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ON THE STRUCTURE OF SPACES WITH RICCI CURVATURE BOUNDED BELOW. I

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0. Introduction

In this paper and in [12], [13], we study the structure of spaces, Y, which are pointed Gromov-Hausdorff limits of sequences, $\{(M_i^n, p_i)\}$, of complete, connected Riemannian manifolds whose Ricci curvatures have a definite lower bound, say $Ric_{M_i^n} \geq -(n-1)$. In Sections 5–7, and sometimes in [12], we also assume a lower volume bound, $\mathrm{Vol}\,(B_1(p_i)) \geq v > 0$. In this case, the sequence is said to be *non-collapsing.* If $\lim_{i\to\infty} \mathrm{Vol}\,(B_1(p_i)) = 0$, then the sequence is said to *collapse.* It turns out that a convergent sequence is noncollapsing if and only if the limit has positive n-dimensional Hausdorff measure. In particular, any convergent sequence is either collapsing or noncollapsing. Moreover, if the sequence is collapsing, it turns out that the Hausdorff dimension of the limit is actually $\leq n-1$; see Sections 3 and 5.

Our theorems on the infinitesimal structure of limit spaces have equivalent statements in terms of (or implications for) the structure on a small but definite scale, of manifolds with $Ric_{M^n} \geq -(n-1)$. Although both contexts are significant, for the most part, it is the limit spaces which are emphasized here. Typically, the relation between corresponding statements for manifolds and limit spaces follows directly from the continuity of the geometric quantities in question under Gromov-Hausdorff limits, together with Gromov's compactness theorem, [37]; Theorems 2.45, 5.12 (see also Remark 5.13), 7.5, 7.6, are examples of

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results concerning Riemannian manifolds, whose proofs depend on results on the infinitesimal structure of limit spaces; see also Remark 4.9.

Our results, most of which were announced in [14], are applications of the "almost rigidity" theorems for manifolds of almost nonnegative Ricci curvature, announced in [14] and proved in [15]. In particular, we use the generalized splitting, "volume cone implies metric cone" and (implicitly) integral Toponogov theorems, together with tangent cone analysis of the sort employed in geometric measure theory.

The continuity of the volume (of balls) under Gromov-Hausdorff limits, $M_i^n \to Y^n$, where $Ric_{M_i^n} \geq -(n-1)$ and Y^n is a manifold, also plays a direct role in the present discussion. The continuity of the volume in the above case was conjectured by Anderson-Cheeger and proved in [26].

The remainder of this paper is divided into 8 sections and two appendices.

1. Renormalized limit measures.
2. Arbitrary limit spaces.
3. $\dim Y \leq n-1$ in the collapsed case.
4. Polar limit spaces.
5. Noncollapsed limit spaces.
6. $\dim\, \mathcal{S}(Y^n) \leq n-2$.
7. Two sided bounds on Ricci curvature and Einstein manifolds.
8. Examples.

Appendix 1. Reifenberg's method and some consequences.
Appendix 2. Remarks on the synthetic treatment of Ricci curvature.

We now describe the contents of the paper in more detail.

Let *dim* denote *Hausdorff dimension.* We write Y^m to indicate that Y has dimension m. Let $\ell \in \mathbb{R}_+$. We say that y is an *ℓ-dimensional point*, if $\lim_{r\to 0} \dim(B_r(y)) = \ell$. We denote the subset of such points by $Y(\ell)$.

Let the complete pointed metric space, (Y^m, y), be the pointed Gromov-Hausdorff limit of a sequence of connected pointed Riemannian manifolds, $\{(M_i^n, p_i)\}$, with $Ric_{M_i^n} \geq -(n-1)$. Of course, $m \leq n$

and Y^m is locally compact; see [37]. By definition, a *tangent cone* at $y \in Y^m$ is a complete pointed Gromov-Hausdorff limit, $\{Y_y, d_\infty, y_\infty\}$ of a sequence of rescaled spaces, $\{(Y^m, r_i^{-1}d, y)\}$, where d, d_∞ are the metrics (distance functions) of Y^m, Y_y respectively, and $\{r_i\}$ is a positive sequence with $r_i \to 0$. It follows from Gromov's compactness theorem that every such sequence has a subsequence, $\{r_j\}$, such that $\{Y^m, r_j^{-1}d, y\}$ is convergent. In particular, tangent cones exist for all $y \in Y^m$, but might depend on the choice of convergent sequence.

It is easy to see that any tangent cone also arises as the pointed limit of some sequence, $\{(M_i^n, q_i)\}$, with $Ric_{M_i^n} \geq -(n-1)\delta_i$, where $\delta_i \to 0$. Thus, tangent cones have nonnegative curvature in a generalized sense. In some arguments, we must also consider *iterated tangent cones* i.e. we take $(Y_y)_z$ at some point, $z \in Y_y$, and iterate this construction finitely many times. It is also easy to see that any iterated cone can be realized as a pointed Gromov-Hausdorff limit of some sequence, $\{(Y^m, r_i^{-1}d, y_i)\}$, and hence, as the limit of some sequence, $\{(M_i^n, q_i)\}$, as well.

Definition 0.1. A point, $y \in Y$, is called *regular*, if for some k, every tangent cone at y is isometric to $\mathbb{R}^k$.

Let $\mathcal{R}_k$ denote the set of k-regular points and put $\mathcal{R} = \cup_k \mathcal{R}_k$, the *regular set*. Note that this definition and notation, as well as certain definitions and notation below, differ somewhat from those of [14]; the theorems of [14] are correct as stated (with the definitions given there).

Definition 0.2. A point, $y \in Y^m$, is called *singular*, if it is not regular.

We denote the singular set by $\mathcal{S}$.

Ideally, we would like to show that $\mathcal{R}$ is connected, $\mathcal{R} = \mathcal{R}_m$, $\dim \mathcal{S} \leq m-1$ and more generally, $\mathcal{S}$ has codimension ≥ 1 with respect to any *natural* measure for which the measure of $\mathcal{R}$ is positive.

In the noncollapsed case, $m = n$, all of the above mentioned properties will be shown to hold; see [12] for the connectedness of $\mathcal{R}$. Moreover, with regard to the dimension of the singular set, we get the stronger assertion, $\dim \mathcal{S} \leq n-2$.

In the collapsed case, $m < n$, without further assumptions, our information at the present is less complete. We do not know that $\mathcal{R}$ is connected, nor do we know that m is an integer, nor that $\mathcal{R}_m$ is nonempty. However, we have partial information on the the latter two issues and strong additional information as well.

An obvious density argument implies $\dim \mathcal{R}_k \leq k$; compare (4.2), (4.3). Moreover, if k is the largest of those integers, ℓ, such that $\mathcal{R}_\ell$ is nonempty, then $\mathcal{H}^k(\mathcal{R}_k) > 0$, where $\mathcal{H}^k(\cdot)$ denotes *k-dimensional Hausdorff measure*; see [13] (and Section 5 for the case $k = n$). We show in Section 4 that this k satisfies $k = m$, for so called *polar* limit spaces, Y^m; see below for the definition. By the results of Section 5, noncollapsed limit spaces are polar, and at present we do not know an explicit example of limit space which is not polar.

For many purposes, the natural measures on our limit spaces are those which are obtained by considering a suitable subsequence, $\{(M_j^n, p_j)\}$, and extracting an appropriate limit of the sequence of renormalized Riemannian measures on the manifolds, M_j^n. Here, the renormalization is such that renormalized volume of the unit ball, $B_1(p_i)$, is equal to 1. These *renormalized limit measures* were constructed in [30]; see also Section 1 and compare [36].

In the noncollapsed case, it turns out that any such measure, ν, is just a multiple of the Hausdorff measure, $\mathcal{H}^n$; see Theorem 5.9. However, in the collapsed case, different Gromov-Hausdorff convergent sequences, $M_i^n \to Y^m$, can lead to different limit measures; see Example 1.24. Thus, a renormalized limit measure encodes information on the collapsing sequence from which it arises; compare [34]. Even in the collapsed case, the renormalized limit measures and Hausdorff measure are closely related; see [13] for further discussion.

Any renormalized limit measure, ν, has the crucial property that $\nu(\mathcal{S}) = 0$ and as a consequence, $\nu(\mathcal{R}) > 0$. In particular it follows that the regular set, $\mathcal{R}$, is dense.

In order to discuss the content of the individual sections, we introduce some additional definitions and notation.

Definition 0.3. A point, $y \in Y^m$, is called *k–weakly Euclidean*, if *some* tangent cone at y splits off a factor, $\mathbb{R}^k$, isometrically.

Let $\mathcal{WE}_k$ denote the set of k-weakly Euclidean points. Then

$$Y^m = \mathcal{WE}_0 \supset \mathcal{WE}_1 \supset \cdots \supset \mathcal{WE}_n = \mathcal{R}_n \supset \mathcal{WE}_{n+1} = \emptyset.$$

Of course, $\mathcal{R}_k \subset \mathcal{WE}_k$,

Definition 0.4. A point, $y \in Y^m$, is called *k-degenerate* if it is *not* $(k+1)$-weakly Euclidean.

Let $\mathcal{D}_k$ denote the set of k-degenerate points. Then

$$\mathcal{D}_0 \subset \mathcal{D}_1 \subset \cdots \subset \mathcal{D}_n = Y^m.$$

If we put $\mathcal{D}_k \setminus \mathcal{R} = \mathcal{S}_k \subset \mathcal{S}$, then $\mathcal{S} = \cup_k \mathcal{S}_k$.

Before proceeding, we mention that in Section 2, a corresponding notion of *k-Euclidean* point is defined, in which the word, "some", in Definition 0.3 is replaced by "every". From the technical standpoint, this distinction is very significant. However, at present, in all known examples, the k-weakly Euclidean and k-Euclidean sets coincide.

In Section 1, we construct renormalized limit measures, ν, on limit spaces, Y^m. Let $\underline{\mathrm{M}}^k_H$ denote the simply connected space of dimension k and curvature $\equiv H$. For $\underline{z} \in \underline{\mathrm{M}}^k_H$, put $V_{k,H}(r) = \mathrm{Vol}\,(B_r(\underline{z}))$. Let μ be a measure. Then by construction, we have, the inequality,

$$\frac{\mu(B_r(z))}{V_{k,H}(r)} \downarrow \tag{0.5}$$

for $z \in Y^m$, $\mu = \nu$, $k = n$, $H = -1$; see [37]. In fact, it follows from [37], that (0.5) holds in the directionally restricted form given in (A.2.1). Presently, we do not know if (0.5) always holds for $\mu = \mathcal{H}^m$, $k = n$, $H = -1$; compare Examples 1.31 and 8.77.

In Section 2, we show that for any renormalized limit measure, ν, we have $\nu(\mathcal{S}) = 0$. We also show that $\mathcal{WE}_k \subset \cup_{i \geq k} \overline{\mathcal{R}_i}$.

In Section 3 we show $\dim Y^m < n$ implies $\dim Y^m \leq n - 1$.

In Section 4 we introduce the class of polar limit spaces. The space Y^m is *polar* if the base point of every iterated tangent cone is a *pole* i.e. if every minimal geodesic segment emanating from the base point is the restriction of some ray. We show that for polar limit spaces, $\dim \mathcal{D}_k \leq k$. Note that from the results of Sections 3 and 5, it follows that $\dim \mathcal{D}_{n-1} \leq n-1$, for all (possibly nonpolar) limit spaces. We also show that in the polar case, $Y(\ell) \subset \cup_{i \geq \{\ell\}} \overline{\mathcal{R}_i}$; compare Section 2.

In Section 5, we discuss the noncollapsed case, $\mathrm{Vol}\,(B_1(p_i)) \geq v > 0$, where $p_i \in M^n_i$, or equivalently, $\dim Y^n = n$. Here, $Y^n = Y(n)$ and it follows easily that $\mathcal{R} = \mathcal{R}_n$. We also show in Section 5 that any tangent cone at $y \in Y^n$ is a metric cone. Thus, in particular, Y^n is polar and so, $\dim \mathcal{D}_k \leq k$. Additionally, we show that the result of [26] on the continuity of the volume (equivalently, n-dimensional Hausdorff measure) can be extended to the general case in which a sequence of limit spaces, $\{Y^n_i\}$, converges to a limit space Y^n. In particular, (0.5) holds for $\mu = \mathcal{H}^n$, $k = n$, $H = -1$, on such limit spaces.

Let d_{GH} denote Gromov–Hausdorff distance and let $B_1(0) \subset \mathbb{R}^k$.

Definition 0.6. The (ε, k)-regular set, $(\mathcal{R}_k)_\varepsilon \supset \mathcal{R}_k(Y^m)$, consists of those points, y, such that every tangent cone, (Y_y, y_∞), satisfies $d_{GH}(B_1(y_\infty), B_1(0)) < \varepsilon$.

Note that $(\mathcal{R}_k)_\varepsilon \cap \mathcal{S}$ need not be empty. If $m = n$, then for ε small, we must have $k = n$ and we will just write $\mathcal{R}_\varepsilon$, rather than $(\mathcal{R}_n)_\varepsilon$.

Using the results of Appendix 1, we show in Section 5 that for $\varepsilon < \varepsilon(n)$, sufficiently small, $\overset{\circ}{\mathcal{R}}_\varepsilon$, the interior of $\mathcal{R}_\varepsilon$, is homeomorphic to a smooth manifold. The homeomorphism is essentially unique and with respect to this parameterization, the metric is bi-Hölder equivalent to a smooth metric, where the exponent, α, satisfies, $\alpha \to 1$ as $\varepsilon \to 0$; possibly, the metric on $\overset{\circ}{\mathcal{R}}_\varepsilon$ is actually bi-Lipschitz equivalent to a smooth metric. The basic idea for constructing a bi-Hölder homeomorphism is to use the results proved in [26]. Recall that in [26], using the solution of a conjecture of Anderson-Cheeger proved there, combined with a conjecture of Anderson-Cheeger and Perelman analogous to the one proved in [24], the following was shown. If $Ric_{M^n} \geq -\varepsilon(n)$ and some ball in M^n is Gromov-Hausdorff close to the corresponding ball in $\mathbb{R}^n$, then every sub-ball (whose center is not very close to the boundary) is close on its own scale to the corresponding ball in $\mathbb{R}^n$. More precisely, for all $\varepsilon > 0$, there exists $\delta > 0$, such that $\mathcal{R}_\delta(Y^n) \subset \overset{\circ}{\mathcal{R}}_\varepsilon(Y^n)$; see Theorem A.1.5. For subsets of $\mathbb{R}^n$, an analog of the condition which defines $\mathcal{R}_\varepsilon$ is known as "Reifenberg's condition", as was pointed out to us by Bruce Kleiner; see [43], [50].

In Section 6 we show that $\mathcal{S}(Y^n) \subset \mathcal{S}_{n-2}(Y^n)$. Thus, in the non-collapsed case, the singular set has codimension at least 2. Obvious 2-dimensional examples show that this result is optimal. In the collapsed case, well-known examples show that the singular set can have codimension 1. For instance, S^3 collapses with bounded sectional curvature, to a closed interval.

In Section 7, we continue to assume dim $Y^n = n$. Using a theorem of Anderson, [4], we show that if $|Ric_{M_i^n}| \leq (n-1)$, then for $\epsilon \leq \epsilon(n)$, in fact $\mathcal{R}_\epsilon = \mathcal{R}$ (i.e., $\mathcal{R}_\epsilon \bigcap \mathcal{S} = \emptyset$). In particular, $\mathcal{R}$ is *open* and $\mathcal{S}$ is *closed.* Clearly, this is not the case if we just assume $Ric_{M_i} \geq -(n-1)$. Moreover, in this case of bounded Ricci curvature, $\mathcal{R}$ has the structure of smooth manifold with $C^{1,\alpha}$ Riemannian metric; the metric is C^∞ if in addition, M_i^n is Einstein. At points of $\mathcal{R}$, the convergence of metrics $g_i \to g_\infty$ takes place in the $C^{1,\alpha}$ (respectively C^∞) topology.

In Section 8, we present a number of examples. The first of these, Example 8.41, illustrates that for all $\epsilon > 0$, there exist Y^n and $y \in \mathcal{R}_\epsilon(Y^n)$, such that the tangent cone at y is not unique.

In the collapsed case, we show that various new phenomena arise. For instance there may exist distinct mutually tangent geodesics and at certain singular points of Y^m, (0.5) can fail to hold, for $\mu = \mathcal{H}^m$, $k = m$, $H = -1$; see Example 8.77. Additionally, there may be points at which there exist distinct tangent cones having different dimensions; see Example 8.80. There exist also collapsed limit spaces containing points at which *no* tangent cone is a *metric cone*; see Example 8.95. However, the spaces in these examples are still polar. Finally (and not surprisingly) we show that if for a sequence, $\{M_i^{n_i}\}$, with $Ric_{M_i^{n_i}} > 0$, we have $n_i \to \infty$, then for the limit space, all good properties (such as the splitting theorem) can fail to hold.

In Appendix 1, we reformulate the theorem of Reifenberg [50] (see also [52]) in an intrinsic setting. In combination with the results of [24]–[26] (in particular with the conjectures of Anderson–Cheeger and Perelman proved there) this implies a sharpening of Perelman's lower bound on the relative contractibility radius in the presence of almost maximal volume; [46]. As a consequence, we obtain sharpenings of most of the results of [24]–[26] and additional new results; see in particular, Theorem A.1.11.

As a specific example, it follows that there exists $\delta(n) > 0$, such that if $Ric_{M^n} \geq n-1$ and $\mathrm{Vol}(M^n) \geq (1-\delta(n))\,\mathrm{Vol}(S_1^n)$, then M^n is *diffeomorphic* to the sphere, S^n. Indeed M^n might be bi-Lipschitz to S^n, but this does not follow from Reifenberg's method.

In Appendix 2, we discuss synthetic treatments of the concept, "Ricci curvature bounded below", in light of the results on limit spaces obtained in the body of the paper and in [12], [13].

We will now give a brief indication of a portion of the contents of [12], [13].

In [12], we will show that the set, $\mathcal{SS} \subset \mathcal{S}$, of so called *strongly singular* points, has codimension ≥ 1 with respect to any renormalized limit measure, ν (in a suitably defined sense). Conjecturally, this holds for $\mathcal{S}$ as well. By definition, the strongly singular set is the complement of the *weakly regular set*.

In [13] we show that on the set of so called *k-strongly regular* points, $\mathcal{SR}_k$, any renormalized limit measure, ν, determines the same measure class as Hausdorff, measure, $\mathcal{H}^k$. Since, the complement of the *strongly regular set*, $\mathcal{SR} = \cup_k \mathcal{SR}_k$, has measure zero with respect to any ν, it follows that the collection, $\{\nu\}$, of all renormalized limit measures determines a well-defined measure class i.e. ν_1 is absolutely continuous with respect to ν_2, for all ν_1, ν_2. The above discussion is based in part on the Poincaré inequality, which is shown to hold for our limit spaces.

For polar limit spaces, we find that $Y(k) \subset \overline{\mathcal{R}_k}$ for $k \in \mathbb{Z}_+$, that $Y(k)$ is empty for $k \notin \mathbb{Z}_+$ and that for ϵ sufficiently small, $(\mathcal{R}_k)_\varepsilon$ is empty for $k > m$. In particular, for polar limit spaces, $m = \dim Y^m$ is an integer.

Additionally, we discuss rectifiability properties of limit spaces. Based in part on this discussion, we show that there is a natural *intrinsically defined* self-adjoint Laplacian on functions, with all of the familiar properties which hold in the smooth case. Moreover, we show that the spectrum of the Laplacian behaves continuously under measured Gromov-Hausdorff convergence.

In [12], in the noncollapsed case, we prove a result on the connectedness of the ϵ-regular set. We show that for all $\epsilon > 0$, there exists $\delta > 0$, such that $\mathcal{R}_\delta(Y^n)$ lies in a single component of $\overset{\circ}{\mathcal{R}}_\epsilon (Y^n)$. Conjecturally, the same holds for arbitrary $m = \dim Y^m$. Under the assumption that (a slightly more technical version of) this condition holds in general, we show the isometry group of any limit space, Y^m, is a Lie group; compare [32]. In particular this is the case for $m = n$. Knowing that the isometry group is always a Lie group would have significant implications for the structure of the fundamental group, for manifolds with diameter bounded above and Ricci curvature bounded below. It would imply that the results of [31], proved there under the assumption that the sectional curvature is bounded below, actually remain valid in the presence of a lower bound on Ricci curvature.

We also show in [12] that certain spaces which closely resemble the "horns" of Example 8.67, but which do not have locally constant dimension, do not arise as limit spaces;

Finally in [12], we specialize the results of the present paper to the case in which the manifolds, M_i^n, are homogeneous spaces; compare [59], [60].

We close this introduction with some additional remarks and conjectures.

Conjecture 0.7. *The interior of $Y^n \setminus \mathcal{S}_{n-4}(Y^n)$ is a topological manifold.*

In a subsequent joint paper with Gang Tian, [17], we prove the stronger statement, $\mathcal{S}(Y^n) \subset \mathcal{S}_{n-4}(Y^n)$, under the additional assumption that for some $p > \frac{3}{2}$, the L_p-norms of the curvature tensors of the manifolds, M_i^n, are uniformly bounded; see also [16].

In case the metrics on the M_i^n are Kähler-Einstein on a fixed complex manifold $M_i^n = M^n$, with fixed Kähler class, a standard argument based on characteristic numbers and curvature identities implies that this uniform bound actually holds, with $p \geq 2$; see [17]. On the other hand, the following conjecture is well known; see [6], Conjecture 2.3.

Conjecture 0.8. *If* $|Ric_{M_i^n}| \leq (n-1)$, *then* $\mathcal{S}(Y^n) \subset \mathcal{S}_{n-4}(Y^n)$ *and* $\mathcal{H}^{n-4}(\mathcal{S}_{n-4}(Y^n)) < \infty$.

Finally, we point out that the results of this paper should be compared to those of [9], [48], which treat analogous questions in the context of a lower bound on sectional curvature, i.e., for Alexandrov spaces. Recall that in [24]–[26], one finds the first theorems on Ricci curvature (integral Toponogov theorems, etc.) that strongly resemble results which play a basic role in Alexandrov space theory.

Moreover, some of the results of Sections 5, 6 of this paper should be contrasted with the theorem of Grove-Petersen [38] (see also [40], [41]) giving a lower bound on the relative contractibility radius at *all* points of M^n, under the assumptions $\mathrm{diam}(M^n) \leq d$, $\mathrm{Vol}\,(M^n) \geq v > 0$, $K_M \geq -1$. Here K_M denotes sectional curvature. Well known examples show that this fails to hold if the bound $K_M \geq -1$ is weakened to $Ric_{M^n} \geq -(n-1)$; see [5]. However, according to Theorem 5.12 and Remark 5.13, the complement of a set of codimension 2 can be written as a union of sets, on each of which, every point has a neighborhood of a definite size diffeomorphic to a standard ball. Conjecturally, a weakened version of this property holds off a set of codimension 4.

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1. Renormalized limit measures

In this section, we construct renormalized limit measures, ν, on (possibly collapsed) spaces, (Y^m, y), which are pointed Gromov-Hausdorff limits of sequences, $\{(M_i^n, p_i)\}$, satisfying

$$Ric_{M_i^n} \geq -(n-1). \tag{1.1}$$

For any sequence, there is a subsequence for which the renormalized limit measure exists. These measures were first constructed by Fukaya, who used a somewhat different argument; see [30].

In the noncollapsed case, the limit measure exists without the necessity of passing to a subsequence, or of renormalizing the measure. The unique limit measure is just Hausdorff measure, $\mathcal{H}^n$; see Theorem 5.9. (If, for the sake of consistency, one does renormalize the measure, then one obtains a multiple of $\mathcal{H}^n$, where as usual, the normalization factor depends on the choice of base point.) However, in the collapsed case, the renormalized limit measure on the limit space can depend on the particular choice of subsequence; see Example 1.24.

The renormalized limit measures play an important role role in [12], [13], for instance in connection with the theory of the Laplace operator on limit spaces; compare [30].

Let M^n satisfy $Ric_{M^n} \geq -(n-1)$. Then by (0.5), for $\mu = \mathrm{Vol}(\cdot)$, $k = n$, $H = -1$ (and the triangle inequality) the following relations hold. For $r_1 \leq r_2$, $\overline{x_1, x_2} = s$,

$$\frac{\mathrm{Vol}\,(B_{r_1}(x_1))}{\mathrm{Vol}\,(B_{r_2}(x_2))} \geq \frac{V_{n,-1}(r_1)}{V_{n,-1}(r_2+s)}, \tag{1.2}$$

$$\frac{\mathrm{Vol}\,(B_{r_2}(x_2))}{\mathrm{Vol}\,(B_{r_1}(x_1))} \geq \frac{V_{n,-1}(r_2)}{V_{n,-1}(r_1+s)} \qquad r_2 \leq r_1 + s, \tag{1.3}$$

$$\frac{\mathrm{Vol}\,(B_{r_2}(x_2))}{\mathrm{Vol}\,(B_{r_1}(x_1))} \geq 1 \qquad r_2 \geq r_1 + s. \tag{1.4}$$

Fix p and define the renormalized volume function,

$$\underline{V}(x,r) = \underline{\mathrm{Vol}}\,(B_r(x))$$

by

$$\underline{V}(x,r) := \underline{\mathrm{Vol}}\,(B_r(x)) := \frac{1}{\mathrm{Vol}\,(B_1(p))}\mathrm{Vol}\,(B_r(x)). \tag{1.5}$$

It follows from (1.2)–(1.4) that on compact subsets, $B_R(p)\times[r_1,r_2]$, the collection of all such functions (i.e., for all (M^n,p) satisfying (1.1)) are uniformly bounded, uniformly bounded away from zero and uniformly equicontinuous.

By combining the proof of Gromov's compactness theorem with an obvious modification of the proof of the theorem of Arzela-Ascoli, we obtain:

Theorem 1.6. *Given any sequence of pointed manifolds, $\{(M_i^n,p_i)\}$, for which $Ric_{M_i^n}\geq -(n-1)$ holds, there is a subsequence,$\{(M_j^n,p_j)\}$, convergent to some (Y^m,y) in the pointed Gromov-Hausdorff sense, and a continuous function $\underline{V}_\infty : Y^m\times\mathbb{R}_+\to\mathbb{R}_+$, such that if $q_j\in M_j^n$, $z\in Y^m$ and $q_j\to z$, then for all $R>0$,*

$$\underline{V}_j(q_j,R)\to\underline{V}_\infty(z,R) \qquad \text{(uniformly on } B_{R_1}(p)\times[0,R_2]). \tag{1.7}$$

Proof. After passing to a subsequence, we can assume $\{(M_i^n,p_i)\}$ converges to (Y^m,y) in the Gromov-Hausdorff sense. Take a countable dense subset, $\{z_k\}\subset Y^m$, and a countable dense subset, $\{R_\ell\}\subset\mathbb{R}_+$. Given the above mentioned bounds implied by (1.2)–(1.4), a standard infinite diagonal argument shows that there exists a subsequence, $\{(M_j^n,p_j)\}$, such that (1.7) holds for $z\in\{z_k\}$, $R\in\{R_\ell\}$. Just as in the Arzela-Ascoli theorem, it follows that (1.7) actually holds in general. This suffices to complete the proof.

Since we can multiply both sides of (0.5) by $(\mathrm{Vol}(B_1(y)))^{-1}$, the function, $\underline{V}_\infty$, satisfies the following inequality for all $z\in Y^m$:

$$\frac{\underline{V}_\infty(z,r_1)}{\underline{V}_\infty(z,r_2)}\geq\frac{V_{n,-1}(r_1)}{V_{n,-1}(r_2)}. \tag{1.8}$$

Indeed, it is clear that (A.2.2), the directionally restricted version of (0.5) holds for $\mu=\nu$, $k=n$, $H=-1$.

Define an outer measure, ν, on subsets of Y^m, by the standard construction,

$$\nu(A)=\lim_{\delta\to 0}\nu_\delta(A), \tag{1.9}$$

where

$$\nu_\delta(A)=\inf\{\sum_i\underline{V}_\infty(z_i,r_i)|B_{r_i}(z_i),r_i\leq\delta\}.$$

By standard measure theory, ν is a *metric outer measure* and the corresponding measure, also denoted by ν, is a Radon measure; see Theorem 13.7 of [44].

Theorem 1.10. *There is a unique Radon measure, ν, such that for all, z, R,*

$$\nu(B_R(z)) = \underline{V}_\infty(z, R). \tag{1.11}$$

In particular ν satisfies the inequality,

$$\frac{\nu(B_{r_1}(z))}{\nu(B_{r_2}(z))} \geq \frac{V_{n,-1}(r_1)}{V_{n,-1}(r_2)} \qquad r_1 \leq r_2. \tag{1.12}$$

Proof. From the definition of $\underline{V}_\infty$, it is clear that for all z, R,

$$\nu(B_R(z)) \geq \underline{V}_\infty(z, R). \tag{1.13}$$

Thus, we must prove the opposite inequality.

Let M^n satisfy (1.1) and let $K \subset B_R(x)$. By a standard covering argument based on (0.5), for $\mu = \mathrm{Vol}(\cdot)$, $k = n$, $H = -1$, it follows that for all $\epsilon > 0$, there exist balls, $B_{r_i}(x_i)$, with $x_i \in K$,

$$1 \leq i \leq N \leq N(\epsilon, n), \tag{1.14}$$

$$\lambda(\epsilon, n) \leq r_i \leq \epsilon, \tag{1.15}$$

such that

$$K \subset \cup_{i=1}^n B_{r_i}(x_i), \tag{1.16}$$

$$\sum_i \mathrm{Vol}\,(B_{r_i}(x_i)) \leq (1+\epsilon)\mathrm{Vol}\,(T_\epsilon(K)). \tag{1.17}$$

Here $T_\epsilon(\cdot)$ denotes the ϵ-tubular neighborhood. Moreover, for some $N' \leq N$ and

$$\frac{1}{2}\lambda(\epsilon, n) \leq r_i' \leq \epsilon \qquad i = 1, \dots N', \tag{1.18}$$

the balls, $\{B_{r_1'}(x_i)\}$, are mutually disjoint and

$$\sum_i \mathrm{Vol}\,(B_{r_i'}(x_i)) \geq (1-\epsilon)\mathrm{Vol}\,(K). \tag{1.19}$$

By dividing both sides of the inequalities in (1.17), (1.19) by $\mathrm{Vol}\,(B_1(p_j))$ and passing to the limit, we obtain corresponding inequalities for Y^m, in which the function, $\mathrm{Vol}\,(B_r(x))$ is replaced by $\underline{V}_\infty$. Clearly, the estimate corresponding to (1.17) implies (1.11), and thus, (1.12) as well.

Finally, let ν' be a second Radon measure satisfying $\nu'(B_R(z)) = \underline{V}_\infty(z,R)$. Since ν,ν' are Radon measures, it suffices to show that they agree on each bounded open set, U. Again since ν,ν' are Borel regular, it follows that for all $\eta>0$, there exists a compact set, $K\subset U$, with

$$\nu(K)\geq(1-\eta)\nu(U), \tag{1.20}$$

$$\nu'(K)\geq(1-\eta)\nu'(U). \tag{1.21}$$

Since ν,ν' agree on balls, with the help of (1.19), we easily conclude that $\nu=\nu'$.

The following is a direct consequence of (1.3).

Proposition 1.22.

$$\nu(B_r(z))\leq c(n,\overline{z,y})r \qquad 0\leq r\leq 1. \tag{1.23}$$

Example 1.24. Consider the sequence of metrics on $\mathbb{R}^2$, each of nonnegative curvature, given in polar coordinates by

$$\{dr^2+[f(nr)/n]^2d\theta^2\}, \tag{1.25}$$

where

$$f|[1,\infty)\equiv 1. \tag{1.26}$$

In this case, the limit space is the Alexandrov space, $[0,\infty)$, with its standard metric and the measure, ν, is 1-dimensional Hausdorff measure, given by integration of the 1-form, dr.

Note however, that there exists a second sequence of metrics on $\mathbb{R}^2$, each of nonnegative curvature, say,

$$\{dr^2+h_n^2(r)d\theta^2\}, \tag{1.27}$$

where

$$h|[\frac{1}{n},\infty)\equiv\frac{r}{n}. \tag{1.28}$$

The manifolds in this sequence look like very thin cones with sharply rounded tips. The limit is again the Alexandrov space, $[0,\infty)$, and the measure, ν, is given by integration of the 1-form,

$$rdr. \tag{1.29}$$

In particular, it follows that for a fixed limit space, which might be an Alexandrov space, the limit measure is not unique even if one fixes the dimension of the approximating sequence. This indicates that even for Alexandrov spaces it is of interest to consider measures other than Hausdorff measure. Many other examples in which the measure is not unique can be constructed e.g. from the examples in Section 8.

Remark 1.30. As mentioned in Section 0, it will be shown in [13] that the collection, $\{\nu\}$, of all renormalized limit measures determines a well-defined measure class i.e. ν_1 is absolutely continuous with respect to ν_2, for all ν_1, ν_2.

Example 1.31. For the sequence of manifolds constructed in Example 8.77, the measure, ν, is given by integration of the 5-form,

$$r^{3(1-\eta)}\omega, \tag{1.32}$$

where ω is the volume form associated to (normalized) Hausdorff measure, $\mathcal{H}^5$, on Y^5. Recall that for balls centered at the origin, one has (0.5), for $\mu = \mathcal{H}^5$, $k = 5 + 4\varepsilon$, $H = -1$. On the other hand, (0.5) holds for $\mu = \nu$, $k = 8 + 4\varepsilon - 3\nu$, $H = -1$. Recall that $4\epsilon - 3\eta < 0$.

Remark 1.33. As a consequence of Proposition 1.22, it follows that any limit measure, ν, is absolutely continuous with respect to Hausdorff measure, in case the limit space is 1-dimensional. It will be shown in [13] that (as a consequence of the results described in Section 0) this result has a suitable generalization to arbitrary limit spaces.

Conjecture 1.34. *For some k, with $m \le k \le n$, one has (0.5), for $\mu = \mathcal{H}^m$, $m \le k \le n$, $H = -1$.*

The phenomena discussed in Example 8.77, are related to Conjecture 1.34.

Proposition 1.35. *Let $\{(M_i^n, p_i)\} \to (Y^m, y)$ satisfy $Ric_{M_i^n} \ge -\varepsilon_i$, where $\varepsilon_i \to 0$. If Y^m splits isometrically, $Y^m = \mathbb{R} \times X$, then any limit measure, ν, is a product measure, $\nu = \mathcal{H}^1_{\mathbb{R}} \times \psi$, for some measure, ψ, on X.*

Proof. It suffices to show that for all x, $r > 0$, $t_1 \le t_2$, we have $\nu(B_r((t_1, x))) = \nu(B_r((t_2, x)))$. After making the change of coordinates, $t \to t - \frac{1}{2}(t_1 + t_2)$, we can assume $t_1 = -t, t_2 = t$, for some fixed $t > 0$. Moreover, by symmetry, it suffices to show

$$\nu(B_r((-t, x))) \le \nu(B_r((t, x))). \tag{1.36}$$

For $w \in B_r((-t, x))$, let γ_s denote a minimal geodesic segment from (s, x) to w. Let

$$I_s = \{u \mid \gamma_s(u) \in B_r((-t, x))\}, \tag{1.37}$$

$$J_s = \{u \mid \gamma_s(u) \in B_r((t, x)), \tag{1.38}$$

and let $|I_s|$, $|J_s|$ denote the 1-dimensional measure of I_s, J_s, respectively. From the isometric splitting, $\mathbb{R} \times X$, it follows that for all $\varepsilon > 0$, there exists $s(\varepsilon)$ such that for $s \ge s(\varepsilon)$, we have

$$|J_s| + \varepsilon \ge |I_s|. \tag{1.39}$$

Clearly, the directionally restricted version of (0.5) (see (A.2.1)) holds for $\mu = \nu, k = n, H = 0$. Therefore, by observing the ball, $B_r((-t, x))$, from the point, (s, x), letting $s \to \infty$ and applying (1.39), we easily obtain (1.36).

Remark 1.40. In [12], [13], we will prove results on generalized volume convergence which are closely related to Proposition 1.35.

2. Arbitrary limit spaces

Let (Y^m, y) be the pointed Gromov–Hausdorff limit of a sequence, $\{(M_i^m, p_i)\}$, such that (1.1) holds. Let ν be a renormalized limit measure as in Section 1.

The main result of this section is

Theorem 2.1. *For any renormalized limit measure, $\nu(\mathcal{S}) = 0$.*

As a consequence of Theorem 2.1, we will deduce Theorem 6.68 of [15], which is restated here as Theorem 2.45. The proof was deferred to the present paper. We mention however, that Theorem 2.45 is considerably weaker than Theorem 2.1 and that a shorter, more direct proof is possible.

In order to prove Theorem 2.1, it will be convenient to introduce some additional concepts and notation; compare Section 0.

Definition 2.2. A point, $y \in Y^m$, is called *k-Euclidean* if *every* tangent cone at y splits off a factor, $\mathbb{R}^k$, isometrically.

We denote by $\mathcal{E}_k$, the set of k-Euclidean points. Of course, $\mathcal{E}_k \subset \mathcal{W}\mathcal{E}_k$.

Definition 2.3. A point, $y \in Y^m$, is called *k-weakly degenerate* if it is not $(k+1)$-Euclidean.

We denote by $\mathcal{W}\mathcal{D}_k$, the set of k-weakly degenerate points.

Let $\underline{\mathcal{W}\mathcal{E}}_k$ denote the set of points for which there exists some tangent cone, *not* isometric to $\mathbb{R}^k$, which splits off a factor, $\mathbb{R}^k$, isometrically. For all $\epsilon > 0$, we also define sets, $(\mathcal{W}\mathcal{E}_k)_\epsilon$, such that $\mathcal{W}\mathcal{E}_k = \cap_\epsilon (\mathcal{W}\mathcal{E}_k)_\epsilon$, as follows. We say that $y \in (\mathcal{W}\mathcal{E}_k)_\epsilon$, if there exists $r > 0$ and X, $(0, x) \in \mathbb{R}^k \times X$, such that

$$d_{GH}(B_r(y), B_r((0,x))) < \epsilon r. \tag{2.4}$$

The strongest assertion in the following Lemma 2.5 is the one concerning the set, $(\mathcal{W}\mathcal{E}_k)_\epsilon$. The proof of this assertion will be given in [13]. However, for the proof of Theorem 2.1, only the assertion concerning the set $\mathcal{W}\mathcal{E}_k$ is required.

We will show in Proposition 2.13 below, that if Y is not a single point, then $\nu(\mathcal{W}\mathcal{D}_0) = 0$. From this and the following two lemmas, we easily obtain Theorem 2.1.

Lemma 2.5. *We have $\nu(\mathcal{W}\mathcal{E}_k \setminus \mathcal{E}_k) = 0$. Moreover, there exists $\epsilon(n) > 0$, such that if $y \in (\mathcal{W}\mathcal{E}_k)_\epsilon$, for $\epsilon < \epsilon(n)$, then for all sufficiently small $r > 0$, we have $\nu(B_r(y) \cap \mathcal{E}_k) > 0$.*

Lemma 2.6.

$$\nu(\underline{\mathcal{W}\mathcal{E}}_k \setminus \mathcal{W}\mathcal{E}_{k+1}) = 0.$$

Proof of Theorem 2.1. We can assume that Y is not a single point. Write $A \sim B$ if A and B coincide off a set of measure zero with respect to ν. Then by Propositon 2.13 below, we have

$$Y^m = \mathcal{W}\mathcal{D}_0 \cup \mathcal{E}_1 \sim \mathcal{E}_1. \tag{2.7}$$

By Lemmas 2.5 and 2.6, we have, $\underline{\mathcal{W}\mathcal{E}}_k \sim \mathcal{E}_{k+1}$, for all k. Thus,

$$\mathcal{E}_k = \mathcal{R}_k \cup (\mathcal{E}_k \cap \underline{\mathcal{W}\mathcal{E}}_k) \sim \mathcal{R}_k \cup \mathcal{E}_{k+1}. \tag{2.8}$$

By induction, we get

$$Y^m \sim \mathcal{R}_1 \cup \mathcal{R}_2 \cup \cdots \cup \mathcal{R}_n. \tag{2.9}$$

Let $y \in Y^m$ and let $\rho_y(z) = \overline{z,y}$ denote the distance function from y.

Definition 2.10. A point, $z \in Y^m$, is *not* a restricted cut point of y, if for all $\varepsilon > 0$, there exists $r(z,\varepsilon) > 0$, such that for $0 < r < r(z,\varepsilon)$, there exists a space, X_r, $(0, x_r) \in \mathbb{R} \times X_r$ (the isometric product) and a pointed εr–Gromov–Hausdorff approximation, $\beta_r : B_r(z) \to B_r((0,x_r))$, such that

$$|\rho_y - t \circ \beta_r| < \varepsilon r \qquad (\text{on } B_r(z)), \tag{2.11}$$

where t denotes the coordinate function on $\mathbb{R} \times X_r$ corresponding to the factor, $\mathbb{R}$.

Let $\mathcal{WD}_0(y)$ denote the set of restricted cut points of y. Note that $\mathcal{WD}_0 \subset \cap_y \mathcal{WD}_0(y)$. We put $\mathcal{E}_1(y) = Y^m \setminus \mathcal{WD}_0(y)$.

Remark 2.12. From the generalized splitting theorem, Theorem 6.62 of [15], it follows that if w is an *interior point* of a minimal geodesic segment, γ, with $\gamma(0) = y$, then $z \in \mathcal{E}_1(y)$. But in principle, $\mathcal{E}_1(y)$ could contain points which lie on no such segment.

Proposition 2.13. *If Y is not a single point, then for all $y \in Y^m$,*

$$\nu(\mathcal{WD}_0(y)) = 0. \tag{2.14}$$

In particular, $\nu(\mathcal{WD}_0) = 0$.

Proposition 2.13 is a direct consequence of the following lemma, which concerns smooth Riemannian manifolds satisfying (1.1). Lemma 2.16 plays a role in Section 3 as well.

Put $A_{s_1,s_2}(p) = B_{s_2}(p) \setminus \overline{B_{s_1}(p)}$. Fix $0 < r_1 < r_2$ and $0 < \eta < r_1$. Let $X(p, r_1, r_2, \eta)$ denote the set of points, $\gamma(t) \in A_{r_1,r_2}(p)$, such that $\gamma(0) = p$, , $\gamma|[0,t]$ is minimal and $\gamma|[0, t+\eta)$ is *not* minimal.

Given $0 < \tau < 1$, put

$$Z(p, r_1, r_2, \eta, \tau) = \{q \in A_{r_1,r_2}(p) \,|\, B_{\tau\eta}(q) \subset X(p, r_1, r_2, \eta)\}. \tag{2.15}$$

Lemma 2.16. *There exist $k = k(n, r_1, r_2)$, $c = c\,(n, r_1, r_2, \tau)$ and $q_1, \ldots, q_N$, such that*

$$Z(p, r_1, r_2, \eta, \tau) \subset \bigcup_{i=1}^{N} \overline{B_{\tau\eta}(q_i)}, \tag{2.17}$$

$$\sum_{i=1}^{N} \underline{\mathrm{Vol}}\left(\overline{B_{\tau\eta}(q_i)}\right) \leq k\eta, \tag{2.18}$$

$$N \leq c\eta^{1-n}. \tag{2.19}$$

Proof. It follows directly from (the directionally restricted version of) (0.5), for $\mu = \mathrm{Vol}(\cdot), k = n, H = -1$ (see (A.2.2)) that

$$\underline{\mathrm{Vol}}\,(X(p, r_1, r_2, \eta)) \leq k(n, r_1, r_2)\eta; \tag{2.20}$$

see (1.5).

Let $q_1, \ldots q_N$ be a maximal set of points in $Z(p, r_1, r_2, \eta, \tau)$ such that $\overline{q_i, q_j} \geq \tau\eta$, if $i \neq j$. Then (2.17) holds and since

$$\cup_i B_{\tau_\eta}(q_i) \subset X(p, r_1, r_2, \eta),$$

it follows that (2.18) holds as well. We have $B_{\frac{1}{2}\tau\eta}(q_i) \cap B_{\frac{1}{2}\tau\eta}(q_j) = \emptyset$, for $i \neq j$. Thus, by (2.20) and (0.5), we get (2.19).

Let $\Psi(\varepsilon_1, \ldots, \varepsilon_i \,|\, c_1, \ldots, c_j)$ denote any function such that for fixed $c_1, \ldots, c_j$,

$$\lim_{\varepsilon_1, \ldots, \varepsilon_i \to 0} \Psi = 0. \tag{2.21}$$

Proof of Proposition 2.13. It follows from the generalized splitting theorem (Theorem 6.62 of [15]) that if $Ric_{M^n} \geq -(n-1)$, $p \in M^n$ and $z \in A_{r_1,r_2}(p) \setminus Z(p, r_1, r_2, \eta, \tau)$, then (2.11) holds with $r = \tau\eta$ and $\varepsilon = \Psi(\eta, \tau | n)$.

Let (Y^m, y) be a limit space such that (1.1) holds and let ν be a renormalized limit measure on Y^m. Let $\{\tau_\ell\}$ be a fixed sequence such that $\tau_\ell \to 0$. Then it follows by an obvious diagonal argument that for all $j, k, \ell \in \mathbb{Z}_+$, the subset, $W(p, j, k, \tau_\ell), \subset A_{2^{-j}, 2^j}(p)$, for which (2.11) fails to hold with $r = \tau_\ell 2^{-(j+k)}$, $\varepsilon = \Psi(\tau_\ell | n)$ admits a covering by balls, $\{B_{\tau_\ell 2^{-(j+k)}}(q_i)\}$, $i = 1, \ldots, N$, such that (2.17)–(2.19) hold with p replaced by y and $\underline{\mathrm{Vol}}(\cdot)$ replaced by $\nu(\cdot)$.

For fixed j, ℓ, we have in particular,

$$\nu\left(\bigcup_{k \geq k_0}^{\infty} W(p, j, k, \tau_\ell)\right) \leq c(n, j, \tau_\ell) 2^{-k_0}, \tag{2.22}$$

$$\nu\left(\bigcap_{k_0=1}^{\infty}\ \bigcup_{k\geq k_0}^{\infty} W(p,j,k,\tau_\ell)\right)=0. \tag{2.23}$$

Note that if

$$z\in A_{2^j,2^{-j}}(p)\setminus \bigcap_{k_0=1}^{\infty}\ \bigcup_{k\geq k_0}^{\infty} W(p,j,k,\tau_\ell),$$

then (2.11) holds for all sufficiently small r and $\varepsilon=\frac{1}{2}\Psi(\tau_\ell|n)$. By considering the sequence, $\tau_\ell\to 0$, we get from (2.23),

$$\nu(\mathcal{WD}_0(y)\cap A_{2^{-j},2^j}(y))=0, \tag{2.24}$$

and letting $j\to\infty$, we obtain (2.14).

Let $(\mathcal{E}_k)_\varepsilon$ denote the set of points, z, such that for *every* tangent cone, Y_z, there exists $X,(0,x)\in\mathbb{R}^k\times X$, such that

$$d_{GH}(B_1(z_\infty),\,B_1((0,x)))<\varepsilon. \tag{2.25}$$

Here, X, might depend on the particular tangent cone, Y_z. Note that $\mathcal{E}_k=\cap_\varepsilon(\mathcal{E}_k)_\varepsilon$.

Recall that given a metric space, Z, and a collection, $\mathcal{B}=\{B_{r_\alpha}(z_\alpha)\}$, of balls, such that $\sup_\alpha r_\alpha<\infty$, then there is a subcollection,

$$\mathcal{B}'=\{B_{r_{\alpha'}}(z_{\alpha'})\}, \tag{2.26}$$

of *mutually disjoint* balls, such that

$$\cup_{\alpha'}B_{6r_{\alpha'}}(z_{\alpha'})\supset\cup_\alpha B_{r_\alpha}(z_\alpha); \tag{2.27}$$

see Chapter 1, Theorem 3.3 of [56]. This statement, which we call the Covering Theorem, has the following standard consequence.

Let ψ be a σ-finite measure on Z, such that for all $R>0$, there exists $c(R)$, such that for all $z\in Z$,

$$\psi(B_{2r}(z))\leq c(R)\psi(B_r(z))\qquad (r\leq R). \tag{2.28}$$

Let $A\subset Z$. If for all $z\in A$, the *lower density*, $\Theta_A(z)$, of A at z (with respect to ψ) satisfies

$$\Theta_A(z):=\underline{\lim}\,\frac{\psi(B_r(z)\cap A)}{\psi(B_r(z))}=0, \tag{2.29}$$

then

$$\psi(A) = 0. \tag{2.30}$$

Proof of Lemma 2.5. In view of (0.5), for $\mu = \nu$, $k = n, H = -1$, the discussion preceding the proof implies that it suffices to show the following. For all $z \in Y^m$ and $\varepsilon > 0$, there exists $\delta > 0$, such that if for some X, $(0, x) \in \mathbb{R}^k \times X$, we have

$$d_{GH}(B_r(z),\, B_r((0,x))) < \delta r, \tag{2.31}$$

then

$$\frac{\nu(B_r(z) \setminus (\mathcal{E}_k)_\varepsilon)}{\nu(B_r(z))} < \varepsilon; \tag{2.32}$$

compare (2.25), (2.29), (2.30).

First consider the case of a Riemannian manifold, M^n, satisfying (1.1). By scaling, it suffices to consider $B_1(p) \subset M^n$, such that for some X, $(0, x) \in X$,

$$d_{GH}(B_1(p),\, B_1((0,x))) < \delta. \tag{2.33}$$

By the proof of Theorem 6.62 of [15] (see also [26, Lemma 1.23]) the following holds. Given

$$q \in B_{1-\Psi_1(\delta|n)}(p), \quad \Psi_2(\delta|n)\Psi_3(\delta|n) < s < \Psi_3(\delta|n),$$

there exist harmonic functions, $\mathbf{b}_1, \ldots, \mathbf{b}_k$ on $B_s(q)$, given by $\mathbf{b}_i|\partial B_s(q) = b_i^+|B_s(q)$, where $b_i^+(x) = \overline{p_i, x} - \overline{p_i, q}$, for suitable p_i, such that

$$\begin{aligned} \frac{1}{\mathrm{Vol(B_s(q))}} \int_{B_s(q)} \Big\{ \sum\nolimits_i |\nabla(\mathbf{b}_i - b_i^+)|^2 & \\ + \sum\nolimits_{i \neq j} |\langle \nabla \mathbf{b}_i,\, \nabla \mathbf{b}_j \rangle| + \sum\nolimits_i |\mathrm{Hess}_{\mathbf{b}_i}|^2 \Big\} & \\ \leq \Psi_4\,(\delta|n). & \end{aligned} \tag{2.34}$$

The collection of points, $\{p_i\}$, is gotten from the splitting in (2.33).

Since apart from a set of measure $\Psi(\delta|n)\mathrm{Vol}\,(B_1(p))$, we can cover, $B_{1-\Psi(\delta|n)}(q)$ by a collection of mutually disjoint balls as in (2.34), it will suffice to consider a single such ball. After rescaling the metric,

$g \to s^{-2}g$ and making the replacement, $\mathbf{b}_i \to s^{-1}\mathbf{b}_i$, we are reduced to considering a ball such that

$$
\begin{aligned}
\frac{1}{\mathrm{Vol}(B_1(q))} \int_{B_1(q)} \Big\{ \sum\nolimits_i \, |\nabla(\mathbf{b}_i - b_i^+)|^2 + \sum\nolimits_{i\neq j} \, |\langle \nabla \mathbf{b}_i, \, \nabla \mathbf{b}_j \rangle \, | \\
+ \sum\nolimits_i \, s^{-2} |\mathrm{Hess}_{\mathbf{b}_i}|^2 \Big\} \\
:= \frac{1}{\mathrm{Vol}(B_1(q))} \int_{B_1(q)} f \\
\leq \Psi(\delta|n).
\end{aligned} \tag{2.35}
$$

As recalled below, relation (2.35) can be used to control the extent to which sub-balls, $B_r(\underline{q}) \subset B_1(q)$, satisfy a condition like (2.33) (possibly with a different constant on the right-hand side and a different space on the left-hand side). Indeed, by the proof of Theorem 6.62 of [15] (see Sections 2, 3, 6) and rescaling, the following holds. If for some $\lambda < 1$,

$$\frac{1}{\mathrm{Vol}(B_r(q))} \int_{B_r(q)} f \leq \lambda^{-1}\Psi(\delta|n)r, \tag{2.36}$$

then there exists X, $(0, x) \in \mathbb{R}^k \times X$ such that

$$d_{GH}(B_r(\underline{q}), \, B_r((0, x))) \leq \Psi(\lambda^{-1}\Psi(\delta|n) \, | \, n)r. \tag{2.37}$$

Now let $\mathcal{B}$ denote the collection of all balls for which (2.36) fails to hold. Thus, $\mathcal{B}$ includes all balls for which (2.37) fails. By the Covering Theorem, there exists a subcollection, $\mathcal{B}' = \{B_{r_{\alpha'}}(q_{\alpha'})\}$, such that (2.27) is valid.

Since the balls of $\mathcal{B}'$ are mutually disjoint, it follows from (2.35), (2.36) that

$$\sum\nolimits_{\alpha'} \underline{\mathrm{Vol}}(B_{r_{\alpha'}}(q_{\alpha'})) \leq \lambda. \tag{2.38}$$

Hence, by (0.5), for $\mu = Vol(\cdot)$, $k = n$, $H = -1$, we have

$$\sum\nolimits_{\alpha'} \underline{\mathrm{Vol}}(B_{6r_{\alpha'}}(q_{\alpha'})) \leq c(n)\lambda, \tag{2.39}$$

where $\cup_{\alpha'} B_{6r_{\alpha'}}(q_{\alpha'})$ contains all balls for which (2.37) fails.

By letting $\delta \to 0$ and then $\lambda \to 0$, we conclude that (2.32) holds for the case of smooth manifolds.

Now, by a straightforward limiting argument, based on (2.37)–(2.39) and the fact that $\cup_{\alpha'} B_{6r_{\alpha'}}(q_{\alpha'})$ contains all balls for which (2.37) fails, we find that (2.32) holds for arbitrary limit spaces, (Y^m, y), satisfying (1.1). This completes the proof.

Remark 2.40. By making stronger use of the L^2-estimate on the Hessians of the harmonic functions, b_i, together with the Poincaré inequality, we can get more detailed information on the regular set. In [13], by using such ideas, we will prove the assertion in Lemma 2.5 concerning $(\mathcal{WE}_k)_\epsilon$. We will also obtain lower estimates on the codimension (suitably defined) of the complement of the set of points, $y \in \mathcal{R}_k$, which satisfy for some r_0, $c > 0$, $1 > \alpha > 0$ and all $r \le r_0$,

$$d_{GH}(B_r(y),\, B_r(0)) \le cr^{1+\alpha}. \tag{2.41}$$

Roughly speaking, at such *α-regular* points, the metric is C^α; see [13] for further discussion. Finally, we mention that the results described in Remarks 1.30 and 1.33 are obtained as applications of this discussion.

Proof of Lemma 2.6. By the discussion preceding the proof of Lemma 2.5, it suffices to show the following. Given $z \in \underline{\mathcal{WE}}_k$, for all $\varepsilon > 0$, there exists $r > 0$, such that

$$\frac{\nu(B_r(z) \setminus (\mathcal{WE}_{k+1})_\varepsilon)}{\nu(B_r(z))} < \varepsilon. \tag{2.42}$$

Suppose, there exists $\varepsilon > 0$ such that (2.42) fails for all r. Then since the sets $(\mathcal{WE}_k)_\varepsilon$ are open, by a standard argument in measure theory, we find a tangent cone $Y_z = \mathbb{R}^k \times X$, where X *is not a point*, such that for $(0, x) \in \mathbb{R}^k \times X$,

$$\frac{\nu_\infty(B_1((0,x)) \setminus (\mathcal{WE}_{k+1})_\varepsilon(\mathbb{R}^k \times X))}{\nu_\infty(B_1((0,x)))} > \varepsilon. \tag{2.43}$$

Here the measure, ν_∞, is a limit measure for some rescaled sequence, $(Y^m, r_j^{-1}d, z)$, which is constructed as in Section 1. Clearly, ν_∞ is itself a renormalized limit measure, in the sense of Section 1.

By Proposition 2.13, we have $\nu_\infty(\mathcal{WD}_0((0,x))) = 0$. Moreover, if $w_\infty \in \mathcal{E}_1\left((0,x)\right) \setminus \mathbb{R}^k \times x$, then

$$w_\infty \in \mathcal{E}_{k+1}(\mathbb{R}^k \times X) \subset \mathcal{WE}_{k+1}(\mathbb{R}^k \times X).$$

Since by Proposition 1.35, $\nu_\infty(\mathbb{R}^k \times x) = 0$ (X is not a point) it follows that for $\mathcal{WD}_k = \mathbb{R}^k \times X \setminus \mathcal{E}_{k+1}$, we have $\nu_\infty(\mathcal{WD}_k) = 0$, which contradicts

(2.41). This completes the proof of Lemma 2.6 and hence of Theorem 2.1 as well.

Remark 2.44. At present, for $m < n$, we are unable to show that the sets, $(\mathcal{R}_k)_\varepsilon(Y^m)$, have nonempty interior. At *interior* points, Reifenberg's theorem, in the intrinsic formulation given in Appendix 1, can be applied to obtain the local topological regularity of $(\mathcal{R}_k)_\varepsilon$; compare Section 5, for the case $m = n$.

We close this section by restating Theorem 6.68 of [15], the proof of which was deffered to the present paper.

Let M^n satisfy (1.1) and let $B_R(p) \subset M^n$.

Theorem 2.45. *For all $\epsilon > 0$, there exists a disjoint union of balls, $\cup_{i=1}^{N_\epsilon} := U_\epsilon \subset B_R(p)$, such that,*

$$\mathrm{Vol}(U_\epsilon) \geq (1-\epsilon)\mathrm{Vol}(B_R(p)), \tag{2.46}$$

$$r_i \geq \lambda(\epsilon, n) > 0, \tag{2.47}$$

$$d_{GH}(B_{r_i}(q_i), B_{r_i}(0)) \leq \epsilon r_i \quad (0 \in \mathbb{R}^{k_i}). \tag{2.48}$$

Moreover, there exist harmonic functions, $\mathbf{b}_{1,i}, \cdots, \mathbf{b}_{k_i,i}$, on $B_{r_i}(q_i)$ and an ϵr_i-Gromov-Hausdorff approximation, $\beta_i : B_{r_i}(q_i) \to B_{r_i}(0)$, such that if $b_{j,i}$ denotes the i'th coordinate function on $\mathbb{R}^{k_i}$, then

$$|\mathbf{b}_{j,i} - b_{j,i} \circ \beta_i| < \epsilon r_i. \tag{2.49}$$

Proof. This follows from an obvious compactness argument based on Theorem 2.1, together with the existence of harmonic functions, $\mathbf{b}_1, \ldots, \mathbf{b}_k$, as in (2.34); compare [15, Section 6].

3. $\dim Y \leq n-1$ in the collapsed case

The main result of this section is

Theorem 3.1. *If (Y^m, y) is the pointed Gromov-Hausdorff limit of a sequence, $\{(M_i^n, m_i)\}$, with $\mathrm{Vol}\,(B_1(p_i)) \to 0$, then $\dim Y^m \leq n-1$. Equivalently, $\dim \mathcal{D}_{n-1} \leq n-1$, for any limit space, Y^m.*

As explained in Section 0, it suffices to prove the first statement. We begin by proving a counterpart for Hausdorff measure, of Proposition 2.13.

Proposition 3.2. dim $\mathcal{WD}_0(z) \le n-1$, *for all* $z \in Y^m$. *In particular,* dim $\mathcal{WD}_0(Y^m) \le n-1$.

Proof. Since the proof is very similar to that of Proposition 2.13, we will be brief.

Let $Z(p, r_1, r_2, \eta, \tau)$ be as in (2.15). Given $\varepsilon > 0$, it follows from (2.19) that

$$(\tau\eta)^{n-1+\varepsilon} N \le c(n, r_1, r_2, \tau)\tau^{n-1+\varepsilon}\eta^{\varepsilon}. \tag{3.3}$$

If we consider sequences, $\eta = 2^{-k}, k = 1, 2, \ldots$, and $\tau_\ell \to 0$, we get the following estimate which corresponds to (2.23):

$$\mathcal{H}^{n-1+\varepsilon}_{\tau_\ell 2^{-k_0}}\left(\bigcup_{k=k_0}^{\infty} W(p, j, k, \tau_\ell)\right) \le c(n, j, \tau_\ell)\tau_\ell^{n-1+\varepsilon} 2^{-(k_0-1)\varepsilon}. \tag{3.4}$$

Here $\mathcal{H}^{\kappa}_{\theta}$ denotes θ-Hausdorff content in dimension κ. Now the proof can be completed as in Proposition 2.13.

Proof of Theorem 3.1. Suppose that $\mathcal{H}^{n-1+\varepsilon}(Y^m) > 0$ for some $\varepsilon > 0$. By a standard lemma in measure theory (see [33, Chapter 11] and compare Section 4) we can write $Y^m = A \cup B$, where $\mathcal{H}^{n-1+\varepsilon}(B) = 0$ and if $z \in A$, then for some tangent cone, Y_z, we have $\mathcal{H}^{n-1+\varepsilon}(B_1(z_\infty)) > 0$. In view of Proposition 3.2, we can assume with no loss of generality, that $z \in \mathcal{E}_1(Y^m)$.

We have $Y_z = \mathbb{R} \times X$, $z_\infty = (0, x)$, for some (X, x). It follows that $\mathcal{E}_1(R \times X) \setminus \mathbb{R} \times x \subset \mathcal{E}_2(Y_y)$. Since $\mathcal{H}^{n-1+\varepsilon}(\mathbb{R} \times x) = 0$, we can repeat the previous argument, starting at some $(0, x') \in \mathcal{E}_2(Y_y)$.

If we repeat this argument, after $n-1$ steps, we arrive at an iterated tangent cone which is isometric to $\mathbb{R}^{n-1} \times W$, for some W, and for which $\mathcal{H}^{n-1+\varepsilon}(B_1((0, w))) \neq 0$. Thus, W is *not a point*, and as above, we have $\mathcal{E}_n(\mathbb{R}^{n-1} \times W) \neq \emptyset$. By using Theorem 5.9, the generalization of the n-dimensional volume convergence theorem of [26], we easily conclude the proof.

4. Polar limit spaces

Throughout this section, we assume (in addition to (1.1)) that Y^m is *polar*, in the sense defined below. Under this assumption, we show that $\mathcal{R}_k$ is *nonempty*, for some $k \geq m$. On the other hand, we show in Part II, that for Y^m an *arbitrary* limit space satisfying (1.1), $\mathcal{R}_k$ is empty, for $k > m$. It follows in particular that if Y^m is polar, then m is an integer.

Additionally, we show that in the polar case, dim $\mathcal{D}_k \leq k$; see Section 0 for the definition of $\mathcal{D}_k$. Since $\mathcal{S}_k \subset \mathcal{D}_k$, in this case, dim $\mathcal{S}_k \leq k$ as well.

As observed in Section 5, if $m = n$, then Y^n is polar. Thus, in particular, dim $\mathcal{S}_k(Y^n) \leq k$.

Let X be a complete length space. We say $x \in X$ is a *pole*, if for all $\underline{x} \neq x$, there is a ray, $\gamma : [0, \infty) \to X$, with $\gamma(0) = x$ and $\gamma(t) = \underline{x}$, for some $t > 0$. Here, as usual, γ is called a *ray* if each finite segment of γ is minimal.

Let $y \in Y^m$ and let Y_y be a tangent cone at y. If $z_\infty \in Y_y$, we can consider a tangent cone to Y_y at z_∞. More generally, any tangent cone obtained by iterating this process will be called an *iterated tangent cone*. Recall that iterated tangent cones played an implicit role in Section 3.

Definition 4.1. The space, Y^m, is called *polar* if for all $y \in Y^m$, the base point of every iterated tangent cone is a pole.

As mentioned in Section 0, currently we do not know an explicit example of a limit space satisfying (1.1) which is *not* polar. If $m = n$, then every tangent cone is a metric cone; see Theorem 5.2. Thus, every limit space, Y^n, is polar.

Fix $0 < t < 1$ (which plays no further role in the discussion). Let X be a metric space and $A \subset X$. We say that $x \in A$ is a *k-density point* of A, if there exists a positive sequence, $r_j \to 0$ such that for all j, and any covering of $B_{r_j}(x) \cap A$ by balls, $\{B_{s_i}(x_i)\}$, we have

$$\sum_i s_i^k \geq \frac{t}{2^k} r_j^k. \tag{4.2}$$

Let $T_k(A)$ denote the set of *k-density points* of A. By an easy lemma in measure theory, (see [33, Chapter 11]),

$$\mathcal{H}^k(A \setminus T_k(A)) = 0. \tag{4.3}$$

Let $Y(k)$ denote the set of k-dimensional points of Y^m (defined in Section 0). Clearly, for all $k' < k$, we have

$$Y(k) \subset \overline{T_{k'}(Y)}. \tag{4.4}$$

Also, if $\underline{y} \in T_{k'}(Y)$, by Gromov's compactness theorem, it is obvious that there exists a tangent cone, $Y_{\underline{y}}$, with $\mathcal{H}^{k'}(Y_{\underline{y}}) > 0$.

Let $\{k\}$ denote the smallest integer $\geq k$.

Theorem 4.5. *If Y^m is polar, then $Y(k) \subset \cup_{i=\{k\}}^n \overline{\mathcal{R}_i}$.*

Proof. As noted above, for all $k' < k$, the set $Y(k)$, is contained in the closure of points, $\underline{y}$ such that there exists $Y_{\underline{y}}$ with $\mathcal{H}^{k'}(Y_{\underline{y}}) > 0$. Let $z_\infty \neq \underline{y}_\infty$ be a k'-density point of $Y_{\underline{y}}$. Then there exists a tangent cone, $(Y_y)_{z_\infty}$, at z_∞, such that, $\mathcal{H}^{k'}((Y_y)_{z_\infty}) > 0$. Since the base point of $(Y_y)_{z_\infty}$ is a pole, from the splitting theorem (Theorem 6.64 of [15]) it follows that $(Y_y)_{z_\infty}$ splits off a factor $\mathbb{R}$, isometrically.

Proceeding by induction, we find that there exists an interated tangent cone at $\underline{y}$ which splits off a factor, $\mathbb{R}^i$ isometrically, where $i \geq \{k\}$.

It follows that for all $\epsilon > 0, Y(k) \subset \overline{(\mathcal{WE}_i)_\epsilon}$. Therefore, using the assertion concerning $(\mathcal{WE}_k)_\epsilon$ in Lemma 2.5 (which will be proved in [13]) our claim follows from an argument like that which was given in the proof of Theorem 2.1; see (2.8), (2.9).

Note that using only the first statement in Lemma 2.5 (the proof of which was given in Section 2) and arguing as in the proof of Theorem 2.1, one obtains from the Baire catagory theorem, the weaker statement, $Y(k) \subset \cup_{i=\{k\}}^n \overline{\mathcal{WR}_i}$.

Let $(\mathcal{WE}_k)_{\varepsilon,r}$ denote the open subset consisting of points, y, such that there exists $X, (0,x) \in \mathbb{R}^k \times X$ and, $s > r$, with

$$d_{GH}(B_s(y), B_s((0,x))) < \epsilon s. \tag{4.6}$$

We denote by $(\mathcal{WR}_k)_{\varepsilon,r} \subset (\mathcal{WE}_k)_{\varepsilon,r}$, the open subset of points, y, such that the space, X, in (4.6), can be taken to consist of a single point. Also, define the *ε-weakly regular set*, $(\mathcal{WR}_k)_\varepsilon$, and *weakly regular set*, $(\mathcal{WR}_k)$, by $(\mathcal{WR}_k)_\varepsilon = \cup_r (\mathcal{WR}_k)_{\varepsilon,r}$, and $(\mathcal{WR}_k) = \cap_\varepsilon (\mathcal{WR}_k)_\varepsilon$, respectively.

On compact subsets of Y, the following holds. For all $\eta > 0$, there exist $\epsilon(\eta), r(\eta) > 0$ such that if $\overline{z, \mathcal{D}_k} \geq \eta$, then $z \in (\mathcal{WE}_{k+1})_{\epsilon,r}$ (where the bar denotes distance). In particular, if we put $(\mathcal{D}_k)_\varepsilon = Y \backslash (\mathcal{WE}_{k+1})_\varepsilon$, then $(\mathcal{D}_k)_\epsilon$ is closed and $\mathcal{D}_k = \cup_{\epsilon>0} (\mathcal{D}_k)_\varepsilon$. Also, $(\mathcal{WE}_n)_\varepsilon = (\mathcal{WR}_n)_\varepsilon$.

Theorem 4.7. *If Y^m is polar, and in particular, if $m = n$, then dim $\mathcal{D}_k \leq k$.*

Proof. If the assertion is false, then for some $k' > k$ and $\epsilon > 0$, there exists a k'-density point, y, of $(\mathcal{D}_k)_\varepsilon$. By Gromov's compactness theorem, we can assume that, $(Y, r_j^{-1}d, y)$ Gromov–Hausdorff converges to a tangent cone, Y_y, where $\{r_j\}$ is as in (4.2).

It follows that for all $\eta > 0$, and j sufficiently large, $z \in (\mathcal{D}_k)_\varepsilon(Y) \cap B_1(y)$, implies

$$d_{GH}(z, (\mathcal{D}_k)_\varepsilon(Y_y) \cap B_1(y_\infty)) < \eta, \tag{4.8}$$

where $B_1(y)$ denotes the unit ball for the rescaled metric, $r_j^{-1}d$. Since y is a k'-density point and $(D_k)_\epsilon(Y_y)$ is closed, we conclude that

$$\mathcal{H}^{k'}((\mathcal{D}_k)_\varepsilon(Y_y) \cap B_1(y_\infty)) > 0.$$

Write $Y_y = \mathbb{R}^\ell \times X$, where $\mathbb{R}^\ell$ is the maximal Euclidean factor. Since $y \in (\mathcal{D}_k)_\varepsilon(Y^m)$, we have $\ell \leq k$. Put $y_\infty = (0, x_\infty)$. Since $\mathcal{H}^{k'}(\mathbb{R}^\ell \times x_\infty) = 0$, it follows that there is a k'-density point, z_∞, of $(\mathcal{D}_k)_\varepsilon(Y_y)$ which does not lie on $\mathbb{R}^\ell \times x_\infty$. Since base point of any tangent cone of Y_y at z_∞ is a pole, it follows that such a tangent cone splits off a factor, $\mathbb{R}^{\ell+1}$ isometrically. Since $z_\infty \in (\mathcal{D}_k)_\varepsilon(Y_y)$, we find that in fact, $\ell + 1 < k$. By repeating this argument sufficiently many times (with successive iterated tangent cones) we conclude $\ell < 0$, a contradiction. By Theorem 5.2, if $m = n$, then Y^m is polar. This suffices to complete the proof.

Remark 4.9. For Riemannian manifolds satisfying $\mathrm{Vol}\,(B_1(p)) \geq v > 0$, Theorem 4.7 can be employed in an obvious fashion to provide an extension of Theorem 2.45.

Remark 4.10. A simple direct argument shows that for polar limit spaces, the set $(\mathcal{D}_0)_\varepsilon$ is actually *finite*, for all $\varepsilon > 0$. In the noncollapsed case, one can give an explicit estimate on the cardinality of this set.

5. Noncollapsed limit spaces

In this section we continue to assume that (Y, y) is the Gromov-Hausdorff limit of a sequence, $\{(M_i^n, p_i)\}$. In addition, we assume

$$\mathrm{Vol}\,(B_1(p_i)) \geq v > 0. \tag{5.1}$$

Before discussing the additional properties of the regular set that hold in this case, we give some results which are more geometric in nature.

Theorem 5.2. *If (Y, y) satisfies (1.1) and (5.1), then for all $\underline{y} \in Y$, every tangent cone at $\underline{y}$ is a metric cone, $C(X)$, on a length space X, with $diam(X) \leq \pi$.*

Proof. Suppose there exists $y \in Y$ for which the assertion fails for some tangent cone, Y_y, which is the pointed Gromov-Hausdorff limit of a sequence $(Y, r_i^{-1}d, y)$, where d is the metric on Y. Since Y_y is not a metric cone, it follows that there exists $\eta > 0, \epsilon > 0, \Omega > 1$ and $r_j \to 0$ such that for any length space, X_j, with $\mathrm{diam}(X_j) \leq \pi$,

$$d_{GH}(A_{r_j,\Omega r_j}, (r_j, \Omega r_j) \times_r X_j) \geq \epsilon r_j. \tag{5.3}$$

Here, as in [15], we understand that the metrics on $A_{r_j,\Omega r_j}$, and $(r_j, \Omega r_j) \times_r X_j$ are measured in $A_{(1-\eta)r_j,(1+\eta)r_j\Omega}$ and

$$((1-\eta)r_j, (1+\eta)\Omega r_j) \times_r X_j$$

respectively. However, according to Theorem 4.91 of [15], for any of the manifolds, M_i^n, with $\lim_{i\to\infty}(M_i^n, p_i) = (Y, y)$ there are at most $\#(\frac{1}{2}\epsilon, \eta, \Omega, n, v)$ annuli which satisfy the above condition with ϵ replaced by $\frac{1}{2}\epsilon$. For i sufficiently large, this is a contradiction.

It follows immediately from Theorem 5.2 that the space, Y^n, is polar in the sense of Definition 4.1.

Next we observe that the result on volume convergence, conjectured by Anderson-Cheeger and proved in [26], can be sharpened as follows.

Theorem 5.4. *For all $d, v, \eta > 0$ there exists*

$$\delta(\eta) = \delta(n, d, v, \eta) > 0,$$

such that if for $i = 1, 2$,

$$\mathrm{diam}\,(M_i^n) \leq d, \tag{5.5}$$

$$\mathrm{Vol}\,(M_i^n) \geq v > 0, \tag{5.6}$$

$$d_{GH}(M_1^n, M_2^n) \leq \delta(\eta), \tag{5.7}$$

then

$$e^{-\eta}\mathrm{Vol}\,(M_1^n) \leq \mathrm{Vol}\,(M_2^n) \leq e^{\eta}\mathrm{Vol}\,(M_1^n). \tag{5.8}$$

Proof. It follows from (5.6) and (0.5), for $\mu = \mathrm{Vol}(\cdot), k = n$, $H = -1$, that there exists $\epsilon(n, v) > 0$, such that if in Theorem 2.45, we take $\epsilon \le \epsilon(n,v)$, then for all i , we have $k_i = n$. With this observation, the theorem follows from the argument of [26].

Let $\mathcal{H}^n$ denote Hausdorff measure. By definition, $\mathcal{H}^n$ is normalized to agree with Lebesgue measure on $\mathbb{R}^n$.

From (the proof of) Theorem 5.4 we immediately obtain the following; compare Theorem 7.5.

Theorem 5.9. *If Y satisfies (1.1) and (5.1) then* $\dim Y = n$. *Moreover, for any $R > 0$ and $q_i \overset{d_{GH}}{\to} z$,*

$$\lim_{i\to\infty} \mathrm{Vol}\,(B_R(q_i)) = \mathcal{H}^n(B_R(z)). \tag{5.10}$$

In particular, any renormalized limit measure, ν, is a multiple of Hausdorff measure, $\mathcal{H}^n$. Thus, on Y^n, (0.5) holds, for $\mu = \mathcal{H}^n, k = n$, $H = -1$.

We now consider the regular set.

Theorem 5.11. *Weakly regular points are regular i.e. $\mathcal{WR}(Y^n) = \mathcal{R}(Y^n) = \mathcal{R}_n(Y^n)$. Moreover, for all $\varepsilon > 0$ there exists $\delta > 0$ such that if $y \in (\mathcal{WR}_n)_\delta$, then $y \in (\mathcal{R}_n)_\varepsilon$.*

Proof. This follows directly from (0.5), for $\mu = \nu, k = n$ and the argument used in [26] to prove the corresponding uniqueness theorem (conjectured by Anderson-Cheeger) for tangent cones at infinity of complete manifolds with $Ric_{M^n} \ge 0$.

Let $(\mathcal{WE}_k)_{\varepsilon,\delta}(M^n)$ be as in Section 4 and put

$$(\mathcal{D}_k)_{\varepsilon,\delta}(M^n) = M^k \setminus (\mathcal{WE}_k)_{\varepsilon,\delta}(M^n).$$

From Theorem 6.2 (see Section 6) we immediately obtain the following.

Theorem 5.12. *Given $k' > n-2$ and $\epsilon, \eta > 0$, there exists $\delta = \delta(n,d,v,k',\epsilon,\eta) > 0$, such that if*

$$Ric_{M^n} \ge -(n-1), \mathrm{diam}\,(M^n) \le d, \mathrm{Vol}\,(M^n) \ge v,$$

then the set $(\mathcal{D}_n)_{\varepsilon,\delta}(M^n)$ admits a covering by balls, $\{B_{s_i}(q_i)\}$, with $\sum_i s_i^{k'} \le \eta$.

Remark 5.13. By Theorem A.1.8, given $\xi > 0$, there exists $\varepsilon = \varepsilon(n,\xi) > 0$, such that for $p \in (\mathcal{WR}_n)_{\varepsilon,\delta}(M^n) = M^n \setminus (\mathcal{D}_n)_{\varepsilon,\delta}(M^n)$, there is a smooth imbedding, $f : B_\delta(0) \to M^n$ (where $0 \in \mathbb{R}^n$) such that

$B_{(1-\xi)\delta}(p) \subset f(B_\delta(0)) \subset B_{(1+\xi)\delta}(p)$. Thus (roughly speaking) off a set of codimension 2, Theorem 5.12 provides a sharpened generalization of the result of [38].

Recall that for all $\varepsilon > 0$, there exists $\delta > 0$, such that $\mathcal{R}_\delta \subset \overset{\circ}{\mathcal{R}}_\varepsilon$. By Theorem A.1.9 we get the following.

Theorem 5.14. *For $\varepsilon \leq \varepsilon(n)$, the set $\overset{\circ}{\mathcal{R}}_\varepsilon(Y^n)$ has a natural smooth manifold structure. Moreover, for this parameterization, the metric on $\overset{\circ}{\mathcal{R}}_\varepsilon(Y^n)$ is bi-Hölder equivalent to a smooth Riemannian metric. The exponent in this bi-Hölder equivalence satisfies $\alpha(\varepsilon) \to 1$ as $\varepsilon \to 0$.*

Remark 5.15. It seems possible that the above parametrization can actually be chosen to be bi-Lipschitz.

6. $\dim \mathcal{S}(Y^n) \leq n-2$

The main result of this section is

Theorem 6.1. *If Y^n satisfies (1.1), (5.1) then $\mathcal{S}(Y^n) \subset \mathcal{S}_{n-2}(Y^n)$ and $\dim \mathcal{S} \leq n-2$.*

By Theorem 4.7, if $\mathcal{S} \subset \mathcal{S}_{n-2}$, then $\dim \mathcal{S} \leq n-2$. Therefore, it suffices to show that a half space, $\mathbb{R}^{n-1} \times \mathbb{R}_+$, does not occur as a limit space (in the present noncollapsing case). Intuitively, the reason why this holds is the following. Consider an interior point, z, in such a half space and a ball, $B_r(z)$, with r greater than the distance from z, to the boundary, $\mathbb{R}^{n-1} \times 0$. The boundary of this ball contains points whose distance from z is strictly less that r, namely $B_r(z) \cap (\mathbb{R}^{n-1} \times 0)$. Note however, that for a ball in a complete Riemannian manifold, this never happens. We will show that in the noncollapsing case, it does not happen for limit spaces as well, i.e., we show that the boundary of the Gromov-Hausodorff limit of a sequence of balls, is the limit of the sequence of boundaries of these balls.

Theorem 6.2. *If Y^n satisfies (1.1), (5.1), then at no point of Y does there exist a tangent cone isometric to $\mathbb{R}^{n-1} \times \mathbb{R}_+$.*

Proof. Put $H^n = \mathbb{R}^{n-1} \times \mathbb{R}_+ \subset \mathbb{R}^n$ and $A_{r_1,r_2}(x) = \overline{B_{r_2}(x)} \setminus B_{r_1}(x)$.

Suppose, that there exists (Y^n, y) with some Y_y isometric to H^n. Let $\{M_i^n\} \to Y^n$ be as in (1.1). Fix $\epsilon > 0$. Then, after rescaling the metrics on M_i, for i sufficiently large, there is a *continuous* ϵ-Gromov-Hausdorff approximation, $f : B_1(p_i) \to B_1(0) \cap H^n$. Here $B_1(p_i) \subset M_i^n$ and $B_1(0) \subset \mathbb{R}^n$.

Using the map f, we will construct an auxiliary map, $\tilde{f}$, and show that the mod 2 degree of $\tilde{f}$ satisfies both $\deg \tilde{f} = 0$ and $\deg \tilde{f} = 1$. This will give the desired contraction.

Without loss of generality, we can assume that 1 is a regular value of the distance function on M_i^n. Thus, $B_1(p_i)$ is a smooth manifold with boundary.

Since f is an ϵ-Gromov-Hausdorff approximation, it follows that $f(\partial B_1(p_i))$ is contained in an ϵ-neighborhood of $\partial B_1(0) \cap H^n$. By adjusting f slightly (using radial projection) we can assume without loss of generality, that

$$f(\partial B_1(p_i)) \subset \partial B_1(0) \cap H^n. \tag{6.3}$$

Note that $\partial B_1(0) \cap H^n$ is a proper subset of $\partial(B_1(0) \cap H^n)$.

Let $q \in \partial B_{\frac{1}{2}}(p_i)$ be a point such that $f(q)$ is at maximal distance from $\mathbb{R}^{n-1} \times \{0\}$. By further adjusting f (using radial projection on say $B_{\frac{2}{5}}(f(q)))$ we can assume without loss of generality that

$$f(B_{\frac{1}{4}}(q)) \subset B_{\frac{1}{4}}(f(q)), \tag{6.4}$$

$$f(A_{\frac{1}{8},\frac{1}{4}}(q)) \subset A_{\frac{1}{8},\frac{1}{4}}(f(q)). \tag{6.5}$$

From Perelman's theorem, [46] (or Appendix 1) together with the result on local volume convergence proved in [26], it follows for say $q_1 \in B_{\frac{1}{6}}(q)$ and $r \leq \frac{1}{6}$, that the ball $B_r(q_1)$ is contractible in the ball $B_{(1+\Psi)r}(q_1)$, where $\Psi = \Psi(\epsilon|n) \to 0$ as $\epsilon \to 0$. Thus, for ϵ sufficiently small, we can construct a *continuous* 2ϵ-Gromov-Hausdorff approximation, $h : B_{\frac{1}{4}}(f(q)) \to B_{\frac{1}{4}}(q)$.

After adjusting h slightly (using radial projection on $B_{\frac{2}{5}}(f(q)))$ we can assume with no loss of generality,

$$h\left(B_{\frac{1}{4}}(f(q))\right) \subset B_{\frac{1}{4}}(q), \tag{6.6}$$

$$h\left(A_{\frac{1}{8},\frac{1}{4}}(f(q))\right) \subset A_{\frac{1}{8},\frac{1}{4}}(f(q)). \tag{6.7}$$

Moreover, for $z \in B_{\frac{1}{4}}(q)$ and $\Psi(\epsilon|n)$ as above,

$$\overline{f \circ h(z), z} \leq \Psi(\epsilon|n). \tag{6.8}$$

Clearly the map of pairs,

$$f \circ h : \left(B_{\frac{1}{2}}(f(q)), A_{\frac{1}{8},\frac{1}{4}}(f(q))\right) \to \left(B_{\frac{1}{4}}(f(q)), A_{\frac{1}{4},\frac{1}{8}}(f(q))\right),$$

has mod 2 degree satisfying, $\deg f \circ g = 1$. Thus, if we let

$$\hat{f} : \left(B_{\frac{1}{4}}(q), A_{\frac{1}{8},\frac{1}{4}}(q)\right) \to \left(B_{\frac{1}{4}}(f(q)), A_{\frac{1}{8},\frac{1}{4}}(f(q))\right),$$

denote the map induced from $\tilde{f}$ by restriction, we have

$$\deg \hat{f} = 1. \tag{6.9}$$

Let N^n denote the closed manifold obtained by doubling $B_1(p_i)$ along its boundary. Let D^n denote the manifold with boundary obtained by doubling $B_1(0) \cap H^n$ along $\partial B_1(0) \cap H^n$. Clearly, D^n is a topological ball. We denote by $\tilde{f}$, the induced map from N^n to D^n.

Since D^n is not a closed manifold, clearly $\deg \tilde{f} = 0$. On the other hand,

$$\deg \tilde{f} = \deg \hat{f} = 1, \tag{6.10}$$

since for say, $z \in B_{\frac{1}{16}}(f(q))$ we have $f^{-1}(z) \subset B_{\frac{1}{4}}(q)$. This contradiction suffices to complete proof.

7. Two sided bounds on Ricci curvature and Einstein manifolds

In this section we continue to assume that (Y^n, y) is the Gromov-Hausdorff limit of a sequence, $\{(M_i^n, p_i)\}$, such that (5.1) holds. However, we strengthen (1.1) to

$$|Ric_{M_i^n}| \leq n - 1. \tag{7.1}$$

Sometimes, we will assume in addition, that each M_i^n is Einstein. In either of these cases, we can replace the intrinsic version of Reifenberg's theorem used in Section 5 by the corresponding estimate on the $C^{1,\alpha}$-harmonic radius (respectively C^∞-harmonic radius) proved in [4]. Then, a more elementary and straightforward version of the discussion of Section 5 yields improved results concerning regularity.

Theorem 7.2. *If Y^n satisfies (5.1), (7.1), then $\mathcal{WR}_\epsilon = \mathcal{R}_\varepsilon = \mathcal{R}$, for some $\epsilon = \epsilon(n) > 0$. In particular, the set $\mathcal{S} \subset \mathcal{S}_{n-2}$ is closed.*

Moreover, $\mathcal{R}$ is a $C^{1,\alpha}$-Riemannian manifold and at points of $\mathcal{R}$, the convergence, $(M_i^n, g_i) \to (Y,^n g_\infty)$, takes place in the $C^{1,\alpha}$-topology.

Theorem 7.3. *If Y^n satisfes (5.1), (7.1) and in addition, M_i^n is Einstein, for all i, then $\mathcal{R}$ is a C^∞-Riemannian manifold and the convergence, $(M_i^n, g_i) \to (Y^n, g_\infty)$, takes place in the C^∞-topology.*

Since, mutadis mutandis, the proof of Theorem 7.3 is the same as that of Theorem 7.2, we will only prove the latter. First, we recall some background from [4].

Recall that there is a constant, $\omega(n) > 0$, and for all $1 > \alpha > 0$, a function, $\lambda(n, \alpha, r, s) : (0, 1) \to \mathbb{R}_+$, with the following properties. Assume M^n satisfies (7.1), $p \in M^n$ and (see Section 1)

$$\frac{\mathrm{Vol}\,(B_r(p))}{V_{n,-1}(r)} > 1 - \omega(n). \tag{7.4}$$

Then for all $q \in B_r(p)$, with $\overline{p,q} = s \cdot r$, there exists $\lambda(n, \alpha, r, s)$ and a harmonic coordinate system on the ball, $B_{\lambda r}(q)$, in which the metric, (g_{ij}), satisfies definite $C^{1,\alpha}$ bounds, say, $|g_{i,j}|_{C^{1,\alpha}} \leq 2$, and $|\det(g_{ij})|^{-1} \leq 2$.

If we assume in addition that M^n is Einstein , then for some $\lambda(n, r, s)$, the same holds with $C^{1,\alpha}$ replaced by C^∞.

According to [26], there exists $\epsilon = \epsilon(n) > 0$ such that if $p \in (\mathcal{WR}_n)_{\varepsilon,\delta}(M^n)$, then (7.4) holds; compare Section 5.

Proof of Theorem 7.2. Write $(\mathcal{WR}_n)_{\frac{1}{2}\epsilon}(Y^n) = \cup_j (\mathcal{WR}_n)_{\frac{1}{2}\epsilon, j^{-1}}(Y^n)$, where $\epsilon = \epsilon(n)$. Let $f_i : M_i^n \to Y^n$ be a Gromov-Hausdorff equivalence realizing the Gromov-Hausdorff distance between M_i^n and Y^n. Then for all fixed j and i sufficiently large, $f_i(q_i) \in (\mathcal{WR}_n)_{\frac{1}{2}, j^{-1}}(Y^n)$ implies $q_i \in (\mathcal{WR}_n)_{\epsilon, j^{-1}}(M_i^n)$. Hence, by the discussion preceding the proof, there is a definite bound on the $C^{1,\alpha}$-harmonic radius at q_i. By using arguments which by now are standard (compare e.g. [11], [7]) and Theorem 6.1, which asserts that $\mathcal{S} \subset \mathcal{S}_{n-2}$, the proof can be concluded.

In view of Theorem 7.2 and 7.3, an obvious application of Gromov's compactness theorem gives

Theorem 7.5. *Given $k' > n - 2$, $0 < \alpha < 1$ and $\eta > 0$, there exist $\delta = \delta(n, d, v, k', \alpha, \eta) > 0$, such that if M^n satisfies (7.1), (5.1), then the set of points contained in a ball of radius d, at which the $C^{1,\alpha}$-harmonic radius is $< \delta$, admits a covering by balls, $\{B_{s_i}(q_i)\}$, with $\sum_i s_i^{k'} \leq \eta$.*

Theorem 7.6. *If M^n satisfy (5.1), (7.1) and in addition, M^n is Einstein, then "$C^{1,\alpha}$-harmonic radius" can be replaced by "C^∞-harmonic radius" in Theorem 7.5.*

8. Examples

In this section we will construct a number of examples of spaces which are Gromov-Hausdorff limits of sequences of pointed Riemannian manifolds, $\{(M_i^n, p_i)\}$, of positive Ricci curvature. The tangent cones of these limit spaces exhibit various phenomena which could not occur if say the sectional curvatures, $K_{M_i^n}$, had a uniform lower bound. Similarly, we will construct complete noncompact manifolds with positive Ricci curvature whose tangent cones at infinity exhibit phenomena which could not occur if the sectional curvature were nonnegative.

The first such example is due to Perelman (unpublished) who constructed a metric on $\mathbb{R}^4$ with positive Ricci curvature, Euclidean volume growth and quadratic curvature decay, for which the tangent cone at infinity is not unique; compare [15] and [22]. In Perelman's example, he views $\mathbb{R}^4 \setminus \{0\}$ as $\mathbb{R}_+ \times S^3$, on which he constructs a metric of triply warped product type, based on the Hopf fibration, $S^1 \to S^3 \xrightarrow{\pi} S^2$. Our examples are based on doubly warped product constructions, and we will begin by briefly reviewing some properties of such metrics; compare [3].

Let I be an interval and let Z be a manifold. Consider a family of Riemannian metrics, $g^Z(r)$ on Z, parameterized by $r \in I$. Assume that for each $p \in Z$, the metrics $g^Z(r)|Z_p$ can all be simultaneously diagonalized with respect to some fixed metric on the tangent space, Z_p. Thus, we can find a local orthonormal basis, $\{\tilde{y}_i\}$, near p, such that for some positive functions, $u_i(r)$, the vector fields, $\{y_i(r)\}$, defined by $y_i = \dfrac{\tilde{y}_i}{u_i}$ are orthonormal for $g^Z(r)$.

Consider the metric

$$g = dr^2 + g^Z(r) \tag{8.1}$$

on $I \times Z$. In order to simplify the discussion, we now recall a sufficient condition for the vector, $n = \frac{\partial}{\partial r}$, to be an eigenvector for the Ricci tensor of g.

If we view the vector fields, y_i, as being defined on $I \times Z$, then their

Lie brackets satisfy

$$[n, y_i] = -\frac{u_i'}{u_i} y_i, \tag{8.2}$$

$$\langle [y_i, y_j], n\rangle = 0. \tag{8.3}$$

Using the standard formula for the Riemannian connection,

$$\langle \nabla_A B, C\rangle = \frac{1}{2}\{\langle [A,B], C\rangle - \langle [A,C], B\rangle - \langle [B,C], A\rangle\}, \tag{8.4}$$

where A, B, C are vector fields whose inner products with respect to the metric, $g = \langle\ ,\ \rangle$, are constant, we find that

$$\nabla_n y_i = 0. \tag{8.5}$$

Thus, we have

$$\langle \nabla_n \nabla_{y_i} y_i, y_j\rangle = n\langle \nabla_{y_i} y_i, y_j\rangle, \tag{8.6}$$

$$\langle \nabla_{y_i} \nabla_n y_i, y_j\rangle = 0, \tag{8.7}$$

$$\langle \nabla_{[n,y_i]} y_i, y_j\rangle = -\frac{u_i'}{u_i}\langle \nabla_{y_i} y_i, y_j\rangle. \tag{8.8}$$

Therefore, if $\langle \nabla_{y_i} y_i, y_j\rangle \equiv 0$, it follows that

$$\langle R(n, y_i) y_i, y_j\rangle = 0, \tag{8.9}$$

and in particular, that n is an eigenvector of the Ricci tensor.

Suppose that the metric $g^Z(r)$ is of the form

$$g^Z(r) = u_1^2(r) k_1 + \ldots + u_d^2(r) k_d, \tag{8.10}$$

where the tangent bundle TZ admits a decomposition,

$$TZ = E_1 \oplus \ldots \oplus E_d,$$

such that, $g^Z(r)|E_i = u_i^2(r) k_i$, and k_i annihilates E_j for $i \neq j$.

Assume further that for the metric $k_1 + \cdots + k_d$, whenever a geodesic, γ, satisfies $\gamma'(0) \subset E_i$ (for some i) then $\gamma'(t) \subset E_i$ for all t. Equivalently, for all i, the symmetric part of the second fundamental form of E_i

vanishes. Then it follows from (8.4) that for fixed r, the same is true for the metric $u_1^2(r)k_1 + \cdots + u_d^2(r)k_d$. Thus, it is clear that the condition $\langle \nabla_{y_i} y_i, y_j \rangle = 0$ holds. So in this case, (8.9) holds as well.

In particular, if E_1, E_2 denote the vertical and horizontal subbundles for a *Riemannian submersion*, $X^\ell \to Z^{\ell+m} \xrightarrow{\pi} W^m$, with *totally geodesic fibres*, then (8.9) holds.

Since the fields, $\tilde{y}_i$, are Jacobi fields, we see that

$$Ric(n,n) = \sum_{i=1}^{n-1} -\frac{u_i''}{u_i}. \tag{8.11}$$

Also, from (8.3)–(8.5), the second fundamental from of a hypersurface, $(\underline{r}, Z)$, is given by

$$II(y_i, y_j) = \begin{cases} -\dfrac{u_i'}{u_i} n & i = j, \\ 0 & i \neq j. \end{cases} \tag{8.12}$$

Thus, if $\widetilde{Ric}$ denotes the Ricci tensor of the induced metric on $(\underline{r}, Z)$, and $\{y_i^\star\}$ denotes the dual basis to $\{y_i\}$, then

$$Ric\,\big|_{(r,Z)} = \widetilde{Ric} - \sum_i \left(\frac{u_i''}{u_i} + \frac{u_i'}{u_i} \sum_{j \neq i} \frac{u_j'}{u_j} \right) (y_i^\star)^2. \tag{8.13}$$

Now let $u : \mathbb{R}_+ \to \mathbb{R}_+$ and put

$$u = \overline{u} r. \tag{8.14}$$

Then

$$\frac{u'}{u} = \frac{\overline{u}'}{\overline{u}} + \frac{1}{r}, \tag{8.15}$$

$$\frac{u''}{u} = \frac{\overline{u}''}{\overline{u}} + \frac{2\overline{u}'}{\overline{u}} \frac{1}{r}. \tag{8.16}$$

From (8.13)–(8.16) it is clear that if for some $\tau > 0$,

$$\widetilde{Ric} > \frac{n-2+\tau}{r^2}, \tag{8.17}$$

and in addition, for sufficiently small $\delta = \delta(\tau, n) > 0$,

$$\left|\frac{\overline{u}_i'}{\overline{u}_i}\right| < \frac{\delta}{r}, \tag{8.18}$$

$$\left|\frac{\overline{u}_i''}{\overline{u}_i}\right| < \frac{\delta}{r^2}, \tag{8.19}$$

then the expression in (8.13) is strictly positive; indeed, it is essentially the same as the corresponding expression for the case of a metric cone on a space with $\widetilde{Ric} > (n-2)$.

For suitable Z, conditions (8.17)–(8.19) are not difficult to achieve. Therefore, to obtain a metric with $Ric > 0$, the essential point is to choose the functions, u_i, such that in addition, the expression in (8.11) is positive.

Our first example is based on repeated application of the following elementary lemma.

Let $\ell_1, \ell_2 \in \mathbb{Z}$ and $C_1, C_2, \tilde{C}_1, \tilde{C}_2 \in \mathbb{R}_+$, satisfy

$$\frac{\tilde{C}_1}{C_1} > 1 > \frac{\tilde{C}_2}{C_2}, \tag{8.20}$$

$$\tilde{C}_1^{\ell_1}\tilde{C}_2^{\ell_2} > C_1^{\ell_1}C_2^{\ell_2}. \tag{8.21}$$

Relation (8.21) corresponds to the inequality in the relative volume comparison theorem ([37]); compare Lemma 8.28 and Example 8.41.

Define α and χ by

$$\left(\frac{\tilde{C}_1}{C_1}\right)^{-\alpha} = \frac{\tilde{C}_2}{C_2}, \tag{8.22}$$

$$\ell_1 - \alpha\ell_2 = \chi > 0. \tag{8.23}$$

Let $\epsilon_1, \epsilon_2 \in \mathbb{R}_+$. Put

$$\epsilon_1 = \beta\epsilon_2 \tag{8.24}$$

and assume

$$-\beta\ell_1 + \ell_2 = \lambda > 0, \tag{8.25}$$

where the left-hand side defines λ.

Given $0 < a < b < 1$, define η_1, η_2 by

$$\frac{a}{b} = \left(\frac{\tilde{C}_1}{C_1}\right)^{-\frac{1}{\eta_1}} = \left(\frac{\tilde{C}_2}{C_2}\right)^{\frac{1}{\eta_2}}. \tag{8.26}$$

Thus,

$$\eta_2 = \alpha\eta_1. \tag{8.27}$$

$\eta_1 \to 0$ is equivalent to $a/b \to 0$.

Let τ, $\delta(\tau, n)$ be as in (8.18), (8.19).

Lemma 8.28. *For $\epsilon_1, \epsilon_2 > 0$ sufficiently small, there exists $\omega = \omega(\ell_1, \ell_2, C_1, C_2, \tilde{C}_1, \tilde{C}_2, \tau, \epsilon_1, \epsilon_2) > 0$, such that if $\omega \geq \frac{a}{b} > 0$, then for $|t_1|, |t_2| \leq 2$, there exist C^1 functions, $u_1, u_2 : [a, b] \to \mathbb{R}_+$, which are C^∞ on $[a, b] \setminus \{\frac{1}{2}b\}$, such that (8.18), (8.19) hold for $n = \ell_1 + \ell_2$ and*

$$\begin{aligned} u_1(b) =& C_1 b(1 + (\frac{1}{4}(\eta_1 + \epsilon_1)) + \tilde{C}_1 a t_1 \eta_1, \\ u_2(b) =& C_2 b(1 - \frac{1}{4}(\eta_2 + \epsilon_2)) + \tilde{C}_2 a t_2 \eta_2, \end{aligned} \tag{8.29}$$

$$u_1'(b) = (1 + \epsilon_1)C_1, \quad u_2'(b) = (1 - \epsilon_2)C_2, \tag{8.30}$$

$$u_1(a) = \tilde{C}_1 a(1 + t_1\eta_1), \quad u_2(a) = \tilde{C}_2 a(1 + t_2\eta_2), \tag{8.31}$$

$$u_1'(a) = (1 - \eta_1)\tilde{C}_1, \quad u_2'(a) = (1 + \eta_2)\tilde{C}_2, \tag{8.32}$$

$$-\left(\ell_1 \frac{u_1''}{u_1} + \ell_2 \frac{u_2''}{u_2}\right) \geq \frac{1}{2r^2}\eta_1\chi \qquad (r \neq \frac{1}{2}b). \tag{8.33}$$

Proof. On $[a, \frac{1}{2}b]$, put

$$u_1(r) = C_1 b^{\eta_1} r^{1-\eta_1} + \tilde{C}_1 a t_1 \eta_1, \quad u_2(r) = C_2 b^{-\eta_2} r^{1+\eta_2} + \tilde{C}_2 a t_2 \eta_2. \tag{8.34}$$

Then by (8.23), (8.27), it follows that (8.31) and (8.32) hold. Also, if η_1 is sufficiently small, where in particular,

$$\eta_1 < \frac{1}{2}\frac{\chi}{\ell_1 + \alpha^2\ell_2}, \tag{8.35}$$

then on $[a, \frac{1}{2}b]$,

$$-\left(\ell_1 \frac{u_1''}{u_1} + \ell_2 \frac{u_2''}{u_2}\right) \geq \frac{1}{2}\eta_1 \chi r^{-2} > 0. \tag{8.36}$$

On $[\frac{1}{2}b, b]$, put

$$\begin{aligned} u_1(r) &= C_1 b^{\eta_1} r^{1-\eta_1} + C_1 \frac{(\eta_1 + \epsilon_1)}{b}(r - \frac{1}{2}b)^2 + \tilde{C}_1 a t_1 \eta_1, \\ u_2(r) &= C_2 b^{-\eta_2} r^{1+\eta_2} - C_2 \frac{(\eta_2 + \epsilon_2)}{b}(r - \frac{1}{2}b)^2 + \tilde{C}_2 a t_2 \eta_2. \end{aligned} \tag{8.37}$$

Then (8.29), (8.30) hold. Additionally, if we fix ϵ_1, ϵ_2 sufficiently small and then take η_1, η_2 sufficiently small, we have (8.18), (8.19). Finally, it is clear that for $\frac{a}{b}$ sufficiently small (i.e., η_1, η_2, sufficiently small) where in particular,

$$\eta_1 < \lambda \epsilon_2 \chi^{-1}, \tag{8.38}$$

we have

$$-\left(\ell_1 \frac{u_1''}{u_1} + \ell_2 \frac{u_2''}{u_2}\right) \geq \frac{1}{2}\epsilon_2 \lambda r^{-2} > 0, \qquad (\text{on } [\frac{1}{2}b, b]). \tag{8.39}$$

This suffices to complete the proof.

Remark 8.40. If in (8.21), one considers values for which the ratio of the left-hand side to the right-hand side tends to 1, while the quantity, $\tilde{C}_1/C_1$, stays bounded away from 1, then the number, χ, of (8.23), tends to zero. Thus, by (8.35), $\eta_1 \to 0$ and hence, $\frac{a}{b} \to 0$ as well; compare the discussion prior to (8.65).

Example 8.41 (Nonuniqueness of tangent cones). We will construct a smooth Riemannian metric of positive Ricci curvature on $\mathbb{R}^4 \setminus \{0\}$ with the following properties. Its metric space completion, Y^4 (which is obtained by adding the origin) is the pointed Gromov-Hausdorff limit of a sequence of smooth complete metrics of positive Ricci curvature on $\mathbb{R}^4$. At the origin, the tangent cone of Y^4 is not unique.

As previously mentioned, the metric on $\mathbb{R}^4 \setminus \{0\} = \mathbb{R}_+ \times S^3$ is of doubly warped product type, with warping functions, f, h. It is the limit as $j \to \infty$, of a sequence of doubly warped product metrics on $\mathbb{R}^4$, with warping functions, f_j, h_j.

Our metrics arise from the Hopf fibration, $S^1 \to S^3 \xrightarrow{\pi} S^2$, which we regard as a Riemannian submersion with totally geodesic fibres. Thus, S^1, S^3, S^2 carry the metrics, $g^{S^1}, g^{S^3}, \frac{1}{4}g^{S^2}$ respectively, where g^{S^n} denotes the canonical metric of curvature $\equiv 1$ on S^n.

The distinct tangent cones which occur at $0 \in Y^4$ can be described as follows.

Fix $0 < \xi < 1$ and C_f, C_h with

$$C_f C_h^2 = \xi^3, \tag{8.42}$$

$$0 < C_h < \xi < C_f, \tag{8.43}$$

$$|C_f - \xi|, |C_h - \xi| < \delta, \tag{8.44}$$

where δ is as in (8.18), (8.19).

Put $g^{S^3} = k_f + k_h$, where $k_h = \pi^*(\frac{1}{4}g^{S^2})$. Then there is a certain 1-parameter family of metrics, $g_t^{S^3}$, joining $\xi g^{S^3} = \xi k_f + \xi k_h$ and $C_f k_f + C_h k_h$, such that for all $0 \le t \le 1$, there exists a tangent cone at $0 \in Y^4$, with cross section isometric to S^3 equipped with the metric $g_t^{S^3}$.

Our construction will be broken into several steps:

i) Our warping functions, f, h, have the property that for a certain sequence, $r_j \to 0$, we have

$$f(r_{2k})/r_{2k} = h(r_{2k})/r_{2k} \to \xi,\ f(r_{2k+1})/r_{2k+1} \to C_f,$$
$$h(r_{2k+1})/r_{2k+1} \to C_h.$$

Since the set of tangent cones is connected, this guarentees the existence of a 1-parameter family of tangent cones as above.

Initial approximations $\underline{f}, \underline{h}$, to the functions f, h, will be constructed by applying Lemma 8.28 inductively (an infinite number of times) where at this stage, we always choose $t_1, t_2 = 0$.

These approximations have jump discontinuities at the points, r_j; see (8.29), (8.31), (8.59). Moreover, at the points, $r_j, \frac{1}{2}r_j$, the left- and right-hand limits of the second derivatives do not agree; see (8.36), (8.39). However, the left- and right-hand limits of the first derivatives do agree at all points; see (8.30), (8.32), (8.60).

The construction can be arranged so that the series of jumps for the functions, $\underline{f}$, $\underline{h}$, can be assumed to converge as fast as we like and in particular, as fast as a given convergent geometric series; see (8.59) and the discussion prior ot (8.65).

ii) For all j, we construct smooth functions, $\tilde{f}_j, \tilde{h}_j$, on the interval, $[0, r_j]$. Each of these functions is asymptotic to $r + O(r^3)$, as $r \to 0$. Next, we define functions, $\underline{f}_j$, $\underline{h}_j$, to be equal to $\tilde{f}_j$, $\tilde{h}_j$, on the interval, $[0, r_j]$ and equal to $\underline{f}$, $\underline{h}$ on (r_j, ∞). These are our initial approximations to the smooth functions f_j, h_j, on $\mathbb{R}^4$, which define the sequence of doubly warped product metrics, the limit of which is the desired metric on Y^4. The functions, $\underline{f}_j$, $\underline{h}_j$, have properties analogous to those of the functions, $\underline{f}$, $\underline{h}$, above.

iii) By adjusting the values of the functions, $\underline{f}_j, \underline{h}_j$, by suitable constants, on each interval, $(r_i, r_{i-1}]$, where $(i \leq j)$ we can remove the jump discontinuities, thereby obtaining functions, $\hat{f}_j, \hat{h}_j$, are C^1; see (8.67). This corresponds to choosing $t_1, t_2 \neq 0$ in Lemma 8.28 (for a suitable inductively determined sequence of choices.)

The functions, $\hat{f}_j, \hat{h}_j$, continue to have the property that the quantity in (8.33) is positive (i.e., positive Ricci curvature in the radial direction for the associated doubly warped product metrics) except at the points, r_i, $\frac{1}{2}r_i$, where the second derivatives are not continuous.

iv) Finally, by modifiying $\hat{f}''_j, \hat{h}''_j$, in sufficiently small intervals containing the points, r_i, where $(i \leq j)$ and $\frac{1}{2}r_i$, where $(i < j)$ we can remove the jumps in the second derivatives. By integration with respect to r, we obtain C^2 functions, f_j, h_j, defining the sequence of doubly warped product metrics on R^4 that we are seeking; of course, it is clear that the functions, f_j, h_j, can actually be chosen to be C^∞. The metric on Y^4 is determined by the corresponding functions, f, h, which are the limits as $j \to \infty$ of the functions, f_j, h_j.

Details of i)—iv):

i) Let ξ be as in (8.42)–(8.44) and let $C_{f,j}, C_{h,j}$, satisfy

$$C_{f,j}C_{h,j}^2 < C_{f,j+1}C_{h,j+1}^2 \qquad j = 0, 1, \ldots \tag{8.45}$$

$$|C_{f,j} - \xi|, |C_{h,j} - \xi| < \delta \qquad j = 0, 1, \ldots \tag{8.46}$$

$$C_{f,2k} = C_{h,2k}, \tag{8.47}$$

$$C_{f,2k},\, C_{f,2k+2} < C_{f,2k+1}, \quad C_{h,2k+1} < C_{h,2k},\ C_{h,2k+2} \qquad k = 0, 1, \dots, \tag{8.48}$$

$$\lim_{k\to\infty} C_{f,2k} = \lim_{k\to\infty} C_{h,2k} = \xi, \tag{8.49}$$

$$\lim_{k\to\infty} C_{f,2k+1} = C_f, \quad \lim_{k\to\infty} C_{h,2k+1} = C_h. \tag{8.50}$$

Put $c = C_{f,0} = C_{h,0}$. Let $d > 0$, and set $\underline{f} = \underline{h} = q(r)$ on $(1, \infty)$, where

$$q(r) = r - (1 - \frac{c}{2})(r^2 + d)^{\frac{1}{2}} - 1 + (1 - \frac{c}{2})(1 + d)^{\frac{1}{2}} + c. \tag{8.51}$$

Note that $q(1) = c$, $q'(1) < 1$, $q'' < 0$. In particular, the corresponding warped product metric on $(1, \infty)$ has positive sectional curvature. Moreover, by taking d sufficiently large, we can make $q'(1)$ as close to 1 as we like. We choose d large enough so that the choices of $\epsilon_{1,1}, \epsilon_{2,1}$ below are sufficiently small (as in Lemma 8.28.)

Let $r_0 = \infty$. We will determine a sequence $1 = r_1 > r_2 > \cdots$, and the restrictions of the functions, $\underline{f}$, $\underline{h}$, to the interval, $(r_{j+1}, r_j]$, by applying Lemma 8.28 inductively, where at this stage, we always choose $t_1, t_2 = 0$. For clarity, when applying Lemma 8.28 to determine the value, r_{j+1}, we will add the subscript, j, to the functions and constants which appear in that lemma.

Specifically, if r_j has already been specified, we determine $\epsilon_{1,j}, \epsilon_{2,j}$ as in (8.56) and apply Lemma 8.28, with $b_j = r_j$ and with remaining data as given below, in order to determine $a_j, \eta_{1,j}, \eta_{2,j}$. We then set $r_{j+1} = a_j$. Thus, in general, $r_{j+1} = a_j = b_{j+1}$.

For $j = 1$, we choose $\epsilon_{1,1}, \epsilon_{2,1}$, such that if for the data as in (8.54), (8.55), we determine $u_{1,1}, u_{2,1}$ as above and set

$$\underline{f} = u_{2,1}, \qquad \underline{h} = u_{1,1} \qquad (\text{on } \ (r_2, 1]), \tag{8.52}$$

then

$$\lim_{r\to 1^-} \underline{f}' = \lim_{r\to 1^+} \underline{f}', \qquad \lim_{r\to 1^-} \underline{h}' = \lim_{r\to 1^+} \underline{h}'. \tag{8.53}$$

For $j = 2k$, $k = 1, \dots$, we use $t_{1,2k}, t_{2,2k} = 0$,

$$C_{1,2k} = C_{f,2k} \left(= \tilde{C}_{2,2k-1}\right), \qquad C_{2,2k} = C_{h,2k} = \left(\tilde{C}_{1,2k-1}\right),$$

$$\tilde{C}_{1,2k} = C_{f,2k+1}, \qquad \tilde{C}_{2,2k} = C_{h,2k+1}, \tag{8.54}$$

For $j = 2k+1$, $k = 0, 1, \ldots$, we use $t_{1,2k+1}, t_{2,2k+1} = 0$,

$$C_{1,2k+1} = C_{h,2k+1} = \left(\tilde{C}_{2,2k}\right), \qquad C_{2,2k+1} = C_{f,2k+1} = \left(\tilde{C}_{1,2k}\right),$$

$$\tilde{C}_{1,2k+1} = C_{h,2k+2}, \qquad \tilde{C}_{2,2k+1} = C_{f,2k+2}. \tag{8.55}$$

Observe that the relations for $j = 2k$, $j = 2k+1$, are consistent with one another (the values in parentheses are redundent; they are provided for the convenience of the reader.)

For all $j > 1$, we determine $\epsilon_{1,j}, \epsilon_{2,j}$ inductively, by setting

$$\epsilon_{1,j} = \eta_{2,j-1}, \qquad \epsilon_{2,j} = \eta_{1,j-1}. \tag{8.56}$$

Thus, we have also, $\beta_j = \alpha_{j-1}$, $\lambda_j = \chi_{j-1}$.

Finally, we put

$$\underline{f} = u_{1,2k}, \qquad \underline{h} = u_{2,2k} \qquad (\text{on } (r_{2k+1}, r_{2k}]), \tag{8.57}$$

and

$$\underline{f} = u_{2,2k+1}, \qquad \underline{h} = u_{1,2k+1}. \qquad (\text{on } (r_{2k+2}, r_{2k+1}]). \tag{8.58}$$

By (8.29), (8.31), (8.56), we have for all j,

$$\Delta_j(\underline{f}) \stackrel{\text{def}}{=} \underline{f}(r_j) - \lim_{r \to r_j^+} \underline{f} = (-1)^j \frac{1}{4} \left(\eta_{j-1,j} + \eta_{j,j-1}\right) C_{f,j} r_j,$$

$$\Delta_j(\underline{h}) \stackrel{\text{def}}{=} \underline{h}(r_j) - \lim_{r \to r_j^+} \underline{h} = (-1)^{j+1} \frac{1}{4} \left(\eta_{j,j} + \eta_{j-1,j-1}\right) C_{h,j} r_j, \tag{8.59}$$

where the first subscript in $\eta_{\cdot,\cdot}$ is to be take mod 2.

By (8.29)–(8.32) and (8.56), we have for all j,

$$\lim_{r \to r_j^-} \underline{f}' = \lim_{r \to r_j^+} \underline{f}', \qquad \lim_{r \to r_j^-} \underline{h}' = \lim_{r \to r_j^+} \underline{h}'. \tag{8.60}$$

Also, by (8.33),

$$-\left(\frac{\underline{f}''}{\underline{f}} + 2\frac{\underline{h}''}{\underline{h}}\right) > 0 \qquad (r \neq r_j, \frac{1}{2} r_j). \tag{8.61}$$

ii) We define nonnegative functions, $\tilde{f}_j$, $\tilde{h}_j$ on $[0, r_j]$ which are strictly positive on $(0, r_j]$, as follows. Let G_C be a smooth nonnegative function on $[0, \infty)$, such that $G_C > 0$ on $(0, \infty)$, $G_C(0) = 0$, $G'_C(0) = 1$, $G''_C(0) = 0$. Assume in addition that $-2C^{-1} < G''_C < 0$ on $(0.\infty)$ and that for some fixed function, G, on $[1, \infty)$, with say $|G'(r) - 1| < e^{-r}$, $|G'(r)| < e^{-r}$, we have

$$G_C(r) = CG(r) \qquad (\text{on } [1, \infty)). \tag{8.62}$$

Now put

$$\tilde{f}_j = r_j^2 G_{C_{f,j}}(r_j^{-2} r) \qquad (\text{on } (0, \frac{1}{2} r_j],$$

$$\tilde{f}_j = r_j^2 G_{C_{f,j}}(r_j^{-2} r) + \frac{C_{f,j}(1 + (-1)^j \eta_{j+1,j}) - G'_{C_{f,j}}(r_j^{-1})}{r_j}(r - \frac{1}{2} r_j)^2,$$
$$(\text{on } (\frac{1}{2} r_j, r_j)),$$

$$\tilde{h}_j = r_j^2 G_{C_{h,j}}(r_j^{-2} r) \qquad (\text{on } ((0, \frac{1}{2} r_j]),$$

$$\tilde{h}_j = r_j^2 G_{C_{h,j}}(r_j^{-2} r) + \frac{C_{h,j}(1 + (-1)^{j+1}) \eta_{j,j} - G'_{C_{h,j}}(r_j^{-1})}{r_j}(r - \frac{1}{2} r_j)^2, \tag{8.63}$$
$$(\text{on } (\frac{1}{2} r_j, r_j)),$$

In (8.63), the first subscript of $\eta_{\cdot,j}$ is to be taken mod 2.

Note that $\tilde{f}'_j(r_j) = \underline{f}(r_j)$, $\tilde{h}'_j(r_j) = \underline{h}(r_j)$.

Since $0 \leq f' < 1, 0 \leq h' < 1$, it follows easily from (8.13) that for the associated doubly warped product metric, the Ricci curvature in directions tangent to the cross-section is positive and bounded away from zero, independent of r_j. Also, from the properties of G, together with (8.23),(8.62), (8.63), we get

$$-\left(\frac{\tilde{f}''_j}{\tilde{f}_j} + 2\frac{\tilde{h}''_j}{\tilde{h}_j}\right) > 0 \qquad (r \neq \frac{1}{2} r_j), \tag{8.64}$$

which implies that the Ricci curvature in the radial direction is positive (though not, of course, uniformly bounded above.)

iii) Note that as a consequence of (8.42), (8.47), (8.49) (which relect the volume cone property for tangent cones of limit spaces with Ricci curvature bounded below) together with Remark 8.40, it follows that $r_{j+1}/r_j \to 0$ (equivalently, $\eta_{\cdot,j} \to 0$.) Indeed, this property is consistent with the fact that for such spaces, tangent cones are metric cones. In particular, we can certainly assume

$$|\Delta_{j+1}(\underline{f})| < \frac{1}{4}|\Delta_j(\underline{f})|, \qquad |\Delta_{j+1}(\underline{h})| < \frac{1}{4}|\Delta_j(\underline{h})|. \tag{8.65}$$

Indeed, since $r_{j+1} < \frac{1}{2}r_j$, this is virtually automatic. Thus, if we set

$$\hat{f}_j|[0, r_j] = \tilde{f}_j, \tag{8.66}$$

and for $i \le j$,

$$\hat{f}_j|(r_i, r_{i-1}] = \underline{f} + \tilde{f}(r_j) - \lim_{r \to r_j^+} \underline{f}(r_j) + \sum_{\ell=i}^{j-1} \Delta_\ell(\underline{f}),$$

$$\hat{h}_j|(r_i, r_{i-1}] = \underline{h} + \tilde{h}(r_j) - \lim_{r \to r_j^+} \underline{h}(r_j) + \sum_{\ell=i}^{j-1} \Delta_\ell(\underline{h}), \tag{8.67}$$

then $\hat{f}_j$, $\hat{h}_j$ are of class C^1. Similarly, we can assume that the properties corresponding to (8.17)–(8.19) hold. By Lemma 8.28 (where now, we no longer have $t_1, t_2 = 0$) we get

$$-\left(\frac{\hat{f}_j''}{\hat{f}_j} + 2\frac{\hat{h}_j''}{\hat{h}_j}\right) > 0 \qquad (r \ne r_j, \frac{1}{2}r_j). \tag{8.68}$$

iv) We can remove the jump discontinuities in the functions, $\hat{f}_j''$, $\hat{h}_j''$, by modifying them by linear interpolation, in arbitrarily small neighborhoods of the points, $\{r_i\}$ (where $i \le j$) and $\{\frac{1}{2}r_i\}$ (where $i < j$). Call the resulting functions f_j'', h_j'', and let the corresponding functions, f_j, h_j, be obtained by integration with respect to r, subject to the conditions, $f_j(0) = h_j(0) = 0$, $f_j'(0) = h_j'(0) = 1$. The modifications in the second derivatives can be performed on intervals whose size decreases rapidly enough to ensure that f_j, h_j satisfy (8.17)–(8.19) and for all $r \in (0, \infty)$, we have

$$-\left(\frac{f_j''}{f_j} + 2\frac{h_j''}{h_j}\right) > 0. \tag{8.69}$$

Moreover, $\lim_{j\to\infty} f_j = f$, $\lim_{j\to\infty} h_j = h$, where f, h satisfy (8.17)–(8.19) and for all $r \in (0,\infty)$, we have

$$-\left(\frac{f''}{f} + 2\frac{h''}{h}\right) > 0. \tag{8.70}$$

Now, by considering the doubly warped product spaces corresponding to $\{(f_j, h_j)\}$ and to (f, h), we obtain the limit space, Y^4, with the properties which were asserted.

When the volume is not subjected to an a priori lower bound, the situation is much less constrained than when such a bound is in force. As a consequence, examples illustrating various new phenomena are not difficult to come by.

Let $S^3 \to S^7 \xrightarrow{\pi} S^4$ denote the Hopf fibration. If S^3, S^7, S^4 carry the metrics, $g^{S^3}, g^{S^7}, \frac{1}{4}g^{S^4}$, then the map, π, is a Riemannian submersion with totally geodesic fibres. Put $g^{S^7} = k_1 + k_2$, where $k_2 = \pi^*(\frac{1}{4}g^{S^4})$.

Recall that the metric, $g_\delta^{S^7} = \delta^2 k_1 + k_2$, has uniformly positive Ricci curvature, for all $0 < \delta \le 1$. To see this, view the fibration, $S^3 \to S^7 \xrightarrow{\pi} S^4$, as a principle bundle. Then we can obtain a Riemannian submersion, $S^7 \times S^3 \to S^7$, by taking the quotient map associated to the diagonal action of the Lie group, S^3, on the product, $S^7 \times S^3$. Here, S^7 has metric $g_\delta^{S^7}$ and $S^7 \times S^3$ has metric $g^{S^7} + \tau^2 g^{S^3}$, where $\tau = \tau(\delta)$ and $\tau(\delta) \to 0$ as $\delta \to 0$. Since the Ricci curvature of $g^{S^7} + \tau^2 g^{S^3}$ is uniformly positive, the above mentioned fact is a direct consequence of O'Neill's formula; see [18].

More precisely, as $\delta \to 0$, the sectional curvatures of planes which are contained in the fibre, S^3, are equal to δ^{-2}, the curvatures of horizontal planes approach 4 and the curvatures of planes spanned by a horizontal vector and a vertical vector approach 0.

We will now give an example of a collapsed limit space which is actually a smooth Riemannian manifold such that it, together with its renormalized limit measure, can be regarded as having positive Ricci curvature in a generalized sense, although not in the classical sense.

Example 8.71 (Smooth limit spaces). Let h be a smooth positive function on $[0,1]$, such that

$$\left|\frac{\overline{h}'}{\overline{h}}\right| < \frac{\delta}{r}, \tag{8.72}$$

$$\left|\frac{\overline{h}''}{h}\right| < \frac{\delta}{r^2}. \tag{8.73}$$

Here, δ is so small that the doubly warped product metric in (8.75) below has positive Ricci curvature in directions tangent to the factor, S^7.

Of course, we can also arrange that $-\frac{h''}{h}$ is somewhere negative. For such values of r, the Ricci curvature of the metric

$$dr^2 + \frac{1}{4}h^2 g^{S^4}, \tag{8.74}$$

is negative in the radial direction.

Now choose $0 < \eta < 1$ such that the doubly warped product metric on $\mathbb{R}^8$,

$$dr^2 + (\chi r^{1-\eta})^2 k_1 + h^2 k_2, \tag{8.75}$$

has strictly positive Ricci curvature for all $r > 0$, provided χ is sufficiently small.

As in Example 8.41 the metric in (8.75) can be truncated at $r = r_\chi$, where

$$\chi r_\chi^{1-\eta} = h(r_\chi), \tag{8.76}$$

and smoothed to produce a metric of positive Ricci curvature on the disk, D^8. This metric can be doubled and smoothed at the equator to give a metric, g_χ, of positive Ricci curvature on S^8. Then as $\chi \to 0$, we have $(S^8, g_\chi) \to (S^5, dr^2 + h^2(\frac{1}{4}g^{S^4}))$.

Although the limit metric in (8.75) does *not* have positive Ricci curvature in the usual sense, it together with its renormalized limit measure, ν, *does* have this property in a generalized sense; compare Appendix 2. The point here is that various properties of manifolds with a definite lower Ricci curvature bound (e.g. existence of ϵ-dense sets of bounded cardinality, Abresch-Gromoll inequality, splitting theorem) remain valid for Gromov-Hausdorff limit spaces and their renormalized limit measures, even in the collapsed case.

The following example shows that objects which are shaped like horns can appear as collapsed limit spaces. These objects contain infinitely many distinct geodesics which are mutually tangent; for Alexandrov spaces, this does not occur.

Example 8.77 (Mutually tangent geodesics, lower dimensional tangent cones, horns). If in the previous example, we replace the function h by $r^{1+\epsilon}$, where

$$-3\eta(1-\eta)+4\epsilon(1+\epsilon)<0, \tag{8.78}$$

then we obtain the limit metric,

$$dr^2+(\frac{1}{2}r^{1+\epsilon})^2 g^{S^4}. \tag{8.79}$$

A warped product metric of this type is called a *metric horn*. For the corresponding (5-dimensional) limit space, Y^5, the tangent cone at the origin is a half line. Thus, geodesics emanating from the origin are all mutually tangent and (0.5), for $\mu=\mathcal{H}^5, k=5$, fails to hold; compare Section 1.

Consider on the other hand, the space, Y_j^5, obtained by adjoining (at the origin) a segment, $[-j,0]$, of the negative real axis, to the space, Y^5, above. Given j sufficiently large and $\delta>0$ sufficiently small, it follows from the quantitative generalization of the splitting theorem, Theorem 6.64 of [15], that the space, Y_j^5, can *not* arise as a Gromov-Hausdorff limit of a sequence, $\{M_i^n\}$, with $Ric_{M_i^n}\geq-(n-1)\delta$.

In fact, one can show that *no* space, Y_j^5, can arise as a limit of any sequence, M_i^n, with $Ric_{M_i^n}\geq(n-1)H>-\infty$. This will be discussed in [12].

Next we construct an example of a limit space, Y^5, with an isolated singular point, for which every metric cone, $dr^2+\frac{1}{4}\lambda^2r^2g^{S^4}$, where $0<\lambda\leq 1$, occurs as a tangent cone. Hence, the half line occurs as a tangent cone as well. The number, $\frac{1}{4}$, which occurs in this metric can be increased, but it can not be replaced by a number larger than $\frac{1}{2}$; see (8.85).

Example 8.80 (Points with nonunique tangent cones of different dimensions). Let $\{C_{h,j}\}$ satisfy

$$C_{h,2i+2}>C_{h,2i}, \tag{8.81}$$

$$\lim_{i\to\infty}C_{h,2i}=1, \tag{8.82}$$

$$C_{h,2i+1}<C_{h,2i-1}, \tag{8.83}$$

$$\lim_{i\to\infty} C_{h,2i-1} = 0. \tag{8.84}$$

From the discussion preceding Example 8.71, it follows that

$$Ric^\delta \geq 12 + o(\delta). \tag{8.85}$$

Here Ric^δ denotes the Ricci tensor of the metric, $g_\delta^{S^7}$. In particular, $Ric^\delta > 6$, the Ricci curvature of S^7. This guarentees that the Ricci curvatures of the metrics constructed below are nonnegative (and thus, bounded away from $-\infty$) in directions tangent to the cross-section. Note that given $\lambda, \eta > 0$, $C_{h,2i} < 1 - \lambda$ and $0 < \epsilon$, $j^{-2} < \delta(\lambda, \eta)$, the metric,

$$(j^{-1}r^{1-\eta})^2 k_1 + (C_{h,2i} b^{-\epsilon} r^{1+\epsilon})^2 k_2, \tag{8.86}$$

has positive Ricci curvature, in directions tangent to the cross-section if

$$j^{-1} r^{-\eta} < C_{h,2i} \left(\frac{r}{b}\right)^\epsilon. \tag{8.87}$$

Moreover, the metric

$$(j^{-1}r^{1-\eta})^2 k_1 + (C_{h,2i+1} b^{\epsilon} r^{1-\epsilon})^2 k_2, \tag{8.88}$$

has positive Ricci curvature in directions tangent to the cross-section if

$$j^{-1} r^{-\eta} < C_{h,2i+1} \left(\frac{b}{r}\right)^\epsilon. \tag{8.89}$$

In what follows, we will consider a sequence of metrics indexed by j.

As in Example 8.41, we can find sequences, $\{\epsilon_\ell\}, \{b_\ell\}$, such that the sequence of metrics given by

$$\begin{aligned} &dr^2 + (j^{-1}r^{1-\eta})^2 k_1 + (C_{h,2i} b_{2i}^{\epsilon_{2i}}\ r^{1+\epsilon_{2i}})^2 k_2 \\ &\qquad\qquad\qquad\qquad (\text{on } (b_{2i+1}, b_{2i}]), \end{aligned} \tag{8.90}$$

$$\begin{aligned} &dr^2 + (j^{-1}r^{1-\eta})^2 k_1 + (C_{h,2i+1} b_{2i+1}^{\epsilon_{2i+1}}\ r^{1-\epsilon_{2i+1}})^2 k_2 \\ &\qquad\qquad\qquad\qquad (\text{on } (b_{2i+2}, b_{2i+1}]), \end{aligned} \tag{8.91}$$

has the following property. Define r_j by

$$j^{-1} r_j^{1-\eta} = \begin{cases} C_{h,2i} b_{2i}^{\epsilon_{2i}}\ r_j^{1-\epsilon_{2i}} & j = 2i, \\ C_{h,2i+1} b_{2i+1}^{-\epsilon_{2i+1}} r_j^{1-\epsilon_{2i+1}} & j = 2i+1. \end{cases} \tag{8.92}$$

Then the restriction of the metric in (8.90), (8.91) to the interval, $[r_j, 1]$, can be smoothed to a metric of positive Ricci curvature.

By considering the limit as $j \to \infty$ of the above sequence of truncations and smoothings, we obtain a limit space, Y^5, with an isolated singular point, for which every metric cone, $dr^2 + \frac{1}{4}\lambda^2 g^{S^4}$, where $0 < \lambda \leq 1$, occurs as a tangent cone. Hence, the half line occurs as a tangent cone as well.

Remark 8.93. With more work one can actually construct limit spaces for which the points with nonunique tangent cones are dense.

Example 8.94 (Complete manifolds). By modifying the construction of the previous example in an obvious fashion, we obtain a complete metric of positive Ricci curvature on $\mathbb{R}^8$, for which any cone $dr^2 + \frac{1}{2}\lambda^2 r^2 g^{S^4}$, $0 < \lambda \leq 1$, occurs as a tangent cone at infinity. Hence the half line occurs as a tangent cone as well.

Example 8.95 (Tangent cones which are not metric cones). Let h be a positive function such that say,

$$|h - \frac{1}{2}| \leq \delta, \tag{8.96}$$

$$|h'|, |h''|, |h'''| < \delta. \tag{8.97}$$

As in our previous examples it is clear that we can construct a metric of positive Ricci curvature on $\mathbb{R}^8$, which, for $r \geq 1$, is of the form

$$dr^2 + (r^{1-\eta})^2 k_1 + (rh(\log r))^2 k_2. \tag{8.98}$$

These metrics have tangent cones at infinity of the form

$$dr^2 + \frac{1}{4}(rh_\infty)^2 g^{S^4}. \tag{8.99}$$

If, for instance, h is periodic, then h_∞ is some translate of h itself. But unless h is constant, *no* tangent cone is a metric cone.

Note that $\mathbb{R}^5$ equipped with metric in (8.99) is itself a limit space, Y^5, with an isolated singular point at which *no* tangent cone is a metric cone.

Example 8.100 (Topologically singular spaces). In the previous example the limit space was topologically nonsingular. However, one can easily produce singular limit spaces with analogous properties, by starting with a suitable ALE space in place of $\mathbb{R}^8$.

Let $\{(M_i^{n_i}, g_i)\}$ be a sequence of manifolds of positive Ricci curvature converging to a finite dimensional limit space. Of course, it may happen that $n_i \to \infty$. In this case (not surprisingly) so much information can be lost in the limit that the limit space cannot be legitimately regarded as a generalized space of nonnegative Ricci curvature. It follows in particular that the estimates in the "almost splitting" theorem (Theorem 6.62 of [15]) cannot be made uniform in the dimension, n, of M^n.

The following example illustrates what can go wrong.

Example 8.101 (dimension $\to \infty$). Consider a warped product space, $\mathbb{R} \times_f S^2$, where S^2 has metric g^{S^2} and

$$f|(-\infty, -1] \equiv \frac{1}{2}, \tag{8.102}$$

$$f|[1, \infty] \equiv 1, \tag{8.103}$$

$$\left|\frac{f'}{f}\right| < 1. \tag{8.104}$$

Let $\{h_j\}$ be a sequence of functions, $h_j : (-j, j) \to \mathbb{R}_+$, such that

$$\lim_{r \to -j} h_j = \lim_{r \to j} h_j = 0, \tag{8.105}$$

$$-\frac{h_j''}{h_j} > 0, \tag{8.106}$$

$$\lim_{j \to \infty} h_j = 0, \qquad \text{(uniformly)} \tag{8.107}$$

and $dr^2 + h_j^2 g^{S^{n_j}}$, defines a smooth metric of strictly positive curvature on S^{n_j+1}.

For n_j sufficiently large, the doubly warped product metric

$$g_j = dr^2 + h_j^2 g^{S^{n_j}} + f^2 g^{S^2}, \tag{8.108}$$

on $S^{n_j+1} \times S^2$, has positive Ricci curvature. As $j \to \infty$, the sequence, $\{(S^{n_j+1} \times S^2, g_j)\}$, converges in the pointed Gromov-Hausdorff sense to the smooth warped product space, $\mathbb{R} \times_f S^2$. Although this space contains geodesic lines, it does *not* split isometrically and thus, cannot be legitimately considered to have nonnegative Ricci curvature in *any* generalized sense.

Appendix 1: Reifenberg's method and some consequences

In this appendix, we formulate an intrinsic version of Reifenberg's theorem, [50], and draw a number of consequences. We thank Bruce Kleiner for bringing Reifenberg's theorem to our attention in connection with the results of [26] and Fred Almgren for some helpful conversations concerning it. We are also grateful to Stephen Semmes, for discussions and for pointing out his paper, [52], and book, [28], which deal with situations closely related to Reifenberg's.

Let $(\mathcal{R}_k)_{\varepsilon,r}$ denote the set of points such that for some $u > r$, (4.6) holds for *all* $s \in (0, u]$ and $\mathbb{R}^k \times X = \mathbb{R}^k$. Thus, $(\mathcal{R}_k)_\varepsilon = \cup_r (\mathcal{R}_k)_{\varepsilon,r}$. Let $0 \in \mathbb{R}^n$. Let the notation, Ψ, be as in previous sections.

Theorem A.1.1. *There exists $\varepsilon(n) > 0$, with the following property. Let (Z, ρ) be a complete metric space such that for some $z \in Z$, and $\varepsilon \le \varepsilon(n)$, we have $z_1 \in (\mathcal{R}_n)_{\varepsilon,r}$ for all $z_1 \in B_1(z)$ and $r \le 1 - \overline{z_1, z}$. Then there exists a topological imbedding, $F : B_1(0) \to B_1(z)$, such $F(B_1(0)) \supset B_{1-\Psi}(z)$, where $\Psi = \Psi(\varepsilon|n)$. Moreover, the maps F, F^{-1} are Hölder continuous, with exponent, $\alpha = 1 - \Psi$. If, in addition, Z is an n-dimensional Riemannian manifold, then F can be taken to be a smooth imbedding.*

Next, we will give some global counterparts of Theorem A.1.1.

Let $\mathcal{M}(n, \varepsilon, r)$ denote the collection of isometry classes of complete separable metric spaces, (Z, ρ), such that $z \in (\mathcal{R}_n)_{\varepsilon,r}$, for all $z \in Z$. Let $[Z]$ denote the isometry class of Z.

Theorem A.1.2. *There exists $\varepsilon(n) > 0$, such that if $[Z] \in \mathcal{M}(n, \varepsilon, r)$, for $\varepsilon < \varepsilon(n)$, then there exists a smooth Riemannian manifold, (W^n, g) and a homeomorphism, $F : W^n \to Z$, such that F, F^{-1} are Hölder continuous, with exponent, $\alpha = 1 - \Psi$, where $\Psi = \Psi(\varepsilon|n)$.*

Let $[Z_1], [Z_2] \in \mathcal{M}(n, \varepsilon, r)$, where $\varepsilon < \varepsilon(n)$. Let (W_1^n, g_1), (W_2^n, g_2) denote the Riemannian manifolds whose existence is asserted in Theorem A.1.2.

Theorem A.1.3. *The number, $\varepsilon(n) > 0$, can be chosen such that if $d_{GH}(Z_1, Z_2) < \varepsilon(n)$, then we can choose $(W_1^n, g_1) = (W_2^n, g_2)$. Moreover, if Z_1, Z_2 are smooth n-dimensional Riemannian manifolds, then for this $W_1^n = W_2^n$, the maps F_1, F_2 can be chosen to be diffeomorphisms. Thus, Z_1 and Z_2 are diffeomorphic in this case.*

Let $\mathcal{M}(n, \varepsilon, r, d)$ denote the subset of $\mathcal{M}(n, \varepsilon, r)$ such that $\mathrm{diam}(Z) \le d$. As a direct consequence of Theorem A.1.3 we obtain:

Theorem A.1.4. *Fix $0 < \varepsilon \leq \varepsilon(n)$ and $r, d > 0$.*

i) There exist at most $N(n, r^{-1}d) < \infty$ bi-Hölder equivalence classes of metric spaces, Z, with $[Z] \in \mathcal{M}(n, \varepsilon, r, d)$.

ii) There exist at most $N(n, r^{-1}d) < \infty$ diffeomorphism classes of n-dimensional Riemannian manifolds, Z^n, with $[Z^n] \in \mathcal{M}(n, \varepsilon, r, d)$.

Apart from some simple (and inessential) technicalities (concerning points near $\partial B_1(z)$) the proof of Theorem A.1.1 is identical to that of Theorem A.1.2. Since the additional statements in Theorems A.1.3, A.1.4 are direct consequences of the proof of Theorem A.1.2, we will only prove Theorem A.1.2.

Before proving Theorem A.1.2, we will collect some consequences of Theorems A.1.1–A.1.3 for Ricci curvature. For these, we need the following result which summarizes some of the conjectures of Anderson–Cheeger that were proved in [26]. (Recall that the relevant sets in parts ii), iii) below were defined prior to Theorem 4.7.)

Theorem A.1.5 ([26]). *For all $\varepsilon > 0$, there exists $r = r(n, \varepsilon)$, $\delta = \delta(n, \varepsilon) > 0$, with the following properties: Let*

$$Ric_{M^n} \geq -(n-1),\ p \in M^n$$

and $r_1 \leq r$.

i) If

$$\mathrm{Vol}(B_{r_1}(p)) \geq (1-\delta)\,\mathrm{Vol}(B_{r_1}(0)), \tag{A.1.6}$$

then $p \in (\mathcal{R}_n)_{\varepsilon, r_1}$.

ii) If $p \in (\mathcal{WR}_n)_{\delta, r_1}$, then

$$\mathrm{Vol}(B_{r_1}(p)) \geq (1-\varepsilon)\,\mathrm{Vol}(B_{r_1}(0)). \tag{A.1.7}$$

iii) If $p \in (\mathcal{WR}_n)_{\delta, r_1}$, then $q \in (\mathcal{R}_n)_{\varepsilon, s}$, for all $q \in B_{r_1}(p)$, $s \leq (1-\varepsilon)r_1 - \overline{q, p}$.

By combining Theorems A.1.1 and A.1.5, we get the following two theorems which sharpen Perelman's theorem; [46].

As usual, let (Y^n, y) be the pointed Gromov–Hausdorff limit of a sequence, $\{(M_i^n, p_i)\}$, satisfying (1.1), (5.1).

Theorem A.1.8. *For all $\varepsilon > 0$, there exists $r = r(n, \varepsilon)$, $\delta = \delta(n, \varepsilon) > 0$, such that if for some $r_1 < r$, either $y \in (\mathcal{WR}_n)_{\delta, r_1}$ or*

$\mathcal{H}^n(B_{r_1}(y)) \geq (1-\delta)\,\mathrm{Vol}(B_{r_1}(0))$, *then there exists a topological imbedding* $F : B_{r_1}(0) \to B_{r_1}(y)$, *such that* $F(B_{r_1}(0)) \supset B_{(1-\Psi)r_1}(y)$, *where* $\Psi = \Psi(\varepsilon|n)$. *The maps,* F, F^{-1} *are Hölder continuous, with exponent,* $\alpha = 1 - \Psi$.

Theorem A.1.9. *Let the assumptions be as in Theorem A.1.8. If* $Y^n = M^n$ *is a smooth Riemannian manifold satisfying* $Ric_{M^n} \geq -(n-1)$, *then the map,* F, *can be taken to be a smooth imbedding.*

By arguing as in [24], [25] (see also [15, Section 5]) but letting Theorem A.1.8 play the role of Perelman's theorem, we obtain the following *differentiable* sphere theorem for Ricci curvature and volume.

Theorem A.1.10. *There exists* $\delta(n) > 0$, *such that if* $Ric_{M^n} \geq n-1$, $\mathrm{Vol}(M^n) \geq (1-\delta(n))\mathrm{Vol}(S^n)$, *then* M^n *is diffeomorphic to* S^n.

Similarly, we have the following noncompact analog of Theorem A.1.10, the proof of which is a minor variation on that of Theorem A.1.2; see Remark A.1.47. As in (0.5) let $V_{n,0}(1)$ denote the volume of the unit ball in $\mathbf{R}^n$.

Theorem A.1.11. *There exists* $\delta(n) > 0$ *such that if* $\mathrm{Ric}_{M^n} \geq 0$ *and* $\mathrm{Vol}\,(B_r(p)) \geq (1-\delta(n))V_{n,0}(1)r^n$, *for all* $p \in M^n, r > 0$, *then* M^n *is diffeomorphic to* $\mathbf{R}^n$.

We also obtain a sharpening of the result of [26] concerning one of the conjectures of Anderson–Cheeger as well as a sharpening of the statement of Gromov's conjecture proved in [26].

Theorem A.1.12. *Let the compact smooth Riemannian manifold,* M^n, *be the Gromov–Hausdorff limit of a sequence,* $\{M_i^n\}$, *satisfying (1.1). Then* M_i^n *is diffeomorphic to* M^n, *for all* i *sufficiently large.*

Theorem A.1.13. *There exists* $\delta(n) > 0$ *such that if* M^n *is a compact* n*-dimensional Riemannian manifold, with* $Ric_{M^n}(\mathrm{diam}(M^n))^2 \geq -\delta(n)$ *and* $b_1(M^n) = n$, *then* M^n *is diffeomorphic to the torus* T^n.

After introducing some notation, we will proceed to the proof of Theorem A.1.2. Let $\mathcal{J} : A \to \mathbb{R}^n$, where $A \subset \mathbb{R}^n$. For some suitable sequence of positive constants, $a_0, a_1, \ldots$, and $t > 0$, write $|\mathcal{J}|_{C^\infty,t} \leq c$, if $|\partial^\beta \mathcal{J}| \leq a_{|\beta|}t^{1-|\beta|}c$, for every multi-index β.

Proof of Theorem A.1.2. By scaling, with no loss of generality, we can assume $r = 40$.

Let $\Psi = \Psi(\varepsilon|n)$, where the particular function with $\lim_{\varepsilon\to 0} \Psi(\varepsilon|n) = 0$ might change from line to line.

We will show that there exists a sequence, $\{(W_i^n, \rho_i)\}$, where W_i^n is a smooth manifold and ρ_i is a nonnegative symmetric function on $W_i^n \times W_i^n$, which vanishes on the diagonal, such that there are sequences of diffeomorphisms, $h_i : W_i^n \to W_{i+1}^n$, and (not necessarily continuous) maps, $f_i : W_i^n \to Z$, such that for all i, the following hold:

i) There exists a Riemannian metric, g_i, on W_i^n, with associated distance function d_i, such that $\rho_i(w_1^i, w_2^i) \leq 2^{-i}$ implies $\rho_i(w_1^i, w_2^i) = d_i(w_1^i, w_2^i)$.

ii) The maps, h_i, satisfy

$$2^{-\Psi}\rho_i \leq \rho_{i+1} \circ h_i \leq 2^{\Psi}\rho_i. \tag{A.1.14}$$

iii) The maps, f_i, satisfy

$$|\rho \circ f_i - \rho_i| \leq \Psi 2^{-i}. \tag{A.1.15}$$

iv) The maps, $f_{i+1} \circ h_i$, f_i, satisfy,

$$\rho(f_{i+1} \circ h_i, f_i) \leq \Psi 2^{-i}. \tag{A.1.16}$$

v) The range of the map, f_i, is $\Psi(\epsilon)$-dense.

vi) If moreover, (Z, ρ) is an n-dimensional Riemannian manifold, then for i sufficiently large (possibly depending on (Z, ρ)) the map, f_i, can be taken to be a diffeomorphism.

Claim. It suffices to construct $\{(W_i^n, \rho_i)\}, \{h_i\}, \{f_i\}$ satisfying i)–vi).

Proof of Claim. We begin by observing that if (Z, ρ) is an n-dimensional Riemannian manifold, then

$$F_i = f_i \circ h_{i-1} \circ \cdots \circ h_0 \tag{A.1.17}$$

is a diffeomorphism, $F_i : W_0^n \to Z$, for i sufficiently large.

Let $w_1^0, w_2^0 \in W_0^n$, with $\rho(w_1^0, w_2^0) \leq 1$. Put

$$s_0 = \rho_0(w_1^0, w_2^0), \tag{A.1.18}$$

$$s_i = \rho_i(h_{i-1} \circ \cdots \circ h_0(w_1^0), h_{i-1} \circ \cdots \circ h_0(w_2^0)). \tag{A.1.19}$$

By iii), iv) we have

$$s_i - \Psi 2^{-i} \le s_{i+1} \le s_i + \Psi 2^{-i}. \tag{A.1.20}$$

In particular, $\lim_{i\to\infty} s_i := s_\infty := \rho_\infty(w_1^0, w_2^0)$ exists.

Since Z is complete, it follows from iv) that $\lim_{i\to\infty} F_i := F$ exists and from iii), we get

$$\rho \circ F = \rho_\infty. \tag{A.1.21}$$

Moreover, by v), F_∞ is surjective. Thus, it will suffice to show that ρ_∞ is a metric on W_0^n and that this metric is bi-Hölder equivalent to d_0. By i), it suffices to compare ρ_∞ to ρ_0.

By ii), we have

$$2^{-\Psi} s_i \le s_{i+1} \le 2^{\Psi} s_i. \tag{A.1.22}$$

If we fix j and use (A.1.22) for $j \le i$ and (A.1.20) for $j > i$, we get (for all j)

$$2^{-\Psi j} s_0 - \Psi 2^{-j} \le s_\infty \le 2^{\Psi j} s_0 + \Psi 2^{-j}. \tag{A.1.23}$$

Clearly, we can chose j such that

$$2^{\Psi j} s_0 \le 2^{-j}, \tag{A.1.24}$$

$$2^{\Psi (j+1)} s_0 \ge 2^{-(j+1)}. \tag{A.1.25}$$

Then (A.1.25) implies

$$2 s_0^{\frac{1}{1+\Psi}} \ge 2^{-j}, \tag{A.1.26}$$

which together with (A.1.23), (A.1.24) gives

$$s_\infty \le 2(1+\Psi) s_0^{\frac{1}{1+\Psi}}. \tag{A.1.27}$$

Similarly, choosing j such that

$$2^{-\Psi (j-1)} s_0 \le 2^{-(j-1)}, \tag{A.1.28}$$

$$2^{-\Psi j} s_0 > 2^{-j}, \tag{A.1.29}$$

we find from (A.1.28),

$$(A.1.30) \qquad s_0^{\frac{1}{1-\Psi}} \geq 2^{-j},$$

and from (A.1.23), (A.1.28), (A.1.30),

$$(A.1.31) \qquad \frac{1}{2}(1-\Psi)s_0^{\frac{1}{1-\Psi}} \leq s_\infty.$$

By (A.1.27), (A.1.31), ρ_∞ is bi-Hölder equivalent to d_0 with exponent, $\alpha = 1-\Psi$.

Before constructing the data in i)–v) above, we recall a well-known fact about sets of points in Euclidean space.

Let $Q = \{q_i\}$ be a minimal η-dense set in $\overline{B_R(0)} \subset \mathbb{R}^n$, $R \geq \eta$. Let $Q_1 \subset Q$ be a maximal subset such that $\overline{q_{i_1}, q_{i_2}} \geq 20\eta$, for $q_{i_1}, q_{i_2} \in Q_1$, $i_1 \neq i_2$. Similarly, let $Q_2 \subset Q \setminus Q_1$ be a maximal subset such that $\overline{q_{i_1}, q_{i_2}} \geq 20\eta$, for $q_{i_1}, q_{i_2} \in Q_2$, $i_1 \neq i_2$. Then if $Q_3, \ldots, Q_N$ are constructed similarly by induction, we have $N \leq N(n)$. Note that if $q_k \in Q_k$, then for any ℓ, we have $B_{8\eta}(q_k) \cap B_{8\eta}(q_\ell) \neq \emptyset$, for at most one $q_\ell \in Q_\ell$.

Clearly, we can assume that the number, $\varepsilon(n)$, of the hypothesis has been chosen such that if $\{z_i\}$ is minimal η-dense subset set of Z, then property of the preceding paragraph holds, for some possibly different $N = N(n)$.

We now construct the data in i)–v), starting with the construction of the manifolds, W_i^n. These will be obtained by gluing together certain balls, $B_{2\cdot 2^{-i}}(0_{i,j})$, where $0_{i,j}$ is the origin in some copy of Euclidean space, $\mathbb{R}^n_{i,j}$. The gluings are determined by diffeomorphisms, $\hat{I}_{i,j_2,j_1}$ (defined for certain pairs of indices, j_1, j_2). The domain of $\hat{I}_{i,j_2,j_1}$ is an open subset of $B_{2\cdot 2^{-i}}(0_{i,j_1})$, and its range is an open subset of $B_{2\cdot 2^{-i}}(0_{i,j_2})$. The consistency condition, $\hat{I}_{i,j_3,j_1} = \hat{I}_{i,j_3,j_2} \circ \hat{I}_{i,j_2,j_1}$, is required to hold on the intersection of the domains of $\hat{I}_{i,j_3,j_1}$ and $\hat{I}_{i,j_3,j_2} \circ \hat{I}_{i,j_2,j_1}$.

It will be clear in what follows that there is a certain degree of freedom in the choice of the numbers (16, 8, 6, 4, 2) which appear in the construction. All that matters is that certain inequalities between these numbers hold.

We begin by successively choosing finite subsets, $X_0 \subset X_1 \subset \cdots$, such that X_i is a minimal 2^{-i}-dense subset of Z. We write $X_i = \bigcup_{j=1}^{N_i} Q_{i,j}$, where $N_i \leq N$ and the sets, $Q_{i,1}, \ldots, Q_{i,N_i}$, are defined analogously to the sets, Q_j, above.

For all i, j, we choose a copy of Euclidean space, $\mathbb{R}^n_{i,j}$, with origin $0_{i,j}$. For all $x_{i,j} \in X_i$, choose $\varepsilon 2^{-i}$–Gromov-Hausdroff approximations,

$$\alpha_{i,j} : B_{16\cdot 2^{-i}}(0_{i,j}) \longrightarrow B_{16\cdot 2^{-i}}(x_{i,j})$$

and

$$\beta_{i,j} : B_{16\cdot 2^{-i}}(x_{i,j}) \longrightarrow B_{16\cdot 2^{-i}}(0_{i,j}),$$

such that $\overline{\beta_{i,j} \circ \alpha_{i,j},\, Id} \leq \Psi 2^{-i}$ and $\overline{\alpha_{i,j} \circ \beta_{i,j},\, Id} \leq \Psi 2^{-i}$.

The maps $\hat{I}_{i,j_2,j_1}$ will be defined only for pairs of indices j_1, j_2, for which

$$B_{6\cdot 2^{-i}}(x_{i,j_1}) \cap B_{6\cdot 2^{-i}}(x_{i,j_2}) \neq \emptyset. \tag{A.1.32}$$

For such pairs, the intersection, $B_{16\cdot 2^{-i}}(x_{i,j_1}) \cap B_{16\cdot 2^{-i}}(x_{i,j_2})$, has a definite size (it contains a ball of radius $10 \cdot 2^{-i}$).

Since the intersection has a definite size and the maps, α_{i,j_1}, β_{i,j_2}, almost preserve distances, it is clear that there exist isometries, $I_{i,j_2,j_1} : \mathbb{R}^n_{i,j_1} \longrightarrow \mathbb{R}^n_{i,j_2}$, such that

$$\overline{I_{i,j_2,j_1},\, \beta_{i,j_2} \circ \alpha_{i,j_1}} \leq \Psi 2^{-i} \qquad (\text{on say } B_{8\cdot 2^{-i}}(0_{i,j_1})). \tag{A.1.33}$$

Now suppose that for some j_1, j_2, j_3, the intersection

$$B_{6\cdot 2^{-i}}(x_{i,j_1}) \cap B_{6\cdot 2^{-i}}(x_{i,j_2}) \cap B_{6\cdot 2^{-i}}(x_{i,j_3})$$

is nonempty. Then

$$B_{16\cdot 2^{-i}}(x_{i,j_1}) \cap B_{16\cdot 2^{-i}}(x_{i,j_2}) \cap B_{16\cdot 2^{-i}}(x_{i,j_3})$$

has a definite size and from (A.1.33), it is clear that

$$\overline{I_{i,j_3,j_2} \circ I_{i,j_2,j_1},\, I_{i,j_3,j_1}} \leq \Psi 2^{-i} \qquad (\text{on say } B_{8\cdot 2^{-i}}(0_{i,j_1})). \tag{A.1.34}$$

We will define maps, $\widetilde{I}_{i,j_2,j_1}$, by suitably modifying the maps, I_{i,j_2,j_1}, in such a way as to guarentee that for j_1, j_2, j_3 as above, the relation, $\widetilde{I}_{i,j_3,j_2} \circ \widetilde{I}_{i,j_2,j_1} = \widetilde{I}_{i,j_3,j_1}$, holds on an appropriate subset.

A given map, I_{i,j_2,j_1}, may have to be modified more than once (but $< N^3$ times) in the course of producing the final map, $\widetilde{I}_{i,j_2,j_1}$. After the first modification has been performed, we use the notation, $\widetilde{I}_{i,j_2,j_1}$, for the resulting map. Thereafter, we refer to all additional modifications as modifications of the map, $\widetilde{I}_{i,j_2,j_1}$. Since any such map undergoes at most a definite number of such modifications, we can assume that ϵ and hence Ψ, are so small that at every stage of the process, we have

$$(A.1.35) \qquad \left|\widetilde{I}_{i,j_2,j_1} - I_{i,j_2,j_1}\right|_{C^\infty,2^{-i}} \leq \Psi \qquad (\text{on } B_{8\cdot 2^{-i}}(0_{i,k_2})).$$

By restricting the domains of the maps, $\widetilde{I}_{i,j_2,j_1}$, to subsets of the balls, $B_{2\cdot 2^{-i}}(0_{i,j_1})$, we will obtain the desired maps, $\hat{I}_{i,j_2,j_1}$.

The modifications are effected by the proceedure of [11], where a situation closely related to the one considered here is treated. Thus, we will refer to [11] for certain details (see also [21]).

If $x_{i,k_1} \in Q_{i,1}$, $x_{i,k_2} \in Q_{i,2}$, are such that

$$I_{i,k_2,k_1}\left(B_{4\cdot 2^{-i}}(0_{i,k_1})\right) \cap B_{4\cdot 2^{-i}}(0_{i,k_2})$$

is nonempty, we put $\widetilde{I}_{i,k_2,k_1} = I_{i,k_2,k_1}$. Note that for x_{i,k_1} fixed, there is at most one such $x_{i,k_2} \in Q_{i,2}$.

Let $x_{i,k_1} \in Q_{i,1}$, $x_{i,k_3} \in Q_{i,3}$ be such that

$$I_{i,k_3,k_1}\left(B_{4\cdot 2^{-i}}(0_{i,k_1})\right) \cap B_{4\cdot 2^{-i}}(0_{i,k_3})$$

is nonempty. Then we put $\widetilde{I}_{i,k_3,k_1} = I_{i,k_3,k_1}$. Suppose that in addition,

$$\widetilde{I}^{-1}_{i,k_2,k_1}\left(B_{(4\cdot 2^{-i}}(0_{i,k_1})\right) \cap \widetilde{I}^{-1}_{i,k_3,k_1}\left(B_{(4\cdot 2^{-i}}(0_{i,k_1})\right) \neq \emptyset.$$

Then, as in [11] (see also [21]) by means of the Isotopy Extension Theorem, we modify the map, I_{i,k_3,k_2}, to obtain a diffeomorphism, $\widetilde{I}_{i,k_3,k_2} : \mathbb{R}^n_{i,k_2} \to \mathbb{R}^n_{i,k_3}$, such that on

$$(\text{A.1.36}) \qquad \begin{aligned} &\widetilde{I}^{-1}_{i,k_2,k_1}\left(\widetilde{I}^{-1}_{i,k_3,k_2}\left(B_{(4-\frac{1}{N^3})2^{-i}}(0_{i,k_1})\right) \cap B_{(4-\frac{1}{N^3})2^{-i}}(0_{i,k_2})\right) \\ &\qquad\qquad \cap \widetilde{I}^{-1}_{i,k_3,k_1}\left(B_{(4-\frac{1}{N^3})2^{-i}}(0_{i,k_1})\right), \end{aligned}$$

we have $\widetilde{I}_{i,k_3,k_2} \circ \widetilde{I}_{i,k_2,k_1} = \widetilde{I}_{i,k_3,k_1}$.

Moreover, it is easy to see that if ϵ and hence Ψ are sufficiently small, then by further decreasing the radii of the relevant balls, we obtain the following. Let j_1, j_2, j_3 by any of the 6 possible permutations of k_1, k_2, k_3. Then on

$$(\text{A.1.37}) \qquad \begin{aligned} &\widetilde{I}^{-1}_{i,j_2,j_1}\left(\widetilde{I}^{-1}_{i,j_3,j_2}\left(B_{(4-\frac{2}{N^3})2^{-i}}(0_{i,j_1})\right) \cap B_{(4-\frac{2}{N^3})2^{-i}}(0_{i,j_2})\right) \\ &\qquad\qquad \cap \widetilde{I}^{-1}_{i,j_3,j_1}\left(B_{(4-\frac{2}{N^3})2^{-i}}(0_{i,j_1})\right), \end{aligned}$$

we have $\widetilde{I}_{i,j_3,j_2} \circ \widetilde{I}_{i,j_2,j_1} = \widetilde{I}_{i,j_3,j_1}$.

Note that the balls occuring in (A.1.37) have radii which are smaller than those in (A.1.36), which in turn, are smaller than those in the relation that preceds it. Indeed, in what follows, at every stage at which our maps are modified, we will shrink by a definite amount, the radii of *all* balls (with centers, $0_{i,j}$) that are used to define the sets on which *all* of our consistency conditions are required to hold. In addition to the reason which we have just mentioned (i.e., to obtain all of the above 6 relations, $\widetilde{I}_{i,j_3,j_2} \circ \widetilde{I}_{i,j_2,j_1} = \widetilde{I}_{i,j_3,j_1}$,) are two additional reasons why the radii must be decreased.

First of all, this is neccessary in order to obtain the new consistency relations which are produced at any given stage of the construction; see [11] for details.

Secondly, decreasing the radii plays a role in ensuring that a modifications performed at a given stage do not destroy any consistency relations which were obtained at earlier stages. This point will be explained at length below.

We now treat the points, $x_{i,k_4} \in Q_{i,4}$, in a fashion similar to that in which the points, $x_{i,k_3} \in Q_{i,3}$, were treated above.

Thus, if x_{i,k_1}, x_{i,k_4} are such that $I_{i,k_4,k_1}(B_{4\cdot 2^{-i}}(0_{i,k_1})) \cap B_{4\cdot 2^{-i}}(0_{i,k_4})$ is nonempty, we put $\widetilde{I}_{i,k_4,k_1} = I_{i,k_4,k_1}$.

Next, we modify all appropriate maps, I_{i,k_4,k_2} to obtain the maps, $\widetilde{I}_{i,k_4,k_2}$ such that on

$$\widetilde{I}^{-1}_{i,k_2,k_1}\left(\widetilde{I}^{-1}_{i,k_4,k_2}\left(B_{(4-\frac{4}{N^3})2^{-i}}(0_{i,k_1})\right) \cap B_{(4-\frac{4}{N^3})2^{-i}}(0_{i,k_2})\right) \cap \widetilde{I}^{-1}_{i,k_4,k_1}\left(B_{(4-\frac{4}{N^3})2^{-i}}(0_{i,k_1})\right),$$

we have $\widetilde{I}_{i,k_4,k_2} \circ \widetilde{I}_{i,k_2,k_1} = \widetilde{I}_{i,k_4,k_1}$.

Then, we modify all appropriate maps, I_{i,k_4,k_3}, so as to obtain maps, $\widetilde{I}_{i,k_4,k_3}$ such that the relations $\widetilde{I}_{i,k_4,k_3} \circ \widetilde{I}_{i,k_3,k_1} = \widetilde{I}_{i,k_4,k_1}$ and finally, $\widetilde{I}_{i,k_4,k_3} \circ \widetilde{I}_{i,k_3,k_2} = \widetilde{I}_{i,k_4,k_2}$, hold on the corresponding subsets (defined by balls whose radii have been appropriately decreased).

In particular, a map, I_{i,k_4,k_3}, may have to be modified *twice* in order to produce the *final* map, $\widetilde{I}_{i,k_4,k_3}$, since there are two consistency relations which must be satisfied.

As previously mentioned, the modification proceedure of [11] guarentees that modifications performed at a given stage, do not destroy consistency conditions which have been obtained at a previous stage. The reasons for this are the following.

First of all since every modifcation can be assumed to be by as small an amount as we like (by making ϵ sufficiently small) and in addition, the radii of *every* ball is shrunk by a definite amount when each modification is performed, it follows easily that the new domain on which a previously established consistency relation is required to hold, is actually a subset of the previous domain of that consistency relation.

Given what has just been explained, it suffices to check the following point which we illustrate by using the indices, k_1, k_2, k_3, k_4, which were considered above; the argument in the general case is precisely the same.

Suppose we have already established the relation,

$$\widetilde{I}_{i,k_4,k_3} \circ \widetilde{I}_{i,k_3,k_1} = \widetilde{I}_{i,k_4,k_1},$$

and must now perform a second modification on the map, $\widetilde{I}_{i,k_4,k_3}$, in order to establish the relation, $\widetilde{I}_{i,k_4,k_3} \circ \widetilde{I}_{i,k_3,k_2} = \widetilde{I}_{i,k_4,k_2}$. Suppose that there exists

$$m_{i,k_1} \in B_{(4-\frac{6}{N^3})2^{-i}}(0_{i,k_1})\,, m_{i,k_2} \in B_{(4-\frac{6}{N^3})2^{-i}}(0_{i,k_2}),$$

such that

$$\widetilde{I}_{i,k_4,k_3} \circ \widetilde{I}_{i,k_3,k_1}(m_{i,k_1}) = \widetilde{I}_{i,k_4,k_1}(m_{i,k_1}),$$
$$\widetilde{I}_{i,k_3,k_2}(m_{i,k_2}) = \widetilde{I}_{i,k_3,k_1}(m_{i,k_1}),$$

and

$$\widetilde{I}_{i,k_4,k_2}(m_{i,k_1}) \in B_{(4-\frac{6}{N^3})2^{-i}}(0_{i,k_4}).$$

We claim that in this case, we actually have, $\widetilde{I}_{i,k_4,k_3} \circ \widetilde{I}_{i,k_3,k_2}(m_{i,k_2}) = \widetilde{I}_{i,k_4,k_2}(m_{i,k_2})$. Granting this for the moment, we note that the modification proceedure of [11] is such that in such an instance, the second modification of the map, $\widetilde{I}_{i,k_4,k_3}$, will leave the value, $\widetilde{I}_{i,k_4,k_3}(m_{i,k_3})$, unchanged; see [11] for details. This suffices to show that the second modification does not destroy the previously established consistency relation, $\widetilde{I}_{i,k_4,k_3} \circ \widetilde{I}_{i,k_3,k_1} = \widetilde{I}_{i,k_4,k_1}$.

To check our claim, note that from the previously established relations, we get $m_{i,k_2} = \widetilde{I}_{i,k_2,k_1}(m_{i,k_1})$ and $\widetilde{I}_{i,k_4,k_2} \circ \widetilde{I}_{i,k_2,k_1}(m_{i,k_1}) = \widetilde{I}_{i,k_4,k_1}(m_{i,k_1})$. These imply $\widetilde{I}_{i,k_4,k_2}(m_{i,k_2}) = \widetilde{I}_{i,k_4,k_1}(m_{i,k_1})$. Additionally, since $\widetilde{I}_{i,k_4,k_3} \circ \widetilde{I}_{i,k_3,k_1}(m_{i,k_1}) = \widetilde{I}_{i,k_4,k_1}(m_{i,k_1})$, by applying $\widetilde{I}_{i,k_4,k_3}$ to both sides of the relation, $\widetilde{I}_{i,k_3,k_2}(m_{i,k_2}) = \widetilde{I}_{i,k_3,k_1}(m_{i,k_1})$, it follows that

$\widetilde{I}_{i,k_4,k_3} \circ \widetilde{I}_{i,k_3,k_2}(m_{i,k_2}) = \widetilde{I}_{i,k_4,k_1}(m_{i,k_1}) = \widetilde{I}_{i,k_4,k_2}(m_{i,k_2})$, which establishes our claim.

By proceeding in as above, with all $x_{i,k_\ell} \in Q_{i,\ell}$, we obtain maps, $\widetilde{I}_{i,j_2,j_1}$, for all j_1, j_2 satisfying (A.1.32), such that

$$\left| \widetilde{I}_{i,j_2,j_1} - I_{i,j_2,j_1} \right|_{C^\infty, 2^{-i}} \leq \Psi \qquad (\text{on } B_{8\cdot 2^{-i}}(0_{i,j_1})) \tag{A.1.38}$$

and such that the following holds:

Let

$$\operatorname{dom} \widehat{I}_{i,j_2,j_1} = \{ w \in B_{2\cdot 2^{-i}}(0_{i,j_1}) \,|\, \widetilde{I}_{i,j_2,j_1}(w) \in B_{2\cdot 2^{-i}}(0)_{i,j_2} \}, \tag{A.1.39}$$

and on this domain, put

$$\widehat{I}_{i,j_2,j_1} = \widetilde{I}_{i,j_2,j_1}. \tag{A.1.40}$$

As usual, put

$$\operatorname{dom} \widehat{I}_{i,j_3,j_2} \circ \widehat{I}_{i,j_2,j_1} = \widehat{I}^{-1}_{i,j_2,j_1} \left(\operatorname{range} \widehat{I}_{i,j_2,j_1} \cap \operatorname{dom} \widehat{I}_{i,j_3,j_2} \right). \tag{A.1.41}$$

Then for $\varepsilon \leq \varepsilon(n)$ sufficiently small and *all* j_1, j_2, j_3, we have

$$\begin{aligned} &\widehat{I}_{i,j_3,j_2} \circ \widehat{I}_{i,j_2,j_1} = \widehat{I}_{i,j_3,j_1} \\ &\qquad (\text{on } \operatorname{dom} \widehat{I}_{i,j_3,j_2} \circ \widehat{I}_{i,j_2,j_1} \cap \operatorname{dom} \widehat{I}_{i,j_3,j_1}). \end{aligned} \tag{A.1.42}$$

It follows from (A.1.42), that the collection, $\{\widehat{I}_{i,j_1,j_2}\}$, determines an atlas, $\{\psi_{i,j}\}$, for a smooth manifold, W_i^n, such that $\psi_{i,j_2} \circ \psi^{-1}_{i,j_1} = \widehat{I}_{i,j_2,j_1}$ and range $\psi_{i,j} = B_{2\cdot 2^{-i}}(0)$.

The manifold, W_i^n is essentially unique. In fact, if $\alpha'_{i,j}$, $\beta'_{i,j}$ are a different set of $\varepsilon \cdot 2^{-i}$-Gromov–Hausdorff equivalences as above, then there exist isometries, $J_{i,j} : \mathbb{R}^n \to \mathbb{R}^n$, with $J_{i,j}(0) = 0$, such that

$$\overline{\alpha'_{i,j}, \alpha_{i,j} \circ J_j} \leq \Psi 2^{-i}, \qquad \text{on } B_{8\cdot 2^{-i}}(0),$$

and

$$\overline{\beta'_{i,j}, J_j \circ \beta_{i,j}} \leq \Psi 2^{-i}, \qquad \text{on } B_{8\cdot 2^{-i}}(x_{i,j}).$$

Let $\{\psi'_{i,j}\}$ denote the atlas for the manifold, $(W')^n_i$, constructed as above from the maps, $\{\alpha'_{i,j}\}$, $\{\beta'_{i,j}\}$. Then for $\varepsilon \leq \varepsilon(n)$ sufficiently small, by modifying the maps in the collection, $\{(\psi'_{i,j})^{-1} \circ J_{i,j} \circ \psi_{i,j}\}$ by a small amount (in the topology induced from $|\ |_{C^\infty, 2^{-i}}$ and the given atlases) as in [11] (see also [21]) one constructs an essentially canonical diffeomorphism from W_i^n to $(W')_i^n$.

Suppose that (Z,ρ) is actually an n-dimensional Riemannian manifold. Then for $\underline{i}$, sufficiently large, the collection of maps, $\{\beta_{\underline{i},j}\}$, can be choosen to be an atlas for the smooth manifold underlying, Z. Then the maps, $\widehat{I}_{\underline{i},j_1,j_2}$, will be slight modifications of the (restrictions of) maps, $\beta_{i,j_2}\circ\beta_{i,j_1}^{-i}$ and as above, it follows that (for $\varepsilon\le\varepsilon(n)$, sufficiently small) there exists an essentially canonical diffeomorphism, $f_{\underline{i}}: W_{\underline{i}}^n\to Z$.

Let X_i be the 2^{-i}-dense set chosen earlier. For each $x_{i+1,j}\in X_{i+1}$, choose $x_{i,k(j)}\in X_i$, such that $x_{i+1,j}\in\overline{B_{2^{-i}}(x_{i,k(j)})}$. If in fact, $x_{i+1,j}\in X_i$, take $x_{i,k(j)}=x_{i+1,j}$. Let $L_{i,j}:\mathbb{R}^n\to\mathbb{R}^n$ be an isometry such that $L_{i,j}\circ\widehat{\psi}_{i,k(j)}(x_{i+1,j})=0$. Let $\{\theta_{i,j}\}$ be the atlas for W_i^n obtained by putting

(A.1.43)

$$\theta_{i,j}=L_{i,j}\circ\psi_{i,k(j)}\,|\,(L_{i,j}\circ\psi_{i,k(j)})^{-1}\ (B_{2^{-i}}(L_{i,j}\circ\psi_{i,k(j)},(x_{i+1,j}))).$$

As above, there exist isometries, say $K_{i,j}:\mathbb{R}^n\to\mathbb{R}^n$, such that by slightly modifying the maps, $\{\psi_{i+1,j}^{-1}\circ K_{i,j}\circ\theta_{i,j}\}$, we obtain an essentially canonical diffeomorphism, $h_i: W_i^n\to W_{i+1}^n$, (provided $\varepsilon\le\varepsilon(n)$, sufficiently small).

Verification of i)–vi):

vi) From, the preceding discussion, it is clear that if Z is an n-dimensional Riemannian manifold, and $\underline{i}$ as above is sufficiently large, then $F_{\underline{i}}$ as defined in (A.1.17), is a diffeomorphism, $F_{\underline{i}}: W_0^n\to Z$.

Consider again the case in which Z is arbitrary.

Let $\{\phi_{i,j}\}$ denote a partition of unity subordinate to the covering, $\{\psi_{i,j}^{-1}(B_{6\cdot 2^{-i}}(0))\}$, constructed in standard fashion (from the pullbacks via the maps, $\psi_{i,j}$, of a standard bump function). Put $g_i=\sum\phi_{i,j}\psi_{i,j}^*(\mathbf{g})$, where $\mathbf{g}$ is the standard flat metric on $\mathbb{R}^n$. Let d_i denote the distance function associated to g_i.

Clearly, the functions, d_i, and maps h_i, satisfy for all i,

$$(A.1.44)\qquad 2^{-\Psi}d_i\le d_{i+1}\circ h_i\le 2^{\Psi}d_i.$$

Starting with the collection of Gromov–Hausdorff approximations, $\{\alpha_{i,j}\}$, in obvious fashion, we can construct a map, $f_i: W_i^n\to Z$ (which might not be continuous if Z is not an n-dimensional Riemannian manifold) such that the following properties hold.

iv) For all i, (A.1.16) holds.

i) Define points, $w_j^i \in W_i^n$ by $w_j^i = \psi_{i,j}^{-1}(0)$. Then for all i, j,

$$(A.1.45) \qquad |\rho \circ f_i - d_i| \le \Psi 2^{-i} \quad (\text{on } B_{2^{-i}}(w_j^i)).$$

Here $B_r(w_j^i)$ denotes the metric ball of radius r with respect to the distance function, d_i.

If we define

$$(A.1.46) \qquad \rho_i(w_1^i, w_2^i) = \begin{cases} d_i(w_1^i, w_2^i) & d_i(w_1^i, w_2^i) \le 2^{-i}, \\ \rho \circ f_i & d_i(w_1^i, w_2^i) > 2^{-i}, \end{cases}$$

then it follows that i) holds.

iii) Moreover, it is clear that (A.1.15) holds.

ii) From (A.1.44), (A.1.46), together with (A.1.15), (A.1.16), we get (A.1.14).

v) Finally, the map, F_∞, is surjective.

This completes the proof.

Remark A.1.47. As previously mentioned, the proof of Theorem A.1.11 is very similar to the proof of the Theorem A.1.2. Let M^n be as in Theorem A.1.11. It follows from [26] (or [15]) that for $1 < c < c(n), 0 < \delta \le \delta(n)$, after rescaling to unit size, every annulus, $A_{R,cR}(p) = B_{cR}(p) \setminus \overline{B_R(p)}$, is $\Psi(\delta|n)$-close to the corresponding annulus $A_{1,c}(0) \subset \mathbf{R}^n$. Write $M^n = B_1(p) \cup (\cup_{i=1}^\infty A_{2^{\ell-1/3},2^{\ell+4/3}}(p))$. In place of the sets, $X_0 \subset X_1 \subset \cdots$, defined prior to (A.1.32), we consider sets, $X_0 \subset X_1 \subset \cdots$, such that $X_i \cap A_{2^{\ell-1/3},2^{\ell+4/3}}(p)$ is $2^{\ell-i}$-dense in $A_{2^{\ell-1/3},2^{\ell+4/3}}(p))$. With this modification, the proof of Theorem A.1.11 can be carried out in a manner strictly analogous to that of Theorem A.1.2.

Appendix 2: Remarks on the synthetic treatment of Ricci curvature

Generalizations of the notion of "smooth function" have long played a very important role in analysis and in questions of a geometric analytic nature; see e.g. [10], [29], [51], [57]. After the pioneering work of Alexandrov and Gromov ([2], [37]) analogous notions of "generalized Riemannian manifold" have begun to play an increasingly significant role in Riemannian geometry. In this appendix, we will discuss some related issues in connection with Ricci curvature.

We begin by fixing some ideas and terminology. Let us consider metric spaces, possibly equipped with some additional structure like a measure. Roughly speaking, we say that a set of conditions which serve to define a subclass of such metric spaces (or associated objects) is *synthetic* if these conditions do not depend on the existence of an underlying smooth structure, or indeed, make any reference to the notion of smoothness. (More generally, our conditions should not entail *any* a priori *structural* assumptions.)

The origins of the synthetic tradition in geometry go back quite far. More recently, questions of a geometric analytic nature, of the sort which classically were studied in $\mathbb{R}^n$, have received considerable attention in more general synthetic contexts; see e.g. [53] and the references therein.

Let $\mathrm{Alex}(n, H)$ denote the class of n-dimensional Alexandrov spaces, X, with Alexandrov-Toponogov curvature $\geq H$. Let $\sec(n, H)$ denote the class of n-dimensional Riemannian manifolds, M^n, with sectional curvature $K_{M^n} \geq H$. Clearly, the former of these classes is defined synthetically while the latter is not. Since $\mathrm{Alex}(n, H)$ contains $\sec(n, H)$ and coincides with it when intersected with the class of smooth Riemannian manifolds, we say that $\mathrm{Alex}(n, H)$ provides a *strict synthetic generalization* of the class, $\sec(n, H)$; see [9], [48], for the general theory of Alexandrov spaces.

Let $\overline{\sec(n, H)}$ denote the closure of $\sec(n, H)$ in the Gromov-Hausdorff topology. This class is *not* defined synthetically, even though it contains members more general than smooth n-dimensional Riemannian manifolds.

In fact, for $H > 0$, it is known that $\overline{\sec(n, H)} \not\subset \mathrm{Alex}(n, H)$; see [49]. However, it is not known whether there exists

$$N(X) < \infty, c(X)H > -\infty,$$

such that if $X \in \mathrm{Alex}(n, H)$, then $X \in \overline{\sec(N(X), c(X)H)}$. If this were known, we would say that the class $\sec(N(\cdot), c(\cdot)H)$ provides a *resolution of singularities* for $\mathrm{Alex}(n, H)$. According to [9], Section 13, this instance of the problem of resolution of singularities "presents difficulties".

If resolution of singularities holds, one can in principle study the synthetically defined class by means of theorems which are proved (initially) in the smooth case by smooth methods, but whose hypotheses and conclusions are phrased in purely synthetic terms and are preserved under Gromov-Hausdorff limits; compare the proof of the Poincaré inequality for limit spaces given in [13].

At present, there are only very few theorems which are known to hold for the class, $\overline{\sec(N(\cdot), c(\cdot)H)} \cap \mathrm{Alex}(n, H)$, but which are not known for $\mathrm{Alex}(n, H)$, itself.

On the other hand, for Ricci curvature, the main rigidity theorems for Gromov-Hausdorff limits are proved in [15] by just this method of resolution of singularities. In this appendix, we will discuss the issue of possible synthetic generaliztions of these theorems and of their consequences.

The generalized splitting theorem provides a particular example. Let $\mathrm{Ric}(n, H)$ denote the class of smooth Riemannian manifolds, M^n, with $Ric_{M^n} \geq (n-1)H$. In [15], the splitting theorem is proved for the class, $\cap_\epsilon \overline{\mathrm{Ric}(n, -\epsilon)}$. Since the splitting theorem itself is not valid for any individual class, $\mathrm{Ric}(n, -\epsilon)$, it is necessary to prove an "almost" or "quantitative" splitting theorem for each ϵ. This implies the corresponding theorem for $\overline{\mathrm{Ric}(n, -\epsilon)}$ and the totality of these theorems yields the splitting theorem for $\overline{\cap_\epsilon \mathrm{Ric}(n, -\epsilon)}$.

For the most part, the theorems of the present paper are formulated and proved purely synthetically (but compare e.g. (2.34)–(2.36)). Thus, most of these results hold for certain *nonstrict* synthetic generalizations of the class, $\mathrm{Ric}(n, H)$, in which various subsets of the relative volume comparison and almost rigidity theorems are assumed (axiomatically) to hold. All of these generalizations contain the class $\overline{\mathrm{Ric}(n, H)}$. If, for example, we assume the integral Toponogov theorem of [15], as formulated with respect to the measure, ν (compare [24]–[26]) we emphasize the connection with Alexandrov space theory and obtain as a particular synthetic consequence, the splitting theorem and (at least a weakened version of) Theorem 2.1.

After recalling some further background, we will point out a particular strict synthetic generalization of the class, $\mathrm{Ric}(n, H)$, for which the almost rigidity and integral Toponogov theorems are not assumed axiomatically to hold. However, it turns out that any theory for which such results are valid, must in one way or another, be based on rather strong additional assumptions.

The idea that there should be a synthetic theory of spaces whose Ricci curvature is bounded below in some generalized sense, goes back to Gromov, whose compactness theorem provides the first nontrivial examples of such spaces; [37], [36].

Fukaya, observed the existence of renormalized limit measures and conjectured the role that they should play in connection with the continuity of the spectrum of the Laplacian under measured Gromov Hausdorff convergence; see [30] and, for the proof of the conjecture, [13].

The first estimate on distances under Ricci curvature bounds is the Abresch-Gromoll inequality; [1]. This estimate automatically passes to Gromov-Hausdorff limits.

In [24]–[26], one finds the first theorems on Ricci curvature in the context of smooth manifolds (integral Toponogov theorems, etc.) that strongly resemble results which for sectional curvature, play a basic role in the theory of Alexandrov spaces.

As noted above, the almost rigidity theorems for Gromov-Hausdorff limit spaces were proved in [15]; it had been conjectured in [31] that the splitting theorem extends to Gromov-Hausdorff limits.

From [30], it is already clear that the basic objects of any synthetic generalization of the class, $\mathrm{Ric}(n,H)$, are pairs, (Y,ν), where Y is a length space, and ν is a Radon measure which plays the role of the renormalized limit measure; compare Sections 1–3. The measure, ν, should satisfy (0.5), for all $z\in Y, \mu=\nu$, and $k=n$, for some $n<\infty$. Consequently, $Y=Y^m$ has Hausdorff dimension $m\leq n$. As indicated by Examples 1.24 and 8.71, neither for Alexandrov spaces, nor even for smooth Riemannian manifolds, should one restrict ν to be m-dimensional Hausdorff measure.

The above mentioned property of ν, which is a strengthened version of what is often refered to as a *doubling condition*, has significant consequences e.g. compactness theorems; compare also [53]. However, to capture more completely the fundamental implication of the condition, "Ricci curvature bounded below", which in the smooth case, is mean curvature comparison, or equivalently (in the smooth case) Laplacian comparison, one needs a version of (0.5) which is localized with respect to direction; [8], [10], [37].

Calabi emphasized that Laplacian comparison holds in a useful generalized sense, even at points where the distance function fails to be smooth i.e. on the cut locus; compare [20], [62].

In terms of mean curvature, this principle can also be formulated as follows. Namely, rate of change of the logarithm of the area of the intersection of *any* (thin) angular sector of minimal geodesics with a family of distance spheres, $\partial B_r(p)$, is less than the corresponding rate of change in the model space, M^n_H.

We now consider certain strict synthetic generalizations of the class, $\mathrm{Ric}(n,H)$, which are based on generalized concepts of mean curvature comparison and Laplacian comparison.

Let X be a length space equipped with a Radon measure, ν. For

$p \in X$ and $0 < r_1 < r_2$, we put $A_{r_1,r_2}(p) = B_{r_2}(p) \setminus \overline{B_{r_1}(p)}$. Given $0 < s_1 < s_2 < r_1$, and an open subset, $U \subset A_{r_1,r_2}(p)$, we set
(A.2.1)

$$S_{s_1,s_2}(p,U) = \{x \in A_{s_1,s_2}(p) \mid \overline{p,x} + \overline{x,z} = \overline{p,x}, \text{ for some } z \in U\}.$$

Thus, $S_{s_1,s_2}(p,U)$ is the intersection with $A_{s_1,s_2}(p)$ of the angular sector consisting of minimal geodesics emanating from p and the ends of which lie in U.

A generalized version of mean curvature comparison can be formulated as follows; compare (0.5). For all $p \in X, 0 < s_1 < s_2 < r_1 < r_2$, we have

$$\frac{\nu(U)}{\nu(S_{s_1,s_2}(p,U))} \leq \frac{V_{n,H}(s_2) - V_{n,H}(s_1)}{V_{n,H}(r_2) - V_{n,H}(r_1)} \quad \text{(for all } U \subset A_{r_1,r_2}(p)). \tag{A.2.2}$$

It is easy to see that condition (A.2.2) already provides a strict synthetic generalization of the class, $\mathrm{Ric}(n,H)$.

In order to formulate Laplacian comparison, we must have a generalized notion of Laplacian for (X,ν).

Let f be a Lipschitz function on X. Given $x \in X$ and $0 < r_1 < r_2$, put

$$\mathrm{Lip}_{r_1,r_2}(x,f) = \sup_{z \in \overline{A_{r_1,r_2}(x)}} \frac{|f(z) - f(x)|}{\overline{z,x}}. \tag{A.2.3}$$

Assume from now on that X is locally compact, from which it follows that $\mathrm{Lip}_{r_1,r_2}(x,f)$ is a continuous function of (x,r_1,r_2). Hence, the function,

$$\mathrm{Lip}(x,f) := \lim_{r_2 \to 0} \lim_{r_1 \to 0} \mathrm{Lip}_{r_1,r_2}(x,f), \tag{A.2.4}$$

is measurable. As in [35], we define a generalized Dirichlet functional by

$$Q(f,f) = \int_X (\mathrm{Lip}(x,f))^2, \tag{A.2.5}$$

whenever the integral is finite. If $\nu(X) < \infty$, then this holds for all Lipschitz functions, f.

Note that at least formally, (A.2.2) implies Theorem 2.11 of [15] and hence a lower bound for the "bottom of the spectrum" of Q in the compact case; compare [15, Remark 2.82]. However, due to the possible lack of regularity in our space, there are technical points to consider in carrying out such a proceedure; compare [13] in which an argument based on resolution of singularities is employed.

If (X,ν) is the limit in the measured Gromov-Hausdorff sense of a sequence, $\{M_i^n\}$, satisfying (1.1), it makes sense to compare the functional, Q, with the corresponding sequence of Dirichlet forms on the manifolds, M_i^n; see [13] for details. However, in order to define the Laplacian of f (in a weak sense) we need an appropriate definition of $Q(f,h)$ for *pairs* of Lipschitz functions (f,h). Given such a definition, we can take $f = f(r)$ and take h to be supported near some point, $z \in X$.

Roughly, for $Q(f,h)$ to be defined, the following should hold. Assume that $\{x_i\}$ satisfies $x_i \to x$ and either $f(x_i) \geq f(x)$ or $f(x_i) \leq f(x)$, for all i. In (A.2.7) below we attach a sign, $+$ or $-$, according to which of these alternatives actually holds. Assume in addition, that

$$(A.2.6) \qquad \lim_{i\to\infty} \frac{|f(x_i) - f(x)|}{\overline{x_i, x}} = q_f(x)$$

and that

$$(A.2.7) \qquad \pm q_f(x) \lim_{i\to\infty} \frac{|h(x_i) - h(x)|}{\overline{x_i, x}} := q_{f,h}(x)$$

exists and is independent of the particular sequence, x_i. If $q_{f,h}(x)$ is defined for almost all x and $q_{f,h}(x) \in L_2(X,\nu)$, then we put

$$(A.2.8) \qquad Q(f,h) = \int_X q_{f,h}(x).$$

Note that in the definition we have given, the roles of f and h are not symmetric. Moreover, the functional, Q, is not bilinear in general.

If $Q(f,h)$ exists for $f = f(r)$ and a suitably dense collection of functions h, then the condition "generalized Laplacian comparison holds with respect to some M_H^n", has an obvious meaning.

The following canonical examples do satisfy generalized Laplacian comparison (as well as (A.2.2)) but *not* the basic rigidity and integral Toponogov theorems. These examples were pointed out to us by Z. Shen; compare [54], [55].

Let X^n denote a normed vector space of dimension n. As usual, we regard X^n as a complete metric space by setting $\overline{v_1, v_2} = |v_1 - v_2|$. Let

ν denote the associated Hausdorff measure. We will assume that unit ball is strictly convex i.e. the norm on X^n satisfies the nondegeneracy condition: If v_1 and v_2 are linearly independent, then

$$(A.2.9) \qquad |tv_1 + (1-t)v_2| < t|v_1| + (1-t)|v_2| \quad (0 < t < 1).$$

In this case, the minimal geodesics are *precisely* the affine line segments and every such segment extends to a *line* in the sense of the splitting theorem.

Since the splitting theorem holds for Alexandrov spaces (see [39] and for the case of Gromov-Hausdorff limit spaces, [61]) it follows that X^n is not an Alexandrov space unless it is isometric to Euclidean space, $\mathbb{R}^n$. (Of course this can also be checked directly). None-the-less, it is easy to verify (A.2.2) (for $H=0$). Also, the functional, Q, is well defined and (for $H = 0$) generalized Laplacian comparison holds.

For convenience, fix an inner product on X^n. Let $\mathcal{L}^n$ denote the corresponding Hausdorff measure. Note that the identity map is bi-Lipschitz from the original normed space, X^n, to X^n equipped with this Euclidean structure. Since the additive group of the underlying vector space acts by isometries with respect to both metrics, it follows in particular that ν is constant multiple of $\mathcal{L}^n$. Thus, the divergence, $\operatorname{div} W$, of a vector field, W, is the same when defined with respect to either of the associated volume forms.

For $t \neq 0$, scalar multiplication by t defines homothety of X^n which scales distances by a factor t and hence scales ν by a factor, t^n. It follows directly that (A.2.2) holds for $H = 0$.

If we regard X as a normed linear space, the nondegeneracy condition, (A.2.8), allows us to define a bijection, L, from the dual space $(X^n)^*$ to X^n (the Legendre transformation). Namely, when restricted to the unit sphere, every linear functional, ℓ^*, takes its maximum at a unique point, v. Then we send ℓ^* to $\ell^*(v)v$. The map, L, is *not* linear unless X^n is isometric to Euclidean space. However, we can define the gradient of a Lipschitz function, f, by putting $\widetilde{\nabla} f = L(df^*)$. Typically, this gradient is not a linear map (from functions to vector fields).

It follows from the previous discussion that the functional, Q, is well defined. Specifically, $q_{f,h} = dh(\widetilde{\nabla} f)$. Of course, if the gradient, $\widetilde{\nabla}$, is not linear, then the functional, Q, will not be bilinear. However, it is easy to verify that Laplacian comparison holds. Indeed, we have

$$(A.2.10) \qquad Q(f,h) = \int_{X^n} \widetilde{\Delta} f \cdot h,$$

where the nonlinear Laplacian, $\widetilde{\Delta}$, is given by

$$\widetilde{\Delta} = \mathrm{div}\widetilde{\nabla}; \tag{A.2.11}$$

(Shen has observed that the above definition extends to arbitrary Finsler manifolds, in which case, one must replace div by $\widetilde{\mathrm{div}}$; see [55]). Then an easy computation gives

$$\widetilde{\Delta}\widetilde{r} = \frac{(n-1)}{\widetilde{r}}. \tag{A.2.12}$$

Assume that there is at least one point, v, on the unit sphere of X^n, at which this sphere is not C^1-smooth. Let w be such that the intersection of the plane spanned by v and w and the the unit sphere is a curve which is not C^1-smooth. Then for arbitrarily thin triangles lying in this plane, with base along v, the excess is bounded below by a definite multiple of the altitude i.e. even in qualitative form, the Abresch-Gromoll inequality, a weak form of the splitting theorem, does not hold.

Note that the nonsmoothness of the unit sphere implies that (A.2.9) is violated for the dual space, $(X^n)^*$. However, if we consider a sequence of norms for which the unit sphere is smooth, which converges to a norm for which the unit sphere is not smooth, it follows that there is no inequality of Abresch-Gromoll type which holds uniformly for such a sequence.

The proof of the Abresch-Gromoll inequality for smooth Riemannian manifolds uses only Laplacian comparison and the maximum principle. Since the Laplacian, $\widetilde{\Delta}$, satisfies the maximum principle, it becomes clear that the failure of the proof in our situation can be traced to the nonlinearity of $\widetilde{\Delta}$. Recall in this connection, that in the proof, it is actually neccesary to apply the Laplacian to a linear combination of two distance functions (which occur in the excess function) and a certain comparison function of the distance from a third point; see [1]. If the Laplacian is nonlinear, Laplacian comparison for individual distance functions does not imply Laplacian comparison for such linear combinations.

Clearly, there is no reason for the generalized Dirichlet form, Q, to be bilinear unless almost all points of the underlying metric space are actually regular. So if we *assume* the bilinearity of the Q, we are in effect making a hidden assumption concerning the local regularity of our space; see [13] for further discussion.

On the other hand, it would be of interest to explore theories (limit space and synthetic) in which the role of Riemannian manifolds is played by Finsler manifolds i.e. in which role of Euclidean spaces is played by Minkowski spaces.

While the bilinearity of Q suffices for the Abresch-Gromoll inequality, in order to obtain the splitting theorem itself, a version of Bochner's formula must be incorporated into the discussion.

Let us make a final remark. In the Riemannian case, the penultimate step in the proof of the splitting theorem produces a vector field of constant norm which is the gradient of a harmonic function; see [20]. Bochner's formula implies that this vector field is parallel, and hence, by the DeRham decomposition theorem, the manifold splits isometrically. As a particular consequence, the gradient field is also Killing. For the space, X^n, considered above, the coordinate functions are harmonic. Moreover, their gradients have constant length and do generate 1-parameter groups of isometries. However, the splitting theorem still fails to hold.

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