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A PRODUCT FORMULA FOR THE SEIBERG-WITTEN INVARIANTS AND THE GENERALIZED THOM CONJECTURE

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1. Introduction

The Thom Conjecture asserts that any compact, embedded surface in $\mathbf{C}P^2$ of degree $d > 0$ must have genus at least as large as the smooth algebraic curve of the same degree, namely $(d-1)(d-2)/2$. More generally, one can ask whether in any algebraic surface a smooth algebraic curve is of minimal genus in its homology class. There was one significant result in this direction. Using $SU(2)$-Donaldson invariants, Kronheimer showed in [3] that this result is true for curves of positive self-intersection in a large class of simply connected surfaces with $b_2^+ > 1$. Unfortunately, for technical reasons, this argument does not extend to cover the case of $\mathbf{C}P^2$. It is the purpose of this paper to prove the general result that a smooth holomorphic curve of non-negative self-intersection in a compact Kähler manifold is genus minimizing.

For any closed, orientable riemann surface C we denote its genus by $g(C)$.

Theorem 1.1 (Generalized Thom Conjecture). *Let X be a compact Kähler surface and let $C \hookrightarrow X$ be a smooth holomorphic curve. Suppose that $C \cdot C \geq 0$. Let $C' \hookrightarrow X$ be a C^∞-embedding of a smooth riemann surface representing the same homology class as C. Then $g(C) \leq g(C')$.*

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In fact, there is a generalization of this result to symplectic manifolds.

Theorem 1.2. *Let X be a compact symplectic four-manifold and let $C \hookrightarrow X$ be a smooth symplectic curve with $C \cdot C \geq 0$. (A symplectic curve is one for which the restriction of the symplectic form is everywhere non-zero.) Let $C' \hookrightarrow X$ be a C^∞ embedding of a riemann surface representing the same homology class as C. Then $g(C') \geq g(C)$.*

The Thom Conjecture and very similar generalizations of it have been established independently by Kronheimer-Mrowka; see [4].

These results are based on the new Seiberg-Witten monopole invariant, [14], flowing from advances in physics [7], [8]. This is a gauge-theory invariant defined using complex line bundles and $Spin^c$-structures on the four-manifold. According to the conjectures of Witten (or rather according to the results deduced by Witten using mathematically non-rigorous physics arguments; see [14]) these invariants should contain equivalent information to the $SU(2)$-invariants defined by Donaldson at least in the case where $b_2^+ > 1$. But from many points of view, these $U(1)$ gauge-theory invariants are much simpler to work with. Hence, in a practical sense, they are more powerful. This result is an example of that power. It is probably true that overcoming a series of technical difficulties, one could establish the Generalized Thom Conjecture and its symplectic generalization using the $SU(2)$-invariants, though this has not been done.

We deduce the Generalized Thom Conjecture from a product formula for the Seiberg-Witten monopole invariants. This is not a general product formula, though it is easy to believe that there is one. Here we deal only with the simplest case of a product formula – one decomposes the manifold along a certain three-manifold ($S^1 \times C$) and we arrange to be in a context in which the 'character variety', i.e., the space of solutions to the corresponding equations on the three-manifold, is a single smooth point. In this context, the analogue of the Floer homology is particularly simple and leads to a particularly simple product formula. The proof of the Generalized Thom Conjecture and its symplectic generalization for curves of genus $g > 1$ is a direct application of this product formula. The case of tori is handled by a different argument using the Seiberg-Witten invariants but not the product formula.

Here we use the product formula to prove non-vanishing results for Seiberg-Witten invariants. It is possible in favorable circumstances to use this product formula together with vanishing results to completely calulate Seiberg-Witten invariants of manifolds obtained by gluing together pieces whose Seiberg-Witten invariants are known. In this paper we treat the case when the three-manifold is $S^1 \times C$, where C is a surface of genus g, and the determinant line bundle of the $Spin^c$-structure has degreee $\pm(2g-2)$ along C. It is possible to generalize to the case of line bundles of other degrees.

In some respects the arguments in [4] are of a similar spirit to ours, relying as they do on the Seiberg-Witten invariants. But instead of using a product formula as we do here, Kronheimer-Mrowka establish a vanishing theorem for the Seiberg-Witten invariants in a related context – a context where there are no solutions to the corresponding equations on the three-manifold.

Acknowledgments

The subject of gauge theory and 4-manifolds was set on its head by the announcement of the new Seiberg-Witten monopole invariants. The first two authors learned of these invariants from Tom Mrowka, who, together with Peter Kronheimer, saw immediately that they were a powerful improvement in a practical sense over the $SU(2)$-Donaldson invariants and, after repeated efforts, convinced us of same. It is a pleasure to thank Tom for his generosity in sharing his enthusiasm for these new invariants as well as explaining to us the technical details of the new theory. It is also a pleasure to thank Bob Friedman for several helpful conversations about Kähler geometry duing the course of this work.

2. Review of the definition of the Seiberg-Witten monopole invariant

2.1. $Spin^c$-structures

Recall that $Spin^c(n) = Spin(n) \times_{\{\pm 1\}} U(1)$ admits a natural map to $SO(n)$ with kernel the central S^1. By a $Spin^c$-structure on an oriented riemannian n-manifold X we mean a lifting of the principal $SO(n)$ bundle associated to the tangent bundle to a principal $Spin^c(n)$-bundle. Given such a lifting $\tilde{P} \to X$, there are the associated complex spin bundles. In the case where $n = 4$, there are two inequivalent spin bundles $S^{\pm}(\tilde{P})$, each of which is a complex two-plane bundle with hermitian metric.

Of course, $Spin^c(n)$ has a natural homomorphism to $U(1)$ given by $[B,\zeta] \mapsto \zeta^2$. Correspondingly, every $Spin^c$-structure $\tilde{P}$ on a riemannian n-manifold X has an associated complex line bundle $\mathcal{L}$ which we call its *determinant line bundle.* Of course, $c_1(\mathcal{L})$ must be characteristic in the sense that its mod two reduction is equal to $w_2(X)$. Fixing a connection A on the determinant line bundle $\mathcal{L}$ induces Dirac operators

$$\partial\!\!\!/_A \colon C^\infty(\mathcal{S}^\pm(\tilde{P})) \to C^\infty(\mathcal{S}^\mp(\tilde{P})).$$

These operators are first-order, linear, elliptic operators and are formal adjoints of each other.

2.2. The Seiberg-Witten equations

Following Seiberg-Witten (see for example [14]) the Seiberg-Witten equations associated to a $Spin^c$-structure $\tilde{P}$ on an oriented, riemannian four-manifold X with a metric g are a pair of non-linear elliptic equations for a unitary connection A on the determinant line bundle $\mathcal{L}$ of $\tilde{P}$ and a plus spinor field ψ, i.e., a section of the plus spin bundle $S^+(\tilde{P})$. The equations are:

$$F_A^+ = q(\psi),$$
$$\partial\!\!\!/_A(\psi) = 0.$$

Here, q is a natural quadratic bundle map from $\mathcal{S}^+(\tilde{P})$ to $\Lambda^2_+(X; i\mathbf{R})$, and $\partial\!\!\!/_A$ is the usual Dirac operator defined using the Levi-Civitá connection on the frame bundle for X and the connection A; see, for example, [5]. These non-linear equations are elliptic and, in the case where X is a closed manifold, the index of the system modulo the action of the gauge group of automorphisms of $\tilde{P}$ covering the identity on the frame bundle is, according to the Atiyah-Singer index formula, given by

$$d(\tilde{P}) = \left(c_1(\mathcal{L})^2 - (2\chi(X) + 3\sigma(X))\right)/4,$$

where $\chi(X)$ and $\sigma(X)$ are respectively the Euler characteristic and the signature of X. Notice that $d(\tilde{P})$ depends only on $c_1(\mathcal{L})$. For this reason we also denote it by $d(\mathcal{L})$. The quotient space of the space of solutions to these equations modulo the action of the group of gauge transformations is the Seiberg-Witten moduli space and is denoted $\mathcal{M}(\tilde{P}, g)$. This moduli space is compact. To obtain a smooth moduli space, it may be necessary to perturb the equations. We take perturbations of the form:

$$F_A^+ = q(\psi) + i\eta^+, \qquad (1)$$
$$\partial\!\!\!/_A(\psi) = 0,$$

where η^+ is a real, self-dual two-form on X. For a generic such η^+, or even for an η^+ generic among self-dual two-forms supported in a small ball in X, the resulting moduli space $\mathcal{M}(\tilde{P}, g, \eta^+)$ is a compact, smooth manifold of dimension equal to $d(\tilde{P})$, [5].

2.3. The definition of the invariant

Fix a closed, oriented, riemannian four-manifold X with metric g, and choose an orientation for $H^2_+(X;\mathbf{R}) \oplus H^1(X;\mathbf{R})$. Let us consider a $Spin^c$-structure $\tilde{P}$ on X. For a generic η^+ the moduli space $\mathcal{M}(\tilde{P}, g, \eta^+)$ is a smooth submanifold of the configuration space, that is to say, of the space of all pairs (A, ψ) modulo the action of the group of gauge transformations. Removing the reducible points consisting of pairs where ψ is identically zero, leaves the space $\mathcal{X}$ of irreducible configurations. The based version $\mathcal{X}^0 \to \mathcal{X}$ is a principal circle bundle whose first Chern class is denoted by $\mu \in H^2(\mathcal{X};\mathbf{Z})$. Provided that the $d(\tilde{P})$ is even (or equivalently provided that $b_1(X) + b_2^+(X)$ is odd), $\mathcal{M}(\tilde{P}, g, \eta^+)$ has a fundamental cycle which represents a homology class of even degree in $\mathcal{X}$. The orientation of $H^1(X;\mathbf{R}) \oplus H^2_+(X;\mathbf{R})$ is necessary in order to orient the moduli space and hence determine the sign of the homology class.

The definition of the Seiberg-Witten invariant of the $Spin^c$-structure in the case where $d(\tilde{P})$ is even, say $2d$, is the value of the integral of μ^d over the fundamental class of $\mathcal{M}(\tilde{P}, g, \eta^+)$. If $d(\tilde{P})$ is odd, then by definition the Seiberg-Witten invariant vanishes. Provided that $b_2^+(X) > 1$ this definition gives a well-defined invariant independent of the choice of metric g and perturbing self-dual form η^+. Thus, for such manifolds X we define the Seiberg–Witten invariant as a function

$$SW_X \colon \{Spin^c\text{–structures}\} \to \mathbf{Z}.$$

It is often convenient to amalgamate this information into a function

$$SW_X \colon \mathcal{C}(X) \to \mathbf{Z},$$

where $\mathcal{C}(X) \subset H^2(Z;\mathbf{Z})$ is the subset of characteristic classes (those whose mod two reduction is the second Stiefel-Whitney class). The value of SW_X on a class k is the sum over the (finite) set of all isomorphism classes of $Spin^c$-structures on X with the given class as the first Chern class of the determinant line bundle. The invariant takes non-zero values on only finitely many classes. Changing the orientation on $H^1(X;\mathbf{R})\oplus H^2_+(X;\mathbf{R})$ reverses the sign of this invariant. By convention this invariant vanishes on any characteristic cohomology class for which this index is negative.

Now suppose that $b_2^+(X) = 1$. Then the value on a cohomology class $k \in H^2(X;\mathbf{Z})$ of the invariant defined using the moduli space $\mathcal{M}(\tilde{P}, g, \eta^+)$ is denoted by $SW_{X,g,\eta^+}(k)$. This invariant is defined only when there are no reducible solutions to the perturbed Seiberg-Witten equations (1); i.e., only when $2\pi k + \eta^+$ has a non-zero L^2-projection onto the space of g-harmonic self-dual two-forms. As we vary (g, η^+) the value of $SW_{X,g,\eta^+}(k)$ depends only on the component of the double cone

$$\{x \in H^2(X;\mathbf{R})\,|x \cdot x > 0\}$$

containing the self-dual projection of $2\pi k + \eta^+$; cf. [5]. (In particular, there are only two possible values for $SW_{X,g,\eta^+}(k)$ as we vary the pair (g,η^+).) Given a class $x \in H^2(X;\mathbf{R} - \{0\})$ of non-negative square we define the *x-negative Seiberg-Witten invariant* of X

$$SW_X^x \colon \mathcal{C}(X) \to \mathbf{Z}$$

as follows. Its value on a characteristic class k is equal to $SW_{X,g,\eta^+}(k)$ for any pair (g,η^+) for which the image of $2\pi k + \eta^+$ under L^2-projection into the self-dual g-harmonic two-forms has negative cup product pairing with x.

3. The product formula

In this section we state the main technical result of this paper, the product formula, and deduce a non-vanishing result for certain generalized connected sum manifolds.

3.1. The statements

Suppose that X and Y are closed, oriented smooth 4-manifolds. Let C be a closed, oriented riemann surface with $g(C) > 1$. Suppose that we have smooth embeddings $C \hookrightarrow X$ and $C \hookrightarrow Y$ representing homology classes of infinite order. Suppose in addition that each of these classes is of square zero. (That is to say, the self-intersection of C is zero in both

X and Y.) Because of this condition there is a regular neighborhood of C in each of X and Y orientation-preserving diffeomorphic to $D^2 \times C$. Let X_0 and Y_0 be the compact manifolds with boundary obtained from X and Y by removing the interiors of these regular neighborhoods. We denote by N the common boundary $S^1 \times C$. There is an obvious orientation-reversing diffeomorphism $\partial X_0 \to \partial Y_0$ which is the identity on the C factor and is complex conjugation on the S^1-factor. We denote by $M = X\#_C Y$ the oriented four-manifold that results from gluing X_0 and Y_0 together via this diffeomorphism. We call it *the sum of* X *and* Y *along* C. Notice that there is an induced embedding of C into M well-defined up to isotopy which represents a homology class of infinite order and of square zero.

Now suppose that $k \in H^2(M;\mathbf{Z})$ is an integral cohomology class whose restriction to $N = S^1 \times C$ is of the form $\rho^*(k_0)$ where $k_0 \in H^2(C;\mathbf{Z})$ is a class and $\rho\colon N \to C$ is the natural projection. Let k_{X_0} and k_{Y_0} denote the restrictions of k to X_0 and Y_0. These classes automatically extend to integral classes k_X and k_Y over X and Y. Each of these extensions is well-defined up to adding an integral multiple of C^*, the class Poincaré dual to the homology class represented by C. If k is characteristic, then exactly half the extensions k_X of k_{X_0} will be characteristic, and similarly for the k_Y. (The characteristic extensions will all differ by even multiples of C^*.)

Here is the statement of the product formula.

Theorem 3.1 (Product Formula). *Let X, Y, C, M, N be as in the previous paragraph. Suppose that $b_2^+(X), b_2^+(Y) \geq 1$. It follows that $b_2^+(M) \geq 1$. Suppose that $k \in H^2(M;\mathbf{Z})$ is a characteristic cohomology class satisfying $k|_N = \rho^* k_0$ where $k_0 \in H^2(C;\mathbf{Z})$ satisfies*

$$\langle k_0, [C]\rangle = 2g-2.$$

Consider the set $\mathcal{K}(k)$ of all characteristic classes $k' \in H^2(M;\mathbf{Z})$ with the property that $k'|_{X_0} = k_{X_0}$, $k'|_{Y_0} = k_{Y_0}$ and $(k')^2 = k^2$. We define $\mathcal{K}_X(k)$ to be all $\ell \in H^2(X;\mathbf{Z})$ which are characteristic and satisfy $\ell|_{X_0} = k_{X_0}$. The set $\mathcal{K}_Y(k)$ is defined analogously. Then for appropriate choices of orientations of $H^1(M), H^1(X), H^1(Y)$ and $H^2_+(M), H^2_+(X), H^2_+(Y)$ determining the signs of the Seiberg-Witten monopole invariants we have

$$(1) \qquad \sum_{k' \in \mathcal{K}(k)} SW_M(k') = (-1)^{b(M,N)} \sum SW_X(\ell_1)\cdot SW_Y(\ell_2),$$

where $b(M,N) = b_1(X_0,N)b_2^{\geq 0}(Y_0,N)$, and the sum on the right-hand-side extends over all pairs $(\ell_1,\ell_2) \in \mathcal{K}_X(k) \times \mathcal{K}_Y(k)$ with the property *that*

$$\ell_1^2 + \ell_2^2 = k^2 - (8g - 8). \tag{2}$$

It is to be understood in Equation (1) that the Seiberg-Witten invariant of any manifold with $b_2^+ = 1$ is the C^*-negative Seiberg-Witten invariant where C^* is the cohomology class Poincaré dual to C.

Remark 3.2. As we have already observed $\mathcal{K}_X(k) \subset H^2(X;\mathbf{Z})$ is a principal homogeneous space for $\mathbf{Z}(2[C]^*)$, and similarly for $\mathcal{K}_Y(k)$. The set $\mathcal{K}_M(k)$ is a principal homogeneous space for the possibly larger lattice $2\mathrm{Im}\big(\delta\colon H^1(N;\mathbf{Z}) \to H^2(M;\mathbf{Z})\big)$.

In general, the sum on the right-hand-side can have more than one non-trivial term. But if $d(k) = 0$, then there is at most one pair $(\ell_1, \ell_2) \in \mathcal{K}_X(k) \times \mathcal{K}_Y(k)$ which satisfies Equality (2) and for which $d(\ell_1) \geq 0$ and $d(\ell_2) \geq 0$. More generally, one can deduce a non-vanishing result for M from non-vanishing results for X and Y. Notice that even when the right-hand-side of the equation has only one non-zero term, it is not evident (and probably not true in general) that the invariants of the glued-up manifold are determined by those of the constituent pieces. The reason is that we have a sum of invariants on the left-hand-side of the equation. There are some cases however, when vanishing theorems allow one to restrict the possible support of the Seiberg-Witten function for the glued-up manifold sufficiently so that one can determine the Seiberg-Witten invariants of the glued-up manifold from this product formula.

Corollary 3.3. *Let X, Y, C be as in the previous theorem and let $M = X\#_C Y$. If there are characteristic classes $\ell_1 \in H^2(X;\mathbf{Z})$ and $\ell_2 \in H^2(Y;\mathbf{Z})$ with $\langle \ell_1, C\rangle = \langle \ell_2, C\rangle = 2g-2$, $SW_X(\ell_1) \neq 0$ and $SW_Y(\ell_2) \neq 0$, then there is a characteristic class $k \in H^2(M;\mathbf{Z})$ with $k|_N = \rho^* k_0$ for $k_0 \in H^2(C;\mathbf{Z})$ satisfying $\langle k_0, [C]\rangle = 2g - 2$ for which $SW_M(k) \neq 0$. (For any of these manifolds with $b_2^+ = 1$ it is understood that the Seiberg-Witten invariant is the C^*-negative Seiberg-Witten invariant.)*

Proof. Without loss of generality, we can assume that $\ell_1 \in H^2(X;\mathbf{Z})$ has $d(\ell_1)$ minimal among all classes satisfying the hypothesis of the corollary. Similarly, for ℓ_2. We set $k \in H^2(M;\mathbf{Z})$ equal to any characteristic class which has the property that $k|_{X_0} = \ell_1|_{X_0}$ and $k|_{Y_0} = \ell_2|_{Y_0}$. (There are such classes since $\ell_1|_N = \ell_2|_N$.) Now adding an appropriate even multiple of the Poincaré dual of $[C]$ to k we arrange that

$$k^2 = \ell_1^2 + \ell_2^2 + 8(g - 1).$$

For this particular choice of k, the sum on the right-hand-side of the product formula has only one non-zero term. The reason is that if $(\ell_1', \ell_2') \in \mathcal{K}_X(\ell_1) \times \mathcal{K}_Y(\ell_2)$ satisfies $(\ell_1')^2 + (\ell_2')^2 = \ell_1^2 + \ell_2^2$ and $(\ell_1', \ell_2') \neq (\ell_1, \ell_2)$, then the minimality of ℓ_1 and ℓ_2 implies that either $SW_X(\ell_1') = 0$ or $SW_Y(\ell_2') = 0$. Thus, in this special case, the sum on the right-hand-side of Equation (1) consists of exactly one non-zero term. Hence, one of the terms on the right-hand-side is non-zero. This completes the proof of the corollary.

4. Genus minimizing curves

In this section we show how to deduce the Generalized Thom Conjecture and its symplectic generalization from the Product Formula and one other result concerning embedded two-spheres in symplectic four-manifolds.

4.1. The general statements

The main application of this product formula is to prove the genus minimizing criterion given below. As the reader can see, this result concerns general four-manifolds not just Kähler surfaces and symplectic four-manifolds. As we go on to state in this section, its application to Kähler surfaces yields the Generalized Thom Conjecture.

Proposition 4.1. *Let X be a closed, oriented four-manifold with $b_2^+(X) + b_1(X)$ odd, and let $C \subset X$ be a C^∞ curve of genus $g > 1$ and square zero. Suppose that there is a characteristic class $k \in H^2(X; \mathbf{Z})$ with the property that $\langle k, [C]\rangle = 2g - 2$ and suppose that the Seiberg-Witten function $SW_X(k) \neq 0$. (It is understood that if $b_2^+(X) = 1$, then this invariant is the C^*-negative Seiberg-Witten invariant, with C^* the class Poincaré dual to C.) Then any C^∞-curve in the same homology class as C has genus at least as large as that of C.*

There is a generalization of this result that covers curves of positive intersection as well. It is deduced from the previous result by blowing up and using the blowup formula from [1] or [2].

Proposition 4.2. *Let X be as above and suppose that $C \subset X$ is a smoothly embedded riemann surface of genus $g > 1$ and with $C \cdot C \geq 0$. Suppose that there is a characteristic class $k \in H^2(X; \mathbf{Z})$ with the property that $\langle k, [C]\rangle = 2g - 2 - C \cdot C$ and $SW_X(k) \neq 0$. (If $b_2^+(X) = 1$, then this invariant is interpreted to be the C^*-negative Seiberg-Witten* *invariant, with C^* the class Poincaré dual to C.) Then any C^∞-curve in the same homology class as C has genus at least as large as that of C.*

4.2. The case of Kähler surfaces and symplectic four-manifolds

Applying this to the case of Kähler surfaces and holomorphic curves yields the following result.

Corollary 4.3. *Let X be a compact Kähler surface and let $C \subset X$ be a smooth holomorphic curve with $C \cdot C \geq 0$ and $g(C) > 1$. Suppose that $C' \subset X$ is a C^∞ riemann surface homologous to C. Then $g(C) \leq g(C')$.*

We also have the analogue for symplectic four-manifolds.

Corollary 4.4. *Let X be a compact symplectic four-manifold and let $C \subset X$ be a smooth symplectic curve with $C \cdot C \geq 0$ and $g(C) > 1$. Suppose that $C' \subset X$ is a C^∞ riemann surface homologous to C. Then $g(C) \leq g(C')$.*

While these corollaries do not cover the case of curves of genus one, this case can be handled by other arguments using Seiberg-Witten invariants.

Proposition 4.5. *Let X be a compact Kähler surface and let $C \subset X$ be a smooth holomorphic curve with $g(C) = 1$. If $C \cdot C \geq 0$, then the homology class of C is not represented by a smoothly embedded sphere.*

Proposition 4.6. *Let X be a compact symplectic four-manifold and let $C \subset X$ be a smooth symplectic curve with $g(C) = 1$. If $C \cdot C \geq 0$, then the homology class of C is not represented by a smoothly embedded sphere.*

Together of course, these results cover the case of all curves of non-negative square in compact Kähler surfaces, thus establishing the Generalized Thom Conjecture and its symplectic generalization as stated in the introduction.

In this section we show that the Product Formula implies Proposition 4.1. We also show that Proposition 4.1 implies Proposition 4.2 and that Proposition 4.2 implies Corollary 4.4 which of course implies Corollary 4.3. Then next five sections are devoted to proving the Product Formula. The last section gives a proof of Proposition 4.6 which of course also implies Proposition 4.5.

4.3. Proof that the product formula implies Proposition 4.1

Suppose that $C \subset X$ is a smooth curve of genus $g > 1$ and suppose that $C \cdot C = 0$. Suppose that there is a characteristic class $k \in H^2(X; \mathbf{Z})$ with the property that $\langle k, [C] \rangle = 2g - 2$ and that $SW_X(k) \neq 0$. (As usual, if $b_2^+(X) = 1$ we use the C^*-negative Seiberg-Witten invariant.)

There is one case we must treat separately.

Lemma 4.7. *If $H_1(C; \mathbf{Z}) \to H_1(X; \mathbf{Z})$ is injective, then the homology class represented by C is not represented by a riemann surface of smaller genus.*

Proof. If $H_1(C; \mathbf{Z}) \to H_1(X; \mathbf{Z})$ is injective, then the skew-symmetric pairing

$$H^1(X; \mathbf{Z}) \otimes H^1(X; \mathbf{Z}) \to \mathbf{Z}$$

given by $a \otimes b \mapsto \langle a \cup b, [C] \rangle$ is of rank $2g(C)$. On the other hand, if $[C]$ is represented as the continuous image of the fundamental class of a riemann surface of genus g', then this pairing has rank at most $2g'$. The result is immediate. q.e.d.

From now on we shall assume that the map $H_1(C; \mathbf{Z}) \to H_1(X; \mathbf{Z})$ has a non-trivial kernel. With this extra hypothesis, we are in a position to prove Proposition 4.1. Let X and C be as in the statement with $H_1(C) \to H_1(X)$ having a non-zero kernel. We double X along C. That is to say, we form $M = X \#_C X$. Of course, $b_2^+(X) \geq 1$ by hypothesis. Since we are assuming that $H_1(C; \mathbf{Z}) \to H_1(X; \mathbf{Z})$ has a non-trivial kernel, it follows easily from the Mayer-Vietoris sequence that $b_2^+(M) \geq 2$. This means that the Seiberg-Witten invariant SW_M is independent of the metric and the perturbing self-dual two-form. According to the Product Formula and the corollary following it, we see that the Seiberg-Witten invariant SW_M is non-trivial.

On the other hand, if the homology class of C is represented by a C^∞ riemann surface of genus less than that of C, then by adding trivial handles we could arrange that C be as before (of square zero and genus g) and in the same homology class, but also with at least one trivial handle. That is to say, in X there is a four-ball meeting C in a punctured torus, that punctured torus being unknotted in the four-ball. We claim that this implies that SW_M vanishes identically. This will give a contradiction and will establish Proposition 4.1 as a consequence of the Product Formula.

There are two different ways to show that SW_M vanishes. We have an embedded S^2 of square zero and a dual torus in a manifold M with

$b_2^+(M) > 1$. The existence of the dual torus implies that the homology class of the sphere is of infinite order in homology. According to Lemma 10.2 this implies the vanishing of SW_M. Alternatively, one can notice that one of the generating circles γ on the torus bounds an embedded disk D in M which is disjoint from S^2, meets the torus only in its boundary, and for which the normal vector field to γ in the torus extends to a nonwhere zero vector field over the disk. We do surgery on the torus inside of M using this disk. This replaces the torus by a two-sphere of square zero geometrically dual to the first sphere. A regular neighborhood of the union of these two dual spheres is diffeomorphic to $S^2 \times S^2 - B^4$. This gives a decomposition of M as a connected sum

$$M \cong M' \# (S^2 \times S^2).$$

Since $b_2^+(M) \geq 2$, it follows that $b_2^+(M') > 0$, and hence by the connected sum theorem [14] or [2] it must be the case that the Seiberg-Witten invariant of M vanishes.

4.4. Proof of that Proposition 4.1 implies Proposition 4.2

Let $C \subset X$ and $k \in \mathcal{C}(X)$ be as in the hypothesis of Proposition 4.2. Fix a $Spin^c$-structure $\tilde{P}$ over X whose determinant line bundle has first Chern class k. Suppose that $C \cdot C = n \geq 0$. We choose a metric g and a self-dual two-form η^+ so that:

- The moduli space $\mathcal{M}(\tilde{P}, g, \eta^+)$ is smooth of the correct dimension.
- The support of η^+ is disjoint from C.
- If $b_2^+(X) = 1$, then the projection of $2\pi k + \eta^+$ into the self-dual g-harmonic two-forms has negative integral over C.

We now blow up at n distinct points along C. That is to say, we form the manifold

$$X_1 = X \#_n \overline{\mathbf{C}P}^2$$

with a metric g_1 which is a connected sum of the metric g with a standard metric on the $\overline{\mathbf{C}P}^2$ factors, connected by sufficiently long tubes. We let η_1^+ be the form which vanishes on the $\overline{\mathbf{C}P}^2$ factors and the tubes, and which agrees with η^+ on the rest of X_1. Let $E_1 \ldots , E_n$ be the exceptional curves in X_1. Let C_1 be the connected sum of C and the exceptional curves $E_1, \ldots , E_n$. This is a smoothly embedded curve in X_1 with $g(C_1) = g(C)$ and $C_1 \cdot C_1 = 0$. Let $k_1 = k + \sum_{i=1}^n E_i$. This is a characteristic class for X_1 with $\langle k_1, C_1\rangle = 2g-2$. If $b_2^+(X_1) = 1$, and the connected sum tubes are sufficiently long, then the projection of $2\pi k_1 + \eta_1^+$ onto the g_1-self-dual harmonic forms will have negative integral over C_1.

There is a unique *Spin*^{c}-structure $\tilde{P}_1$ for X_1, which agrees with $\tilde{P}$ on the complement of the exceptional curves and has determinant line bundle with first Chern class k_1. The blow-up formula [1] tells us that if the connected sum tubes are sufficiently long, then

$$SW_{X,g,\eta^+}(\tilde{P}) = SW_{X_1,g_1,\eta_1^+}(\tilde{P}_1),$$

when we use compatible orientations on $H^1(X) = H^1(X_1)$ and $H^2_+(X) = H^2_+(X_1)$. Since this is true for all $\tilde{P}$, it follows that

$$SW_{X,g,\eta^+}(k) = SW_{X_1,g_1,\eta_1^+}(k_1).$$

Consequently, $0 \neq SW_X^{C^*}(k) = SW_{X_1}^{C_1^*}(k_1)$, where if $b_2^+(X) = 1$, then we use the C^*- and C_1^*-negative Seiberg-Witten invariants of X and X_1, respectively.

Thus, we see that X_1, C_1, k_1 satisfy all the hypotheses of Proposition 4.1. By that proposition, it follows that C_1 is genus minimizing in its homology class. This implies that the same is true for C.

4.5. Proof of that Proposition 4.2 implies Corollary 4.4

Now we are ready to apply this general analysis to symplectic four-manifolds and symplectic curves. The first step is to recall the results of [12] about the value of the Seiberg-Witten function on the canonical class of a symplectic four-manifold.

Lemma 4.8 (Taubes [12]). *Let X be a symplectic four-manifold with symplectic form ω. Let $K_X \in H^2(X;\mathbf{Z})$ be the canonical class of the symplectic structure. If $b_2^+(X) > 1$, then $SW_X(K_X) = \pm 1$. If $b_2^+(X) = 1$ then $SW_X^\omega(k) = \pm 1$.*

Let X be a symplectic four-manifold, and $C \subset X$ a symplectic curve. Let $K_X \in H^2(X;\mathbf{Z})$ be the canonical class of the symplectic structure. By the adjunction formula we have

$$\langle K_X, C\rangle + C\cdot C = 2g-2.$$

Thus, Corollary 4.4 is immediate from the previous lemma and Proposition 4.2 in the case where $b_2^+(X) > 1$.

Let us suppose that $b_2^+(X) = 1$. For any symplectic curve $C \subset X$ with $C \cdot C \geq 0$, we have $\int_C \omega > 0$. This means that the class C^* Poincaré dual to C and the class of ω lie in the same component of the double cone

$$\{x \in H^2(X; \mathbf{R} - \{0\}) \,|\, x \cdot x \geq 0\}.$$

Thus, $SW_X^\omega = SW_X^{C^*}$, and Corollary 4.4 follows in this case as well from Lemma 4.8 and Proposition 4.2.

5. The Seiberg-Witten equations for the three-manifold $N = S^1 \times C$

The next five sections of this paper are devoted to the proof of the product formula. This formula follows from a gluing theorem for moduli spaces, which compares the product of moduli spaces for two cylindrical-end manifolds with the moduli space for the glued-up manifold. The proof of this gluing theorem follows the pattern laid down in the proofs of the various product formulae in the case of Donaldson $SU(2)$-invariants. We begin in this section with the analogue of the Floer homology; that is to say with the theory of the Seiberg-Witten equations for the three-manifold $N = S^1 \times C$.

Recall that associated to the $Spin^c$-structure $\tilde{P}_N$ on a riemannian three-manifold N there is an irreducible complex spin bundle $S(\tilde{P}_N)$ unique up to isomorphism. In what follows we shall use the structure of $S(\tilde{P}_N)$ as a module over the entire Clifford algebra $Cl(N)$. There are two possibilities for this module structure and we choose to work with the one that factors through the projection to $Cl(N)^+$. The bundle $S(\tilde{P}_N)$ is a two-dimensional complex bundle with a hermitian inner product. If we have a hermitian connection A_0 on the determinant line bundle $\mathcal{L}_N$ of this $Spin^c$-structure, then there is the associated self-adjoint first-order elliptic Dirac operator $\partial\!\!\!/_{A_0}$. As in the four-dimensional case, the hermitian metric induces an anti-linear isomorphism

$$S(\tilde{P}_N) \to S^*(\tilde{P}_N).$$

We denote this map by $\psi \mapsto \psi^*$. Also, there is the exact sequence of vector bundles:

$$0 \to \Lambda^2(N) \otimes \mathbf{C} \to S(\tilde{P}_N) \otimes S^*(\tilde{P}_N) \to \mathbf{C} \to 0,$$

where the first map is the adjoint of Clifford multiplication, and the second map is the evaluation pairing (i.e., the trace of the endomorphism).

These structures allow us to define a quadratic map

$$q\colon C^{\infty}\left(S(\tilde{P}_N)\right) \to \Omega^2(N;\mathbf{C})$$

by associating to ψ the element

$$q(\psi) = \psi \otimes \psi^* - \frac{\|\psi\|^2}{2}\mathrm{Id}.$$

This later element is in the kernel of the evaluation mapping and hence defines an element $q(\psi) \in \Omega^2(N;\mathbf{C})$. As in the four-dimensional case, this element is purely imaginary, i.e., it lies in $\Omega^2(N;i\mathbf{R})$.

The 3-dimensional Seiberg-Witten (or monopole) equations for a $Spin^c$-structure $\tilde{P}_N \to N$ are equations for a pair (A,ψ), where A is a unitary connection on the determinant line bundle $\mathcal{L}_N$ of $\tilde{P}_N$, and ψ is a section of the complex spin bundle $S(\tilde{P}_N)$. The equations are:

$$(SW^3): F_A = q(\psi),$$

$$\partial\!\!\!/_A(\psi) = 0.$$

The case of particular interest for us here is the case where $N = S^1 \times C$ and the determinant of the $Spin^c$-structure on N has degree $2g-2$ on C. Since the tangent bundle of N is naturally decomposed as a product of the tangent bundle of C with a trivial real line bundle, a $Spin^c$-structure on C induces one on N.

Proposition 5.1. *Let $N = S^1 \times C$. Consider all $Spin^c$-structures $\tilde{P}$ on N, whose determinant line bundles $\mathcal{L}$ have degree $2g-2$ on C. As we range over all these $Spin^c$-structures there is exactly one solution (A_0,ψ_0) to the equations SW^3 up to gauge automorphisms. The $Spin^c$-structure for which this solution exists is induced via the projection from a $Spin^c$-structure on C.*

Proof. As in the case of the equations on the four-manifold, there is a natural involution on the solutions to the three-dimensional equations. This involution sends the determinant line bundle to its inverse. Thus, it suffices to consider the case where the degree of the determinant line bundle of α on C is $2-2g$.

Let us first consider the case of a $Spin^c$-structure $\tilde{P}_0$ with the property that the determinant line bundle $\mathcal{L}_0$ is induced from a line bundle on C. Since $H_1(N;\mathbf{Z})$ has no two-torsion, a $Spin^c$-structure on N is determined up to isomorphism by its determinant line bundle. This means that the $Spin^c$-structure $\tilde{P}_0$ is in fact an extension from $SO(2)$ to $SO(3)$ of a $Spin^c$-structure P_C on C. Let S_C^+, S_C^- be the complex line bundles over C, which are the plus and minus spinors for P_C. Let e_1, e_2 be an orthonormal, oriented basis for TC at a point. By definition ie_1e_2 acts on $S_C^\pm$ by ± 1, so that e_1e_2 acts on $S_C^\pm$ by $\mp i$. Given a unitary connection A on the determinant line bundle $\mathcal{L}_C \to C$ there are induced operators: $\not{\partial}_A^+$ from sections of S_C^+ to sections of S_C^-, and its adjoint $\not{\partial}_A^-$ from sections of S_C^- to sections of S_C^+. These operators are identified with

$$(\sqrt{2})\overline{\partial}_A \colon \Omega^0(C; (K_C \otimes \mathcal{L}_C)^{1/2}) \to \Omega^{0,1}(C; (K_C \otimes \mathcal{L}_C)^{1/2})$$

and its adjoint $(\sqrt{2})\overline{\partial}_A^*$.

There are two irreducible representations of $Cl(\mathbf{R}^3)$, factoring through $Cl(\mathbf{R}^3)^\pm$, the plus and minus one eigenspaces for $-e_1e_2e_3$. We choose to work with the one $S_{\mathbf{R}^3}$ factoring through $Cl(\mathbf{R}^3)^+$. Thus, the bundle of spinors on N for the induced $Spin^c$-structure are simply $\rho^*(S_C^+) \oplus \rho^*(S_C^-)$ where $\rho\colon N \to C$ is the natural projection. Given a unitary connection A on $\rho^*(\mathcal{L}_C)$, the induced Dirac operator is given by

$$\not{\partial}_A = \begin{pmatrix} -i\nabla_\theta & (\sqrt{2})\overline{\partial}_{A|_C}^* \\ (\sqrt{2})\overline{\partial}_{A|_C} & i\nabla_\theta \end{pmatrix},$$

where ∇_θ denotes the covariant derivative with respect to the connection induced by A on $S^\pm$ evaluated on the unit tangent vector to the circle in the positive direction.

In general, for a $Spin^c$-structure $\tilde{P}_N$ whose determinant line bundle $\mathcal{L}_N$ is of degree $2-2g$ on C there is a line bundle $\mathcal{L}_1$ of degree 0 on C such that $\mathcal{L}_N = \mathcal{L}_0 \otimes \mathcal{L}_1^2$. It follows that

$$S^+(\tilde{P}_N) = S^+(\tilde{P}_0) \otimes \mathcal{L}_1 = \left(\rho^*(S_C^+) \otimes \mathcal{L}_1\right) \oplus \left(\rho^*(S_C^-) \otimes \mathcal{L}_1\right).$$

Writing the unitary connection A on $\mathcal{L}_N$ as the product of a connection A_0 on $\mathcal{L}_0$ and a connection and A_1 on $\mathcal{L}_1$, we see that

$$\not{\partial}_A = \not{\partial}_{A_0} \otimes A_1.$$

Thus, once again, the Dirac operator is given by the two-by-two matrix

$$\not{\partial}_A = \begin{pmatrix} -i\nabla_\theta & (\sqrt{2})\overline{\partial}_{A|_C}^* \\ (\sqrt{2})\overline{\partial}_{A|_C} & i\nabla_\theta \end{pmatrix},$$

where as before ∇_θ denotes the covariant derivative of A in the circle direction. To simplify the notation we write $\overline{\partial}_A$ for $\overline{\partial}_{A|_C}$.

Now suppose that (A,ψ) is a solution to the monopole equations for a general $Spin^c$-structure $\tilde{P}_N$. We write $\psi=(\alpha,\beta)$. The Dirac equation becomes

$$\begin{aligned} -i\nabla_\theta(\alpha)+(\sqrt{2})\overline{\partial}_A^*(\beta) &= 0,\\ (\sqrt{2})\overline{\partial}_A(\alpha)+i\nabla_\theta(\beta) &= 0. \end{aligned} \tag{4}$$

Applying $\frac{1}{\sqrt{2}}\overline{\partial}_A$ to the first equation yields

$$\frac{-i}{\sqrt{2}}\overline{\partial}_A\nabla_\theta(\alpha)+\overline{\partial}_A\overline{\partial}_A^*(\beta)=0. \tag{5}$$

Suppose that in local holomorphic coordinates $z=x+iy$ on C we have $F_A=F_{x,y}dx\wedge dy+F_{x,\theta}dx\wedge d\theta+F_{y,\theta}dy\wedge d\theta$. Since we are using the plus spin bundle over $Cl(N)$, the action of F_A by Clifford multiplication is the same as the action of

$$\begin{aligned} &F_{x,y}d\theta-F_{x,\theta}dy+F_{y,\theta}dx\\ &\quad=F_{x,y}d\theta+\frac{1}{2}\left(F_{y,\theta}-iF_{x,\theta}\right)d\overline{z}+\frac{1}{2}\left(F_{y,\theta}+iF_{x,\theta}\right)dz. \end{aligned}$$

This means that Clifford multiplication by F_A is given by the matrix

$$\begin{pmatrix} -iF_{x,y} & \left(\frac{1}{2}\left(F_{y,\theta}-iF_{x,\theta}\right)d\overline{z}\wedge(\cdot)\right)^*\\ \frac{1}{2}\left(F_{y,\theta}-iF_{x,\theta}\right)d\overline{z}\wedge(\cdot) & iF_{x,y} \end{pmatrix}.$$

It follows that the curvature equation of SW^3 reads:

$$\begin{aligned} -iF_{x,y}&=\frac{|\alpha|^2-|\beta|^2}{2},\\ \frac{1}{2}\left(F_{y,\theta}-iF_{x,\theta}\right)d\overline{z}&=\overline{\alpha}\beta. \end{aligned}$$

Now we commute ∇_θ and $\overline{\partial}_A$ in Equation (5) introducing a curvature term. Notice that

$$\left(\overline{\partial}_A\circ\nabla_\theta-\nabla_\theta\circ\overline{\partial}_A\right)(\ )=(F_{x,\theta}-iF_{y,\theta})d\overline{z}\wedge(\ ).$$

Plugging this in to Equation (5) gives

$$\frac{-i}{\sqrt{2}}\nabla_\theta\overline{\partial}_A(\alpha)+\frac{1}{\sqrt{2}}\left(-iF_{x,\theta}+F_{y,\theta}\right)d\overline{z}\cdot\alpha+\overline{\partial}_A\overline{\partial}_A^*(\beta)=0.$$

Using the second part of Equation (4) we have

$$-\frac{1}{2}\nabla_\theta\nabla_\theta(\beta)+\frac{1}{\sqrt{2}}(-iF_{x,\theta}+F_{y,\theta})d\overline{z}\cdot\alpha+\overline{\partial}_A\overline{\partial}_A^*(\beta)=0.$$

Now applying the curvature equation yields

$$\frac{1}{\sqrt{2}}\left(-iF_{x,\theta}+F_{y,\theta}\right)d\overline{z}=\sqrt{2}\beta\otimes\overline{\alpha},$$

and hence,

$$-\frac{1}{2}\nabla_\theta\nabla_\theta(\beta)+\sqrt{2}|\alpha|^2\beta+\overline{\partial}_A\overline{\partial}_A^*(\beta)=0.$$

Since rotation in the circle direction acts by isometries on N, we have $\nabla_\theta^*=-\nabla_\theta$, and therefore

$$\frac{1}{2}\nabla_\theta^*\nabla_\theta(\beta)+\sqrt{2}|\alpha|^2\beta+\overline{\partial}_A\overline{\partial}_A^*(\beta)=0.$$

Taking the inner product with β then gives

$$\frac{1}{2}\|\nabla_\theta(\beta)\|_{L^2}^2+\sqrt{2}\|\overline{\alpha}\beta\|_{L^2}^2+\|\overline{\partial}_A^*(\beta)\|_{L^2}^2=0.$$

Hence, $\overline{\alpha}\beta=0$. Plugging the fact that one of α or β equals zero into the curvature equation, we see that

$$\frac{i}{2\pi}F_A=\frac{\|\beta\|^2-\|\alpha\|^2}{4\pi}\rho^*dvol(C).$$

Since the line bundle $\mathcal{L}_N$ has negative degree on C, it must be the case that $\beta=0$. It now follows that α is covariantly constant in the direction of the circle. In the end we have shown that $\beta=0$, that $\nabla_\theta(\alpha)=0$, that $\overline{\partial}_A(\alpha)=0$, and that

$$\int_C\|\alpha\|^2dvol(C)=4\pi(2g-2). \tag{6}$$

Also, we have seen that the curvature F_A is a two-form which is at each point induced from a two-form on C. Since F_A is also covariantly constant in the direction of the circle, F_A is the pullback of a two-form on C under the natural projection $N\to C$. Lastly, since α is non-zero on an open dense subset of N and is covariantly constant in the circle direction, it follows that parallel translation with respect to the connection A on the plus spin bundle has trivial holonomy around the circles. This implies that the holonomy of A on $\mathcal{L}$ is also trivial around the circles. Thus, up to gauge equivalence, the triple $(\mathcal{L}, A, \alpha)$ is induced from corresponding triple $(\mathcal{L}_C, A_C, \alpha_C)$ over C. Of course, α_C must be a harmonic plus spinor for the $Spin^c$-structure on C with determinant line bundle $\mathcal{L}_C$. Since the degree of $\mathcal{L}_C$ is $2-2g$, the bundle of plus spinors, $(K_C \otimes \mathcal{L}_C)^{1/2}$, is of degree zero. That is to say it is a topologically trivial bundle. The connection A_C induces a holomorphic structure on $\mathcal{L}_C$ and a holomorphic structure on $(K_C \otimes \mathcal{L}_C)^{1/2}$. With respect to this holomorphic structure, α_C is a non-trivial holomorphic section. This implies that the holomorphic bundle $(K_C \otimes \mathcal{L}_C)^{1/2}$ is holomorphically trivial and α_C is a constant section. Its norm is determined by Equation (6). Such a triple $(\mathcal{L}_C, A_C, \alpha_C)$ then is clearly unique up to gauge equivalence.

This proves that there is exactly one solution up to isomorphism to the monopole equations on $N = S^1 \times C$ among all $Spin^c$ structures whose determinant line bundle has degree $2-2g$ on C. Furthermore, the $Spin^c$-structure for which the solution exists is pulled back from a $Spin^c$-structure on C. By symmetry the result follows when the degree is $2g-2$. q.e.d.

We need to fit SW^3 into a non-linear elliptic context. Let us consider the context of an arbitrary compact, oriented, riemannian three-manifold N and a $Spin^c$-structure $\tilde{P}_N$ over it. We let $\mathcal{B}^*(\tilde{P}_N)$ be the space of L^2_1-configurations modulo the action of L^2_2-changes of gauge. We consider the equations SW^3 as defining a section ξ_{SW} of the L^2-version of the tangent bundle of $\mathcal{B}^*(\tilde{P}_N)$. The tangent space to $\mathcal{B}^*(\tilde{P}_N)$ at $x = [A, \psi]$ is the cokernel of the linear map

$$L^2_2(X; i\mathbf{R}) \xrightarrow{D_x} L^2_1\left((T^*X \otimes i\mathbf{R}) \oplus S(\tilde{P}_N)\right),$$

where the map D_x is given by

$$D_x(if) = (2idf, -if \cdot \psi).$$

These quotient spaces fit together to give the tangent bundle of $\mathcal{B}^*(\tilde{P}_N)$. The L^2 version of the tangent bundle is the bundle whose fiber over x is the cokernel of the map

$$D_x\colon L^2_1(X; i\mathbf{X}) \to L^2\left((T^*X \otimes i\mathbf{R}) \oplus S(\tilde{P}_N)\right).$$

The smooth section of this bundle given by the SW^3 equations is

$$[A, \psi] \mapsto [*(F_A - q(\psi)), \partial\!\!\!/_A(\psi)].$$

The first thing to notice is that the zeros of this section are exactly the gauge equivalence classes of irreducible configurations satisfying the equations SW^3. It is also an easy exercise to show that the differential of this section is Fredholm at every point. Thus, viewed in this way there is a finite dimensional Kuranishi picture in a neighborhood of each point of the moduli space of solutions to the equations SW^3. The Zariski tangent space is the kernel of the differential of the section and the obstruction space is the cokernel of this differential.

Now let us turn to our special case.

Lemma 5.2. *Let N and $\mathcal{L}$ be as in Proposition 5.1. Then the unique solution of the equations SW^3 in $\mathcal{B}^*(\tilde{P}_N)$ is a smooth point in the sense that the Zariski tangent space and the obstruction space are trivial.*

Proof. What this means is that for a solution $x = (A_0, \psi_0)$ the sequence

$$0 \to L^2_2(X; i\mathbf{R}) \xrightarrow{D_x} L^2_1\left(T^*X \otimes i\mathbf{R} \oplus S(\tilde{P}_N)\right)$$
$$\xrightarrow{D\xi_{SW}} L^2\left(T^*X \otimes i\mathbf{R} \oplus S(\tilde{P}_N)\right) / D_x\left(L^2_1(X; i\mathbf{R})\right)$$

is exact. This is a direct computation along the same lines as the proof of Proposition 5.1, but simpler.

5.1. Perturbations of the Seiberg-Witten equations on a 3-manifold

Let $N = S^1 \times C$ and let $\tilde{P}_N$ be a $Spin^c$-structure on N whose determinant line bundle $\mathcal{L}$ has degree $\pm(2-2g)$ on C.

Corollary 5.3. *Under the hypotheses of the previous lemma, for any sufficiently small closed real two-form h on N there is a unique solution to the perturbed Seiberg-Witten equations (SW^3_h):*

$$F_A = q(\psi) + ih,$$
$$\not{\partial}_A(\psi) = 0.$$

This solution represents a smooth point of the moduli space in the sense that its Zariski tangent space is trivial.

Remark 5.4. As in the unperturbed case we view the equations SW^3_h as defining a smooth section of the L^2-version of the tangent bundle of $\mathcal{B}^*(\tilde{P}_N)$. Once again the space of solutions to the equations modulo gauge equivalence is identified with the zero set of this section, and the section is a non-linear Fredholm section which is a perturbation of the section associated to the equations SW^3.

Proof. This result is immediate From Lemma 5.2 and the fact that transversality is an open condition on a section. q.e.d.

Actually, in one special case we can identify the solution.

Corollary 5.5. *With N and $\mathcal{L}$ as above, let n be a harmonic one-form on C and let $*$ be the (complex-linear) Hodge star operator for N. We write $n = \eta + \overline{\eta}$ where η is a holomorphic one-form on C. Then there is a unique solution (up to gauge) to the perturbed Seiberg-Witten equations SW^3_{*n}. This solution is gauge equivalent to one pulled up from (A, ψ) on C where $\psi = (\alpha, \beta)$ with α being a constant real section $r > 0$ of S^+_C and β being the section of $S^-_C = \Lambda^{0,1}(T^*C)$ given by $\frac{-i\overline{\eta}}{r}$; the constant r is determined by the fact that*

$$\int_C \left(\frac{\|\overline{\eta}\|^2}{r^2} - r^2 \right) dvol(C) = 4\pi(2 - 2g).$$

Proof. We proceed as in the case of the unperturbed equation. We find

$$\frac{1}{2}\|\nabla_\theta(\beta)\|^2_{L^2} + \|\overline{\partial}^*_A\beta\|^2_{L^2} + \sqrt{2}\langle \overline{\alpha}\beta + i\overline{\eta}, \overline{\alpha}\beta \rangle_{L^2} = 0.$$

On the other hand, the fact that $\overline{\eta}$ is a closed form on C and that $\mathcal{L}$ is induced from a line bundle on C gives that

$$0 = \int_N i\overline{\eta} \wedge F_A = \langle i\overline{\eta}, \overline{\alpha}\beta + i\overline{\eta} \rangle_{L^2}.$$

Adding the first equation to $\sqrt{2}$ times this one yields

$$\frac{1}{2}\|\nabla_\theta(\beta)\|^2 + \|\overline{\partial}^*_A\beta\|^2 + \sqrt{2}\|\overline{\alpha}\beta + i\overline{\eta}\|^2_{L^2} = 0.$$

We conclude that $\overline{\alpha}\beta + i\overline{\eta} = 0$, that β is covariantly constant in the circle direction, and that $\overline{\partial}^*_A(\beta) = 0$. Plugging these back into the Dirac equation we see that α is covariantly constant in the circle direction and that $\overline{\partial}_A\alpha = 0$.

From this everything else is a direct computation. q.e.d.

Remark 5.6. A perturbation of this type was introduced by Witten in [14] in order to study solutions to the Seiberg-Witten equations over a Kähler surface.

6. The gradient flow equation for solutions on a cylinder

6.1. The Seiberg-Witten equations on a cylinder $I \times N$

Let us consider a smooth, oriented riemannian four-manifold X which is orientation-preserving isometric to $I \times N$ where N is a closed oriented three-manifold and I is a (possible infinite) open interval. Our purpose here is to rewrite the Seiberg-Witten equations on X as gradient flow equations for a path in the space of configurations on N. Let $\pi\colon X \to N$ be the natural projection. We have a natural isomorphism of bundles of algebras $\pi^*(Cl(N)) \to Cl_0(X)$. This isomorphism sends $\alpha_0 + \alpha_1$ in $\pi^*(Cl(N))$ to $\alpha_0 + V \cdot \alpha_1$ where V is the unit vector field in the positive direction along I.

Suppose that $\tilde{P} \to X$ is a $Spin^c$-structure for X. Let $Q \to X$ be the $U(1)$-bundle which is the determinant of $\tilde{P}$. There is a $U(1)$-bundle $Q_N \to N$ and an isomorphism $j\colon \pi^*Q_N \to Q$. The double covering $\tilde{P} \to P_{SO(4)}X \times_X Q$ induces a double covering $\tilde{P}_N \to P_{SO(3)}N \times_N Q_N$ (which is then a $Spin^c$-structure on N) and an embedding $\pi^*\tilde{P}_N \hookrightarrow \tilde{P}$ covering the obvious embedding

$$\pi^*(P_{SO(3)}N \times_N Q_N) \hookrightarrow P_{SO(4)}X \times_X Q.$$

The spinor bundle $S(\tilde{P})$ is an irreducible module over $Cl(X)$. As a module over $Cl_0(X) = \pi^*(Cl(N))$, it splits as $S^+(\tilde{P}) \oplus S^-(\tilde{P})$ with each of $S^\pm(\tilde{P})$ being an irreducible module over $\pi^*(Cl(N))$. Thus we have an isomorphism of $U(2)$-bundles

$$\rho^+\colon \pi^*(S(\tilde{P}_N)) \to S^+(\tilde{P}),$$

where ρ^+ carries the action of the bundle of Clifford algebras $\pi^*(Cl(N))$ to the action of $Cl_0(X)$. This means that, letting V be the unit vector field in the positive I-direction, for any section $\sigma \in S(\tilde{P})$ and any tangent vector e in the N-direction we have

$$\rho^+(e \cdot \sigma) = Ve \cdot \rho^+(\sigma).$$

It follows that if F is a two-form on N then

$$\rho^+(F \cdot \sigma) = F \cdot \rho^+(\sigma),$$

so that

$$\rho^+(\omega_{\mathbf{C}}(N) \cdot \sigma) = \omega_{\mathbf{C}}(X)\rho^+(\sigma) = \rho^+(\sigma).$$

Hence $\omega_{\mathbf{C}}(N)$ acts as the identity on $S(\tilde{P}_N)$.

Claim 6.1. *Let $\psi(t)$ be a section of $\pi^*\left(S(\tilde{P}_N)\right)$ and let $A(t)$ be a one-parameter family of connections on Q_N. We can view the $A(t)$ as defining a connection A on Q via the isomorphism $\pi^*Q_N = Q$. The resulting connection A is temporal with respect to the given product structure in the I-direction in the sense that the A-parallel translation in the I-direction gives the product structure in this direction. Then we have*

$$\not{\partial}_A(\rho^+(\psi))(t) = V \cdot \rho^+\left(-\not{\partial}_{A(t)}(\psi(t)) + \frac{\partial\psi}{\partial t}\right).$$

Proof. Let e_1, e_2, e_3 be an orthonormal basis for TN at a point $n \in N$. We compute

$$\not{\partial}_A(\rho^+(\psi))(n,t) = \sum_{i=1}^{3} e_i \nabla_{e_i}(\rho^+(\psi)(n,t)) + V\nabla_t(\rho^+(\psi)(n,t)).$$

Clearly, $\nabla_{e_i}(\rho^+(\psi(n,t))) = \rho^+(\nabla_{e_i}(\psi(n,t)))$. Since A is temporal with respect to the given product structure in the I-direction, we have

$$\nabla_t(\rho^+(\psi)) = \rho^+\left(\frac{\partial\psi}{\partial t}\right).$$

Hence,

$$\begin{aligned}
\not{\partial}_A(\rho^+(\psi))(t) &= \sum_{i=1}^{3} e_i \nabla_{e_i}(\rho^+(\psi)) + V\nabla_t(\rho^+(\psi)) \\
&= \sum_{i=1}^{3} -VVe_i\nabla_{e_i}(\rho^+(\psi)) + V\nabla_t(\rho^+(\psi)) \\
&= -V\sum_{i=1}^{3} Ve_i\rho^+\left(\nabla_{e_i}(\psi)\right) + V\nabla_t(\rho^+(\psi)) \\
&= -V\sum_{i=1}^{3} \rho^+(e_i\nabla_{e_i}(\psi)) + V\nabla_t(\rho^+(\psi)) \\
&= V \cdot \rho^+\left(-\not{\partial}_{A(t)}(\psi(t)) + \frac{\partial\psi}{\partial t}\right). \qquad \text{q.e.d.}
\end{aligned}$$

Claim 6.2. *Let $F(t)$ be a one-parameter family of complex-valued two-forms on N. It determines a two-form F on X. Let $\psi(t)$ be a section of $\pi^*(S(\tilde{P}_N))$. Then we have*

$$\frac{1}{2}(F + *_4 F) \cdot \rho^+(\psi(t)) = F \cdot \rho^+(\psi)(t) = \rho^+(F(t) \cdot \psi(t)),$$

where $*_4$ is the complex-linear Hodge $*$-operator on the four-manifold X.

Proof. This is immediate from the fact that $\rho^+(\psi)$ is a section of $S^+(\tilde{P})$ and that ρ^+ commutes with Clifford multiplication and the embedding $Cl(N) = Cl_0(X) \subset Cl(X)$. q.e.d.

Lemma 6.3. *With the above isomorphisms, the Seiberg-Witten equations on X, written in terms of a path $(A(t), \psi(t))$ in the configuration space of $\tilde{P}_N$, are*

$$\left(F_{A(t)} + *\frac{\partial A}{\partial t}\right) = q(\psi(t)),$$

$$-\not{\partial}_{A(t)}(\psi) + \frac{\partial \psi}{\partial t} = 0,$$

where $*$ is the complex-linear Hodge star operator on N. We can rewrite these equations as

$$\frac{\partial A(t)}{\partial t} = * \left(q(\psi(t)) - F_{A(t)}\right),$$

$$\frac{\partial \psi(t)}{\partial t} = \not{\partial}_{A(t)}(\psi(t)),$$

where, once again, $*$ is the complex-linear $*$-operator on N.

Proof. We have already seen that the second equation is the Dirac equation. The curvature equation on X is:

$$(F_{A(t)} + dt \wedge \frac{\partial A}{\partial t})^+ = \rho^+ \circ q(\psi(t)) \circ (\rho^+)^{-1},$$

where the left-hand-side is interpreted as the automorphism induced by Clifford multiplication. Equivalently, we can write the equations as an automorphism of $S(\tilde{P}_N)$:

$$(\rho^+)^{-1} \circ (F_{A(t)} + dt \wedge \frac{\partial A}{\partial t})^+ \circ \rho^+ = q(\psi(t)).$$

Of course,

$$(F_{A(t)} + dt \wedge \frac{\partial A}{\partial t})^+ = \frac{1}{2}\left(F_{A(t)} + *\frac{\partial A}{\partial t} + dt \wedge *F_{A(t)} + dt \wedge \frac{\partial A}{\partial t}\right),$$

where $*$ denotes the complex-linear Hodge star operator on N. According to Claim 6.2 then the composition

$$(\rho^+)^{-1} \circ (F_{A(t)} + dt \wedge \frac{\partial A}{\partial t})^+ \circ \rho^+$$

is simply Clifford multiplication by

$$F_{A(t)} + *\frac{\partial A}{\partial t}.$$

Thus, we can rewrite the curvature equation as

$$F_{A(t)} + *\frac{\partial A}{\partial t} = q(\psi(t)).$$

q.e.d.

6.2. The gradient flow equation

Now we are ready to show that the Seiberg-Witten equations on X are equivalent to a gradient flow equation. We choose to work with the L^2_1-version of the configuration space for $\tilde{P}_N$. Thus, the space $\mathcal{C}(\tilde{P}_N)$ is the space of pairs (A, ψ) where A is an L^2_1-connection on the determinant line bundle of $\tilde{P}_N$, and ψ is an L^2_1-section of the associated spin bundle. The group of gauge automorphisms is the group of L^2_2-maps from N to S^1. Notice that every element of the group of gauge automorphisms is a continuous map. To make the expressions come out on the nose, we choose a slightly non-standard inner product on the tangent bundle to $\mathcal{C}(\tilde{P}_N)$. The tangent space is identified with the space of L^2_1-sections of $(T^*N \otimes i\mathbf{R}) \oplus S^+(\tilde{P}_N)$. The inner product we take is the standard L^2-inner product on the first factor and is twice the real part of the L^2-hermitian inner product on the second factor. We denote this inner product by $\langle \cdot, \cdot \rangle'_{L^2}$. Notice that for tangent vectors $a, b \in \Omega^1(X; i\mathbf{R})$ the L^2 inner product is given by

$$\langle a, b \rangle_{L^2} = -\int_N a \wedge *b,$$

where $*$ is the complex-linear Hodge $*$-operator. The reason for the change of sign is that the forms are purely imaginary and the positive definite product is given by the complex anti-linear Hodge $*$-operator.

Fix a background C^∞ connection A_0 on $\mathcal{L}_N$. Using this choice we define a function

$$f\colon \mathcal{C}(\tilde{P}_N) \to \mathbf{R}$$

by setting

$$f(A,\psi) = \int_N F_{A_0} \wedge a + \frac{1}{2}\int_N a \wedge da + \int_N \langle \psi, \not{\partial}_A \psi \rangle dvol,$$

where $a = A - A_0$. It is easy to see that f is a smooth function in the L_1^2-topology, and if we replace A_0 by a different background connection A_1, then we simply change the function f by a constant.

Lemma 6.4. *There is a natural homomorphism*

$$c\colon \mathcal{G}(\tilde{P}_N) \to H^1(N;\mathbf{Z}),$$

which assigns to each L_2^2-map $\sigma\colon X \to S^1$ the pull back under σ of the fundamental cohomology class for the circle. This map is surjective and its kernel is the component of the identity $\mathcal{G}_0(\tilde{P}_N)$ of $\mathcal{G}(\tilde{P}_N)$. If $\sigma \in \mathcal{G}(\tilde{P}_N)$, then

$$f(\sigma \cdot (A,\psi)) = f(A,\psi) + 2\pi\langle c(\sigma) \cup c_1(\mathcal{L}), [N]\rangle.$$

In particular, $f\colon \mathcal{C}(\tilde{P}_N) \to \mathbf{R}$ descends to a map

$$f\colon \mathcal{B}(\tilde{P}_N) \to \mathbf{R}/2\pi\mathbf{Z}.$$

Proof. This lemma is a direct computation. q.e.d.

In particular, we can use the function f to define a vector field on $\mathcal{B}^*(\tilde{P}_N)$. Notice that the inner product that we have chosen on the tangent bundle to $\mathcal{C}(\tilde{P}_N)$ descends to an inner product on the tangent bundle to $\mathcal{B}^*(\tilde{P}_N)$.

Definition 6.5. We denote by $\widetilde{\mathcal{B}}^*(\tilde{P}_N)$ the quotient of $\mathcal{C}^*(\tilde{P}_N)$ by $\mathcal{G}_0(\tilde{P}_N)$. This is a normal covering space of $\mathcal{B}^*(\tilde{P}_N)$ with covering group $H^1(N;\mathbf{Z})$. Notice that f descends to a function

$$f\colon \widetilde{\mathcal{B}}^*(\tilde{P}_N) \to \mathbf{R}.$$

By the gradient $\nabla f(A,\psi)$ we mean the formal tangent vector to $\mathcal{C}(\tilde{P}_N)$ for which the following equation holds for any L_1^2-tangent vector τ to $\mathcal{C}(\tilde{P}_N)$ at (A,ψ):

$$\langle \nabla f(A,\psi), \tau\rangle'_{L^2} = \frac{\partial f(A,\psi)}{\partial \tau} \equiv \frac{df((A,\psi)+s\tau)}{ds}\Big|_{s=0}\,.$$

From this description it seems that $\nabla f(A,\psi)$ is only an L^2_{-1} tangent vector. Actually, as we shall see below, it is an L^2-tangent vector. As such it is a tangent vector to a bigger space of configurations defined by a weaker norm.

Proposition 6.6. *Fix an open interval I. If a configuration $(A(t),\psi(t))$ in a temporal gauge for the Spin^c-structure $I\times\tilde{P}_N\to I\times N$ satisfies the Seiberg-Witten equations, then it gives a C^∞-path in $\mathcal{C}(\tilde{P}_N)$ satisfying the gradient flow equation*

$$\frac{\partial(A,\psi)}{\partial t}=\nabla f(A,\psi).$$

Two solutions to the Seiberg-Witten equations are gauge equivalent if and only if the paths in $\mathcal{C}(\tilde{P}_N)$ that they determine in temporal gauges are gauge equivalent under the action of the group $\mathcal{G}(\tilde{P}_N)$.

Remark 6.7. The gradient flow equation makes sense for any C^1-path in $\mathcal{C}(\tilde{P}_N)$, where we view the equations as equations of continuous functions on I with values in the space of L^2-sections of $\left(T^*X\otimes i\mathbf{R}\oplus S(\tilde{P}_N)\right)$. Along any path $(A(t),\psi(t))$ in $\mathcal{C}(\tilde{P}_N)$ which comes from a solution of the Seiberg-Witten equations the gradient of f is in fact a smooth section of $(T^*N\otimes i\mathbf{R})\oplus S(\tilde{P}_N)$, as can be seen using reasonable standard methods in elliptic regularity theory. In particular, at such points the gradient of f is an actual tangent vector to the infinite dimensional Hilbert manifold $\mathcal{C}(\tilde{P}_N)$.

Proof. Let us compute $\langle\nabla f(A,\psi),\tau\rangle'_{L^2}$ for a tangent vector $\tau=(\eta,\lambda)$ to $\mathcal{C}(\tilde{P}_N)$ at (A,ψ). We first consider the case where $\lambda=0$, i.e., the vector is a tangent vector in the connection direction; i.e., $\eta\in L^2_1(T^*X\otimes i\mathbf{R})$. Computing directly from the definition we see that

$$\frac{\partial f}{\partial\eta}=\int_N F_{A_0}\wedge\eta+\frac{1}{2}(\eta\wedge da+a\wedge d\eta)+\frac{1}{2}\int_N\langle\psi,\eta\cdot\psi\rangle dvol.$$

Of course, by Stokes' theorem we have

$$\int_N a\wedge d\eta=\int_N\eta\wedge da$$

and hence we can rewrite the above expression as

$$\begin{aligned}\frac{\partial f}{\partial\eta}&=\int_N F_{A_0}\wedge\eta+\eta\wedge da+\frac{1}{2}\int_N\langle\psi,\eta\cdot\psi\rangle dvol\\&=\int_N\eta\wedge F_A+\frac{1}{2}\int_N\langle\psi,\eta\cdot\psi\rangle dvol.\end{aligned}$$

The following claim is established by a direct computation in a local basis.

Claim 6.8. *Let μ, ν be purely imaginary two-forms on N and let $\langle \mu, \nu \rangle$ be the pointwise hermitian inner product between them. Then*

$$\langle \mu, \nu \rangle = \frac{1}{2} \mathrm{Tr}\left(Cl(\nu) \circ Cl(\mu) \right),$$

where $Cl(x)$ is the endomorphism of $S(\tilde{P}_N)$ which is Clifford multiplication by x.

Claim 6.9. *Let μ be the purely imaginary two-form on N such that Clifford multiplication by μ is the automorphism $q(\psi)$ and let η be a purely imaginary one-form. Then the pointwise inner product $\langle \psi, \eta \cdot \psi \rangle$ is given by:*

$$\langle \psi, \eta \cdot \psi \rangle = -2 * (\eta \wedge \mu) = 2 \langle *\eta, \mu \rangle,$$

where $*$ is the complex-linear extension of the Hodge star operator.

Proof. Fix a unitary basis for $S(\tilde{P}_N)$ at a point. Then the value of $\langle \psi, \eta \cdot \psi \rangle$ at that point is given by the matrix product

$$\overline{\psi}^T \cdot Cl(\eta) \cdot \psi,$$

where T indicates the transpose. This product is of course simply the trace of the matrix product

$$Cl(\eta) \cdot \psi \cdot \overline{\psi}^T,$$

which is the trace of the composition of Clifford multiplication by η and $q(\psi) + \frac{|\psi|^2}{2}\mathrm{Id}$. Since η is a one-form, the trace of Clifford multiplication by η is zero. Thus, we have that the trace of the composition $(\eta \cdot) \circ \frac{|\psi|^2}{2}\mathrm{Id}$ is zero, and hence

$$\langle \psi(x), \eta(x) \cdot \psi(x) \rangle = \mathrm{Tr}\left((Cl(\eta) \circ q(\psi(x)) \right).$$

Of course, since $S(\tilde{P}_N)$ is a module over $Cl^+(T^*N)$ it follows that $Cl(\eta) = Cl(*\eta)$ where $*$ is the complex linear extension of the Hodge star-operator. According to the previous claim this trace is equal to $2\langle \mu, *\eta \rangle$. Since μ and η are purely imaginary, this last inner product is equal to $-2 * (\mu \wedge \eta)$ and is also equal to $2 \langle *\eta, \mu \rangle$. q.e.d.

It follows immediately from the claim that

$$\int_N \langle \psi, \eta\psi \rangle dvol = -2 \int_N \eta \wedge q(\psi),$$

where we view $q(\psi)$ as a purely imaginary two-form on N through the inverse of Clifford multiplication. Thus, we see that

$$\frac{\partial f(A,\psi)}{\partial \eta} = \int_N \eta \wedge (F_A - q(\psi)) = \langle \eta, *(F_A - q(\psi))\rangle'_{L^2},$$

where $*$ is the complex anti-linear Hodge $*$-operator. This means that at least for the tangent vectors in the connection directions we have that $\nabla f(A,\psi)$ is given by

$$*(F_A - q(\psi)).$$

(Let us emphasize once again that here the $*$-operator is the complex anti-linear one.)

Now let us compute in the directions tangent to the spinor field. Let λ be a section of $S(\tilde{P}_N)$. We have

$$\frac{\partial f(A,\psi)}{\partial \lambda} = \int_N \left(\langle \lambda, \partial\!\!\!/_A(\psi)\rangle + \langle \psi, \partial\!\!\!/_A(\lambda)\rangle\right) dvol.$$

Since the Dirac operator is self-adjoint we can rewrite this as

$$\begin{aligned}\frac{\partial f(A,\psi)}{\partial \lambda} &= \int_N \left(\langle \lambda, \partial\!\!\!/_A(\psi)\rangle + \langle \partial\!\!\!/_A(\psi), \lambda\rangle\right) dvol \\ &= 2\mathrm{Re}\left(\langle \lambda, \partial\!\!\!/_A(\psi)\rangle\right)_{L^2} = \langle \lambda, \partial\!\!\!/_A(\psi)\rangle'_{L^2}.\end{aligned}$$

Thus, the component of $\nabla f(A,\psi)$ in the direction of the spinor fields is $\partial\!\!\!/_A(\psi)$.

Notice that the critical set of f, i.e., the subset of (A,ψ) for which $\nabla f(A,\psi) = 0$, is exactly the set of solutions to the Seiberg-Witten equations on N.

Furthermore, the equation

$$\frac{\partial(A(t),\psi(t))}{\partial t} = \nabla(f(A,\psi)$$

reads

$$\frac{\partial A(t)}{\partial t} = *(F_{A(t)} - q(\psi(t)))$$

and

$$\frac{\partial \psi(t)}{\partial t} = \partial\!\!\!/_{A(t)}(\psi(t)),$$

where in the first equation the $*$-operator is the complex anti-linear Hodge $*$-operator, which is minus the complex-linear Hodge $*$-operator. Rewriting the first equation with the complex-linear $*$-operator yields precisely the Seiberg-Witten equations.

Now let us consider solutions up to gauge equivalence. Since we are always working in a temporal gauge, the only freedom we have is to vary a solution $(A(t), \psi(t))$ by a gauge transformation $\sigma(t)$ which is constant in t, that is to say $\sigma(t) = \sigma \in \mathcal{G}(\tilde{P}_N)$ for all t. Clearly, such an operation changes f by a constant and hence leaves the gradient of f invariant. Hence, it takes gradient flows to gradient flows. This completes the proof of the proposition. q.e.d.

Since any solution of the Seiberg-Witten equation is gauge equivalent to a C^∞ solution, we see that if $(A(t), \psi(t))$ is a solution, then the function $f(A(t), \psi(t))$ is a C^∞ function of t.

6.2.1. Estimates related to the function f

We finish this subsection with two lemmas pertaining to the function f which will be used later in establishing exponential decay of solutions on the tubes. Throughout this subsubsection we assume that $N = S^1 \times C$, with C being a curve of genus $g > 1$, and that the determinant line bundle of $\tilde{P}_N$ is induced from a line bundle on C which has degree $2 - 2g$.

The first lemma is closely related to the fact that f satisfies Smale's Condition C. On each tangent space to $\mathcal{B}^*(\tilde{P}_N)$ there is an L^2_1-metric. We identify the tangent space at the point $[A, \psi] \in \mathcal{B}^*(\tilde{P}_N)$ with the slice in $\mathcal{C}(\tilde{P}_N)$ through (A, ψ). The square of the L^2_1-norm is identified with the restriction to this slice of the sum of the usual L^2_1-norm on one-forms and the L^2_1-norm on sections of $S(\tilde{P}_N)$ computed using the connection A. It is easy to see that this metric is independent of the choice of representative (A, ψ) for the point in $\mathcal{B}^*(\tilde{P}_N)$.

Lemma 6.10. *For any $\epsilon > 0$ there is $\lambda > 0$ such that if $x = (A, \psi) \in \mathcal{B}^*(\tilde{P}_N)$ has L^2_1-distance at least ϵ from the critical point $[A_0, \psi_0]$, then*

$$\|\nabla f(x)\|_{L^2} \geq \lambda.$$

Proof. If this result does not hold for some $\epsilon > 0$, then there is a sequence $x_i = (A_i, \psi_i)$ in $\mathcal{B}^*(\tilde{P}_N)$ whose L^2_1-distance from x_i to the critical point is at least ϵ for which

$$\|\nabla f(x_i)\|_{L^2} \mapsto 0$$

as $i \mapsto \infty$. This means that

$$\|F_{A_i} - q(\psi_i)\|_{L^2} \mapsto 0$$

and

$$\|\partial\!\!\!/_{A_i}(\psi_i)\|_{L^2} \mapsto 0$$

as $i \mapsto \infty$. Thus there is a constant $C > 0$ such that

$$\int_N |F_{A_i} - q(\psi_i)|^2 + 2|\partial\!\!\!/_{A_i}(\psi_i)|^2 \leq C.$$

Using the Weitzenbock formula for $\partial\!\!\!/_{A_i} \circ \partial\!\!\!/_{A_i}$ we can rewrite this inequality as

$$\int_N \left(|F_{A_i}|^2 + \frac{1}{2}|\psi_i|^4 - 2\langle F_{A_i}, q(\psi_i)\rangle + 2|\nabla_{A_i}(\psi_i)|^2 + \frac{s}{2}|\psi_i|^2 + \langle F_{A_i}\psi_i, \psi_i\rangle \right) \leq C.$$

By Claim 6.9 and the fact that by construction $S(\tilde{P}_N)$ is a module over $Cl(T^*N)^+$, we have

$$2\langle F_{A_i}, q(\psi)\rangle = \langle F_{A_i}\psi, \psi\rangle.$$

Thus, this expression simplifies to

$$\|F_{A_i}\|_{L^2}^2 + \frac{1}{2}\|\psi_i\|_{L^4}^4 + 2\|\nabla_{A_i}(\psi_i)\|_{L^2}^2 \leq C + \max_{x\in N}(-\frac{s}{2}(x))\|\psi_i\|_{L^2}^2.$$

It follows easily from this inequality that $\|F_{A_i}\|_{L^2}, \|\psi_i\|_{L^4}$, and $\|\nabla_{A_i}(\psi_i)\|_{L^2}$ are all bounded independent of i.

Since the F_{A_i} are bounded in L^2, this means after appropriate changes of gauge, we can arrange that the A_i are uniformly bounded in L^2_1. Thus the ψ_i are bounded in L^2_1. We fix a base C^∞ connection A_0 and write $A_i = A_0 + \alpha_i$. Then

$$\|\partial\!\!\!/_{A_0}(\psi_i) + \frac{1}{2}\alpha_i \cdot \psi_i\|_{L^2} \mapsto 0.$$

Since the α_i are bounded in L^2_1 and the ψ_i are bounded in L^2_1, after passing to a subsequence, we can assume that $\alpha_i \cdot \psi_i$ converges in L^2, so that $\partial\!\!\!/_{A_0}(\psi_i)$ converges in L^2. This means that the component of ψ_i which is L^2-orthogonal to the harmonic spinors converges in L^2_1. The L^4-bound on the ψ_i implies that the harmonic projection of the ψ_i are bounded in every norm. Hence, after replacing the sequence by a subsequence, we can assume that the ψ_i converge in L^2_1 to a limit ψ_∞. This implies that the $q(\psi_i)$ converge in L^2 to $q(\psi_\infty)$, and hence that the F_{A_i} converge in L^2 to $q(\psi_\infty)$. It now follows that the A_i converge in L^2_1 to a limit A_∞. Of course, the limit (A_∞, ψ_∞) is a solution to the Seiberg-Witten Equations SW^3 for $\tilde{P}_N$. Thus, we have shown that there is a subsequence of the x_i which converges, to an element $x_\infty \in \mathcal{C}(\tilde{P}_N)$. This limiting element is a solution to the Equations SW^3 and hence is irreducible. But its image in $\mathcal{B}^*(\tilde{P}_N)$ has L^2_1-distance at least $\epsilon > 0$ from the critical point. This is a contradiction. q.e.d.

We also need estimates near the critical point.

Lemma 6.11. *There is a constant $K > 0$ such that if the L^2_1-distance from $[A,\psi]$ to the critical point is sufficiently small, then the L^2_1-distance from $[A,\psi]$ to the critical point is bounded by $K\|\nabla f(A(t),\psi(t))\|_{L^2}$.*

Proof. A direct computation shows that $\nabla(f(A,\psi))$ is L^2-orthogonal to the tangent space to the gauge orbit through (A,ψ). Thus, $\|\nabla f(A,\psi)\|_{L^2}$ is equal to the L^2-norm of the value at $[A,\psi]$ of the section ξ_{SW}. Since ξ_{SW} is smooth and is transverse to zero at the critical point of f, the lemma is immediate. q.e.d.

6.3. Preliminary estimates on tubes

Let us begin with an elementary estimate for any tube $T = [a,b]\times N$.

Claim 6.12. *Let $\gamma(t) = (A(t),\psi(t))$ be a solution to the Seiberg-Witten equation on the tube $[a,b]\times N$. Let $\ell = b-a$ be the length of the tube, and let*

$$E^2 = \int_a^b \|\nabla f(\gamma(t))\|^2_{L^2(N_t)}dt = \|\dot{A}\|^2_{L^2(T)} + \|\dot{\psi}\|^2_{L^2(T)}$$

be the square of the energy of the solution. Finally, let $-s_0$ be a lower bound for the scalar curvature of N, with $s_0 \geq 0$. Then

$$\|\psi\|^2_{L^4(T)} \leq 2s_0\sqrt{\mathrm{vol}(N)\ell} + 2\sqrt{2}E.$$

Proof. We have

$$E^2 \geq \int_a^b \frac{1}{2}\|F_A - q(\psi)\|^2_{L^2(N_t)}dt + \int_a^b \|\partial\!\!\!/_A(\psi)\|^2_{L^2(N_t)}dt.$$

Expanding the the right-hand-side of this expression using the Bockner-Weitzenbock formula and Claim 6.9 we see that

$$E^2 \geq \int_a^b \left(\frac{1}{2}\|F_A\|^2_{L^2(N_t)} + \frac{1}{8}\|\psi\|^4_{L^4(N_t)} + \|\nabla_A(\psi)\|^2_{L^2(N_t)} - \frac{s_0}{4}\|\psi\|^2_{L^2(N_t)}\right)dt,$$

and hence that

$$\|\psi\|^4_{L^4(T)}dt \leq 2s_0\|\psi\|^2_{L^2(T)} + 8E^2.$$

By Cauchy-Schwartz we have

$$\|\psi\|^2_{L^2(T)} \leq \|\psi\|^2_{L^4(T)}\sqrt{\mathrm{vol}(N)\ell}.$$

Putting this together proves the claim.

Corollary 6.13. *Under the hypothesis and notation of the previous lemma we have*

$$\|F_A^+\|_{L^2(T)} \leq s_0\sqrt{\mathrm{vol}(N)\ell} + \sqrt{2}E.$$

Proof. Since (A,ψ) is a solution to the Seiberg-Witten equations we have $F_A^+ = q(\psi)$ and hence $|F_A^+| = |q(\psi)| = \frac{1}{2}|\psi|^2$. q.e.d.

The next result is a standard type of bootstrapping result in the elliptic context.

Lemma 6.14. *There are constants $E_0, K > 0$ depending only on N such that for any $T \geq 1$ and for $\gamma(t) = (A(t),\psi(t))$ any solution to the Seiberg-Witten equation in a temporal gauge on $[-1, T+1] \times N$ satisfying*

$$\int_{-1}^{T+1} \|\nabla f(\gamma(t))\|^2_{L^2(N)}dt \leq E_0^2,$$

we have

$$\int_0^T \|\nabla f(\gamma(t))\|^2_{L^2_1(N)} \leq K\int_{-1}^{T+1} \|\nabla f(\gamma(t))\|^2_{L^2(N)}dt.$$

Proof. Let T be a four-manifold of the form $[a,b]\times N$ and let $\gamma(t) = (A(t),\psi(t))$ be a solution to the Seiberg-Witten equations in a temporal gauge on T for the $Spin^c$-structure $\tilde{P} = [a,b] \times \tilde{P}_N$. We denote by $\ell = b - a$ the length of the tube T. Let us denote by the square of the energy

$$\begin{aligned} E^2 &= \int_T |\nabla f(\gamma(t))|^2 d\mathrm{vol}_N dt \\ &= \int_T \left(|F_A - q(\psi)|^2 + |\partial\!\!\!/^3_A(\psi)|^2\right) d\mathrm{vol}_N dt \\ &= \int_T \left(|\dot{A}|^2 + |\dot{\psi}|^2\right) d\mathrm{vol}_N dt. \end{aligned}$$

Let $B(\cdot,\cdot)$ be the bilinear form on the space of sections of $S^+(\tilde{P})$ with values in $\Omega^2_+(T; i\mathbf{R})$ which is associated to the quadratic form q. Differentiating the Seiberg-Witten equations gives

$$\begin{aligned} P_+d(\dot{A}) - B(\dot{\psi},\psi) &= 0, \\ \partial\!\!\!/_A(\dot{\psi}) + \frac{1}{2}\dot{A}\cdot\psi &= 0. \end{aligned}$$

The fact that ∇f is L^2-orthogonal to the gauge orbit at each point implies that

$$d_N^*\dot{A} + \mathrm{Im}\langle\dot{\psi},\psi\rangle = 0.$$

Since A and hence $\dot{A}$ have no dt-component we can rewrite this last equation as

$$d^*\dot{A} + \mathrm{Im}\langle\dot{\psi},\psi\rangle = 0,$$

where d^* is the adjoint of d on the four-manifold T.

We fix a C^∞-function $\xi\colon T \to [0,1]$ which is identically one on the middle third T' of T and identically zero near the ends of T. We do this so that $|d\xi|$ is at most M/ℓ for some universal constant M. We set $V = \xi\dot{A}$ and $\lambda = \xi\dot{\psi}$. Let us consider the operator

$$E(V,\lambda) = \left(P_+d(V) - B(\lambda,\psi), \partial\!\!\!/_A(\lambda) + \frac{1}{2}V\cdot\psi, d^*V + \mathrm{Im}\langle\lambda,\psi\rangle\right).$$

Since $E(\dot{A},\dot{\psi}) = 0$, it is easy to see that there is a universal constant C_0 such that

$$\|E(V,\lambda)\|^2_{L^2(T)} \le \frac{C_0}{\ell^2}\left(\|\dot{A}\|^2_{L^2(T)} + \|\dot{\psi}\|^2_{L^2(T)}\right). \tag{7}$$

On the other hand, direct computation shows that

$$\begin{aligned} &\|E(V,\lambda)\|^2_{L^2(T)} \\ &\ge \frac{1}{2}\left(\|P_+d(V)\|^2_{L^2(T)} + \|\partial\!\!\!/_A(\lambda)\|^2_{L^2(T)} + \|d^*(V)\|^2_{L^2(T)}\right) \\ &\quad - 2\left(\|B(\lambda,\psi)\|^2_{L^2(T)} + \frac{1}{2}\|V\cdot\psi\|^2_{L^2(T)} + \|\mathrm{Im}\langle\lambda,\psi\rangle\|^2_{L^2(T)}\right). \end{aligned} \tag{8}$$

For the moment, let us assume that $\ell \le 1$. By the Sobolev multiplication theorem we see that there is a constant C_1 depending only on N such that

$$\|B(\lambda,\psi)\|^2_{L^2(T)} \le C_1\|\lambda\|^2_{L^4(T)}\|\psi\|^2_{L^4(T)}.$$

By Lemma 6.14 and the Sobolev embedding theorem, this implies that there are constants C_2, C_3 depending only on N such that

$$\|B(\lambda,\psi)\|^2_{L^2(T)} \leq \left(C_2\sqrt{\ell} + C_3E\right) \|\lambda\|^2_{L^2_1(T)}.$$

Similarly, choosing C_2, C_3 appropriately we can also arrange that

$$\|V\cdot\psi\|^2_{L^2(T)} \leq \left(C_2\sqrt{\ell} + C_3E\right) \|V\|^2_{L^2_1(T)}$$

and

$$\|\mathrm{Im}\langle\lambda,\psi\rangle\|^2_{L^2(T)} \leq \left(C_2\sqrt{\ell} + C_3E\right) \|\lambda\|^2_{L^2_1(T)}.$$

Putting all this together we have constants C_4, C_5 independent of ℓ such that

$$\begin{aligned} &\|B(\lambda,\psi)\|^2_{L^2(T)} + \frac{1}{2}\|V\cdot\psi\|^2_{L^2(T)} + \|\mathrm{Im}\langle\lambda,\psi\rangle\|^2_{L^2(T)} \\ &\qquad \leq \left(C_4\sqrt{\ell} + C_5E\right)\left(\|V\|^2_{L^2_1(T)} + \|\lambda\|^2_{L^2_1(T)}\right). \end{aligned} \tag{9}$$

Lastly, from the Weitzenbock-Bockner formula for the Dirac operator it follows that

$$\|\partial\!\!\!/_A(\lambda)\|^2_{L^2} \geq \|\nabla_A(\lambda)\|^2_{L^2(T)} - \frac{s_0}{4}\|\lambda\|^2_{L^2(T)} + \frac{1}{2}\int_T \langle F_A^+\cdot\lambda,\lambda\rangle,$$

where $-s_0$ is a non-positive lower bound for the scalar curvature of N. Thus, we see that there is a constant C_6 such that, again replacing C_2, C_3 by larger constants if necessary, we have

$$\|\nabla_A(\lambda)\|^2_{L^2(T)} \leq \|\partial\!\!\!/_A(\lambda)\|^2_{L^2(T)} + C_6\|\lambda\|^2_{L^2(T)} + \left(C_2\sqrt{\ell} + C_3E\right)\|\lambda\|^2_{L^2_1(T)}.$$

From this and Equations (7), (8), and (9) we conclude that given any $\delta > 0$, there are constants $0 < \ell_0 < 1$ and $E_0 > 0$ such that if $0 < \ell \leq \ell_0$ and $0 < E \leq E_0$ then there is a constant C depending only on N such that

$$\begin{aligned} &\frac{1}{2}\left(\|P_+\, d(V)\|^2_{L^2(T)} + \|\nabla_A(\lambda)\|^2_{L^2(T)} + \|d^*(V)\|^2_{L^2(T)}\right) \\ &\qquad \leq \frac{C}{\ell^2}\left(\|\dot{A}\|^2_{L^2(T)} + \|\dot{\psi}\|^2_{L^2(T)}\right) - \delta\left(\|V\|^2_{L^2_1(T)} + \|\lambda\|^2_{L^2_1(T)}\right). \end{aligned}$$

Standard elliptic estimates for the operator (P_+d, d^*) show that, provided that $\delta > 0$ is sufficiently small, this inequality implies that there is a constant K_0 depending only on N such that if $E \leq E_0$ and the length of the tube T is equal to ℓ_0, then

$$\left(\|V\|^2_{L^2_1(T)} + \|\lambda\|^2_{L^2_1(T)}\right) \leq \frac{K_0}{\ell_0^2}\left(\|\dot{A}\|^2_{L^2(T)} + \|\dot{\psi}\|^2_{L^2(T)}\right).$$

Since $V|_{T'} = \dot{A}|_{T'}$ and $\lambda|_{T'} = \dot{\psi}|_{T'}$, this yields that for any tube T of length ℓ_0 and any solution $(A(t), \psi(t))$ on T whose energy is at most E_0^2 we have

$$\|\dot{A}\|^2_{L^2_1(T')} + \|\dot{\psi}\|^2_{L^2_1(T')} \leq \frac{K_0}{\ell_0^2}\left(\|\dot{A}\|^2_{L^2(T)} + \|\dot{\psi}\|^2_{L^2(T)}\right).$$

Now let us drop the assumption that the length of T is ℓ_0. By adding up over a sequence of middle third tubes of length ℓ_0, we establish the statement of the proposition easily from this inequality. The constants are E_0 and $K = 3K_0/\ell_0^2$. q.e.d.

6.4. Exponential decay in tubes

In this subsection we restrict to the Case when $N = S^1 \times C$ where C is a closed riemann surface of genus $g > 1$ and where $\tilde{P}_N$ is induced from a $Spin^c$-structure on C whose determinant line bundle has degree $\pm(2-2g)$ on C. According to Proposition 5.1 and Lemma 5.2, the character variety of solutions to the Seiberg-Witten equations is one point, this point being a non-degenerate solution. It is the purpose of this section to use this non-degeneracy to establish two fundamental exponential decay results. But before we get to these results we establish some elementary estimates for solutions in the tube.

Our first exponential decay result is a fairly standard one. It concerns flow lines which are near the critical point. It is a consequence of the fact that the critical points are non-degenerate, see, for example [10] or [6], for similar estimates for the $SU(2)$ anti-self-dual equations.

Lemma 6.15. *With $N = S^1 \times C$ and $\tilde{P}_N$ a $Spin^c$-structure whose determinant line bundle $\mathcal{L}$ has degree $\pm(2-2g)$ on C, there are positive constants $\epsilon, \delta > 0$ such that for any $T \geq 1$ if $(A(t), \psi(t))$ is a solution to the Seiberg-Witten equations on $[0,T] \times N$ in a temporal gauge and if for each t, $0 \leq t \leq T$, the equivalence class of $(A(t), \psi(t))$ is within ϵ in the L^2_1-topology on $\mathcal{B}^*(\tilde{P}_N)$ of the solution $[A_0, \psi_0]$ of the Seiberg-Witten equations on N, then the distance from $[A(t), \psi(t)]$ to $[A_0, \psi_0]$ in the L^2_1-topology is at most*

$$d_0 \exp(-\delta t) + d_T \exp(-\delta(T-t)),$$

where d_0 (resp. d_T) is the L^2_1-distance from $[A(0), \psi(0)]$ to $[A_0, \psi_0]$ (resp., the L^2_1-distance from $[A(T), \psi(T)]$ to $[A_0, \psi_0]$).

The other exponential decay result is more delicate. This is a special case of the 'small energy implies small length' results established by Simon in [9], see [6] or [10], for similar results in the $SU(2)$-context. They apply without the assumption that the critical point is isolated and non-degenerate. Here, the results are stronger than the general results and can be established fairly directly using the previous exponential decay.

Proposition 6.16. *There is a constant $\delta > 0$, and given any $\lambda > 0$ there is $E_0 > 0$ such that for any solution $(A(t), \psi(t))$ in temporal gauge to the Seiberg-Witten equations on $[-1, T+1] \times N$, with $T \geq 1$, the following holds. Let $\gamma\colon [-1, T+1] \to \mathcal{C}(\tilde{P}_N)$ be the associated path to the solutions. If $f(\gamma(T+1)) - f(\gamma(-1)) \leq E_0^2$, then for $0 \leq t \leq T$, the L^2_1-distance from $[A(t), \psi(t)]$ to the static solution $[A_0, \psi_0]$ is at most*

$$\lambda\big(\exp(-\delta t) + \exp(-\delta(T-t))\big).$$

Proof. First notice that

$$f(\gamma(T+1)) - f(\gamma(-1)) = \int_{-1}^{T+1} \|\nabla f(\gamma(t))\|^2_{L^2} dt.$$

We fix $0 < \epsilon_1 << \epsilon_2$ with ϵ_2 being less than the constant $\epsilon > 0$ given in the statement of Lemma 6.15. It follows from Lemma 6.10 that, if E_0 is sufficiently small, then the total length of the open subset of $t \in [-1, T+1]$ for which $[A(t), \psi(t)]$ has L^2_1-distance at least $\epsilon_1 > 0$ from the critical point is less than $1/2$, and in particular, there must be $t_1 \in [0, T]$ such that the L^2_1-distance from $[A(t_1), \psi(t_1)]$ to the critical point is at most ϵ_1. Suppose that there is $t_2 \in [0, T]$ so that the L^2_1-distance from $[A(t_2), \psi(t_2)]$ to the critical point is $\geq \epsilon_2$. By symmetry we can assume that $t_1 < t_2$. We can then choose the first such t_2, so that $\gamma([t_1, t_2])$ is contained in the closed L^2_1-neighborhood of diameter ϵ_2 of the critical point and the distance from $[A(t_2), \psi(t_2)]$ to the critical point is exactly ϵ_2. By Lemma 6.15 we have that for any $t \in [t_1, t_2]$, the L^2_1 distance from $[A(t), \psi(t)]$ to the critical point is at most

$$\epsilon_1 e^{-\delta(t-t_1)} + \epsilon_2 e^{-\delta(t_2-t)}.$$

We consider a $u \in [t_1, t_2]$ for which the L^2_1-distance from $[A(u), \psi(u)]$ to the critical point is exactly $\epsilon_2/2$. For this u we have

$$\epsilon_1 e^{-\delta(u-t_1)} + \epsilon_2 e^{-\delta(t_2-u)} \geq \epsilon_2/2.$$

Since $\epsilon_1 << \epsilon_2$, we must have

$$\epsilon_2 e^{-\delta(t_2-u)} \geq \epsilon_2/3,$$

or equivalently

$$\delta(t_2 - u) \leq \log(3).$$

We conclude that

$$0 < (t_2 - u) \leq \frac{\log(3)}{\delta}.$$

Since the L^2_1-distance from $[A(t_2), \psi(t_2)]$ to the critical point is ϵ_2, we know that the L^2_1-distance from $[A(u), \psi(u)]$ to $[A(t_2), \psi(t_2)]$ is at least $\epsilon_2/2$, and consequently,

$$\int_u^{t_2} \|\nabla f(\gamma(v))\|_{L^2_1(N)} \geq \epsilon_2/2.$$

Of course, since the volume of the tube $[u, t_2] \times N$ is bounded, the Cauchy-Schwartz inequality tells us that there is a positive constant ϵ_3 depending only on N such that

$$\int_u^{t_2} \|\nabla f(\gamma(v))\|^2_{L^2_1(N)} \geq \epsilon_3.$$

By Lemma 6.14, this implies that there is a positive constant ϵ_4 such that

$$\int_{u-1}^{t_2+1} \|\nabla f(\gamma(t))\|^2_{L^2(N)} dt \geq \epsilon_4.$$

But

$$\int_{u-1}^{t_2+1} \|\nabla f(\gamma(t))\|^2_{L^2(N)} dt \leq \int_{-1}^{T+1} \|\nabla f(\gamma(t))\|^2_{L^2(N)} dt \leq E_0^2.$$

Provided that E_0 is sufficiently small, this is a contradiction.

It follows from this contradiction that given $\epsilon_2 > 0$, with $\epsilon_2 < \epsilon$, if $E_0 > 0$ is sufficiently small then for all $t \in [0, T]$ the point $\gamma(t)$ is contained in the L^2_1-neighborhood of diameter ϵ_2 centered at the critical point. Hence, if $E_0 > 0$ is sufficiently small, by Lemma 6.15 for all $t \in [0, T]$, the L^2_1-distance from $[A(t), \psi(t)]$ is at most

$$d_0 e^{-\delta(t)} + d_T e^{-\delta(T-t)},$$

where d_0 (resp. d_T) is L^2_1-distance from $[A(0), \psi(0)]$ (respectively $[A(T), \psi(T)]$) to the critical point. Of course, since the whole path is contained in the ϵ_2-neighborhood of the critical point, we have $d_0, d_T \leq \epsilon_2$. The result now follows by taking $\epsilon_2 \leq \lambda$. q.e.d.

Of course, the ellipticity allows us to bootstrap these L_1^2-estimates into C^∞ estimates.

Corollary 6.17. *There is a constant $\delta > 0$, and given any $\lambda > 0$ there is $E_0 > 0$ such that for any solution $(A(t), \psi(t))$ in temporal gauge to the Seiberg-Witten equations on $[-1, T+1] \times N$, with $T \geq 1$, the following holds. Let $\gamma\colon [-1, T+1] \to \mathcal{C}(\tilde{P}_N)$ be the associated path. If $f(\gamma(T+1)) - f(\gamma(-1)) \leq E_0^2$, then for $0 \leq t \leq T$, the C^∞-distance from $[A(t), \psi(t)]$ to the static solution $[A_0, \psi_0]$ is at most*

$$\lambda\big(\exp(-\delta t) + \exp(-\delta(T - t))\big).$$

6.5. The space of all solutions on the cylinder

In this section we shall describe the space of all finite energy solutions to the Seiberg-Witten equations on $\mathbf{R} \times N$ in the context of the previous section: $N = S^1 \times C$ with C being a riemann surface of genus $g > 1$ and $\mathcal{L}$ having degree $2g - 2$ on C.

Lemma 6.18. *There is a constant K depending only on the riemannian 3-manifold $N = S^1 \times C$ such that the following holds. Let $\mathbf{R} \times \tilde{P}_N$ on $\mathbf{R} \times N$ be a Spin^c-structure for which the degree of the determinant line bundle of $\tilde{P}_N$ on C is $(2-2g)$. Let (A, ψ) be a solution to the Seiberg-Witten equations for $\mathbf{R} \times N$. Write $(A, \psi) = (A(t), \psi(t))$ in a temporal gauge and let $\gamma\colon \mathbf{R} \to \mathcal{C}(\tilde{P}_N)$ be the path determined by this solution. Suppose that the function $f(\gamma(t))$ has finite limits as $t \mapsto \pm\infty$. Then for every $x \in \mathbf{R} \times N$ we have*

$$|\psi(x)| \leq K.$$

Remark 6.19. This corollary holds for any closed three-manifold and any *Spin^c*-structure on it. The proof uses the weaker energy-length results alluded to before.

Proof. In an appropriate gauge any finite energy solution is C^∞. An immediate application of Corollary 6.17 to longer and longer finite tubes shows that the solution $(A(t), \psi(t))$ decays exponentially in the C^∞-topology to the static solution $[A_0, \psi_0]$ as $t \mapsto \pm\infty$. Since there is clearly K_1 with the property that $|\psi_0(x)| \leq K_1$ for every $x \in N$, either $|\psi(x)| \leq K_1$ for every $x \in \mathbf{R} \times N$ or $|\psi(x)|$ achieves its maximum at some $x_0 \in \mathbf{R} \times N$. At a local maximum x_0 an easy maximum principle argument (see [4], for example) shows that

$$|\psi(x_0)| \leq -\kappa(x_0),$$

where κ is the scalar curvature. Since the scalar curvature of $\mathbf{R} \times N$ is bounded, the lemma follows. q.e.d.

Lemma 6.20. *Suppose that (A, ψ) is a solution to the Seiberg-Witten equations for $\mathbf{R} \times \tilde{P}_N$ on $\mathbf{R} \times N$ and that the degree of the determinant line bundle of $\tilde{P}_N$ on C is $(2-2g)$. Write $(A, \psi) = (A(t), \psi(t))$ in a temporal gauge and let $\gamma \colon \mathbf{R} \to \mathcal{C}(\tilde{P}_N)$ be the path determined by this solution. Suppose that the function $f(\gamma(t))$ has finite limits as $t \mapsto \pm\infty$. Then A is a holomorphic connection with respect to the natural complex structure on $X = \mathbf{R} \times N = (\mathbf{R} \times S^1) \times C = \mathbf{C}^* \times C$, and $\psi = (\alpha, 0)$ where α is a holomorphic section of $(\mathcal{L} \otimes K_X)^{1/2}$. Furthermore, if the solution is not a static solution, then the formal dimension of the moduli space at this solution is negative.*

Proof. As before, the condition on the function $f(\gamma(t))$ implies that as $t \mapsto \pm\infty$ the configuration $(A(t), \psi(t))$ decays exponentially in the C^∞-topology to a static solution. Writing the spinor field ψ as (α, β) as before and applying $\overline{\partial}_A$ to the Dirac equation give

$$\overline{\partial}_A \overline{\partial}_A(\alpha) + \overline{\partial}_A \overline{\partial}_A^*(\beta) = 0.$$

Invoking the curvature equation we get

$$|\alpha|^2 \beta + \overline{\partial}_A \overline{\partial}_A^*(\beta) = 0.$$

Because the determinant line bundle has negative degree on C, it follows that for the static solution $\beta = 0$. Thus, the field β is decaying in L_1^2 to the trivial field, and since A is decaying in L_1^2 to a fixed connection pulled back from C, we see that we can take the pointwise inner product with β, integrate by parts over $X = \mathbf{R} \times N$ and conclude that

$$\int_X \left(|\alpha|^2 |\beta|^2 + |\overline{\partial}_A^*(\beta)|^2 \right) dvol = 0.$$

Thus, as before, we have that one of α and β is zero. In fact, it must be β that is zero because of the nature of the limits as $t \mapsto \pm\infty$. This implies that A determines a holomorphic structure on $\mathcal{L}$ with respect to which α becomes a holomorphic section of $\mathcal{L}_0$.

At this point we have shown that any solution must be holomorphic. Lastly, let us consider the dimension of the moduli space at the solution. The formula for the dimension is the same as in the closed case. Namely, the dimension is

$$\frac{1}{4}\left(2\langle c_1(\mathcal{L}),[C]\rangle\frac{i}{2\pi}\left(\int_{\mathbf{R}\times S^1}F_A\right)\right)=(1-g)\frac{i}{2\pi}\int_{\mathbf{R}\times S^1}F_A.$$

(Since A is decaying exponentially to a connection pulled back from C at each end, the restriction of A to $\mathbf{R}\times S^1$ decays exponentially to a product connection at each end. Thus,

$$\frac{i}{2\pi}\int_{\mathbf{R}\times S^1}F_A$$

is well-defined and finite and in fact is an integer.) Since the metric is the standard flat metric in the $\mathbf{R}\times S^1$ direction, this means that the connection A_0 on $\mathcal{L}_0=\sqrt{\mathcal{L}\otimes K_X}$ induced by A and the holomorphic metric connection on K_X decays exponentially to a product connection at each end, and

$$\int_{\mathbf{R}\times S^1}F_A=2\int_{\mathbf{R}\times S^1}F_{A_0}.$$

It follows that the formal dimension of the moduli space is

$$2(1-g)\frac{i}{2\pi}\int_{\mathbf{R}\times S^1}F_{A_0}.$$

But α is a holomorphic section of $\mathcal{L}_0$ which converges exponentially fast at each end to a constant non-zero section. Hence, we can extend the connection $A_0|_{\mathbf{R}\times S^1}$ to a holomorphic connection on a line bundle over S^2, and extend α to a holomorphic section which does not vanish at either of the points added at the ends. This means that $\frac{i}{2\pi}\int_{\mathbf{R}\times S^1}F_{A_0}$ is equal to the number of zeros (counted with multiplicity) of the holomorphic section α. In particular, this integral is non-negative, and is zero if and only if α is a constant section. Thus, the formal dimension of the moduli space is non-positive and is zero only when α is constant along each $\mathbf{R}\times S^1$, and hence constant on X. This means that our solution is the static solution. q.e.d.

Remark 6.21. Notice that in the course of this proof we have shown that for any finite energy solution to the Seiberg-Witten equations on the infinite cylinder $\mathbf{R}\times N$ then the spinor field is identified with a holomorphic section α of $\mathcal{L}_0$ and

$$\frac{i}{2\pi}\int_{\mathbf{R}\times S^1} F_A = 2(\#\text{ zeros of }\alpha).$$

Definition 6.22. We call $(i/2\pi)\int_{\mathbf{R}\times S^1} F_A$ the *degree of A along* $\mathbf{R}\times S^1$, and similarly we define the degree of A_0 along $\mathbf{R}\times S^1$.

Let $\gamma\colon \mathbf{R}\to \mathcal{B}^*(\tilde{P}_N)$ be a path with $\lim_{t\mapsto-\infty}(\gamma(t)) = \lim_{t\mapsto\infty}(\gamma(t))$ being the solution for $\tilde{P}_N$ of the three-dimensional Seiberg-Witten equations for $\tilde{P}_N$. Say $\gamma(t) = (A(t),\psi(t))$. Denote by A the connection on $\mathbf{R}\times\mathcal{L}\to\mathbf{R}\times N$ determined by the path $A(t)$ on connections on $\mathcal{L}$. Lift γ to a path $\tilde{\gamma}$ in $\widetilde{\mathcal{B}}^*(\tilde{P}_N)$. This path has endpoints which differ by a lattice point in $H^1(N;\mathbf{Z})$. Let $\delta(\gamma)$ be this difference. Then for any one-cycle W in N we have:

$$\frac{i}{2\pi}\int_{\mathbf{R}\times W} F_A = \langle\delta(\gamma), W\rangle.$$

Lemma 6.23. *For any solution $\gamma(t)$, $-\infty < t < \infty$, to the gradient flow equation with the property that $\lim_{t\mapsto-\infty}f(\gamma(t))$ and $\lim_{t\mapsto\infty}f(\gamma(t))$ are both finite, the difference element $\delta(\gamma)$ is a non-negative integral multiple of the Poincaré dual of $[C]$ in $H^1(N;\mathbf{Z})$.*

Proof. The condition on the limits of $f(\gamma(t))$ as $t\mapsto\pm\infty$ implies that the gauge equivalence classes of $\lim_{t\mapsto-\infty}\gamma(t)$ and $\lim_{t\mapsto\infty}\gamma(t)$ are equal to the solution to the three-dimensional Seiberg-Witten equations on $\tilde{P}_N$, so that the difference element $\delta(\gamma)$ is defined. For any $x\in C$ the value of $\delta(\gamma)$ on $[S^1\times\{x\}]$ is equal to the integral of $iF_A/2\pi$ over $\mathbf{R}\times S^1$. Since the metric is flat on this factor, this integral is equal to twice the degree of $\mathcal{L}_0$ over $\mathbf{R}\times S^1\times\{x\}$ which is equal to the number of zeros of the section α on this factor. Since α is a holomorphic section, its number of zeros is non-negative.

To complete the proof we need to see that $\langle\delta(\gamma),\{\theta_0\}\times W\rangle = 0$ if W is a one-cycle in C and $\theta_0\in S^1$. Consider the restriction of A_0 to any slice $\{t\}\times\theta\times C$. This is a connection on a complex line bundle of degree zero. It, of course, determines a holomorphic structure for this line bundle. The restriction of the section α to this slice is then a holomorphic section with respect to this holomorphic structure. Provided that α is not identically zero on the given slice, this means that the holomorphic structure determined by A_0 over this slice is trivial. Under the same proviso it follows that the holomorphic structure determined by A over this slice is isomorphic to the canonical holomorphic structure on K_C^{-1}. Now α vanishes identically on only finitely many slices. Hence, for all by finitely many points in $x \in \mathbf{R} \times S^1$ the holomorphic structure determined by A over this slice is in fact isomorphic to the canonical holomorphic structure on K_C^{-1}. Since the holomorphic structure determined by A over the slice $\{x\} \times C$ varies continuously with x, it follows that over all slices $\{x\} \times C$ the connection A determines the canonical holomorphic structure on K_C^{-1}. But this means that the gauge equivalence class of the restriction of A over the slices $\{x\} \times C$ is independent of x. Of course the connection varies continuously as we vary the point $y \in \mathbf{R} \times S^1$. Thus the restrictions of A to the various slices $\{y\} \times C$ are in fact gauge equivalent by a gauge transformation in the component of the identity. Consequently, for each $y \in \mathbf{R} \times S^1$

$$\int_{\{y\}\times W} A$$

is independent of $\{y\}$. But

$$\langle \delta(\gamma), \{\theta_0\} \times W \rangle = \lim_{t\mapsto\infty} \int_{\{t\}\times\{\theta_0\}\times W} A - \lim_{t\mapsto -\infty} \int_{\{t\}\times\{\theta_0\}\times W} A.$$

Since these integrals are independent of t, it follows that the difference is zero. q.e.d.

6.6. Perturbation of the equations

In this subsection we shall consider the same topological set-up: $N = S^1 \times C$ with C being a curve of genus $g > 1$ and $\mathcal{L}$ having degree $2 - 2g$ on C. All the structure established in the last two subsections holds for a sufficiently small perturbation of the Seiberg-Witten equations. Fix a harmonic one-form $n \in \Omega^1(C; \mathbf{R})$. We write $n = \eta + \overline{\eta}$ with η being a holomorphic one-form on C.

We consider the equations on $\mathbf{R} \times N$ denoted (SW_h) :

$$F_A^+ = q(\psi) + i(*n + dt \wedge n),$$
$$\not{\partial}_A(\psi) = 0,$$

where $*$ is the complex-linear Hodge $*$-operator on N, and we have set $h = *n + dt \wedge n$.

Claim 6.24. *The associated equations on N are (SW^3_{*n}) :*

$$F_{A(t)} = q(\psi(t)) + i(*n),$$
$$\not\partial_{A(t)}(\psi(t)) = 0$$

in the sense that the static solutions to the Equations SW_h are simply the solutions to SW^3_{*n}. Furthermore, the associated function on $\mathcal{C}(\tilde{P}_N)$ is

$$f_n(A,\psi) = \int_N F_{A_0} \wedge a + \frac{1}{2}\int_N a \wedge da - \int_N i(*n) \wedge a + \int_N \langle \psi, \not\partial_A(\psi)\rangle,$$

(where as before $a = A - A_0$) in the sense that solutions to the Equations (SW_h) written in a temporal gauge yield C^∞-paths in $\mathcal{C}(\tilde{P}_N)$ which satisfy the gradient flow equations for f_n. Lastly, the analogues of Lemma 6.15, Corollary 6.17 and Lemma 6.18 hold for solutions to the perturbed equations.

Remark 6.25. The gradient flow equation for f_n is interpreted exactly as in Remark 6.7.

Proof. All of this follows from the same computations given in the unperturbed case. q.e.d.

We have already seen above that for all n sufficiently small there is a unique solution to the Equations (SW^3_{*n}). It is non-degenerate, and we have explicitly identified the solution. As before the set of solutions to SW^3_{*n} in $\tilde{\mathcal{B}}^*(\tilde{P}_N)$ is a lattice associated to $H^1(N;\mathbf{Z})$ and for any two solutions there is a difference element δ in this lattice. Given any smooth path $\gamma\colon \mathbf{R} \to \mathcal{B}^*(\tilde{P}_N)$ with $\lim_{t\mapsto\infty}\gamma(t)$ and $\lim_{t\mapsto-\infty}\gamma(t)$ both equal to the solution to SW^3_{*n}, the difference element $\delta(\gamma) \in H^1(N;\mathbf{Z})$ is defined. As before, if $\gamma(t) = (A(t),\psi(t))$, and A is the connection on $\mathbf{R}\times\mathcal{L} \to \mathbf{R}\times N$ determined by the path $A(t)$ of connections on $\mathcal{L}$, then for any one-cycle W in N we have

$$\frac{i}{2\pi}\int_{\mathbf{R}\times W} F_A = \langle \delta(\gamma), W\rangle.$$

Notice that if $\gamma(t)$, $-\infty < t < \infty$, is a solution to the gradient flow equation for f_n with $f_n(\gamma(t))$ having finite limits at $\pm\infty$, then the difference element $\delta(\gamma)$ is defined.

Proposition 6.26. *If $n \neq 0$ is a sufficiently small harmonic one-form on C, then the only solutions $(A(t),\psi(t)) = \gamma(t)$ to the perturbed*

Seiberg-Witten equations SW_h (where $h = *n + dt \wedge n$) on $\mathbf{R} \times N$ which satisfy:

- *the limits as $t \mapsto \pm\infty$ of $f_n(\gamma(t))$ are both finite, and*
- *the difference element $\delta(\gamma)$ is a multiple of the Poincaré dual of $[C]$ in $H^1(N; \mathbf{Z})$*

are holomorphic in the sense that the connection A defines a holomorphic structure on $\mathcal{L}$ and hence on $\mathcal{L}_0$ and $\psi = (\alpha, \beta)$ with α being a holomorphic section of $\mathcal{L}_0$ and with $\overline{\beta}$ being a holomorphic two-form with values in $\mathcal{L}_0$.

Remark 6.27.

- In fact, this result holds for any non-zero holomorphic one-form, and the proof in the general case is similar to the one given here.
- Notice that the same argument as in the unperturbed case shows that if A is a holomorphic connection then the difference element is indeed a non-negative multiple of the Poincaré dual of $[C]$. This means that the second condition in the statement is necessary.

Proof. The condition on the limits of $f_n(t)$ as $t \mapsto \pm\infty$ and the analogue of Corollary 6.17 imply that at the two ends the solution decays exponentially in the C^∞-topology to the static solution. As before we write $\psi = (\alpha, \beta)$ with $\alpha \in \Omega^0(X; \mathcal{L}_0)$ and $\beta \in \Omega^{0,2}(X; \mathcal{L}_0)$. Then the equations are:

$$\overline{\partial}_A(\alpha) + \overline{\partial}_A^*(\beta) = 0$$

and

$$F_A^+ = q(\alpha, \beta) + i(*n + dt \wedge n).$$

Arguing as before with the Dirac equation and the curvature equation we have

$$F_A^{0,2} \cdot \alpha + \overline{\partial}_A \overline{\partial}_A^*(\beta) = 0.$$

Now we take L^2-inner product with β and integrate. The result is

$$\langle F_A^{0,2}, \overline{\alpha}\beta \rangle_{L^2} + \|\overline{\partial}_A^*(\beta)\|_{L^2}^2 = 0. \tag{10}$$

Of course, from the curvature equation it follows that

$$F_A^{0,2} = \overline{\alpha}\beta + i(*n + dt \wedge n)^{0,2}.$$

Claim 6.28. *The second condition in the statement of the proposition implies that*

$$\langle F_A^{0,2}, i(*n + dt \wedge n)^{0,2}\rangle_{L^2} = 0.$$

Proof. By hypothesis F_A decays exponentially to the same constant closed two-form at the two ends of X, a closed two-form pulled up from a form on C, whose $(0,2)$-component is zero. This means that the L^2-inner product is finite. Since $*n + dt \wedge n$ is self-dual and annihilates the Kähler form we have

$$\langle F_A^{0,2}, i(*n + dt \wedge n))^{0,2}\rangle_{L^2} = \frac{1}{2}\langle F_A, i(*n + dt \wedge n))\rangle_{L^2}.$$

Now let us subtract from F_A the pullback of the form on C which is the limit of F_A at each end. This does not change the inner product since this form is pointwise orthogonal to both $*n$ and $dt \wedge n$. Let the difference form be Δ. Then Δ exponentially decays to zero at each end of X, and hence represents a relative cohomology class. The inner product

$$\frac{1}{2}\langle F_A, i(*n + dt \wedge n)\rangle_{L^2}$$

is then the cohomological product

$$\frac{-i}{2}\int_X \Delta \wedge (*n + dt \wedge n).$$

The second term vanishes since $d(tn) = dt \wedge n$ and Δ is a relative class. The first term is equal to

$$\frac{-i}{2}\int_{\mathbf{R}\times W} \Delta = \frac{-i}{2}\int_{\mathbf{R}\times W} F_A = -\pi\langle\delta(\gamma), W\rangle,$$

where $W \subset C$ is the real cycle Poincaré dual in N to $*n$. By hypothesis $\delta(\gamma)$ is Poincaré dual to $[C]$ in $H^1(N;\mathbf{Z})$ and hence vanishes on any one-cycle W in C. This completes the proof of the claim. q.e.d.

Thus, adding $\langle F_A^{02,}, i(*n + dt \wedge n)\rangle_{L^2} = 0$ to Equation (10) we obtain

$$\|F_A^{0,2}\|_{L^2}^2 + \|\overline{\partial}_A^*(\beta)\|_{L^2}^2 = 0.$$

It then follows that $F_A^{0,2} = 0$ and hence that A is a holomorphic connection. It also follows that α is a holomorphic section and that $\overline{\beta}$ is a holomorphic two-form. Lastly, we have

$$\overline{\alpha}\beta + i(*n + dt \wedge n)^{0,2} = 0$$

or

$$\overline{\alpha}\beta = -2i(*n)^{0,2}.$$

This completes the proof of the proposition. q.e.d.

Now we come to the crucial lemma which deals with the difference element.

Lemma 6.29. *Fix a constant $K > 0$. For any harmonic one-form n on C which is sufficiently small the following holds. Set $h = *n + dt \wedge n$. Then for any solution $\gamma(t)$, $-\infty < t < \infty$, to the Equations SW_h which has finite limits as $t \mapsto \pm\infty$ for $f_n(\gamma(t))$ and for which the difference of these limits is at most K, the difference element $\delta(\gamma)$ in $H^1(N; \mathbf{Z})$ is a multiple of the Poincaré dual of $[C]$.*

This lemma will be proved in the next subsection.

Lastly, we have our result:

Proposition 6.30. *Fix a constant $K > 0$. For any non-zero harmonic one-form n on C which is sufficiently small the following holds. Set $h = *n + dt \wedge n$. Then any solution $\gamma(t)$, $-\infty < t < \infty$, to the Equations SW_h for which the limits for $f_n(\gamma(t))$ as $t \mapsto \pm\infty$ are finite and the difference of these limits is at most K is a static solution.*

Proof. Putting together Proposition 6.26 and the previous lemma, we conclude that, for any n sufficiently small, any solution $(A, (\alpha, \beta))$ to SW_h has the following properties:

- A is a holomorphic connection for $\mathcal{L}$,
- α is a holomorphic section of $\mathcal{L}_0 = \sqrt{\mathcal{L} \otimes K_X}$,
- $\overline{\beta}$ is a holomorphic two-form with values in $\mathcal{L}_0$, and
- $\overline{\alpha}\beta$ is equal to the $(0, 2)$-component of $-2i(*n)$.

The line bundle $\mathcal{L}_0$ is trivial on each slice $\{t\} \times \{\theta\} \times C$, and hence the section α is constant along each of these slices. Notice that the $(0, 2)$-component of $*n$ is invariant under the natural action of S^1 and $\mathbf{R}$, and that it does not vanish identically on any slice $\{t\} \times \{\theta\} \times C$. This implies that α is never zero, and thus, the bundle $\mathcal{L}_0$ is holomorphically trivial on each $\mathbf{R} \times S^1$ with α being a constant section. Since the product $\overline{\alpha}\beta$ is also constant on each $\mathbf{R} \times S^1$-slice, it follows also that β is constant on each $\mathbf{R} \times S^1$. Thus, the solution is invariant under the translation action of $\mathbf{R}$ and hence is a static solution. q.e.d.

6.7. Limits

Now we turn to the proof of Lemma 6.29. The fact that the critical points of f in $\widetilde{\mathcal{B}}^*(\tilde{P}_N)$ are non-degenerate and form a discrete subset easily yields the following.

Lemma 6.31. *There are constants $K_1, \delta > 0$ depending only on N such that the following holds. For $\epsilon > 0$ sufficiently small there is a contractible open neighborhood ν of the critical point for f in $\mathcal{B}^*(\tilde{P}_N)$ with contractible closure, such that the following hold: Let $\gamma(t)$, $0 \leq t \leq T$ be a C^1-path in $\mathcal{C}(\tilde{P}_N)$ which solves the gradient flow equations for f.*

- *If the image of γ in $\widetilde{\mathcal{B}}^*(\tilde{P}_N)$ is a path with endpoints in different components of the preimage of $\overline{\nu} \subset \widetilde{\mathcal{B}}^*(\tilde{P}_N)$ then*

$$f(\gamma(T)) - f(\gamma(0)) \geq K_1.$$

- *If the image of γ to $\widetilde{\mathcal{B}}^*(\tilde{P}_N)$ has endpoints in the same component of the preimage of $\overline{\nu}$, then the L^2_1-distance from $\gamma(t)$ to the critical point x_0 of f is at most*

$$d_0 \exp(-\delta t) + d_T \exp(-\delta(T - t)),$$

  where d_0, d_T are respectively the L^2_1-distances of $\gamma(0), \gamma(T)$ from x_0.

- *If $a, b \in \widetilde{\mathcal{B}}^*(\tilde{P}_N)$ are in the same component of the preimage of $\overline{\nu}$, then $|f(b) - f(a)| < \epsilon$.*

- *For any harmonic one-form $n \in \Omega^1(C; \mathbf{R})$ which is sufficiently small these results hold for the function f_n replacing f.*

Now we need a related, but slightly more delicate estimate.

Corollary 6.32. *Fix a neighborhood ν as in Lemma 6.31. Then for any $e > 0$ the following holds for any harmonic one-form $n \in \Omega^1(C; \mathbf{R})$ sufficiently small. If $x = (A, \psi) \in \mathcal{C}(\tilde{P}_N)$ is in the complement of the preimage of ν, then*

$$\|\nabla f(x) - \nabla f_n(x)\|_{L^2} \leq e \|\nabla f(x)\|_{L^2}.$$

Proof. According to Lemma 6.10 there is $\lambda > 0$ depending on ν such that for x as in the statement $\|\nabla f(x)\|_{L^2} \geq \lambda$. We simply require n to be sufficiently small such that $\|n\|_{L^2} \leq e\lambda$. Since $\nabla f - \nabla f_n = *in$, the result follows immediately. q.e.d.

Proposition 6.33. *Fix a constant $K > 0$. For any sufficiently small harmonic one-form n on C the following holds. If $\gamma(t)$, $-\infty < t < \infty$, is a solution to the gradient flow equation for f_n with*

$$\lim_{t\mapsto\infty}(f_n(\gamma(t))) - \lim_{t\mapsto-\infty}(f_n(\gamma(t))) \leq K,$$

then the difference element $\delta(\gamma)$ is a multiple of the Poincaré dual of $[C]$.

Proof. Fix $\epsilon > 0$ with $\epsilon << K_1$, the constant given in Lemma 6.31. Construct the neighborhood $\nu \subset \mathcal{B}^*(\tilde{P}_N)$ of the critical point for f for ϵ as in Lemma 6.31. Let $\tilde{\nu}$ be the preimage of ν in $\widetilde{\mathcal{B}}^*(\tilde{P}_N)$. Now fix $0 < e << 1$. Lastly, fix a harmonic one-form n on C, sufficiently small so that the last item in Lemma 6.31 holds for it and such that Corollary 6.32 holds for ν and e and this n.

Now consider an open interval I, a closed subinterval $[A, B] \subset I$, and a flow line $\gamma\colon I \mapsto \widetilde{\mathcal{B}}^*(\tilde{P}_N)$, to the gradient flow equation for f_n.

Claim 6.34. *If $A \leq a < c < b \leq B$, if $\gamma(a)$ and $\gamma(b)$ are in the same component, say $\tilde{\nu}_0$ of $\tilde{\nu}$, and $\gamma(c)$ is in the closure of $\tilde{\nu}$, then $\gamma(c)$ is in the closure of the component $\tilde{\nu}_0$.*

Proof. The function $f_n(\gamma(t))$ is an increasing function of t. According to Lemma 6.31 the fact that $\gamma(a)$ and $\gamma(b)$ are in the closure of the same component of $\tilde{\nu}$ means that $0 \leq f_n(\gamma(b)) - f_n(\gamma(a)) \leq \epsilon$. If $\gamma(c)$ is in the closure of a different component of $\tilde{\nu}$, then the same lemma implies that

$$|f_n(\gamma(c)) - f_n(\gamma(a))| \geq K_1.$$

This is impossible given the first inequality and the fact that $f_n(\gamma(t))$ is a monotone increasing function of t and that $\epsilon << K_1$.

q.e.d.

Now we consider the maximal intervals $b_1, \ldots, b_t$ in $[A, B]$ with the property that the endpoints of each b_i map into the same component of $\partial\tilde{\nu}$. According to the previous lemma, these intervals are disjoint. We number them from left to right. Let $a_0, \ldots, a_t$ be the complementary set of intervals in $[A, B]$, also numbered from left to right. We call the

a_i *spanning intervals* and the b_j *local intervals.* We allow the possibility that a_0 and/or a_t is empty, but when either of these is non-empty we call it an *extremal spanning interval.* The other spanning intervals are called *regular spanning intervals.*

For any subarc $\mu \subset [A, B]$ we define $\delta f(\mu)$ (resp. $\delta f_n(\mu)$) to be the value of f (resp. f_n) at the final point of μ minus the value of f (resp. f_n) and the initial point of μ.

Claim 6.35. *Fix $\epsilon > 0$ and small. Then there exists an open neighborhood ν of the critical point for f such that when n is sufficiently small and for any C^1-path $\gamma\colon I \to \mathcal{C}(\tilde{P}_N)$ which solves the gradient flow equation for the function f_n and any closed subinterval $[A, B] \subset I$, the following hold:*

- $\delta f([A, B]) \geq -\epsilon.$
- *If* $\max(\delta f([A, B]), \delta f_n([A, B])) \geq K_1$, *then*

$$(1 - \epsilon) \leq \frac{\delta f_n([A, B])}{\delta f([A, B])} \leq (1 + \epsilon).$$

Proof. Let K_1 be the constant given in Lemma 6.31, and fix a neighborhood ν so that Lemma 6.31 holds with ϵ replaced by $\epsilon/6$. Fix a positive $e << \epsilon$ and require n to be sufficiently small so that Corollary 6.32 holds for the given neighborhood ν and the given constant e. Then for any spanning interval $a \subset [A, B]$ we have

$$|\delta f(a) - \delta f_n(a)| \leq \frac{e}{1 - e} \delta f_n(a),$$

and hence,

$$(1 - e) \leq \frac{\delta f_n(a)}{\delta f(a)} \leq (1 + 2e). \tag{11}$$

In particular, for any spanning interval $a \subset [A, B]$ we have $\delta f(a) \geq 0$. Furthermore, if a is a regular spanning interval, then we have $\min(\delta f(a), \delta f_n(a)) \geq K_1$. Of course, by the third condition in Lemma 6.31, if $b \subset [A, B]$ is a local interval, then $\delta f(b) \geq -\epsilon/6$. Since the spanning intervals and local intervals alternate and since $\epsilon << K_1$, it follows easily that the only way that $\delta f([A, B])$ can be negative is for there to be no regular spanning interval in $[A, B]$. If this is the case, then there is at most one local interval in $[A, B]$ and $-\epsilon$ is a lower bound for $\delta f([A, B])$.

Now let us consider the second statement. It follows immediately from Inequality (11) that

$$(1-e) \leq \frac{\sum_i \delta f_n(a_i)}{\sum_i \delta f(a_i)} \leq (1+2e).$$

Also, for any local interval b_j we have

$$|\delta f(b_j)|, |\delta f_n(b_j)| \leq \epsilon/6.$$

Using the fact that the a_i and the b_i alternate and that for each regular spanning interval a_i we have $\min(\delta f(a_i), \delta f_n(a_i)) \geq K_1$, it is easy to see, provided that

$$\max(\delta f([A,B]), \delta f_n([A,B])) \geq K_1,$$

we have

$$\frac{\epsilon}{3K_1} \sum_i \delta f(a_i) \geq \sum_j |\delta f(b_j)|$$

and

$$\frac{\epsilon}{3K_1} \sum_i \delta f_n(a_i) \geq \sum_j \delta f_n(b_j) \geq 0.$$

Thus,

$$\frac{\sum_i \delta f_n(a_i)}{(1+(\epsilon/3K_1)) \sum_i \delta f(a_i)} \leq \frac{\delta f_n([A,B])}{\delta f([A,B])} \leq \frac{(1+(\epsilon/3K_1)) \sum_i \delta f_n(a_i)}{(1-(\epsilon/3K_1)) \sum_i \delta f(a_i)},$$

and therefore

$$\frac{1-e}{1+(\epsilon/3K_1)} \leq \frac{\delta f_n([A,B])}{\delta f([A,B])} \leq \frac{1+(\epsilon/3K_1)}{1-(\epsilon/3K_1)}(1+2e).$$

Since $\epsilon << K_1$ and $e << \epsilon$, this yields

$$(1-\epsilon) \leq \frac{\delta f_n([A,B])}{\delta f([A,B])} \leq (1+\epsilon).$$

This completes the proof of the claim. q.e.d.

Now let us return to the proof of the proposition. We let

$$\gamma \colon (-\infty, \infty) \to \mathcal{C}^*(\tilde{P}_N)$$

be the flow line corresponding to a solution of the perturbed Seiberg-Witten equations satisfying the hypothesis of the proposition. It follows from the previous claim that there is a constant K' depending only on K such that

$$-\epsilon \le \lim_{t\mapsto\infty} f(\gamma(t)) - \lim_{t\mapsto-\infty} f(\gamma(t)) \le K'.$$

As shown in Lemma 6.10 there is a positive lower bound to $\|\nabla f(\gamma(t))\|_{L^2}$ for any t for which $\gamma(t)$ is not contained in the preimage of ν. This bound is independent of γ and n. This and the previous inequality yield that there is an a priori bound to the total length of the domains of all the spanning intervals for γ. Also, since the change of f along each spanning interval is at least K_1, this implies that there is an a priori bound to the number of spanning intervals in γ.

Now suppose that we have a sequence of solutions $\gamma^j(t)$ for the gradient flow equations for f_{n^j} where the $n^j \mapsto 0$ as $j \mapsto \infty$. By passing to a subsequence we can suppose that the number of spanning intervals for each of the paths γ^j is constant, say equal to k. We denote the spanning intervals for γ^j by $a_1^j, \ldots, a_k^j$. We can also assume that for each $i \le k$ the lengths ℓ_i^j of a_i^j converge to a finite limit ℓ_i as $j \mapsto \infty$. Let $b_0^j, \ldots, b_k^j$ be the local intervals for γ^j. We also arrange that for each i, the lengths m_i^j of the b_i^j converge as $j \mapsto \infty$. Some of the limiting lengths may be finite and others may be infinite. For each j we form a new set of intervals by taking the components of the subset of $\mathbf{R}$ which is the union of all the $\{a_i^j\}$ with the union of the set of $\{b_t^j\}$ for which $\lim_{i\mapsto\infty} m_t^j < \infty$. In this way for each j we construct a finite set of intervals $s_1^j, \ldots, s_f^j$ in $\mathbf{R}$ whose lengths converge to finite limits as $j \mapsto \infty$ and which contain all the spanning intervals. Let $r_0^j, \ldots, r_f^j$ be the complementary set to the $\{s_i^j\}$. The r_i^j have lengths going to ∞ as $j \mapsto \infty$. (Notice that each r_i^j is a local interval and that r_0^j and r_f^j are semi-infinite intervals but that all the other r_i^j are finite intervals.)

The endpoints of each $\tilde{r}_i^j$ are mapped into the same component of the preimage of $\overline{\nu}$. We denote the difference element in $H^1(N;\mathbf{Z})$ between the component of the preimage of ν containing the limit at $-\infty$ and the component of $\overline{\nu}$ containing the endpoints of $\tilde{r}_i^j$ by δ_i^j. Notice that $\delta_0^j = 0$ for all j and that $\delta_f^j = \delta(\gamma^j)$ for all j.

We pass to a subsequence of the γ^j such that, for each $i \le f$, the geometric limit of $\gamma^j(t)$ centered at a midpoint of s_i^j exists. (This means that for each i, up to gauge equivalence, for every $T > 0$, the following sequence of configurations on $[-T,T] \times N$, indexed by $j \ge 1$, converges: For each j let t_i^j be the midpoint of s_i^j. Translate the restriction of

γ^j to $[-T + t_i^j, t_i^j + T]$ to the left by t_i^j to form a path on $[-T, T]$, or equivalently a configuration (B^j, λ^j) on $[-T, T] \times N$.) Clearly, for each i, this geometric limit is a solution on $\mathbf{R} \times N$ to the Seiberg-Witten equations and hence in a temporal gauge gives a C^∞-path which solves the gradient flow equation for f. Furthermore, the limits at both $\pm\infty$ of this path is the critical point of f. Thus, for all i sufficiently large, the difference element of this solution is equal to the element measuring the difference of the components of the preimage of $\overline{\nu}$ containing the endpoints of $\tilde{s}_i^j$. This element is of course equal to $\delta_i^j - \delta_{i-1}^j$. By Lemma 6.23 we see that this limiting difference element is a multiple of the Poincaré dual of $[C]$. It then follows for all j sufficiently large that we have $\delta_i^j - \delta_{i-1}^j$ is a multiple of the Poincaré dual of $[C]$. Hence, by induction on i, for all j sufficiently large, we see that the δ_i^j is a multiple of the Poincaré dual of $[C]$ for all i. In particular for all j sufficiently large we have $\delta_f^j = \delta(\gamma^j)$ is a multiple of the Poincaré dual of $[C]$.

This completes the proof of the proposition.

7. Boundedness of the gradient flow line as we stretch out the neck

7.1. The case of the unperturbed equation

Let us formulate the context precisely. Let N be a closed oriented, riemannian three-manifold. Let M be a smooth four-manifold and suppose that $N \subset M$ is a smoothly embedded three-manifold dividing M into two pieces Y and X. We fix an orientation on M and we take the orientation on N induced by requiring that the orientation on N preceded by the unit normal vector to N pointing into Y gives the orientation of M. Let ν be a product neighborhood $[-1, 1] \times N$. We consider a family of metrics $\{g_s\}$ on M, parametrized by s in the interval $[1, \infty)$ which are all the same on $M - \nu$ but stretch out the product neighborhood of N. That is to say, we suppose that $g_s|_\nu = \lambda_s(t)^2 dt^2 + d\theta^2 + d\sigma^2$ where dt^2 is the usual metric on $[-1, 1]$, $d\theta^2$ is the usual metric on length 2π on S^1, $d\sigma^2$ is a fixed (say, constant curvature) metric on C, and $\lambda_s(t)$ is a positive smooth function on $[-1, 1]$ which is identically equal to one on $[-1, -1/2] \cup [1/2, 1]$ and satisfies

$$\int_{-1/2}^{1/2} \lambda_s(t) dt = s.$$

We denote by M_s the riemannian manifold (M, g_s), by N_- and N_+ the submanifolds $\{-1/2\} \times N, \{+1/2\} \times N$ inside M_s, and by $T_s \subset M_s$ the cylinder that they bound.

Fix a $Spin^c$-structure $\tilde{P}$ on M. Let $\tilde{P}_N$ be the restriction of $\tilde{P}$ to N. If (A, ψ) is a solution to the Seiberg-Witten equations for $\tilde{P}$ over M_s, then the restriction $(A, \psi)|_T$ gives a C^∞-flow line for the gradient flow equation

$$\gamma'(t) = \nabla f(\gamma(t))$$

in the space of configurations $\mathcal{C}^*(\tilde{P}_N)$ defined on an interval of length s. We have an isometry from $[0, s] \times N$ with T_s. Let N_+, N_- be the boundary components $\{s\} \times N, \{0\} \times N$, and denote by S the difference $M_s - T_s$. This is a riemannian manifold with boundary which is independent of s. The next two lemmas are of crucial importance.

Lemma 7.1. *There is a constant E depending only on M such that for any $s \geq 1$ and any solution (A, ψ) to the Seiberg-Witten equations for $\tilde{P}_s$ over M_s and any $x \in M_s$ we have*

$$|\psi(x)| \leq E.$$

Proof. Since there is a uniform bound to the scalar curvatures of the M_s for all $s \geq 1$, the result is an easy application of the maximum principle; cf. [4]. q.e.d.

Lemma 7.2. *There is a constant $K > 0$ depending only on M and the isomorphism class of the $Spin^c$-structure $\tilde{P}$ such that the following holds for any $s \geq 1$. If (A, ψ) is a solution to the Seiberg-Witten equations for $\tilde{P}_s$ over M_s, and $\gamma\colon [0, s] \to \mathcal{C}(\tilde{P}_N)$ is the gradient flow line associated to the solution, then*

$$0 \leq f(\gamma(s)) - f(\gamma(0)) \leq K.$$

Proof. According to the previous lemma there is a uniform pointwise bound for $|\psi|$ and hence for $|F_A^+|$ independent of s and the solution (A, ψ). Of course, we have

$$\int_{M_s} iF_A \wedge iF_A = \frac{1}{4\pi^2}\langle c_1(\mathcal{L})^2, [M]\rangle.$$

This means that

(12)
$$\|F_A^+\|^2_{L^2(S)} - \|F_A^-\|^2_{L^2(S)} + \int_{T_s} iF_A \wedge iF_A = \frac{1}{4\pi^2}\langle c_1(\mathcal{L})^2, [M]\rangle.$$

The right-hand-side of this equation is determined by $\tilde{P}$. Of course, the pointwise bound on F_A^+ implies that there is a constant $K' > 0$ depending only on S and $\tilde{P}|_S$ such that

$$\|F_A^+\|_{L^2(S)}^2 \leq K. \tag{13}$$

Claim 7.3. *There is a constant* $K'' > 0$ *depending only on* M *and* $\tilde{P}$ *such that*

$$\int_{T_s} F_A \wedge F_A \geq -K''.$$

Proof. By Stokes' theorem we have

$$\begin{aligned} f(\gamma(s))-f(\gamma(0)) &= \int_{T_s} F_A \wedge F_A + \int_{N_+} \langle \psi, \partial\!\!\!/_A(\psi)\rangle \\ &\quad - \int_{N_-} \langle \psi, \partial\!\!\!/_A(\psi)\rangle. \end{aligned} \tag{14}$$

Since γ is a flow line from the gradient flow equation

$$\gamma'(t) = \nabla f(\gamma(t)),$$

we see that

$$f(\gamma(s)) - f(\gamma(0)) \geq 0.$$

Thus, to complete the proof, we need only show that there is a bound depending only on M and $\tilde{P}$ to both

$$\int_{N_+} \langle \psi, \partial\!\!\!/_A(\psi)\rangle$$

and

$$\int_{N_-} \langle \psi, \partial\!\!\!/_A(\psi)\rangle.$$

But these bounds are immediate from the *a priori* pointwise bound on $|\psi|$ and the L^2-bound on $\nabla\psi$. q.e.d.

As an immediate consequence of this claim we have that

$$\int_{T_s} iF_A \wedge iF_A \leq K''. \tag{15}$$

It follows from Equation (12) and Inequalities (13) and (15) that $\|F^-\|^2_{L^2(S)}$ is bounded above by a constant depending only on M and $\tilde{P}$. But once we know that both $\|F^{\pm}\|^2_{L^2}$ are bounded by a constant depending only on M and $\tilde{P}$, the same is true for

$$\left|\int_{T_s} iF_A \wedge iF_A\right|.$$

Invoking Stokes' theorem once again we see that $f(\gamma(s)) - f(\gamma(0))$ is bounded by a constant depending only on M and $\tilde{P}$. q.e.d.

7.2. The case of the perturbed equation when $N = S^1 \times C$ and $\mathcal{L}$ has degree $\pm(2-2g)$ on C

Now let us suppose that $N = S^1 \times C$, with C being a riemann surface of genus $g > 1$. Suppose that $\tilde{P} \to M$ is a $Spin^c$-structure whose determinant line bundle $\mathcal{L}$ has degree $2-2g$ on C. Let $M_s = (M, g_s)$ be the family of riemannian manifolds discussed in the last section. Let $\varphi_s\colon M_s \to [0,1]$ be a C^∞ function which is identically 1 on T_s and whose support is contained in $[-1,1] \times N \subset M$. We choose the φ_s so that they are all the same on $M_s - T_s$ under the obvious identification of these spaces. As before, we let N_- and N_+ be the copies of N which make up ∂T_s. Fix a real harmonic one-form n on C. For each s let h_s be the self-dual two-form $\varphi_s(*n + dt \wedge n)$ where $*$ is the Hodge $*$-operator for N. We consider the perturbed Seiberg-Witten equations (SW_{h_s}) :

$$F_A^+ = q(\psi) + ih_s,$$
$$\partial\!\!\!/_A(\psi) = 0.$$

The restriction to the tube T_s of a solution to the perturbed Seiberg-Witten equations SW_{h_s} gives a C^∞-path γ which solves the gradient flow equation:

$$\gamma'(t) = \nabla f_n(\gamma(t))$$

defined on an interval of length s. We denote by $t_\pm$ the values of the parameter corresponding to $N_\pm$.

Lemma 7.4. *There are constants E, K_2 depending only on M and $\tilde{P}$ such that for any $s \geq 1$ and any sufficiently small harmonic one-form $n \in \Omega^1(C;\mathbf{R})$ the following hold. If (A,ψ) is a solution to SW_{h_s}, and $\gamma\colon [0,s] \to \mathcal{C}(\tilde{P}_N)$ is the gradient flow line for f_n associated to this solution, then*

$$0 \leq f_n(\gamma(s)) - f_n(\gamma(0)) \leq K_2,$$

and for every $x \in M_s$

$$|\psi(x)| \leq E.$$

Proof. The pointwise bound E for the spinor field ψ follows just as in the unperturbed case. Also, Claim 6.35 shows that if n is sufficiently small, then there is a universal lower bound $(-\epsilon)$ to $f(\gamma(s)) - f(\gamma(0))$. Consequently, the arguments given in the unperturbed case show that there is a constant K_0 depending only on M and $\tilde{P}$ such that for any $s \geq 1$ and any solution (A, ψ) to the perturbed equations SW_{h_s} we have

$$\left| \int_{T_s} F_A \wedge F_A \right| \leq K_0.$$

It follows that for the functional f associated with the unperturbed equations we have

$$|f(\gamma(s)) - f(\gamma(0))| \leq K_1$$

for some constant C_1 depending only on M and $\tilde{P}$.

By Claim 6.35, provided that n is sufficiently small, we see that

$$f_n(\gamma(s)) - f_n(\gamma(0))$$

is bounded from above by a constant K_2 depending only on M and $\tilde{P}$. It is positive since $\gamma(t)$ is a gradient flow line for f_n. q.e.d.

Corollary 7.5. *There is a constant $K > 0$ depending only on M and $\tilde{P}$ such that for any harmonic one-form $n \neq 0$ in $\Omega^1(C; \mathbf{R})$ sufficiently small and for any $s \geq 1$ and any solution (A, ψ) to the perturbed Seiberg-Witten equations SW_{h_s} on M_s, the restriction of (A, ψ) satisfies the following. For any $t \in [0, s]$, we have that the L^2_1 distance from $(A(t), \psi(t))$ to a static solution is at most*

$$K\exp(-\delta d(t)),$$

where δ is the constant in Lemma 6.15 and where $d(t) = \min(t, s - t)$.

Proof. This is immediate from Lemma 7.4 and Proposition 6.30 and standard limit arguments.

8. Definition of the moduli spaces for cylindrical-end manifolds

The gluing theorem will describe all solutions to the monopole equations on M whose determinant line bundle has degree $\pm(2-2g)$ on C in terms of solutions on the two sides. First, we need to define and study the moduli spaces of solutions to the monopole equations on non-compact 4-manifolds with ends isometric to $[0,\infty) \times N$. Of course, the same equations make sense over a non-compact four-manifold. We consider only solutions to the equations which are of finite energy on the cylindrical end, as we make precise below.

Now we are ready to define the moduli space for a cylindrical end four-manifold. For the moment fix an arbitrary, closed, oriented, riemannian 3-manifold N and an complete riemannian 4-manifold X whose end is orientation-preserving isometric to $[-1,\infty) \times N$.

Let $f\colon \mathcal{C}^*(\tilde{P}_N) \to \mathbf{R}$ be the function introduced in Subsection 6.2.

Definition 8.1. Fix a $Spin^c$-structure $\tilde{P}$ on X whose restriction to N is denoted by $\tilde{P}_N$. For any C^∞ solution (A,ψ) to the Seiberg-Witten equations with respect to this $Spin^c$-structure there is a temporal gauge for $\tilde{P}$ restricted to the cylindrical end so that the flow line $\gamma\colon [0,\infty) \to \mathcal{C}^*(\tilde{P}_N)$ determined by the solution satisfies the gradient flow equation. Such a temporal gauge is unique up to an automorphism of $\tilde{P}_N$. A finite energy solution to the Seiberg-Witten equations is a C^∞-solution for which an associated flow line $\gamma\colon [0,\infty) \to \mathcal{C}^*(\tilde{P}_N)$ satisfies

$$\lim_{t\mapsto\infty} f(\gamma(t)) - f(\gamma(0)) < \infty.$$

(Notice that this condition is independent of the choice of temporal gauge.)

Actually, we are mainly interested here in the case where $N = S^1 \times C$. In this case it will be convenient to work with solutions to a perturbed equation with the same finite energy condition.

Definition 8.2. Let $N = S^1 \times C$, and let $n \in \Omega^1(C;\mathbf{R})$ be a harmonic form. Fix a C^∞ function $\varphi\colon [-1,\infty) \to [0,1]$ which is identically zero near -1 and identically 1 on $[0,\infty)$. We can view φ as a function from X to $\mathbf{R}$ by defining it on the end by projecting onto to $[-1,\infty)$ factor and extending it to be identically zero on the rest of X. Consider the modified Seiberg-Witten equations for a pair (A,ψ)

$$F_A^+ = q(\psi) + i\varphi(*n + dt \wedge n), \tag{16}$$
$$\partial\!\!\!/_A(\psi) = 0.$$

Here $*n$ represents the dual of n in the three-manifold $N = S^1 \times C$.

Analogously to what we did in the unperturbed case, we consider only C^∞-solutions to these equations which satisfy

$$\lim_{t\mapsto\infty} f_n(\gamma(t)) - f_n(\gamma(0)) < \infty,$$

where f_n is the function introduced in Subsection 6.6, and $\gamma\colon [0,\infty) \to \mathcal{C}^*(\tilde{P}_N)$ is the gradient flow line for f_n determined by the restriction of the solution to the cylindrical end of X in a temporal gauge. As before, we call such solutions finite energy solutions to the perturbed equations.

The first result to establish is that any finite energy solution to the Seiberg-Witten equations or the perturbed Seiberg-Witten equations in fact has exponential decay to a static solution in an appropriate gauge.

The following is an immediate consequence of Lemma 6.15, Claim 6.24 and the argument given in the proof of Corollary 6.18.

Theorem 8.3. *Let $N = S^1 \times C$, and let X be a complete riemannian manifold with cylindrical-end isomorphic to $[-1,\infty) \times N$. Let $\tilde{P}$ be a Spin^c-structure whose restriction to N is isomorphic to the pullback from C of a Spin^c-structure whose determinant line bundle has degree $\pm(2-2g)$. Then the following holds for any sufficiently small harmonic one-form $n \in \Omega^1(C;\mathbf{R})$. Let (A,ψ) be an finite energy solution to the perturbed Seiberg-Witten equations (16) associated to $\tilde{P}$. Then there is a C^∞-product structure for $\tilde{P}|_{[0,\infty)\times N}$ such that in this product structure (A,ψ) converges exponentially fast to a static solution. The exponent of the decay δ depends only on the riemannian metric on N. Furthermore, there is a constant E_1 depending only on X such that if (A,ψ) is a finite energy solution to the above equations, then $|\psi(x)| \leq E_1$ for all $x \in X$.*

Remark 8.4. Notice that since in an appropriate gauge A decays exponentially fast to a static solution and for the static solution B we have $F_B \wedge F_B = 0$, we see that $\int_X F_A \wedge F_A$ is finite. We call

$$\frac{-1}{4\pi^2}\int_X F_A \wedge F_A$$

the *Chern integral* of the solution and denote it by $c(A,\psi)$.

Let $\tilde{\mathcal{M}}(\tilde{P})$ be the set of all finite energy solutions to the Seiberg-Witten equations SW. We give $\tilde{\mathcal{M}}(\tilde{P})$ a topology as follows. Let (A,ψ) be a finite energy solution. Then a basis for the open neighborhoods of (A,ψ) are determined by choosing $T \in [0,\infty)$, $\epsilon > 0$, and $k \in \mathbf{Z}^+$ such that, letting γ be the flow line associated to (A, ψ) in a temporal gauge, we have

$$\lim_{t\mapsto\infty} f(\gamma(t)) - f(\gamma(T)) < \epsilon.$$

We then define $U(T, \epsilon)$ to be the subset of all finite energy solutions (A', ψ') to the equations such that

$$\mathrm{dist}_{C^k(X_T)}\left((A, \psi)|_{X_T}, (A', \psi')_{X_T}\right) < \epsilon$$

and

$$\lim_{t\mapsto\infty} f(\gamma'(t)) - f(\gamma'(T)) < \epsilon,$$

where X_T is the complement of the cylinder $(T, \infty) \times N$ in X, and γ' is the flow line associated to (A', ψ') in a temporal gauge. The group of gauge transformations $\mathcal{G}(\tilde{P})$ is simply the group of all C^∞-changes of gauge. It clearly acts continuously on $\tilde{\mathcal{M}}(\tilde{P})$. We denote the quotient by $\mathcal{M}(\tilde{P})$. As with the case of $SU(2)$-ASD connection, the moduli space $\mathcal{M}(\tilde{P})$ is given by the zeros of a map with Fredholm differential modulo the action of the group of changes of gauge (c.f., [10], [6]). The index of the Fredholm complex is

$$\text{(17)} \qquad \frac{1}{4}\left(\frac{-1}{4\pi^2}\int_X F_A \wedge F_A - 2\chi(X) - 3\sigma(X)\right) = \frac{1}{4}[c(A, \psi) - 2\chi(X) - 3\sigma(X)].$$

For a generic compactly supported, real, self-dual two-form η^+ the perturbed Seiberg-Witten equations, where the curvature equation is replaced by

$$F_A^+ = q(\psi) + i\eta^+,$$

determine a moduli space $\mathcal{M}(\tilde{P}, \eta^+)$ which is a smooth manifold whose dimension at any point (A, ψ) is given by Equation (17).

It is clear from the definition of the topology that the Chern integral is a continuous function on $\mathcal{M}(\tilde{P})$ and $\mathcal{M}(\tilde{P}, \eta^+)$. On the other hand, the values taken by the Chern integral form a discrete set. Hence, the Chern integral gives a locally constant function on $\mathcal{M}(\tilde{P})$. We denote by $\mathcal{M}_c(\tilde{P})$ the union of components where the value of the Chern integral is c. Similarly, we define $\mathcal{M}_c(\tilde{P}, \eta^+)$.

The same topology and group action in the case of the perturbed equation leads to a moduli space $\mathcal{M}(\tilde{P}, n)$ and the subspaces $\mathcal{M}_c(\tilde{P}, n)$ of a given Chern integral. As before, these subspaces are each a union of components. The same formula gives the formal dimension of this moduli space for the perturbed equations. As before, a further perturbation of the equations leads to:

$$F_A^+ = q(\psi) + i\varphi(*n + dt \wedge n) + i\eta^+,$$

$$\not{\partial}_A(\psi) = 0,$$

for a generic compactly supported, self-dual real two-form η leads to moduli spaces $\mathcal{M}_c(\tilde{P}, n, \eta^+)$ for all c which are smooth and of the correct dimension.

Notice that for a generic compactly supported purely imaginary self-dual two-form $i\eta^+$ the moduli space $\mathcal{M}_c(\tilde{P}, n, \eta)$ is empty if $c < 2\chi(X) + 3\sigma(X)$.

Compactness results. Here is the basic compactness result in our context.

Proposition 8.5. *Let X be a complete riemannian four-manifold with cylindrical end isometric to $[-1, \infty) \times N$, with $N = S^1 \times C$ for a curve C of genus $g > 1$. Let $\tilde{P}$ be a Spin^c-structure on X with determinant line bundle $\mathcal{L}$ whose restriction to the end is isomorphic to the pullback from C of a bundle of degree $\pm(2-2g)$. Fix c_0. Then for all sufficiently small harmonic one-form $n \neq 0$ on C the following holds for every $c \leq c_0$: Let $\mathcal{M}_c(\tilde{P}, n)$ be the moduli space of gauge equivalence classes of finite energy solutions to the perturbed Seiberg-Witten equations SW_h (16), which also satisfy*

$$\frac{-1}{4\pi}\int_X F_A \wedge F_A = c. \tag{18}$$

(We take $h = \varphi(*n + dt \wedge n)$.) Then this space is compact. A further perturbation of the first equation by adding a generic compactly supported self-dual, purely imaginary, two-form $i\eta^+$ on X to the right-hand-side leads to a compact moduli space $\mathcal{M}(\tilde{P}, n, \eta^+)$ which is smooth of the correct dimension at each solution.

Proof. The usual maximum principle arguments as in the compact case [2] show that for any solution (A, ψ) to the perturbed Seiberg-Witten equations SW_h there is a bound to the pointwise norm of ψ and F_A^+. These bounds are independent of n and η^+, provided only that these forms are sufficiently small. Let us consider the gradient flow path $\gamma\colon [0, \infty) \to \mathcal{C}(\tilde{P}_N)$ for f_n associated to a solution of the perturbed equations. Arguing as in Section 7 we see that this implies that there is an upper bound $K(X, \tilde{P}, c_0)$ to

$$\lim_{t \mapsto \infty} f(\gamma(t)) - f(\gamma(0)),$$

which depends only on X and $\tilde{P}$ and the upper bound c_0 for the Chern integral.

Now we require that $n \neq 0$ to be sufficiently small so that Proposition 6.30 holds for $\dot{n}$ with the constant K of that proposition being $K(X, \tilde{P}, c_0)$.

Now suppose that we have a sequence $\{(A_i, \psi_i)\}$ of solutions to SW_h of Chern integral $c \leq c_0$. Using the bounds described in the previous paragraph and a standard diagonalization argument, after passing to a subsequence we can suppose that there is a configuration (A, ψ) such that, up to gauge, the sequence converges C^∞ on each compact subset of X to (A, ψ). Of course, (A, ψ) is a solution to the perturbed Seiberg-Witten equations SW_h. It will be the limit of the sequence in the topology of the moduli space if and only if its Chern integral is equal to c. If its Chern integral is not c, then this means that there is a sequence of $T_i \mapsto \infty$ and $\epsilon > 0$ such that

$$\left| \int_{T_i}^{\infty} F_A \wedge F_A \right| \geq \epsilon.$$

Because of the exponential decay result in fact this means that there is a sequence $T_i \mapsto \infty$ and $\epsilon > 0$ such that

$$\left| \int_{T_i}^{T_i+1} F_A \wedge F_A \right| \geq \epsilon.$$

Now we take a geometric limit of the configurations on $[-T_i, T_i] \times N$ given by translating the solution $(A_i, \psi_i)|_{[0,2T_i] \times N}$ by $-T_i$ in the first factor. The result is a solution to the perturbed Seiberg-Witten equations on $\mathbf{R} \times N$. The curvature equation on $\mathbf{R} \times N$ is

$$F_A^+ = q(\psi) + i(*n + dt \wedge n).$$

Clearly, this limiting solution satisfies:

$$\lim_{t \mapsto \infty} f(\gamma(t)) - \lim_{t \mapsto -\infty} f(\gamma(t)) \leq K(X, \tilde{P}, c_0),$$

$$\left| \int_0^1 F_A \wedge F_A \right| \geq \epsilon.$$

This contradicts Proposition 6.30 which says that all solutions to the perturbed equations satisfying the first condition must be static.

The last statement about the compactly supported perturbation to achieve smoothness goes just as in the compact case. q.e.d.

This compactness result does not hold for $\mathcal{M}_c(\tilde{P})$ and $\mathcal{M}_c(\tilde{P}, \eta^+)$. The reason is that there are non-static finite solutions to the Seiberg-Witten equations on the cylinder $\mathbf{R} \times N$. This is in fact the reason that we were led to consider the perturbation of the equations by adding a term in the cylindrical end.

As an immediate corollary of the compactness result we have the following uniformity result.

Corollary 8.6. *Let $N = S^1 \times C$, and let X be a complete riemannian manifold with cylindrical-end isomorphic to $[-1, \infty) \times N$. Let $\tilde{P}$ be a Spin^c-structure whose restriction to N is isomorphic to the pull-back from C of a Spin^c-structure whose determinant line bundle has degree $\pm(2-2g)$. Then for any c_0 the following holds for any sufficiently small harmonic one-form $n \neq 0 \in \Omega^1(C; \mathbf{R})$ and every $c \leq c_0$: There is a constant $T \geq 1$ such that if (A, ψ) is a finite energy solution to the equations SW_h with Chern integral c, then for every $t \geq T$ the restriction $(A(t), \psi(t))$ is within $\exp(-z(t-T))$ in the L^2_1-topology of a solution to the equations SW_{*n} on N, where the constant z depends only on N. The same result holds when the curvature equation is replaced by*

$$F_A^+ = q(\psi) + ih + i\eta^+$$

for any sufficiently small, compactly supported, self-dual two-form η^+ on X.

9. The Gluing Theorem

Now we come to the gluing theorem in our context. Let M be a closed oriented four-manifold, and let $N = S^1 \times C$. Suppose that $N \subset M$ is a smooth embedding with $M - N = X \coprod Y$. Fix complete metrics on X and Y which have cylindrical ends with orientation-preserving isometries to $[-1, \infty) \times N$. For any $s \geq 1$ we denote by X_s and Y_s the compact manifolds with boundary obtained by truncating X and Y at $\{s\} \times N$. For any $s \geq 1$ let M_s be the closed riemannian four-manifold obtained by identifying X_s and Y_s along their boundaries by the identification which is the identity on C and is complex conjugation on S^1. Fix $Spin^c$-structures $\tilde{P}_X$ and $\tilde{P}_Y$ whose determinant line bundles restricted to N are both isomorphic to the pullback from C of a line bundle of degree $(2-2g)$ on C. Also fix an integer e. Let $c_0(X) = e - 2\chi(Y) - 3\sigma(Y)$ and let $c_0(Y) = e - 2\chi(X) - 3\sigma(X)$. We choose small perturbations of the monopole equations for X and Y so that the equations on X are $SW_{h_X+\eta_X^+}$:

$$F_A^+ = q(\psi) + i\varphi_X(*n + dt \wedge n) + i\eta_X^+,$$
$$\partial\!\!\!/_A(\psi) = 0,$$

where n is a harmonic one-form on C, $h_X = *n + dt \wedge n$, η_X^+ is a compactly supported self-dual two-form, and φ_X is a C^∞ function which is identically 1 on $[0,\infty) \times N$ and vanishes off of $[-1,\infty) \times N$. For any $c(X)$ let $\mathcal{M}_{c(X)}(\tilde{P}_X, n, \eta_X^+)$ be the moduli space of finite energy solutions to the perturbed equations with Chern integral $c(X)$. In a completely analogous fashion we fix φ_Y and define the moduli space $\mathcal{M}_{c(Y)}(\tilde{P}_Y, n, \eta_Y^+)$ of finite energy solutions to $SW_{h_Y+\eta_Y^+}$ with Chern integral $c(Y)$. By choosing the η_X^+ and η_Y^+ generically, we arrange that these are smooth, compact moduli spaces for all $c(X) \le c_0(X)$ and $c(Y) \le c_0(Y)$.

Let $\mathcal{S}$ be the set of isomorphism classes of $Spin^c$ structures $\tilde{P}$ on M with the property that $\tilde{P}|_X \cong \tilde{P}_X$ and $\tilde{P}|_Y \cong \tilde{P}_Y$. We denote by $\mathcal{S}_e$ the subset of those $Spin^c$-structures in $\mathcal{S}$ whose determinant line bundle $\mathcal{L}$ satisfies $c_1(\mathcal{L})^2 = e$. Let $\tilde{P}$ represent an element of $\mathcal{S}_e$. For each $s \ge 0$ we have the corresponding $Spin^c$-structure $\tilde{P}_s$ over M_s. For any s sufficiently large we denote by η^+ the self-dual form on M_s which is $\eta_X^+ + \eta_Y^+$. (Notice that if s is sufficiently large the support of η_X^+ and η_Y^+ are contained in M_s and are disjoint in M_s.) We define $\mathcal{M}(\tilde{P}_s, h_s, \eta^+)$ the moduli space of solutions to the perturbed Seiberg-Witten equations $SW_{h_s+\eta^+}$:

$$F_A^+ = q(\psi) + i\varphi_s(*n + dt \wedge n) + i\eta^+,$$
$$\partial\!\!\!/_A(\psi) = 0$$

for $\tilde{P}_s$. Here, $\varphi_s \colon M_s \to [0,1]$ is the function which agrees with φ_X on $X_s \subset M_s$, and with φ_Y on $Y_s \subset M_s$, and $h_s = \varphi_s(*n + dt \wedge n)$.

Theorem 9.1. The Gluing Theorem. *With the notation and assumptions above, suppose that n is sufficiently small and generic, and that η_X^+ and η_Y^+ are generic. Then for all s sufficiently large and for each $\tilde{P} \in \mathcal{S}_e$, the moduli space $\mathcal{M}(\tilde{P}_s, n, \eta^+)$ is a smooth manifold of* *the correct dimension. Furthermore, gluing together configurations on X and Y and deforming slightly so as to solve the perturbed Seiberg-Witten equations determine a diffeomorphism*

$$\coprod_{c_1+c_2=e} \mathcal{M}_{c_1}(\tilde{P}_X, n, \eta_X^+) \times \mathcal{M}_{c_2}(\tilde{P}_Y, n, \eta_Y^+) \xrightarrow{\cong} \coprod_{\tilde{P}\in\mathcal{S}_e} \mathcal{M}(\tilde{P}_s, n, \eta^+).$$

In addition, there is a constant $T > 0$ depending only on X and Y, and for given an integer $k > 0$, there are constants $K, \delta > 0$ such that the following holds for all s sufficiently large: If $[A, \psi] \in \mathcal{M}(\tilde{P}_s, n, \eta^+)$ corresponds to the pair

$$([A_X, \psi_X], [A_Y, \psi_Y]) \in \mathcal{M}_{c_1}(\tilde{P}_X, n, \eta_X^+) \times \mathcal{M}_{c_2}(\tilde{P}_Y, n, \eta_Y^+)$$

under the gluing isomorphism, then up to change of gauge, the restrictions to X_{s-1} of $[A, \psi]$ and $[A_X, \psi_X]$ are within a distance $Ke^{-\delta s}$ in the C^∞-topology on X_{s-1}. There is the analogous statement for the restrictions to Y_{s-1}. Lastly, with appropriate conventions on orienting the moduli spaces for X and Y and for M, then the diffeomorphism acts by $(-1)^{b^1(X,N)b^2_{\geq 0}(Y,N)}$ on the orientations, where, $b^2_{\geq 0}(Y, N)$ is the dimension of any maximal positive semi-definite subspace for the intersection pairing on $H^2(Y, N; \mathbf{R})$.

Notice that $\mathcal{M}_{c_1}(\tilde{P}_X, n, \eta^+)$ is empty if $c_1 < 2\chi(X) + 3\sigma(X)$, and similarly for c_2. It then follows that $c_1 \leq c(X)$ and $c_2 \leq c(Y)$, so that the disjoint union of the products of moduli spaces for X and Y is finite and that the compactness results of the previous section hold for all the moduli spaces of X and Y that appear in this union of products.

The fact that the gluing map induces diffeomorphism follows by the, by now standard, gluing arguments and limiting arguments (see [2], [11]) from the following facts:

- The moduli space of solutions over the three-manifold consists of a single point, that point being smooth (Corollary 5.3.).
- The fact that the moduli spaces of the cylindrical-end four-manifolds which appear in the disjoint union of products are smooth and compact (Proposition 8.5) and have the uniform decay as described in Corollary 8.6.
- The decay result in the center of the tube T_s for solutions over the manifold M_s, results which are uniform in s (Corollary 7.5.).

The next paragraph discusses the orientations and proves the statement concerning orientations.

9.1. Orientations

Let us compute the elliptic complex associated with the Seiberg-Witten monopole equations on a four-manifold Z with cylindrical end. As we have seen, we can work in the context where the connections and changes of gauge are exponentially decaying with a given exponent of decay $\delta > 0$. Thus, the tangent space at the identity to the group of changes of gauge is $\Omega^0_\delta(Z; i\mathbf{R})$, the tangent space to the space of connections at any point is $\Omega^1_\delta(Z; i\mathbf{R})$, and tangent space to the affine space where the curvature lies is $\Omega^2_{+,\delta}(Z; i\mathbf{R})$. Similarly, we can view the sections of $S^+(\tilde{P}_Z)$ as differing from a fixed section which is in the kernel of the Dirac operator at least in the cylindrical end by a δ-decaying section. Thus, the elliptic complex associated to a solution (A, ψ) of the Seiberg-Witten equations on a cylindrical-end manifold is

$$0 \longrightarrow \Omega^0_\delta(Z; i\mathbf{R}) \xrightarrow{2d+m_\psi} \Omega^1_\delta(Z; i\mathbf{R}) \oplus C^\infty_\delta(S^+(\tilde{P}_Z))$$
$$\longrightarrow \Omega^2_{+,\delta}(Z; i\mathbf{R}) \oplus C^\infty(S^-(\tilde{P}_Z)) \longrightarrow 0$$

where $m_\psi(f)$ is the section $-if\psi$ of $S^+(\tilde{P}_Z)$, and the last map is given by the matrix

$$\begin{pmatrix} d^+ & \mathrm{Re}(<\psi, \cdot>) \\ ic_\psi & \not{\partial}_A \end{pmatrix}.$$

Here $<,>$ is the bilinear form associated to the quadratic form q, and c_ψ is Clifford multiplication against ψ.

It follows easily that orienting the determinant line bundle of this operator is equivalent to orienting the vector space

$$H^1_\delta(Z; \mathbf{R}) \oplus H^2_{+,\delta}(Z; \mathbf{R}) \oplus H^0_\delta(Z; \mathbf{R}).$$

An easy computation shows that $H^0_\delta(Z; \mathbf{R}) = 0$, that $H^1_\delta(Z; \mathbf{R}) = H^1(Z, T; \mathbf{R})$ and that $H^2_{+,\delta}(Z; \mathbf{R}) = H^2_{\geq 0}(Z, T; \mathbf{R})$ where T is a cylindrical neighborhood of infinity in Z, and $H^2_{\geq 0}(Z, T)$ is the maximal subspace of exponentially decaying harmonic forms of which the intersection pairing is positive semi-definite.

Corollary 9.2. *To orient the moduli space of finite energy solutions to Seiberg-Witten equations on a cylindrical-end manifold Z, it suffices to orient $H^1(Z, T; \mathbf{R}) \oplus H^2_{\geq 0}(Z, T; \mathbf{R})$ where T is a cylindrical neighborhood of infinity in Z.*

Now let us see how the orientations compare when we glue. Let $M - N = X \coprod Y$. We can ignore the spinor fields since the spaces of these fields are complex linear vector spaces and hence have canonical orientations. It becomes a question of comparing the sum of the δ complexes of forms for X and Y with the usual complex of forms for M. We can replace the δ-complex for X and Y with the complexes of forms which vanish in cylindrical ends $T_\pm$ without changing the cohomology. Let $T \subset M$ be the image of the union of T_+ and T_- glued up manifold M. We have an exact sequence of operators:

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
\Omega^1(M,T) & \xrightarrow{d^+ + d^*} & \Omega^2_+(M,T) \oplus \Omega^0(M,T) \\
\downarrow & & \downarrow \\
\Omega^1(M) & \xrightarrow{d^+ + d^*} & \Omega^2_+(M) \oplus \Omega^0(M) \\
\downarrow & & \downarrow \\
\Omega^1(T,\partial T) & \xrightarrow{d^+ + d^*} & \Omega^2_+(T,\partial T) \oplus \Omega^0(T,\partial T) \\
\downarrow & & \downarrow \\
0 & & 0.
\end{array}$$

Let us denote these operators by $D_{M,T}, D_M, D_{T,\partial T}$ respectively. Clearly, in light of the above exact sequence, there is a natural isomorphism $\det D_M \equiv \det D_{M,T} \otimes \det D_{T,\partial T}$. Of course, $H^i(T,\partial T) = H^{i-1}(N) \otimes H^1(I,\partial I)$. Thus, $H^0(T,\partial T) = 0$ and $H^1(T,\partial T) = H^1(I,\partial I)$. Orienting I so that it points toward the Y-side orients this last group. It also gives an orientation for N. Since C is oriented, it follows that the circle direction receives an orientation. The orientation on C induces one on $H^1(C)$; that together with the orientation on the circle gives an orientation to $H^1(N)$, and hence to $H^2_{\geq 0}(T,\partial T) = H^1(N)$. Thus, we see that with these conventions, there is a natural isomorphism $\det D_{M,T} \equiv \det D_M$.

Lastly, we need to compare the orientation of $\det D_{M,T}$ with the tensor product $\det D_{Y,T_+} \otimes \det D_{X,T_-}$. The comparison of these determinants involves switching the order of $H^1(X,T_-)$ and $H^2_{\geq 0}(Y,T_+)$ and hence introduces a sign which is $(-1)^{b^1(X,T_-) b^2_{\geq 0}(Y,T_+)}$.

9.2. Relative invariants

Definition 9.3. Let X be an oriented, complete, riemannian four-manifold with a cylindrical end T isometric to $[0,\infty)\times S^1\times C$ where C is a riemann surface of genus $g>1$. Let $\tilde{P}_X$ be a $Spin^c$-structure whose restriction to the end of X is isomorphic to the pullback of a $Spin^c$-structure on C with determinant line bundle of degree $\pm(2-2g)$. We choose a sufficiently small generic harmonic one-form $n\in\Omega^1(C;\mathbf{R})$ and a generic compactly supported self-dual two-form η^+. We then form the moduli space $\mathcal{M}_c(\tilde{P}_X,n,\eta^+)$ where the Chern integral is equal to c. As we have seen this is a compact, smooth moduli space. We choose an orientation for $H^1(X,T;\mathbf{R})\oplus H^2_{\geq 0}(X,T;\mathbf{R})$. This determines an orientation for the above moduli spaces. If the dimension of the moduli space is even, say equal to $2d$, then we define the *relative Seiberg-Witten invariant* $SW_c(\tilde{P}_X)$ by integrating the d^{th} power of the first Chern class of the universal circle bundle over this moduli space.

The construction of the moduli spaces can be made over the parameter space of all n and η. The result is a smooth infinite dimensional moduli space with a smooth map with Fredholm differential to the parameter space. Furthermore, all the fibers over $n\neq 0$ are compact. It follows easily that the relative invariant as defined above is independent of the choice of generic forms n and η^+.

If the dimension of the moduli space is odd, then we define the relative Seiberg-Witten invariant to be zero.

Though this is not a direction that we will pursue much further in this paper, we wish to point out that in this case the relative invariants satisfy the analogue of Seiberg-Witten simple type.

Proposition 9.4. *Let X be a cylindrical-end four manifold with end isometric to $[0,\infty)\times S^1\times C$ where C is a riemann surface of genus $g>1$. Let $\tilde{P}_X\to X$ be a $Spin^c$-structure whose determinant line bundle has degree $2g-2$ along C. Then for any c, if the dimension of $\mathcal{M}_c(\tilde{P}_X,n,\eta^+)=2d>0$, then the value of the relative Seiberg-Witten invariant $SW_c(\tilde{P}_X)$ is zero.*

Proof. The moduli space is compact and every point $[A,\psi]$ in the moduli space is aysmptotic at infinity to the same irreducible configuration $[A_0,\psi_0]$ on N. It follows immediately that the base point fibration $\mathcal{M}^0_c(\tilde{P}_X,n,\eta^+)\to\mathcal{M}_c(\tilde{P}_X,n,\eta^+)$ is trivial. From this the proposition is immediate. q.e.d.

Let us compute the possible values for c. Let $\hat{X}$ be the compact four-manifold

$$\hat{X} = X \cup (D^2 \times C),$$

where X and $D^2 \times C$ are glued together along $(0, \infty) \times S^1 \times C$ via the identification

$$(0, \infty) \times S^1 \cong (D^2 - \{0\})$$

sending $(t, \theta) \mapsto \exp(-t + i\theta)$. The condition on $\tilde{P}_X \to X$ implies that it has an extension to a $Spin^c$-structure over $\hat{X}$. The possible determinant line bundles $\hat{\mathcal{L}}$ of such extensions all differ by even multiples of the Poincaré dual of $[\{0\} \times C] \in H_2(\hat{X}; \mathbf{Z})$. The possible values for c so that the relative invariant $SW_c(\tilde{P}_X)$ are defined are simply

$$\langle c_1(\hat{\mathcal{L}})^2, [\hat{X}] \rangle,$$

as $\hat{\mathcal{L}}$ ranges over the determinant line bundles of the extensions of $\tilde{P}_X$. These numbers differ by integral multiples of $(8 - 8g)$.

9.3. A first product formula

From the Gluing Theorem and the definition of the relative invariants we have the following result.

Theorem 9.5. *Let $M, N, X, Y, \tilde{P}_X, \tilde{P}_Y$ be as above in this section. Let $\mathcal{S}_c$ be the set of equivalence classes of $Spin^c$ structures on M whose restrictions to X and Y agree up to isomorphism with $\tilde{P}_X$ and $\tilde{P}_Y$ and whose determinant line bundles $\mathcal{L}$ satisfy $c_1(\mathcal{L})^2 = c$. Choose orientations for $H^1(X, T; \mathbf{R}) \oplus H^2_{\geq 0}(X, T; \mathbf{R})$ and $H^1(Y, T; \mathbf{R}) \oplus H^2_{\geq 0}(Y, T; \mathbf{R})$. This determines the sign of the relative Seiberg-Witten invariants for X and Y. It also determines an orientation for $H^1(M; \mathbf{R}) \oplus H^2_+(M; \mathbf{R})$ and hence a sign for the Sieberg-Witten invariants of M. With these choices of orientations we have the following product formula:*

$$\sum_{\tilde{P} \in \mathcal{S}_c} SW(\tilde{P}) = (-1)^{b^1(X,N) b^2_{\geq 0}(Y,N)} \sum_{c_1 + c_2 = c} SW_{c_1}(\tilde{P}_X) SW_{c_2}(\tilde{P}_Y).$$

Notice that because all the relative invariants are of simple type, the sum on the right-hand-side of the equation in the theorem is in fact at most one term, the term when c_1 and c_2 are such that that the dimension of the cylindrical-end moduli spaces for X and Y are zero dimensional.

9.4. Computation of the Invariants for $D^2 \times C$

To complete the proof of the Product Formula we need to identify the relative invariants of a cylindrical-end manifold X with the invariants of the closed manifold

$$\hat{X} = X \cup D^2 \times C.$$

This identification is achieved using the product formula for relative invariants and an evaluation of the relative invariants of $D^2 \times C$. It is the purpose of this subsection to evaluate these latter relative invariants.

We give $D^2 \times C$ a product metric – the metric on the C-factor is any metric (but for definiteness let us assume the metric is of constant curvature). The metric on the disk is a complete metric with cylindrical end.

Suppose that (A, ψ) is a finite energy solution to the Seiberg-Witten equations on $D^2 \times C$ (with as usual $\psi = (\alpha, \beta)$) whose determinant line bundle $\mathcal{L}$ has degree $2 - 2g$ on C. Then the analogue of Remark 6.21 is the following: The section β is zero and the section α is a holomorphic section of $\mathcal{L}_0$ and we have

$$\frac{i}{2\pi} \int_{D^2} F_A = 1 + 2(\# \text{ zeros of } \alpha).$$

The reason for the extra one in this formula as compared to the formula in Remark 6.21 is that $A = A_X + 2A_0$ where A_X is the natural connection on the canonical bundle. Because of our choice of metric on D^2, it is easy to see that

$$\frac{i}{2\pi} \int_{D^2} F_{A_X} = 1.$$

The Chern integral $c(A, \psi)$ is given by

$$c(A, \psi) = \frac{-1}{4\pi^2} \int_{D^2 \times C} F_A \wedge F_A = 4(1 - g) \frac{i}{2\pi} \int_{D^2} F_A.$$

The formal dimension of the moduli space $\mathcal{M}(\tilde{P})$ of finite energy solutions at $[A, \psi]$ is

$$\frac{1}{4} c(A, \psi) + (g - 1) = 2(1 - g)(\# \text{ zeros of } \alpha).$$

Since α is a holomorphic section, it follows that the formal dimension of $\mathcal{M}(\tilde{P})$ at any point is ≤ 0. The only solution (A, ψ) at which the dimension is zero is when $c(A, \psi) = 4 - 4g$, which is the case when α is a constant section. This solution is induced from a configuration on C and is unique up to gauge equivalence.

Thus, we have proved the following lemma.

Lemma 9.6. *Let $\tilde{P} \to D^2 \times C$ be a Spin^c-structure whose determinant line bundle has degree $2-2g$ on C. If $\mathcal{M}_c(\tilde{P})$ is non-empty, then $c \leq 4-4g$ and the formal dimension of $\mathcal{M}_c(\tilde{P})$ is non-positive. If the formal dimension of the moduli space is zero, then $c = 4-4g$ and $\mathcal{M}_c(\tilde{P})$ is a single point, that point being a smooth point of the moduli space. The orientation on $\mathcal{M}_c(\tilde{P})$ induced from the complex structure on $D^2 \times C$ makes this point a plus point.*

Remark 9.7. Notice that for any $\ell > 0$, the moduli space $\mathcal{M}_c(\tilde{P})$ for $c = 4(1-g)(2\ell+1)$ is a smooth manifold of dimension 2ℓ, and in fact this moduli space is diffeomorphic to the ℓ-fold symmetric product of D^2. Nevertheless, the formal dimension of $\mathcal{M}_c(\tilde{P})$ at each of its points is $2\ell(1-g)$.

Now we consider the perturbed equations on $D^2 \times C$. We fix an isometric parameterization $[-1,\infty) \times S^1 \hookrightarrow D^2$, and we fix a function $\varphi\colon D^2 \to [0,1]$ whose support is contained in $[-1,\infty) \times S^1$ and which is identically one on $[0,\infty) \times S^1$.

For any harmonic one-form $n \in \Omega^1(C;\mathbf{R})$, we set $h = \varphi(*n + dt \wedge n)$ and let $\mathcal{M}(\tilde{P}, n)$ be the moduli space of gauge equivalence classes of finite energy solutions to the perturbed equations SW_h. As before, the formal dimension of this moduli space at $[A,\psi]$ is given by

$$(1-g)\frac{i}{2\pi}\left(\int_{D^2} F_A - 1\right),$$

and the Chern integral is

$$c = 4(1-g)\frac{i}{2\pi}\int_{D^2} F_A.$$

Lemma 9.8. *For any sufficiently small non-zero harmonic one-form $n \in \Omega^1(C;\mathbf{R})$ let $\mathcal{M}_c(\tilde{P}, n)$ denote the moduli space of gauge equivalence classes of finite energy solutions to the equations*

$$F_A^+ = q(\psi) + i\varphi(*n + n \wedge dt),$$

$$\partial\!\!\!/_A(\psi) = 0,$$

which have Chern integral c. Then $\mathcal{M}_c(\tilde{P}, n)$ is empty unless $c \leq 4-4g$. For $c = 4-4g$ the moduli space consists of one point, that point being a *smooth point and (with the orientation on the moduli space determined by the complex structure on $D^2 \times C$) is a plus point. Finally, for $c < 4-4g$ the formal dimension of the moduli space is negative. Thus, for a further deformation by a generic compactly supported, self-dual, two-form η^+ the moduli space $\mathcal{M}_c(\tilde{P}, n, \eta^+)$ is empty unless $c = 4-4g$ in which case the moduli space is a single smooth point.*

Proof. Let K_1 be the constant as in Lemma ?? Fix $\epsilon > 0$ sufficiently small and ν as in Lemma ?? Let $\{n_j\}$ be a sequence of harmonic one-forms on C tending to 0 as $j \mapsto \infty$. Let $h_j = \varphi(*n_j + dt \wedge n_j)$. Fix c and suppose that (A^j, ψ^j) are finite energy solutions to the perturbed equations SW_{h_j} with Chern integral c. Of course,

$$n(A^j, \psi^j) = \frac{i}{2\pi}\int_{D^2} F_{A^j} = \frac{c}{4-4g}.$$

Let $\gamma^j\colon [0,\infty) \to \mathcal{C}(\tilde{P}_N)$ be the path associated to (A^j, ψ^j). It is a gradient flow line for f_{n_j}. Using the notation and terminology from Subsection 6.7 we find the spanning intervals $a_1^j, \dots, a_{k_j}^j$ and the local intervals $b_1^j, \dots, b_{k_j}^j$ for γ^j and ν. As before the total L^2-length

$$\sum_{i=1}^{k_j} \ell_{L^2}(a_i^j)$$

is bounded independent of j. Since each regular spanning interval has L^2-length which is at least K_1, this implies that the number, k_j, of spanning intervals is bounded independent of j. Passing to a subsequence we can suppose that all the k_j are equal, say equal to k, and that for each $i \le k$ we have that

$$\lim_{j\mapsto\infty} \ell_{L^2}(a_i^j)$$

converges to a finite limit ℓ_i. In a similar manner we can assume that for each $i \le k$

$$\lim_{j\mapsto\infty} \ell_{L^2}(b_i^j) = m_i \le \infty.$$

For each j, we adjoin to $\coprod_i a_i^j$ the union of the $\{b_t^j\}$ for which $m_t < \infty$. This union is a finite disjoint union of closed intervals $s_1^j, \dots, s_f^j$ with the property that $\lim_{j\mapsto\infty}\ell_{L^2}(s_i^j)$ exists and is finite. The complementary set of intervals $r_1^j \dots, r_u^j$ in $[0,\infty)$ have the property that for each i the limit $\lim_{j\mapsto\infty}\ell_{L^2}(r_i^j) = \infty$. (For each j the number of r_i^j is either the same as the number of s_i^j or is one more.) Passing to a further subsequence, we can suppose that either $0 \in s_1^j$ for all j or $0 \notin s_1^j$ for all j. If the first case prevails, we simply delete s_1^j from the set of s-intervals and renumber the others. (We make no change to the r_i^j. In this way we arrive at a situation in which r_1^j lies to the left of s_1^j and for each j the number of r_i^j is one more than the number of s_i^j. By construction each r_i^j is a local interval for γ^j and $\lim_{j\mapsto\infty}\ell_{L^2}(r_i^j) = \infty$.)

Now passing to a further subsequence we can arrange that for each i there is a geometric limit for (A^j, ψ^j) based at the center point of s_i^j and that there is a geometric limit for (A^j, ψ^j) based at $\{0\} \times C$. We call these limits (A_i, ψ_i) and (A_0, ψ_0) respectively. For each i, $0 < i \leq t$, (A_i, ψ_i) is a non-static, finite energy solution to the unperturbed Seiberg-Witten equations on $\mathbf{R} \times S^1 \times C$, and (A_0, ψ_0) is a finite energy solution to the Seiberg-Witten equations on $D^2 \times C$. Clearly,

$$n(A^j, \psi^j) = n(A_0, \psi_0) + \sum_{i>0}^{t} n(A_i, \psi_i).$$

By Lemma 9.8 we have that $n(A_0, \psi_0) \geq 0$. By Lemma 6.20 we have $n(A_i, \psi_i) > 0$. Since we are assuming that the formal dimension of $\mathcal{M}_c(\tilde{P}, n^j)$ is non-negative, it follows that $c \geq 4 - 4g$ and hence that $n(A^j, \psi^j) \leq 1$. This implies that $t = 0$ and that $n(A_0, \psi_0) = 1$. This proves that the only c for which $\mathcal{M}_c(\tilde{P}, n^j)$ is non-empty and of non-negative formal dimension is $c = 4 - 4g$. This moduli space is of formal dimension 0, and we see that for all j sufficiently large the solution (A^j, ψ^j) is arbitrarily close on a fixed compact subset to the static solution of the unperturbed equations.

The last thing to establish is that for all j sufficiently large $\mathcal{M}_{4-4g}(\tilde{P}, n^j)$ consists of a single point, a smooth point which with the orientation induced by the complex structure on $D^2 \times C$ is a plus point. This follows from the fact that these statements are true for the unperturbed equations, and the fact that the moduli spaces vary smoothly with the parameters n and η^+.

Now for a generic compactly supported self-dual two form η^+, the moduli spaces $\mathcal{M}_c(\tilde{P}, n, \eta^+)$ which are of formal negative dimension will be empty. If η^+ is sufficiently small, then it will still be true that $\mathcal{M}_c(\tilde{P}, n, \eta^+)$ will be empty for $c > 4 - 4g$ and will consist of a single smooth point for $c = 4 - 4g$. q.e.d.

Corollary 9.9. *Let X be an oriented riemannian four-manifold with a cylindrical end isometric $[0,\infty) \times S^1 \times C$. Let $\hat{X}$ be the closed four-manifold obtained by filling in X with $D^2 \times C$. Then for Spin^c-structure $\tilde{P} \to \hat{X}$ which has the property that the determinant line bundle $\mathcal{L}$ of $\tilde{P}$ has degree $(2-2g)$ on $\{0\} \times C$ we have*

$$SW(\tilde{P}) = SW_c(\tilde{P}|_X),$$

where

$$c + (4g-4) = \langle c_1(\mathcal{L})^2, [\hat{X}]\rangle.$$

Proof. This is immediate from the Product Formula, Theorem 9.5 and the computation in the previous lemma. q.e.d.

Notice that for $\hat{X}$ and $\tilde{P}$ as in the previous corollary, it follows that if $SW(\tilde{P})$ is non-zero, then the formal dimension $d(\tilde{P})$ is zero.

Corollary 9.10. *Let M, N, X, Y be as in Theorem ?? Let $\tilde{P}_X \to X$ and $\tilde{P}_Y \to Y$ be Spin^c-structures whose determinant line bundles have degree $(2-2g)$ on C. Let $\hat{X}, \hat{Y}$ be the compactifications of X and Y obtained by filling in $D^2 \times C$. Fix $c \in \mathbf{Z}$. We set $\mathcal{S}_c$ equal to the set of isomorphism classes of pairs $(\hat{P}_X, \hat{P}_Y)$ of Spin^c-structures on $\hat{X}$ and $\hat{Y}$ extending $\tilde{P}_X$ and $\tilde{P}_Y$ with the property that*

$$c_1(\hat{\mathcal{L}}_X)^2 + c_1(\hat{\mathcal{L}}_Y)^2 = c + (8g-g).$$

Similarly, set $\mathcal{P}_c$ equal to the set of isomorphism classes of Spin^c-structures on M which restrict to X and Y to give $\tilde{P}_X$ and $\tilde{P}_Y$, up to isomorphism and with the property that $c_1(\mathcal{L})^2 = c$. Fix orientations for $H^1(X,N), H^1(Y,N), H^2_{\geq 0}(X,N)$, and $H^2_{\geq 0}(Y,N)$ inducing orientations on the moduli spaces for X Y and M and hence signs for the Seiberg-Witten invariants. Then we have

$$\sum_{\tilde{P}\in\mathcal{P}_c} SW(\tilde{P}) = (-1)^{b_1(X,N)b^2_{\geq 0}(Y,N)} \sum_{(\hat{P}_X,\hat{P}_Y)\in\mathcal{S}_c} SW(\hat{P}_X)SW(\hat{P}_Y).$$

Proof. This is immediate from Theorem 9.5 and the previous corollary. q.e.d.

It follows from Proposition 9.4 that unless $c = 2\chi(M) + 3\sigma(M)$ all terms in the summation on the right-hand-side of this equation are zero. In the case where $c = 2\chi(M) + 3\sigma(M)$ there is at most one non-zero term on the right-hand-side. In fact, one can also show that if $c \neq 2\chi(M) + 3\sigma(M)$, then all the terms in the summation on the left-hand-side of the equation also vanish.

10. Proof of Proposition 4.6

In this section we complete the proof of the two main theorems of the Introduction by showing that a symplectic torus T of non-negative square in symplectic four-manifold X is genus minimizing in its homology class. Of course, this simply means that the homology class of T is not represented by a smoothly embedded sphere. Were there a sphere $S \subset X$ representing the same homology class as T then that class would be of infinite order. By the adjunction formula and the fact that T and S are homologous we also have that

$$\langle K_X, S\rangle = \langle K_X, T\rangle = -T\cdot T = -S\cdot S,$$

where K_X is the canonical class of the symplectic structure of X. Thus, Proposition 4.6 follows from the main result of this section.

Proposition 10.1. *Let X be a closed symplectic four-manifold. Then there is no smoothly embedded sphere $S \subset X$ with the following properties:*

- $S\cdot S \geq 0$
- *The homology class represented by S is of infinite order in $H_2(X;\mathbf{Z})$.*
- *If $b_2^+(X) = 1$, then, letting K_X be the canonical class of the symplectic structure of X, we have $\langle K_X, S\rangle + S\cdot S = 0$.*

10.1. First reductions in the proof of Proposition 10.1

Blowing up X at points along S, shows that in order to prove Proposition 10.1 it suffices to consider the case when $S \cdot S = 0$.

We fix a compact manifold X and a smoothly embedded sphere $S \subset X$ of square zero. We write

$$X = X_0 \cup \big([0,1]\times S^1 \times S\big) \cup D^2 \times S,$$

and we fix a one-parameter family of metrics g_t, $1 \leq t$ on X satisfying:

- The family is constant on X_0.
- The family is constant on $D^2 \times S$ and this metric is the product of a constant positive curvature metric on the sphere with a non-negative curvature metric on D^2.

- The metric g_t on $[0,1] \times S^1 \times S$ is isometric to a product of an interval of length t^2 with the standard product metric on $S^1 \times S$.

Eventually, we shall work with the riemannian manifold (X, g_R) for some sufficiently large R.

Here is the basic lemma.

Lemma 10.2. *Let X be a closed, oriented four-manifold, and suppose that $S \subset X$ is a smoothly embedded sphere of square zero representing a homology class of infinite order in $H_2(X;\mathbf{Z})$. If $b_2^+(X) > 1$, then the Seiberg-Witten invariant SW_X vanishes identically. If $b_2^+(X) = 1$, letting S^* denote the cohomology class Poincaré dual to S, we have that the S^*-negative Seiberg-Witten invariant $SW_X^{S^*}$ vanishes on any characteristic class k with the property that $\langle k, S\rangle = 0$.*

Remark 10.3. In the case where $b_2^+(X) = 1$, the symmetry of this result under replacing S by $-S$ implies that the change in $SW_X(k)$ as we cross the wall of reducibles must be zero. One can check directly that under the given topological conditions that the skew-symmetric form

$$H^1(X;\mathbf{R}) \otimes H^1(X;\mathbf{R}) \to \mathbf{R}$$

given by $(a,b) \mapsto \langle a \cup b \cup k, [X]\rangle$ is degenerate. This implies directly that the wall-crossing formula for $SW_X(k)$ is trivial.

Let us show that this lemma implies Proposition 10.1. Suppose that X is symplectic with symplectic form ω. In the case where $b_2^+(X) = 1$, at the expense of reversing the orientation on S, we can assume that the S^*-negative Seiberg-Witten invariant is the same as the ω-negative Seiberg-Witten invariant. Thus, in both cases we can apply Taubes non-vanishing result for the value of the Seiberg-Witten invariant on the canonical class of a symplectic manifold (Lemma 4.8) to establish Proposition 10.1 from this lemma. The rest of this section is devoted to the proof of this lemma.

The argument is divided into two cases depending on whether $b_2^+(X) > 1$ or not. First we consider the easier case where $b_2^+(X) > 1$.

10.2. The case $b_2^+(X) > 1$

Let (Z, g_Z) be the complete riemannian four-manifold isometric to

$$X_0 \cup \left([0,\infty) \times S^1 \times S\right).$$

Since the end of Z has postive scalar curvature, given any constant E there are constants $K, \delta > 0$ such that for any finite energy solution

(A, ψ) to the Seiberg-Witten equations on Z with

$$\frac{-1}{4\pi} \int_Z F_A \wedge F_A \leq E,$$

after modifying by a gauge transformation, we can assume that for any $(t, x) \in [0, \infty) \times (S^1 \times S)$ we have the following pointwise C^∞-bounds:

- $|\psi(t, x)|_{C^\infty} \leq Ke^{-\delta t}$.
- There is a flat connection A_0 on $S^1 \times S$ such that $|A(t, x) - A_0(x)|_{C^\infty} \leq Ke^{-\delta t}$.

It follows that for any $Spin^c$-structure and any e the moduli space $\mathcal{M}_e(\tilde{P}, g_Z)$ is compact.

For any finite energy solution (A, ψ) to the Seiberg-Witten equations for a $Spin^c$-structure over Z the connection A decays exponentially in a temporal gauge to a flat connection at infinity. The space of gauge equivalence classes of flat connections on $S^1 \times S$ is S^1. Thus, for any $Spin^c$-structure $\tilde{P} \to Z$ and any constant e, there is a well-defined boundary map, a smooth map

$$\partial \colon \mathcal{M}_e(\tilde{P}, g_Z) \to S^1,$$

which assigns to each solution the limiting flat connection at infinity. Adding a generic compactly supported self-dual form η^+ we can arrange that $\mathcal{M}_e(\tilde{P}, g_Z, \eta^+)$ is smooth as well as compact. The exponential decay results still hold for the configurations representing points of this moduli space so that there is a boundary map $\partial \colon \mathcal{M}_e(\tilde{P}, g_Z, \eta^+) \to S^1$. For generic η^+ this map is transverse to $-1 \in S^1$.

Now fix a $Spin^c$-structure $\tilde{P}_X \to X$. Let $\mathcal{L}$ be its determinant line bundle and set

$$e = \int_X c_1(\mathcal{L})^2.$$

Let $\tilde{P}$ be the restriction of $\tilde{P}_X$ to Z. Choose a generic, compact supported η^+ so that $\mathcal{M}_e(\tilde{P}, g_Z, \eta^+)$ is smooth and so that ∂ is transverse to $-1 \in S^1$. For all $R >> 0$, the form η^+ induces a self-dual two-form η_R^+ on (X, g_R). Under these conditions, for all $R >> 0$, the gluing theorem identifies $\mathcal{M}(\tilde{P}', g_R, \eta_R^+)$ with the codimension-one submanifold

$$\partial^{-1}(-1) \subset \mathcal{M}_e(\tilde{P}, g_Z, \eta^+).$$

The fundamental class of the codimension-one subset $\partial^{-1}(-1)$ is Poincaré dual to $\mu(S^1) \in H^1(\mathcal{M}_e(\tilde{P}, g_Z, \eta^+); \mathbf{Z})$. Thus, supposing that the dimension of $\mathcal{M}(\tilde{P}_X, g_R, \eta_R^+)$ is $2d$ we have

$$\int_{\mathcal{M}(\tilde{P}_X, g_R, \eta_R^+)} \mu^d = \int_{\mathcal{M}_e(\tilde{P}, g_Z, \eta^+)} \mu^d \cup \mu(S^1). \tag{19}$$

So far we have not used the fact that the homology class of S is of infinite order. The relevance of this condition is that it implies that the class represented by S^1 in $H_1(Z;\mathbf{Z})$ is of finite order. (Its order is given by the minimal positive intersection number of a class in $H_2(X;\mathbf{Z})$ with the class of S.) But if S^1 is of finite order in $H_1(Z;\mathbf{Z})$, it follows that $\mu(S^1) \in H^1(\mathcal{M}_e(\tilde{P}, g_Z, \eta^+); \mathbf{Z})$ is also of finite order and hence the integral on the right-hand-side of Equation 19 is zero.

This completes the proof that the Seiberg-Witten invariant for $\tilde{P}_X$ vanishes. Since $\tilde{P}_X$ was an arbitrary *Spin*^{c}-structure on X, this completes the proof of Lemma 10.2 in the case where $b_2^+(X) > 1$.

10.3. The case when $b_2^+(X) = 1$

For any $R > 1$, let ω_R^+ be the g_R-self-dual form on X of norm one with positive integral over S.

Claim 10.4. *As $R \mapsto \infty$ the forms ω_R^+ converge to zero on $(X_0 \coprod D^2 \times S) \subset X$.*

Proof. First let us show that the forms ω_R^+ are pointwise universally bounded on $A = X_0 \coprod D^2 \times S$. Take a point $x \in A$ and let B be a small ball containing x. The L^2-norm of the forms $\omega_R^+|_B$ are universally bounded, and hence we have pointwise bounds on any smaller ball to the C^∞-norm of ω_R^+. Given these pointwise bounds, it is possible to extract a subsequence of the ω_R^+ which converges to a harmonic form on the cylindrical-end manifold $Z \coprod T$ where T is diffeomorphic to $D^2 \times S$ and has a cylindrical end isometric to $[0,\infty) \times S^1 \times S$. This limit form is self-dual and its L^2-norm is at most one. But $b_2^+(Z \coprod T) = 0$ so there are no non-zero self-dual L^2-forms. This means that the limit is the trivial form, proving that the ω_R^+ must go to zero pointwise on any compact subset of $X_0 \coprod D^2 \times S$. q.e.d.

Corollary 10.5. *If λ is any closed form supported on X_0, then*

$$\lim_{R \mapsto \infty} \int_X \omega_R^+ \wedge \lambda = 0.$$

Now let us consider the nature of ω_R^+ in the tube $[0, R^2] \times S^1 \times S$. This form exponentially decays to a constant self-dual harmonic-form, i.e., it decays exponentially fast to a multiple of the $dt \wedge d\theta + d\mathrm{vol}_S$. The exponent is universal depending only on X not on R. Since

$$\int_X \omega_R^+ \wedge \omega_R^+ = 1,$$

since the length of the tube is R^2 and since the integral of $\omega_R^+ \wedge \omega_R^+$ is going to zero on the complement of the tube, it follows that the multiple is of the form $C/R + o(1/R)$ where C^{-2} is twice the product of the length of the circle and the area of S.

In particular, there are constants $L, C' > 0$ such that for all $R >> 0$ and any $L \leq t \leq R^2 - L$ and any $x \in S^1 \times S$, measuring the norms with respect to g_R, we have

$$|\omega_R^+(t,x)| \leq C'/R. \tag{20}$$

In light of this and Claim 10.4 we also have that for $R >> 0$ and all $p \in X$, measuring norms with respect to g_R, we have

$$|\omega_R^+(p)| \leq 1. \tag{21}$$

Fix a $Spin^c$-structure $\tilde{P} \to X$ and consider the following perturbed Seiberg-Witten equations for $\tilde{P}$ and g_R:

$$\begin{aligned} F_A^+ &= q(\psi) - ir\omega_R^+, \\ \partial\!\!\!/_A(\psi) &= 0 \end{aligned} \tag{22}$$

for some $r > 0$.

Since $\int_S \omega_R^+ > 0$, for all $r >> 0$ (how large may depend on R), the moduli space of solutions for these equations computes the S^*-negative Seiberg-Witten invariant of $\tilde{P}$. We shall show that for $R >> 0$ and $r >> R$, if the degree of the determinant line bundle $\mathcal{L}$ of $\tilde{P}$ on S is zero, then there are no solutions to these equations, implying that the $SW_X^{S^*}(\tilde{P}) = 0$. This will establish Lemma 10.2 in the case where $b_2^+(X) = 1$.

We fix $R >> 0$ so that Inequalities (20) and (21) hold. Since ω_R^+ is a harmonic form, it vanishes only on a set of measure zero. We denote the function $|\omega_R^+|$ by u_R. Then we have

$$\begin{aligned} u_R &\leq 1 \\ C'/R &\leq u_R(t,x) \quad \text{for all } L \leq t \leq R^2 - L \end{aligned}$$

On the open dense subset where $\omega_R^+ \neq 0$, we can decompose the spin bundle $S^+(\tilde{P})$ into two complex line bundles, $L^- \oplus L^+$, where the action of Clifford multiplication by ω_R^+ on a section of $L^\pm$ is given by multiplication by $\pm 2u_R i$. Using this decomposition we write any spinor field $\psi = (\alpha, \beta)$ with α being the component in L^- and β being the component in L^+.

Let (A, ψ) be a solution to the Equations (22) for a $Spin^c$-structure $\tilde{P} \to X$ whose determinant line bundle $\mathcal{L}$ has degree zero along S. The Bochner-Weitzenbock formula tells us that

$$0 = \partial\!\!\!/_A \partial\!\!\!/_A(\psi) = \nabla_A^* \nabla_A(\psi) + \frac{s}{4}\psi + \frac{F_A^+}{2}\psi,$$

where s is the scalar curvature of (X, g_R). Using the curvature equation, decomposing $\psi = (\alpha, \beta)$ and taking the L^2-inner product with ψ we get

$$0 = \|\nabla_A(\psi)\|_{L^2}^2 + \int_X \frac{s}{4}(|\alpha|^2 + |\beta|^2) + \frac{(|\alpha|^2 + |\beta|^2)^2}{4} + ru_R(|\beta|^2 - |\alpha|^2)dvol(g_R).$$

In particular,

$$0 \geq \int_X \frac{s}{4}(|\alpha|^2 + |\beta|^2) + \frac{(|\alpha|^2 + |\beta|^2)^2}{4} + ru_R(|\beta|^2 - |\alpha|^2)dvol(g_R).$$

We can rewrite the integrand in this expression as

$$\left(\frac{|\alpha|^2}{2} - 2ru_R + \frac{s}{4}\right)^2 + \left(\frac{|\beta|^2}{2} + \frac{s}{4}\right)^2 + \frac{|\alpha|^2|\beta|^2}{2} + |\beta|^2 ru_R - \frac{s^2}{8} + ru_R s - 4r^2u_R^2 + |\alpha|^2 ru_R.$$

Removing first four terms, each of which is obviously non-negative, we conclude that

$$0 \geq \int_X \frac{-s^2}{8} + ru_R s + 4ru_R\left(\frac{|\alpha|^2}{4} - ru_R\right) dvol(g_R).$$

A fortiori, we have

(23)
$$0 \geq \int_X \frac{-s^2}{8} + ru_R s + 4ru_R\left(\frac{|\alpha|^2 - |\beta|^2}{4} - ru_R\right) dvol(g_R).$$

Let us estimate each of the terms in this inequality.

Clearly since s is bounded independent of R, and the volume of (X, g_R) is equal to $C_0 + C_1 R^2$ for some constants independent of R, we conclude that for all $R >> 0$

$$\int_X \frac{-s^2}{8} dvol_{g_R} \geq -C_2 R^2 \tag{24}$$

for some $C_2 \geq 0$ independent of R. From the Inequalities (20) and (21) and the fact that s is constant and positive on $[0, R^2] \times S^1 \times S$, it follows that for all $R >> 0$

$$\int_X r u_R s \; dvol(g_R) \geq C_3 r R - C_4 r \tag{25}$$

for some constants $C_3, C_4 > 0$ independent of R.

Finally, using the curvature equation once again, it is easy to see that

$$\begin{aligned}\int_X 4 r u_R \left(\frac{|\alpha|^2 - |\beta|^2}{4} - r u_R \right) dvol(g_R) &= r \int_X -4iF_A^+ \wedge \omega_R^+ \\ &= r \int_X -8\pi c_1(\mathcal{L}) \wedge \omega_R^+.\end{aligned}$$

By hypothesis we have $\langle c_1(\mathcal{L}), S \rangle = 0$. Thus, the class $c_1(\mathcal{L})$ is represented by a closed 2-form λ on X with support contained in X_0. It then Corollary 10.5 yields that

$$\lim_{R \mapsto \infty} \int_X c_1(\mathcal{L}) \wedge \omega_R^+ = 0.$$

Consequently, there is a constant $C_5 > 0$ independent of R such that for all $R >> 0$,

$$\left| \int_X -4iF_A^+ \wedge \omega_R^+ \right| \leq C_5. \tag{26}$$

From Equations (24), (25), and (26) it follows immediately that for $R >> 0$ we have

$$\begin{aligned}\int_X \frac{-s^2}{8} + r u_R s + 4 r u_R \left(\frac{|\alpha|^2 - |\beta|^2}{4} - r u_R \right) dvol(g_R) \\ \geq C_3 r R - C_4 r - C_5 r - C_2 R^2.\end{aligned}$$

Clearly, if $R >> 0$ and $r >> R$, then the right-hand-side of this expression is positive. This contradicts Inequality (23) and establishes that there are no solutions to Equations (22) for the given $Spin^c$-structure and $r >> R >> 0$.

This proves that $SW_X^{S^*}(k) = 0$ for any characteristic class with the property that $\langle k, S\rangle = 0$, and thus establishes the lemma in the case that $b_2^+(X) = 1$.

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