J. DIFFERENTIAL GEOMETRY
45 (1997) 319-335

A SPLITTING THEOREM FOR ISOPARAMETRIC SUBMANIFOLDS IN HILBERT SPACE

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1. Introduction

Recall that for a submanifold M of a Hilbert space V, the *end point map* $\eta : \nu(M) \longrightarrow V$ is defined by $\eta(v) = x + v$ for $v \in \nu(M)_x$, where $\nu(M)$ is the normal bundle of M. M is called *proper Fredholm* if it has finite codimension and the end point map restricted to any finite normal disk bundle is a proper Fredholm map. A proper Fredholm submanifold M is called *isoparametric* if its normal bundle is globally flat and the shape operators along any parallel normal vector field are conjugate. We will always assume that M is complete. An isoparametric submanifold M of V is called *decomposable* if there exist two proper closed affine subspaces V_1, V_2 of V and isoparametric submanifolds M_i in V_i for $i = 1, 2$ such that $V = V_1 \oplus V_2$ and $M = M_1 \times M_2$, otherwise, M is called *indecomposable*. To every isoparametric submanifold, there is a cannonical way to associate a Coxeter group (cf. [10]). A Coxeter group is called *decomposable* if its Coxeter diagram is not connected. The main purpose of this paper is to prove the following decomposition theorem which was conjectured in [4].

Theorem A. *An isoparametric submanifold is decomposable if its Coxeter group is decomposable.*

It is easy to see that if an isoparametric submanifold M is decomposed into the product of two isoparametric submanifolds and each component has nontrivial Coxeter group, then the Coxeter group of M is decomposable. Note that an isoparametric submanifold has trivial Coxeter group if and only if it is a closed affine subspace of the ambient

Received January 17, 1996.

Hilbert space with finite codimension. The theorem above has been proved by Terng (cf. [9]) for isoparametric submanifolds of finite dimensional Euclidean spaces.

The above theorem can be applied to study hyperpolar actions on symmetric spaces. Recall that an isometric action of a compact Lie group H on a Riemannian manifold M is called *hyperpolar* if there exists a closed, flat, connected submanifold Σ of M that meets all H-orbits orthogonally. Such a Σ is called a *section*. Define $N(\Sigma) = \{h \in H \mid h(\Sigma) \subset \Sigma\}$ and $Z(\Sigma) = \{h \in H \mid h(x) = x \text{ for all } x \in \Sigma\}$. Let $W(\Sigma) = N(\Sigma)/Z(\Sigma)$. $W(\Sigma)$ is called the *generalized Weyl group* associated to the H-action. Two isometric actions of groups H_i on Riemannian manifolds M_i for $i = 1, 2$ are said to be *ω-equivalent* if there exists an isometry $f : M_1 \longrightarrow M_2$ such that $f(H_1 \cdot x) = H_2 \cdot f(x)$ for all $x \in M_1$. An isometric action of H on M is said to be *decomposable* if there exist Riemannian H_i-manifolds M_i for $i = 1, 2$ such that the action of H on M is ω-equivalent to the product action of $H_1 \times H_2$ on $M_1 \times M_2$. In this case, we say that the action on M decomposes as the product of two isometric actions. Moreover if one of the H_i-actions is transitive, then we say that this decomposition is *trivial*. If G is a compact Lie group equipped with a biinvariant metric and H is a closed subgroup of $G \times G$, then H has a natural isometric action on G defined by $(h_1, h_2) \cdot g = h_1 g h_2^{-1}$ for $(h_1, h_2) \in H$ and $g \in G$. One application of Theorem A is to the following decomposion theorem of hyperpolar actions which was also conjectured in [4].

Theorem B. *Let G be a compact, simply connected Lie group, and H a closed subgroup of $G \times G$. Suppose the action of H on G is hyperpolar, A is a section through e, and $W(A)$ is the generalized Weyl group associated with the H-action. Then the following statements are equivalent:*

(a) the H-action on G decomposes nontrivially as product of two isometric actions,

(b) the isometric action of $W(A)$ on A is decomposable.

Let A be a section through e of a hyperpolar action on a compact Lie group G, and $\mathfrak{a}$ the Lie algebra of A. Then the Weyl group $W(A)$ acts on the unit lattice $\Lambda = \exp^{-1}(e) \cap \mathfrak{a}$. The corresponding semidirect product $\hat{W} = W(A) \ltimes \Lambda$ is an affine Weyl group. Another application of Theorem A is the following:

Theorem C. *If G is a compact simple Lie group, then the affine*

Weyl group associated with any hyperpolar action on G is irreducible.

Note that in this theorem, we do not assume G to be simply connected. Using this theorem, we can improve a result of [4] (Corollary 3.14) which states that a group whose action on a compact irreducible symmetric space is hyperpolar is often maximal in terms of ω-equivalence. More precisely, we have

Corollary D. *Let $X = G/K$ be a compact, connected, irreducible symmetric space, and $H \subset L \subset G$ closed, connected subgroups. If the H-action on X is hyperpolar, then the L-action on X is either transitive or ω-equivalent to the H-action.*

Proof of the corollary: If X is a compact simple Lie group, then this follows from Theorem C above and Corollary 3.14 of [4]. If X is a symmetric space of the first type, then G is a simple Lie group, and the statement follows from the first case by considering the $H \times K$ and $L \times K$ actions on G.

We would like to make a remark on another conjecture in [4] which states that if G is a compact Lie group and the action of H on G is hyperpolar and indecomposable, then G is simple. This conjecture is not true in general. A counterexample is the following. Let G_0 be a compact simple Lie group. Let $G = G_0 \times G_0$ and $\Delta G_0 = \{(g, g) \mid g \in G_0\}$ the diagonal subgroup of G. Let $H = \Delta G_0 \times \Delta G_0$. Then the action of H on G is hyperpolar, since $(G, \Delta G_0)$ is a compact symmetric pair (cf. [4] Examples 3.1(4) Hermann's examples). On one hand, $G_0 \times G_0$ is the only nontrivial way to decompose G as the product of two Riemannian manifolds. On the other hand, the orbit of the H-action through e is ΔG_0 which does not respect the decomposition of G. Therefore the H-action on G is indecomposible. However, G is not simple.

This paper is organized as follows. Preliminary knowledge about isoparametric submanifolds and some of their properties will be given in Section 2. In Section 3, we prove Theorem A, while Theorem B and Theorem C are proved in Section 4.

The second author would like to thank the University of Augsburg for hospitality.

2. Preliminaries

We refer to [7] and [6] for the foundations of Riemannian Hilbert manifolds. In this section we review some basic facts about isoparametric submanifolds in Hilbert space. Suppose M is an isoparametric submanifold in a Hilbert space V. Since the normal bundle νM is flat, the tangent bundle of M splits as $TM = \overline{\oplus\{E_i \mid i \in I\}}$ into the direct sum of the simultaneous eigenspaces E_i of the shape operators, where I is a countable index set. $\{E_i \mid i \in I\}$ are called the curvature distributions of M. Let $\{n_i \mid i \in I\}$ be the corresponding curvature normals of M, i.e., the globally defined parallel normal vector fields such that for any parallel normal vector field v on M, the restriction of the shape operator to each E_i is

$$A_v|_{E_i} = < v, n_i > \mathrm{Id}.$$

We will always denote the zero curvature normal by n_0 and the corresponding curvature distribution by E_0. We will always assume that M is full, i.e., not contained in any proper closed affine subspace of V. This is equivalent to saying that for any point $x \in M$ the cuvature normals $\{n_i(x) \mid i \in I\}$ span the normal space $\nu_x M$ [10, Proposition 6.7]. It is known that each curvature distribution is integrable. If $n_i \neq 0$, the rank of E_i is finite and for any $x \in M$, the leaf of E_i passing through x is a round sphere centered at $x + (n_i(x)/\|n_i\|^2)$ with radius $1/\|n_i\|$. The leaves of E_0 are closed affine subspaces of the Hilbert space V (cf. [10]). The following lemma is useful in proving the integrability of direct sums of curvature distributions.

Lemma 2.1. *For all $i, j, k \in I$ and vector fields $X_i \in E_i$, $X_j \in E_j$, $X_k \in E_k$,*

$$< \nabla_{X_i} X_j, X_k > (n_j - n_k) \ = < \nabla_{X_j} X_i, X_k > (n_i - n_k).$$

Here ∇ is the Levi-Civita connection on M.

Proof. Let α be the second fundamental form of M, and $\nabla^\perp$ the induced connection on the normal bundle νM. Then the Codazzi equation is equivalent to

$$(\nabla_{X_i} \alpha)(X_j, X_k) = (\nabla_{X_j} \alpha)(X_i, X_k)$$

for all $X_i \in E_i$, $X_j \in E_j$, and $X_k \in E_k$. Now

$$\begin{aligned}
(\nabla_{X_i}\alpha)(X_j, X_k) &= \nabla^{\perp}_{X_i}(\alpha(X_j, X_k)) \\
&\quad -\alpha(\nabla_{X_i}X_j, X_k) - \alpha(X_j, \nabla_{X_i}X_k) \\
&= \nabla^{\perp}_{X_i}(< X_j, X_k > n_j) - < \nabla_{X_i}X_j, X_k > n_k \\
&\quad - < X_j, \nabla_{X_i}X_k > n_j \\
&= < \nabla_{X_i}X_j, X_k > n_j + < X_j, \nabla_{X_i}X_k > n_j \\
&\quad - < \nabla_{X_i}X_j, X_k > n_k - < X_j, \nabla_{X_i}X_k > n_j \\
&= < \nabla_{X_i}X_j, X_k > (n_j - n_k),
\end{aligned}$$

which implies the lemma.

Although in general, we cannot exchange the order of derivative and infinite sum, the following simple fact will be good enough for later calculations.

Lemma 2.2. *Given a Riemannian Hilbert manifold W. If $\{X_i \mid i \in \mathbb{N}\}$ is an orthonormal frame for the tangent bundle TW over an open subset of W, then for all differentiable functions $f_i \in C^1(W)$, with $i \in \mathbb{N}$, and vector fields Y on W,*

$$\nabla_Y(\sum_i f_i X_i) = \sum_i \nabla_Y(f_i X_i).$$

Proof. This can be shown by taking inner product with X_j on both sides of the equation for all $j \in \mathbb{N}$.

Now we are able to prove the following integrability theorem.

Proposition 2.3. *Fix an arbitrary point $x_0 \in M$. For any affine subspace P of the normal space $\nu_{x_0}M$, let*

$$D_P = \overline{\oplus\{E_i \mid n_i(x_0) \in P\}}.$$

Then D_P is a totally geodesic distribution on M, i.e., $\nabla_X Y \in D_P$ whenever X, $Y \in D_P$. In particular, D_P is integrable and each leaf of D_P is a totally geodesic submanifold of M.

Proof. Because of Lemma 2.2, we only need to show that if $n_i(x_0)$, $n_j(x_0) \in P$ but $n_k(x_0) \notin P$, then for any vector fields $X_s \in E_s$, where $s = i, j, k$, we have

$$< \nabla_{X_i}X_j, X_k >= 0.$$

Since $< X_i, X_k >\equiv 0$,

$$< \nabla_{X_j}X_i, X_k > (n_i - n_k) \ = < \nabla_{X_j}X_k, X_i > (n_k - n_i).$$

Applying Lemma 2.1 to both sides of this equation, we have

$$< \nabla_{X_i} X_j, X_k > (n_j - n_k) \ = < \nabla_{X_k} X_j, X_i > (n_j - n_i).$$

Since either $n_j - n_k$ and $n_j - n_i$ are linearly independent, or $n_j - n_i = 0$ but $n_j - n_k \neq 0$, we must have $< \nabla_{X_i} X_j, X_k >= 0$. This proves the proposition.

Next, we review the definiton of the Coxeter group associated with M. For any $x_0 \in M$, the intersection of the affine normal space $\nu(x_0) = x_0 + \nu_{x_0} M$ with the singular point set of the end point map η is the union of a locally finite set of affine hyperplanes of $\nu(x_0)$. More precisely, it is equal to

$$\cup \{ l_i(x_0) \mid i \in I, i \neq 0 \},$$

where

$$l_i(x_0) = \{ x_0 + v \mid v \in \nu_{x_0} \ \text{and} \ \ < v, n_i(x_0) >= 1 \}$$

is the focal affine hyperplane associated with the curvature normal n_i. It is obvious that n_i is perpendicular to l_i. The Coxeter group of M is defined to be the discrete group generated by the set of reflections along all l_i's in $\nu(x_0)$ (cf. [10]). It is easy to see that the following lemma is true (cf. [1, Proposition 6, p.84])

Lemma 2.4. *The Coxeter group of M is decomposible if and only if there exist two proper linear subspaces P_1 and P_2 of $\nu_{x_0} M$ such that $P_1 \perp P_2$ and $P_1 \cup P_2$ contains all curvature normals of M at x_0. If M is full, we always have $\nu_{x_0} M = P_1 \oplus P_2$.*

3. Proof of Theorem A

We will use the notation of Section 2. Suppose M is an isoparametric submanifold of a Hilbert space V. Define

$$V' = \overline{\mathrm{Span}\{ v(x) \mid x \in M, \ \text{and} \ v(x) \in \nu_x M \}}.$$

V' is a closed linear subspace of V. Let $V_0 = (V')^{\perp}$.

Lemma 3.1. *If the dimension of V_0 is larger than 0, then there exists a submanifold $M' \subset V'$ such that $M = M' \times V_0$ and M' is an isoparametric submanifold of V'. In fact $M' = M \cap V'$.*

Proof. We first show that $V_0 \subset E_0(x)$ for all $x \in M$. Recall that E_0 is the curvature distribution of M corresponding to the zero curvature normal. For any $x \in M$, since $V_0 \perp \nu_x M$, $V_0 \subset T_x M$. We only need to show that for any $Y \in E_i(x)$, where $n_i \neq 0$, $Y \perp V_0$. Consider the leaf S of E_i through x. Then $Y \in T_x S$. It is known that S is a round sphere centered at $c(x) = x + (n_i(x)/\|n_i\|^2)$. By Proposition 2.3, S is also totally geodesic in M. Let γ be a curve in S with $\gamma(0) = x$ and $\gamma'(0) = Y$. Then $\gamma(t) - c(x) \in \nu_{\gamma(t)} M$. Therefore $\gamma(t) - c(x) \perp V_0$. Taking derivative with respect to t, we have $\gamma'(t) \perp V_0$. In particular, $Y \perp V_0$. This shows that $V_0 \subset E_0(x)$.

Let W be the leaf of E_0 through x. It is known that W is an affine subspace. Therefore, $x + V_0 \subset W$, since $V_0 \subset E_0(x)$. This implies that $x + V_0 \subset M$ for all $x \in M$. Let $M' = M \cap V'$. Then $M = M' \times V_0$ and M' is a submanifold of V'. Observe that for any $x \in M'$, the normal space of M' at x in V' is the same as the normal space of M at x in V. If v is a parallel normal vector field on M, then $v|_{M'}$ is again a parallel normal vector field on M' in V'. So the normal bundle of M' in V' is globally flat and the shape operator of M' along $v|_{M'}$ is equal to the restriction of the shape operator of M along v to TM'. Therefore M' is isoparametric in V'.

Based on Lemma 3.1, in the rest of this section, we will assume that

$$V = \overline{\mathrm{Span}\{v(x) \mid x \in M, \text{ and } v(x) \in \nu_x M\}}.$$

We will also assume M to be full. Therefore at every point $x \in M$, the curvature normals $\{n_i(x) \mid i \in I\}$ of M at x span the normal space $\nu_x M$.

Fix an arbitrary point $x_0 \in M$. Let Δ be an open Weyl chamber in $x_0 + \nu_{x_0} M$ containing x_0. Since the normal bundle of M is flat, $M \times \Delta$ can be identified with an open subset of the normal bundle of M in a cannonical way. Therefore the end point map η induces a map from $M \times \Delta$ to V. Abusing notation, we denote this map also by η. Let $U = \eta(M \times \Delta)$. Then U is a connected open dense subset of V consisting of non-focal points of M, and η is a diffeomorphism from $M \times \Delta$ onto U (cf. [10, Theorem 9.6]). Let P be a linear subspace of $\nu_{x_0} M$, which is spanned by some curvature normals. For every $x \in M$, let $P(x)$ be the linear subspace of $\nu_x M$, which is obtained by parallel translating P from x_0 to x in the normal bundle. Let D_P be the distribution on M, which is defined as in Proposition 2.3. Note that D_P contains E_0 since P is a linear space and we treat $n_0 = 0$ as a curvature normal. Let $\tilde{D}_P$ be the distribution on U defined by

$$\tilde{D}_P(\eta(x, z)) = P(x) \oplus \eta_*|_{(x,z)}(D_P(x)),$$

where $x \in M$ and $z \in \Delta$.

Lemma 3.2. *$\tilde{D}_P$ is a totally geodesic distribution on U, and each leaf of $\tilde{D}_P$ is an open subset of a closed affine subspace of V.*

Proof. We identify U with $M \times \Delta$ via η. If X is a tangent vector field and v is a normal vector field on M, we define $\tilde{X}(x,z) = X(x)$ and $\tilde{v}(x,z) = v(x)$ as vector fields on U. For any $z \in \Delta$, let M_z be the parallel submanifold of M along z, i.e., $M_z = \{\eta(x,z) \mid x \in M\}$. It is known that M_z is also an isoparametric submanifold of V. If $X_i \in E_i$, then $\tilde{X}_i(\cdot, z) \in E_i^z$ where E_i^z is the curvature distribution of M_z with curvature normal $1/(1- < z, n_i(x_0) >) \cdot \tilde{n}_i(\cdot, z)$. If v is a parallel normal vector field on M, then $\tilde{v}(\cdot, z)$ is also a parallel normal vector field on M_z (cf. [8, p.113]). Let $\tilde{W}_i$, $i = 1, 2$, be subdistributions of $\tilde{D}_P$ defined by

$$\tilde{W}_1(x,z) = \{\tilde{X}(x,z) \mid X(x) \in D_P(x)\},$$

and

$$\tilde{W}_2(x,z) = \{\tilde{v}(x,z) \mid v(x) \in P(x)\}$$

for $x \in M$ and $z \in \Delta$. Then $\tilde{D}_P = \tilde{W}_1 \oplus \tilde{W}_2$. Now $\tilde{W}_2$ is an integrable distribution with

$$\{x+v(x) \mid v \text{ is a parallel normal vector field on } M, \text{ and } v(x_0) \in P \cap \Delta\},$$

where $x \in M$, as leaves. By Proposition 2.3, $\tilde{W}_1$ is also integrable and each leaf of $\tilde{W}_1$ is a totally geodesic submanifold of M_z for some $z \in \Delta$.

Let $\tilde{\nabla}$ be the Levi-Civita connection of the ambient Hilbert space V. Let ∇^z and α_z be the Levi-Civita connection and second fundmental form of M_z respectively. To prove the lemma, we need to show that for any vector fields $\tilde{X}$, $\tilde{Y} \in \tilde{D}_P$, which are defined as above, $\tilde{\nabla}_{\tilde{X}} \tilde{Y} \in \tilde{D}_P$. This is trivial if $\tilde{X} \in \tilde{W}_2$ since $\tilde{Y}$ does not depend on $z \in \Delta$. Therefore we assume that $\tilde{X} \in \tilde{W}_1$. If $\tilde{Y} \in \tilde{W}_1$, by Proposition 2.3, $\nabla^z_{\tilde{X}} \tilde{Y} \in \tilde{W}_1$. By the definition of D_P, $\alpha_z(\tilde{X}, \tilde{Y}) \in \tilde{W}_2$. Therefore

$$\tilde{\nabla}_{\tilde{X}} \tilde{Y} = \nabla^z_{\tilde{X}} \tilde{Y} + \alpha_z(\tilde{X}, \tilde{Y}) \in \tilde{D}_P.$$

If v is a parallel normal vector field on M and $X_i \in E_i$ with $n_i(x_0) \in P$, then

$$\tilde{\nabla}_{\tilde{X}_i} \tilde{v} = -A^z_{\tilde{v}}(\tilde{X}_i) \in E_i^z \subset \tilde{W}_1 \subset \tilde{D}_P,$$

where A^z is the shape operator of M_z. This proves that $\tilde{\nabla}_{\tilde{X}} \tilde{Y} \in \tilde{D}_P$ if $\tilde{Y} \in \tilde{W}_2$. Therefore $\tilde{D}_P$ is totally geodesic. The rest of the lemma follows from the fact that any connceted totally geodesic submanifold of a Hilbert space, whose tangent spaces are closed subspaces of the Hilbert space, is an open subset of a closed affine subspace.

For any point $x \in M$, let $L_P(x)$ and $\tilde{L}_P(x)$ be the leaves of D_P and $\tilde{D}_P$ through x respectively. Let $W_P(x)$ be the closed affine subspace of V which contains $\tilde{L}_P(x)$ as an open subset. Then for any $y \in \tilde{L}_P(x)$,

$$W_P(x) = y + T_y\tilde{L}_P(x) = y + \tilde{D}_P(y).$$

Lemma 3.3. *$L_P(x)$ is a full isoparametric submanifold of $W_P(x)$. Moreover, if $P \cup P^\perp$ contains all the curvature normals of M at x_0, then $\tilde{L}_P(x)$ is equal to the set of all non-focal points of $L_P(x)$ in $W_P(x)$, and therefore it is open and dense in $W_P(x)$.*

Proof. Let v be a parallel normal vector field on M such that $v(x_0) \in P$. Then $v|_{L_P(x)}$ is a parallel normal vector field of $L_P(x)$ in $W_P(x)$. Since the codimension of $L_P(x)$ in $W_P(x)$ is equal to the dimension of P, this shows that the normal bundle of $L_P(x)$ in $W_P(x)$ is globally flat. Since the shape operator of $L_P(x)$ along $v|_{L_P(x)}$ is the restriction of the shape operator of M along v to the tangent space of $L_P(x)$, it follows that $L_P(x)$ is an isoparametric submanifold of $W_P(x)$. $L_P(x)$ is full since its normal space in $W_P(x)$ is spanned by curvature normals of $L_P(x)$.

Let R be the set of all non-focal points of $L_P(x)$ in $W_P(x)$. If $y \in W_P(x)$ is a focal point of $L_P(x)$, then it is also a focal point of M. Since every point in $\tilde{L}_P(x)$ is a non-focal point of M, we have $\tilde{L}_P(x) \in R$. On the other hand, if $P \cup P^\perp$ contains all the curvature normals of M, then for any $y \in W_P(x)$, if y is a focal point of M, it is also a focal point of $L_P(x)$. To see this, take $y_0 \in L_P(x)$ such that $y - y_0 \perp T_{y_0}L_P(x)$. Since $y \in W_P(x) = y_0 + (T_{y_0}L_P(x) \oplus P(y_0))$, we have $y - y_0 \in P(y_0) \subset \nu_{y_0}M$. If y is a focal point of M, then there exists a curvature normal n_i of M such that $< y - y_0, n_i(y_0) >= 1$. $n_i(y_0)$ must lie in $P(y_0)$ since otherwise $< y - y_0, n_i(y_0) >$ would be zero by the assumption. Hence $n_i(y_0)$ is also a curvature normal of $L_P(x)$. This shows that y is a focal point of $L_P(x)$. Therefore $R \subset U$. It follows that $R = \tilde{L}_P(x)$. This finishes the proof of the lemma.

Lemma 3.4. *Let v be a parallel normal vector field on M which satisfies $v(x_0) \perp P$. Then $\tilde{v}|_{\tilde{L}_P(x)} \equiv constant$.*

Proof. Since $\tilde{v}$ does not depend on $z \in \Delta$, it is equivalent to showing that $v|_{L_P(x)} \equiv constant$. To prove this, we need to show that for any curve γ in $L_P(x)$, $\dot{v}(t) \equiv 0$, where $v(t) = v(\gamma(t))$. Since $v(x_0) \perp P$, the shape operator $A_v|_{D_P} = 0$. Therefore, we have $\dot{v}(t) = -A_{v(t)}\dot{\gamma}(t) = 0$, since $\dot{\gamma}(t) \in D_P$.

For every linear subspace P of ν_{x_0}, we can define a closed linear subspace, V_P, of the ambient Hilbert space V as follows:

$$V_P = \overline{Span\{v(x) \mid x \in M, \text{ and } v(x) \in P(x)\}}.$$

Lemma 3.5. *If P_1 and P_2 are two linear subspaces of $\nu_{x_0}M$ such that $P_1 \perp P_2$ and $P_1 \cup P_2$ contains all curvature normals of M at x_0 (this implies $\nu_{x_0}M = P_1 \oplus P_2$ since M is full), then $V_{P_1} \perp V_{P_2}$.*

Proof. Let v_1 and v_2 be two parallel normal vector fields on M such that $v_1(x_0) \in P_1$ and $v_2(x_0) \in P_2$. We need to show that for any two points x_1, $x_2 \in M$, $v_1(x_1) \perp v_2(x_2)$. In fact, we will prove that for any two points y_1, $y_2 \in U$, $\tilde{v}_1(y_1) \perp \tilde{v}_2(y_2)$. Note that this is trivial if $y_1 = y_2$.

Let $\tilde{L}_1$ be the leaf of $\tilde{D}_{P_1}$ through y_2, and W_1 the closed affine subspace which contains $\tilde{L}_1$ as an open subset. By Lemma 3.4, $\tilde{v}_2|_{\tilde{L}_1} \equiv$ *constant*. For any $y \in \tilde{L}_1$, let $\tilde{L}_2(y)$ be the leaf of the distribution $\tilde{D}_{P_2}$ through y, and $\tilde{W}_2(y)$ the closed affine subspace which contains $\tilde{L}_2(y)$ as an open subset. By Lemma 3.4, $\tilde{v}_1|_{\tilde{L}_2(y)} \equiv$ *constant* for all $y \in \tilde{L}_1$. Since at the point $y \in \tilde{L}_1 \cap \tilde{L}_2(y)$, $\tilde{v}_1(y) \perp \tilde{v}_2(y)$, we have $\tilde{v}_1(y') \perp \tilde{v}_2(y_2)$ for all $y' \in \tilde{L}_2(y)$. Let $U' = \cup \tilde{L}_2(y)$ where y runs through $\tilde{L}_1$. Then we have $\tilde{v}_1(y') \perp \tilde{v}_2(y_2)$ for all $y' \in U'$. By Lemma 3.3, $\tilde{L}_1$ is dense in $\tilde{W}_1$ and $\tilde{L}_2(y)$ is dense in $\tilde{W}_2(y)$ for all $y \in \tilde{L}_1$. Therefore U' is dense in V. By continuity of $\tilde{v}_1$, we have $\tilde{v}_1(y') \perp \tilde{v}_2(y_2)$ for all $y' \in U$. In particular, $\tilde{v}_1(y_1) \perp \tilde{v}_2(y_2)$. This finishes the proof of the lemma.

By assumption, V is spanned by the set of all normal vectors of M. Therefore we have

Corollary 3.6. *Let P_1 and P_2 be as in Lemma 3.5. Then $V = V_{P_1} \oplus V_{P_2}$.*

From the proof of Lemma 3.5, we also have

Corollary 3.7. *Let P_i, $i = 1, 2$, be as in Lemma 3.5. For any $x \in M$, let $L_{P_i}(x)$ and $W_{P_i}(x)$ be defined as in Lemma 3.3. Then*

$$V_{P_i} = \overline{Span\{v(y) \mid y \in L_{P_i}(x), \text{ and } v(y) \in \nu_y L_{P_i}(x)\}},$$

where $\nu_y L_{P_i}(x)$ is the normal space, at y, of $L_{P_i}(x)$ considered as a submanifold of $W_{P_i}(x)$.

Let P_i, $i = 1, 2$, be as in Lemma 3.5. For any $x \in M$, define $M_i(x) = (x + V_{P_i}) \cap M$.

Lemma 3.8. *For $i = 1, 2$ and $x \in M$, $M_i(x)$ is an isoparametric submanifold of $x + V_{P_i}$. Let $F_i(x) = T_x M_i(x)$. Then F_i defines a totally geodesic distribution on M whose leaf through x is $M_i(x)$. Moreover $F_1(x) \perp F_2(x)$ and $T_x M = F_1(x) \oplus F_2(x)$.*

Proof. We use the notation in Lemma 3.3. Applying Lemma 3.1 to $L_{P_i}(x)$ in $W_{P_i}(x)$, and using Corollary 3.7, we know that $M_i(x)$ is an isoparametric submanifold of $x + V_{P_i}$. $M_i(x)$ is totally geodesic in M since it is totally geodesic in $L_{P_i}(x)$ and $L_{P_i}(x)$ is totally geodesic in M. From the definition of F_i, we know that the F_i's are integral distributions on M with $M_i(x)$, $x \in M$, as leaves. They are also totally geodesic since their leaves are totally geodesic. $F_1(x) \perp F_2(x)$ since $V_{P_1} \perp V_{P_2}$. Since the codimension of $F_i(x)$ in $x + V_{P_i}$ is equal to the dimension of P_i, by Corollary 3.6, the codimension of $F_1(x) \oplus F_2(x)$ in V is equal to the codimension of M. Therefore $T_x M = F_1(x) \oplus F_2(x)$.

To prove Theorem A, it remains to show that $M = M_1(x) \times M_2(x)$ for some $x \in M$. If M is a simply connected, finite dimensional Riemannian manifold, this follows immediately from the de Rham decomposition theorem. However, in our case, we do not assume that M is simply connected. We also do not know whether the de Rham decomposition theorem is true for infinite dimensional manifolds. So we give a complete proof below. We need

Lemma 3.9. *Let M_1 be a leaf of F_1. Let γ be an arbitrary curve in M_1. If β_s is a one-parameter family of geodesics in M such that $\beta_s(0) = \gamma(s)$, $\dot{\beta}_0(0) \perp M_1$, and $\nabla_{\dot{\gamma}(s)} \dot{\beta}_s(0)|_{s=0} = 0$, then for every t,*

$$\frac{\partial}{\partial s} \beta_s(t)|_{s=0} \in F_1(\beta_0(t)).$$

Proof. Let $J(t) = \frac{\partial}{\partial s} \beta_s(t)|_{s=0}$. Let $J_i(t)$, $i = 1, 2$, be the orthogonal projection of $J(t)$ to $F_i(\beta_0(t))$. We need to show that $J_2 \equiv 0$.

In fact $J(0) = \dot{\gamma}(0) \in F_1(\gamma(0))$ and

$$J'(0) = \nabla_{\dot{\beta}_0(t)} J(t)|_{t=0} = \nabla_{\dot{\gamma}(s)} \dot{\beta}_s(0)|_{s=0} = 0.$$

Since F_2 is totally geodesic, we have $J_2(0) = J_2'(0) = 0$. Since J is a Jacobi field and F_2 is totally geodesic, $J_2(t)$ is a Jacobi field as well. Therefore $J_2 \equiv 0$.

Lemma 3.10. *Let $M_1(x)$ and $M_2(x)$ be the leaves of F_1 and F_2 through x respectively. Then for any $y \in M_1(x)$ and $z \in M_2(x)$, the leaves $M_2(y)$ and $M_1(z)$ have nonempty intersection.*

Proof. We prove this lemma in two steps.

Step 1, if x and z can be connected by a geodesic in $M_2(x)$, then $M_2(y) \cap M_1(z) \neq \emptyset$.

In fact, let β_0 be a geodesic in $M_2(x)$ such that $\beta_0(0) = x$ and $\beta_0(1) = z$. Let γ be an arbitray curve in $M_1(x)$ such that $\gamma(0) = x$ and $\gamma(1) = y$. Let $X(s)$ be the parallel translation of $\dot{\beta}_0(0)$ along $\gamma(s)$. Since $M_1(x)$ is totally geodesic, $X(s) \in F_2(\gamma(s))$ for all $s \in [0, 1]$. Let β_s be the geodesic in M with initial velocity $X(s)$. Since $M_2(\gamma(s))$ is totally geodesic, $\beta_s(t) \in M_2(\gamma(s))$ for all t. In particular, $\beta_1(1) \in M_2(y)$. On the other hand, by Lemma 3.9, $\frac{\partial}{\partial s}\beta_s(1) \in F_1(\beta_s(1))$ for all s. Therefore the curve $s \longmapsto \beta_s(1)$ is contained in $M_1(z)$. In particular, $\beta_1(1) \in M_1(z)$. Hence $\beta_1(1) \in M_2(y) \cap M_1(z) \neq \emptyset$.

Step 2, for an arbitrary $z \in M_2(x)$, there exists a piecewise geodesic β in $M_2(x)$ with finitely many singular points $0 = t_0 < t_1 < \cdots < t_n = 1$ such that $\beta(0) = x$ and $\beta(1) = z$. By assumption, $y \in M_2(y) \cap M_1(\beta(t_0)) \neq \emptyset$. Assume $M_2(y) \cap M_1(\beta(t_k)) \neq \emptyset$. Choose $y_k \in M_2(y) \cap M_1(\beta(t_k))$. Applying Step 1 to $x_k = \beta(t_k)$, y_k, and $z_k = \beta(t_{k+1})$, we have $M_2(y_k) \cap M_1(z_k) \neq \emptyset$. Since $M_2(y_k) = M_2(y)$, we have $M_2(y) \cap M_1(\beta(t_{k+1})) \neq \emptyset$. By induction, the lemma is therefore proved.

Corollary 3.11. *Let M_1 be a leaf of F_1. Then for any x, $y \in M_1$, the translation map $f(z) = z + y - x$ of the ambient Hilbert space V maps $M_2(x)$ isometrically onto $M_2(y)$.*

Proof. First we prove that if $z \in M_2(x)$, then

$$f(z) = z + y - x \in M_2(y).$$

By Lemma 3.10, $M_2(y) \cap M_1(z) \neq \emptyset$. It is obvious that

$$(y + V_{P_2}) \cap (z + V_{P_1}) = \{z + y - x\}.$$

Since $M_2(y) \subset y + V_{P_2}$ and $M_1(z) \subset z + V_{P_1}$, we must have $M_2(y) \cap M_1(z) = \{z + y - x\}$. In particular, this implies that $f(z) \in M_2(y)$. Therefore f maps $M_2(x)$ into $M_2(y)$. Similarly, we can show that f^{-1} maps $M_2(y)$ into $M_2(x)$. Since f is an isometry of the ambient Hilbert space, it follows that f maps $M_2(x)$ isometrically onto $M_2(y)$.

Corollary 3.12. *For any $x \in M$, $M = M_1(x) \times M_2(x)$.*

Proof. Define

$$\begin{array}{rccc} \tilde{f}: & (x+V_{P_1})\times(x+V_{P_2}) & \longrightarrow & V, \\ & (y,z) & \longmapsto & y+z-x. \end{array}$$

It is clear that $\tilde{f}$ is an isometry. From Corollary 3.11, we know $\tilde{f}$ maps $M_1(x)\times M_2(x)$ into M. Since $F_1\oplus F_2=TM$, $\tilde{f}(M_1(x)\times M_2(x))$ is an open and closed submanifold of M, and is therefore equal to M since M is connected.

Proof of Theorem A. This theorem follows from Lemma 2.4 and Corollary 3.12.

4. Proofs of Theorem B and Theorem C

First we review the relationship between hyperpolar actions on compact Lie groups and isoparametric submanifolds in Hilbert spaces, which was discovered by Terng (cf. [11]). Let G be a compact, connected, semisimple Lie group, equipped with a bi-invariant metric, and $\mathfrak{g}$ its Lie algebra. Let $V=H^0([0,1],\mathfrak{g})$ be the Hilbert space of H^0-maps from $[0,1]$ to $\mathfrak{g}$. One can think of V as the space of connections of the trivial principal G-bundle over $[0,1]$. For every $u\in V$, let $E_u(t)$ be the parallel translation corresponding to the connection defined by u. More precisely, $E_u:[0,1]\longrightarrow G$ is the unique solution to the initial value problem

$$\begin{cases} E^{-1}E'=u, \\ E(0)=e. \end{cases}$$

Let $\Phi(u)=E_u(1)$ be the holonomy of the connection u. Φ defines a fibration of V over G. For any $x\in\mathfrak{g}$, let $\hat{x}$ denote the constant path in $\mathfrak{g}$ with value x. The map $x\longmapsto\hat{x}$ defines an embedding of the Lie algebra $\mathfrak{g}$ into V. The restriction of Φ to the image of $\mathfrak{g}$ is just the exponential map on G.

Let $\hat{G}=H^1([0,1],G)$ be the Hilbert Lie group of H^1-paths from $[0,1]$ to G. $\hat{G}$ acts on V isometrically via gauge transformations:

$$g\cdot u=gug^{-1}-g'g^{-1},$$

where $g\in\hat{G}$ and $u\in V$. It is easy to see that this action is transitive. Let H be a closed, connected subgroup of $G\times G$. Define

$$P(G,H)=\{g\in\hat{G}\mid(g(0),g(1))\in H\}.$$

The action of $P(G,H)$ on V via gauge transformation is proper and Fredholm. The holonomy map Φ maps the orbit of the $P(G,H)$-action on V onto the orbit of the H-action on G. Suppose that the H-action on G is hyperpolar, and A is a section of the H-action through e with Lie algebra $\mathfrak{a}$. Then the $P(G,H)$-action on V is also hyperpolar with section $\hat{\mathfrak{a}} = \{\hat{a} \mid a \in \mathfrak{a}\}$. Moreover $u \in V$ is a singular point of the $P(G,H)$-action if and only if $\Phi(u)$ is a singular point of the H-action (cf. [11, Theorem 1.2]). Without loss of generality, we will always assume e to be a regular point of the H-action on G. Then $\hat{0}$, which is the origin of V, is a regular point of the $P(G,H)$-action. Let $M = P(G,H)\cdot\hat{0}$. Then M is an isoparametric submanifold of V with normal space $\nu_{\hat{0}}M = \hat{\mathfrak{a}}$. The Coxeter group of M as an isoparametric submanifold is isomorphic to the affine Weyl group $\hat{W}$ of the H-action on G which was defined in the introduction (cf. [10, Theorem 8.10] and [11, Theorem 1.2]).

Lemma 4.1. *Assume that the H-action on G is hyperpolar, and e is a regular point. Let $\mathfrak{a} \subset \mathfrak{g}$ be the Lie algebra of the section of the H-action through e. If there exist two proper linear subspaces $\mathfrak{a}_1$ and $\mathfrak{a}_2$ of $\mathfrak{a}$, such that $\mathfrak{a}_1 \perp \mathfrak{a}_2$ and $\hat{\mathfrak{a}}_1 \cup \hat{\mathfrak{a}}_2$ contains all curvature normals of $M = P(G,H)\cdot\hat{0}$ at $\hat{0}$, then the ideals of $\mathfrak{g}$ generated by $\mathfrak{a}_1$ and $\mathfrak{a}_2$ are perpendicular to each other.*

Proof. For any $b \in \mathfrak{a}$, we define a normal vector field $\tilde{b}$ on M by $\tilde{b}(g\cdot\hat{0}) = g\hat{b}g^{-1}$, where $g \in P(G,H)$. Since the $P(G,H)$-action is hyperpolar, $\tilde{b}$ is a parallel normal vector field on M. Let $b_i \in \mathfrak{a}_i$ for $i = 1,2$. By Lemma 3.5, for any $g_i \in P(G,H)$, $i = 1,2$, $\tilde{b}_1(g_1\cdot\hat{0}) \perp \tilde{b}_2(g_2\cdot\hat{0})$. Therefore $g_1\hat{b}_1g_1^{-1} \perp g_2\hat{b}_2g_2^{-1}$. In particular, we have $g\hat{b}_1g^{-1} \perp \hat{b}_2$ for all $g \in P(G,H)$. So we have

$$0 = < g\hat{b}_1g^{-1}, \hat{b}_2 > = \int_0^1 < g(t)b_1g(t)^{-1}, b_2 > \,\mathrm{d}t$$

for all $g \in P(G,H)$. Since any path in G which is obtained by a reparametrization of $g \in P(G,H)$ is also in $P(G,H)$, the above equation holds for all parametrizations of g. In case g is a continuous path, this implies that $< g(t)b_1g(t)^{-1}, b_2 > = 0$ for all t. Since for any point in G, we can choose a smooth curve $g \in P(G,H)$ which passes through this point, we conclude that b_2 is perpendicular to the adjoint orbit of b_1 for any $b_i \in \mathfrak{a}_i$, $i = 1,2$. Therefore $\mathfrak{a}_2$ is perpendicular to the ideal generated by $\mathfrak{a}_1$. Since the orthogonal complement of an ideal is an ideal as well, the lemma follows.

Proof of Theorem C. Without loss of generality, we may assume

e is a regular point of the H-action on G. Let $M = P(G, H) \cdot \hat{0}$. By Lemma 2.4, if $\hat{W}$ is decomposable, then there exists two proper linear subspaces $\hat{\mathfrak{a}}_1$ and $\hat{\mathfrak{a}}_2$ of $\hat{\mathfrak{a}} = \nu_{\hat{0}} M$ such that $\hat{\mathfrak{a}}_1 \perp \hat{\mathfrak{a}}_2$ and $\hat{\mathfrak{a}}_1 \cup \hat{\mathfrak{a}}_2$ contains all curvature normals of M at $\hat{0}$. By Lemma 4.1, the ideals generated by $\mathfrak{a}_i$, $i = 1, 2$, are proper ideals of the Lie algebra of G. This contradicts to the assumption that G is simple.

Before proving Theorem B, we first prove the following general fact.

Lemma 4.2. *Assume that M is a compact Riemannian H-manifold and the H-action on M is hyperpolar with flat tori as sections. If the H-action is decomposed into the product of two isometric actions, then each component is hyperpolar.*

Proof. Suppose that M_i is an H_i-manifold for $i = 1, 2$ such that the H-action on M is ω-equivalent to the product action of $H_1 \times H_2$ on $M_1 \times M_2$. Then the $(H_1 \times H_2)$-action is hyperpolar with flat tori as sections. Let $x = (x_1, x_2)$, where $x_i \in M_i$, be a regular point of the $(H_1 \times H_2)$-action. We identify M_1 with $M_1 \times \{x_2\}$ and M_2 with $\{x_1\} \times M_2$. Let Σ be the section of the $(H_1 \times H_2)$-action through x. Then $T_x\Sigma = \nu_1 \oplus \nu_2$ for some subspaces ν_i, $i = 1, 2$, of $T_x M_i$. Let Σ_i, $i = 1, 2$, be the closure of the set $\exp_x \nu_i$. Since both Σ and M_i are totally geodesic in $M_1 \times M_2$, $\Sigma_i \subset M_i \cap \Sigma$ for $i = 1, 2$. Moreover, since Σ is a flat torus, Σ_i, $i = 1, 2$, is a flat, totally geodesic submanifold of M_i. Since $\Sigma_1 \cap \Sigma_2 \subset M_1 \cap M_2 = \{x\}$, comparing the dimensions of Σ_i's with that of Σ, we know that $\Sigma_i = \exp_x \nu_i$ for $i = 1, 2$ and $\Sigma = \Sigma_1 \times \Sigma_2$. It then follows that Σ_i is the section of the H_i-action on M_i. Therefore the H_i-action, $i = 1, 2$, is hyperpolar.

Proof of Theorem B. $(a \Rightarrow b)$. We first notice that if G is a product of two Riemannian manifolds, then each component is a compact Lie group due to the fact that the holonomy representation of G is just the adjoint representation. The statement then follows trivially from Lemma 4.2.

$(b \Rightarrow a)$. Without loss of generality, we assume that e is a regular point of the H-action on G. Suppose that the action of the generalized Weyl group $W(A)$ on A is ω-equivalent to the product action of $W_1 \times W_2$ on $A_1 \times A_2$, where W_1 and W_2 are two finite groups and $A_1 \times A_2$ is isometric to A. We may identify $A_1 \times A_2$ with A. By the definition of ω-equivalence, the $W(A)$-action and the $(W_1 \times W_2)$-action have the same singular point sets. Let S be the set of all singular points of the $W(A)$-action on A, and S_i, $i = 1, 2$, the set of all singular points of the W_i-action on A_i. Then $S = (S_1 \times A_2) \cup (A_1 \times S_2)$. It is known that S

is a union of finitely many totally geodesic hypersurfaces of A (cf. [2]).

Now consider the $P(G,H)$-action on $V = H^0([0,1],\mathfrak{g})$. It is known that the holonomy map Φ maps singular points of the $P(G,H)$-action on V to singular points of the H-action on G (cf. [11, Theorem 1.2]). Let $M = P(G,H)\cdot\hat{0}$. The set of all singular points of the $P(G,H)$-action is also the set of all focal points of M. Let $\mathfrak{a}$, $\mathfrak{a}_1$, and $\mathfrak{a}_2$ be the Lie algebras of A, A_1, and A_2 respectively. The restriction of the holonomy map to $\hat{\mathfrak{a}} = \nu_{\hat{0}}M$, i.e., $\Phi|_{\hat{\mathfrak{a}}} : \hat{\mathfrak{a}} \longrightarrow A$, is locally isometric. It maps the union of focal hyperplanes in $\hat{\mathfrak{a}}$ locally isometrically to S (cf, [11, Theorem 1.10]). Consequently, the normal vector of each focal hyperplane in $\hat{\mathfrak{a}}$ is contained either in $\hat{\mathfrak{a}}_1$ or $\hat{\mathfrak{a}}_2$. Let $\mathfrak{g}_1$ be the ideal (of $\mathfrak{g}$) generated by $\mathfrak{a}_1$, and $\mathfrak{g}_2$ the orthogonal complement of $\mathfrak{g}_1$ in $\mathfrak{g}$. By Lemma 4.1, $\mathfrak{g}_2$ is a proper ideal of $\mathfrak{g}$ which contains $\mathfrak{a}_2$. Let G_i, $i = 1,2$, be the closed connected Lie subgroup of G whose Lie algebra is $\mathfrak{g}_i$. Since G is simply connected, $G = G_1 \times G_2$. Therefore $G \times G$ also splits as a product of $G_1 \times G_1$ and $G_2 \times G_2$. Let H_i, $i = 1,2$, be the projection of H to $G_i \times G_i$. It is easy to see that $\mathfrak{a}_i$, $i = 1,2$, is perpendicular to the orbit of the H_i-action on G_i at the unit element. Since $H \subset H_1 \times H_2$, this shows that the orbits through e of the H-action and the $(H_1 \times H_2)$-action on $G = G_1 \times G_2$ have the same normal space at e. Consequently, these two orbits coincide with each other. By a well known lemma of Hermann [5], all orbits of the $(H_1 \times H_2)$-action are perpendicular to A. Hence the $(H_1 \times H_2)$-action is hyperpolar with the same principal orbits as those of the H-action. Since the principal orbits of a hyperpolar action determine the other orbits, the H-action is ω-equivalent to the product action of $H_1 \times H_2$.

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