting holonomy perturbations for A. Then $\psi\equiv 0$ on $B(x_j,R_0)$.*

Proof. By hypothesis, there are holonomy sections $\mathfrak{m}_l:=\mathfrak{m}_{j,l,\alpha}(A)$, $l=1,2,3$, which span $\mathfrak{su}(E)|_y$, for any point $y\in B(x_j,R_0)$, and so

$\{\mathfrak{m}_l\}_{l=0}^3$ spans $\mathfrak{u}(E)|_y$, where $\mathfrak{m}_0 := i\cdot\mathrm{id}_E$. Let $\delta\vartheta_l := \delta\vartheta_{j,l,\alpha} \in \Omega^1(X,\mathbb{C})$, $l = 1,2,3$, denote the corresponding coefficients, and let $\delta\vec{\vartheta}$ be a perturbation sequence with all other coefficients equal to zero.

By the hypothesis of Proposition 5.8, we have

$$(D\underline{\mathfrak{S}}(\delta\tau_0, \delta\vartheta_0, \delta\vec{\tau}, \delta\vec{\vartheta}, a, \phi), (v,\psi))_{L^2} = 0$$

for all $(\delta\tau_0, \delta\vartheta_0, \delta\vec{\tau}, \delta\vec{\vartheta}, a, \phi)$, and so

$$\begin{aligned} 0 &= (D\underline{\mathfrak{S}}(0, \delta\vartheta_0, 0, \delta\vec{\vartheta}, 0, 0), (v,\psi))_{L^2} \\ &= (D\underline{\mathfrak{S}}_2(0, \delta\vartheta_0, 0, \delta\vec{\vartheta}, 0, 0), \psi)_{L^2} \\ &= (\rho(\delta\vartheta_0)\Phi + \delta\vec{\vartheta}\cdot\vec{\mathfrak{m}}(A)\Phi, \psi)_{L^2} \\ &= \sum_{l=0}^3 (\rho(\delta\vartheta_l)\otimes\mathfrak{m}_l\Phi, \psi)_{L^2}. \end{aligned}$$

Taking a sequence of $\delta\vartheta_l$'s which approximate $\delta\vartheta_{l,y}\delta(\cdot,y)$, where $\delta(\cdot,y)$ is the Dirac delta distribution supported at y and $\delta\vartheta_{l,y} \in T^*X|_y \otimes \mathbb{C}$, we obtain the pointwise identity

$$\sum_{l=0}^3 \langle\rho(\delta\vartheta_{l,y})\otimes\mathfrak{m}_l\Phi|_y, \psi|_y\rangle = 0,$$

for all $\delta\vartheta_{l,y} \in T^*X|_y\otimes\mathbb{C}$, $l = 0,1,2,3$. Lemma 5.16 then implies that either $\Phi|_y = 0$ or $\psi|_y = 0$. If $\psi|_y \neq 0$, then Φ would be zero on the nonempty open subset $\{\psi \neq 0\}\cap B(x_j, R_0)$. Consequently, Lemma 5.12 would imply that $\Phi \equiv 0$ on X, again contradicting our assumption that $\Phi \not\equiv 0$. Thus, $\psi \equiv 0$ on $B(x_j, R_0)$, as desired. q.e.d.

We can now conclude the proof of Proposition 5.8:

Proof of Proposition 5.8. If (v,ψ) is in the cokernel of $D\underline{\mathfrak{S}}$ and $B(x_j, R_0)$ is a ball supporting holonomy perturbations for A, then $(v,\psi) \equiv 0$ on $B(x_j, R_0)$ by Lemmas 5.15 and 5.17. q.e.d.

5.3. Local reducibility implies global reducibility. The goal of this section is to prove Theorem 5.11. The argument has two main ingredients: a local extension result for stabilizers of pairs which are reducible on a ball and a description of how these local stabilizers fit together to give a stabilizer and thus a reducible pair on the whole manifold.

Remark 5.18. The fact that an anti-self-dual connection which is reducible on an open subset is necessarily reducible on all of X is an essential part of Donaldson and Kronheimer's proof of transversality for the moduli space of anti-self-dual connections in [20, §4.3]. The original argument of Freed and Uhlenbeck [30, pp. 57-58] constructs a parallel section ζ of $\mathfrak{su}(E)$ on the set $\{F_A \neq 0\}$. Because the connection A is anti-self-dual and therefore Yang-Mills, so $d_A^* F_A = 0 = d_A F_A$, the set $\{F_A \neq 0\}$ is open, dense and connected. The section ζ cannot develop any holonomy on $\{F_A = 0\}$, so it extends across all of X, showing that A is globally reducible. This argument does not work in the case of PU(2) monopoles because the connection A is not necessarily Yang-Mills and our argument does not show that the existence of a nonzero element (v, ψ) in $\operatorname{Coker} D\underline{\mathfrak{S}}$ implies that the connection A is reducible on a dense open subset of X.

We first state the local extension result for pair stabilizers and defer its lengthy proof until after that of Theorem 5.11. Generalizations due to Taubes of the analogous result for anti-self-dual connections, namely Lemma 4.3.21 in [20], appear as Theorems 4 and 5 in [88]. As Taubes points out in [88, p. 35], unique continuation theorems for solutions to the anti-self-dual equation do not seem to follow from standard results for elliptic partial differential equations (such as those of Aronszajn [5]) since the anti-self-dual equation does not linearize as an elliptic equation for the connection. Because the PU(2) monopole equations do not linearize as an elliptic system for pairs, the same remarks apply here as well. Rather than rely on the Agmon-Nirenberg theorem for the unique continuation property for a general class of ordinary differential equations (Theorem 5.25), Taubes proves the required unique continuation property directly for the ordinary differential equation induced by the anti-self-dual equation on a cylinder. (As Mrowka pointed out to us, it should also be possible to deduce the unique continuation results of [88] by studying the anti-self-dual equation on a ball and applying the Fredholm theory of [6].) Recall that $B(x_0, r_0) \subset X$ denotes an open geodesic ball with center at the point x_0 and radius r_0. Also, recall that if $[A, \Phi]$ is a point in $M_{W,E}$, then Proposition 3.7 implies that it has a 'smooth' (that is, C^r) representative (A, Φ) solving (2.27).

Proposition 5.19. *Let X be an oriented, smooth four-manifold with C^r Riemannian metric g and injectivity radius $\varrho = \varrho(x_0)$ at a point x_0. Suppose that $0 < r_0 < r_1 \leq \frac{1}{2}\varrho$. Let (A, Φ) be a C^r pair solving the* PU(2) *monopole equations* (2.27) *on X. If u is a C^{r+1} gauge* *transformation of $E|_{B(x_0,r_0)}$ satisfying $u(A,\Phi) = (A,\Phi)$ on $B(x_0,r_0)$, and if either $B(x_j,R_0) \cap B(x_0,r_1) = \emptyset$ or $B(x_j,R_0) \subset B(x_0,r_0)$, for all balls $B(x_j,R_0)$ for which $\beta_j[A] > 0$, then there is an extension of u to a C^{r+1} gauge transformation $\hat{u}$ of $E|_{B(x_0,r_1)}$ with $\hat{u}(A,\Phi) = (A,\Phi)$ on $B(x_0,r_1)$.*

Remark 5.20. This extension result only holds on domains $B(x_0,r_1)\backslash\bar{B}(x_0,r_0)$ where the perturbations vanish because the perturbation terms $\mathfrak{m}_{j,l,\alpha}(A)$ depend not just on the connection A and its derivatives at a point, but rather on the connection A over open neighborhoods in X. Although the unique continuation theorem of [4] does allow certain integral terms, it still does not cover the perturbations we consider here because of their non-local dependence on A.

We digress briefly to introduce some useful facts about stabilizer subgroups of ${}^\circ\mathcal{G}_E$.

Lemma 5.21. *If $u \in \mathrm{Stab}_\Phi$ for $\Phi \in C^0(X, W^+ \otimes E)$ and $u \neq \mathrm{id}_E$, then Φ is rank one. If $u \in S^1_Z$ and $u \neq \mathrm{id}_E$, then $\Phi \equiv 0$ on X.*

Proof. Because u and Φ are continuous, the equality $u\Phi = \Phi$ holds at each point $x \in X$ and so $u|_x \in \mathfrak{gl}(E)|_x$ must be the identity on the image of Φ in $E|_x$. If $\Phi|_x$ is rank two, then $u|_x$ must be the identity on $E|_x$, while if $u|_x \neq \mathrm{id}_{E_x}$, then $\Phi|_x$ can be at most rank one. If $u \in S^1_Z$, then $u\Phi = e^{i\theta}\Phi = \Phi$ and so $\Phi = 0$. q.e.d.

Next we consider the stabilizers of reducible pairs. Recall from [44, Chapter II], [61, §III.3.3] that the stabilizer $\mathrm{Stab}_A \subset {}^\circ\mathcal{G}_E$ may be identified with a subgroup of $\mathrm{Aut}(E|_x)$, for any point $x \in X$, by parallel translation with respect to the connection A and hence identified with a subgroup of $\mathrm{U}(2)$ by choosing an orthonormal frame for $E|_x$; these subgroups are again denoted by Stab_A. If $E = L_1 \oplus L_2$, let $S^1_{L_1}$ denote the group of gauge transformations given by $e^{i\theta}\mathrm{id}_{L_1} \oplus \mathrm{id}_{L_2}$.

Lemma 5.22. *Let (A,Φ) be a $\mathrm{PU}(2)$ monopole in C^r on $(E, W^+ \otimes E)$. Let $U \subset X$ be a connected open set, with $U \cap B(x_j,R_0)$ empty or $B(x_j,2R_0) \subset U$ for all balls $B(x_j,R_0)$ such that $\beta_j[A] > 0$. If $\Phi \not\equiv 0$ and $A|_U$ is reducible, then $A|_U$ is reducible with respect to a splitting $E|_U = L_1 \oplus L_2$ where Φ is rank one on U, with image contained either in L_1 or in L_2. In the first case, $\mathrm{Stab}_{(A,\Phi)|_U} = S^1_{L_2}$, while in the second case, $\mathrm{Stab}_{(A,\Phi)|_U} = S^1_{L_1}$.*

Proof. Because $A|_U$ is reducible, all the holonomy sections $\mathfrak{m}_{j,l,\alpha}(A)$ vanish on U if $B(x_j,2R_0) \subset U$. If $B(x_j,R_0) \cap U$ is empty, the holonomy sections also vanish on U, as they are supported on $B(x_j, R_0)$. Since $\Phi \not\equiv 0$, we have $(F_A^+)_0 \not\equiv 0$ on U by Lemma 2.19 and the equation $(F_A^+)_0 = (\mathrm{id} + \tau_0)\rho^{-1}(\Phi \otimes \Phi^*)_{00}$ of (2.27), so $A|_U$ is not projectively flat. Therefore, $\mathrm{Stab}_{A|_U} \simeq T^2$ and $A|_U$ is reducible with respect to a splitting $E|_U = L_1 \oplus L_2$ by Lemma 2.6.

Because the connection $A|_U$ is reducible with respect to the splitting $E|_U = L_1 \oplus L_2$, we can write $A|_U = A_1 \oplus A_2$, where A_1, A_2 are unitary connections on L_1, L_2, so $F_A|_U = F_{A_1} \oplus F_{A_2}$ and

$$(F_A^+)_0 = \begin{pmatrix} \frac{1}{2}(F_{A_1}^+ - F_{A_2}^+) & 0 \\ 0 & -\frac{1}{2}(F_{A_1}^+ - F_{A_2}^+) \end{pmatrix} = \varpi \otimes \sigma_1,$$

where $\varpi = -\frac{i}{2}(F_{A_1}^+ - F_{A_2}^+) \in \Omega^+(U, \mathbb{R})$ and $\sigma_1 \in \mathfrak{su}(2)$ is one of the Pauli matrices (2.14). Hence, $(F_A^+)_0$ is rank one on U, and the equation $(F_A^+)_0 = (\mathrm{id} + \tau_0)\rho^{-1}(\Phi \otimes \Phi^*)_{00}$ implies that $(\Phi \otimes \Phi^*)_{00}$ is rank one on U. Lemma 2.21 then implies that Φ is also rank one on U and so $\Phi|_U = \phi \otimes \xi$ for some $\phi \in C^r\Omega^0(U, W^+)$ and $\xi \in C^r\Omega^0(U, E)$, and

$$(\Phi \otimes \Phi^*)_{00} = -i(\phi \otimes \phi^*)_0 \otimes i(\xi \otimes \xi^*)_0.$$

Writing $\xi = \xi_1 + \xi_2$ for $\xi_j \in C^r\Omega^0(U, L_j)$, we see that

$$(\xi \otimes \xi^*)_0 = \begin{pmatrix} \frac{1}{2}(|\xi_1|^2 - |\xi_2|^2) & \xi_1 \otimes \xi_2^* \\ \xi_2 \otimes \xi_1^* & -\frac{1}{2}(|\xi_1|^2 - |\xi_2|^2) \end{pmatrix}.$$

Since $(\mathrm{id} + \tau_0)\rho^{-1}(\Phi \otimes \Phi^*)_{00} = -(\mathrm{id} + \tau_0)\rho^{-1}(i(\phi \otimes \phi^*)_0) \otimes i(\xi \otimes \xi^*)_0$, we have

$$\begin{aligned}(\mathrm{id} + \tau_0)&\rho^{-1}(\Phi \otimes \Phi^*)_{00} \\ &= -(\mathrm{id} + \tau_0)\rho^{-1}(i(\phi \otimes \phi^*)_0) \otimes \begin{pmatrix} \frac{i}{2}(|\xi_1|^2 - |\xi_2|^2) & i(\xi_1 \otimes \xi_2^*) \\ i(\xi_2 \otimes \xi_1^*) & -\frac{i}{2}(|\xi_1|^2 - |\xi_2|^2) \end{pmatrix},\end{aligned}$$

and by comparison with our matrix expression for $(F_A^+)_0$, we see that $\xi_1 \otimes \xi_2^* = 0$ on U and thus at each point of U, either $\xi_1 = 0$ or $\xi_2 = 0$. Since $\nabla_A \xi = \nabla_{A_1}\xi_1 \oplus \nabla_{A_2}\xi_2$ and because the perturbations vanish on U, the equation $(D_A + \vec{\vartheta} \cdot \vec{\mathfrak{m}}(A))\Phi = 0$ reduces to $(D_{A_1} + \rho(\vartheta_0))(\phi \otimes \xi_1) = 0$ and $(D_{A_2} + \rho(\vartheta_0))(\phi \otimes \xi_2) = 0$. The unique continuation result for the perturbed Dirac operator, Lemma 5.12, implies that if $\phi \otimes \xi_1$ vanishes on an open subset of U, then $\phi \otimes \xi_1 \equiv 0$ on U, and similarly for $\phi \otimes \xi_2$. If ξ_1 is non-zero at a point and thus non-zero on an open neighborhood, ξ_2 vanishes on this open set and by unique continuation, $\xi_2 \equiv 0$ on all of the connected set U. Symmetrically, if ξ_2 is non-zero at a point, then $\xi_1 \equiv 0$ on U. Thus, $\Phi = \phi \otimes \xi_1$ or $\Phi = \phi \otimes \xi_2$.

The stabilizer of $A|_U$ is $S^1_{L_1} \times S^1_{L_2}$. If $\xi_1 = 0$, then $\mathrm{Stab}_{\Phi|_U} = \mathrm{Map}(U, S^1_{L_1})$ while if $\xi_2 = 0$, then $\mathrm{Stab}_{\Phi|_U} = \mathrm{Map}(U, S^1_{L_2})$. q.e.d.

We see that elements of the stabilizer of a pair cannot exhibit holonomy, in the sense of the following lemma.

Lemma 5.23. *Suppose (A,Φ) is a* PU(2) *monopole with $\Phi \not\equiv 0$ and that U_1, U_2 are connected open subsets of X. If there are gauge transformations $u_i \in {}^\circ\mathcal{G}_{E|U_i}$, $i = 1,2$, such that $u_i \in \mathrm{Stab}_{A|_{U_i}}$, $u_i \in \mathrm{Stab}_{(A,\Phi)|_{U_1 \cap U_2}}$, $u_i \neq \mathrm{id}$, and there is a point $x \in U_1 \cap U_2$ such that $u_1 = u_2$ on $E|_x$, then $u_1 = u_2$ on $E|_{U_1 \cap U_2}$.*

Proof. Let $V \subset U_1 \cap U_2$ be the dense open subset of points $\{\Phi \neq 0\} \cap U_1 \cap U_2$. Because there is a gauge transformation $u_i \in {}^\circ\mathcal{G}_{E|U_i}$ with $u_i\Phi = \Phi$ over U_i, then $\Phi|_{U_1\cap U_2}$ must be rank one by Lemma 5.21 and there is an orthogonal decomposition $E|_V = \mathrm{Im}\,\Phi \oplus (\mathrm{Im}\,\Phi)^\perp$. Since $u_i\Phi = \Phi$ on V, both u_i respect this decomposition and must be the identity on $\mathrm{Im}\,\Phi|_V$. Thus, on V we can write

$$u_i = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\theta_i} \end{pmatrix}$$

with respect to this decomposition. Now $\det u_i = e^{i\theta_i}$ and because $u_i \in {}^\circ\mathcal{G}_{E|U_i}$, the function $\det u_i$ is constant on U_i and $e^{i\theta_i} \in S^1$. Hence, if $u_1 = u_2$ on $E|_x$, then $u_1 = u_2$ on all points in V which can be connected to x by a path in U_1 and a path in U_2 (note that these need not be the same paths). Because U_1 and U_2 are connected, $u_1 = u_2$ over all of V. But V is dense in $U_1 \cap U_2$ and the u_i are continuous, so $u_1 = u_2$ over all of $U_1 \cap U_2$. q.e.d.

Theorem 5.11 now follows from Proposition 5.19:

Proof of Theorem 5.11, given Proposition 5.19. Let (A,Φ) be a C^r solution to the PU(2) monopole equations (2.27) with $\Phi \not\equiv 0$ and A reducible on a non-empty open set $U \subset X$. Let $J(A) = \{j : 1 \leq j \leq N_G \text{ and } \beta_j[A] > 0\}$ and let $\bar{B}_J(A) = \cup_{j\in J(A)}\bar{B}(x_j, R_0)$. Let U° be a connected component of $U - \bar{B}_J(A)$. Lemma 5.22 then implies that $\mathrm{Stab}_{(A,\Phi)|_{U^\circ}} \simeq S^1 \neq S^1_Z$, and so there is a C^{r+1} gauge transformation u of $E|_{U^\circ}$ such that $u(A,\Phi) = (A,\Phi)$ on U°. Proposition 5.19 allows the extension of u to open subsets of $X \backslash \bar{B}_J(A)$ containing U°. To be explicit, let $x \in U^\circ$ and $r_0 = \mathrm{dist}_g(x, \partial U^\circ)$ and choose $r_1 > r_0$ such that $r_1 \leq \frac{1}{2}\varrho$ and $r_1 \leq \min_{j\in J(A)} \operatorname{dist}_g(x, B(x_j, R_0))$. By Proposition 5.19 there is an extension of $u \in \operatorname{Stab}_{(A,\Phi)|_{B(x,r_0)}}$ to an element $\hat{u} \in \operatorname{Stab}_{(A,\Phi)|_{B(x,r_1)}}$. By Lemma 5.23 we have $\hat{u} = u$ on $U^\circ \cap B(x_0, r_1)$ and not just $B(x_0, r_0)$. This gives an extension of the stabilizer u for (A, Φ) over U° to a stabilizer $\hat{u}$ over the slightly larger open set $U^\circ \cup B(x, r_1)$. Since we do not assume that X is simply connected, we must check that the extension obtained by repeating this process yields a single-valued gauge transformation over $X - \bar{B}_J(A)$.

The consistency of two extensions follows from Lemma 5.23. Let u_i, $i = 1, 2$, be two extensions of u to connected open sets U_i° containing U°, so we have $u_i \in \operatorname{Stab}_{(A,\Phi)|_{U_i^\circ}}$ with $u_i = u$ on $E|_{U^\circ}$. Because $u_1 = u = u_2$ on $E|_{U^\circ}$, Lemma 5.23 implies that $u_1 = u_2$ on $E|_{U_1^\circ \cap U_2^\circ}$. Therefore, the extensions of $u \in \operatorname{Stab}_{(A,\Phi)|_{U^\circ}}$ fit together to form a global gauge transformation $\hat{u} \in \operatorname{Stab}_{A,\Phi}$ on $X - \bar{B}_J(A)$ such that $\hat{u} = u$ on $E|_{U^\circ}$.

Thus, given that A is reducible on an open set U containing $\bar{B}_J(A)$, the above argument produces an element $u \in \operatorname{Stab}_{(A,\Phi)|_{X-\bar{B}_J(A)}}$ with $u \notin S^1_Z$. Consequently, A is reducible on $X - \bar{B}_J(A)$ by Lemma 2.6. For $j = 1, \ldots, N_b$, let V_j be an open, connected subset of $U \cap B(x_j, 2R_0)$ containing $\bar{B}(x_j, R_0)$, such that $V_j \cap (X - B_J(A))$ is connected, and set

$$\begin{aligned}
X_1 &= X - \bar{B}_J(A), \\
X_j &= (X - \bar{B}_J(A)) \cup \bigcup_{1\leq k\leq j-1} V_k, \quad 2 \leq j \leq N_b + 1.
\end{aligned}$$

Each subset X_j and $V_j \cap X_j$ is connected, and $X_1 \subset X_2 \subset \cdots \subset X_{N_b+1} = X$. We extend $u \in \operatorname{Stab}_{(A,\Phi)|_{X-\bar{B}_J(A)}}$ inductively over each X_j. Plainly, we have $u \in \operatorname{Stab}_{A|_{V_j\cap X_j}}$ and $\operatorname{Stab}_{A|_{V_j}} \subset \operatorname{Stab}_{A|_{V_j\cap X_j}}$, so we first check that $\operatorname{Stab}_{A|_{V_j}} = \operatorname{Stab}_{A|_{V_j\cap X_j}}$. By hypothesis, $A|_U$ is reducible and so for each subset $V_j \subset U$, we have that $A|_{V_j}$ is reducible. Because $\Phi|_{V_j\cap X_j} \not\equiv 0$ and thus $F_A \not\equiv 0$ on $V_j \cap X_j$ by (2.27) — so $A|_{V_j\cap X_j}$ is not flat — then Lemma 2.6 implies that $\operatorname{Stab}_{A|_{V_j\cap X_j}} \simeq T^2$. The same argument yields $\operatorname{Stab}_{A|_{V_j}} \simeq T^2$. (We use the assumptions on the connectedness of V_j and $V_j \cap X_j$ here: if the sets were not connected, the stabilizers would be $\oplus T^2$, a direct sum over connected components.) Thus, $\operatorname{Stab}_{A|_{V_j}} = \operatorname{Stab}_{A|_{V_j\cap X_j}} \simeq T^2$.

Hence, there is an element $u'' \in \operatorname{Stab}_{A|_{V_j}}$ such that $u'' = u'$ on $V_j \cap X_j$, where $u' \in \operatorname{Stab}_{(A,\Phi)|_{X_j}}$ with $u' \neq \mathrm{id}$. Together, u' and u'' give an element $u \in \operatorname{Stab}_{A|_{X_{j+1}}}$ which is not in S^1_Z. The connection A is then reducible on X_{j+1}, which implies that all the holonomy perturbations vanish on X_{j+1}. Lemma 5.22 therefore shows that the pair (A, Φ) is reducible on X_{j+1} and we obtain a stabilizer $u \in \mathrm{Stab}_{(A,\Phi)|_{X_{j+1}}}$, with $u \neq \mathrm{id}$. The construction of $u \in \mathrm{Stab}_{A,\Phi}$, $u \neq \mathrm{id}$, is thus completed by induction on j. q.e.d.

Remark 5.24. The analogue of Theorem 5.11 (namely, that local reducibility implies global reducibility) does not hold for anti-self-dual connections without further restrictions on the topology of X. For example, in [30] it is assumed that the four-manifold X is simply connected (see Lemma 4.3.21). As described by Kronheimer and Mrowka in [47] one can have *locally reducible* anti-self-dual connections (called 'twisted reducibles' in [47]); see their Lemma 2.4 for a sharp version of the Freed-Uhlenbeck generic metrics theorem (Corollaries 4.3.15, 4.3.18, and 4.3.19 in [20]) which holds when the requirement that X be simply connected is dropped. In our case, we see from the proof of Theorem 5.11 that globally irreducible, locally reducible solutions (A, Φ) to (2.27) do not exist (at least when $\Phi \not\equiv 0$) because the stabilizer $u \in \mathrm{Stab}_{A,\Phi}$ must stabilize the section Φ and not just the connection A.

The proof of Proposition 5.19 takes up the remainder of this section.

5.3.1. The Agmon-Nirenberg unique continuation theorem. As in the case of the anti-self-dual equation [20, Lemma 4.3.21], our proof of the unique continuation property for PU(2) monopoles in radial gauge relies on the following special case of a more general result due to Agmon and Nirenberg for an ordinary differential equation on a Hilbert space [4]:

Theorem 5.25 *[4, Theorem 2 (ii)]. Let $\mathfrak{H}$ be a Hilbert space and let $\mathcal{P} : \mathrm{Dom}(\mathcal{P}(r)) \subset \mathfrak{H} \to \mathfrak{H}$ be a family of symmetric linear operators for $r \in [r_0, R)$. Suppose that $\eta \in C^1([r_0, R), \mathfrak{H})$, with $\eta(r) \in \mathrm{Dom}(\mathcal{P}(r))$ and $\mathcal{P}\eta \in C^0([r_0, R), \mathfrak{H})$ such that*

$$\left\| \frac{d\eta}{dr} - \mathcal{P}(r)\eta(r) \right\| \leq c_1 \|\eta(r)\|, \tag{5.10}$$

for some positive constant c_1 and all $r \in [r_0, R)$. If the function $r \mapsto (\eta(r), \mathcal{P}(r)\eta(r))$ is differentiable for $r \in [r_0, R)$ and satisfies

$$\frac{d}{dr}(\eta, \mathcal{P}\eta) - 2\,\mathrm{Re}\left(\frac{d\eta}{dr}, \mathcal{P}\eta\right) \geq -c_2 \|\mathcal{P}\eta\| \, \|\eta\| - c_3 \|\eta\|^2, \tag{5.11}$$

for positive constants c_2, c_3 and every $r \in [r_0, R)$, then the following holds: If $\eta(r_0) = 0$, then $\eta(r) = 0$ for all $r \in [r_0, R)$.

Applications of [4, Theorem 2] to the proof of unique continuation results for first-order elliptic and parabolic partial differential equations were considered by Agmon in [3, Chapter II]. In our application, $\mathcal{P}(r)$ will be a family of first-order partial differential operators which are self-adjoint over the closed manifold X. Theorem 5.25 has also been applied by D. Salamon to prove the unique continuation property for harmonic spinors [77, Appendix E]. One of the difficulties in applying the Agmon-Nirenberg theorem to the PU(2) monopole equations (in Gaussian polar coordinates, (r,θ)) is the requirement that the one-parameter family of operators be *self-adjoint* with respect to a *fixed inner product* on a *fixed vector space*. The additional complication, not present in [20, §4.3.4], is that the induced spin^c structures on geodesic spheres in X vary with the induced family of r-dependent metrics.

Remark 5.26 [4, Remark, p. 209]. If $\operatorname{Dom}(\mathcal{P}(r)) = D$ is independent of r, and $\mathcal{P}(r)$, $r\in[r_0,R)$, is a differentiable family of self-adjoint operators, then the left-hand side of (5.11) simplifies, of course, to give the condition

$$\left(\eta, \frac{d\mathcal{P}}{dr}\eta\right) \geq -c_2\,\|\mathcal{P}\eta\|\,\|\eta\| - c_3\|\eta\|^2, \tag{5.12}$$

since, noting that $\mathcal{P}$ is self-adjoint,

$$\begin{aligned}\frac{d}{dr}(\eta,\mathcal{P}\eta) - 2\operatorname{Re}\left(\frac{d\eta}{dr},\mathcal{P}\eta\right) &= \left(\frac{d\eta}{dr},\mathcal{P}\eta\right) + \left(\eta,\frac{d\mathcal{P}}{dr}\eta\right) + \left(\eta,\mathcal{P}\frac{d\eta}{dr}\right)\\ &\quad - \left(\frac{d\eta}{dr},\mathcal{P}\eta\right) - \left(\mathcal{P}\eta,\frac{d\eta}{dr}\right)\\ &= \left(\eta,\frac{d\mathcal{P}}{dr}\eta\right).\end{aligned}$$

If $\mathcal{P}(r)$, $r\in[r_0,R)$, is a differentiable family of self-adjoint operators, then it is easy to see that (5.12) follows from the simpler condition

$$\left\|\frac{d\mathcal{P}}{dr}\eta\right\| \leq c_2\|\mathcal{P}\eta\| + c_3\|\eta\|, \tag{5.13}$$

provided $\mathcal{P}\eta\in C^0([r_0,R),\mathfrak{H})$.

5.3.2. The PU(2) monopole equations in Gaussian polar coordinates. Our first task is to write the pair of PU(2) monopole equations (2.27) as an ordinary differential equation with respect to

Gaussian polar coordinates (r, θ) centered at a point $x_0 \in X$. For the analogous ordinary differential equation in the case of the anti-self-dual equation, see [20], [63], [88], and for the Seiberg-Witten equations; see [47], [77].

Recall that ϱ is the injectivity radius of (X, g) at the point x_0, so $\exp_{x_0} : B(0, \varrho) \subset (TX)_{x_0} \to B(x_0, \varrho) \subset X$ is a diffeomorphism. For each ξ in the unit sphere $S^3 \subset (TX)_{x_0}$, let $\{e_i(\xi)\}_{i=1}^3$ be an oriented, orthonormal basis for $(\mathbb{R}\xi)^\perp = (TS^3)_\xi \subset (TX)_{x_0}$; that is, let $\{e_i\}$ be an orthonormal frame for TS^3. Let $\gamma_\xi(r)$ be the geodesic $\exp_{x_0}(r\xi)$, $r \in [0, \varrho)$, so that $\gamma_\xi(0) = x_0$ and $|\gamma'_\xi(r)| = |\xi|$, and let $\tau_\xi(r) : (TX)_{x_0} \to (TX)_{\gamma_\xi(r)}$ denote parallel translation with respect to the metric's Levi-Civita connection along $\gamma_\xi(r)$. Let $e_i(r, \xi) := \tau_\xi(r) e_i(\xi)$ for $r \geq 0$, so that $\{\gamma'_\xi(r), e_i(r, \xi)\}$ is an orthonormal frame for $(TX)_{\gamma_\xi(r)}$ which is parallel along the radial geodesics $\gamma_\xi(r)$ and satisfies $\gamma'_\xi(0) = \xi$ and $e_i(0, \xi) = e_i(\xi)$. Denote the radial vector $\gamma'_\xi(r) \in (TX)_{\gamma_\xi(r)}$ by $\frac{\partial}{\partial r} := \frac{\partial}{\partial r}|_{\gamma_\xi(r)}$ when no confusion can arise. Thus, $\{\frac{\partial}{\partial r}, e_i\}$ is an oriented, orthonormal frame for TX over $B(x_0, \varrho) \setminus \{0\}$, which is parallel along radial geodesics; let $\{dr, e^i\}$ be the corresponding dual frame for T^*X over $B(x_0, \varrho) \setminus \{0\}$.

With respect to the parametrization $S^3 \times (0, \varrho) \simeq B(x_0, \varrho) - \{x_0\}$, given by $(r, \xi) \mapsto \exp_{x_0}(r\xi)$, the metric g on $B(x_0, \varrho) - \{x_0\}$ pulls back to

$$g = (dr)^2 + g_r,$$

where g_r is the metric on S^3 pulled back from the restriction $g|_{S^3(x_0, r)}$ to the geodesic sphere $S^3(x_0, r) := \{x \in X : \mathrm{dist}_g(x, x_0) = r\}$. Let $*_{g_r}$ denote the Hodge star operator for the metric g_r on S^3 and, for emphasis, we write $*_g$ for the Hodge star operator for the metric g on X.

Suppose that a pair (A, Φ) on $(\mathfrak{su}(E), W^+ \otimes E)$ is a C^r solution to the PU(2) monopole equations (2.27) over X,

$$\begin{aligned} \rho(F_A^+) &= \rho\tau\rho^{-1}(\Phi \otimes \Phi^*)_{00}, \\ D_{A,\vartheta_0}\Phi &= 0, \end{aligned}$$

where $\tau := \mathrm{id}_{\Lambda^+} + \tau_0$ is an automorphism of Λ^+ and

$$D_{A,\vartheta_0}\Phi := D_A\Phi + \rho(\vartheta_0)\Phi.$$

We have not included the holonomy perturbations $\vec{\tau} \cdot \vec{\mathfrak{m}}(A)$ and $\vec{\vartheta} \cdot \vec{\mathfrak{m}}(A)$ because they vanish near x_0 by hypothesis.

We obtain an isomorphism $E|_{B(x_0,\varrho)} \simeq E_0 \times B(x_0,\varrho)$ of complex two-plane bundles by choosing a unitary frame for $E_0 := E|_{x_0}$ and using parallel translation via the U(2) connection on E defined by A and $A_{\det E}$ along radial geodesics emanating from x_0. Let $A = B + Cdr$ denote the induced SO(3) connection on the bundle $\mathfrak{su}(E_0) \times S^3 \times (0,\varrho)$ over $S^3 \times (0,\varrho)$ and note that A is in radial gauge with respect to the point x_0, so $C := A(\frac{\partial}{\partial r}) = 0$. We let $B = B(r)$, $r \in (0,\varrho)$, denote the resulting one-parameter family of SO(3) connections on the bundle $\mathfrak{su}(E_0) \times S^3$ over S^3. In exactly the same way, we obtain an induced one-parameter family of U(2) connections on the bundle $W_0^+ \times S^3$ over S^3, $r \in (0,\varrho)$, induced by the isomorphism $W^+|_{B(x_0,\varrho)-\{x_0\}} \simeq W_0^+ \times S^3 \times (0,\varrho)$.

A section Φ of the bundle $W^+ \otimes E$ over $B(x_0,\varrho)$ pulls back, via the isomorphism $(W^+ \otimes E)|_{B(x_0,\varrho)-\{x_0\}} \simeq (W_0^+ \otimes E_0) \times S^3 \times (0,\varrho)$, to a one-parameter family of sections $\Psi(r)$ of the bundle $W_0^+ \otimes E_0 \times S^3$ over S^3. The automorphism τ of $\Lambda^{+,g}$ pulls back to a one-parameter family of automorphisms $\sigma(r)$ of T^*S^3, for $r \in (0,\varrho)$, using the isomorphism

$$(0,\varrho) \times T^*S^3 \to \Lambda^{+,g}(T^*X), \qquad (r,\alpha) \mapsto *_{g_r}\alpha + dr \wedge \alpha.$$

The g-compatible Clifford map $\rho : T^*X \to \mathrm{Hom}(W^+, W^-)$ and the isomorphism $W^+|_{B(x_0,\varrho)-\{x_0\}} \simeq W_0^+ \times S^3 \times (0,\varrho)$ define a family of g_r-compatible Clifford maps $\gamma(r) : T^*S^3 \to \mathrm{End}(W_0^+)$ by setting

$$\gamma(r) := \rho(dr)\rho(\,\cdot\,).$$

Indeed, to see this, observe that $g(dr,dr) = 1$ and so for a family of one-forms $\alpha(r)$ on S^3, defined by the isomorphism

$$B(x_0,\varrho) - \{x_0\} \simeq S^3 \times (0,\varrho)$$

and a one-form α on $B(x_0,\varrho)$, we have

$$\begin{aligned}\gamma_r(\alpha)^\dagger\gamma_r(\alpha) &= \rho(\alpha)^\dagger\rho(dr)^\dagger\rho(dr)\rho(\alpha) = \rho(\alpha)^\dagger\rho(\alpha)\\ &= g_r(\alpha,\alpha)\,\mathrm{id}_{W_0^+},\end{aligned}$$

as required. The map γ extends to a one-parameter family of Clifford maps $\gamma(r) : \Lambda^\bullet(T^*S^3) \otimes \mathbb{C} \to \mathrm{End}(W_0^+)$ in the usual way. For example, $\gamma_r(\alpha \wedge \beta) := \gamma_r(\alpha)\gamma_r(\beta)$, for $\alpha,\beta \in \Omega^1(S^3)$, in which case we see that $\gamma_r(\alpha \wedge \beta) = \rho(\alpha)\rho(\beta)$.

With the above understood, we can proceed to rewrite the PU(2) monopole equations (2.27) over the ball $B(x_0,\varrho)$ as an ordinary differential equation for a one-parameter family of pairs $(B(r),\Psi(r))$ on

$(\mathfrak{su}(E_0), W_0^+ \otimes E_0)$ over S^3. The curvature F_A of the connection A over $B(x_0, \varrho)$ is given by

$$F_A = F_B - \frac{dB}{dr} \wedge dr.$$

For any $\omega \in \Omega^2(X, \mathbb{R})$, we have $\rho_+(\omega) = \rho_+(*_g\omega)$, since $\rho_+|_{\Lambda^-} = 0$. If the radial component of ω vanishes and we consider $\omega|_{B(x_0,\varrho)}$ as a one-parameter family of two-forms $\omega(r)$ on S^3, then we see that $*_g\omega = -(*_{g_r}\omega) \wedge dr$. Combining these observations yields

$$\begin{aligned}
\rho(F_A^+) = \rho_+(F_A) &= \rho_+\left(F_B - \frac{dB}{dr} \wedge dr\right) \\
&= \rho_+\left(-(*_{g_r}F_B) \wedge dr - \frac{dB}{dr} \wedge dr\right) \\
&= \rho_-(dr)\rho_+\left(*_{g_r}F_B + \frac{dB}{dr}\right),
\end{aligned}$$

and therefore,

$$\rho(F_A^+) = \gamma\left(*_{g_r}F_B + \frac{dB}{dr}\right).$$

The section $\rho\tau\rho^{-1}(\Phi \otimes \Phi^*)_{00}$ of $\mathfrak{su}(W^+) \otimes \mathfrak{su}(E)$ over $B(x_0, \varrho)$ pulls back to the one-parameter family of sections $\gamma\sigma\gamma^{-1}(\Psi \otimes \Psi^*)_{00}$ of

$$\mathfrak{su}(W_0^+) \otimes \mathfrak{su}(E_0) \times S^3$$

over S^3 via the isomorphism

$$W^+|_{B(x_0,\varrho)-\{x_0\}} \simeq W_0^+ \times S^3 \times (0, \varrho),$$

and similarly for E.

Let $\theta(r)$ be the induced one-parameter family of complex one-forms on S^3 defined by ϑ_0 on $B(x_0, \varrho)$ and the isomorphism

$$B(x_0, \varrho) - \{x_0\} \simeq S^3 \times (0, \varrho),$$

so $\vartheta_0 = -f dr + \theta$ on $S^3 \times (0, \vartheta)$, where $f(r) \in \Omega^0(S^3, \mathbb{C})$. Given the preceding identifications, the Dirac operator term in (2.27) can then be written over $S^3 \times \{r\}$ as

$$\begin{aligned}
D_{A,\vartheta_0}\Phi &= \rho(dr)\nabla^A_{\frac{\partial}{\partial r}}\Psi + \sum_{i=1}^3 \rho(e^i)\nabla^A_{e_i}\Psi + \rho(\vartheta_0)\Psi \\
&= \rho(dr)\frac{d\Psi}{dr} + \sum_{i=1}^3 \rho(e^i)\nabla^B_{e_i}\Psi + \rho(\theta)\Psi - f\rho(dr)\Psi \\
&= \rho(dr)\left(\frac{d\Psi}{dr} - \sum_{i=1}^3 \rho(dr)\rho(e^i)\nabla^B_{e_i}\Psi - \rho(dr)\rho(\theta)\Psi - f\Psi\right) \\
&= \rho(dr)\left(\frac{d\Psi}{dr} - \sum_{i=1}^3 \gamma(e^i)\nabla^B_{e_i}\Psi - \gamma(\theta)\Psi - f\Psi\right),
\end{aligned}$$

and therefore,

$$D_{A,\vartheta_0}\Phi = \rho(dr)\left(\frac{d\Psi}{dr} - D_{B,\theta,f}\Psi\right).$$

Hence, the PU(2) monopole equations (2.27) can be written as

$$\begin{aligned}
\gamma\left(\frac{dB}{dr} + *_{g_r}F_B\right) &= \gamma\sigma\gamma^{-1}(\Psi\otimes\Psi^*)_{00}, \\
\rho(dr)\frac{d\Psi}{dr} &= \rho(dr)D_{B,\theta,f}\Psi.
\end{aligned}$$

We use $D_B : \Omega^0(S^3, W_0^+ \otimes E_0) \to \Omega^0(S^3, W_0^+ \otimes E_0)$ to denote the one-parameter family of Dirac operators defined by the family of $g(r)$-compatible Clifford maps $\gamma(r) : T^*S^3 \to \operatorname{End} W_0^+$, the family of U(2) connections on $W_0^+ \times S^3$, the family of SO(3) connections $B(r)$ on $\mathfrak{su}(E_0) \times S^3$ over S^3, and the family of determinant connections on $\det E_0 \times S^3$ over S^3. Since the Clifford map gives an isomorphism $\gamma : T^*S^3 \simeq \mathfrak{su}(W_0^+) \times S^3$, the PU(2) monopole equations then take the shape

$$\begin{aligned}
\frac{dB}{dr} + *_{g_r}F_B &= \sigma\gamma^{-1}(\Psi\otimes\Psi)_{00}, \\
\frac{d\Psi}{dr} &= D_{B,\theta,f}\Psi,
\end{aligned} \tag{5.14}$$

for a one-parameter family $(B(r), \Psi(r))$, $r \in (0, \varrho)$, on $(E_0, W_0^+ \otimes E_0)$ over S^3.

5.3.3. The ordinary differential equation for the difference pair. Let (A,Φ) be a C^r solution to the PU(2) monopole equations (2.27) over X. Suppose, as in the hypothesis of Proposition 5.19, that there is a C^{r+1} gauge transformation u of $E|_{B(x_0,r_0)}$ such that $u(A,\Phi)=(A,\Phi)$ on $B(x_0,r_0)$. Let (A,Φ) again denote the induced pair defined by the isomorphism $E|_{B(x_0,\varrho)}\simeq E_0\times S^3\times(0,\varrho)$ (given by a choice of unitary frame for $E_0=E_{x_0}$ and parallel, radial translation via A) and let u be the induced gauge transformation on $E_0\times S^3\times(0,r_0)$. Then $u(A)=uAu^{-1}-(d_Au)u^{-1}$ and $A=B+C\,dr$, where $C=A(\frac{\partial}{\partial r})=0$, and so $du/dr=0$. We now extend u by parallel translation via A along radial geodesics emanating from x_0 to a gauge transformation $\hat{u}$ on $E_0\times S^3\times(0,\varrho)$.

Let $(\hat{A},\hat{\Phi})=\hat{u}(A,\Phi)$ be the gauge-equivalent pair on $S^3\times(0,\varrho)$, so $(\hat{A},\hat{\Phi})$ is a C^r solution to (2.27) over $S^3\times(0,\varrho)$, with $\hat{A}=\hat{u}(A)$ in radial gauge. In particular, $(\hat{A},\hat{\Phi})=(A,\Phi)$ over $S^3\times(0,r_0)$: we need to show that $(\hat{A},\hat{\Phi})=(A,\Phi)$ over $S^3\times(0,r_1)$ in order to prove Proposition 5.19.

The one-parameter family of pairs $(\hat{B}(r),\hat{\Psi}(r))$ on $(\mathfrak{su}(E_0),W_0^+\otimes E_0)$ over S^3 also satisfies the ordinary differential equation in (5.14) so, subtracting these two pairs of ordinary differential equations, we obtain for $r\geq r_0$,

$$\frac{d(\hat{B}-B)}{dr}+*_{g_r}(F_{\hat{B}}-F_B)=\sigma\gamma^{-1}(\hat{\Psi}\otimes\hat{\Psi}^*-\Psi\otimes\Psi^*)_{00},$$
$$\frac{d(\hat{\Psi}-\Psi)}{dr}=D_{B,\theta,f}\Psi-D_{\hat{B},\theta,f}\hat{\Psi}.$$

Since

$$F_B=d(B-\Gamma)+(B-\Gamma)\wedge(B-\Gamma)$$

and

$$D_{B,\theta,f}=D+\gamma(B-\Gamma)+\gamma(\theta)+f\,\mathrm{id}_{W_0^+\otimes E_0},$$

we obtain an ordinary differential equation for the difference pair

$$(b,\psi):=(\hat{B}-B,\hat{\Psi}-\Psi)\in\Omega^1(S^3,\mathfrak{su}(E_0))\oplus\Omega^0(S^3,W_0^+\otimes E_0),$$

so that,

$$\tag{5.15}\begin{aligned}\frac{db}{dr}&=-*_{g_r}db+*_{g_r}(\hat{B}\wedge b+b\wedge B)\\&\quad+\sigma\gamma^{-1}(\hat{\Psi}\otimes\psi^*+\psi\otimes\Psi^*)_{00},\\\frac{d\psi}{dr}&=D\psi+\gamma(\hat{B}-\Gamma)\psi+\gamma(b)\Psi+\gamma(\theta)\psi+f\psi,\end{aligned}$$

where $r \in (r_0, \varrho)$, and Γ is the product connection on $\mathfrak{su}(E_0) \times S^3$ over S^3 defined by our trivialization. The above system has the schematic form

$$\frac{d(b,\psi)}{dr} = \begin{pmatrix} -*_{g_r} d & 0 \\ 0 & D \end{pmatrix} (b,\psi) + \mathcal{Z}_r(b,\psi), \tag{5.16}$$

where $\mathcal{Z}_r$ is a one-parameter family of zeroth-order operators with coefficients depending on g_r, B, σ, and Ψ.

5.3.4. Reduction to the Agmon-Nirenberg theorem. We first observe that the operator $-*_{g_r} d$ on $\Omega^1(S^3, \mathfrak{su}(E_0))$ is self-adjoint with respect to the L^2 inner product induced by the metric g_r on S^3. Indeed, as $*_{g_r}^2 = 1$ on $\Omega^1(S^3)$ and $d\,\mathrm{tr}(b \wedge b') = \mathrm{tr}(db \wedge b') - \mathrm{tr}(b \wedge db')$, we have

$$\begin{aligned}
\int_{S^3} \langle b, -*_{g_r} db' \rangle_r \, d\,\mathrm{vol}_r &= -\int_{S^3} \mathrm{tr}(b \wedge *_{g_r}(-*_{g_r} db')) \\
&= \int_{S^3} \mathrm{tr}(b \wedge db') = \int_{S^3} \mathrm{tr}(db \wedge b') \\
&= \int_{S^3} \mathrm{tr}(b \wedge db') = \int_{S^3} \mathrm{tr}(b' \wedge *_{g_r}^2 db) \\
&= -\int_{S^3} \langle b', *_{g_r} db \rangle_r \, d\,\mathrm{vol}_r,
\end{aligned}$$

and therefore,

$$\int_{S^3} \langle b, -*_{g_r} db' \rangle_r \, d\,\mathrm{vol}_r = \int_{S^3} \langle -*_{g_r} db, b' \rangle_r \, d\,\mathrm{vol}_r \, .$$

The Dirac operators $D : \Omega^1(S^3, W_0^+ \otimes E_0) \to \Omega^1(S^3, W_0^- \otimes E_0)$ are defined by the one-parameter family of g_r-compatible Clifford maps $\gamma_r : T^*S^3 \to \mathrm{End}(W_0^+)$, the one-parameter family of U(2) connections on $W_0^+ \times S^3$, the one-parameter family of determinant connections on $\det E_0 \times S^3$, and the product SO(3) connection on $\mathfrak{su}(E_0) \times S^3$. Then, by [57, Proposition II.5.3] we have

$$\langle D\psi, \psi' \rangle = \langle \psi, D\psi' \rangle + \mathrm{div}_{g_r} \xi,$$

where $\xi(r)$ is the one-parameter family of vector fields on S^3 defined by $\alpha(\xi) = -\langle \psi, \gamma(\alpha)\psi' \rangle$, for all $\alpha \in \Omega^1(S^3)$. Hence, the divergence theorem for the metric g_r implies that D is self-adjoint with respect the the family of L^2 inner products on $\Omega^1(S^3, W_0^+ \otimes E_0)$ defined by g_r:

$$\int_{S^3} \langle D\psi, \psi' \rangle \, d\,\mathrm{vol}_r = \int_{S^3} \langle \psi, D\psi' \rangle \, d\,\mathrm{vol}_r \, .$$

The Agmon-Nirenberg theorem is not immediately applicable to the ordinary differential equation (5.16) since the differential operator $\mathcal{P}_r := -*_{g_r} d \oplus D$ is only self-adjoint on the Hilbert space

$$\mathfrak{H}_r := L^2(S^3, \Lambda^1 \otimes \mathfrak{su}(E_0)) \oplus L^2(S^3, W_0^+ \otimes E_0),$$

with L^2 inner products $(\cdot\ ,\ \cdot)_r$ defined by the family of metrics g_r on S^3, rather than a fixed inner product.

If $d\,\mathrm{vol}_r$ is the volume form on S^3 defined by the metric g_r, then we may write

$$d\,\mathrm{vol}_r = h_r^2 d\,\mathrm{vol}, \qquad r \in (0, \varrho),$$

for some positive function h_r on S^3, where $d\,\mathrm{vol}$ is the volume form on S^3 defined by the standard metric. However,

$$\mathcal{Q}_r := h_r \mathcal{P}_r h_r^{-1}, \qquad r \in (0, \varrho), \tag{5.17}$$

is a differentiable path of self-adjoint, first-order, elliptic differential operators on the fixed, real Hilbert space underlying

$$\mathfrak{H} := L^2(S^3, \Lambda^1 \otimes \mathfrak{su}(E_0)) \oplus L^2(S^3, W_0^+ \otimes E_0),$$

with L^2 inner product $(\cdot\ ,\ \cdot)$ defined by the standard metric on S^3. Indeed, if we define a Hilbert-space isomorphism $\mathfrak{H}_r \to \mathfrak{H}$ by $(b, \psi) \mapsto (\beta, \varphi) := h_r(b, \psi)$, then

$$\begin{aligned}
\int_{S^3} \langle \mathcal{Q}_r(\beta, \varphi), (\beta', \varphi')\rangle\, d\,\mathrm{vol} &= \int_{S^3} \langle h_r \mathcal{P}_r h_r^{-1} h_r(b, \psi), h_r(b', \psi')\rangle\, d\,\mathrm{vol} \\
&= \int_{S^3} \langle \mathcal{P}_r(b, \psi), (b', \psi')\rangle\, d\,\mathrm{vol}_r \\
&= \int_{S^3} \langle (b, \psi), \mathcal{P}_r(b', \psi')\rangle\, d\,\mathrm{vol}_r \\
&= \int_{S^3} \langle h_r(b, \psi), h_r \mathcal{P}_r h_r^{-1} h_r(b', \psi')\rangle\, d\,\mathrm{vol},
\end{aligned}$$

and therefore,

$$\int_{S^3} \langle \mathcal{Q}_r(\beta, \varphi), (\beta', \varphi')\rangle\, d\,\mathrm{vol} = \int_{S^3} \langle (\beta, \varphi), \mathcal{Q}_r(\beta', \varphi')\rangle\, d\,\mathrm{vol}, \qquad r \in (0, \varrho).$$

The operators $\mathcal{Q}_r$ have dense domain L_1^2. Since $(b, \psi) = h_r^{-1}(\beta, \varphi)$, we see that

$$\frac{d(b, \psi)}{dr} = -h_r^{-2} \frac{dh_r}{dr}(\beta, \varphi) + h_r^{-1} \frac{d(\beta, \varphi)}{dr},$$

and so, subsituting into (5.16) gives

$$\frac{d(\beta,\varphi)}{dr} = \mathcal{Q}_r(\beta,\varphi) + h_r\mathcal{Z}_r h_r^{-1}(\beta,\varphi) + h_r^{-1}\frac{dh_r}{dr}(\beta,\varphi), \qquad r \in (0,\varrho), \tag{5.18}$$

with (β,φ) a path in $\mathfrak{H}$.

5.3.5. Verification of the Agmon-Nirenberg conditions. We can now conclude the proof of our unique continuation result for reducible PU(2) monopoles:

Completion of proof of Proposition 5.19. The estimates (5.10) and (5.12) are, in principle, straightforward to check since we only need them for r varying in the compact interval $[\frac{1}{2}r_0, r_1]$; the second condition, (5.12), requires a little more explanation since we need an estimate for $d\mathcal{Q}/dr$. We first check condition (5.10). Comparing (5.15), (5.16), and (5.18), we find that

$$\left\|\frac{d(\beta,\varphi)}{dr} - \mathcal{Q}_r(\beta,\varphi)\right\| \le c_1\|(\beta,\varphi)\|, \qquad \tfrac{1}{2}r_0 \le r \le \tfrac{1}{2}\varrho,$$

for some positive constant $c_1 = c_1(g, r_0, A, \Phi)$, and so the ordinary differential equation (5.18) obeys the estimate (5.10) on the interval $[\frac{1}{2}r_0, r_1]$. To check condition (5.12), observe that the definition of $\mathcal{Q}_r$ in (5.16) and (5.17) yields the pointwise bound

$$\left|\frac{d\mathcal{Q}_r}{dr}(\beta,\varphi)\right| \le C\left(|(\nabla\beta,\nabla\varphi)| + |(\beta,\varphi)|\right),$$

where $\frac{1}{2}r_0 \le r \le \frac{1}{2}\varrho$, for some positive constant $C = C(g,\tau,A_W,r_0)$, and ∇ denotes covariant derivatives on $\Lambda^1\otimes\mathfrak{su}(E_0)$ and $W_0^+\otimes E_0\times S^3$ which are independent of r. Thus, using the standard elliptic estimate for $\mathcal{Q}_r$ we obtain the L^2 bound

$$\left\|\frac{d\mathcal{Q}_r}{dr}(\beta,\varphi)\right\| \le C\left(\|\mathcal{Q}_r(\beta,\varphi)\| + \|(\beta,\varphi)\|\right),$$

and so, for $\frac{1}{2}r_0 \le r \le \frac{1}{2}\varrho$,

$$\left|\left((\beta,\varphi), \frac{d\mathcal{Q}_r}{dr}(\beta,\varphi)\right)\right| \le C\left(\|\mathcal{Q}_r(\beta,\varphi)\|\|(\beta,\varphi)\| + \|(\beta,\varphi)\|^2\right).$$

Therefore, (5.12) is obeyed with $c_2 = c_3 = C$ on the interval $[\frac{1}{2}r_0, r_1]$. By Theorem 5.25 and Remark 5.26, the solution $(\beta(r),\varphi(r))$ vanishes for $r \in (0, r_1)$ since it vanishes for $r \in (0, r_0)$. Hence, $(a(r), \phi(r)) = 0$ for $r \in (0, r_1)$ and $\hat{u}(A, \Phi) = (A, \Phi)$ on the ball $B(x_0, r_1)$. This completes the proof of Proposition 5.19. q.e.d.

Appendix A. Holonomy perturbations and regularity

When defining our holonomy perturbations in §2.5.2 we deferred a detailed discussion of several important regularity issues which arise in their construction. The first concerns the regularity of sections of E and $\mathfrak{su}(E)$ which are constructed by parallel transport via Sobolev connections and is described in §A.1. The second concerns the regularity of the $\mathcal{G}_E$ equivariant maps $\mathfrak{h}_{j,l,\alpha} : \mathcal{A}_E(X) \to L^2_{k+1}(X, \mathfrak{su}(E))$ and we show in §A.2 that these maps are C^∞. (All of the $\mathcal{G}_E$ equivariant maps discussed here are ${}^\circ\mathcal{G}_E$ equivariant since S^1_Z acts trivially on connections in $\mathcal{A}_E$; if the connection on $\det E$ is not fixed, then all of the $\mathcal{G}_E$ equivariant maps are $\operatorname{Aut} E$ equivariant.)

The definition of the maps $\mathfrak{m}_{j,l,\alpha}$ also makes use of the existence of a locally finite, C^∞ 'positive partition' on a paracompact manifold modelled on a separable Hilbert space, in the sense of Proposition A.12. The differentials of the cutoff functions in these partitions need not be bounded *a priori* for the usual reason that in an infinite-dimensional Hilbert manifold, closed and bounded sets are not compact. Nonetheless, as we shall see in §A.3, it is possible to modify the standard proof [55] of the existence of a C^∞ partition of unity on a paracompact C^∞ manifold to produce C^∞ cutoff functions all of whose differentials are bounded. The sums defining the perturbations $\vec{\tau} \cdot \vec{\mathfrak{m}}$ and $\vec{\vartheta} \cdot \vec{\mathfrak{m}}$ of §2.5.2 are finite when restricted to any of the open neighborhoods $(\pi \circ r_{Y_j})^{-1}(U_{j,\alpha}) \subset \mathcal{A}^*_E(X)$, where $r_{Y_j} : \mathcal{A}_E(X) \to \mathcal{A}_E(Y_j)$ is the restriction map and $\pi : \mathcal{A}_E(X) \to \mathcal{B}_E(Y_j)$ is the canonical projection. However, the number of terms in these sums may be infinite in the neighborhood of a reducible connection: it is for this reason that they are required to converge with respect to a suitable choice of weights, as described in §2.5.2. We specify the choice of weights in §A.4 and explain why the $\mathcal{G}_E$ equivariant maps $\vec{\tau} \cdot \vec{\mathfrak{m}}$ and $\vec{\vartheta} \cdot \vec{\mathfrak{m}}$ are C^∞ on $\mathcal{A}_E(X)$.

A.1. Parallel transport for Sobolev connections. The fact that parallel transport is well-defined for L^2_2 connections has been pointed out in [61]. However, to construct L^2_{k+1} sections using parallel transport via L^2_k connections with $k \geq 2$, we need to regularize the sections, taking care to do this gauge equivariantly. (Given an L^2_k

connection A, it does not follow that local sections constructed by A-parallel transport are in L^2_{k+1}.) If we ignored the issue of equivariance, then it would suffice to use the standard smoothing kernel described in [33, §7.2 & §7.3], yielding a C^∞ section irrespective of whether the connection is L^2_k or C^∞. Instead, we use the kernel

$$K_t(A)(x,y) \in \mathrm{Hom}(\mathfrak{su}(E)|_y, \mathfrak{su}(E)|_x),$$

$t \in (0,\infty)$, of the heat operator

$$K_t(A) = \exp(-td_A^* d_A) : L^2(X, \mathfrak{su}(E)) \to L^2_{k+1}(X, \mathfrak{su}(E)),$$

defined by the C^∞ metric g on TX and L^2_k connection A on E [34, §1.6]; related constructions are described in [84, p. 339], [85, p. 177]. The key well-known properties [12], [34, §1.6] of the heat kernel which we need are summarized below:

Lemma A.1. *Continue the above notation and suppose that $\zeta \in L^2(X, \mathfrak{su}(E))$.*

(1) As $t \to 0$, the heat operator $\exp(-td_A^* d_A)$ converges to the L^2 orthogonal projection onto $\mathrm{Ker}(d_A^* d_A)^\perp$.

(2) If A on $\mathfrak{su}(E)$ is irreducible, then $K_t(A)(x,\cdot)$ converges to the Dirac delta distribution $\delta(x,\cdot)$.

(3) If A is L^2_k, then $K_t(A)\zeta$ is L^2_{k+1}; if A is C^∞, then $K_t(A)\zeta$ is C^∞.

(4) If $\zeta \in C^0(\widetilde{U}, \mathfrak{su}(E))$, for an open set $\widetilde{U} \subset X$, and A is irreducible on $\mathfrak{su}(E)$, then $K_t(A)\zeta \to \zeta$ in $C^0(U, \mathfrak{su}(E))$ as $t \to 0$, for any open subset $U \Subset \widetilde{U}$.

Lemma A.1 continues to hold for unitary connections A on Hermitian bundles E over compact C^∞ manifolds with boundary $\overline{Y} = Y \cup \partial Y$; we use the Neumann boundary conditions of [20, p. 192], [87, Proposition 2.1] to obtain an L^2 self-adjoint Laplacian $d_A^* d_A$. Our main application will be to sections defined by parallel translation:

Lemma A.2. *Let $k \geq 2$ be an integer and let A be an L^2_k unitary connection on a Hermitian two-plane bundle E over X. Let x_0 be a point in X, let $\zeta_0 \in \mathfrak{su}(E)|_{x_0}$, and let $B(x_0, r)$ be a geodesic ball with center x_0 and radius $r > 0$.*

(1) Parallel transport with respect to A is well-defined along C^∞ paths in X.

(2) *If ζ is the section of $\mathfrak{su}(E)|_{B(x_0,r)}$ obtained by A-parallel transport of ζ_0 along radial geodesics originating at x_0, and A is C^∞, then ζ is C^∞ on $B(x_0,r)$. If A is L^2_k, then ζ is C^0 on $B(x_0,r)$ and its mollification $K_t(A|_{B(x_0,r)})\zeta$ is L^2_{k+1} on $B(x_0,r)$, for any $t>0$.*

Proof. Let $\gamma:[0,1]\to X$ be a C^∞ path such that $\gamma(0)=x_0$ and let $U\subset X$ be an open neighborhood of $\gamma([0,1])$. We may suppose without loss of generality that $A\in L^2_k(U,\Lambda^1\otimes\mathfrak{u}(2))$ is a connection matrix and consider the parallel transport of $\eta_0\in\mathbb{C}^2$. Note that we have a continuous Sobolev embedding $L^2_2(U)\to L^2([0,1])$ [2, Theorem V.4]. We wish to solve the ordinary differential equation

$$\frac{d\eta}{dt}+a\eta=0,\qquad t\in[0,1], \tag{A.1}$$

where $a=\gamma^*A\in L^2([0,1],\mathfrak{u}(2))$, with initial condition $\eta(0)=\eta_0$, for a solution $\eta\in(L^2_1\cap C^0)([0,1],\mathbb{C}^2)$.

If A is C^∞ then there is a unique solution $\eta\in C^\infty([0,1],\mathbb{C}^2)$ to (A.1) (see, for example, [36]) and it obeys the inequality [36, Lemma IV.4.1]

$$\begin{aligned}|\eta(t)|&\le|\eta_0|\exp\left(\int_0^t|a(s)|\,ds\right)\\&\le|\eta_0|\exp\left(\|a\|_{L^1([0,1])}\right)\le|\eta_0|\exp(c\|A\|_{L^2_2(U)}),\end{aligned} \tag{A.2}$$

for $t\in[0,1]$. If A is only L^2_2, we may choose any sequence $\{A_\alpha\}$ of C^∞ connections which converge to A in L^2_2 and let $\{\eta_\alpha\}$ be the corresponding sequence of C^∞ solutions to (A.1). From (A.2) it follows that $\{\eta_\alpha\}$ is C^0-Cauchy, and then (A.1) implies that $\{\eta_\alpha\}$ is L^2_1-Cauchy with limit $\eta\in(L^2_1\cap C^0)([0,1],\mathbb{C}^2)$ solving (A.1). This proves Assertion (1).

From the proof of Assertion (1) we see that $\zeta\in C^0(U,\mathfrak{su}(2))$. If the connection A is C^∞ on $B(x_0,r)$, then by differentiating (A.1) with respect to local coordinates (regarded as parameters [36, Chapter 5]) on $B(x_0,r)$ and solving the resulting first order ordinary differential equations for the derivatives, it follows that the section ζ is C^∞ on $B(x_0,r)$. The rest of Assertion (2) is an easy consequence of the properties of the smoothing kernel $K_t(A)(x,y)$ given in Lemma A.1. q.e.d.

A.2. Regularity of holonomy maps. We consider the regularity of the holonomy maps from $A\in\mathcal{A}_E$ to $h_{\gamma,x_0}(A)\in\mathfrak{u}(E)|_{x_0}$, to $\hat{\mathfrak{h}}_\gamma(A)\in C^0(B(x_0,r_0),\mathfrak{su}(E))$, and to $\mathfrak{h}_\gamma(A)\in L^2_{k+1}(B(x_0,r_0),\mathfrak{su}(E))$. See also [8, §3.3], [29, Lemma 1b.1], [86, §8a], and [90, §5A] for related calculations.

Let $\gamma : [0,1] \to X$ denote a parametrization for an oriented path $\gamma \subset X$ with $\gamma(0) = x_0$. The general Sobolev embedding [2, Theorem V.4] implies that we have a continuous restriction map $L^2_k(X, \Lambda^1 \otimes \mathfrak{su}(E)) \to L^2_{k-2}(\gamma, \Lambda^1 \otimes \mathfrak{su}(E))$, for $k \geq 2$. We fix an isomorphism $E|_\gamma \simeq \gamma \times \mathbb{C}^2$ of U(2) bundles and denote the connection matrix corresponding to a unitary connection A on E again by $A \in L^2_{k-2}(\gamma, \Lambda^1 \otimes \mathfrak{u}(2))$. Let $A + sa$, $s \in \mathbb{R}$, be a one-parameter family of nearby connections on E, with connection matrices $A(t) + sa(t) \in L^2_{k-2}(\gamma, \Lambda^1 \otimes \mathfrak{u}(2))$. Let $\xi_0 \in \mathbb{C}^2$ correspond to a point in the fiber $E|_{x_0}$ and, with respect to the connection $A + sa$, let $\xi(t; s)$ be the parallel transport of ξ_0 along $\gamma(t)$, so $\xi(t; s)$ solves

$$\left(\frac{d}{dt} + A(t) + sa(t)\right) \xi(t; s) = 0, \qquad \xi(0; s) = \xi_0. \tag{A.3}$$

Thus, if $P_\gamma(t; A) \in \mathrm{Isom}(E|_{\gamma(0)}, E_{\gamma(t)}) \simeq \mathrm{U}(2)$ denotes parallel transport along γ from $\gamma(0)$ to $\gamma(t)$ with respect to the connection A, we have

$$\xi(t; s) = P_\gamma(t; A + sa)\xi_0, \quad t \in [0,1].$$

Therefore, our task is to compute

$$(DP_\gamma(t; \cdot))_A(a)\xi_0 = \frac{d}{ds} P_\gamma(t; A + sa)\xi_0 \bigg|_{s=0} = \frac{d\xi}{ds}(t; s)\bigg|_{s=0}, \quad t \in [0,1].$$

Differentiating (A.3) with respect to s, we see that $d\xi(t; s)/ds$ solves

$$\left(\frac{d}{dt} + A(t) + sa(t)\right) \frac{d\xi}{ds} = -a(t)\xi(t; s), \qquad \frac{d\xi}{ds}(0; s) = 0. \tag{A.4}$$

Let $Y(t; s)$ be the fundamental matrix solution for the linear differential operator on the left-hand side. Then the solution $d\xi(t; s)/ds$ can be written as [36, Corollary IV.2.1]

$$\frac{d\xi}{ds}(t; s) = -Y(t; s) \int_0^t Y^{-1}(\tau; s) a(\tau) \xi(\tau; s)\, d\tau.$$

Since $\xi(t; s) = Y(t; s)\xi_0 = P_\gamma(t; A + sa)\xi_0$ for any $\xi_0 \in E|_{\gamma(0)}$, setting $s = 0$ above gives

$$(DP_\gamma(t; \cdot))_A(a) = -P_\gamma(t; A) \int_0^t P_\gamma^{-1}(\tau; A) a(\tau) P_\gamma(\tau; A)\, d\tau, \tag{A.5}$$

and so, by the Sobolev embedding theorem, a derivative bound

$$|(DP_\gamma(t;\cdot))_A(a)| \le \|a\|_{L^1(\gamma)} \le c\|a\|_{L^2_{2,A}(X)}, \quad t \in [0,1]. \tag{A.6}$$

The estimate (A.6) implies that the right-hand side of (A.5) is well-defined whenever $k \ge 2$. In particular, when $t = 1$ we have $P_\gamma(1;A) = h_{\gamma,x_0}(A)$, and so (A.5) gives

$$(Dh_{\gamma,x_0})_A(a) = -h_{\gamma,x_0}(A) \int_\gamma P_\gamma^{-1}(A) a P_\gamma(A). \tag{A.7}$$

Thus, we have a well-defined differential

$$(Dh_{\gamma,x_0})_A : L^2_k(X, \Lambda^1 \otimes \mathfrak{u}(E)) \to T_{h_{\gamma,x_0}(A)} \mathrm{U}(E)|_{x_0} \simeq \mathfrak{u}(E)|_{x_0}$$

of the holonomy map $h_{\gamma,x_0} : \mathcal{A}_E(X) \to \mathrm{U}(E)|_{x_0}$ at the point $A \in \mathcal{A}_E(X)$.

A similar argument shows that all higher derivatives exist by repeatedly applying the derivative formulas (A.5) and (A.7) (see, for example, [86, §8a]) and so we may conclude:

Lemma A.3. *For $k \ge 2$, the holonomy map*

$$h_{\gamma,x_0} : \mathcal{A}_E(X) \to \mathrm{U}(E)|_{x_0}$$

is C^∞.

In the same vein, the holonomy map $h_{\gamma,x_0} : \mathcal{A}_E(X) \to \mathrm{SO}(\mathfrak{su}(E))|_{x_0}$ is C^∞ and we have:

Lemma A.4. *For $k \ge 2$, the following holonomy maps are C^∞:*

$$\mathcal{A}_E(X) \ni A \mapsto \hat{\mathfrak{h}}_\gamma(A) \in C^0(B(x_0,r_0), \mathfrak{su}(E)),$$
$$\mathcal{A}_E(X) \ni A \mapsto \mathfrak{h}_\gamma(A) \in L^2_{k+1}(B(x_0,r_0), \mathfrak{su}(E)).$$

Proof. The fact that $\hat{\mathfrak{h}}_\gamma$ is a C^∞ map follows from the proof of Lemma A.3. The map $\mathfrak{h}_\gamma$ is defined by $\mathfrak{h}_\gamma(A) = K_t(A)\hat{\mathfrak{h}}_\gamma(A)$ and so the fact that the map

$$\mathcal{A}_E(X) \ni A \mapsto K_t(A) \in \mathrm{Hom}(L^2(B(x_0,r_0), \mathfrak{su}(E)), L^2_{k+1}(B(x_0,r_0), \mathfrak{su}(E)))$$

is C^∞ yields the second conclusion. q.e.d.

A.3. Positive partitions and cutoff functions with bounded differentials on Hilbert manifolds. We modify Lang's proof of the existence of a C^∞ *partition of unity* on a paracompact C^∞ manifold modelled on a separable Hilbert space [55, Corollary II.3.8] to produce a C^∞ *positive partition* whose cutoff functions have bounded differentials of all orders; see Proposition A.12. Recall:

Proposition A.5 [55, Corollary II.3.8]. *Let $\mathcal{X}$ be a paracompact C^∞ manifold modelled on a separable Hilbert space $\mathfrak{H}$. Then $\mathcal{X}$ admits locally finite, C^∞ partitions of unity: For any open cover of $\mathcal{X}$ there is a countable, locally finite open subcover $\{U_\alpha\}_{\alpha=1}^\infty$ and a family of C^∞ functions $\psi_\alpha : \mathcal{X} \to [0,1]$ such that*

- $\operatorname{supp}\psi_\alpha \subset U_\alpha$,
- $\sum_{\alpha=1}^\infty \psi_\alpha(x) = 1$ *for all* $x \in \mathcal{X}$.

We shall need the following generalizations to a Banach space setting of the analogous familiar facts from analysis on finite-dimensional spaces.

Lemma A.6 [1, Proposition 1.3.10]. *Let M be a topological space and let (N, d) be a complete metric space. Then the set $C(M, N)$ of all bounded continuous maps is a complete metric space with respect to the metric $D(f, g) := \sup\{d(f(x), g(x)) : x \in M\}$.*

Let $\mathbf{E}, \mathbf{F}$ be Banach spaces and let $U \subset \mathbf{E}$ be an open subset. By analogy with the usual definitions in finite dimensions, we let $C^s(U, \mathbf{F})$ be the set of C^s maps $f : U \to \mathbf{F}$ with norm

$$\|f\|_{C^s(U)} := \max_{0 \le p \le s} \sup_{x \in U} \|(D^p f)_x\| < \infty,$$

where $\|(D^p f)_x\|$ is the norm of $(D^p f)_x \in \operatorname{Hom}(\otimes^p \mathbf{E}, \mathbf{F})$. Lemma A.6 implies that $C^0(U, \mathbf{F})$ is a Banach space. In general we have:

Lemma A.7. *For any integer $s \ge 0$, the set $C^s(U, \mathbf{F})$ is a Banach space.*

Proof. For the case $s = 1$ this follows from Lemma A.6 and [37, Theorem 1.1.5]. The general case [1, pp. 113–114] is proved in the same way using Taylor's Theorem. q.e.d.

By analogy with the finite-dimensional case we set $C^\infty(U, \mathbf{F}) = \cap_{s \ge 0} C^s(U, \mathbf{F})$.

The C^∞ cutoff functions produced by Proposition A.5 may not necessarily have bounded differentials on Hilbert manifolds, as their supports are not compact, so we now describe a modified procedure which does produce bounded cutoff functions.

Lemma A.8 [55, Lemma II.3.5]. *Let M be a metric space and let $\{B_\alpha\}_{\alpha=1}^\infty$ be a covering of a subset $W \subset M$ by open balls. Then there exists a locally finite open covering $\{V_\alpha\}_{\alpha=1}^\infty$ of W such that $V_\alpha \subset B_\alpha$ for all α and*

$$V_\alpha = B_\alpha \cap {}^c\bar{B}(x_{\alpha,1}, r_{\alpha,1}) \cap \cdots \cap {}^c\bar{B}(x_{\alpha,\alpha-1}, r_{\alpha,\alpha-1}),$$

where ${}^c\bar{B} = M - \bar{B}$.

Lemma A.9. *Let $B_0, B_1 \dots, B_m$ be open balls in a Hilbert space $\mathfrak{H}$ and let V be a* scalloped *open subset:*

$$V = B_0 \cap {}^c\bar{B}_1 \cap \cdots \cap {}^c\bar{B}_m.$$

Then there exists a C^∞ function $\omega : \mathfrak{H} \to [0,1]$ such that $\omega(x) > 0$ if $x \in V$ and $\omega(x) = 0$ otherwise, while the differentials $D^p\omega$ are bounded for all $p \in \mathbb{N}$, with bounds depending only on $p, r_0, \dots, r_m$.

Proof. Let $\varphi : \mathbb{R} \to [0,1]$ be a C^∞ function such that $\varphi(t) = 1$ for $t \leq 1$, $0 < \varphi(t) < 1$ for $1 < t < 2$, and $\varphi(t) = 0$ for $t \geq 2$. Suppose $B_\alpha = B(x_\alpha, r_\alpha)$ for $\alpha = 0, \dots, m$. Set

$$\varphi_\alpha(x) := \varphi(r_\alpha^{-2}\|x - x_\alpha\|^2), \qquad x \in \mathfrak{H}, \quad \alpha = 1, \dots, m,$$

so $0 \leq \varphi_\alpha < 1$ on ${}^c\bar{B}_\alpha$ and $\varphi_\alpha = 1$ on $\bar{B}_\alpha$. Set $\varphi_0(x) = \varphi(2r_0^{-2}\|x - x_0\|^2)$, so $0 < \varphi_0 \leq 1$ on B_0 and $\varphi_0 = 0$ on ${}^cB_0$. Then

$$\omega := \varphi_0 \prod_{\alpha=1}^m (1 - \varphi_\alpha)$$

is positive on V and zero on ${}^cV = \mathfrak{H} - V$, while

$$\|D^p\omega\| \leq C,$$

for some positive constant $C = C(\varphi, p, r_0, \dots, r_m)$ and all $p \in \mathbb{N}$.

q.e.d.

Lemma A.10. *Let A_1, A_2 be non-empty, closed disjoint subsets of a separable Hilbert space $\mathfrak{H}$. Then there exists a C^∞ function $\chi : \mathfrak{H} \to [0,1]$ such that $\chi(x) = 0$ if $x \in A_1$ and $\chi(x) > 0$ if $x \in A_2$, with bounded differentials of all orders.*

Proof. By Lindelöf's Theorem there is a countable collection of balls $\{B_\alpha\}_{\alpha=1}^\infty$ covering A_2 such that $B_\alpha \subset {}^c A_1 = \mathfrak{H} - A_1$. Let $W = \cup_\alpha B_\alpha$ and find a locally finite refinement $\{V_\alpha\}$ of scalloped open subsets $V_\alpha \subset B_\alpha$, using Lemma A.8, which covers W. Using Lemma A.9 we find a C^∞ function $\omega_\alpha : \mathfrak{H} \to \mathbb{R}$ such that ω_α is positive on V_α, zero on ${}^c V_\alpha$, and has bounded differentials of all orders. The sum

$$\chi := \sum_{\alpha=1}^{\infty} \frac{2^{-\alpha}\omega_\alpha}{1 + \|\omega_\alpha\|_{C^\alpha(\mathfrak{H})}}$$

is finite on any neighborhood V_α. The function $\chi : \mathfrak{H} \to \mathbb{R}$ is positive on $\cup_\alpha V_\alpha = W \supset A_2$ and zero on A_1. Moreover, χ has bounded differentials of all orders, as desired. q.e.d.

Remark A.11. It is at this point that the usual proof of existence of a partition of unity encounters difficulties if we wish to ensure that the cutoff function χ has bounded differentials of all orders. In the proof of Theorem II.3.7 of [55], Lang first constructs a function ω obeying the conclusions of Lemma A.10 and then a C^∞ function $\sigma : \mathfrak{H} \to \mathbb{R}$ such that $\sigma > 0$ on the closed set ${}^c U$ disjoint from A_2, where U is the open set where $\omega > 0$, and $\sigma = 0$ on A_2. He then defines $\chi = \omega/(\omega + \sigma)$, so $\chi = 1$ on A_2. In finite dimensions, the compactness of closed bounded sets ensures that χ will have bounded differentials of all orders if A_2 is bounded, but this obviously fails in infinite dimensions.

Proposition A.12. *Let $\mathcal{X}$ be a paracompact C^∞ manifold modelled on a separable Hilbert space $\mathfrak{H}$. Then $\mathcal{X}$ admits a locally finite, C^∞* positive partition: *For any open cover of $\mathcal{X}$ there is a countable, locally finite open subcover by parametrizations $\{U_\alpha, \pi_\alpha\}_{\alpha=1}^\infty$, where*

$$\pi_\alpha : \mathfrak{H} \supset \mathbf{U}_\alpha \simeq U_\alpha \subset \mathcal{X},$$

and a family of C^∞ functions $\chi_\alpha : \mathcal{X} \to [0,1]$ such that

- $\operatorname{supp} \chi_\alpha \subset U_\alpha$,
- $\sum_{\alpha=1}^\infty \chi_\alpha(x) > 0$ *for all* $x \in \mathcal{X}$,
- $\chi_\alpha \circ \pi_\alpha : \mathfrak{H} \to [0,1]$ *has bounded differentials of all orders.*

Proof. Let $\{B_x\}$ be an open covering of $\mathcal{X}$ by balls and, using paracompactness and Lemma A.8, let $\{U_\alpha\}$ be a countable, locally finite refinement such that each open subset U_α is contained in some $B_{x(\alpha)}$, where $\pi_\alpha^{-1} : \mathcal{X} \supset B_{x(\alpha)} \to \mathfrak{H}$ is a coordinate chart and $\pi_\alpha^{-1}(U_\alpha) = \mathbf{U}_\alpha$. We now find open refinements $\{V_\alpha\}$ and then $\{W_\alpha\}$ such that

$$\overline{W}_\alpha \subset V_\alpha \subset \overline{V}_\alpha \subset U_\alpha,$$

the bar denoting closure in $\mathcal{X}$. For each α, Lemma A.10 and the identification $\pi_\alpha : \mathfrak{H} \supset \mathbf{U}_\alpha \simeq U_\alpha \subset \mathcal{X}$ provide a C^∞ cutoff function $\chi_\alpha : \mathcal{X} \to \mathbb{R}$ such that $\chi_\alpha > 0$ on $\overline{W}_\alpha$ and $\chi_\alpha = 0$ on $\mathcal{X} - V_\alpha$. The C^∞ map $\chi_\alpha \circ \pi_\alpha : \mathbf{U}_\alpha \to [0,1]$ extends by zero to a C^∞ map $\chi_\alpha \circ \pi_\alpha : \mathfrak{H} \to [0,1]$ having bounded differentials of all orders. Hence, $\{U_\alpha, \chi_\alpha\}$ is the desired positive partition. q.e.d.

Let $\overline{Y} = Y \cup \partial Y \subset X$ be a smooth submanifold-with-boundary. For any open subset $U \subset \mathcal{A}_E(Y)$ and C^s function $f : U \to \mathbb{C}$, we define

$$\|f\|_{C^s(U)} := \sup_{A \in U} \sup_{\substack{1 \le i \le s \\ \|a_i\|_{L^2_{k,A}(Y)} \le 1}} |(D^s f)_A(a_1, \dots, a_s)|. \tag{A.8}$$

Let $\{B([A_\alpha], r_\alpha)\}$ be a countable covering of $\mathcal{B}^*_E(Y)$ by L^2_k balls with subordinate locally finite subcover $\{U_\alpha\}$ and C^∞ positive partition $\{\chi_\alpha\}$, with $\operatorname{supp} \chi_\alpha \subset U_\alpha$. We may suppose that $B([A_\alpha], r_\alpha) = \pi(\mathbf{B}(A_\alpha, r_\alpha))$, where $\mathbf{B}(A_\alpha, r_\alpha) \subset \mathcal{A}^*_E(Y)$ is an open L^2_k ball in the Coulomb slice $A_\alpha + \operatorname{Ker} d^*_{A_\alpha}$ with center A_α and radius r_α, and $\pi : \mathcal{A}^*_E(Y) \to \mathcal{B}^*_E(Y)$ is the canonical projection. Let $\mathbf{U}_\alpha = \pi^{-1}(U_\alpha) \cap \mathbf{B}(A_\alpha, r_\alpha)$. Then $\mathbf{U}_\alpha \subset A_\alpha + \operatorname{Ker} d^*_{A_\alpha}$ and

$$\operatorname{supp} \chi_\alpha \circ \pi \subset \pi^{-1}(U_\alpha) = \mathcal{G}_E(Y) \cdot \mathbf{U}_\alpha \simeq \mathcal{G}_E(Y) \times \mathbf{U}_\alpha.$$

Now $\chi_\alpha \circ \pi(u(A)) = \chi_\alpha \circ \pi(A)$ for all $u \in \mathcal{G}_E(Y)$, and by the construction of Proposition A.12 the C^∞ map

$$\chi_\alpha \circ \pi : A_\alpha + \operatorname{Ker} d^*_{A_\alpha} \subset L^2_{k+1}(Y, \mathfrak{su}(E)) \to [0,1] \tag{A.9}$$

has bounded differentials of all orders with respect to the *fixed* L^2_{k+1,A_α} norm on $L^2_{k+1}(Y, \mathfrak{su}(E))$, that is, it has bounded differentials of all orders in the sense of Proposition A.12. Now for any $A \in \mathbf{U}_\alpha$, the L^2_{k+1,A_α} and $L^2_{k+1,A}$ norms on $L^2_{k+1}(Y, \mathfrak{su}(E))$ compare uniformly with constants depending only on $r_\alpha \ge \|A - A_\alpha\|_{L^2_{k;A_\alpha}}$ (since $k \ge 2$), and so the maps (A.9) have bounded differentials of all orders in the sense of equation (A.8). Moreover, the same holds for the maps

$$\chi_\alpha \circ \pi : \mathcal{A}^*_E(Y) \to [0,1].$$

Finally, we have a restriction map $r_Y : \mathcal{A}^*_E(X) \to \mathcal{A}^*_E(Y)$ given by $A \mapsto A|_Y$ and the composition

$$\chi_\alpha \circ \pi \circ r_Y : \mathcal{A}^*_E(X) \to [0,1]$$

again has bounded differentials of all orders in the sense of equation (A.8).

A.4. Uniform convergence of holonomy perturbations on neighborhoods of reducibles. In §2.5.2 we constrained the sequences perturbations $\vec{\tau}$ and $\vec{\vartheta}$ to vary in certain weighted ℓ^1_δ spaces. We now show that a sequence of positive weights $\delta \in \ell^\infty(\mathbb{R}^+)$ may be chosen in such a way that the sums $\vec{\tau} \cdot \vec{\mathfrak{m}}$ and $\vec{\vartheta} \cdot \vec{\mathfrak{m}}$ and all their differentials converge uniformly on $\mathcal{A}_E(X)$.

For any open subset $U \subset \mathcal{A}_E(X)$ and C^s map

$$\mathfrak{f} : U \to L^2_{k+1}(X, \mathfrak{su}(E)),$$

we define

$$\|\mathfrak{f}\|_{C^s(U)} := \sup_{A \in U} \sup_{\substack{1 \le i \le s \\ \|a_i\|_{L^2_{k,A}(X)} \le 1}} \|(D^s \mathfrak{f})_A(a_1, \dots, a_s)\|_{L^2_{k+1,A}(X)}. \tag{A.10}$$

We then have:

Proposition A.13. *Continue the notation of §2.5.2 and let $k \ge 2$ be an integer. Then there exists a sequence $\delta = (\delta_\alpha)_{\alpha=1}^\infty \in \ell^\infty((0,1])$ of positive weights such that the $\mathcal{G}_E$ equivariant maps*

$$\begin{aligned}
\vec{\tau} \cdot \vec{\mathfrak{m}} &: \mathcal{A}_E(X) \to L^2_{k+1}(X, \mathfrak{gl}(\Lambda^+) \otimes_{\mathbb{R}} \mathfrak{so}(\mathfrak{su}(E))), \\
\vec{\vartheta} \cdot \vec{\mathfrak{m}} &: \mathcal{A}_E(X) \to L^2_{k+1}(X, \mathrm{Hom}(W^+, W^-) \otimes_{\mathbb{C}} \mathfrak{sl}(E)),
\end{aligned}$$

are C^∞, with uniformly bounded differentials of all orders in the sense of (A.10)*. In particular, they satisfy the following C^0 estimates for $r \ge k+1$,*

$$\begin{aligned}
\sup_{A \in \mathcal{A}_E} \|\vec{\vartheta} \cdot \vec{\mathfrak{m}}(A)\|_{L^2_{k+1,A}(X)} &\le C \|\vec{\vartheta}\|_{\ell^1_\delta(C^r(X))}, \\
\sup_{A \in \mathcal{A}_E} \|\vec{\tau} \cdot \vec{\mathfrak{m}}(A)\|_{L^2_{k+1,A}(X)} &\le C \|\vec{\tau}\|_{\ell^1_\delta(C^r(X))},
\end{aligned}$$

for some positive constant $C = C(g,k)$, and more generally they satisfy the C^s estimates of (A.11) *and* (A.12)*, for every integer $s \ge 0$.*

Remark A.14. As should be clear from their definition, the maps $\vec{\tau}\cdot\vec{\mathfrak{m}}$ and $\vec{\vartheta}\cdot\vec{\mathfrak{m}}$ are not analytic, although this will cause no difficulty in practice.

Proof. It suffices to consider $\vec{\vartheta}\cdot\vec{\mathfrak{m}}$, since the argument for $\vec{\tau}\cdot\vec{\mathfrak{m}}$ is obviously identical. We first observe that the sum $\vec{\vartheta}\cdot\vec{\mathfrak{m}}(A)$ is finite for each connection $A\in\mathcal{A}_E(X)$, and so defines an element of $L^2_{k+1}(X,\mathrm{Hom}(W^+,W^-)\otimes\mathfrak{gl}(E))$, while it is identically zero if A is reducible on X. However, on any open neighborhood of a reducible connection $A\in\mathcal{A}_E(X)$, the number of terms in the sum $\vec{\vartheta}\cdot\vec{\mathfrak{m}}$ may be infinite, and so our task is to choose a sequence of weights δ such that this sum and all its differentials converge uniformly on $\mathcal{A}_E(X)$. The sum $\vec{\vartheta}\cdot\vec{\mathfrak{m}}$ will then define a C^∞ map. We begin with a couple of preparatory lemmas.

Lemma A.15. *Let $\overline{Y}=Y\cup\partial Y\subset X$ be a smooth submanifold with boundary, let $B(A_0,r_0)\subset\mathcal{A}^*_E(Y)$ be an open L^2_k ball with $k\geq 2$, and let $\eta\in L^2(Y,\mathfrak{su}(E))$. Then the maps*

$$B(A_0,r_0)\ni A\mapsto\mathfrak{f}(A):=e^{-td^*_A d_A}\eta\in L^2_{k+1}(Y,\mathfrak{su}(E))$$

have bounded differentials of all orders, in the sense of (A.10)*, with constants $c=c(k,r_0,s,t)$:*

$$\|\mathfrak{f}\|_{C^s(B(A_0,r_0))}\leq c\|\eta\|_{L^2(Y)}.$$

Proof. The Sobolev multiplication theorems and the fact that $k\geq 2$ imply that the norms

$$\|\zeta\|_{L^2_{k+1,A}(Y)}\quad\text{and}\quad\|\zeta\|'_{L^2_{k+1,A}(Y)}:=\|(1+\Delta_A)^{(k+1)/2}\zeta\|_{L^2(Y)}$$

on $L^2_{k+1}(Y,\mathfrak{su}(E))$ are equivalent for $A\in B(A_0,r_0)\subset\mathcal{A}^*_E(Y)$, with constants depending at most on k,r_0.

Since $\Delta_A=d^*_A d_A$, its derivative with respect to A in the direction $\delta A=a$ is given by $\delta\Delta_A=[a,\cdot]^*d_A+d^*_A[a,\cdot]$; we use the abbreviation $\delta\Delta_A=(D\Delta_{(\cdot)})_A(a)$. The connection A_0 is irreducible by hypothesis and so for a small enough open L^2_k ball $B(A_0,r_0)$, there is a positive constant $\lambda_0>0$ such that $\lambda[A]\geq\lambda_0>0$ for all $A\in B(A_0,r_0)$, where $\lambda[A]$ is the least eigenvalue of Δ_A. Thus,

$$\mathrm{Spec}(\Delta_A)\subset[\lambda_0,\infty),\qquad A\in B(A_0,r_0),$$

and, for any holomorphic function f on an open neighborhood $\Omega \subset \mathbb{C}$ with $\operatorname{Spec}(\Delta_A) \subset \Omega$ and Γ any contour that surrounds $\operatorname{Spec}(\Delta_A)$ in Ω, we have [76]

$$\begin{aligned} f(\Delta_A) &= \frac{1}{2\pi i}\oint_\Gamma f(\lambda)(\lambda - \Delta_A)^{-1}\,d\lambda, \\ (Df)_{\Delta_A}(\delta\Delta_A) &= \frac{1}{2\pi i}\oint_\Gamma f(\lambda)(\lambda - \Delta_A)^{-1}(\delta\Delta_A)(\lambda - \Delta_A)^{-1}\,d\lambda, \end{aligned}$$

and similarly for all higher-order derivatives. Note that

$$\begin{aligned} &(1+\Delta_A)^{(k+1)/2} f(\Delta_A) \\ &\qquad = \frac{1}{2\pi i}\oint_\Gamma (1+\lambda)^{(k+1)/2} f(\lambda)(\lambda - \Delta_A)^{-1}\,d\lambda, \\ &(1+\Delta_A)^{(k+1)/2}(Df)_{\Delta_A}(\delta\Delta_A) \\ &\qquad = \frac{1}{2\pi i}\oint_\Gamma (1+\lambda)^{(k+1)/2} f(\lambda)(\lambda - \Delta_A)^{-1}(\delta\Delta_A)(\lambda - \Delta_A)^{-1}\,d\lambda. \end{aligned}$$

We can fix Ω and $\Gamma \subset \Omega$ such that $\operatorname{dist}(\Gamma, \operatorname{Spec}(\Delta_A)) \geq d_0 > 0$, for some positive constant d_0 and all $A \in B(A_0, r_0)$ (see [34, §1.6]). Now choose $f(z) = e^{-tz}$ and $f_{k+1}(z) = (1+z)^{(k+1)/2} f(z)$, and estimate as in [34, p. 53]:

$$\begin{aligned} \|e^{-t\Delta_A}\eta\|_{L^2_{k+1,A}(Y)} &\leq c\|(1+\Delta_A)^{(k+1)/2} e^{-t\Delta_A}\eta\|_{L^2(Y)} \\ &\leq c\oint_\Gamma \left|(1+\lambda)^{(k+1)/2} e^{-t\lambda}\,d\lambda\right| \sup_{\lambda\in\Gamma}\|(\lambda - \Delta_A)^{-1}\eta\|_{L^2(Y)} \\ &\leq c\|\eta\|_{L^2(Y)}, \end{aligned}$$

which gives the desired C^0 bound

$$\sup_{A\in B(A_0,r_0)} \|e^{-t\Delta_A}\eta\|_{L^2_{k+1,A}(Y)} \leq c\|\eta\|_{L^2(Y)},$$

where $c = c(d_0, k, r_0, t)$.

To obtain the C^1 bound, observe that

$$\begin{aligned} &\|(Df)_{\Delta_A}(\delta\Delta_A)\eta\|_{L^2_{k+1,A}(Y)} \\ &\leq c\|(1+\Delta_A)^{(k+1)/2}(Df)_{\Delta_A}(\delta\Delta_A)\eta\|_{L^2(Y)} \\ &\leq c\oint_\Gamma \left|(1+\lambda)^{(k+1)/2} f(\lambda)\,d\lambda\right| \sup_{\lambda\in\Gamma}\|(\lambda - \Delta_A)^{-1}(\delta\Delta_A)(\lambda - \Delta_A)^{-1}\eta\|_{L^2(Y)} \\ &\leq \sup_{\lambda\in\Gamma} c\|(\lambda - \Delta_A)^{-1}(\delta\Delta_A)(\lambda - \Delta_A)^{-1}\eta\|_{L^2(Y)}. \end{aligned}$$

Our expression for $\delta\Delta_A$ gives

$$\begin{aligned}
&\|(\lambda-\Delta_A)^{-1}(\delta\Delta_A)(\lambda-\Delta_A)^{-1}\eta\|_{L^2(Y)} \\
&\le \|(\lambda-\Delta_A)^{-1}([a,\cdot]^*d_A+d_A^*[a,\cdot])(\lambda-\Delta_A)^{-1}\eta\|_{L^2(Y)} \\
&\le c\left(\|[a,\cdot]^*d_A(\lambda-\Delta_A)^{-1}\eta\|_{L^2(Y)}+\|d_A^*[a,\cdot](\lambda-\Delta_A)^{-1}\eta\|_{L^2(Y)}\right) \\
&\le c\|a\|_{L^2_{2;A}(Y)}\|(\lambda-\Delta_A)^{-1}\eta\|_{L^2_{2;A}(Y)} \\
&\le c\|a\|_{L^2_{k,A}(Y)}\|\eta\|_{L^2(Y)},
\end{aligned}$$

where $c=c(r_0,d_0)$. Combining these estimates yields

$$\|\delta(e^{-t\Delta_A})\eta\|_{L^2_{k+1,A}(Y)} \le c\|a\|_{L^2_{k,A}(Y)}\|\eta\|_{L^2(Y)},$$

and so we have the desired C^1 bound

$$\sup_{A\in B(A_0,r_0)}\ \sup_{\|a\|_{L^2_{k,A}(X)}\le 1}\|(De^{-t\Delta_{(\cdot)}})_A(a)\eta\|_{L^2_{k+1,A}(Y)} \le c\|\eta\|_{L^2(Y)},$$

for some $c=c(d_0,k,r_0,t)$. The analysis can be repeated, essentially unchanged, for all higher differentials and is left to the reader. q.e.d.

Lemma A.16. *The $\mathcal{G}_E$ equivariant holonomy maps*

$$\mathfrak{m}_{j,l,\alpha}:\mathcal{A}_E(X)\to L^2_{k+1}(X,\mathfrak{su}(E))$$

of (2.20) *are C^∞ with bounded differentials of all orders in the sense of* (A.10).

Proof. Recall from (2.17) and (2.20) that

$$\begin{aligned}
\mathfrak{m}_{j,l,\alpha}(A) &= \beta_j[A]\chi_{j,\alpha}[A|_{B(x_j,2R_0)}]\varphi_j\mathfrak{h}_{\gamma_{j,l,\alpha}}(A), \\
\varphi_j\mathfrak{h}_{\gamma_{j,l,\alpha}}(A) &= \varphi_j K_t(A|_{B(x_0,2R_0)})\hat{\mathfrak{h}}_{\gamma_{j,l,\alpha}}(A).
\end{aligned}$$

The cutoff functions $\chi_{j,\alpha}\circ\pi\circ r_{B(x_j,2R_0)}:\mathcal{A}_E(X)\to[0,1]$ have bounded differentials of all orders, in the sense of (A.8), by the remarks following Proposition A.12; moreover, they are supported in $\mathcal{A}_E^*(X)$. The functions $\beta_j:\mathcal{A}_E(X)\to[0,1]$ have bounded differentials of all orders, as is clear from their definition in (2.18). Finally, from the proofs of Lemmas A.3 and A.4, together with Lemma A.15, one can see that the maps

$$\varphi_j K_t(\cdot|_{B(x_0,2R_0)})\hat{\mathfrak{h}}_{\gamma_{j,l,\alpha}}:\mathcal{A}_E^*(X)\to L^2_{k+1}(X,\mathfrak{su}(E))$$

have bounded differentials of all orders in the sense of (A.10) on open subsets $\mathcal{G}_E \cdot \mathbf{B}(A_0, r_0) \subset \mathcal{A}_E^*(X)$, where $\mathbf{B}(A_0, r_0)$ is an open L_k^2 ball in $\mathbf{K}_{A_0} \subset \mathcal{A}_E^*(X)$. q.e.d.

Given Lemma A.16, we have

$$M_{\alpha,s} := \max_{j,l} \|\mathfrak{m}_{j,\alpha,l}\|_{C^s(\mathcal{A}_E)} < \infty,$$

where we recall that $1 \le l \le 3$ and $1 \le j \le N_b$. Now choose a sequence of positive weights $\delta = (\delta_\alpha)_{\alpha=1}^\infty \in \ell^\infty(\mathbb{R}^+)$ by setting

$$\delta_\alpha := \left(1 + \max_{0 \le s \le \alpha} M_{\alpha,s}\right)^{-1}, \qquad \alpha \in \mathbb{N},$$

and suppose $\vec{\vartheta} \in \ell_\delta^1(\mathbb{A}, C^r(X))$. Then

$$\|\vec{\vartheta}\|_{\ell_\delta^1(C^r(X))} = \sum_{j,l,\alpha} \delta_\alpha^{-1} \|\vartheta_{j,l,\alpha}\|_{C^r(X)} < \infty,$$

and therefore,

$$\begin{aligned}
\|\vec{\vartheta} \cdot \vec{\mathfrak{m}}\|_{C^s(\mathcal{A}_E)} &\le \sum_{j,l,\alpha} \|\vartheta_{j,l,\alpha} \mathfrak{m}_{j,l,\alpha}\|_{C^s(\mathcal{A}_E)} \\
&\le c \sum_{j,l,\alpha} \|\vartheta_{j,l,\alpha}\|_{L^2_{k+1}(X)} \|\mathfrak{m}_{j,l,\alpha}\|_{C^s(\mathcal{A}_E)} \quad \text{by (A.10)} \\
&\le c \sum_{j,l,\alpha} \|\vartheta_{j,l,\alpha}\|_{C^r(X)} M_{\alpha,s},
\end{aligned}$$

since $r \ge k+1$. Here, $c = c(g,k)$ is a universal positive constant coming from the continuous Sobolev multiplication $L^2_{k+1} \times L^2_{k+1,A} \to L^2_{k+1,A}$. Hence, using the facts that $M_{\alpha,s} \le 1 + \max_{0 \le t \le \alpha} M_{\alpha,t} = \delta_\alpha^{-1}$ for $\alpha \ge s$

and $1 \leq \delta_\alpha^{-1}$ for $1 \leq \alpha \leq s-1$, we get

$$
\begin{aligned}
\|\vec{\vartheta} \cdot \vec{\mathfrak{m}}\|_{C^s(\mathcal{A}_E)} &\leq c \sum_{j,l} \left(\sum_{\alpha=1}^{s-1} M_{\alpha,s} \|\vartheta_{j,l,\alpha}\|_{C^r(X)} + \sum_{\alpha \geq s} \delta_\alpha^{-1} \|\vartheta_{j,l,\alpha}\|_{C^r(X)} \right) \\
&\leq c \left(1 + \max_{1 \leq \alpha \leq s-1} M_{\alpha,s} \right) \sum_{j,l} \left(\sum_{\alpha=1}^{s-1} \|\vartheta_{j,l,\alpha}\|_{C^r(X)} \right. \\
&\qquad \left. + \sum_{\alpha \geq s} \delta_\alpha^{-1} \|\vartheta_{j,l,\alpha}\|_{C^r(X)} \right) \\
&\leq c \left(1 + \max_{1 \leq \alpha \leq s-1} M_{\alpha,s} \right) \sum_{j,l,\alpha} \delta_\alpha^{-1} \|\vartheta_{j,l,\alpha}\|_{C^r(X)} \\
&= C \|\vec{\vartheta}\|_{\ell^1_\delta(C^r(X))} < \infty,
\end{aligned} \tag{A.11}
$$

where $C = C(g, s, k)$ is defined by the last equality above. In particular, we see that $\vec{\vartheta} \cdot \vec{\mathfrak{m}}$ is a C^s map on $\mathcal{A}_E(X)$, for every integer $s \geq 0$. The same argument gives

$$
\|\vec{\tau} \cdot \vec{\mathfrak{m}}\|_{C^s(\mathcal{A}_E)} \leq C \|\vec{\tau}\|_{\ell^1_\delta(C^r(X))} < \infty, \tag{A.12}
$$

and $\vec{\tau} \cdot \vec{\mathfrak{m}}$ is a C^s map on $\mathcal{A}_E(X)$, for every integer $s \geq 0$. q.e.d.

References

[1] R. Abraham, J. E. Marsden & T. Ratiu, *Manifolds, tensor analysis, and applications*, 2nd ed., Springer, New York, 1988.

[2] R. A. Adams, *Sobolev spaces*, Academic Press, Orlando, FL, 1975.

[3] S. Agmon, *Unicité et convexité dans les problèmes differentiels*, Sém. Math. Sup., 1965, Univ. of Montreal Press, 1966.

[4] S. Agmon & L. Nirenberg, *Lower bounds and uniqueness theorems for solutions of differential equations in Hilbert spaces*, Comm. Pure Appl. Math. **20** (1967) 207–229.

[5] N. Aronszajn, *A unique continuation theorem for solutions of elliptic partial differential equations or inequalities of the second order*, J. Math. Pures Appl. **36** (1957) 235–249.

[6] M. F. Atiyah, V. K. Patodi & I. M. Singer, *Spectral asymmetry in Riemannian geometry. I*, Math. Proc. Cambridge Philos. Soc. **77** (1975) 43–69.

[7] T. Aubin, *Nonlinear analysis on complex manifolds. Monge-Ampère Equations*, Springer, New York, 1982.

[8] P. J. Braam & S. K. Donaldson, *Floer's work on instanton homology, knots, and surgery*, The Floer Memorial Volume, (eds. H. Hofer et al.), Birkhäuser, Boston, 1995, 196–256.

[9] S. B. Bradlow & G. D. Daskalopoulos, *Moduli of stable pairs for holomorphic bundles over Riemann surfaces*, Internat. J. Math. **2** (1991) 477–513.

[10] S. B. Bradlow & O. Garcia-Prada, *Non-abelian monopoles and vortices*, Proc. Conference on Geom. and Phys. (Aarhus, 1995), Lecture Notes in Pure and Appl. Math. **184**, Dekker, New York, 1997, 567–589; `alg-geom/9602010`.

[11] T. Bröcker & T. tom Dieck, *Representations of compact Lie groups*, Springer, New York, 1985.

[12] I. Chavel, *Eigenvalues in Riemannian geometry*, Academic Press, Orlando, FL, 1984.

[13] R. Constantinescu, MIT Thesis (Mathematics), 1996.

[14] S. K. Donaldson, *An application of gauge theory to four-dimensional topology*, J. Differential Geom. **18** (1983) 279–315.

[15] ———, *Connections, cohomology and the intersection forms of four-manifolds*, J. Differential Geom. **24** (1986) 275–341.

[16] ———, *Irrationality and the h-cobordism conjecture*, J. Differential Geom. **26** (1987), 141–168.

[17] ———, *The orientation of Yang-Mills moduli spaces and 4-manifold topology*, J. Differential Geom. **26** (1987) 397–428.

[18] ———, *Polynomial invariants for smooth four-manifolds,* Topology **29** (1990) 257–315.

[19] ———, *The Seiberg-Witten equations and four-manifold topology*, Bull. Amer. Math. Soc. **33** (1996) 45–70.

[20] S. K. Donaldson & P. B. Kronheimer, *The geometry of four-manifolds*, Oxford University Press, Oxford, 1990.

[21] G. Ellingsrud & L. Göttsche, *Wall-crossing formulas, Bott residue formula and the Donaldson invariants of rational surfaces*, `alg-geom/9506019`.

[22] P. M. N. Feehan, *Critical-exponent Sobolev norms and the slice theorem for the quotient space of connections*, submitted to a print journal, October, 1996; `dg-ga/9711004`.

[23] ______, *Generic metrics, irreducible rank-one* PU(2) *monopoles, and transversality*, submitted to a print journal, August 27, 1997.

[24] P. M. N. Feehan & T. G. Leness, PU(2) *monopoles and relations between four-manifold invariants*, Topology Appl. **88** (1988) 111–145; `dg-ga/9709022`.

[25] ______, PU(2) *monopoles, II: Highest-level singularities and relations between four-manifold invariants*, submitted to a print journal, December 8, 1997; `dg-ga/9712005`.

[26] ______, PU(2) *monopoles, III: Existence of gluing and obstruction maps*, in preparation.

[27] ______, PU(2) *monopoles, IV: Surjectivity of gluing maps*, in preparation.

[28] R. Fintushel & R. Stern, *Donaldson invariants of 4-manifolds with simple type*, J. Differential Geom. **42** (1995) 577–633.

[29] A. Floer, *An instanton-invariant for 3-manifolds*, Comm. Math. Phys. **118** (1988) 215–240.

[30] D. Freed & K. K. Uhlenbeck, *Instantons and four-manifolds*, 2nd ed., Springer, New York, 1991.

[31] R. Friedman & J. W. Morgan, *Smooth four-manifolds and complex surfaces*, Springer, Berlin, 1994.

[32] R. Friedman & Z. Qin, *Flips of moduli spaces and transition formulas for Donaldson polynomial invariants of rational surfaces*, Comm. Anal. Geom. (1995) 11–83.

[33] D. Gilbarg & N. Trudinger, *Elliptic partial differential equations of second order*, 2nd ed., Springer, New York, 1983.

[34] P. B. Gilkey, *Invariance theory, the heat equation, and the Atiyah-Singer index theorem*, Publish or Perish, Wilmington, DE, 1984.

[35] L. Göttsche, *Modular forms and Donaldson invariants for 4-manifolds with* $b_+ = 1$, J. Amer. Math. Soc. **9** (1996), 827–843.

[36] P. Hartman, *Ordinary differential equations*, 2nd ed., Birkhäuser, Boston, 1982.

[37] L. Hörmander, *The analysis of linear partial differential operators*, Vol. III, Springer, New York, 1983.

[38] S. Hyun, J. Park & J-S. Park, *Topological QCD*, Nuclear Phys. B **453** (1995) 199–224; `hep-th/9503201`.

[39] ______, $N = 2$ *Supersymmetric QCD and four-manifolds, I: the Donaldson and the Seiberg-Witten invariants*, `hep-th/9508162`.

[40] R. V. Kadison & J. R. Ringrose, *Fundamentals of the theory of operator algebras*, Vol. I, Academic Press, New York, 1983.

[41] T. Kato, *Perturbation theory for linear operators*, 2nd ed., Springer, New York, 1984.

[42] J. Kazdan, *Unique continuation in geometry*, Comm. Pure Appl. Math. **41** (1988) 667–681.

[43] S. Kobayashi *Differential geometry of complex vector bundles*, Princeton University Press, Princeton, NJ, 1987.

[44] S. Kobayashi & K. Nomizu, *Foundations of differential geometry*, Vols. I & II, Wiley, New York, 1963 & 1968.

[45] D. Kotschick & J. W. Morgan, SO(3) *invariants for four-manifolds with* $b^+ = 1$. *II*, J. Differential Geom. **39** (1994) 433–456.

[46] P. B. Kronheimer, *Seiberg-Witten invariants for four-manifolds*, Lectures at Oxford University, Spring 1995.

[47] P. B. Kronheimer & T. S. Mrowka, *The genus of embedded surfaces in the projective plane*, Math. Res. Lett. **1** (1994) 797–808.

[48] ———, *Embedded surfaces and the structure of Donaldson's polynomial invariants*, J. Differential Geom. **43** (1995) 573–734.

[49] ———, *Monopoles and contact structures*, Invent. Math. **130** (1997) 209–255.

[50] J. M. F. Labastida & C. Lozano, *Mathai-Quillen formulation of twisted N=4 supersymmetric Gauge theories in four dimensions*, Nuclear Phys. B **502** (1997) 741–790; `hep-th/9702106`.

[51] J. M. F. Labastida & M. Mariño, *Non-abelian monopoles on four-manifolds*, Nuclear Phys. B **448** (1995) 373–395; `hep-th/9504010`.

[52] ———, *Polynomial invariants for* $SU(2)$ *monopoles*, Nuclear Phys. B **456** (1995) 633–668; `hep-th/9507140`.

[53] ———, *Twisted* $N = 2$ *supersymmetry with central charge and equivariant cohomology*, Comm. Math. Phys. **185** (1997) 37–71; `hep-th/9603169`.

[54] O. A. Ladyzhenskaya & N. N. Ural'tseva, *Linear and quasilinear elliptic equations*, Academic Press, New York, 1968.

[55] S. Lang, *Differential and Riemannian manifolds*, 3rd ed., Springer, New York, 1995.

[56] H. B. Lawson, *The theory of gauge fields in four dimensions*, Conf. Board Math. Sci. **58**, Amer. Math. Soc., Providence, RI, 1985.

[57] H. B. Lawson & M-L. Michelsohn, *Spin geometry*, Princeton Univ. Press, Princeton, NJ, 1988.

[58] T. G. Leness, *Donaldson wall-crossing formulas via topology*, Forum Math., to appear; `dg-ga/9603016`.

[59] J. Li & G. Tian, *Virtual moduli cycles and Gromov-Witten invariants of general symplectic manifolds*, `alg-geom/9602007`.

[60] G. Moore & E. Witten, *Integration over the u-plane in Donaldson theory*, Adv. Theor. Math. Phys. **1** (1997) 298–387; `hep-th/9709193`.

[61] J. W. Morgan, *Gauge theory and the topology of smooth four-manifolds*, Lecture Notes, Harvard University, 1988.

[62] ———, *The Seiberg-Witten equations and applications to the topology of smooth four-manifolds*, Princeton University Press, Princeton, NJ, 1996.

[63] J. W. Morgan, T. S. Mrowka & D. Ruberman, *L^2 moduli spaces and a vanishing theorem for Donaldson polynomial invariants*, Internat. Press, Cambridge, MA, 1994.

[64] C. B. Morrey, *Multiple integrals in the calculus of variations*, Springer, New York, 1966.

[65] T. S. Mrowka, *Seiberg-Witten invariants for four-manifolds*, Lectures at Harvard University, Spring 1995.

[66] T. S. Mrowka, P. S. Ozsváth & B. Yu, *Seiberg-Witten monopoles on Seifert-fibered spaces*, Comm. Anal. Geom. **5** (1997) 685–791.

[67] C. Okonek & A. Teleman, *The coupled Seiberg-Witten equations, vortices, and moduli spaces of stable pairs*, Internat. J. Math. **6** (1995) 893–910; `alg-geom/9505012`.

[68] ———, *Quaternionic monopoles*, Comm. Math. Phys. **180** (1996) 363–388; `alg-geom/9505029`.

[69] ———, *Recent developments in Seiberg-Witten theory and complex geometry*, `alg-geom/9612015`.

[70] T. H. Parker, *Gauge theories on four dimensional Riemannian manifolds*, Comm. Math. Phys. **85** (1982) 563–602.

[71] V. Y. Pidstrigach, *From Seiberg-Witten to Donaldson:* SO(3) *monopole equations*, Lecture at Newton Institute, Cambridge, December 1994.

[72] ———, *Seiberg-Witten and Donaldson invariants*, Lecture at Oberwolfach, May 1996.

[73] V. Y. Pidstrigach & A. N. Tyurin, *Invariants of the smooth structure of an algebraic surface arising from the Dirac operator*, Russian Acad. Sci. Izv. Math. **40** (1993) 267–351.

[74] ———, *Localization of Donaldson invariants along the Seiberg-Witten classes*, `dg-ga/9507004`.

[75] Y. Ruan, *Virtual neighborhoods and pseudoholomorphic curves*, `alg-geom/9611021`.

[76] W. Rudin, *Functional analysis*, McGraw-Hill, New York, 1973.

[77] D. Salamon, *Spin geometry and Seiberg-Witten invariants*, Birkhäuser, Boston, to appear.

[78] S. Sedlacek, *A direct method for mimizing the Yang-Mills functional*, Comm. Math. Phys. **86** (1982) 515–527.

[79] B. Siebert, *Gromov-Witten invariants for general symplectic manifolds*, `alg-geom/9608005`.

[80] S. Smale, *An infinite-dimensional version of Sard's theorem*, Amer. J. Math. **87** (1973) 213–221.

[81] C. H. Taubes, *Self-dual Yang-Mills connections on non-self-dual 4-manifolds*, J. Differential Geom. **17** (1982) 139–170.

[82] ———, *Self-dual connections on 4-manifolds with indefinite intersection matrix*, J. Differential Geom. **19** (1984) 517–560.

[83] ———, *Path-connected Yang-Mills moduli spaces*, J. Differential Geom. **19** (1984) 337–392.

[84] ———, *A framework for Morse theory for the Yang-Mills functional*, Invent. Math. **94** (1988) 327–402.

[85] ———, *The stable topology of self-dual moduli spaces*, J. Differential Geom. **29** (1989) 162–230.

[86] ———, *Casson's invariant and gauge theory*, J. Differential Geom. **31** (1990) 547–599.

[87] ———, *L^2 moduli spaces on 4-manifolds with cylindrical ends*, Internat. Press, Cambridge, MA, 1993.

[88] ———, *Unique continuation theorems in gauge theories*, Comm. Anal. Geom. **2** (1994) 35–52.

[89] ———, *The Seiberg-Witten invariants and symplectic forms*, Math. Res. Lett. **1** (1994) 809–822.

[90] ———, *Floer theory for twisted circle bundles*, Internat. Press, Cambridge, MA, to appear.

[91] A. Teleman, *Non-abelian Seiberg-Witten theory and stable oriented pairs*, Internat. J. Math. **8** (1997) 507–535; `alg-geom/9609020`.

[92] ———, *Moduli spaces of* PU(2) *monopoles*, `dg-ga/9702006`.

[93] ———, Untitled preprint.

[94] K. K. Uhlenbeck, *Removable singularities in Yang-Mills fields*, Comm. Math. Phys. **83** (1982) 11–29.

[95] ———, *Connections with L^p bounds on curvature*, Comm. Math. Phys. **83** (1982) 31–42.

[96] ———, *The Chern classes of Sobolev connections*, Comm. Math. Phys. **101** (1985) 449–457.

[97] V. S. Varadarajan, *Lie groups, Lie algebras, and their representations*, Springer, New York, 1984.

[98] E. Witten, *Monopoles and four-manifolds*, Math. Res. Lett. **1** (1994) 769–796.

[99] H-J. Yang, *Transition functions and a blow-up formula for Donaldson polynomials*, Columbia University Thesis, 1992.

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