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GEOMETRIC EXPANSION OF CONVEX PLANE CURVES

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1. Introduction

In the last several years, there has been considerable interest in the deformation of Euclidean hypersurfaces in the direction of their normal vector field with speed various functions of the principal curvatures. In particular, for contracting flows, Gage-Hamilton [12] and Grayson [14] studied the curve shortening flow, Brakke [3] and Huisken [19] studied the mean curvature flow and Tso [22] studied the Gauss curvature flow (see also Chow [6], [7], Hamilton [17], [18] and Andrews [2]). For more general homogeneous contracting flows, see Andrews [1]. Besides contracting flows, there has also been recent interest in expanding flows. Similar results have been proved by Urbas [23], [24], Huisken [19] and Gerhardt [13]. More recently, Andrews [1] has studied more general expanding flows, especially flows with anisotropic speeds.

In each of the above papers, the hypersurfaces are evolving with speed a *homogeneous* increasing function of the principal radii. For expanding flows, one generally assumes in addition that the function is *concave*. In a series of papers, of which this is the first, we investigate expanding flows with speed an increasing function of the principal radii. In particular, we shall not assume the function is homogeneous. Our results generalize most of the previous results on expanding flows.

In this paper we study the motion of a smooth, strictly convex, embedded closed curve in $\mathbb{R}^2$ expanding in the direction of its outward normal vector with speed given by an *arbitrary* positive increasing function G of its principal radius of curvature. Our result is that there exists a unique one-parameter family of smooth, strictly convex curves

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satisfying the above equation, which expand to infinity. Moreover, the shapes of the curves become round asymptotically in the sense that if one rescales the equation appropriately, the support function of the rescaled curves converge uniformly to the constant 1 in C^2-norm. Under additional hypotheses on the function G, we prove that the convergence is in C^∞-norm.

In later papers we shall study expanding convex compact hypersurfaces of dimension at least two. In Chow-Tsai [11] we study the case where the speed is a *non-homogeneous* function of the principal radii. Under certain assumptions on the curvature function analogous to those considered by Urbas, etal., we prove that hypersurfaces remain smooth, strictly convex and expand to infinity while their shapes asymptotically become round. In particular, after an appropriate rescaling, their support functions converge to the constant 1 in C^1-norm. In Chow-Liou-Tsai [10] we consider the equation $u_t = F(\Delta u + nu)$ on S^n which corresponds to deforming a hypersurface $M^n \subset \mathbb{R}^{n+1}$ in the direction of its outward normal with speed a function of the harmonic mean curvature.

2. Main result

Let γ be a convex embedded closed curve in $\mathbb{R}^2$ parametrized by a smooth embedding $X_0 : S^1 \to \mathbb{R}^2$. We consider the initial value problem

$$\frac{\partial X}{\partial t} = G\left(\tfrac{1}{k}\right)N, \tag{1}$$

$$X(x,0) = X_0(x), \qquad x \in S^1, \tag{2}$$

where $k(x,t)$ is the curvature of the curve given by $X(\cdot,t)$ at the point x, $G : \mathbb{R}_+ \to \mathbb{R}_+$ is a positive smooth function with $G' > 0$ everywhere, and $N(\cdot,t)$ is the outward unit normal vector field to $X(\cdot,t)$.

Without loss of generality, we can assume that γ encloses the origin of $\mathbb{R}^2$ initially. Similar to Tso [22] and Urbas [24], we can reduce the initial value problem (1)-(2) to an initial value problem for the support function. The support function u_0 of γ is defined by

$$u_0(x) = \langle x, F(x)\rangle, \qquad x \in S^1,$$

where $F : S^1 \to \gamma \subseteq \mathbb{R}^2$ is the inverse Gauss map of γ. Since F is smooth, $u_0(x)$ will be a smooth function on S^1, and if γ is a circle of radius r centered at $(c_1, c_2) \in \mathbb{R}^2$, then $u_0(x) = r + c_1 \cos x + c_2 \sin x$ for all $x \in S^1$.

We compute the curvature of γ in terms of its support function $u_0(x)$. The principal radius of γ is given as

$$\frac{1}{k(x)} = (u_0)_{xx}(x) + u_0(x), \qquad x \in S^1,$$

and equations (1)-(2) are equivalent to

$$\frac{\partial u}{\partial t} = G(u_{xx} + u), \tag{3}$$

$$u(x,0) = u_0(x), \qquad x \in S^1, \tag{4}$$

together with the condition

$$u_{xx}(x,t) + u(x,t) > 0, \tag{5}$$

whenever the solution exists. The main result of this paper is the following.

Theorem 1. *Let G be an arbitrary positive function with $G' > 0$ everywhere. For any smooth function $u_0 : S^1 \to \mathbb{R}$ with $(u_0)_{xx}(x) + u_0(x) > 0$ for all $x \in S^1$, there exists a unique solution $u \in C^\infty(S^1 \times [0,T))$ of equations (3)-(4) satisfying (5), where $0 < T \leq \infty$, such that $\lim_{t\to T} u_{min}(t) = \infty$. Moreover, there exists a constant C depending only on u_0 such that*

$$|u_{xx}(x,t)| \leq C$$

for all $x \in S^1$ and $t \in [0,T)$. As a consequence, there exists a solution $R(t)$ to the ODE $dR/dt = G(R)$ on $[0,T)$ such that

$$u_{\min}(t) \leq R(t) \leq u_{\max}(t),$$

and the support functions $\tilde{u}$ of the rescaled curves $\tilde{\gamma} = \gamma/R$ satisfy

$$\|\tilde{u}(\cdot,t) - 1\|_{C^2(S^1)} \leq \frac{C}{R(t)}$$

for all $t \in [0,T)$.

In the rest of the paper, we will consider equations (3)-(4), and prove the long time existence of a solution satisfying condition (5).

3. A lower bound for the principal radius

Standard parabolic theory guarantees the existence of a unique smooth solution $u(x,t)$ of (3)-(4) on $S^1 \times [0,T)$ for some small $T > 0$. Because our initial curve γ is uniformly convex and encloses the origin, we have $(u_0)_{xx} + (u_0) \geq \delta$ and $u_0 \geq \delta$ on S^1 for some $\delta > 0$. Geometrically, if our initial curve γ is convex, it will remain convex during the evolution. Therefore, we prove

Lemma 1. *Let $u(x,t)$ be solution to (3)-(4) with $(u_0)_{xx}+u_0 \geq \delta > 0$ for all $x \in S^1$. Then*

$$u_{xx}(x,t) + u(x,t) \geq \delta > 0 \tag{7}$$

for all $(x,t) \in S^1 \times [0,T)$.

Proof. By continuity, we have $u_{xx} + u > 0$ on $S^1 \times [0,\varepsilon]$ for some $\varepsilon > 0$. Let $v(x,t) = G(u_{xx}+u)$ on $S^1 \times [0,T)$. Then $v > 0$ on $S^1 \times [0,\varepsilon]$ and $v(x,0) \geq G(\delta) > 0$.

We compute

$$\frac{\partial v}{\partial t} = G'(u_{xx} + u) \cdot (v_{xx} + v), \qquad \text{on } S^1 \times [0,\varepsilon].$$

Since $G' > 0$, by the weak maximum principle we have

$$v(x,t) \geq G(\delta) > 0 \qquad \text{on } S^1 \times [0,\varepsilon].$$

and hence

$$u_{xx}(x,t) + u(x,t) \geq \delta > 0 \qquad \text{on } S^1 \times [0,\varepsilon].$$

Repeating the same process implies the above inequality in $S^1 \times [0,T)$. q.e.d.

4. The gradient estimate

The next result concerns the uniform estimate of the gradient of u, which was proved in a previous paper as a special case of a more general theorem based upon a variant of the Aleksandrov reflection method. See Chow-Gulliver [8, Theorem 2.1]. However, for completeness, a brief proof is given below.

Proposition 1. *Let $u(x,t)$ be a smooth solution to (3)-(4) on $S^1 \times [0,T)$ where $G : \mathbb{R}_+ \to \mathbb{R}_+$ is a smooth function with $G' \geq 0$ everywhere.*

Then there exists a constant $\lambda \geq 0$ *depending only on* u_0 *such that*

$$|u(x_1,t) - u(x_2,t)| \leq \lambda \left|\sin\left(\frac{x_1 - x_2}{2}\right)\right| \tag{8}$$

for all $x_1, x_2 \in S^1 = \mathbb{R}/2\pi\mathbf{Z}$ *and* $t \in [0,T)$.

Proof. Given $\theta \in S^1$, define $w_\theta(x) = u_0(x) - u_0(2\theta - x)$. Then w_θ is a Lipschitz function on the half circle $[\theta - \pi, \theta]$ with $w_\theta(\theta - \pi) = w_\theta(\theta) = 0$. This implies that there exists a $\lambda(\theta) \in \mathbb{R}$ such that $\lambda(\theta)\sin(\theta - x) \geq w_\theta(x)$ for all $\theta - \pi \leq x \leq \theta$. Define $u^{\lambda(\theta)}(x,t) = u(2\theta - x, t) + \lambda(\theta)\sin(\theta - x)$. Then $u^{\lambda(\theta)}$ is a solution to (3) with initial condition

$$u^{\lambda(\theta)}\big|_{t=0} = u_0^{\lambda(\theta)},$$

where $u_0^{\lambda(\theta)} : S^1 \to \mathbb{R}$ is given by

$$u_0^{\lambda(\theta)}(x) = u_0(2\theta - x) + \lambda(\theta)\sin(\theta - x).$$

Since both $u^{\lambda(\theta)}$ and u are solutions to (3) with $G' \geq 0$, $u^{\lambda(\theta)} = u$ on $(\{\theta - \pi\} \cup \{\theta\}) \times [0,T)$, and $u^{\lambda(\theta)} \geq u_0$ on $[\theta - \pi, \theta]$, by the weak maximum principle for parabolic equations of the second order, we conclude that $u^{\lambda(\theta)} - u \geq 0$ in $[\theta - \pi, \theta] \times [0,T)$. Therefore there exists $\lambda \geq 0$ depending only on u_0 such that

$$u(2\theta - x, t) + \lambda \sin(\theta - x) \geq u(x,t)$$

for all $\theta \in S^1$, $x \in [\theta - \pi, \theta]$, $t \in [0,T)$. Setting $x = x_1$ and $\theta = (x_1 + x_2)/2$. We conclude that

$$u(x_2,t) + \lambda \sin\left(\frac{x_2 - x_1}{2}\right) \geq u(x_1,t) \tag{9}$$

for all $x_1, x_2 \in S^1$, $t \in [0,T)$. Proposition 1 follows easily. q.e.d.

As an immediate consequence of Proposition 1, we have the following uniform gradient estimate for u (Chow-Gulliver [8, Corollary 2.3].)

Corollary 1. *Suppose* $u : S^1 \times [0,T) \to \mathbb{R}$ *satisfies the hypotheses of Proposition 1, and let* $\lambda \geq 0$ *be the constant given in the conclusion of Proposition 1. Then*

$$|u_x| \leq \frac{\lambda}{2} \qquad \text{in } S^1 \times [0,T). \tag{10}$$

For each $t \in [0, T)$, let

$$u_{\max}(t) = \max_{x \in S^1} u(x,t), \quad u_{\min}(t) = \min_{x \in S^1} u(x,t).$$

Since G is a positive function, $u_{\max}(t)$ and $u_{\min}(t)$ are increasing on $[0\ T)$ and $u_{\min}(t) < u_{\max}(t)$ for all $t \in [0, T)$ unless we are in the trivial case $u(x,t) = f(t)$ with $\dot{f}(t) = G(f(t))$ on $[0, T)$. Proposition 1 implies

$$u_{\max}(t) - u_{\min}(t) \le C \qquad \text{on } [0, T), \tag{11}$$

where C is a constant depending only on u_0.

5. The second derivative estimate

In this section we shall show that the second derivative of u is uniformly bounded, independent of time. This is the main estimate, which, together with certain standard results, implies the long time existence of a solution to (3)-(4). Before proceeding, we need some results on functions which are not necessarily differentiable. The following discussion on Lipschitz functions is based on Hamilton (1986) (see also Urbas (1991)). Let $f(t)$ be a Lipschitz function on some interval $[a, b]$. Then we define

$$\begin{aligned}
\frac{d^+ f}{dt} \le C \quad &\text{if} \quad \limsup_{h \searrow 0} \frac{f(t+h) - f(t)}{h} \le C,\\
\frac{d^+ f}{dt} \ge C \quad &\text{if} \quad \liminf_{h \searrow 0} \frac{f(t+h) - f(t)}{h} \ge C,\\
\frac{d^- f}{dt} \le C \quad &\text{if} \quad \limsup_{h \searrow 0} \frac{f(t) - f(t-h)}{h} \le C,\\
\frac{d^- f}{dt} \ge C \quad &\text{if} \quad \liminf_{h \searrow 0} \frac{f(t) - f(t-h)}{h} \ge C.
\end{aligned}$$

Lemma 2. *Let $f(t)$ be a Lipschitz function on $[a, b]$.*

(i) If $f(a) \le 0$ and $\frac{d^+ f}{dt} \le 0$ whenever $f \ge 0$ on $[a, b]$, then $f(b) \le 0$.

(ii) If $f(a) \ge 0$ and $\frac{d^+ f}{dt} \ge 0$ whenever $f \le 0$ on $[a, b]$, then $f(b) \ge 0$.

(iii) If $f(b) = 0$ *and* $\frac{d^- f}{dt} \leq 0$ *whenever* $f \leq 0$ *on* $[a, b]$*, then* $f(a) \geq 0$.

(iv) If $f(b) = 0$ *and* $\frac{d^- f}{dt} \geq 0$ *whenever* $f \geq 0$ *on* $[a, b]$*, then* $f(a) \leq 0$.

Proof. For (i) and (ii) see Hamilton [16, Lemma 3.1]. Parts (iii) and (iv) are analogous, so we omit the details of the proof. q.e.d.

Let Y be a compact set in Euclidean space and $g : [a, b] \times Y \to \mathbb{R}$ be a smooth function. Define $f(t) = \sup\{g(t, y) : y \in Y\}$ and $h(t) = \inf\{g(t, y) : y \in Y\}$. Both $f(t)$ and $h(t)$ are Lipschitz functions on $[a, b]$, and we have the following estimate on their derivatives.

Lemma 3. *Let* $Y_1(t) = \{y : g(t, y) = f(t)\}$ *and* $Y_2(t) = \{y : g(t, y) = h(t)\}$*. Then:*

(i) $\frac{d^+ f}{dt}(t) \leq \sup\{\frac{\partial}{\partial t} g(t, y) : y \in Y_1(t)\}$*, for all* $t \in [a, b)$.

(ii) $\frac{d^- f}{dt}(t) \leq \sup\{\frac{\partial}{\partial t} g(t, y) : y \in Y_1(t)\}$*, for all* $t \in (a, b]$.

(iii) $\frac{d^+ h}{dt}(t) \geq \inf\{\frac{\partial}{\partial t} g(t, y) : y \in Y_2(t)\}$*, for all* $t \in [a, b)$.

(iv) $\frac{d^- h}{dt}(t) \geq \inf\{\frac{\partial}{\partial t} g(t, y) : y \in Y_2(t)\}$*, for all* $t \in (a, b]$.

Proof. The proof of (i) is given in Hamilton [16, Lemma 3.5]. The proof of the rest is similar. q.e.d.

We are now ready to estimate the second derivative of $u(x, t)$. The right quantity to estimate is $w = \frac{1}{2}(u_x^2 + u_{xx}^2)$, which is a constant independent of space and time when initial curve γ is a round circle.

Lemma 4. *Let* $u(x, t)$ *be a smooth solution to (3)-(4) on* $S^1 \times [0, T)$*. Then*

$$|u_{xx}(x, t)| \leq C \qquad on\ S^1 \times [0, T),$$

where C *is a constant depending only on* u_0.

Proof. Set $w = \frac{1}{2}(u_x^2 + u_{xx}^2)$. We compute

$$\begin{aligned}
\frac{\partial w}{\partial t} =& u_{xx} \cdot (\partial_t u)_{xx} + u_x \cdot (\partial_t u)_x \\
=& u_{xx} \cdot [G(u_{xx} + u)]_{xx} + u_x \cdot [G(u_{xx} + u)]_x \\
=& u_{xx} \cdot [G'(u_{xx} + u) \cdot (u_{xx} + u)_{xx} + G''(u_{xx} + u) \cdot (u_{xx} + u)_x^2] \\
&+ u_x \cdot G'(u_{xx} + u) \cdot (u_{xx} + u)_x.
\end{aligned}$$

Using

$$w_x = u_{xx} \cdot (u_{xx} + u)_x$$

and

$$w_{xx} = u_{xx} \cdot (u_{xx} + u)_{xx} + u_{xxx} \cdot (u_{xx} + u)_x.$$

we can rewrite the above equation as

$$\begin{aligned} \frac{\partial w}{\partial t} =& G'(u_{xx} + u) \cdot w_{xx} + G''(u_{xx} + u) \cdot (u_{xx} + u)_x \cdot w_x \\ &+ G'(u_{xx} + u) \cdot (u_{xx} + u)_x (u - u_{xx})_x. \end{aligned} \tag{12}$$

In the argument below, we shall apply the maximum principle to equation (12) in a slightly unconventional way to obtain a uniform bound for w. Initially, we have

$$w(x, 0) = \tfrac{1}{2}[(u_0)_x^2 + (u_0)_{xx}^2] \leq C_1 \quad \text{on } S^1,$$

where C_1 is a constant depending only on u_0 such that $\frac{1}{2}(\frac{\lambda}{2})^2 < C_1$. Here $\frac{\lambda}{2}$ is the constant given in the conclusion of Corollary 1. For each $t \in [0, T)$, define $f(t) = \max_{x \in S^1} w(x, t) = w(p_t, t)$ for some $p_t \in S^1$. At (p_t, t), we have $w_x = 0$ and $w_{xx} \leq 0$, which imply either $u_{xx} = 0$ or $(u_{xx} + u)_x = 0$ at (p_t, t). We now have $f(0) \leq C_1$ and whenever $f(t) \geq C_1$ for some $t \in (0, T)$, u_{xx} can not be zero at (p_t, t) since if $u_{xx} = 0$ at (p_t, t) we will have $f(t) = \frac{1}{2}u_x^2(p_t, t) \leq \frac{1}{2}(\frac{\lambda}{2})^2 < C_1$, which is a contradiction. This in turn implies $(u_{xx} + u)_x = 0$ at (p_t, t). As a consequence of this, we obtain, from (12), that

$$\frac{\partial w}{\partial t} \leq 0 \quad \text{at } (p_t, t),$$

whenever $f(t) \geq C_1$. By Lemma 2 (i) and Lemma 3 (i) we obtain

$$f(t) \leq C_1 \quad \text{for any } t \in [0, T).$$

Therefore

$$w(x, t) = \tfrac{1}{2}(u_x^2 + u_{xx}^2) \leq C_1 \quad \text{on } S^1 \times [0, T),$$

and the conclusion of Lemma 4 follows. q.e.d

6. Higher derivatives

In this section we prove time-dependent estimates for the higher derivatives of u. These estimates also follow from the standard results, given the C^2-estimate for u of the previous section.

Lemma 5. *If $u \le M$ on $S^1 \times [0,T)$ for some positive constant M, then*

$$|u_{xxx}(x,t)| \le C \quad on\ S^1 \times [0,T), \tag{13}$$

where $C < \infty$ is a constant depending only on M, G, u_0 and t.

Proof. We shall let C denote any constant depending only on M, G, u_0 and t, where C may change from line to line. Let $w = \partial_t u = G(u_{xx} + u)$. Then

$$\partial_t w = H(w) \cdot (w_{xx} + w),$$

where $H = G' \cdot G^{-1}$ and $H(w) = G'(u_{xx} + u) > 0$. We compute

$$\begin{aligned} \partial_t(w_x) = \ & H(w)(w_x)_{xx} + H'(w)w_x \cdot (w_x)_x \\ & +[H(w) + H'(w) \cdot w]w_x. \end{aligned} \tag{14}$$

Since $0 < \frac{1}{C} \le w \le C$, we have $H(w) + H'(w)w \le C$ and hence the maximum principle implies

$$w_x \le C.$$

Therefore

$$u_{xxx} + u_x \le \frac{C}{G'(u_{xx} + u)},$$

and

$$u_{xxx} \le \frac{C}{G'(u_{xx} + u)} - u_x \le C.$$

The proof that $u_{xxx} \ge -C$ is similar and Lemma 5 follows. q.e.d.

Lemma 6. *Under the same assumption as in Lemma 5, we have*

$$|u^{(4)}(x,t)| \le C \quad on\ S^1 \times [0,T), \tag{15}$$

where $u^{(4)} = u_{xxxx}$, and C is a constant depending only on M, G, u_0 and t.

Proof. Since

$$w_{xx} = G'(u_{xx} + u)(u^{(4)} + u_{xx}) + G''(u_{xx} + u)(u_{xxx} + u_x)^2$$

by Lemma 5, it suffices to show $w_{xx} \le C$. We compute

$$\begin{aligned} \partial_t(w_{xx}) =& [\partial_t(w_x)]_x \\ =& [H(w)(w_x)_{xx} + H'(w)w_x(w_x)_x + (H(w) + H'(w)w)w_x]_x \\ =& H(w)(w_{xx})_{xx} + [2H'(w)w_x](w_{xx})_x + H'(w)(w_{xx})^2 \\ & + [H''(w)(w_x)^2 + H(w) + H'(w)w]w_{xx} \\ & + [2H'(w) + H''(w)w](w_x)^2. \end{aligned} \tag{16}$$

Since $w \le C$ and $w_x \le C$, the only bad term in (16) is $H'(w) \cdot (w_{xx})^2$, which is not bounded above by $C \cdot w_{xx}$. To cancel off the bad term, consider the evolution equation

$$\begin{aligned}\partial_t[\tfrac{1}{2}(w_x)^2] =& H(w)[\tfrac{1}{2}(w_x)^2]_{xx} - H(w)(w_{xx})^2 + H'(w) \cdot (w_x)^2 \cdot w_{xx} \\ &+ [H(w) + H'(w)w](w_x)^2,\end{aligned}$$

and estimate

$$\begin{aligned}&\partial_t[w_{xx} + \tfrac{\alpha}{2}(w_x)^2] \\ &\quad = H(w)[w_{xx} + \tfrac{\alpha}{2}(w_x)^2]_{xx} + [H'(w) - \alpha H(w)](w_{xx})^2 \\ &\qquad + 2H'(w) \cdot w_x \cdot [w_{xx} + \tfrac{\alpha}{2}(w_x)^2]_x \\ &\qquad + [H''(w)(w_x)^2 + H(w) + H'(w)w - \alpha H'(w)(w_x)^2]w_{xx} \\ &\qquad + [H''(w) \cdot w + 2H'(w) + \alpha H'(w)w + \alpha H(w)](w_x)^2,\end{aligned}$$

where α is a positive constant.

We can choose constant α large enough, depending on M, G, u_0, such that

$$H'(w) - \alpha H(w) \le 0 \quad \text{on } S^1 \times [0, T).$$

Let $B = w_{xx} + \frac{\alpha}{2}(w_x)^2$. We conclude that

$$\partial_t B \le H(w)B_{xx} + 2H'(w)w_x \cdot B_x + C\, w_{xx} + C,$$

whenever $B \ge 0$. Here we have used the estimate $|w_x| \le C$ from Lemma 5. Again, by the maximum principle we obtain $B \le C$ and hence

$$w_{xx} \le C.$$

The proof of the lower bound is similar and Lemma 6 follows. q.e.d.

Lemma 7. *Let $u(x,t)$ be a solution to (3)-(4) on $S^1 \times [0,T)$, for $n \ge 5$ we have*

$$\partial_t(u^{(n)}) = H \cdot (u^{(n)})_{xx} + H \cdot (u^{(n)})_x \quad + H \cdot u^{(n)} + H \quad \text{on } S^1 \times [0,T), \tag{17}$$

where each H is some expression involving only G, u, $u^{(1)}, \cdots, u^{(n-1)}$.

Proof. For $n = 5$, this follows from a straightforward computation. The case $n > 5$ can be proved by an induction argument. q.e.d.

Remark. Lemma 7 does not hold for $n \le 4$.

Based on Lemma 7, we now have

Lemma 8. *Under the same assumption as in Lemma 5, we have*

$$\left|\frac{\partial^\ell}{\partial t^\ell}\frac{\partial^k}{\partial x^k}u(x,t)\right| \leq C \quad on\ S^1\times[0,T), \tag{18}$$

where C is a constant depending only on M, G, u_0, t, k, ℓ.

Proof. Apply the maximum principle to (17) and use Lemma 6, an induction argument, and the equation $\partial_t u = G(u_{xx}+u)$. Lemma 8 then follows. q.e.d.

As an immediate consequence of Lemma 8, we infer the following existence and uniqueness result.

Proposition 2. *There exists a unique smooth solution $u(x,t)$ to equations (3)-(4) on some maximal time interval $[0,T)$ and*

$$\lim_{t\to T} u_{\min}(t) = \infty.$$

Proof. We know $u_{\min}(t)$ is increasing on $[0,T)$, and if $\lim_{t\to T} u_{\min}(t)$ is finite we will have $u(x,t)\leq M$ on $S^1\times[0,T)$ for some constant M since $u_{\max}(t) - u_{\min}(t)$ is uniformly bounded. By Lemma 8, we can extend the solution $u(x,t)$ smoothly up to $t=T$ and hence, by the short-time existence theorem, extend $u(x,t)$ beyond $t=T$, which is a contradiction. This takes care of the case where $T<\infty$. If $T=\infty$, we note that

$$\frac{\partial u}{\partial t} = G(u_{xx}+u) \geq G(\delta) > 0,$$

where δ is the constant in Lemma 1. We therefore have $\lim_{t\to\infty} u_{\min}(t) = \infty$.

The uniqueness of the solution is standard and hence the proof of Proposition 2 is complete. q.e.d.

7. Rescaling the equation and convergence in C^2

In order to understand the asymptotic behavior of the solution it will be convenient to work with the rescaled solution $\tilde{u}(x,t)$ defined below, rather than $u(x,t)$ itself. We shall see that the quantitative behavior of $u(x,t)$ is same as some solution $R(t)$ to the ODE

$$\frac{dR}{dt} = G(R(t)). \tag{20}$$

Lemma 9. *There exists a solution $R(t)$ to (20) on $[0,T)$ such that*

$$u_{\min}(t) \leq R(t) \leq u_{\max}(t),\ \forall t\in[0,T), \tag{21}$$

where $u(x,t)$ is the unique solution to (3)-(4) on the maximum time interval $[0,T)$.

Proof. First note that if $u_{\max}(t) = u(p_t,t)$ for some $p_t \in S^1$, then

$$\frac{\partial u}{\partial t}(p_t,t) = G(u_{xx}+u) \le G(u_{\max}(t)) \quad \text{at } (p_t,t) \tag{22}$$

for any $t \in [0,T)$. Similarly

$$\frac{\partial u}{\partial t}(q_t,t) \ge G(u_{\min}(t)), \tag{23}$$

where $u(q_t,t) = u_{\min}(t)$.

Take an increasing sequence $T_i \in (0,T)$ with $\lim_{i\to\infty} T_i = T$ and let $R_i^+(t)$ be the solution to

$$\frac{dR_i^+}{dt} = G(R_i^+), \tag{24}$$

$$R_i^+(T_i) = u_{\max}(T_i). \tag{25}$$

The domain of $R_i^+(t)$ will be at least $(T_i-\varepsilon_i, T_i]$ for some $\varepsilon_i > 0$. Set $f(t) = u_{\max}(t) - R_i^+(t)$ on $(T_i-\varepsilon_i, T_i]$. It will be Lipschitz on $(T_i-\varepsilon_i, T_i]$ and $f(T_i) = 0$. For any $t \in (T_i-\varepsilon_i, T_i]$, compute

$$\begin{aligned}\frac{d^-f}{dt}(t) &= \limsup_{h\searrow 0} \frac{f(t)-f(t-h)}{h}\\ &\le \limsup_{h\searrow 0} \frac{u_{\max}(t)-u_{\max}(t-h)}{h} - G(R_i^+(t)),\end{aligned}$$

and by Lemma 3 (ii) and (22) we know

$$\frac{d^-f}{dt}(t) \le G(u_{\max}(t)) - G(R_i^+(t)) \le 0, \quad \text{whenever } f(t) \le 0. \tag{26}$$

Lemma 2 (iii) now implies that

$$f(t) \ge 0 \quad \text{for all } t \in (T_i-\varepsilon_i, T_i]. \tag{27}$$

Similarly we consider $g(t) = u_{\min}(t) - R_i^+(t)$ and use Lemma 3 (iv), (23) and Lemma 2 (iv) to conclude

$$g(t) \le 0 \quad \text{for all } t \in (T_i-\varepsilon_i, T_i]. \tag{28}$$

Thus

$$u_{\min}(t) \le R_i^+(t) \le u_{\max}(t) \quad \text{on } (T_i - \varepsilon_i, T_i]. \tag{29}$$

From (29) and the basic theory of ODE's, we deduce that the domain of $R_i^+(t)$ will be at least $[0, T_i]$, and for each i we have

$$u_{\min}(t) \le R_i^+(t) \le u_{\max}(t) \quad \text{on } [0, T_i]. \tag{30}$$

Consider the sequence $\{R_i^+\}_{i=1}^{\infty}$, on any compact subinterval $[0, T-\delta]$ of $[0, T)$, the domain of R_i^+ will cover $[0, T-\delta]$ for i large enough. We may assume all R_i^+ are defined on $[0, T-\delta]$. Observe that each $R_i^+(t)$ is convex on $[0, T-\delta]$ and

$$u_{\min}(t) \le R_j^+(t) \le R_i^+(t) \le u_{\max}(t) \quad \text{on } [0, T-\delta] \tag{31}$$

for $j > i$.

Define $R^+(t) = \lim_{i\to\infty} R_i^+(t)$, $t \in [0, T-\delta]$. Then $R^+(t)$ is convex, and hence continuous on $[0, T-\delta]$, and $R_i^+ \to R^+$ uniformly on $[0, T-\delta]$. We finally conclude that $R^+(t)$ is differentiable on $[0, T-\delta]$, and $\frac{dR^+}{dt} = G(R^+)$. Let $\delta \to 0$. Then (21) follows when $T < \infty$. The case where $T = \infty$ is also clear. q.e.d.

From now on we will choose one $R(t)$ satisfying (21) and use it to rescale the solution $u(x,t)$. Define the rescaled solution $\tilde{u}(x,t)$ as

$$\tilde{u}(x,t) = \frac{u(x,t)}{R(t)}.$$

Then we have

Lemma 10. *Let $[0, T)$ be the maximal time interval of existence for $u(x,t)$. Then*

(i) $|\tilde{u}(x,t) - 1| \le \frac{C}{R(t)}$ *on* $S^1 \times [0, T)$,

(ii) $|\tilde{u}_x(x,t)| \le \frac{C}{R(t)}$ *on* $S^1 \times [0, T)$,

(iii) $|\tilde{u}_{xx}(x,t)| \le \frac{C}{R(t)}$ *on* $S^1 \times [0, T)$,

where C is a constant depending only on u_0 and $R(t) \to \infty$ as $t \to T$.

Proof. (i) By (11) we have

$$|\tilde{u}(x,t) - 1| = \left|\frac{u(x,t) - R(t)}{R(t)}\right| \le \frac{u_{\max}(t) - u_{\min}(t)}{R(t)} \le \frac{C}{R(t)}.$$

(ii) and (iii) are consequences of Corollary 1 and Lemma 4, respectively. q.e.d.

Remark. The significance of Lemma 10 is that we get convergence to the unit circle in the C^2-norm, that is,

$$\|\tilde{u}(\cdot, t) - 1\|_{C^2(S^1)} \leq \frac{C}{R(t)} \quad \forall t \in [0, T),$$

for an arbitrary positive function G with $G' > 0$ everywhere.

8. The case where G is concave

So far we have only assumed that the function G is positive and $G' > 0$. We shall now consider the case where G is a concave function. Under an additional hypothesis we shall prove that the rescaled support functions $\tilde{u}$ converge in C^∞ to the constant 1.

It is a simple exercise to verify that when G is concave, the solution $R(t)$ to the ODE $\frac{dR}{dt} = G(R)$ with $R(0) > 0$ will not blow up in finite time. Therefore the maximal time interval of existence for $u(x, t)$ is $[0, \infty)$.

Lemma 11. *If $G : \mathbf{R}_+ \to \mathbf{R}_+$ is concave with $G' > 0$ everywhere, then*

(i) $\dfrac{u_{\max}(t)}{R(t)}$ *is decreasing on* $(0, \infty)$*,*

(ii) $\dfrac{u_{\min}(t)}{R(t)}$ *is increasing on* $(0, \infty)$*,*

(iii) $\dfrac{u_{\max}(t)}{u_{\min}(t)}$ *is decreasing on* $(0, \infty)$*.*

Proof. (i) $u_{\max}(t)/R(t)$ is Lipschitz on any compact interval $[a, b] \subset (0, \infty)$. Using Lemma 3 (i), we find

$$\frac{d^+}{dt}\left(\frac{u_{\max}(t)}{R(t)}\right) \leq \frac{R(t)G(u_{\max}(t)) - u_{\max}(t)G(R(t))}{R^2(t)}. \tag{32}$$

Since G is concave, we have

$$\lambda G(x) \leq G(\lambda x) \tag{33}$$

for all $x \in \mathbb{R}_+ = (0, \infty)$, $0 \leq \lambda \leq 1$. Applying (33) together with $R(t)/u_{\max}(t) \leq 1$ to (32), we get

$$\frac{d^+}{dt}\left(\frac{u_{\max}(t)}{R(t)}\right) \leq 0 \quad \text{for all } t \in (0, \infty), \tag{34}$$

which and Lemma 2 (i) imply (i). The proof of (ii) is analogous to that of (i). (iii) is an easy consequence of (i) and (ii). q.e.d.

In the following discussion, we shall use the Banach spaces of k-times (Hölder) continuously differentiable functions on S^1 and $S^1 \times I$, $C^k(S^1)$, $C^{k,\alpha}(S^1)$, $\tilde{C}^k(S^1 \times I)$, and $\tilde{C}^{k,\alpha}(S^1 \times I)$, where $I = [a, b] \subset \mathbf{R}$, equipped with the standard norms. See, for example, Urbas [24] for detailed definitions.

Lemma 12. *For any $0 \leq T_0 < \infty$, we have*

$$\|\tilde{u} - 1\|_{\tilde{C}^2(S^1 \times [T_0, \infty))} \leq \frac{C}{R(T_0)}, \tag{35}$$

where C is a constant depending only on u_0 and G.

Proof. By Lemma 10 we have

$$\|\tilde{u}(x, t) - 1\|_{C^2(S^1)} \leq \frac{C}{R(t)} \leq \frac{C}{R(T_0)} \tag{36}$$

for all $t \in [T_0, \infty)$. Hence we only need to estimate the time-derivative of $\tilde{u}$ on $S^1 \times [T_0, \infty)$. The equation for the rescaled solution $\tilde{u}(x, t)$ is

$$\frac{\partial \tilde{u}}{\partial t} = -\frac{1}{R} G(R)\tilde{u} + \frac{1}{R} G(R(\tilde{u}_{xx} + \tilde{u})). \tag{37}$$

Applying (36) to (37) yields

$$\begin{aligned} \frac{\partial \tilde{u}}{\partial t}(x, t) \leq & - \frac{G(R(t))}{R(t)}\left(1 - \frac{C}{R(t)}\right) + \frac{1}{R(t)} G(R(t) + C) \\ = & \frac{G(R(t) + C) - G(R(t))}{R(t)} + C \cdot \frac{G(R(t))}{R(t)^2}. \end{aligned}$$

Since G is concave, $G(y + C) - G(y) \leq C'$ independent of $y \in [R(0), \infty)$, and $G(y)/y$ is a decreasing function of y on $(0, \infty)$. Therefore

$$\frac{\partial \tilde{u}}{\partial t}(x, t) \leq \frac{C}{R(t)} \leq \frac{C}{R(T_0)}$$

for all $t \in [T_0, \infty)$, where C depends only on u_0. Similarly, one can show that

$$\frac{\partial \tilde{u}}{\partial t}(x,t) \geq -\frac{C}{R(T_0)}$$

for all $t \in [T_0, \infty)$. This completes the proof of the lemma. q.e.d.

Unfortunately, the linearized equation of (37) will not be uniformly parabolic on the domain $S^1 \times [0, \infty)$. This makes us unable to quote some standard theorems for parabolic differential equations. To remedy this we need to rescale time also.

Let

$$\tau(t) = \log G(R(t)) - \log G(R(0))$$

for all $t \in [0, \infty)$. We have $\tau(0) = 0$, $d\tau/dt = G'(R(t)) > 0$ and $\lim_{t\to\infty} \tau(t) = \infty$ if we assume $\lim_{x\to\infty} G(x) = \infty$.

Set $\hat{u}(x,\tau) = \tilde{u}(x, t(\tau))$. We have $\hat{u}_x(x,\tau) = \tilde{u}_x(x, t(\tau))$, $\hat{u}_{xx}(x,\tau) = \tilde{u}_{xx}(x, t(\tau))$ and

(38)
$$\frac{\partial \hat{u}}{\partial \tau} = \frac{\partial \tilde{u}}{\partial t}\frac{\partial t}{\partial \tau} = \frac{1}{G'(R)}\left[-\frac{1}{R}G(R)\cdot \tilde{u} + \frac{1}{R}G(R(\tilde{u}_{xx} + \tilde{u}))\right].$$

Hence

$$\frac{\partial \hat{u}}{\partial \tau} = \frac{1}{G'(R)}\left[-\frac{1}{R}G(R)\cdot \hat{u} + \frac{1}{R}G(R(\hat{u}_{xx} + \hat{u}))\right]. \tag{39}$$

If we linearize the above equation at $\hat{u}$, we obtain the equation

(40)
$$\begin{aligned}\frac{\partial v}{\partial t} &= \frac{1}{G'(R)}\left[-\frac{1}{R}G(R)\cdot v + \frac{1}{R}G'(R(\hat{u}_{xx} + \hat{u}))\cdot R(v_{xx} + v)\right] \\ &= \frac{G'(R(\hat{u}_{xx} + \hat{u}))}{G'(R)} v_{xx} + \left[\frac{G'(R(\hat{u}_{xx} + \hat{u}))}{G'(R)} - \frac{G(R)}{RG'(R)}\right] v.\end{aligned}$$

Lemma 13. *If there exists a constant $\varepsilon > 0$ such that $y \cdot G'(y) \geq \varepsilon > 0$ for y sufficiently large, then*

$$\|\hat{u}(x,\tau)\|_{\tilde{C}^2(S^1\times[0,\infty))} \leq C, \tag{41}$$

where C is a constant depending only on u_0, and ε.

Proof. Since $\hat{u}$ differs from $\tilde{u}$ only by a change in the time variable, by Lemma 12, it suffices to estimate $|\partial\hat{u}/\partial\tau|$. Using Lemma 12 again together with the hypothesis on G, we obtain

$$\left|\frac{\partial \hat{u}}{\partial \tau}(x,\tau)\right| = \frac{\partial t}{\partial \tau}(\tau)\cdot\left|\frac{\partial \tilde{u}}{\partial t}(x,t(\tau))\right| \leq \frac{1}{G'(R(t))}\cdot\frac{C}{R(t)} \leq \frac{C}{\varepsilon}. \quad \text{q.e.d.}$$

Remark. The hypothesis on G in Lemma 13 is equivalent to the condition that there exists $\varepsilon > 0$ such that $G(y) - \varepsilon \log y$ is an increasing function of y on some interval $[T_0, \infty)$.

Lemma 14. *Let C be a constant depending only on u_0 such that*

$$R(t) - C \leq R(t)(\tilde{u}_{xx}(x,t) + \tilde{u}(x,t)) \leq R(t) + C,$$

for all $x \in S^1$, $t \geq 0$, as given by Lemma 12. If there exists a constant $\delta > 0$ such that

$$\frac{G'(y+C)}{G'(y)} \geq \delta > 0$$

for y sufficiently large, then equation (40) is uniformly parabolic on $S^1 \times [0, \infty)$, and the coefficient of v is uniformly bounded from above.

Proof. By our hypothesis, using the concavity of G, we have for t sufficiently large

$$\frac{1}{\delta} \geq \frac{G'(R-C)}{G'(R)} \geq \frac{G'(R(\hat{u}_{xx} + \hat{u}))}{G'(R)} \geq \frac{G'(R+C)}{G'(R)} \geq \delta > 0.$$

The upper bound for the coefficient of v also follows since $G(R)/(RG'(R)) > 0$. q.e.d.

Remark. The hypothesis on G in Lemma 14 is equivalent to the condition that there exists a constant $\delta > 0$ such that $G(y+C) - \delta G(y)$ is an increasing function of y on some interval $[T_0, \infty)$.

Now we are in a position to use the result of Krylov and Safonov [20] and standard parabolic theory to conclude (see also Urbas[24], Lemmas 3.9 and 3.10).

Lemma 15. *Suppose that there exist $\varepsilon, \delta > 0$ such that both $G(y) - \varepsilon \log y$ and $G(y+C) - \delta G(y)$ are increasing functions of y for y sufficiently large. Let $\hat{u}(x, \tau)$ be the solution of (38) on $S^1 \times [0, \infty)$. Then for any $\tau \in (0, \infty)$, any positive integer k, and any $\alpha \in (0,1)$ we have*

$$\|\hat{u}\|_{\tilde{C}^{k,\alpha}(S^1 \times [\tau, \infty))} \leq C,$$

where C is a constant depending only on $k, \alpha, \varepsilon, \delta, G, 1/\tau$ and $\|\hat{u}\|_{\tilde{C}^2(S^1 \times [0,\infty))}$.

To obtain the convergence of $\hat{u}(x, \tau)$ to 1 in the C^k norm, we follow the arguments in Urbas [24], apply an interpolation inequality of Hamilton [16] and the estimate in Lemma 10 (i) to Lemma 15, and conclude (we refer to Urbas [24] for the details).

Proposition 3. *Under the same assumption as in Lemma 15, we have*

$$\|\hat{u}(x,\tau) - 1\|_{C^k(S^1)} \leq \frac{C}{R(t(\tau))^\sigma}$$

for any $\sigma < 1$, where C is a constant depending only on k, σ, G, u_0.

Added in proof. Recently the second author generalized the results of this paper to starshaped plane curves (see Tsai [21]). Even more recently the results have been further generalized to embedded plane curves with turning angle greater than $-\pi$ (see Chow-Liou-Tsai [9]). See K. S. Chou and X. P. Zhu [4], [5] for recent results for anisotropic flows of plane curves.

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