J. DIFFERENTIAL GEOMETRY
48 (1998) 445-495

LIE GROUP VALUED MOMENT MAPS

ANTON ALEKSEEV, ANTON MALKIN
& ECKHARD MEINRENKEN

Abstract

We develop a theory of "quasi"-Hamiltonian G-spaces for which the moment map takes values in the group G itself rather than in the dual of the Lie algebra. The theory includes counterparts of Hamiltonian reductions, the Guillemin-Sternberg symplectic cross-section theorem and of convexity properties of the moment map. As an application we obtain moduli spaces of flat connections on an oriented compact 2-manifold with boundary as quasi-Hamiltonian quotients of the space $G^2 \times \cdots \times G^2$.

1. Introduction

The purpose of this paper is to study Hamiltonian group actions for which the moment map takes values not in the dual of the Lie algebra but in the group itself.

For the circle group S^1 this situation has been studied in literature (see e.g. [18], [24]). The standard example is the real 2-torus $T^2 = S^1 \times S^1$, with its standard area form and the circle acting by rotation of the first S^1; the moment map is given by projection to the second S^1. It is in fact known that every symplectic S^1-action on a symplectic manifold for which the 2-form has integral cohomology class admits an S^1-valued moment map.

In this paper we consider group valued moment maps for general non-abelian compact Lie groups G. One theory encountering group valued moment maps is the theory of Poisson-Lie group actions on symplectic manifolds [16], [17], where G is a Poisson Lie group and the target of the moment map is the dual Poisson-Lie group G^*. In [1] it was shown that for compact, connected, simply connected Lie groups G this theory is equivalent to the standard theory of Hamiltonian actions.

Received July 31, 1997.

In this paper we introduce the notion of "quasi"-Hamiltonian (q-Hamiltonian) G-spaces consisting of a G-manifold M, an invariant 2-form ω and a group valued moment map $\mu : M \to G$ satisfying certain natural compatibility conditions. It turns out that for non-abelian G these spaces differ in many respects from Hamiltonian G-spaces. In particular, the conditions that the 2-form ω be non-degenerate and closed have to be replaced by somewhat more complicated conditions. In spite of these differences, Hamiltonian reductions of these spaces are defined and result in spaces with symplectic forms.

Basic examples for q-Hamiltonian G-spaces are conjugacy classes in G. Another example is the "double" $D(G) = G \times G$ generalizing the above T^2-example. Note that for G compact and simply-connected, $D(G)$ does not admit a symplectic structure because its second cohomology is trivial. Yet, it admits a (minimally degenerate) q-Hamiltonian structure.

As an application we obtain the moduli space $M(\Sigma)$ of flat connections on a closed 2-manifold Σ of genus k as a q-Hamiltonian quotient of the space G^{2k}. Our construction is a reinterpretation of the construction due to Jeffrey[12] and Huebschmann[11] (see also [10]) which represents $M(\Sigma)$ as a symplectic quotient of a certain finite dimensional non-compact symplectic space X (which is in fact an open subset of G^{2k}). Their construction is based on the group cohomology approach of Goldman[9], Karshon[14] and Weinstein[23]. For another closely related construction see King-Sengupta[15].

We show that the space G^{2k} admits a q-Hamiltonian structure with a G-valued moment map corresponding to the G-action by simultaneous conjugations. Then $M(\Sigma)$ is obtained as a q-Hamiltonian quotient of G^{2k}. The closedness and non-degeneracy of the 2-form on the moduli space follows from the basic properties of the reduction procedure. More generally, if Σ has a boundary and $\mathcal{C} = \{\mathcal{C}_j\}$ are conjugacy classes of holonomies associated to the boundary components, the corresponding moduli space $M(\Sigma, \mathcal{C})$ is a q-Hamiltonian reduction of the space $G^{2(r+k)}$ where $r + 1$ is the number of boundary components. We will explicitly describe the 2-form on $G^{2(r+k)}$ that gives rise to the symplectic form on moduli space and check that the answer coincides with Atiyah-Bott's [3] gauge theoretic construction of the symplectic form on $M(\Sigma, \mathcal{C})$.

The paper is organized as follows. The definition of a q-Hamiltonian G-space with G-valued moment map is given in Section 2. Section 3 contains basic examples of q-Hamiltonian G-spaces, and Section 4 discusses some of their basic properties. In Section 5 we show that Hamiltonian reduction extends to the present setting. In Section 6 we define the "fusion product" of two G-Hamiltonian spaces, with moment map the pointwise product of the two moment maps. In Section 7 we prove a q-Hamiltonian version of the Guillemin-Sternberg symplectic cross-section theorem and discuss convexity properties of the moment map. In Section 8 we explain the relation of our theory to Hamiltonian loop group spaces, and prove that there is a natural one-to-one correspondence between compact q-Hamiltonian G-spaces and Hamiltonian LG-spaces with proper moment map. Section 9 contains the application of our results to moduli spaces of flat connections. In Section 10 we explain the relation to the Lu-Weinstein theory of Poisson-Lie group actions on symplectic manifolds.

Acknowledgements

A.A. is grateful to Department of Physics, Yale University and to Institute for Theoretical Physics, ETH Zürich for hospitality during the period when this paper was written. During his stay at Yale the research of A.A. was supported in part by DOE Outstanding Young Investigator Award. A.A. thanks B. Enriquez, G. Felder, J.-C. Hausmann and A. Knutson for useful discussions.

E.M. was supported by a Feodor Lynen Fellowship from the Humboldt Foundation. He is grateful to C. Woodward for many interesting suggestions.

2. Quasi-Hamiltonian G-spaces

In this section we recall the definition of Hamiltonian G-spaces with $\mathfrak{g}^*$-valued moment maps and then present our definition for "quasi"-Hamiltonian G-spaces with G-valued moment maps.

2.1. Hamiltonian G-spaces. Throughout this paper G denotes a compact Lie group with Lie algebra $\mathfrak{g}$. A G-manifold is a manifold M together with an action $\mathcal{A} : G \times M \to M$. Given $g \in G$, $x \in M$ we will often write

$$\mathcal{A}(g, x) = x^g,$$

and for $\xi \in \mathfrak{g}$ we denote by v_ξ the generating vector field on M. Given a closed invariant 2-form ω on a G-space $(M, \mathcal{A})$, the contraction $\iota(v_\xi)\omega$

is closed because

$$\mathrm{d}\iota(v_\xi)\omega = \mathcal{L}_{v_\xi}\omega - \iota(v_\xi)\mathrm{d}\omega = 0.$$

In symplectic geometry, one is mainly interested in the case that $\iota(v_\xi)\omega$ is exact and ω is non-degenerate:

Definition 2.1. A Hamiltonian G-space $(M, \mathcal{A}, \omega, \mu)$ is a G-manifold $(M, \mathcal{A})$, together with an invariant 2-form $\omega \in \Omega^2(M)^G$ and an equivariant *moment map* $\mu \in C^\infty(M, \mathfrak{g}^*)^G$ such that:

(A1) The form ω is closed: $\mathrm{d}\omega = 0$.

(A2) The moment map satisfies

$$\iota(v_\xi)\omega = \mathrm{d}\langle \mu, \xi\rangle \quad \text{for all} \quad \xi \in \mathfrak{g}.$$

(A3) The form ω is non-degenerate.

Simple consequences of the axioms are the following description of the kernel and image of the derivative of the moment map:

$$\mathrm{im}(\mathrm{d}_x\mu) = \mathfrak{g}_x^0, \quad \ker(\mathrm{d}_x\mu)^\omega = \{v_\xi(x), \xi \in \mathfrak{g}\}. \tag{0.1}$$

Here $\mathfrak{g}_x$ is the isotropy algbra of x and $\mathfrak{g}_x^0$ its annihilator in $\mathfrak{g}^*$, and for any subspace $E \subset T_xM$ the subspace

$$E^\omega := \{v \in T_xM,\, \omega(v, w) = 0 \text{ for all } w \in E\}$$

is its ω-orthogonal complement.

2.2. q-Hamiltonian G-spaces. Let us try to develop the notion of a "quasi"-Hamiltonian G-space $(M, \mathcal{A}, \omega, \mu)$ with a G-valued moment map $\mu : M \to G$. We denote by $(\cdot, \cdot)$ some choice of an invariant positive definite inner product on $\mathfrak{g}$ which we use to identify $\mathfrak{g} \cong \mathfrak{g}^*$, and by $\theta, \overline{\theta} \in \Omega^1(G, \mathfrak{g})$ the left- and right- invariant Maurer-Cartan forms. (In a faithful matrix representations for G, $\theta = g^{-1}\mathrm{d}g$ and $\overline{\theta} = \mathrm{d}g\, g^{-1}$.) If G is abelian (so that $\theta = \overline{\theta}$), the natural replacement for the moment map condition is

$$\iota(v_\xi)\omega = \mu^*(\theta, \xi). \tag{0.2}$$

If G is non-abelian, condition (0.2) does not work as it is incompatible with the anti-symmetry of ω. One is forced to replace it by

$$\iota(v_\xi)\omega = \frac{1}{2}\mu^*(\theta + \overline{\theta}, \xi). \tag{0.3}$$

If we want ω to be G-invariant, this condition is no longer compatible with $\mathrm{d}\omega = 0$: Indeed,

$$0 \stackrel{!}{=} \mathcal{L}_{v_\xi}\omega = (\mathrm{d}\iota(v_\xi) + \iota(v_\xi)\mathrm{d})\omega = \frac{1}{2}\mu^*\mathrm{d}\,(\theta + \overline{\theta}, \xi) + \iota(v_\xi)\mathrm{d}\omega. \tag{0.4}$$

This equation can be rewritten as follows. Let $\chi \in \Omega^3(G)$ denote the canonical closed bi-invariant 3-form on G:

$$\chi = \frac{1}{12}(\theta, [\theta, \theta]) = \frac{1}{12}(\overline{\theta}, [\overline{\theta}, \overline{\theta}]). \tag{0.5}$$

Denote by v_ξ^r and v_ξ^l the right and left-invariant vector field on G generated by ξ. The fundamental vector field for the adjoint action is $v_\xi = v_\xi^r - v_\xi^l$ so that

$$\iota(v_\xi)\theta = \mathrm{Ad}_{g^{-1}}\,\xi - \xi,\ \iota(v_\xi)\overline{\theta} = \xi - \mathrm{Ad}_g\,\xi, \tag{0.6}$$

which together with the structure equations $\mathrm{d}\theta = -\frac{1}{2}[\theta, \theta]$ and $\mathrm{d}\overline{\theta} = \frac{1}{2}[\overline{\theta}, \overline{\theta}]$ gives

$$\iota(v_\xi)\chi = \frac{1}{2}\,\mathrm{d}(\overline{\theta} + \theta, \xi). \tag{0.7}$$

Using these formulas condition (0.4) becomes

$$0 \stackrel{!}{=} \iota(v_\xi)(\mathrm{d}\omega + \mu^*\chi),$$

so that we are lead to require $\mathrm{d}\omega = -\mu^*\chi$. However, the moment map condition (0.3) is in general also incompatible with non-degeneracy of ω since it implies that all generating vectors $v_\xi(x)$ with ξ a solution of $\mathrm{Ad}_{\mu(x)}\,\xi = -\xi$ have to lie in the kernel of ω_x.

We are therefore lead to the following "minimal" definition.

Definition 2.2. A quasi-Hamiltonian G-space is a G-manifold $(M, \mathcal{A})$ together with an invariant 2-form $\omega \in \Omega(M)^G$ and an equivariant map $\mu \in C^\infty(M, G)^G$ such that:

(B1) The differential of ω is given by:

$$\mathrm{d}\omega = -\mu^*\chi.$$

(B2) The map μ satisfies

$$\iota(v_\xi)\omega = \frac{1}{2}\mu^*(\theta + \overline{\theta}, \xi).$$

(B3) At each $x \in M$, the kernel of ω_x is given by

$$\ker \omega_x = \{v_\xi(x),\, \xi \in \ker(\mathrm{Ad}_{\mu(x)} + 1)\}.$$

We will refer to μ as a moment map.

In the following section we will give some examples of q-Hamiltonian G-spaces. Let us, however, first make a remark on the definition.

Remark 2.1. For ordinary Hamiltonian G-spaces, the defining conditions can be expressed elegantly in terms of the de Rham model for equivariant cohomology. Let $\Omega^*_G(M) = \bigoplus_k \Omega^k_G(M)$ be the complex of equivariant differential forms,

$$\Omega^k_G(M) = \bigoplus_{2l+j=k} (\Omega^j(M) \otimes S^l\mathfrak{g}^*)^G$$

with differential

$$(\mathrm{d}_G\alpha)(\xi) = \mathrm{d}(\alpha(\xi)) - \iota(v_\xi)\alpha(\xi).$$

Conditions (A1) and (A2) can be summarized by the requirement that $\omega_G(\xi) = \omega + \langle\mu,\xi\rangle$ be an equivariantly closed form in $\Omega^2_G(M)$:

$$\mathrm{d}_G\omega_G(\xi) = \mathrm{d}\omega + \mathrm{d}\langle\mu,\xi\rangle - \iota(v_\xi)\omega = 0. \tag{0.8}$$

However, one can also take a slightly different point of view and rewrite (0.8) as follows:

$$\mathrm{d}_G\omega(\xi) = \mathrm{d}\omega - \iota(v_\xi)\omega = -\mathrm{d}\langle\mu,\xi\rangle = -\mu^*\chi_G(\xi).$$

Here $\chi_G \in \Omega^3_G(\mathfrak{g}^*)$ is an equivariant 3-form defined as $\chi_G(\xi) = \mathrm{d}\langle a,\xi\rangle$, where $a : \mathfrak{g}^* \to \mathfrak{g}^*$ is the identity map. This latter formulation extends to q-Hamiltonian G-spaces; the relevant d_G-closed equivariant 3-form $\chi_G \in \Omega^3_G(G)$ is

$$\chi_G(\xi) = \chi + \frac{1}{2}(\theta + \overline{\theta}, \xi).$$

3. Examples of q-Hamiltonian G-spaces

3.1. Conjugacy classes in G. Basic examples of Hamiltonian G-spaces are provided by coadjoint orbits $\mathcal{O} \subset \mathfrak{g}^*$. Their q-Hamiltonian counterparts are conjugacy classes $\mathcal{C} \subset G$.

Proposition 3.1 (Conjugacy classes). *For every conjugacy class $\mathcal{C} \subset G$ there exists a unique invariant 2-form $\omega \in \Omega^2(\mathcal{C})^G$ for which $\mathcal{C}$ becomes a q-Hamiltonian G-space, with moment map the embedding $\mu : \mathcal{C} \hookrightarrow G$. The value of ω at $f \in \mathcal{C}$ is given on fundamental vector fields v_ξ, v_η by*

$$\omega_f(v_\xi, v_\eta) = \frac{1}{2}\Big((\eta, \mathrm{Ad}_f\,\xi) - (\xi, \mathrm{Ad}_f\,\eta)\Big). \tag{0.9}$$

The 2-form (0.9) on conjugacy classes plays an important role in [10].

Proof. Clearly ω is G-invariant. The fact that (0.9) satisfies condition (B2) is tautological because

$$\omega_f(v_\xi, v_\eta) = \frac{1}{2}\big(\,\mathrm{Ad}_{f^{-1}}\,\eta - \mathrm{Ad}_f\,\eta, \xi\big) = \frac{1}{2}\,\iota(v_\eta)\big(\theta_f + \overline{\theta}_f, \xi\big).$$

To check (B1) consider the fibration $\pi_f : G \to \mathcal{C}$ defined by $\pi_f(u) = \mathrm{Ad}_u\, f$. The pull-back of ω is the left-invariant 2-form

$$\pi_f^*\omega = \frac{1}{2}(\mathrm{Ad}_f\,\theta, \theta).$$

We have

$$\pi_f^*\mathrm{d}\omega = \mathrm{d}\pi_f^*\omega = -\frac{1}{2}(\mathrm{Ad}_f[\theta,\theta],\theta) + \frac{1}{2}(\mathrm{Ad}_f\,\theta, [\theta,\theta]).$$

Using $\pi_f^*\theta = \mathrm{Ad}_{uf^{-1}}\,\theta - \overline{\theta}$ one verifies that this last expression is equal to $-\pi_f^*\chi$ which shows (B1). Suppose next that $\xi \in \mathfrak{g}$ is such that $v_\xi(f)$ is in the kernel of ω_f. By definition of ω this means

$$\mathrm{Ad}_f\,\xi - \mathrm{Ad}_{f^{-1}}\,\xi = 0,$$

or $\xi \in \ker(\mathrm{Ad}_{f^2} - 1)$. The kernel of $(\mathrm{Ad}_{f^2} - 1)$ is a direct sum

$$\ker(\mathrm{Ad}_{f^2} - 1) = \ker(\mathrm{Ad}_f - 1) \oplus \ker(\mathrm{Ad}_f + 1).$$

For any $\xi \in \ker(\mathrm{Ad}_f - 1)$ we have $v_\xi(f) = 0$. This shows that the kernel of ω_f is given by (B3). Uniqueness of ω is a consequence of (B2) since $\mathcal{C}$ is a homogeneous space. q.e.d.

Remark 3.1. Since the kernel of the 2-form ω on $\mathcal{C}$ has constant rank, it defines a distribution on $\mathcal{C}$. It is integrable, with leaves given by the orbits $\mathrm{Ad}(Z_{f^2}) \cdot f \subset \mathrm{Ad}(G) \cdot f$ as f ranges over $\mathcal{C}$. In particular, if f^2 is contained in the center $Z(G)$, then the 2-form ω on $\mathcal{C}$ is identically zero.

3.2. Double $D(G)$. Our next example is a q-Hamiltonian $G \times G$-space which plays the same role as the cotangent bundle T^*G in the category of Hamiltonian G-spaces. Since this new q-Hamiltonian space is a product of two copies of G, we call it the double of G and denote it by $D(G)$ (alluding to the definition of the double in the theory of Quantum Groups). Let

$$D(G) := G \times G,$$

and let a and b be projections to the first and second factors in the direct product. Introduce a $G \times G$-action $\mathcal{A}_D$ on $D(G)$ via

$$(a,b)^{(g_1,g_2)} = (g_1 a g_2^{-1}, g_2 b g_1^{-1}).$$

Define a moment map $\mu_D = (\mu_1, \mu_2) : D(G) \to G^2$ as

$$\mu_1(a,b) = ab, \quad \mu_2(a,b) = a^{-1}b^{-1},$$

and let the 2-form ω_D be defined by

$$\omega_D = \frac{1}{2}(a^*\theta,\, b^*\overline{\theta}) + \frac{1}{2}(a^*\overline{\theta},\, b^*\theta).$$

Proposition 3.2 (The double $D(G)$). *The quadruple $(D(G), \mathcal{A}_D, \mu_D, \omega_D)$ is a q-Hamiltonian $G \times G$-space.*

Proof. The equivariance of the moment map is immediate from the definition. Using

$$\mu_1^*\theta = \mathrm{Ad}_{b^{-1}}\, a^*\theta + b^*\theta,\ \mu_2^*\theta = -\,\mathrm{Ad}_b\, a^*\overline{\theta} - b^*\overline{\theta} \tag{0.10}$$

we compute

$$\mu_1^*\chi = a^*\chi + b^*\chi - \frac{1}{2}\, d(a^*\theta, b^*\overline{\theta}), \quad \mu_2^*\chi = -a^*\chi - b^*\chi - \frac{1}{2}\, d(a^*\overline{\theta}, b^*\theta),$$

which gives the required property $d\omega_D = -\mu^*\chi$. Next, let

$$\xi = (\xi_1, \xi_2) \in \mathfrak{g} \oplus \mathfrak{g},$$

and v_ξ be the corresponding vector field on $D(G)$. If $\xi_2 = 0$, using (0.10) we find

$$\begin{aligned} \iota(v_\xi)\omega_D =& \frac{1}{2}(\, a^*\overline{\theta} + b^*\theta + \mathrm{Ad}_{b^{-1}}\, a^*\theta + \mathrm{Ad}_a\, b^*\overline{\theta}, \xi_1) \\ =& \frac{1}{2}\mu_1^*(\theta + \overline{\theta}, \xi_1). \end{aligned}$$

A similar calculation gives the condition for $\xi_1 = 0$.

Suppose now that $v \in T_{(a,b)}D(G)$ is in the kernel of ω. Let (ξ, η) be such that v is the value at (a, b) of the right-invariant vector field $v = (v^r_\xi, v^r_\eta)$. The condition

$$0 = \iota(v)\omega_D|_{(a,b)} = (\mathrm{Ad}_a\, b^*\overline{\theta} + b^*\theta,\, \xi) + (\mathrm{Ad}_{b^{-1}}\, a^*\theta + a^*\overline{\theta},\, \eta)$$

gives two equations

$$\xi + \mathrm{Ad}_{ab}\, \xi = 0, \qquad \eta + \mathrm{Ad}_{ab}\, \eta = 0.$$

Setting

$$\alpha = \frac{1}{2}(\xi + \eta), \quad \beta = -\frac{1}{2}\, \mathrm{Ad}_b(\xi - \eta).$$

we obtain

$$\xi = \alpha + \mathrm{Ad}_a\, \beta\ ,\ \ \eta = \alpha + \mathrm{Ad}_{b^{-1}}\, \beta. \tag{0.11}$$

Equation (0.11) says that v is the value at (a, b) of the fundamental vector field for (α, β). Moreover,

$$\alpha + \mathrm{Ad}_{ab}\, \alpha = 0\ ,\ \ \beta + \mathrm{Ad}_{a^{-1}b^{-1}}\, \beta = 0,$$

which shows that (B3) is satisfied. q.e.d.

Remark 3.2. We will also make use of different coordinates on $D(G)$, corresponding to the left trivialization of T^*G: setting $u = a$ and $v = ba$ the action reads

$$(u, v)^{g_1, g_2} = (g_1\, u\, g_2^{-1},\ \mathrm{Ad}_{g_2}\, v),$$

the moment map is

$$\mu_1(u, v) = \mathrm{Ad}_u\, v,\ \ \mu_2(u, v) = v^{-1}$$

and the 2-form is given by

$$\omega_D = \frac{1}{2}(\mathrm{Ad}_v\, u^*\theta, u^*\theta) + \frac{1}{2}(u^*\theta, v^*\theta + v^*\overline{\theta}).$$

3.3. From Hamiltonian G-spaces to q-Hamiltonian G-spaces. In this section we show how to construct a q-Hamiltonian G-space from a usual Hamiltonian G-space $(M, \mathcal{A}, \sigma, \Phi)$. The basic Lemma is (see also Jeffrey [12])

Lemma 3.3. *For $s \in \mathbb{R}$ let $\exp_s : \mathfrak{g} \to G$ be defined by $\exp_s(\eta) = \exp(s\eta)$. The 2-form on the Lie algebra $\mathfrak{g}$ given by*

$$\varpi = \frac{1}{2} \int_0^1 (\exp_s^* \overline{\theta}, \frac{\partial}{\partial s} \exp_s^* \overline{\theta}) ds$$

is G-invariant and satisfies $d\varpi = -\exp^* \chi$. If v_ξ is a fundamental vector field for the adjoint G-action on $\mathfrak{g}$, we have

$$\iota(v_\xi)\varpi = -d(\cdot, \xi) + \frac{1}{2} \exp^*(\theta + \overline{\theta}, \xi). \tag{0.12}$$

We omit the proof at this point since Lemma 3.3 is a consequence of Proposition 8.1 below.

Proposition 3.4. *Let $(M, \mathcal{A}, \sigma, \Phi)$ be a Hamiltonian G-space. Then M with 2-form $\omega = \sigma + \Phi^* \varpi$ and moment map $\mu = \exp(\Phi)$ satisfies all axioms of a q-Hamiltonian G-space except possibly the non-degeneracy condition, (B3). If the differential $d_\xi \exp$ is bijective for all $\xi \in \Phi(M)$, then (B3) is satisfied as well, and $(M, \mathcal{A}, \omega, \mu)$ is a q-Hamiltonian G-space.*

Proof. Equivariance of μ is clear, and invariance of ω follows from equivariance of exp and Φ and invariance of ϖ. Condition (B1) is obtained from

$$\mathrm{d}\omega = \mathrm{d}\sigma + \mathrm{d}\Phi^* \varpi = -\Phi^* \exp^* \chi = -\mu^* \chi,$$

and (B2) from the calculation

$$\iota(v_\xi)\omega = \mathrm{d}(\Phi, \xi) + \frac{1}{2} \Phi^* \exp^*(\theta + \overline{\theta}, \xi) - \mathrm{d}(\Phi, \xi) = \frac{1}{2} \mu^*(\theta + \overline{\theta}, \xi).$$

To check the non-degeneracy condition (B3) suppose that $v \in T_x M$ is in the kernel of ω. Then

$$\iota(v)\sigma_x = -\iota(v)(\Phi^* \varpi)_x. \tag{0.13}$$

Since the 1-form on the right-hand side annihilates the kernel of $d_x \Phi$, equation (13) implies that v is σ-orthogonal to $\ker(d_x \Phi)$. By (0.1) the σ-orthogonal complement to $\ker(d_x \Phi)$ is equal to the span of the fundamental vector fields $v_\xi(x)$, $\xi \in \mathfrak{g}$. Letting $v = v_\xi(x)$ and using (B2) we arrive at the condition

$$\Phi^* \exp^*(\theta + \overline{\theta}, \xi)_x = 0 \tag{0.14}$$

at x. Pairing with a fundamental vector field $v_\eta(x)$, we find

$$\big(\eta, \big(\operatorname{Ad}_{\mu(x)^2} - 1\big)\xi\big) = 0$$

for all η, which shows

$$\xi \in \ker(\operatorname{Ad}_{\mu(x)^2} - 1) = \ker(\operatorname{Ad}_{\mu(x)} - 1) \oplus \ker(\operatorname{Ad}_{\mu(x)} + 1).$$

As we remarked in Section 2.2, solutions of $\operatorname{Ad}_{\mu(x)} \xi = -\xi$ lead to elements in the kernel. Therefore it suffices to consider the case $\operatorname{Ad}_{\mu(x)} \xi = \xi$, where the condition (0.14) reads

$$(\Phi^*(\exp^* \theta, \xi))_x = 0.$$

If $\Phi(x) \in \mathfrak{g}^* \cong \mathfrak{g}$ is not a singular value of exp, this equation implies that ξ annihilates the image of the tangent map $\mathrm{d}_x\Phi$. By (0.1) this means $\xi \in \mathfrak{g}_x$, that is $v_\xi(x) = 0$, and the proof is complete. q.e.d.

Remark 3.3. Suppose conversely that $(M, \mathcal{A}, \omega, \mu)$ is a q-Hamiltonian G-space. Assume also that there exists $U \subset \mathfrak{g}$ such that exp is a diffeomorphism from U onto some subset $V \subset G$ containing $\mu(M)$, and let $\log : V \to U$ be the inverse. Reversing the argument in the proof of Proposition 3.4 we see that $(M, \mathcal{A}, \omega - \mu^* \log^* \varpi, \log(\mu))$ is a Hamiltonian G-space in the usual sense.

4. Properties of q-Hamiltonian G-spaces

The following result summarizes a number of consequences of Definition 2.2, analogous to (0.1) for Hamiltonian G-spaces.

Proposition 4.1. *Let $(M, \mathcal{A}, \omega, \mu)$ be a q-Hamiltonian G-space and $x \in M$. For any subspace $E \subset T_xM$ let $E^\omega = \{v \in T_xM,\ \omega(v, w) = 0$ for all $w \in E\}$ denote its ω-orthogonal complement. Then the following hold:*

1. *The map*

$$\ker(\operatorname{Ad}_{\mu(x)} + 1) \to \ker \omega_x, \quad \xi \mapsto v_\xi(x) \tag{14a}$$

is an isomorphism.

1. $\ker(d_x\mu) \cap \ker \omega_x = \{0\}$.

2. $\mathrm{im}(\mu^*\theta)_x = \mathfrak{g}_x^\perp$.

3. $(\ker d_x\mu)^\omega = \{v_\xi(x), \xi \in \mathfrak{g}\}$.

Proof. Observe first that there is an orthogonal splitting of $\mathfrak{g}$,

$$\begin{aligned}\mathfrak{g} &= \ker(\mathrm{Ad}_{\mu(x)} + 1) \oplus \mathrm{im}(\mathrm{Ad}_{\mu(x)} + 1) \\ &= \ker(\mathrm{Ad}_{\mu(x)} + 1) \oplus \ker(\mathrm{Ad}_{\mu(x)} - 1) \oplus \mathrm{im}(\mathrm{Ad}_{\mu(x)^2} - 1)\end{aligned}$$

and that $\mathfrak{g}_{\mu(x)} = \ker(\mathrm{Ad}_{\mu(x)} - 1)$.

1. By the non-degeneracy condition (B3) the map (14a) is surjective. On the other hand if $v_\xi(x) = 0$, then $v_\xi(\mu(x)) = 0$ by equivariance of the moment map or equivalently $\xi \in \ker(\mathrm{Ad}_{\mu(x)} - 1) \subset \mathrm{im}(\mathrm{Ad}_{\mu(x)} + 1)$. This shows injectivity.

2. Let $v \in \ker(\mathrm{d}_x\mu) \cap \ker\omega_x$. Using Property 1, we can write $v = v_\eta$ with $\eta \in \ker(\mathrm{Ad}_{\mu(x)} + 1)$. By equivariance of the moment map,

$$0 = \mathrm{d}_x\mu(v_\eta(x)) = v_\eta(\mu(x)),$$

which shows $\eta \in \mathfrak{g}_{\mu(x)} = \ker(\mathrm{Ad}_{\mu(x)} - 1)$. Thus $\eta = 0$ and consequently $v = 0$.

3. Using the defining equation for the moment map,

$$\omega(v_\xi, v) = \frac{1}{2}\iota(v)\mu^*(\theta + \overline{\theta}, \xi) = \frac{1}{2}\big((\mathrm{Ad}_\mu + 1)\iota(v)\mu^*\theta, \xi\big)$$

and Property 1, we have

$$\begin{aligned}&\mathrm{im}(\mu^*\theta)_x \cap \mathrm{im}(\mathrm{Ad}_{\mu(x)} + 1) \\ &\quad = (\mathrm{Ad}_{\mu(x)} + 1)\,\mathrm{im}(\mu^*\theta)_x = \{\xi, v_\xi(x) \in \ker\omega_x\}^\perp \\ &\quad = (\mathfrak{g}_x \oplus \ker(\mathrm{Ad}_{\mu(x)} + 1))^\perp = \mathfrak{g}_x^\perp \cap \mathrm{im}(\mathrm{Ad}_{\mu(x)} + 1).\end{aligned}$$

On the other hand, equivariance of the moment map together with (0.6) shows that

$$\mathrm{im}(\mu^*\theta)_x \supseteq \mathrm{im}(\mathrm{Ad}_{\mu(x)^{-1}} - 1) = \ker(\mathrm{Ad}_{\mu(x)} - 1)^\perp \supseteq \ker(\mathrm{Ad}_{\mu(x)} + 1).$$

Since $\mathfrak{g}_x \subseteq \mathfrak{g}_{\mu(x)} = \ker(\mathrm{Ad}_{\mu(x)} - 1) \subseteq \mathrm{im}(\mathrm{Ad}_{\mu(x)} + 1)$, we also have

$$\mathfrak{g}_x^\perp \supseteq \ker(\mathrm{Ad}_{\mu(x)} + 1);$$

these two equations prove 3.

4. The inclusion $\supseteq$ is a direct consequence from the defining property (B2) of the moment map. Equality follows by dimension count: Using 2. and 3, we obtain

$$\begin{aligned}\dim(\ker \mathrm{d}_x\mu)^\omega &= \dim M - \dim\ker(\mathrm{d}_x\mu) = \dim\mathrm{im}(\mathrm{d}_x\mu)\\ &= \dim\mathfrak{g} - \dim\mathfrak{g}_x.\end{aligned}$$

q.e.d.

Remark 4.1. We shall also need the following refinement of Property 4.: Suppose that G is a product $G = G_1 \times G_2$, and let $\mu = (\mu_1, \mu_2)$ be the components of the moment map. Then

$$(\ker \mathrm{d}_x\mu_1)^\omega = \{v_\xi, \xi \in \mathfrak{g}_1\} + \ker\omega_x.$$

To see this note that as a direct consequence of Property 3.,

$$\mathrm{im}(\mathrm{d}_x\mu_1) = \mathfrak{g}_x^\perp \cap \mathfrak{g}_1 = (\mathfrak{g}_1)_x^\perp \cap \mathfrak{g}_1$$

which implies the above equation by dimension count.

Proposition 4.2. *Let $(M_j, \mathcal{A}_j, \omega_j, \mu_j)$ $(j = 1, 2)$ be q-Hamiltonian G-spaces, and $F : M_1 \to M_2$ an equivariant smooth map such that $F^*\omega_2 = \omega_1$ and $F^*\mu_2 = \mu_1$. Then F is an immersion.*

Proof. Let $x \in M$. Since $F^*\omega_2 = \omega_1$ we have $\ker \mathrm{d}_xF \subset \ker(\omega_1)_x$. Since $F^*\mu_2 = \mu_1$, Lemma 4.1 shows that d_xF restricts to an isomorphism $\ker(\omega_1)_x \cong \ker(\omega_2)_{F(x)}$. Thus $\ker \mathrm{d}_xF \cap \ker(\omega_1)_x = \{0\}$. q.e.d.

Corollary 4.3. *If $(M, \mathcal{A}, \omega, \mu)$ is a q-Hamiltonian G-space on which G-acts transitively, the moment map μ is a covering onto a conjugacy class $\mathcal{C} \subset G$.*

Proof. Let $x \in M$ and $f = \mu(x)$. By equivariance μ is a submersion from M onto $\mathcal{C} = \mathrm{Ad}(G) \cdot f$. (B2) shows that ω is the pull-back by μ of the 2-form on $\mathcal{C}$. Therefore μ is an immersion by Proposition 4.2.

q.e.d.

Proposition 4.4 (Inversion). *If $(M, \mathcal{A}, \omega, \mu)$ is a q-Hamiltonian G-space, then the quadruple $(M, \mathcal{A}, -\omega, \mu^{-1})$ is also a q-Hamiltonian G-space.*

Proof. This is an immediate consequence of the fact that under the inversion map $\mathrm{Inv} : G \to G$, $g \mapsto g^{-1}$,

$$\mathrm{Inv}^*\,\theta = -\overline{\theta},\ \mathrm{Inv}^*\,\overline{\theta} = -\theta,\ \mathrm{Inv}^*\,\chi = -\chi.$$

q.e.d.

We denote the q-Hamiltonian G-space given in Proposition 4.4 by M^-. Note that if $\mathcal{C}$ is the conjugacy class of $f \in G$, then $\mathcal{C}^-$ is the conjugacy class of f^{-1}.

All of the above results are natural analogues of the well-known facts about Hamiltonian G-spaces. The following Theorem introduces a nontrivial automorphism of q-Hamiltonian G-spaces that does not have a counterpart for Hamiltonian G-spaces. As we shall see later, this automorphism corresponds to Dehn twists of 2-manifolds.

Theorem 4.5 (Twist automorphism). *Let $(M, \mathcal{A}, \omega, \mu)$ be a q-Hamiltonian G-space. The map*

$$Q:\, M \to M,\ x \mapsto x^{\mu(x)}$$

is a diffeomorphism satisfying $Q^*\omega = \omega$, $Q^*\mu = \mu$.

Proof. By equivariance of the moment map, $Q^*\mu = \mathrm{Ad}(\mu)\mu = \mu$. The map $x \mapsto x^{\mu(x)^{-1}}$ is an inverse to Q so that Q is a diffeomorphism. The tangent map to Q is given by

$$\mathrm{d}_x Q(v) = \mathrm{d}_x\mathcal{A}_{\mu(x)} v + v_\xi(Q(x)),$$

where $\xi = \iota(v)(\mu^*\overline{\theta})_x \in \mathfrak{g}$. Letting $v_j \in T_xM$ be two tangent vectors and $\xi_j = \iota(v_j)(\mu^*\overline{\theta})_x$, we find that the expression

$$Q^*\omega_x(v_1, v_2) = \omega_{Q(x)}(\mathrm{d}_x Q(v_1), \mathrm{d}_x Q(v_2))$$

is the sum of the following four terms:

$$\omega_{Q(x)}(\mathrm{d}_x\mathcal{A}_{\mu(x)} v_1, \mathrm{d}_x\mathcal{A}_{\mu(x)} v_2) = \omega_x(v_1, v_2)$$

plus

$$\begin{aligned}
\frac{1}{2}\omega_{Q(x)}(v_{\xi_1}(Q(x)), \mathrm{d}_x\mathcal{A}_{\mu(x)} v_2) &= \frac{1}{2}\iota(\mathrm{d}_x\mathcal{A}_{\mu(x)} v_2)\mu^*(\theta + \overline{\theta}, \xi_1)_{Q(x)} \\
&= \frac{1}{2}\iota(v_2)\mu^*(\theta + \overline{\theta}, \mathrm{Ad}(g^{-1})\xi_1)_x \\
&= \frac{1}{2}(\mathrm{Ad}(\mu(x)^{-1})\xi_2 - \xi_2, \mathrm{Ad}(\mu(x)^{-1})\xi_1) \\
&= \frac{1}{2}(\xi_1, \xi_2 - \mathrm{Ad}(\mu(x))\xi_2)
\end{aligned}$$

minus the same term with v_1, v_2 exchanged, plus

$$\frac{1}{2}\iota\big(v_{\xi_2}(Q(x))\big)\mu^*(\theta+\overline{\theta},\xi_1)_{Q(x)} = \frac{1}{2}(\xi_1, \mathrm{Ad}(\mu(x)^{-1})\xi_2 - \mathrm{Ad}(\mu(x))\xi_2).$$

Adding up all contributions we find $Q^*\omega_x(v_1,v_2) = \omega_x(v_1,v_2)$. q.e.d.

Remark 4.2. If G is a product $G = G_1 \times G_2$, one has a twist automorphism for every factor: That is, both maps $x \mapsto x^{\mu_j(x)}$ are equivariant diffeomorphisms preserving μ and ω.

On q-Hamiltonian G-spaces one can define G-invariant Hamiltonian vector fields.

Proposition 4.6 (Hamiltonian dynamics). *Let $(M, \mathcal{A}, \omega, \mu)$ be a q-Hamiltonian G-space. Then for every G-invariant function $F \in C^\infty(M,\mathbb{R})^G$ there is a unique smooth vector field v_F satisfying the following conditions:*

$$\iota(v_F)\omega = dF \ , \ \iota(v_F)\mu^*\theta = 0; \tag{0.15}$$

the vector field v_F is G-invariant and preserves ω and μ. Thus $\{F_1, F_2\} = \omega(V_{F_1}, V_{F_2})$ defines a Poisson structure on $C^\infty(M,\mathbb{R})^G$.

Proof. By Proposition 4.1, Property 1 the map

$$A : TM \to T^*M \oplus \mathfrak{g},\ v \mapsto (\iota(v)\omega, \iota(v)\mu^*\theta).$$

is injective. Its image defines a smooth sub-bundle $E \subset T^*M \oplus \mathfrak{g}$. We need to show that $x \mapsto (\mathrm{d}_xF, 0)$ defines a section of E; the corresponding vector field v_F is then just the pre-image under A. Since F is G-invariant, d_xF annihilates the space $\{v_\xi(x), \xi \in \mathfrak{g}\} \subset T_xM$ and in particular the kernel $\ker\omega_x$. This shows that d_xF is contained in the image of the map $T_xM \to T_x^*M$, $v \mapsto \iota(v)\omega_x$. Let $v_0 \in T_xM$ with $\mathrm{d}_xF = \iota(v_0)\omega_x$. and set $\xi := \iota(v_0)(\mu^*\theta)_x$. For all $\eta \in \mathfrak{g}$, using once again that F is invariant, we have

$$\begin{aligned} 0 = \iota(v_\eta)\mathrm{d}_xF = -\,\iota(v_0)\iota(v_\eta)\omega_x &= -\frac{1}{2}\iota(v_0)\mu^*(\theta+\overline{\theta},\eta)_x \\ &= -\,\frac{1}{2}(\xi + \mathrm{Ad}_{\mu(x)^{-1}}\,\xi, \eta), \end{aligned}$$

which shows $\xi \in \ker(\mathrm{Ad}_{\mu(x)} + 1)$. Consequently $v_1 := v_\xi(x) \in \ker \omega_x$ and $v = v_0 + \frac{1}{2} v_1$ still solves $\iota(v)\omega_x = \mathrm{d}_x F$. By (B2) we obtain

$$\iota(v)(\mu^*\theta)_x = \iota(v_0 + \frac{1}{2} v_1)(\mu^*\theta)_x = \xi + \frac{1}{2}(\mathrm{Ad}_{\mu^{-1}} \xi - \xi) = \xi - \xi = 0,$$

so that v solves $A(v) = (\mathrm{d}_x F, 0)$. This shows that $(\mathrm{d}_x F, 0)$ is in the image of A.

G-invariance of v_F follows from the G-invariance of its defining equations. The equation

$$\mathcal{L}_{v_F}\omega = (d\iota(v_F) + \iota(v_F)d)\omega = ddF - \iota(v_F)\mu^*\chi = 0$$

shows that the 2-form ω is v_F-invariant. The invariance of μ is equivalent to the invariance of the G-valued 1-form $\mu^*\theta$:

$$\mathcal{L}_{v_F}\mu^*\theta = (d\iota(v_F) + \iota(v_F)d)\mu^*\theta = -\frac{1}{2}\,\iota(v_F)\mu^*[\theta, \theta] = 0.$$

q.e.d.

5. q-Hamiltonian reduction

In this Section we show that the usual Hamiltonian (Meyer-Marsden-Weinstein) reduction procedure can be carried out for q-Hamiltonian G-spaces. We assume that G is a product $G = G_1 \times G_2$, and we consider reductions with respect to the first factor. Given $f \in G_1$ let $Z_f \subseteq G_1$ be its centralizer and $\mathfrak{z}_f$ the Lie algebra. Let M be a q-Hamiltonian $G_1 \times G_2$-space, with moment map (μ_1, μ_2). Suppose that $f \in G_1$ is a regular value of μ_1, so that $\mu_1^{-1}(f)$ is a smooth submanifold. Proposition 4.1 shows that for all $x \in \mu_1^{-1}(f)$, the isotropy algebra $(\mathfrak{g}_1)_x$ is trivial, i.e., $(G_1)_x \subseteq Z_f$ is a discrete subgroup. It follows that the *reduced space* $M_f = \mu_1^{-1}(f)/Z_f$ is an G_2-equivariant orbifold. Not surprisingly M_f is a q-Hamiltonian G_2-space:

Theorem 5.1 (q-Hamiltonian reduction). *Let M be a q-Hamiltonian $G_1 \times G_2$-space and let $f \in G_1$ be a regular value of the moment map $\mu_1 : M \to G_1$. Then the pull-back of the 2-form ω to $\mu_1^{-1}(f)$ descends to the* reduced space

$$M_f = \mu_1^{-1}(f)/Z_f$$

and makes it into a q-Hamiltonian G_2-space. In particular, if $G_2 = \{e\}$ is trivial, then M_f is a symplectic orbifold.

Proof. Let $\iota : \mu_1^{-1}(f) \hookrightarrow M$ denote the embedding, and $\pi : \mu_1^{-1}(f) \to M_f$ the projection. The form $\iota^*\omega$ is $Z_f \times G_2$-invariant because ω is $G_1 \times G_2$-invariant. Moreover if $\xi \in \mathfrak{z}_f$, then

$$\iota(v_\xi)\iota^*\omega = \iota^*\iota(v_\xi)\omega = \iota^*\mu_1^*(\theta + \overline{\theta}, \xi) = 0$$

(because $\mu_1 \circ \iota : \mu_1^{-1}(f) \to \{f\}$) implies that $\iota^*\omega$ is Z_f-basic. Let $\omega_f \in \Omega^2(M_f)^{G_2}$ be the unique 2-form such that $\pi^*\omega_f = \iota^*\omega$. The restriction $\iota^*\mu_2$ is $Z_f \times G_2$-invariant and descends to an equivariant map $(\mu_2)_f \in C^\infty(M_f, G_2)^{G_2}$ satisfying (B2). Letting χ_1, χ_2 be the canonical 3-forms for G_1, G_2, we have

$$\pi^*\mathrm{d}\omega_f = \mathrm{d}\iota^*\omega = \iota^*\mathrm{d}\omega = -\iota^*(\mu_1^*\chi_1 + \mu_2^*\chi_2) = -\iota^*\mu_2^*\chi_2 = -\pi^*(\mu_2)_f^*\chi_2,$$

so that ω_f satisfies (B1). Finally, we need to check the non-degeneracy condition (B3) for ω_f. The kernel of ω_f at a point $\pi(x) \in M_f$ is just the projection $\mathrm{d}_x\pi \ker(\iota^*\omega)_x$. Thus Remark 4.1 yields

$$\begin{aligned}
\ker(\iota^*\omega)_x &= \ker(\mathrm{d}_x\mu_1) \cap \ker(\mathrm{d}_x\mu_1)^\omega \\
&= \ker(\mathrm{d}_x\mu_1) \cap \big(\{v_\xi(x), \xi \in \mathfrak{g}_1\} + \ker\omega_x\big) \\
&= T_x(Z_f \cdot x) + \{v_\eta(x) |\, \eta \in \ker(\mathrm{Ad}_{\mu_2(x)} + 1)\}.
\end{aligned}$$

q.e.d.

Remark 5.1. The proof shows that if M satisfies the conditions for a q-Hamiltonian $G_1 \times G_2$-space except for (B3), the reduced space M_f is still well-defined and satisfies (B1) and (B2).

Let us consider a few examples of q-Hamiltonian reduction.

Example 5.1.

1. For a conjugacy class $M = \mathcal{C} \subset G$, the reduced space M_f is a point if $f \in \mathcal{C}$, and is empty otherwise.

2. Let $M = D(G)$ be the double defined in the previous section, with moment map $\mu_D = (\mu_1, \mu_2)$. Consider the reduction with respect to the second G-factor, $\mu_2^{-1}(f)/Z_f$. Since $\mu_2^{-1}(f) = \{(a, f^{-1}a^{-1}) |\, a \in G\}$, with Z_f acting diagonally from the right and the first G acting diagonally from the left, we find that the reduced space is the conjugacy class through the element f^{-1}.

3. It follows from the proof that if M satisfies (B1) and (B2) but not necessarily (B3), the reduced space M_f is still well-defined and satisfies (B1),(B2). For example let $(M, \mathcal{A}, \sigma, \Phi)$ be a Hamiltonian G-space, and let $(M, \mathcal{A}, \omega, \mu)$ be the q-Hamiltonian G-space obtained from it, with $\mu = \exp(\Phi)$ and $\omega = \sigma + \Phi^*\varpi$. Recall that even if σ is non-degenerate, ω can fail to satisfy condition (B3). Suppose f is a regular value of both $\mu : M \to G$ and $\exp : \mathfrak{g} \to G$. Then all pre-images $\mu \in \exp^{-1}(f)$ are regular values of Φ, and the q-Hamiltonian reduction M_f is symplectic and is a disjoint union

$$M_f = \coprod_{\exp \mu = f} M_\mu.$$

This follows from the fact that the pull-back of the extra term $\Phi^*\varpi$ to $\mu^{-1}(f)$ vanishes.

Remark 5.2. The q-Hamiltonian structure on M_f can also be obtained as follows. Let τ be the 2-form on the conjugacy class $\mathcal{C} = \mathrm{Ad}(G) \cdot f$ and let $\iota : \mu_1^{-1}(\mathcal{C}) \to M$ and $\pi : \mu_1^{-1}(\mathcal{C}) \to \mu_1^{-1}(\mathcal{C})/G$ be the embedding and projection. Then

$$\pi^*\omega_f = \iota^*(\omega - \mu^*\tau).$$

Remark 5.3 (Hamiltonian Dynamics commutes with reduction). The Hamiltonian dynamics defined on M by a $G_1 \times G_2$-invariant Hamiltonian F descends to the reduced space M_f. Indeed, the corresponding vector field v_F descends to $\mu_1^{-1}(f)$ because it is G-invariant and tangent to $\mu_1^{-1}(f)$. Moreover, the restriction of the $G_1 \times G_2$-invariant function F to $\mu_1^{-1}(f)$ is $Z_f \times G_2$-invariant and descends to an G_2-invariant function F_f on M_f satisfying

$$\pi^*\mathrm{d}F_f = \iota^* dF = \iota^*\iota(v_F)\omega = \pi^*\iota(\pi(v_{F_f}))\omega_f$$

and

$$\pi^*\iota(v_{F_f})(\mu_2)_f^*\theta = \iota(v_F)\iota^*\mu_2^*\theta = 0.$$

It follows that $G_1 \times G_2$-invariant Hamiltonian dynamics commutes with reduction.

6. Fusion product

In this section we introduce a ring structure on the category of q-Hamiltonian G-spaces. We call it a fusion product because it provides a finite-dimensional "classical analogue" to fusion products of representations of quantum groups at roots of unity, and to fusion products of positive energy representations of loop groups (see [19] and Section 8 below).

Theorem 6.1 (Fusion product). *Let M be a q-Hamiltonian $G \times G \times H$-space, with moment map $\mu = (\mu_1, \mu_2, \mu_3)$. Let $G \times H$ act by the diagonal embedding $(g, h) \to (g, g, h)$. Then M with 2-form*

$$\tilde{\omega} = \omega + \frac{1}{2}(\mu_1^*\theta, \mu_2^*\overline{\theta}) \tag{0.16}$$

and moment map

$$\tilde{\mu} = (\mu_1 \cdot \mu_2, \mu_3) : \, M \to G \times H \tag{0.17}$$

is a q-Hamiltonian $G \times H$-space.

Proof. The moment map $\tilde{\mu}$ is equivariant because the group multiplication is an equivariant map with respect to the action by conjugations. Property (B1) follows from

$$(g_1 g_2)^*\chi = g_1^*\chi + g_2^*\chi - \frac{1}{2}\mathrm{d}\,(g_1^*\theta,\, g_2^*\overline{\theta}), \tag{0.18}$$

which shows that

$$d\omega = -\mu_1^*\chi - \mu_2^*\chi - \mu_3^*\chi + \frac{1}{2}\mathrm{d}(\mu_1^*\theta, \mu_2^*\overline{\theta}) = -\tilde{\mu}^*\chi.$$

For $\xi \in \mathfrak{g}$ and $\eta \in \mathfrak{h}$ let v_ξ^1, v_ξ^2 and v_η denote the fundamental vector fields for the action of the respective factors of $G \times G \times H$. The fundamental vector field for the diagonal G-action is just the sum $v_\xi = v_\xi^1 + v_\xi^2$. Clearly

$$\iota(v_\eta)\tilde{\omega} = \iota(v_\eta)\omega = \frac{1}{2}\mu_3^*(\theta + \overline{\theta}, \eta),$$

which verifies (B2) for the H-factor. Moreover,

$$\begin{aligned}
\iota(v_\xi)\tilde{\omega} &= \iota(v^1_\xi)\omega + \iota(v^2_\xi)\omega + \frac{1}{2}(\mu_1^*\iota(v_\xi)\theta, \mu_2^*\overline{\theta}) - \frac{1}{2}(\mu_1^*\theta, \mu_2^*\iota(v_\xi)\overline{\theta}) \\
&= \frac{1}{2}\mu_1^*(\theta + \overline{\theta}, \xi) + \frac{1}{2}\mu_2^*(\theta + \overline{\theta}, \xi) \\
&\quad + \frac{1}{2}(\mathrm{Ad}_{\mu_1^{-1}}\xi - \xi, \mu_2^*\overline{\theta}) - \frac{1}{2}(\mu_1^*\theta, \xi - \mathrm{Ad}_{\mu_2}\xi) \\
&= \frac{1}{2}(\mu_1\mu_2)^*(\theta + \overline{\theta}, \xi).
\end{aligned}$$

Finally, we need to check that $\tilde{\omega}$ satisfies the minimal degeneracy condition (B3). Suppose the vector $v \in T_xM$ is in the kernel of $\tilde{\omega}_x$ (we will omit the basepoint x to simplify notation):

$$0 = \iota(v)\tilde{\omega} = \iota(v)\omega + \frac{1}{2}(\iota(v)\mu_1^*\theta, \mu_2^*\overline{\theta}) - \frac{1}{2}(\mu_1^*\theta, \iota(v)\mu_2^*\overline{\theta}). \tag{0.19}$$

Let $\zeta \in \ker(\mathrm{Ad}_{\mu_1} + 1)$, so that $v^1_\zeta \in \ker \omega_x$. Contracting (0.19) with v_ζ we find

$$0 = \iota(v^1_\zeta)\ (\mu_1^*\theta, \iota(v)\mu_2^*\overline{\theta}) = ((\mathrm{Ad}_{\mu_1^{-1}} - 1)\zeta, \iota(v)\mu_2^*\overline{\theta}) = -2(\zeta, \iota(v)\mu_2^*\overline{\theta}).$$

This shows

$$\iota(v)\mu_2^*\overline{\theta} \in \ker(\mathrm{Ad}_{\mu_1} + 1)^\perp = \mathrm{im}(\mathrm{Ad}_{\mu_1} + 1).$$

A similar argument applies to $\iota(v)\mu_1^*\theta$. We can therefore choose $\xi_1, \xi_2 \in \mathfrak{g}$ with

$$\iota(v)\mu_2^*\overline{\theta} = (\mathrm{Ad}_{\mu_1^{-1}} + 1)\xi_1, \quad \iota(v)\mu_1^*\theta = -(\mathrm{Ad}_{\mu_2} + 1)\xi_2, \tag{0.20}$$

which turns (0.19) into

$$\iota(v)\omega = \frac{1}{2}\mu_1^*(\theta + \overline{\theta}, \xi_1) + \frac{1}{2}\mu_2^*(\theta + \overline{\theta}, \xi_2) = \iota(v^1_{\xi_1})\omega + \iota(v^2_{\xi_2}).$$

Changing ξ_1, ξ_2 if necessary, it follows that $v = v^1_{\xi_1} + v^2_{\xi_2} + v_\eta$ for suitable $\xi_1, \xi_2 \in \mathfrak{g}$ and $\eta \in \ker(\mathrm{Ad}_{\mu_3} + 1)$. Re-inserting this into equations (0.20) we find

$$(1 - \mathrm{Ad}_{\mu_2})\xi_2 = (1 + \mathrm{Ad}_{\mu_1^{-1}})\xi_1, \quad (1 + \mathrm{Ad}_{\mu_2})\xi_2 = (1 - \mathrm{Ad}_{\mu_1^{-1}})\xi_1.$$

Adding these equations gives $\xi_1 = \xi_2$, and then either equation shows the non-degeneracy condition

$$\mathrm{Ad}_{\mu_1\mu_2}\,\xi = -\xi.$$

q.e.d.

We call the operation of replacing the $G \times G \times H$-action by the $G \times H$-action on a manifold M internal fusion, and denote the resulting space by M_{12} (in particular if there are more G-factors involved). Given two q-Hamiltonian $G \times H_j$-spaces M_j we define their fusion product $M_1 \circledast M_2$ to be the q-Hamiltonian $G \times H_1 \times H_2$-space obtained from the q-Hamiltonian $G \times H_1 \times G \times H_2$-space $M_1 \times M_2$ by fusing the G-factors.

Remark 6.1.

1. Let $\{\mathrm{pt}\}$ denote the trivial G-space, with moment map $\mathrm{pt} \mapsto e$. Then $M \circledast \{\mathrm{pt}\} = M = \{\mathrm{pt}\} \circledast M$ for every q-Hamiltonian G-space M.

2. Suppose M is a q-Hamiltonian $G \times G \times H$-space. Then $(M^-)_{12} = (M_{21})^-$.

The fusion operation is associative: Given q-Hamiltonian $G \times H_j$-spaces M_j we have

$$(M_1 \circledast M_2) \circledast M_3 = M_1 \circledast (M_2 \circledast M_3).$$

More generally if M is a q-Hamiltonian $G \times G \times G \times H$-space, the two q-Hamiltonian $G \times H$-spaces $M_{(12)3}$ obtained by first fusing the first two G-factors and $M_{1(23)}$ obtained by first fusing the last two G-factors are identical. The new 2-form on M in either case is given by

$$\omega + \frac{1}{2}(\mu_1^*\theta, \mu_2^*\overline{\theta}) + \frac{1}{2}(\mu_2^*\theta, \mu_3^*\overline{\theta}) + \frac{1}{2}(\mu_1^*\theta, \mathrm{Ad}_{\mu_2}\,\mu_3^*\overline{\theta}).$$

We shall now show that the fusion product is also commutative on *isomorphism classes* of *q*-Hamiltonian G-spaces. Switching the two G-factors in Theorem 6.1 before fusing we obtain a Hamiltonian $G \times H$-space M_{21} with the same action, but moment map $(\mu_2 \cdot \mu_1, \mu_3)$ and 2-form

$$\omega + \frac{1}{2}(\mu_2^*\theta, \mu_1^*\overline{\theta}).$$

If G is non-commutative, the identity map $M \to M$ does not provide an isomorphism. However, we have the following result. Let

$$\mathcal{A}^1, \mathcal{A}^2 :\, G \to \mathrm{Diff}(M)$$

denote the actions of the two G-factors.

Theorem 6.2 (Commutativity of the Fusion Product). *Under the hypotheses of Theorem 6.1 the map*

$$R :\, M \to M,\, x \mapsto \mathcal{A}^2_{\mu_1(x)}(x)$$

is a $G \times H$-equivariant diffeomorphism satisfying

$$R^*(\mu_2\mu_1) = \mu_1\mu_2,\, R^*\mu_3 = \mu_3, \tag{0.21}$$

$$R^*\Big(\omega + \frac{1}{2}(\mu_2^*\theta, \mu_1^*\overline{\theta})\Big) = \omega + \frac{1}{2}(\mu_1^*\theta, \mu_2^*\overline{\theta}). \tag{0.22}$$

Thus R gives an isomorphism $R :\, M_{12} \to M_{21}$ of q-Hamiltonian $G \times H$-spaces.

Proof. By equivariance of the moment map, R is $G \times H$-equivariant, and we have

$$R^*\mu_1 = \mu_1,\ R^*\mu_2 = \mathrm{Ad}(\mu_1)\mu_2,\ R^*\mu_3 = \mu_3.$$

proving (0.21). To prove (0.22) we note that the tangent map to R is

$$\mathrm{d}_x R(v) = \mathrm{d}_x \mathcal{A}^2_{\mu_1(x)}(v) + v^2_\xi(R(x)),$$

where v^2_ξ is the fundamental vector field of $\xi := \iota(v)(\mu_1^*\overline{\theta})_x$ with respect to $\mathcal{A}^2$. By a calculation similar to that in the proof of Theorem 4.5 we find

$$\begin{aligned}(R^*\omega)(v_1, v_2) \quad &= \quad \omega(v_1, v_2) + \frac{1}{2}\Big(\,\mathrm{Ad}(\mu_1^{-1})\xi_1, \xi_2 + \mathrm{Ad}(\mu_2)\xi_2\Big) \\ &\qquad -\frac{1}{2}\Big(\,\mathrm{Ad}(\mu_1^{-1})\xi_2, \xi_1 + \mathrm{Ad}(\mu_2)\xi_1\Big) \\ &\qquad -\frac{1}{2}(\xi_1, \mathrm{Ad}(\mu_2)\xi_2 - \mathrm{Ad}(\mu_2^{-1})\xi_2)\end{aligned}$$

with $\xi_j = \iota(v_j)(\mu_1^*\overline{\theta})$, and thus

$$R^*\omega = \omega + \frac{1}{2}(\mu_1^*\theta, \mu_2^*(\theta + \overline{\theta}) - \mathrm{Ad}(\mu_2)\mu_1^*\theta). \tag{0.23}$$

Furthermore

$$\begin{aligned} R^*\mu_1^*\overline{\theta} &= \mu_1^*\overline{\theta}, \\ R^*\mu_2^*\theta &= -\mu_1^*\overline{\theta} + \mathrm{Ad}(\mu_1\mu_2^{-1})\mu_1^*\theta + \mathrm{Ad}(\mu_1)\mu_2^*\theta, \end{aligned}$$

so that

$$\begin{aligned} R^*(\mu_2^*\theta, \mu_1^*\overline{\theta}) =& (\mathrm{Ad}(\mu_2^{-1})\mu_1^*\theta + \mu_2^*\theta, \mu_1^*\theta) \\ =& (\mu_1^*\theta,\, \mathrm{Ad}(\mu_2)\mu_1^*\theta - \mu_2^*\theta). \end{aligned} \tag{0.24}$$

Adding (0.23) $+\frac{1}{2}$ (0.24) proves the theorem. q.e.d.

We leave it to the reader to check that the map

$$R' : M \to M,\ x \mapsto \mathcal{A}^1_{\mu_2(x)^{-1}}(x)$$

has just the same properties (0.21), (0.22) as the map R. We will call R, R' *braid isomorphisms.*

Example 6.1. Let us apply internal fusion to the double $D(G)$. Fusing the two G-factors we get a q-Hamiltonian G-space $\mathbf{D}(G) := D(G)_{12}$ which is just G^2, with G-action

$$(a, b)^g = (\mathrm{Ad}_g\, a, \mathrm{Ad}_g\, b),$$

moment map

$$\mu(a, b) = aba^{-1}b^{-1} \equiv [a, b],$$

and 2-form

$$\omega = \frac{1}{2}(a^*\theta,\, b^*\overline{\theta}) + \frac{1}{2}(a^*\overline{\theta},\, b^*\theta) + \frac{1}{2}((ab)^*\theta, (a^{-1}b^{-1})^*\overline{\theta}).$$

The braid isomorphisms in this case are given by

$$R : \mathbf{D}(G) \to \mathbf{D}(G),\ (a, b) \mapsto (ab^{-1}a^{-1},\ ab^2),$$

$$R' : \mathbf{D}(G) \to \mathbf{D}(G),\ (a, b) \mapsto (a^{-1}b^{-1}a,\ b^2a).$$

Remark 6.2. It is interesting to re-examine q-Hamiltonian reduction in connection with fusion. Suppose M is a q-Hamiltonian $G \times G \times H$-space, with moment map (μ_1, μ_2, μ_3), and e is a regular value of $\mu := \mu_1\mu_2$. Then the diagonal action of G on $\mu^{-1}(e)$ is locally free, hence $M/\!\!/\operatorname{diag}(G) := \mu^{-1}(e)/\operatorname{diag}(G)$ is an H-equivariant orbifold and μ_3 descends to $M/\!\!/\operatorname{diag}(G)$. We claim that the pull-back of ω to $\mu^{-1}(e)$ is basic and that the induced form $\omega/\!\!/\operatorname{diag}(G)$ makes $M/\!\!/\operatorname{diag}(G)$ into a q-Hamiltonian H-space. In fact, since the pull-back of the additional term in (0.16) to $\mu^{-1}(e)$ vanishes, the claim follows immediately from

$$M/\!\!/\operatorname{diag}(G) = (M_{12})_e.$$

Let us also observe that just as in the category of Hamiltonian G-spaces there is a shifting-trick for q-Hamiltonian reduction: If $(M, \mathcal{A}, \omega, \mu_1, \mu_2)$ is a q-Hamiltonian $G \times H$-space, then f is a regular value for μ_1 if and only if the identity e is a regular value for the moment map on $M \circledast \mathcal{C}^-$ where $\mathcal{C} = \operatorname{Ad}(G) \cdot f$, and in this case there is a canonical isomorphism

$$M_f \cong (M \circledast \mathcal{C}^-)_e = (M \times \mathcal{C}^-)/\!\!/\operatorname{diag}(G).$$

7. Cross-sections and convexity

One of the basic tools in the study of Hamiltonian G-spaces is the cross-section Theorem of Guillemin-Sternberg, which is a method of reducing problems to subgroups of G.

In this section we prove a cross-section Theorem for q-Hamiltonian G-spaces and explain its relation to convexity theorems for the moment map.

Let $(M, \mathcal{A}, \omega, \mu)$ be a q-Hamiltonian G-space and let $f \in G$. Since the centralizer $Z_f \subset G$ is transversal to the conjugacy class $\mathcal{C} = \operatorname{Ad}(G) \cdot f$ there exists an open Z_f-invariant subset $U \subset Z_f$ containing f and an equivariant diffeomorphism

$$G \times_{Z_f} U \to G,\ [g, u] \mapsto g\,u$$

onto an open subset of G. By equivariance of μ, the pre-image $\mu^{-1}(Z_f)$ is a smooth Z_f-equivariant submanifold $Y \subset M$, and there is a natural diffeomorphism $G \times_{Z_f} Y \to M$ onto an open subset. In analogy to the cross-section theorem of Guillemin-Sternberg we have

Proposition 7.1 (Cross-section theorem). *Let $(M, \mathcal{A}, \omega, \mu)$ be a q-Hamiltonian G-space, and $f \subset G$, $U \subset Z_f$ as above. Then the* cross-section *$Y := \mu^{-1}(U)$ is a smooth Z_f-invariant submanifold, and* *is a q-Hamiltonian Z_f-space with the restriction of μ as a moment map. In particular, if Z_f is abelian the cross-section is symplectic.*

Proof. All conditions for a q-Hamiltonian Z_f-space are immediate except the non-degeneracy condition (B3). Let $\iota : Y \to M$ be the inclusion. For all $y \in Y$ the tangent space $T_{\mu(y)}G$ splits into a direct sum

$$T_{\mu(y)}G = T_y\mathcal{C} \oplus \{v_\xi(\mu(y)) \,|\, \xi \in \mathfrak{z}_f^\perp\}.$$

Consequently

$$T_yM = T_yY \oplus \{v_\xi(y) \,|\, \xi \in \mathfrak{z}_f^\perp\}. \tag{0.25}$$

The second summand is mapped under $\mathrm{d}_y\mu$ to a subspace of the tangent space to conjugacy class $\mathrm{Ad}(G)\cdot\mu(y)$. If $v \in T_yY$, $\xi \in \mathfrak{z}_f^\perp$, we have

$$\omega(v_\xi, v) = \frac{1}{2}\iota(v)\mu^*(\theta + \overline{\theta}, \xi) = 0,$$

because $\iota(d_y\mu(v))(\theta + \overline{\theta}) \in \mathfrak{z}_f$, so that the decomposition (0.25) is ω-orthogonal. Thus if $v \in \ker \iota^*\omega$, then also $v \in \ker \omega$. Since M satisfies (B3), this shows $v = v_\xi$ for some $\xi \in \mathfrak{z}_f$ satisfying $\mathrm{Ad}(\mu(x))\xi = -\xi$. q.e.d.

Remark 7.1. Instead of the restriction $\mu|Y$, one can also use the shifted moment map $\hat{\mu} = f^{-1}(\mu|Y)$. It satisfies $\hat{\mu}(x) = e$ for all $x \in \mu^{-1}(f)$. Remark 3.3 applies and shows that Y is locally equivalent near $\mu^{-1}(f) \subset Y$ to a Hamiltonian Z_f-space in the usual sense.

Suppose now that the group G is in addition connected and simply connected. Then there are canonical choices for the cross-sections constructed as follows. Let $T \subset G$ be a maximal torus of G, with Lie algebra $\mathfrak{t}$, and $\mathfrak{t}_+$ some choice of a positive Weyl chamber. Every $\mathrm{Ad}(G)$-orbit in $\mathfrak{g}$ passes through a unique point of $\mathfrak{t}_+$ so that $\mathfrak{t}_+ = \mathfrak{g}/\mathrm{Ad}(G)$. Let $\mathfrak{A} \subset \mathfrak{t}_+$ be the fundamental Weyl alcove. Every conjugacy class $\mathcal{C} \subset G$ contains a unique point of $\exp(\mathfrak{A}) \subset T$ so that we can identify

$$\mathfrak{A} = G/\mathrm{Ad}(G)$$

as the space of conjugacy classes. For every open face $\sigma \subset \mathfrak{A}$ the centralizer $Z_{\exp(\xi)}$ with $\xi \in \sigma$ is independent of ξ and will be denoted Z_σ. For σ in the interior of $\mathfrak{A}$, $Z_\sigma = T$. Introducing a partial order by setting $\sigma \succeq \tau$ if $\overline{\sigma} \supseteq \tau$ we have $\sigma \succeq \tau \Rightarrow Z_\sigma \subseteq Z_\tau$, in particular every Z_σ

contains T. Let us write

$$\mathfrak{A}_\sigma = \bigcup_{\tau \succeq \sigma} \tau$$

and

$$U_\sigma = \mathrm{Ad}(Z_\sigma)\exp(\mathfrak{A}_\sigma).$$

Then $U_\sigma \subset Z_\sigma \subseteq G$ is smooth, and is a slice for the $\mathrm{Ad}(G)$-action at points in σ. In particular, for every $g \in U_\sigma$ we have $\mathrm{Ad}(G)g \cap U_\sigma = \mathrm{Ad}(Z_\sigma)g$.

Remark 7.2. Let us consider in particular the cross-section $Y = Y_\sigma$ for $\sigma = \mathrm{int}\,\mathfrak{A}$. The T-action over Y extends by equivariance to a G-equivariant T-action over the open subset $G \cdot Y \subseteq M$. In fact, it is Hamiltonian in the sense of Proposition 4.6. To see this let $q : G \to G/\mathrm{Ad}(G) \cong \mathfrak{A}$ be the quotient map. Note that q is smooth over $G \cdot U_\sigma$ for $\sigma = \mathrm{int}\,\mathfrak{A}$. Using 7.1, it is clear that the components of $q \circ \mu$ generate the T-action just described. In the case of moduli spaces these T-actions are known as the Goldman flows [9]. See e.g. [19] for a discussion and references.

The above fact that the cross-sections are equivalent, after shift of the moment map, to Hamiltonian spaces in the usual sense implies in particular that Kirwan's theorem on convexity and connectedness of fibers of moment maps applies to every connected component of the cross-section.

It turns out that the cross-sections $Y_\sigma = \Phi^{-1}(U_\sigma)$ for a connected, simply connected compact Lie group are necessarily connected, which then has the following consequence.

Theorem 7.2 (Convexity Theorem). *Let $(M, \mathcal{A}, \omega, \mu)$ be a connected q-Hamiltonian G-space, where G is compact, connected and simply connected. Then all fibers of the moment map μ are connected, and the image of $\mu(M) \subseteq G$ under the projection $G \to G/\mathrm{Ad}(G) = \mathfrak{A}$ is a convex polytope.*

We will not give the detailed argument here since, as the following section shows, it is indeed just a translation of the proof given in [19] into the terminology of q-Hamiltonian G-spaces. Note that the fibers of μ are not necessarily connected if G is for instance a torus.

8. Relation to Hamiltonian LG-spaces

In this Section we prove that there exists a one-to-one correspondence between q-Hamiltonian G-spaces and Hamiltonian LG-spaces with proper moment map (see [19], [20]). So, one always has a choice either to work with infinite-dimensional objects (LG-spaces) and more conventional definitions (Hamiltonian spaces) or to use finite-dimensional objects and the new definitions (q-Hamiltonian G-spaces).

8.1. Loop group LG: notation. Let G be a compact Lie group. We define the loop group LG as a space of maps

$$LG = \mathrm{Map}(S^1, G)$$

of a fixed Sobolev class $\lambda > 1/2$. Then LG consists of continuous maps and the group multiplication is defined pointwise. Its Lie algebra is the space of maps $L\mathfrak{g} = \Omega^0(S^1, \mathfrak{g})$ of Sobolev class λ. We define $L\mathfrak{g}^*$ as the space of 1-forms

$$L\mathfrak{g}^* = \Omega^1(S^1, \mathfrak{g})$$

of Sobolev class $\lambda - 1$. The natural pairing of $L\mathfrak{g}^*$ and $L\mathfrak{g}$ given by

$$\langle A, \xi \rangle = \oint_{S^1} (A, \xi)$$

makes $L\mathfrak{g}^*$ into a subset of the topological dual $(L\mathfrak{g})^*$.

We view $L\mathfrak{g}^*$ as the affine space of connections on the trivial bundle $S^1 \times G$ and let the loop group LG act by gauge transformations:

$$A^g = \mathrm{Ad}_g\, A - g^*\overline{\theta}.$$

Let

$$\mathrm{Hol} : L\mathfrak{g}^* \to G$$

denote the holonomy map. Recall that if we identify $S^1 \cong \mathbb{R}/\mathbb{Z}$ and let $s \in \mathbb{R}$ denote the local coordinate, $\mathrm{Hol} = \mathrm{Hol}_1$ where $\mathrm{Hol}_s : L\mathfrak{g}^* \to G$ is defined as the unique solution of the differential equation

$$\mathrm{Hol}_s(A)^{-1} \frac{\partial}{\partial s} \mathrm{Hol}_s(A) = A, \quad \mathrm{Hol}_0(A) = e.$$

On constant connections $A = \xi \mathrm{d}s$ (for $\xi \in \mathfrak{g}$) the holonomy map restricts to the exponential map: $\mathrm{Hol}_s(\xi \mathrm{d}s) = \exp(s\xi)$. The map Hol_s satisfies the equivariance condition

$$\mathrm{Hol}_s(A^g) = g(0)\, \mathrm{Hol}_s(A)\, g(s)^{-1}, \tag{0.26}$$

in particular the holonomy map $\mathrm{Hol} = \mathrm{Hol}_1$ is equivariant with respect to the evaluation homomorphism $LG \to G$, $g \mapsto g(1)$ and the adjoint action of G on itself.

Let the *based loop group* $\Omega G \subset LG$ be defined as the kernel of the evaluation mapping $LG \to G$, $g \mapsto g(1)$. Then LG is a semi-direct product $LG = \Omega G \rtimes G$. The action of ΩG on $L\mathfrak{g}^*$ is free, and the quotient map is just the holonomy map. Thus $\mathrm{Hol} : L\mathfrak{g}^* \to G$ is the universal ΩG-principal bundle. Consider now the closed three-form $\chi \in \Omega^3(G)$. Since $L\mathfrak{g}^*$ is an affine space the pull-back $\mathrm{Hol}^* \chi$ to $L\mathfrak{g}^*$ is exact. An explicit potential for $\mathrm{Hol}^* \chi$ is given as follows. Define the following 2-form on $L\mathfrak{g}^*$:

$$\varpi = \frac{1}{2} \int_0^1 \mathrm{d}s\, (\mathrm{Hol}_s^* \overline{\theta}, \frac{\partial}{\partial s} \mathrm{Hol}_s^* \overline{\theta}).$$

Proposition 8.1. *The 2-form ϖ is LG-invariant and its differential is given by*

$$d\varpi = -\, \mathrm{Hol}^* \chi. \tag{0.27}$$

Its contraction with a fundamental vector field v_ξ ($\xi \in L\mathfrak{g}$) is given by

$$\iota(v_\xi)\varpi = -d \oint_{S^1} (A, \xi) + \frac{1}{2} \mathrm{Hol}^*(\theta + \overline{\theta}, \xi(0)). \tag{0.28}$$

The proof of this proposition is deferred to the appendix.

8.2. Equivalence theorem.

Definition 8.2. A Hamiltonian LG-space is a Banach manifold N together with an LG-action $\mathbf{A}$, an invariant 2-form $\sigma \in \Omega^2(M)^{LG}$, and an equivariant map $\Phi \in C^\infty(N, L\mathfrak{g}^*)^{LG}$ such that:

(C1) The 2-form σ is closed.

(C2) The map Φ is a moment map for the LG-action $\mathbf{A}$:

$$\iota(v_\xi)\sigma = d \oint_{S^1} (m, \xi).$$

(C3) The form σ is weakly non-degenerate, that is, the induced map $\sigma_x^\flat : T_xN \to T_x^*N$ is injective.

We will show in this section that every Hamiltonian LG-space with proper moment map determines a q-Hamiltonian G-space and vice versa. Since the action of the based loop group $\Omega G \subset LG$ on $L\mathfrak{g}^*$ is free, its action on N is free as well and we can form the quotient

$$M = \mathrm{Hol}(N) := N/\Omega G.$$

If the moment map Φ is proper, then $\mathrm{Hol}(N)$ is a smooth finite-dimensional manifold. We denote by Hol the projection $\mathrm{Hol} : N \to \mathrm{Hol}(N)$. Since $G = LG/\Omega G$, the diagram

$$\begin{array}{ccc} LG \times N & \longrightarrow & N \\ \downarrow & & \downarrow \\ G \times \mathrm{Hol}(N) & \longrightarrow & \mathrm{Hol}(N) \end{array} \tag{0.29}$$

defines a G-action $\mathrm{Hol}(\mathbf{A})$ on $\mathrm{Hol}(N)$, and the diagram

$$\begin{array}{ccc} N & \longrightarrow & L\mathfrak{g}^* \\ \downarrow & & \downarrow \\ \mathrm{Hol}(N) & \longrightarrow & G \end{array} \tag{0.30}$$

a G-equivariant map $\mathrm{Hol}(\Phi) : \mathrm{Hol}(N) \to G$. The following result says that holonomy manifolds of Hamiltonian LG-spaces with proper moment maps carry canonically the structure of q-Hamiltonian G-spaces.

Theorem 8.3 (Equivalence Theorem). *Let $(N, \mathbf{A}, \sigma, \Phi)$ be a Hamiltonian LG-space with proper moment map and let $M = \mathrm{Hol}(N)$ be its holonomy manifold, with G-action $\mathcal{A} = \mathrm{Hol}(\mathbf{A})$ and map $\mu := \mathrm{Hol}(\Phi)$. The 2-form on N*

$$\sigma + \Phi^* \varpi \tag{0.31}$$

is basic with respect to the projection $\mathrm{Hol} : N \to M$, and is therefore the pull-back $\mathrm{Hol}^* \omega$ of a unique 2-form ω on M. Then $(M, \mathcal{A}, \omega, \mu)$ is *a q-Hamiltonian G-space. Conversely, given a q-Hamiltonian G-space $(M, \mathcal{A}, \omega, \mu)$ there is a unique Hamiltonian LG-space $(N, \mathbf{A}, \sigma, \Phi)$ such that M is its holonomy manifold.*

Proof. Since ϖ is LG-invariant the 2-form (0.31) is LG-invariant. Moreover if $\xi \in L\mathfrak{g}$, by (0.28) we have

$$\begin{aligned} \iota(v_\xi)(\sigma + \Phi^*\varpi) &= \mathrm{d}\langle \Phi, \xi\rangle + \Phi^*\iota(v_\xi)\varpi \\ &= \frac{1}{2}\Phi^* \operatorname{Hol}^*(\theta + \overline{\theta}, \xi(0)) \\ &= \frac{1}{2}\operatorname{Hol}^* \mu^*(\theta + \overline{\theta}, \xi(0)). \end{aligned} \tag{0.32}$$

This shows that $\sigma + \Phi^*\varpi$ is basic, and that the 2-form ω on M defined by it satisfies condition (B2). Condition (B1) is a consequence of $\mathrm{d}\sigma = 0$ and (0.27):

$$\operatorname{Hol}^* \mathrm{d}\omega = \mathrm{d} \operatorname{Hol}^* \omega = \mathrm{d}\sigma + \Phi^*\mathrm{d}\varpi = -\Phi^* \operatorname{Hol}^* \chi = -\operatorname{Hol}^* \mu^*\chi.$$

It remains to check the non-degeneracy condition (B3). The kernel of ω is the image under the tangent map $d\operatorname{Hol}$ of the kernel of the form $\sigma + \Phi^*\varpi$. Suppose $v \in T_yN$ is in the kernel. Then

$$\iota(v)\sigma_y = -\iota(v)\Phi^*\varpi.$$

Since the 1-form on the right-hand side annihilates the kernel of $\mathrm{d}_y\Phi$, this equation says $v \in \ker(d_y\Phi)^\sigma$. Using cross-sections for Hamiltonian LG-actions [19], [20] one sees that $\ker(d_y\Phi)^\sigma = T_y(LG\cdot y)$, just as for finite dimensional Hamiltonian G-spaces. Hence there exists $\xi \in L\mathfrak{g}$ with $v = v_\xi(y)$. By (0.32) we arrive at the condition

$$\Phi^* \operatorname{Hol}^*(\theta + \overline{\theta}, \xi(0)) = 0.$$

Applying an arbitrary vector field v_η to the left-hand side one finds that

$$\xi(0) \in \ker(\operatorname{Ad}_{f^2} - 1) = \ker(\operatorname{Ad}_f - 1) \oplus \ker(\operatorname{Ad}_f + 1),$$

where $f := \operatorname{Hol}(\Phi(y))$. For every ξ with $\xi(0) \in \ker(\operatorname{Ad}_f + 1)$, the vector v_ξ is in the kernel of $\tilde{\sigma}$, and the projection by $\mathrm{d}\operatorname{Hol}$ of such vectors is the space

$$\{v_\eta(\operatorname{Hol}(f)) \in T_{\operatorname{Hol}(y)}M,\ \eta \in \ker(\operatorname{Ad}_f - 1)\}.$$

It remains to show that if $\xi(0) \in \ker(\operatorname{Ad}_f - 1)$ and $v_\xi \in \ker\tilde{\sigma}$, then $\xi(0) = 0$, so that $v_\xi(y) \in T_y(\Omega G \cdot y)$. Let $\eta \in L\mathfrak{g}$ be defined by

$$\eta(s) := \operatorname{Ad}\big(\operatorname{Hol}_s(\Phi(y))\big)\xi(0).$$

Since $\eta - \xi \in \Omega\mathfrak{g}$, the fundamental vector field $v_\eta(y)$ still lies in $\ker \tilde{\sigma}_y$. On the other hand, using Proposition 8.1 and Lemma A.1 in the Appendix the value at $\Phi(y)$ of the fundamental vector field v_η on $L\mathfrak{g}^*$ lies in $\ker \varpi_{\Phi(y)}$. Therefore $v_\eta \in \ker \sigma_y$, and by non-degeneracy of σ_y finally $v_\eta(y) = 0$. This completes the proof that $(M, \mathcal{A}, \omega, \mu)$ is a q-Hamiltonian G-space.

Suppose conversely that we are given a q-Hamiltonian G-space $(M, \mathcal{A}, \omega, \mu)$. Define N as the pull-back of the universal ΩG-principal bundle $\mathrm{Hol} : L\mathfrak{g}^* \to G$ by the map μ. In other words, N is given by the fiber product diagram (0.30). Define Φ as the upper horizontal arrow in (0.30). There is a unique LG-action $\mathbf{A}$ on N such that Φ is equivariant and such that the diagram (0.29) commutes. Explicitly, it is induced from the LG-action on $M \times L\mathfrak{g}^*$ given by

$$g : (x, A) \to (x^{g(0)}, A^g).$$

The 2-form σ is reconstructed from equation (0.31):

$$\sigma = \mathrm{Hol}^* \omega - \Phi^* \varpi,$$

where Hol is the left vertical projection in (0.30). The above argument, read backwards shows that N is a Hamiltonian LG-space with proper moment map, and in fact $M = \mathrm{Hol}(N)$. q.e.d.

The most basic examples of Hamiltonian LG-spaces are coadjoint LG-orbits $\mathcal{O}$ for the affine action on $L\mathfrak{g}^*$, equipped with the Kirillov-Kostant-Souriau symplectic structure. All such orbits are preimages $\mathcal{O} = \mathrm{Hol}^{-1}(\mathcal{C})$ of conjugacy classes in $\mathcal{C} \subset G$, and conversely the holonomy manifold of the orbit $\mathcal{O}$ is just the conjugacy class $\mathcal{C}$. By construction, the corresponding G-action is the restriction of the adjoint action of G, and the moment map μ is the embedding into G. From Theorem 8.3 it follows that the conjugacy class $\mathcal{C}$ inherits from $\mathcal{O}$ a quasi-Hamiltonian structure. By Proposition 3.1 such a structure is unique and is described by formula (0.9). We have proved the following proposition:

Proposition 8.4. *Let $\mathcal{O} = \mathrm{Hol}^{-1}(\mathcal{C})$ be an orbit of gauge action in $L\mathfrak{g}^*$. Then the corresponding holonomy manifold coincides with the conjugacy class $\mathcal{C} \subset G$, and the induced quasi-Hamiltonian structure is given by* (0.9).

9. Moduli spaces of flat connections on 2-manifolds

In this section we apply our techniques to describe the symplectic structure on the moduli spaces of flat connections on Riemann surface. Suppose Σ is an oriented 2-manifold with non-empty boundary $\partial\Sigma = \coprod_{j=0}^{r} V_j$. The space $\mathcal{A}_{flat}(\Sigma) \subset \Omega^1(\Sigma, \mathfrak{g})$ of flat G-connections on $\Sigma \times G$ is invariant under the action of the gauge group $\mathcal{G}(\Sigma)$. Let $p_j \in V_j$ be given base points on the boundary components, and let the restricted gauge group $\mathcal{G}_{res}(\Sigma)$ consist of gauge transformations that are the identity at the p_j's. The quotient $M(\Sigma) = \mathcal{A}_{flat}(\Sigma)/\mathcal{G}_{res}(\Sigma)$ is a smooth, finite dimensional manifold – in fact $M(\Sigma) = G^{2(r+k)}$ where k is the genus. It carries a residual G^{r+1} action, and the holonomies around the V_j descend to a smooth equivariant map $\mu : M(\Sigma) \to G^{r+1}$. Given a collection of conjugacy classes $\mathcal{C} = (\mathcal{C}_0, \dots, \mathcal{C}_r) \subset G^{r+1}$ the quotient

$$M(\Sigma, \mathcal{C}) = \mu^{-1}(\mathcal{C})/G^{r+1} \tag{0.33}$$

is the moduli space of flat connections with prescribed holonomies, which according to Atiyah-Bott [3] carries a natural symplectic structure. In this section we show that there exists a natural q-Hamiltonian structure on the moduli space of flat connections $M(\Sigma)$. The moduli spaces $M(\Sigma, \mathcal{C})$ with holonomies in prescribed conjugacy classes $\mathcal{C}$ are obtained by q-Hamiltonian reduction from $M(\Sigma)$. The main result of this section is Theorem 9.3 which identifies $M(\Sigma)$ as a fusion product of a number of copies of the double $D(G)$. The upshot is that we arrive at an explicit finite-dimensional description of the symplectic form on the moduli space. For similar constructions see e.g. [12], [11], [15], and [10]. For *complex* Lie groups there is an alternative finite dimensional construction due to Fock-Rosly [8], using ideas from Poisson geometry.

9.1. Gauge-theoretic description. We start by recalling the gauge-theory construction of the symplectic 2-form on $M(\Sigma, \mathcal{C}_0, \dots, \mathcal{C}_r)$, following Atiyah-Bott [3]. Let Σ be a compact connected oriented 2-manifold with boundary components $V_0, \dots, V_r$ and $P \to \Sigma$ a principal G-bundle. For simplicity we assume that P is the trivial bundle $\Sigma \times G$ although the following discussion goes through for non-trivial bundles as well[1]. Fix $\lambda > 1$ and let $\Omega^j(\Sigma, \mathfrak{g})$ denote $\mathfrak{g}$-valued differential forms of Sobolev class $\lambda - j$. Then the gauge $\mathcal{G}(\Sigma) = \mathrm{Map}(\Sigma, G)$ is a Banach Lie group modeled on $\mathrm{Lie}(\mathcal{G}(\Sigma)) = \Omega^0(\Sigma, \mathfrak{g})$. Consider the space of

[1] Recall that if G is simply connected, every G-principal bundle over Σ is trivial.

connections $\mathcal{A}(\Sigma) = \Omega^1(\Sigma, \mathfrak{g})$ as an affine Banach space, with smooth $\mathcal{G}(\Sigma)$-action given by

$$A^g = \mathrm{Ad}(g)A - g^*\overline{\theta}.$$

The space $\mathcal{A}(\Sigma)$ carries a natural symplectic form

$$\sigma(a, b) = \int_\Sigma a \wedge b, \qquad a, b \in T_A\mathcal{A}(\Sigma) \cong \Omega^1(\Sigma, \mathfrak{g})$$

for which the $\mathcal{G}(\Sigma)$-action is Hamiltonian, with moment map

$$\langle \Psi(A), \xi \rangle = \int_\Sigma (\mathrm{curv}(A), \xi) + \int_{\partial\Sigma} (A, \xi), \qquad \xi \in \Omega^0(\Sigma, \mathfrak{g}), \tag{0.34}$$

where $\mathrm{curv}(A) \in \Omega^2(\Sigma, \mathfrak{g})$ denotes the curvature, and we have taken the orientation on $\partial\Sigma$ opposite to the induced orientation. Let us choose a base point p_j for every boundary component V_j, and let $\mathrm{Hol}^j(A) \in G$ denote the holonomy of A along the loop based at p_j and winding once around B_j. Given conjugacy classes $\mathcal{C}_0, \ldots, \mathcal{C}_r \subset G$ let

$$\begin{aligned} &M(\Sigma, \mathcal{C}_0, \ldots, \mathcal{C}_r) \\ &\qquad = \{A \in \mathcal{A}(\Sigma) |\, \mathrm{curv}(A) = 0,\ \mathrm{Hol}^j(A) \in \mathcal{C}_j\}/\mathcal{G}(\Sigma) \end{aligned} \tag{0.35}$$

be the moduli space of flat connections with holonomies in $\mathcal{C}_j$. Since the holonomy of a G-connection on S^1 is determined up to conjugacy by its gauge equivalence class, $M(\Sigma, \mathcal{C}_0, \ldots, \mathcal{C}_r)$ is a symplectic quotient. It does not depend on the choice of Sobolev class λ and is finite-dimensional and compact, but sometimes singular. (For technical details, see e.g. [3], [5].)

Let us suppose for the rest of this section that $r > 0$, i.e., that Σ has at least one boundary component. This assumption can be made without loss of generality because if $\partial\Sigma = \emptyset$ and $\hat{\Sigma}$ is the 2-manifold obtained from Σ by removing a disk, there is a natural identification $M(\Sigma) \cong M(\hat{\Sigma}, \{e\})$.

The condition $\partial\Sigma \neq \emptyset$ implies that the subset $\mathcal{A}_{flat}(\Sigma)$ of flat connections is a smooth Banach submanifold of finite codimension. Let

$$\iota : \mathcal{A}_{flat}(\Sigma) \hookrightarrow \mathcal{A}(\Sigma)$$

denote the embedding. Define the restricted gauge group $\mathcal{G}_{res}(\Sigma)$ as the kernel of the evaluation mapping

$$\mathcal{G}(\Sigma) \to G^{r+1},\ g \mapsto (g(p_0), \ldots, g(p_r)). \tag{0.36}$$

In other words $\mathcal{G}_{res}(\Sigma)$ consists of gauge transformations that are the identity at the given base points. Set

$$M(\Sigma) := \mathcal{A}_{flat}(\Sigma)/\mathcal{G}_{res}(\Sigma).$$

Since the action of $\mathcal{G}_{res}(\Sigma)$ is free, it is not very hard to check that $M(\Sigma)$ is a finite dimensional, smooth manifold – in fact we will see below that it is isomorphic to $G^{2(k+r)}$ where k is the genus of Σ. From the original $\mathcal{G}(\Sigma)$-action we have a residual action $\mathcal{A}$ of $\mathcal{G}(\Sigma)/\mathcal{G}_{res}(\Sigma) \cong G^{r+1}$, and the collection of holonomy maps $\mathrm{Hol}^j(A)$ descends to an equivariant map $\mu : M(\Sigma) \to G^{r+1}$. We have

$$M(\Sigma, \mathcal{C}_0, \ldots, \mathcal{C}_r) = \mu^{-1}(\mathcal{C}_0, \ldots, \mathcal{C}_r)/G^{r+1}.$$

Theorem 9.1. *There is a natural G^{r+1}-invariant 2-form ω on $M(\Sigma)$ for which the quadruple $(M(\Sigma), \mathcal{A}, \omega, \mu)$ is a q-Hamiltonian G^{r+1}-space.*

Proof. Choose orientation and base point preserving parametrizations $V_j \cong S^1$ of the boundary circles, and let

$$R_j : \mathcal{A}(\Sigma) \to \Omega^1(S^1, \mathfrak{g}) \cong L\mathfrak{g}^*$$

denote the restriction mapping to the jth boundary component. For any $g \in \mathcal{G}_{res}(\Sigma)$ the restriction to the jth boundary component is in the based loop group ΩG. Equation (0.34) and Proposition 8.1 show that pull-back $\iota^*\tilde{\sigma}$ to $\mathcal{A}_{flat}(\Sigma)$ of the 2-form

$$\tilde{\sigma} = \sigma + \sum_{j=0}^{r} R_j^* \varpi \tag{0.37}$$

on $\mathcal{A}(\Sigma)$ is $\mathcal{G}_{res}(\Sigma)$-basic, and is therefore the pull-back of a unique invariant 2-form $\omega \in \Omega^2(M(\Sigma))$ which satisfies (B1) and (B2). The minimal degeneracy condition (B3) is verified along the lines of the proof of Theorem 8.3. Let $\pi : \mathcal{A}_{flat}(\Sigma) \to M(\Sigma)$ denote the projection. The kernel of ω at $\pi(A)$ is the image under $\mathrm{d}\pi$ of the kernel of $\iota^*\tilde{\sigma}$.

Suppose $v \in T_A\mathcal{A}_{flat}(\Sigma)$ is in the kernel of $\iota^*\tilde{\sigma}$. By definition of $\tilde{\sigma}$ this implies that $\iota(v)\sigma_A$ annihilates $\ker \mathrm{d}_A\Psi \subset T_A\mathcal{A}_{flat}(\Sigma)$. Consequently $v \in \ker(\mathrm{d}_A\Psi)^\sigma$. The σ-orthogonal complement of $\ker(\mathrm{d}_A\Psi)$ is equal to the tangent space to the orbit through A. Hence $v = v_\xi$ for some $\xi \in \mathrm{Lie}(\mathcal{G}(\Sigma))$. As in the proof of Theorem 8.3 this implies $\xi(p_j) \in \ker(\mathrm{Ad}(\mathrm{Hol}(\Psi(p_j))^2) - 1)$ for all p_j, and we conclude $\xi(p_j) \in \ker(\mathrm{Ad}(\mathrm{Hol}(\Psi(p_j))) + 1)$, proving the non-degeneracy condition. q.e.d.

Theorem 9.2. *The moduli space $M(\Sigma, \mathcal{C}_0, \ldots, \mathcal{C}_r)$ is a q-Hamiltonian reduction of $M(\Sigma)$ corresponding to the conjugacy classes $(\mathcal{C}_0, \ldots, \mathcal{C}_r) \subset G^{r+1}$.*

Proof. Let $\mathcal{O}_j = \mathrm{Hol}^{-1}(\mathcal{C}_j) \subset L\mathfrak{g}^*$ denote the coadjoint LG-orbits corresponding to the conjugacy classes $\mathcal{C}_j$, and Ω_j their Kirillov-Kostant-Souriau symplectic forms. Let $\hat{\iota} : Z \subset \mathcal{A}_{flat}(\Sigma)$ denote the set of flat connections such that $\mathrm{Hol}^j(A) \in \mathcal{C}_j$, i.e., $R_j^* A \in \mathcal{O}_j$. Let

$$\hat{\pi} : Z \to M(\Sigma, \mathcal{C}_0, \ldots, \mathcal{C}_r)$$

be the projection. By definition of the symplectic form σ_{red} on $M(\Sigma, \mathcal{C}_0, \ldots, \mathcal{C}_r)$,

$$\hat{\pi}^* \sigma_{red} = \hat{\iota}^* \Big(\sigma - \sum_{j=0}^{r} R_j^* \Omega_j \Big) = \hat{\iota}^* \Big(\tilde{\sigma} - \sum_{j=0}^{r} R_j^* (\Omega_j + \varpi) \Big).$$

By Proposition 8.4, $\Omega_j + \varpi = \mathrm{Hol}^* \omega_j$, where ω_j is the unique 2-form which defines a quasi-Hamiltonian structure to $\mathcal{C}_j$. This implies

$$\pi^* \sigma_{red} = \iota^* (\tilde{\sigma} - (\mathrm{Hol}^j)^* \omega_j),$$

since $\mathrm{Hol} \circ R_j = \mathrm{Hol}^j$. By the Remark in Section 5.2 this Equation shows that σ_{red} coincides with the 2-form obtained by q-Hamiltonian reduction from the space $M(\Sigma)$ equipped with the 2-form ω. q.e.d.

Remark 9.1. Suppose that G is simply connected. Then the restriction mapping $\mathcal{G}(\Sigma) \to \mathcal{G}(\partial\Sigma) \cong LG^{r+1}$ is surjective. Let $\mathcal{G}_0(\Sigma)$ be its kernel. Then $\mathcal{M}(\Sigma) := \mathcal{A}_{flat}(\Sigma)/\mathcal{G}_0(\Sigma)$ is a Banach manifold and is a Hamiltonian LG^{r+1}-space with proper moment map (see e.g. [4], [19]). Its holonomy manifold is the finite-dimensional G^{r+1}-space $M(\Sigma)$. See [20] for applications of $M(\Sigma)$ in this context.

9.2. Holonomy description. Our next goal is to make the q-Hamiltonian structure of $M(\Sigma)$ more explicit. We start by introducing coordinates on $M(\Sigma)$. Choose a system of smooth oriented paths U_j from p_j to p_0 $(j = 1, \ldots, r)$ on Σ and loops A_i, B_i $(i = 1, \ldots, k)$ based at p_0, such that:

1. The paths U_j, V_j, A_i, B_i meet only at p_0.
2. The fundamental group of the closed 2-manifold $\hat{\Sigma}$ obtained by capping off the boundary components of Σ, is the group generated by the A_i, B_i, modulo the relation $\prod_{i=1}^{k} [A_i, B_i] = 1$.

3. The path V_0 is obtained by catenation:

$$V_0^{-1} = U_1 V_1 U_1^{-1} \cdots U_r V_r U_r^{-1}\,[A_1, B_1] \cdots [A_k, B_k].$$

Up to $\mathcal{G}_{res}(\Sigma)$-gauge equivalence, every flat connection on Σ is completely determined by the holonomies a_i, b_i, u_j, v_j along A_i, B_i, U_j, V_j ($i = 1, \ldots, k, j = 1, \ldots, r$), and conversely every collection a_i, b_i, u_j, v_j is realized by some flat connection. Consequently we have

$$M(\Sigma) = G^{2(r+k)}$$

with coordinates (a_i, b_i, u_j, v_j). The action of $(g_0, \ldots, g_r) \in G^{r+1}$ is given by

$$a_j \mapsto \mathrm{Ad}_{g_0} a_j,\ b_j \mapsto \mathrm{Ad}_{g_0} b_j,\ u_j \mapsto g_0 u_j g_j^{-1},\ v_j \mapsto \mathrm{Ad}_{g_j} v_j \tag{0.38}$$

and the components of the moment map μ are

$$\begin{aligned} \mu_j(a, b, u, v) =& v_j^{-1}, \quad (j = 1, \ldots r), \\ \mu_0(a, b, u, v) =& \mathrm{Ad}_{u_1} v_1 \cdots \mathrm{Ad}_{u_r} v_r\,[a_1, b_1] \cdots [a_k, b_k]. \end{aligned} \tag{0.39}$$

We now construct a q-Hamiltonian structure on $M(\Sigma)$ as follows. Take r copies of the double $D(G) \cong G^2$, with coordinates (u_j, v_j) as in Remark 3.2, and k copies of its "internal fusion" $\mathbf{D}(G) \cong G^2$, with coordinates (a_i, b_i) as in Example 6.1. Recall that $D(G)$ is a q-Hamiltonian $G \times G$-space with moment maps $(v_j^{-1}, \mathrm{Ad}_{u_j} v_j)$, while $\mathbf{D}(G)$ is a G-space with moment map $[a_i, b_i]$. Fusing the $D(G)$'s with respect to the second component of the $G \times G$-action in each copy together with all $\mathbf{D}(G)$'s we obtain a q-Hamiltonian G^{r+1}-space with action given by (0.38) and moment map by (0.39).

9.3. Equivalence of the gauge theory construction and the holonomy construction. We will now prove that the fusion product from the preceeding subsection gives indeed the correct 2-form.

Theorem 9.3. *Let Σ be a smooth 2-dimensional orientable manifold of genus k with $r+1$ boundary components. Then the moduli space $M(\Sigma) = \mathcal{A}_{flat}(\Sigma)/\mathcal{G}_{res}(\Sigma)$ is canonically isomorphic to the fusion product*

$$\underbrace{D(G) \circledast \cdots D(G)}_{r\ \text{times}} \circledast \underbrace{\mathbf{D}(G) \circledast \cdots \mathbf{D}(G)}_{k\ \text{times}}. \tag{0.40}$$

Proof. Let P denote the polyhedron obtained by cutting Σ along the paths U_j, A_i and B_i. The boundary ∂P consists of $3r+4k+1$ segments:

$$\partial P = U_1 V_1 U_1^{-1} \dots U_r V_r U_r^{-1} A_1 B_1 A_1^{-1} B_1^{-1} \dots A_k B_k A_k^{-1} B_k^{-1} V_0. \tag{0.41}$$

Since P is contractible, every flat connection A on Σ determines a unique function $\psi \in \mathrm{Map}(P, G)$ such that

$$A = \psi^* \theta, \quad \psi(p_0) = e. \tag{0.42}$$

Choose an orientation-preserving parametrization $\partial P \cong [0,1]$ such that $p_0 = 0$, and let ψ_s denote the value of $\psi|\partial P$ at $s \in \partial P$. As before let $\iota : \mathcal{A}_{flat}(\Sigma) \hookrightarrow \mathcal{A}(\Sigma)$ denote the inclusion.

Lemma 9.4. *The pull-back of symplectic form σ to the submanifold of flat connections is given by the formula*

$$\iota^* \sigma = \frac{1}{2} \int_0^1 ds (\psi_s^* \overline{\theta}, \frac{\partial}{\partial s} \psi_s^* \overline{\theta}).$$

For a proof of this result see e.g. [2].

To proceed we introduce some more notation. Suppose $\Delta = [s_0, s_1] \subset [0,1]$ is a subinterval, and let $c := \psi_{s_0}$ and $d := c^{-1} \psi_{s_1}$. Then

$$\psi^{s_0} := c^{-1} \psi$$

satisfies Equation (0.42) with initial condition $\psi(s_0) = e$ instead of $\psi(p_0) = e$. Set $\psi_s^{s_0} = c^{-1} \psi_s$ and define

$$\varpi_\Delta = \frac{1}{2} \int_{s_0}^{s_1} \mathrm{d}s \big((\psi_s^{s_0})^* \overline{\theta}, \frac{\partial}{\partial s} (\psi_s^{s_0})^* \overline{\theta} \big).$$

Then

$$\begin{aligned} \frac{1}{2} \int_{s_0}^{s_1} ds (\psi_s^* \overline{\theta}, \frac{\partial}{\partial s} \psi_s^* \overline{\theta}) =& \frac{1}{2} \int_{s_0}^{s_1} ds \big((c \psi_s^{s_0})^* \overline{\theta}, \frac{\partial}{\partial s} (c \psi_s^{s_0})^* \overline{\theta} \big) \\ =& \varpi_\Delta + \frac{1}{2} (c^* \theta, d^* \overline{\theta}). \end{aligned} \tag{0.43}$$

Lemma 9.4 asserts that $\iota^* \sigma = \varpi_{\partial P}$, while the 2-form $\tilde{\sigma}$ on $\mathcal{A}(\Sigma)$ that gives rise to the q-Hamiltonian structure on the moduli space $M(\Sigma)$ satisfies

$$\iota^* \tilde{\sigma} = \varpi_{\partial P} - \sum_{j=0}^{r} \varpi_{V_j^{-1}}. \tag{0.44}$$

Using (0.43) we evaluate this equation as follows. Combine the segments (0.41) of the boundary ∂P to the following $r+1+k$ loops on Σ:

$$\Delta_i = \begin{cases} V_0 & : \quad i=0, \\ A_i\,B_i\,A_i^{-1}B_i^{-1} & : \quad 1 \le i \le k, \\ U_{i-k}V_{i-k}U_{i-k}^{-1} & : \quad k+1 \le i \le k+r. \end{cases}$$

For $i = 0, \dots k+r+1$ let $t_i \in [0,1]$ such that $\Delta_i = [t_i, t_{i+1}]$, and let $c_i := \psi_{t_i}$ and $d_i = c_i^{-1} c_{i+1}$. Then

$$\varpi_{\partial P} = \omega_{V_0} + \sum_{i=1}^{k+r} \Big(\varpi_{\Delta_i} + \frac{1}{2}(c_i^* \theta, d_i^* \overline{\theta}) \Big). \tag{0.45}$$

Let us first consider the contribution of the loops Δ_{k+j} for $j = 1, \dots, r$. By another application of (0.43) we have

$$\begin{aligned} \varpi_{\Delta_{k+j}} &= \varpi_{U_j} + \varpi_{V_j} + \varpi_{U_j^{-1}} + \frac{1}{2}(u_j^*\theta, (v_j u_j^{-1})^* \overline{\theta}) + \frac{1}{2}(v_j^* \theta, (u_j^{-1})^* \overline{\theta})\} \\ &= \varpi_{V_j} + \frac{1}{2}(\mathrm{Ad}_{v_j}\, u_j^* \theta, u_j^* \theta) + \frac{1}{2}(u_j^* \theta, v_j^* \theta + v_j^* \overline{\theta})\}. \end{aligned}$$

In the first line, the contributions ϖ_{U_j} and $\varpi_{U_j^{-1}}$ cancel each other due to the difference in orientation. The term ϖ_{V_j} cancels the corresponding contribution to (0.44). The remaining expression gives the 2-form on the double $D(G)$.

Next, we analyze the contribution of $\Delta_i = A_i\,B_i\,A_i^{-1}B_i^{-1}$ for $i = 1, \dots, k$:

$$\begin{aligned} \varpi_{\Delta_i} &= \varpi_{A_j B_j} + \varpi_{A_j^{-1} B_j^{-1}} + \frac{1}{2}((a_j b_j)^* \theta, (a_j^{-1} b_j^{-1})^* \overline{\theta}) \\ &= \varpi_{A_j} + \varpi_{B_j} + \varpi_{A_j^{-1}} + \varpi_{B_j^{-1}} \\ &\quad + \frac{1}{2}\big((a_j^* \theta, b_j^* \overline{\theta}) + (a_j^* \overline{\theta}, b_j^* \theta) + ((a_j b_j)^* \theta, (a_j^{-1} b_j^{-1})^* \overline{\theta})\big) \\ &= \frac{1}{2}\big((a_j^* \theta, b_j^* \overline{\theta}) + (a_j^* \overline{\theta}, b_j^* \theta) + ((a_j b_j)^* \theta, (a_j^{-1} b_j^{-1})^* \overline{\theta})\big). \end{aligned}$$

The last line reproduces the 2-form on $\mathbf{D}(G)$. Finally, the contribution ϖ_{V_0} cancels the corresponding term in (0.44). We have shown that the 2-form $\tilde{\sigma}$ is a pull-back of the sum of r copies of the 2-form on $D(G)$, along the maps u_i, v_i, and of k copies of the 2-form on $\mathbf{D}(G)$, along the maps a_i, b_i. The cross terms $\frac{1}{2}(c_i^* \theta, d_i^* \overline{\theta})$ are precisely the extra terms coming from fusion, cf. equation (0.16). This completes the proof of Theorem 9.3. q.e.d.

Suppose now that we are given a tuple of conjugacy classes $\mathcal{C} = (\mathcal{C}_0, \dots, \mathcal{C}_r)$. Recall that the reduction of $D(G)$ at $\mathcal{C}_i$ is equal to $\mathcal{C}_i^-$. Theorem 9.3 together with Theorem 9.2 shows that the moduli space $M(\Sigma, \mathcal{C})$ is a q-Hamiltonian reduction,

$$M(\Sigma, \mathcal{C}) = (\mathcal{C}_0^- \circledast \cdots \circledast \mathcal{C}_r^- \circledast \mathbf{D}(G) \circledast \cdots \circledast \mathbf{D}(G))_e.$$

In particular, the moduli space for the sphere with d holes is the reduction of a d-fold fusion product of conjugacy classes. This fits nicely with the well-known similarities of this space with the symplectic reductions of a d-fold product of coadjoint G-orbits.

9.4. Action of the mapping class group. Let $\mathrm{Diff}(\Sigma)$ denote the group of orientation preserving diffeomorphisms of Σ, with the C^1-topology. Its Lie algebra is the space of vector fields that are tangent to the boundary. The action

$$\mathrm{Diff}(\Sigma) \times \mathcal{A}(\Sigma) \to \mathcal{A}(\Sigma), \ A^\phi = (\phi^{-1})^* A$$

preserves the 2-form σ, and is Hamiltonian with moment map

$$\langle \Phi(A), X \rangle = -\int_\Sigma (\mathrm{curv}(A), \iota(X)A) - \int_{\partial\Sigma} (A, \iota(X)A)$$

for $X \in \mathrm{Lie}(\mathrm{Diff}(\Sigma))$. It also acts on the gauge group $\mathcal{G}(\Sigma)$ by $g^\phi = (\phi^{-1})^* g$, and combines with the gauge group action to an action of the semi-direct product $\mathcal{G}(\Sigma) \rtimes \mathrm{Diff}(\Sigma)$.

Consider the smaller group $\mathrm{Diff}_{res}(\Sigma)$ consisting of all $\phi \in \mathrm{Diff}(\Sigma)$ that preserve the base points $\{p_0, \dots, p_r\}$ up to permutation. Its action descends to $M(\Sigma)$, and combines with the action of G^{r+1} to an action of the semi-direct product $G^{r+1} \rtimes \mathrm{Diff}_{res}(\Sigma)$, where $\mathrm{Diff}_{res}(\Sigma)$ acts on G^{r+1} by permuting factors. Clearly the $\mathrm{Diff}_{res}(\Sigma)$-action preserves ω and permutes the components of μ. However, the action of the identity component $\mathrm{Diff}^0_{res}(\Sigma) = \mathrm{Diff}_{res}(\Sigma) \cap \mathrm{Diff}^0(\Sigma)$ on $M(\Sigma)$ is trivial because every $\phi \in \mathrm{Diff}^0(\Sigma)$ is connected to the identity by a smooth path ϕ_t, and because the fundamental vector field v_X of $X \in \mathrm{Lie}(\mathrm{Diff}_{res}(\Sigma))$ is equal to v_ξ for $\xi = -\iota(X)A \in \mathrm{Lie}(\mathcal{G}(\Sigma))$. All that remains is therefore the action of the mapping class group $\Gamma(\Sigma) = \mathrm{Diff}_{res}(\Sigma)/\mathrm{Diff}^0_{res}(\Sigma)$, and we obtain an action of the semi-direct product

$$G^{r+1} \rtimes \Gamma(\Sigma)$$

preserving ω. The action of the "pure" mapping class group, i.e., the kernel of the homomorphism $\Gamma(\Sigma) \to S(p_0, \dots, p_r)$ to the permutation group, descends to a symplectomorphism of the reduced spaces $M(\Sigma, \mathcal{C}_0, \dots, \mathcal{C}_r)$.

The action of $\Gamma(\Sigma)$ can be described explicitly in terms of coordinates on $M(\Sigma)$.

Example 9.1. According to Theorem 9.3 the moduli space $M(\Sigma)$ for $\Sigma = \Sigma_0^2$ the 2-holed sphere is just the double $D(G)$ considered in the previous section. The element a is interpreted as parallel transport along a path from $p_1 \in V_1$ to $p_2 \in V_2$, while ab is the holonomy around the boundary component V_2. The map

$$S : D(G) \to D(G),\ (a, b) \mapsto (a^{-1}, b^{-1})$$

corresponds to a diffeomorphism exchanging V_1 and V_2: Indeed it satisfies $S^*\omega = \omega$, but switches the G-factors so that $S^*(\mu_1, \mu_2) = (\mu_2, \mu_1)$ and $S((a, b)^{(g_1, g_2)}) = (S(a, b))^{(g_2, g_1)}$. Another interesting action is given by

$$Q : D(G) \to D(G),\ (a, b) \mapsto (aba, a^{-1}).$$

This action by Q is equivariant and preserves both the 2-form and the moment map. It corresponds to a diffeomorphism which rotates one of the boundary circles by 2π while leaving the other one fixed. Since $D(G)$ acts as the identity under diagonal reduction (that is, $(M \circledast D(G))_e \cong M$) this explains the existence of the twist automorphism, Theorem 4.5.

Example 9.2. The fusion operation $M_1 \circledast M_2$ can be viewed as a diagonal G^2-reduction

$$\big(M_1 \times M_2 \times M(\Sigma_0^3)\big)/\!\!/G^2.$$

with respect to two of the three boundary circles of Σ_0^3. Choosing an element of the mapping class group exchanging these two boundary circles we obtain a q-Hamiltonian isomorphism $M(\Sigma_0^3) \to M(\Sigma_0^3)$ exchanging two components of the moment map. It descends to a q-Hamiltonian isomorphism $M_1 \circledast M_2 \to M_2 \circledast M_1$. This is the origin for the braid isomorphisms discussed in Section 6.

In [20], a fusion operation was introduced for Hamiltonian LG-manifolds with proper moment maps. Letting $\mathcal{M}(\Sigma_0^3)$ be the Hamiltonian LG^3-manifold associated to Σ_0^3, the fusion product of two Hamiltonian LG-manifolds $\mathcal{M}_1$ and $\mathcal{M}_2$ is the diagonal LG^2-reduction

$$\mathcal{M}_1 \circledast \mathcal{M}_2 = (\mathcal{M}_1 \times \mathcal{M}_2 \times \mathcal{M}(\Sigma_0^3))/\!\!/LG^2;$$

it is a Hamiltonian LG-space with proper moment map. (By contrast, the moment map for the direct product $\mathcal{M}_1 \times \mathcal{M}_2$ with diagonal LG-action is not proper and does not have the correct equivariance property.) The holonomy manifold of $\mathcal{M}_1 \circledast \mathcal{M}_2$ is

$$\mathrm{Hol}(\mathcal{M}_1 \circledast \mathcal{M}_2) = \mathrm{Hol}(\mathcal{M}_1) \circledast \mathrm{Hol}(\mathcal{M}_2).$$

10. Relation to Poisson-Lie G-spaces

In this Section we establish a connection between the theory of Poisson-Lie groups and the theory of q-Hamiltonian G-spaces. Although these two theories are not equivalent to each other, the definition of a Poisson-Lie G-space can be rewritten in a form very similar to the definition of a q-Hamiltonian G-space.

10.1. Poisson-Lie G-spaces. We begin with a short exposition of the theory of Poisson-Lie G-spaces of J.-H. Lu and A. Weinstein [16], [17]. As for q-Hamiltonian spaces, the target of the moment map is a non-abelian Lie group.

Throughout this section G denotes a connected and simply connected compact Lie group and $T \subset G$ a maximal torus. The inner product $(\,,\,)$ on $\mathfrak{g}$ induces a complex-bilinear form, still denoted $(\,,\,)$ on its complexification $\mathfrak{g}^{\mathbb{C}}$. We will regard $\mathfrak{g}^{\mathbb{C}}$ as a real Lie algebra, and let $G^{\mathbb{C}}$ be the corresponding Lie group. Let $\mathbf{n} \subset \mathfrak{g}^{\mathbb{C}}$ be the sum of root spaces for the positive roots and $\mathfrak{a} := \sqrt{-1}\,\mathfrak{t}$. Write $A = \exp(\mathfrak{a})$ and $N = \exp(\mathbf{n})$. We have the Iwasawa decompositions

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{g} \oplus \mathfrak{a} \oplus \mathbf{n}, \quad G^{\mathbb{C}} = G\,A\,N = A\,N\,G.$$

The pairing of the subalgebra $\mathfrak{a} \oplus \mathbf{n}$ with $\mathfrak{g}$ given by the imaginary part of $(\,,\,)$

$$\langle \zeta, \eta \rangle = \mathrm{Im}(\zeta, \eta)\ ,\ \zeta \in \mathfrak{a} \oplus \mathbf{n}\ ,\ \eta \in \mathfrak{g}.$$

is nondegenerate, and identifies $\mathfrak{a} \oplus \mathbf{n} \cong \mathfrak{g}^*$. Let $G^* = AN \subset G^{\mathbb{C}}$ be the corresponding simply connected subgroup. We denote the projection to the first factor by $\alpha : G^* \to A$. According to Drinfeld the isomorphism $\mathfrak{g}^{\mathbb{C}} = \mathfrak{g} \oplus \mathfrak{g}^*$ means that G is a *Poisson-Lie group*, and G^* its dual Poisson-Lie group. By the Iwasawa decomposition any element $G^{\mathbb{C}}$ can be uniquely written as a product of elements of G^* and G:

$$G^{\mathbb{C}} = G^* G.$$

Left-multiplication of G on $G^{\mathbb{C}}$ induces an action of G on $G^* = G^{\mathbb{C}}/G$ which is known as the (left) dressing action (this terminology is due to Semenov-Tian-Shansky). We denote the fundamental vector fields for the dressing action by $v_\xi^\sharp$.

We will use the same notation $\theta, \overline{\theta}$ for the Maurer-Cartan forms on $G^{\mathbb{C}}$ and its subgroups. This does not lead to ambiguities since the Maurer-Cartan form on a subgroup of a group is just the pull-back of the Maurer Cartan-form on the group. Sometimes we denote the Maurer-Cartan forms on G^* by $\theta_{G^*}, \overline{\theta}_{G^*}$ for clarity. Let

$$\chi_{\mathbb{C}} = \frac{1}{12}(\theta, [\theta, \theta]) \in \Omega^3(G^{\mathbb{C}}, \mathbb{C}).$$

Definition 10.1 (Lu). A Poisson-Lie G-space is a G-manifold $(M, \mathcal{A})$ together with a 2-form $\omega \in \Omega^2(M)$, and an equivariant map $\mu \in C^\infty(M, G^*)$ such that the following conditions are fulfilled:

(D1) The form ω is closed.

(D2) For all $\xi \in \mathfrak{g}$,

$$\iota(v_\xi^\sharp)\omega = 2\mu^* \langle \overline{\theta}_{G^*}, \xi \rangle.$$

(D3) The form ω is non-degenerate.

The map μ is called a Poisson-Lie moment map.

The factor of 2 is introduced into (D2) in order to simplify the comparison of the definition of a Poisson-Lie G-space to the definition of a q-Hamiltonian G-space with P-valued moment map.

Remark 10.1.

1. The 2-form ω is not invariant. Rather, the point of the definition is that the action map $\mathcal{A}$ becomes a Poisson map [16].

2. Just as for q-Hamiltonian G-spaces, there is a ring structure on the category of Poisson-Lie spaces. Given two Lie-Poisson G-spaces M_1, M_2 there exists the structure of a Lie-Poisson G-space on $M_1 \times M_2$ with symplectic form the sum $\omega_1 + \omega_2$ and moment map the pointwise product $\mu_1 \cdot \mu_2$. The G-action is not simply the diagonal G-action but is "twisted". See e.g. [7].

3. The moment map for Poisson-Lie G-spaces has the properties

$$\mathrm{Im}(\mu^*\theta)_x = \mathfrak{g}_x^0, \quad \ker(\mathrm{d}_x\mu)^\omega = \{v_\xi^\sharp(x),\, \xi \in \mathfrak{g}\}. \tag{0.46}$$

The proof is analogous to that for Hamiltonian G-spaces.

10.2. q-Hamiltonian G-space with P-valued moment maps. As we explained in Remark 2.1, the possibility of choosing $X = \mathfrak{g}^*$ or $X = G$ as target space for a generalized moment map relies on the existence of a natural equivariantly closed equivariant 3-form χ_G on X. In this subsection we present another example of a target X with this property.

Let $\tau : G^{\mathbb{C}} \to G^{\mathbb{C}}$ denote the Cartan involution, defined by exponentiating the complex conjugation mapping $\mathfrak{g}^{\mathbb{C}} \to \mathfrak{g}^{\mathbb{C}}$, and let $I : G^{\mathbb{C}} \to G^{\mathbb{C}}$ denote the map $Ig = \tau(g^{-1})$. We will also use the notation $Ig = g^\dagger$ since for $G = \mathrm{SU}(N)$ the complexification is $G^{\mathbb{C}} = \mathrm{Sl}(N, \mathbb{C})$ and the map I is Hermitian conjugation. Let P denote the symmetric space

$$P = \{g \in G^{\mathbb{C}},\, g = g^\dagger\}.$$

The adjoint action of G on $G^{\mathbb{C}}$ leaves P invariant. Let $p : P \hookrightarrow G^{\mathbb{C}}$ denote the embedding. Set

$$\theta_P = p^*\theta,\ \overline{\theta}_P = p^*\overline{\theta} \in \Omega^1(P, \mathfrak{g}^{\mathbb{C}}),$$

and define a 3-form $\chi_P \in \Omega^3(P)$ by

$$\chi_P = \frac{1}{12} p^* \,\mathrm{Im}(\overline{\theta}, [\overline{\theta}, \overline{\theta}]).$$

For all $\xi \in \mathfrak{g}$, the complex conjugate of the 1-form $(\theta, \xi) \in \Omega^1(G^{\mathbb{C}}, \mathbb{C})$ is $-(I^*\overline{\theta}, \xi)$. Therefore the 1-form $(\theta + I^*\overline{\theta}, \xi)$ is purely imaginary. On P the map I is trivial, so that $(\theta_P + \overline{\theta}_P, \xi)$ is purely imaginary. Put differently, $\theta_P + \overline{\theta}_P$ takes values in $\sqrt{-1}\mathfrak{g}$. The equivariantly closed extension of χ_P is the 3-form $\chi_{P,G} \in \Omega_G^3(P)$ defined by

$$\chi_{P,G}(\xi) = \chi_P + \frac{1}{2\sqrt{-1}} \mu_P^*(\theta_P + \overline{\theta}_P, \xi).$$

Definition 10.2. A q-Hamiltonian G-space with P-valued moment map is a manifold M equipped with a G-action $\mathcal{A}$, a 2-form $\omega_P \in \Omega^2(M)$ and an equivariant map $\mu_P \in C^\infty(M, P)^G$ such that:

(E1) The differential of ω_P is given by:

$$\mathrm{d}\omega_P = -\mu_P^* \chi_P.$$

(E2) For all $\xi \in \mathfrak{g}$,

$$\iota(v_\xi)\omega_P = \frac{1}{2\sqrt{-1}}\,\mu_P^*(\theta_P + \overline{\theta}_P, \xi).$$

(E3) The form ω_P is non-degenerate.

The map μ_P is called a P-valued moment map.

Since G by assumption is connected, equivariance of the moment map μ_P together with conditions (E1) and (E2) implies invariance of the 2-form ω_P:

$$\begin{aligned}\mathcal{L}_{v_\xi}\omega_P &= \iota(v_\xi)\mathrm{d}\omega_P + \mathrm{d}\iota(v_\xi)\omega_P \\ &= \mu_P^*\Big(- \iota(v_\xi)\chi_P + \frac{1}{2\sqrt{-1}}\,\mathrm{d}(\theta_P + \overline{\theta}_P, \xi)\Big) = 0.\end{aligned}$$

10.3. Equivalence of Poisson-Lie G-spaces and q-Hamiltonian G-spaces with P-valued moment map. In this subsection we prove the equivalence of the definitions of a Poisson-Lie G-space and of a q-Hamiltonian G-space with P-valued moment map. Consider the map

$$j : G^{\mathbb{C}} \to P\ ,\ j(b) = b\, b^\dagger.$$

It turns the left G-action on $G^{\mathbb{C}}$ into the adjoint action on P and restricts to an equivariant diffeomorphism $G^* \cong P$. Let $\kappa : G^* \hookrightarrow G^{\mathbb{C}}$ denote the embedding.

Proposition 10.3. *Let $(M, \mathcal{A}, \omega, \mu)$ be a Poisson-Lie G-space. Then the manifold M equipped with the same G-action $\mathcal{A}$, moment map*

$$\mu_P = j \circ \mu : M \to P$$

and 2-form

$$\omega_P = \omega + \frac{1}{2}\mu^*\kappa^* \operatorname{Im}\ (I^*\overline{\theta}, \theta)$$

is a q-Hamiltonian G-space with P-valued moment map.

Proof. The map μ_P is equivariant because it is the composition of two equivariant maps. To check (E1) observe first that

$$j^*\chi_{\mathbb{C}} = \chi_{\mathbb{C}} + I^*\chi_{\mathbb{C}} - \frac{1}{2}\mathrm{d}(I^*\overline{\theta}, \theta). \tag{0.47}$$

Taking imaginary parts this shows $j^* \operatorname{Im} \chi_{\mathbb{C}} = -\frac{1}{2} \operatorname{Im} \mathrm{d}(I^*\overline{\theta}, \theta)$. Therefore

$$\mathrm{d}\omega_P = \mathrm{d}\omega + \frac{1}{2}\mathrm{d}\mu^* \operatorname{Im}(I^*\overline{\theta}, \theta) = -\mu^* j^* \operatorname{Im} \chi_{\mathbb{C}} = -\mu_P^* \chi_P$$

in accordance with (E1). By Lemma 10.4 below the imaginary part of the 1-form $\iota(v_\xi^\sharp)\kappa^*(\theta, I^*\overline{\theta})$ on G^* is given by

$$\iota(v_\xi^\sharp)\kappa^* \operatorname{Im}(\theta, I^*\overline{\theta}) = 4\kappa^* \operatorname{Im}(\overline{\theta}, \xi) - \kappa^* j^* \operatorname{Im}(\theta + \overline{\theta}, \xi).$$

Using this fact together with (D2) we compute

$$\begin{aligned}
\iota(v_\xi)(\omega - \frac{1}{2}\mu^* \operatorname{Im}\ \kappa^*(I^*\overline{\theta}, \theta)) &= \frac{1}{2}\mu^*\kappa^* j^* \operatorname{Im}\left((\theta + \overline{\theta}, \xi)\right) \\
&= \frac{1}{2}\mu_P^* \operatorname{Im}(\theta + \overline{\theta}, \xi),
\end{aligned}$$

which gives (E2). To verify (E3) let $v \in \ker(\omega_P)_x$, that is,

$$\iota(v)\omega = -\iota(v)\frac{1}{2} \operatorname{Im} \mu^*\kappa^* \, (I^*\overline{\theta}_{G^*}, \theta_{G^*}).$$

Since the right-hand side of this equation annihilates the kernel of $d_x\mu$, this shows that $v \in \ker(d_x\mu)^\omega$. By (0.46) this implies that $v = v_\xi(x)$ for some $\xi \in \mathfrak{g}$. Using (E2) we arrive at the condition

$$0 = \iota(v_\xi)\omega_P = \frac{1}{2\sqrt{-1}}\, \mu_P^*(\theta_P + \overline{\theta}_P, \xi). \tag{0.48}$$

Contracting this Equation with v_η for $\eta \in \mathfrak{g}$ yields

$$(\eta, \operatorname{Ad}_{\mu_P(x)} \xi - \operatorname{Ad}_{\mu_P(x)^{-1}} \xi) = 0$$

for all η, or

$$\xi \in \ker(\operatorname{Ad}_{\mu_P(x)^2} - 1) = \ker(\operatorname{Ad}_{\mu_P(x)} - 1) \oplus \ker(\operatorname{Ad}_{\mu_P(x)} + 1).$$

Since the eigenvalues of $\operatorname{Ad}_{\mu_P(x)}$ are nonegative real numbers, the space $\ker(\operatorname{Ad}_{\mu_P(x)} + 1)$ is trivial. Hence $\operatorname{Ad}_{\mu_P(x)} \xi = \xi$. For such ξ the above equation becomes

$$0 = \mu_P^*(\theta_P + \overline{\theta}_P, \xi) = 2\mu_P^*(\theta_P, \xi).$$

By (D2) this equation implies $\iota(v_\xi)\omega = 0$, and finally $v_\xi = 0$ by non-degeneracy of ω, proving (E3). q.e.d.

One can easily reverse the argument and show that the structure of a q-Hamiltonian G-space with P-valued moment map defines the structure of a Poisson-Lie G-space on the same manifold. In the proof we used the following Lemma:

Lemma 10.4. *The contraction of the fundamental vector field $v_\xi^\sharp$ for the dressing action with the 2-form $\kappa^*(I^*\overline{\theta}, \theta)$ on G^* is given by the formula*

$$\iota(v_\xi^\sharp)\kappa^*(I^*\overline{\theta}, \theta) = 2\kappa^*(\overline{\theta} + I^*\theta, \xi) - \kappa^* j^*(\theta + \overline{\theta}, \xi). \tag{0.49}$$

Proof. We compute:

$$\begin{aligned} \iota(v_\xi^\sharp)\kappa^*(I^*\overline{\theta}, \theta) &= (\iota(v_\xi^\sharp)\kappa^* I^*\overline{\theta}, \kappa^*\theta) - (\kappa^* I^*\overline{\theta}, \iota(v_\xi^\sharp)\kappa^*\theta) \\ &= (\iota(v_\xi^\sharp)\kappa^*(\theta + I^*\overline{\theta}), \kappa^*\theta) \\ &\quad -(\kappa^* I^*\overline{\theta}, \iota(v_\xi^\sharp)\kappa^*(I^*\overline{\theta} + \theta)) \\ &\quad -(\iota(v_\xi^\sharp)\kappa^*\theta, \kappa^*\theta) + (\kappa^* I^*\overline{\theta}, \iota(v_\xi^\sharp)\kappa^* I^*\overline{\theta}). \end{aligned} \tag{0.50}$$

The last two terms in this expression cancel, for the following reason. It is easy to see that one of them is a complex conjugate of the other. We will show that the first one is real which ensures the cancellation. Indeed, the subalgebra $\mathbf{n}$ equals the kernel of the bilinear form $(,)$ restricted to $\mathfrak{a} \oplus \mathbf{n}$. Hence

$$(\iota(v_\xi^\sharp)\kappa^*\theta, \kappa^*\theta) = (\iota(v_\xi^\sharp)\alpha^*\kappa^*\theta, \alpha^*\kappa^*\theta),$$

which is real since the restriction of $(,)$ to $\mathfrak{a}$ is real-valued. To compute the first two terms in (0.50) observe that

$$I^*\overline{\theta} + \theta = \mathrm{Ad}_{b^\dagger}\, j^*\overline{\theta},$$

which gives

$$\iota(v_\xi^\sharp)\kappa^*(\theta + I^*\overline{\theta}) = \mathrm{Ad}_{b^{-1}}\,\xi - \mathrm{Ad}_{b^\dagger}\,\xi. \tag{0.51}$$

Therefore

$$\iota(v_\xi^\sharp)\kappa^*(I^*\overline{\theta}, \theta) = \kappa^*(\overline{\theta} + I^*\theta - \mathrm{Ad}_{(b^\dagger)^{-1}}\,\theta - \mathrm{Ad}_b\, I^*\overline{\theta}, \xi).$$

Combining this with

$$j^*(\theta+\overline{\theta}) = \mathrm{Ad}_{(b^\dagger)^{-1}}\,\theta + \mathrm{Ad}_b\, I^*\overline{\theta} + \overline{\theta} + I^*\theta \tag{0.52}$$

completes the proof of the Lemma. q.e.d.

10.4. Equivalence of q-Hamiltonian G-spaces with P-valued moment map and Hamiltonian G-spaces. Let us note that the space P has two important properties:

1. P is contractible. Hence, the closed 3-form χ_P is exact.
2. The restriction of the exponential map $\exp : \mathfrak{g}^{\mathbb{C}} \to G^{\mathbb{C}}$ to $\sqrt{-1}\,\mathfrak{g} \subset \mathfrak{g}^{\mathbb{C}}$ is invertible and has image P. Let $\varkappa : P \to \sqrt{-1}\,\mathfrak{g}$ be the inverse map, and identify $\sqrt{-1}\,\mathfrak{g} \cong \mathfrak{g}^*$ by means of the pairing $\mathrm{Im}(\,,\,)$.

Using these facts one can convert a q-Hamiltonian G-space with P-valued moment map into a usual Hamiltonian G-space.

Proposition 10.5.

1. *There exists a canonical G-invariant 2-form τ on P such that:*

$$d\tau = \chi_P, \quad \iota(v_\xi)\tau = d(\varkappa,\xi) - \frac{1}{2\sqrt{-1}}\,(\theta_P + \overline{\theta}_P, \xi).$$

2. *Let $(M, \mathcal{A}, \mu_P, \omega_P)$ be a q-Hamiltonian G-space with P-valued moment map. Then the manifold M with the same G-action $\mathcal{A}$, moment map*

$$\mu = \varkappa \circ \mu_P : M \to \mathfrak{g}^*$$

and 2-form

$$\omega = \omega_P + \mu_P^* \tau$$

is a Hamiltonian G-space.

This was proved in [1]. In fact, the proof is essentially as that of Proposition 3.4 since the restriction of the exponential map to $\sqrt{-1}\mathfrak{g}$ is invertible. Propositions 10.3 and 10.5 reduce the theory of Poisson-Lie G-spaces to the usual theory of Hamiltonian G-spaces.

Appendix A. Properties of the form ϖ

In this Appendix we give the proof of Proposition 8.1 concerning the properties of the 2-form

$$\varpi = \frac{1}{2}\int_0^1 \mathrm{d}s\,(\mathrm{Hol}_s^*\overline{\theta},\ \frac{\partial}{\partial s}\mathrm{Hol}_s^*\overline{\theta})$$

on $L\mathfrak{g}^*$. We will need the following properties of the pull-back of $\overline{\theta}$ under Hol_s:

Lemma A.1 (Properties of $\mathrm{Hol}_s^*\overline{\theta}$). *Let $\mathcal{A}_g : L\mathfrak{g}^* \to L\mathfrak{g}^*$ the action defined by $g \in LG$. Then*

$$\mathcal{A}_g^*\,\mathrm{Hol}_s^*\overline{\theta} = \mathrm{Ad}(g(0))\,\mathrm{Hol}_s^*\overline{\theta}. \tag{0.53}$$

The contractions with fundamental vector fields v_ξ (for $\xi \in L\mathfrak{g}$) are

$$\iota(v_\xi)\,\mathrm{Hol}_s^*\overline{\theta} = \xi(0) - \mathrm{Ad}(\mathrm{Hol}_s(A))\xi(s). \tag{0.54}$$

For any $\zeta \in L\mathfrak{g}^*$, viewed as a constant vector field on $L\mathfrak{g}^*$, one has

$$\iota(\zeta)\,\mathrm{Hol}_s^*\overline{\theta} = \int_0^s \mathrm{Ad}(\mathrm{Hol}_u(A))\zeta(u)\,du. \tag{0.55}$$

Proof. For $h \in G$ let $R_h, L_h : G \to G$ be the left-resp. right multiplication by h. By the equivariance property (0.26) and right-invariance of $\overline{\theta}$, we have

$$\mathcal{A}_g^*\,\mathrm{Hol}_s^*\overline{\theta} = \mathrm{Hol}_s^*\,L_{g(0)}^*R_{g(s)^{-1}}^*\overline{\theta} = \mathrm{Hol}_s^*\,\mathrm{Ad}(g(0))\overline{\theta} = \mathrm{Ad}(g(0))\,\mathrm{Hol}_s^*\overline{\theta}.$$

By another application of (0.26), the tangent map $d_A\,\mathrm{Hol}_s$ satisfies

$$(d_A\,\mathrm{Hol}_s)(v_\xi) = \xi(0)\,\mathrm{Hol}_s(A) - \mathrm{Hol}_s(A)\xi(s),$$

which implies (0.54). For the third identity write

$$\mathrm{Hol}_s(A + t\zeta) = \phi_s(t\zeta)\,\mathrm{Hol}_s(A)$$

so that

$$\iota(\zeta)\,\mathrm{Hol}_s^*\overline{\theta} = (d_A\,\mathrm{Hol}_s)(\zeta)\ \mathrm{Hol}_s(A)^{-1} = \frac{\partial}{\partial t}\Big|_{t=0}\phi_s(t\zeta).$$

Differentiating the defining identity

$$\mathrm{Hol}_s(A)^{-1}\phi_s(t\zeta)^{-1}\frac{\partial}{\partial s}\Big(\phi_s(t\zeta)\,\mathrm{Hol}_s(A)\Big) = A + t\zeta$$

with respect to t at $t=0$ leads to

$$\mathrm{Ad}(\mathrm{Hol}_s^{-1})\frac{\partial}{\partial s}\big(\iota(\eta)\,\mathrm{Hol}_s^*\overline{\theta}\big) = \zeta.$$

Applying $\mathrm{Ad}(\mathrm{Hol}_s)$ and integrating from 0 to s give (0.55). q.e.d.

We now give the proof of Proposition 8.1.

Proof. LG-invariance of ϖ follows immediately from (0.53). The differential is computed as follows:

$$\begin{aligned}
\mathrm{d}\varpi &= \frac{1}{4}\int_0^1 \mathrm{d}s\,(\mathrm{Hol}_s^*[\overline{\theta},\overline{\theta}],\,\frac{\partial}{\partial s}\,\mathrm{Hol}_s^*\,\overline{\theta}) - \frac{1}{4}\int_0^1 \mathrm{d}s\,(\mathrm{Hol}_s^*\,\overline{\theta},\,\frac{\partial}{\partial s}\,\mathrm{Hol}_s^*[\overline{\theta},\overline{\theta}])\\
&= \frac{1}{2}\int_0^1 \mathrm{d}s\,(\mathrm{Hol}_s^*[\overline{\theta},\overline{\theta}],\,\frac{\partial}{\partial s}\,\mathrm{Hol}_s^*\,\overline{\theta}) - \frac{1}{4}\,\mathrm{Hol}^*([\overline{\theta},\overline{\theta}],\overline{\theta})\\
&= \frac{1}{6}\int_0^1 \mathrm{d}s\,\frac{\partial}{\partial s}\,\mathrm{Hol}_s^*([\overline{\theta},\overline{\theta}],\overline{\theta}) - \frac{1}{4}\,\mathrm{Hol}^*([\overline{\theta},\overline{\theta}],\overline{\theta})\\
&= -\frac{1}{12}\,\mathrm{Hol}^*([\overline{\theta},\overline{\theta}],\overline{\theta}) = -\,\mathrm{Hol}^*\,\chi.
\end{aligned}$$

Given $\xi\in L\mathfrak{g}$ by partial integration we have

$$\begin{aligned}
\iota(v_\xi)\varpi =& \frac{1}{2}\int_0^1 \mathrm{d}s\,(\iota(v_\xi)\,\mathrm{Hol}_s^*\,\overline{\theta},\,\frac{\partial}{\partial s}\,\mathrm{Hol}_s^*\,\overline{\theta})\\
&- \frac{1}{2}\int_0^1 \mathrm{d}s\,(\mathrm{Hol}_s^*\,\overline{\theta},\,\frac{\partial}{\partial s}\iota(v_\xi)\,\mathrm{Hol}_s^*\,\overline{\theta})\\
=& \int_0^1 \mathrm{d}s\,(\iota(v_\xi)\,\mathrm{Hol}_s^*\,\overline{\theta},\,\frac{\partial}{\partial s}\,\mathrm{Hol}_s^*\,\overline{\theta}) - \frac{1}{2}\,\mathrm{Hol}^*(\overline{\theta},\,\iota(v_\xi)\overline{\theta})\\
=& \int_0^1 \mathrm{d}s\,(\xi(0) - \mathrm{Ad}(\mathrm{Hol}_s)\xi(s),\,\frac{\partial}{\partial s}\,\mathrm{Hol}_s^*\,\overline{\theta})\\
&- \frac{1}{2}\,\mathrm{Hol}^*(\overline{\theta}-\theta,\,\xi(0))\\
=& \frac{1}{2}(\mathrm{Hol}^*(\theta+\overline{\theta}),\xi(0)) - \int_0^1 (\xi(s),\mathrm{Ad}(\mathrm{Hol}_s^{-1})\,\frac{\partial}{\partial s}\,\mathrm{Hol}_s^*\,\overline{\theta}).
\end{aligned}$$

From (0.55) it follows that for all $\zeta\in T_A L\mathfrak{g}^*\cong L\mathfrak{g}^*$,

$$\iota(\zeta)\int_0^1 (\xi(s),\,\mathrm{Ad}(\mathrm{Hol}_s^{-1})\,\frac{\partial}{\partial s}\,\mathrm{Hol}_s^*\,\overline{\theta}) = \int_0^1 (\xi(s),\zeta(s)) = \iota(\zeta)\mathrm{d}\oint(A,\,\xi),$$

which concludes the proof. q.e.d.

References

[1] A. Yu. Alekseev, *On Poisson actions of compact Lie groups on symplectic manifolds*, J. Differential Geom. **45** (1997) 241–256.

[2] A. Yu. Alekseev & A. Z. Malkin, *Symplectic structure of the moduli space of flat connections on a Riemann surface*, Comm. Math. Phys. **169** (1995) 99–119.

[3] M. F. Atiyah & R. Bott, *The Yang-Mills equations over Riemann surfaces*, Philos. Trans. Roy. Soc. London **308** (1982) 523–615.

[4] S. K. Donaldson, *Boundary value problems for Yang-Mills fields*, J. Geom. Phys. **8** (1992) 89–122.

[5] S. K. Donaldson & P. Kronheimer, *The geometry of four-manifolds*, Oxford Math. Monographs, 1990.

[6] V. G. Drinfeld, *Hamiltonian structures on Lie groups, Lie bialgebras and the geometric meaning of the classical Yang-Baxter equation*, Soviet. Math. Dokl. **27** (1983) 68-71.

[7] H. Flaschka & T. Ratiu, *A convexity theorem for Poisson actions of compact Lie groups*, Ann. Sci. École Norm. Sup. **29** (1996) 787–80

[8] V. V. Fock & A. A. Rosly, *Poisson structure on moduli of flat connections on Riemann surfaces and r-matrix*, Preprint, 1992.

[9] W. Goldman, *The symplectic nature of fundamental groups of surfaces*, Adv. Math. **54** (1984) 200–225.

[10] K. Guruprasad, J. Huebschmann, L. Jeffrey, &A. Weinstein, *Group systems, groupoids, and moduli spaces of parabolic bundles*, Preprint, 1995.

[11] J. Huebschmann, *Poisson structure on certain moduli spaces for bundles on a surface*, Ann. Inst. Fourier (Grenoble) **45** (1995) 65–91.

[12] L. Jeffrey, *Extended moduli spaces of flat connections on Riemann surfaces,* Math. Ann. **298** (1994) 667–692.

[13] ———, *Group cohomology construction of the cohomology of moduli spaces of flat connections on 2-manifolds,* Duke Math. J. **77** (1995) 407–429.

[14] Y. Karshon, *An algebraic proof for the symplectic structure of moduli space*, Proc. Amer. Math. Soc. **116** (1992) 591–605.

[15] C. King & A. Sengupta, *A symplectic structure on surfaces with boundary*, Comm. Math. Phys. **175** (1996) 657–671.

[16] J.-H. Lu, *Momentum mappings and reduction of Poisson actions*, Symplectic geometry, groupoids, and integrable systems (Berkeley, CA, 1989), Math. Sci. Res. Inst. Publ., Vol. 20, Springer, New York, 1991, 209–226.

[17] J.-H. Lu & A. Weinstein, *Poisson-Lie groups, dressing transformations and Bruhat decompositions*, J. Differential Geom. **31** (1990) 501–526.

[18] D. McDuff, *The moment map for circle actions on symplectic manifolds*, J. Geom. Phys. **5** (1988) 149–160.

[19] E. Meinrenken & C. Woodward, *A symplectic proof of Verlinde factorization*, Preprint, August 1996.

[20] ———, *Fusion of Hamiltonian loop group manifolds and cobordism*, Preprint, July 1997.

[21] A. Pressley & G. Segal, *Loop groups*, Oxford University Press, Oxford, 1988.

[22] E. Verlinde, *Fusion rules and modular transformations in 2D conformal field theory*, Nuclear Phys. B **300** (1988) 360–376.

[23] A. Weinstein, *The symplectic structure on moduli space*, The Floer memorial volume, Progr. Math., 133, Birkhäuser, Basel, 1995, 627–635.

[24] J. Weitsman, *A Duistermaat-Heckman formula for symplectic circle actions*, Internat. Math. Res. Notices **12** (1993) 309–312.

[25] E. Witten, *On quantum gauge theories in two dimensions*, Comm. Math. Phys. **141** (1991) 153–209.

Uppsala University, Sweden
Yale University
University of Toronto