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DEHN SURGERY ON ARBORESCENT KNOTS

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1. Introduction

A knot K is called an *arborescent knot* if it can be obtained by summing and gluing several rational tangles together; see [7] or below for more detailed definitions. Recall that a 3-manifold is called a Haken manifold if it is irreducible and contains an incompressible surface. Following Hatcher [14] we say that a 3-manifold M is *laminar* if it contains an essential lamination. The purpose of this paper is to study Dehn surgeries on arborescent knots, and to see which of these surgered manifolds are laminar, Haken, or hyperbolic.

There has been some study on these problems for Montesinos knots. Denote by $K = K(p_1/q_1, \dots, p_n/q_n)$ a Montesinos knot obtained by gluing rational tangles corresponding to the rational numbers p_i/q_i together in a cyclic way; see for example [24] for more details. To avoid the trivial case, we always assume that $|q_i| \geq 2$. We call n the length of K. Oertel [24] showed that if $n \leq 3$, then there are no closed essential surfaces in the knot exterior $E(K) = S^3 - \operatorname{Int} N(K)$, and if $n \geq 4$ and $|q_i| \geq 3$, then there are incompressible surfaces which remain incompressible after all nontrivial surgeries. Delman [4], [5] studied essential laminations in $E(K)$, the exterior of K, showing that for most Montesinos knots there are essential laminations in $E(K)$ which remain essential after all nontrivial surgeries. The result is particularly interesting for those K with $n \leq 3$, because by the results of Oertel [24] and Hatcher [13] most of these surgered manifolds are nonHaken manifolds.

For our purpose we divide arborescent knots into three types. Type I knots are those Montesinos knots which have length at most 3. A knot is of type II if it is of the form shown in Figure 1.1, where $R(p_i/q_i)$ are rational tangles with $|q_i| \geq 2$, and B is any 4-string braid from the left

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to the right such that the resulting link is a knot. In other words, a knot is of type II if it is the union of two tangles, each of which is a sum of a (1/2)-tangle and a rational tangle. All the other arborescent knots are called type III knots. We will mainly study surgeries on type II or III knots.

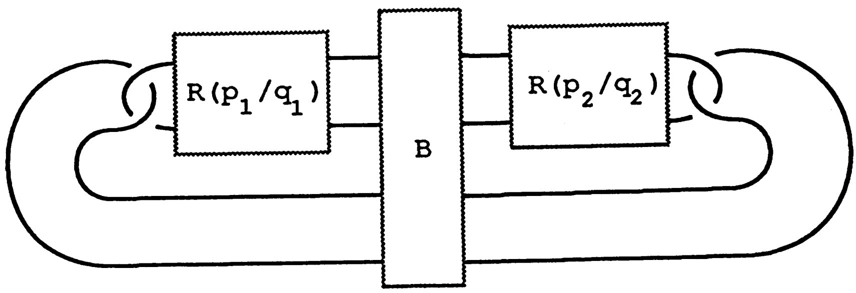

FIGURE 1.1

Theorem 2.4. *Let K be an arborescent knot. If K is not a Montesinos knot of length at most 3, then $K(\gamma)$ is laminar for all non-trivial slopes γ.*

Remark. The following knots in the knot table [25] satisfy the hypothesis of Theorem 2.4: 8_{16}, 8_{17}, 9_{29}, 9_{32}, 9_{33}, 9_{38}, 10_{79} – 10_{97}, and 10_{148} – 10_{154}.

Corollary 2.5. *All arborescent knots K have property P, i.e, $\pi_1(K(\gamma)) \neq 1$ for all nontrivial γ.*

Corollary 2.6. *The cabling conjecture is true for arborescent knots, that is, if K is a nontorus arborescent knot, then $K(\gamma)$ is irreducible for all γ.*

Remark. The property P conjecture says that all nontrivial knots have property P. Modulo the Poincaré conjecture, this would follow from the Gordon-Luecke theorem that knots are determined by their complements [11]. Other classes of knots for which the conjecture has been proved include satellite knots [8], and symmetric knots [3]. Recently Delman and Roberts proved it for alternating knots.

The cabling conjecture says that if K is not a cable knot or torus knot, then all surgeries on K produce irreducible manifolds. It has been proved for satellite knots [26], alternating knots [22], strongly invertible knots [6], and those knots with bridge number at most 4 [12].

In most cases, a stronger result than Theorem 2.4 holds.

Theorem 3.6. *If K is a type III arborescent knot, then $K(\gamma)$ is a hyperbolic Haken manifold for all nontrivial γ. In particular, this is true for all Montesinos knots $K = K(p_1/q_1, \dots, p_n/q_n)$ with $q_i \geq 2$ and $n \geq 4$.*

Theorem 4.4. *If K is a type II arborescent knot, then $K(\gamma)$ is a hyperbolic Haken manifold for all non-integral slopes γ.*

Theorem 4.4 is not true for integral surgeries on type II knots. There are infinitely many isotopy classes of connected, closed, incompressible surfaces in any type II knot complement, but none of them can survive under any integral surgery.

Theorem 4.8. *If K is a type II arborescent knot, then all closed incompressible surfaces in $E(K)$ are compressible in $K(\gamma)$ for all integral slopes γ.*

Remark. Theorem 4.8 was proved by Lopez [20] for a subclass of type II knots. The proof there is not complete, as the author does not seem to have noticed that there are infinitely many incompressible surfaces in the knot complement.

Combining Theorem 4.8 with a theorem of Hatcher [13], we see that all but finitely many integral surgeries on a type II knot produce non Haken laminar manifolds.

We will use tangles to prove the above theorems. Theorem 2.4 follows from a more general result: If K is the union of two nonsplit tangles, then either K is some $(2, q)$ cable of a composite knot, or $E(K)$ has essential laminations which remain essential after all nontrivial surgeries. Note that in the first case an incompressible torus in $E(K)$ remains incompressible after all nontrivial surgeries, but the surgery along the cabling slope produces a reducible manifold, so it is not laminar. But clearly this is the only "bad" surgery besides the trivial one. Theorem 3.6 is a consequence of Theorem 3.3, which states that if K is the union of two nontrivial atoroidal tangles, and at least one of the tangles is ∂-irreducible, then all surgeries on K are hyperbolic and Haken.

The purpose of the remaining part of this section is to give some definitions and conventions. We refer the reader to [17] for basic concepts about 3-manifolds.

If X is a subset of a 3-manifold M, we use $N(X)$ and $|X|$ to denote a regular neighborhood of X and the number of components in X respectively. Let K be a knot in M. A slope γ is an isotopy class of simple closed curves on $\partial N(K)$. A slope γ is *nontrivial* if it is not the meridional slope of K. It is called an integral slope if it intersects the meridional slope of K just once. We use $(M, K; \gamma)$ to denote the manifold obtained from M by surgery on K along γ, that is, $(M, K; \gamma) = (M - \operatorname{Int} N(K)) \cup (S^1 \times D^2)$, where γ bounds a disk in the solid torus $S^1 \times D^2$. When $M = S^3$, the surgered manifold $(M, K; \gamma)$ is simply denoted by $K(\gamma)$.

We define a *tangle* to be a pair (B, T), where B is a 3-ball, and $T = t_1 \cup t_2$ is a pair of arcs, called strings, properly embedded in B. When there is no confusion we also call T a tangle. T is called a *trivial tangle* if it is properly isotopic to a pair of arcs on ∂B. Denote by $E(T)$ the *tangle space* $B - \operatorname{Int} N(T)$. We say that T is *∂-reducible* if $E(T)$ has compressible boundary, otherwise it is *∂-irreducible*. Recall that a closed or properly embedded surface in a 3-manifold M is called an *essential surface* if it is incompressible, ∂-incompressible, and not parallel to a surface on ∂M. A 3-manifold M is atoroidal if it contains no essential tori. A tangle T is said to be *atoroidal* if $E(T)$ is atoroidal.

A *marked tangle* is a triple (B, T, Δ), where (B, T) is a tangle, and Δ is a disk on ∂B containing two endpoints of T. A marked tangle is called a rational tangle if its underlying tangle (B, T) is trivial. We assign a rational number or ∞ to the tangle as follows. Suppose the string t_1 of T is rel ∂t_1 isotopic (in $B - t_2$) to an arc α on ∂B. Let F be a torus whose double branch covers ∂B with the branch set ∂T. Let m be a component of the lifting of $\partial \Delta$, and let l be a curve on F intersecting m once. Orient m, l so that the intersection number of m with l is $+1$ with respect to the orientation of F induced from a fixed orientation of ∂B. Then the lifting of α represents some $pl + qm$ in $H_1(F)$. We say that (B, T, Δ) is a p/q rational tangle, and use $R(p/q)$ to denote it. Because of the ambiguity of the choice of l, the number p/q is defined mod $\mathbb{Z}$. Thus $R(r) = R(r')$ if and only if $r = r'$ mod $\mathbb{Z}$. The tangles in Figure 1.2 are the 0-tangle, ∞-tangle and $(1/5)$-tangle, respectively.

One can check that if a tangle is a (p,q) rational tangle in the usual sense (see e.g. [2] or [16]), and we choose the left-hand side disk as the disk Δ, then it is an $R(p/q)$ according to our definition.

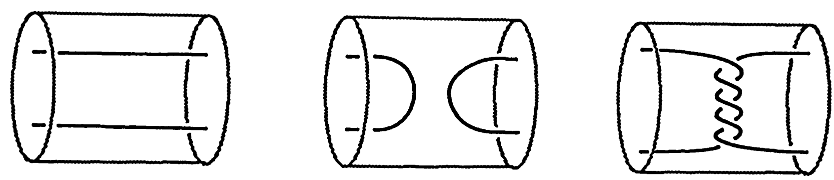

FIGURE 1.2

Given two tangles (B_1, T_1) and (B_2, T_2), we can choose a disk Δ_i on ∂B_i to form marked tangles (B_i, T_i, Δ_i), and then glue the two disks Δ_i together to form a new tangle (B, T). We say that (B, T) is the sum of (B_1, T_1, Δ_1) and (B_2, T_2, Δ_2), and write $(B, T) = (B_1, T_1, \Delta_1) + (B_2, T_2, \Delta_2)$ or simply $T = T_1 + T_2$. This process depends on the choice of Δ_i and the gluing map. When neither of (B_i, T_i, D_i) is $R(0)$ or $R(\infty)$, we say that the sum is a nontrivial sum. A tangle is called an *algebraic tangle* if it is obtained by nontrivially summing rational tangles together in various ways. Thus a sum of algebraic tangles is still an algebraic tangle. Define the length $L(T)$ of an algebraic tangle T as follows. $L(T) = 1$ if T is a rational tangle. In general, if $T = T_1 + T_2$ is a nontrivial sum, then $L(T) = L(T_1) + L(T_2)$. It can be shown that the length of an algebraic tangle is well defined.

Given two tangles (B_1, T_1) and (B_2, T_2), we may glue the boundaries of the B_i together to get a knot or link K in S^3. In this case K is called a *union* of T_1 and T_2, and we write $K = T_1 \cup T_2$. Again, K depends on the gluing map $\partial B_1 \to \partial B_2$. From Figure 1.1 one can see that an arborescent knot K is of type II if and only if it is a union of two tangles T_1 and T_2, and each T_i is a sum $R(1/2) + R(p_i/q_i)$.

A knot K is called an arborescent knot if it is the union of two algebraic tangles. This is equivalent to the definition given in [7]. Note that Montesinos knots [23], which are also called star knots [24], are a special kind of arborescent knots. A Montesinos knot $K(p_1/q_1, \dots, p_n/q_n)$ is obtained by gluing n rational tangles with associated rational numbers $p_1/q_1, \dots, p_n/q_n$ together in a cyclic way, where $q_i \geq 2$. We call n the length of K.

2. Essential laminations after surgery

A tangle (B, T) is a *split tangle* if there is a disk in B separating the two strings. (B, T) is called a *parallel tangle* if T is a pair of parallel knotted arcs. Suppose a knot K in S^3 is a union of two nonsplit tangles T_1 and T_2. In this section we will show that in most cases there are essential laminations in $K(\gamma)$. More explicitly, if K is not a $(2, q)$ cable of a composite knot, then all nontrivial surgeries on K produce laminar manifolds. See [9] for definitions and properties of essential laminations.

We first consider the case that one of the T_i is toroidal.

Lemma 2.1. *Suppose $K = T_1 \cup T_2$, where T_i are non-split tangles. If T_1 is toroidal, then one of the following holds.*

(a) Both T_i are parallel tangles, so K is some $(2, q)$ cable of a composite knot;

(b) $K(\gamma)$ is a Haken manifold for all $\gamma \neq \infty$.

Proof. As before, we use $E(T_i)$ to denote the tangle space $B^3 - \operatorname{Int} N(T_i)$. Let P be the punctured sphere $E(T_1) \cap E(T_2)$ in the knot exterior $E(K) = S^3 - \operatorname{Int} N(K)$. Let F be an essential torus in $E(T_1)$. Since the T_i are nonsplit, P is incompressible in $E(K)$, so F is also incompressible in $E(K)$. Let V be the (knotted) solid torus in S^3 bounded by F. Then K is a knot in V, and P is an incompressible surface in $V - \operatorname{Int} N(K)$. One can show that this implies that K is not a closed braid in V. By a theorem of Gabai [8], F remains incompressible after all nontrivial surgeries on K. Hence (b) follows unless $(V, K; \gamma)$ is reducible. If $(V, K; \gamma)$ is reducible, by a theorem of Scharlemann [26], K is some (p, q) cable of a knot K' in V. Let V' be a regular neighborhood of K' containing K. Isotope P to minimize its intersection with $\partial V'$. Since K is a closed braid in V', P cannot lie in V', so $k = |P \cap \partial V'| \geq 2$. Each component of $P \cap V'$ intersects $\partial N(K)$ just p times, so

$|P \cap \partial N(K)| = kp$. As P is a four-punctured sphere, we have $k = p = 2$. It is now easy to see that conclusion (a) holds.

Lemma 2.2. *Suppose (B, T) is a nontrivial atoroidal tangle with t_1, t_2 as the strings. Let m_i be a meridian of the string t_i on $\partial E(T)$. Then at least one of the $\partial E(T) - m_j$ $(j = 1, 2)$ is incompressible.*

Proof. If $\partial E(T)$ is incompressible, then both $\partial E(T) - m_j$ are incompressible. So assume $\partial E(T)$ is compressible. Cutting along a compressing disk D, we get a manifold with one or two tori as boundary depending on whether D is separating. Since T is assumed atoroidal and $E(T)$ is irreducible, each of the tori bounds a solid torus. Therefore, $E(T)$ is a handlebody of genus two.

Suppose $\partial E(T) - m_1$ is compressible. After cutting along a compressing disk, we get one or two solid tori, so m_1 lies on the boundary of a solid torus V. We claim that m_1 is a primitive curve of ∂V, i.e, it intersects a meridian disk of V just once. For if m_1 were not primitive, by attaching a 2-handle along m_1 we would get a manifold W with $\partial W = S^2$ and $\pi_1 W \neq 1$. But attaching a 2-handle along m_1 is the same as to refill $N(t_1)$ back into V, so W would be a summand of the 3-ball B, which is absurd.

Thus, if both $\partial E(T) - m_1$ and $\partial E(T) - m_2$ are compressible, then m_1 and m_2 are primitive curves on the boundary of the handlebody $E(T)$. Moreover, when attaching 2-handles to both m_1 and m_2, we get the 3-ball B. From Lemma 2.3.2 of [3] or Theorem 1 of [10] it now follows that the set $m_1 \cup m_2$ is standard, in the sense that there is a disk D cutting $E(T)$ into two solid tori, each containing an m_i as a primitive curve. But this implies that T is a trivial tangle, contradicting the assumption of the lemma.

Theorem 2.3. *Let $K \subset S^3$ be the union of two nonsplit tangles T_1 and T_2. Suppose that at least one of the T_i is not a parallel tangle. Then there is an essential lamination $\mathcal{L}$ in $E(K)$ which remains essential after all nontrivial surgeries on K.*

Proof. If one of the tangles T_i is toroidal, the result follows from Lemma 2.1 because a Haken manifold is laminar. So we assume that both T_i are atoroidal. Let t_1, t_2 be the strings of T_1. Let U_i be the annulus $\partial N(t_i) \cap \partial E(T_1)$. Similarly, let V_i be the annulus $\partial N(s_i) \cap \partial E(T_2)$, where s_1, s_2 are the strings of T_2. By Lemma 2.2, one of the $\partial E(T_1) - U_j$ (resp. $\partial E(T_2) - V_j$) is incompressible. Without loss of generality we may assume that $\partial E(T_1) - U_1$ and $\partial E(T_2) - V_1$ are incompressible.

The proof of Theorem 2.3 is divided into four steps. In Step 1 we construct a branched surface $\mathcal{B}$ in $E(K)$. Step 2 shows that it fully carries a lamination. We then prove in Step 3 that $\mathcal{B}$ is essential in $E(K)$. Finally in Step 4 it will be shown that $\mathcal{B}$ remains essential after all nontrivial surgeries on K. This will complete the proof of Theorem 2.3 because by [9] any lamination fully carried by an essential branched surface is essential.

Step 1. Construction of essential branched surfaces.

Figure 2.1 indicates a part of $N(K)$ and the part of the surface P in a neighborhood of $N(K)$. The surface P cuts $E(K)$ into $E(T_1)$ and $E(T_2)$, and cuts the torus $\partial N(K)$ into the four annuli U_1, V_1, U_2, V_2, as shown in Figure 2.1.

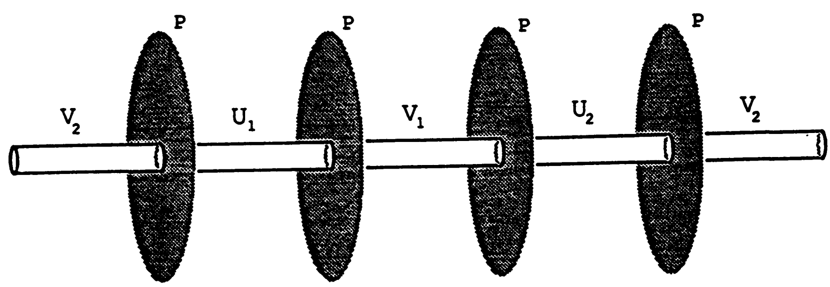

FIGURE 2.1

We take the branched surface $\mathcal{B}$ to be the same as P outside of some neighborhood of $N(K)$. Inside of this neighborhood $\mathcal{B}$ is as shown in Figure 2.2. It can be constructed as follows. Take the union of P with $U_1 \cup V_1 \cup U_2$. There are two branch curves c_1 and c_2, where $c_1 = U_1 \cap V_1 \cap P$, and $c_2 = V_1 \cap U_2 \cap P$. Smooth this branched surface so that at c_1 the cusp is in the corner between P and U_1, and the cusp at c_2 is in the corner between P and V_1. We then push the resulting branched surface into the interior of $E(K)$ to obtain the required branched surface $\mathcal{B}$.

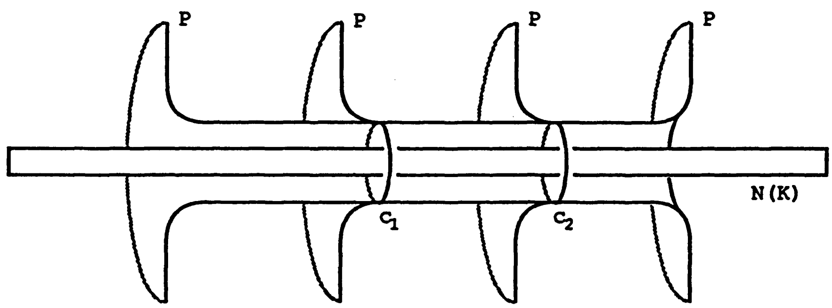

FIGURE 2.2

Step 2. $\mathcal{B}$ fully carries a lamination.

Cutting the branched surface $\mathcal{B}$ along the branch curves c_1 and c_2, we get a surface F which is homeomorphic to the disjoint union of P and V_1. We can construct a regular neighborhood $N(\mathcal{B})$ as follows. Let $F \times I$ be a product neighborhood of F. The three branches at c_1 give rise to three boundary components of F, which in turn determine three annulus components H_1, H_2, H_3 of $\partial F \times I$. Write H_i as $S^1 \times I_i$. Let H_3 be the component on $\partial V_1 \times I$. Choose an injective map $\varphi : I_1 \cup I_2 \to I_3$. Then using $\mathrm{id} \times \varphi : H_1 \cup H_2 \to H_3$ we can glue the two annuli $H_1 \cup H_2$ to H_3. Gluing the three annuli near c_2 together in a similar way, we obtain a manifold homeomorphic to a regular neighborhood of $\mathcal{B}$. Clearly, the I-bundle structure of $F \times I$ gives rise to the I-bundle structure of $N(\mathcal{B})$.

Now let $\mathcal{L}'$ be the set $F \times K \subset F \times I$, where K is a Cantor set in I. On the annulus H_i, $\mathcal{L}'$ is a product $S^1 \times K_i$, where K_i is a Cantor set in I_i. By the property of Cantor set, we can choose the map $\varphi : I_1 \cup I_2 \to I_3$ in such a way that $\varphi(K_1 \cup K_2) = K_3$. Choose the gluing map near c_2 in a similar way. Then the quotient of $\mathcal{L}'$ in $N(\mathcal{B})$ is a lamination $\mathcal{L}$ which is transverse to the I-bundle structure, and intersects all I-fibers. Hence $\mathcal{L}$ is a lamination fully carried by the branched surface $\mathcal{B}$.

Step 3. $\mathcal{B}$ is essential in $E(K)$.

Recall the construction of $\mathcal{B}$. Outside of a neighborhood of $N(K)$ $\mathcal{B}$ is the same as P, and inside of the neighborhood $\mathcal{B}$ is as shown in

Figure 2.2. There is a torus T parallel to $\partial N(K)$, containing the part of $\mathcal{B}$ in Figure 2.2, which is parallel to $\partial N(K)$. The surface $\mathcal{B} \cup T$ is topologically the same as the surface $P \cup \partial N(K)$ shown in Figure 2.1. Let V_2' be the part of T which does not lie on $\mathcal{B}$. Let X be the manifold obtained by cutting $E(K)$ along the branched surface $\mathcal{B}$. Then topologically X is obtained by first cutting $E(K)$ along $\mathcal{B} \cup T$, and then gluing back along the annulus V_2'. In the first step we cut $E(K)$ into three pieces. The part inside of T is a product $T \times I$. The other two components are homeomorphic to $E(T_1)$ and $E(T_2)$, and will still be denoted by $E(T_1)$ and $E(T_2)$ respectively. In the second step we glue $T \times I$ to $E(T_2)$ along the annulus V_2'. Note that V_2' is identified with V_2 on $\partial E(T_2)$, and is a meridional annulus on $T \times I$ (i.e, an essential curve of V_2' is isotopic to a meridian of K.) Thus, X has two components: $E(T_1)$, and $Y = E(T_2) \cup_{V_2} (T \times I)$.

Let F_h and F_v be the horizontal and vertical surfaces on ∂X respectively; see [9] for definitions. Since $\mathcal{B}$ has two branch loci c_1 and c_2, F_v has two components. One can see from Figure 2.2 that the component corresponding to c_1 lies on $\partial E(T_1)$ and is isotopic to U_1, while the one corresponding to c_2 lies on $E(T_2) \subset Y$ and is isotopic to V_1. By definition the horizontal surface is $F_h = \partial N(\mathcal{B}) - F_v$. Therefore $F_h \cap E(T_1) = \partial E(T_1) - U_1$, and $F_h \cap Y$ is the component of $\partial Y - V_1$ other than $\partial N(K)$.

According to [9], $\mathcal{B}$ is essential if the following conditions hold. (We split condition (ii) of [9] into (ii) and (ii′) below.)

(i) $\mathcal{B}$ has no disk of contact;

(ii) F_h is incompressible, and has no sphere component;

(ii′) F_h has no monogons;

(iii) X is irreducible;

(iv) $\mathcal{B}$ contains no Reeb branched surface;

(v) $\mathcal{B}$ fully carries a lamination.

We remark that condition (ii′) can be replaced by

(ii″) No component X' of X is a solid torus with $F_v \cap X'$ a longitudinal annulus.

One is referred to the proposition in section 2 of [1] for a proof of this fact. (ii″) is much easier than (ii′) to check.

If $\mathcal{B}$ had a disk of contact, the central curve of some component of

F_v would bound a disk in $E(K)$. In our case both components of F_v are isotopic to a meridional annulus in $E(K)$, so their central curves are homotopically nontrivial. This proves (i).

Since $E(T_1)$ is a tangle space, it is irreducible. Also, by our assumption at the beginning of the section, $\partial E(T_1) - U_1$ is incompressible. Therefore (ii) and (iii) are true for the component $E(T_1)$ of X. To prove them for the component Y of X, we use the following well known fact: If W is a 3-manifold, and S is an essential surface in W, then W is irreducible and ∂-irreducible provided that the manifold obtained by cutting W along S is. Consider the (noncompact) manifold $Y - V_1$. Since $F_h \cap Y$ is a component of $\partial(Y - V_1)$, conditions (ii) and (iii) will follow if $Y - V_1$ is irreducible and ∂-irreducible. Now $Y - V_1 = (E(T_2) - V_1) \cup_{V_2} (T \times I)$. One can easily show that V_2 is essential in $Y - V_1$. Since both $E(T_2) - V_1$ and $T \times I$ are irreducible and ∂-irreducible, (ii) and (iii) are proved.

In our case, both components of X have some genus-two boundary components, so they cannot be solid tori. This proves (ii$''$).

Since no component of F_h is a disk, by Remark 1.3 of [9] (iv) is true.

(v) was proved in Step 2.

Step 4. $\mathcal{B}$ remains essential after surgery.

As before, we use $K(\gamma)$ to denote the manifold obtained from S^3 by Dehn surgery on K along the slope γ. Let $X(\gamma)$ (resp. $Y(\gamma)$) be the manifold obtained by Dehn filling on X (resp. Y) with slope γ. Thus $X(\gamma) = K(\gamma) - \operatorname{Int} N(\mathcal{B})$. We want to show that $\mathcal{B}$ is essential as a branched surface in $K(\gamma)$. Some of the conditions listed in Step 3 are quite easy to check. Conditions (i) and (v) depend only on the branched surface $\mathcal{B}$, not on the manifold in which it is embedded, so they still hold for $\mathcal{B}$ in $K(\gamma)$. (ii$''$) is also obvious because each component of $X(\gamma)$ still has a non torus boundary component. (iv) again follows from Remark 1.3 of [9] and the fact that F_h has no disk components. The component $E(T_1)$ of X is unchanged in $X(\gamma)$, so (ii) and (iii) are true for this component of $X(\gamma)$. It remains to show that $Y(\gamma)$ is irreducible, and $F' = F_h \cap Y(\gamma)$ is incompressible in $Y(\gamma)$. Note that $F' = (\partial Y - V_1) - \partial N(K) = \partial Y(\gamma) - V_1$.

Consider the trivial surgery $Y(m)$, where m is the meridional slope of K. Since $Y = E(T_2) \cup_{V_2} (T \times I)$ and V_2 is a meridional annulus on $T \times I$, a meridian disk of K extends to a compressing disk D' of F' in $Y(m)$. Since $|D' \cap K| = 1$, K cannot be a cable knot in $Y(m)$. By a theorem of Scharlemann [26], $Y(\gamma)$ is irreducible for all $\gamma \neq m$. There is an annulus in Y with one boundary on F' and the other a meridian m on $\partial N(K)$, so by [28] the surface F' is incompressible in $Y(\gamma)$ if $\Delta(\gamma, m) \geq 2$.

It remains to show that F' is incompressible in $Y(\gamma)$ if $\Delta(\gamma, m) = 1$. Recall that $Y = E(T_2) \cup_{V_2} (T \times I)$. We have $Y(\gamma) = E(T_2) \cup_{V_2} ((T \times I)(\gamma))$. Clearly, $(T \times I)(\gamma)$ is a solid torus. Since the central curve of V_2 is isotopic to the meridian m of K, it intersects a meridian of the new solid torus $(T \times I)(\gamma)$ just once, so V_2 is a longitudinal annulus on $(T \times I)(\gamma)$. Therefore gluing $(T \times I)(\gamma)$ to $E(T_2)$ does not affect the manifold. In other words, $Y(\gamma)$ is homeomorphic to $E(T_2)$. Under this homeomorphism, the surface $F' = \partial Y(\gamma) - V_1$ is mapped to $\partial E(T_2) - V_1$. By the assumption at the beginning of the section, $\partial E(T_2) - V_1$ is incompressible in $E(T_2)$. Therefore, F' is incompressible in $Y(\gamma)$.

This completes the proof of Theorem 2.3.

Theorem 2.4. *Let K be an arborescent knot. If K is not a Montesinos knot of length at most 3, then $K(\gamma)$ is laminar for all non-trivial slopes γ.*

Proof. We claim that if K is not a Montesinos knot of length at most 3, then it is a union of two nontrivial algebraic tangles.

By definition K is the union of two algebraic tangles T_1 and T_2. If T_1 is trivial, and T_2 has length at most 2, then K is a Montesinos knot of length at most 3. If T_2 has length at least 3, then T_2 can be written as $T' + T''$, with $2 \leq L(T'') < L(T_2)$. Since T_1 intersects T' at two points, we have $K = (T_1 + T') + T''$. If $T_1 + T'$ is still a trivial tangle, we can proceed by induction, since $L(T'') < L(T_2)$. This proves the claim.

Now suppose $K = T_1 \cup T_2$ and both T_i are nontrivial. By Lemma 3.2 of [31], a sum of atoroidal tangles is still atoroidal. Thus all algebraic tangles are atoroidal. In particular, they cannot be parallel tangles. Since a nontrivial split tangle is toroidal, T_i must be nonsplit, so the result follows from Theorem 2.3.

Corollary 2.5. *All arborescent knots K have property P, i.e, $\pi_1(K(\gamma)) \neq 1$ for all nontrivial γ.*

Proof. If K is of type I, then it is a Montesinos knot, which admits an involution. Hence K is a symmetric knot. By Corollary 7 of [3], K has property P. If K is of type II or III, $K(\gamma)$ is laminar by Theorem 2.4, so it has infinite fundamental group [9].

Corollary 2.6. *The cabling conjecture is true for arborescent knots,* *that is, if K is a non-torus arborescent knot, then $K(\gamma)$ is irreducible for all γ.*

Proof. If K is of type II or III, this follows from Theorem 2.3 because a laminar manifold is irreducible. So suppose $K = K(\frac{p_1}{q_1}, \frac{p_2}{q_2}, \frac{p_3}{q_3})$. Delman [5] shows that if all q_i are odd, then $K(\gamma)$ is laminar for all nontrivial γ. If one of the q_i is even, K is strongly invertible, in which case the result has been proved by Eudave-Muñoz [6].

3. Surgery on type III knots

Suppose (B, T) is a tangle. We use $\partial_0 E(T)$ and $\partial_1 E(T)$ to denote the punctured sphere $\partial B \cap E(T)$ and the two annuli $\partial N(T) \cap \partial E(T)$ respectively. Thus $\partial E(T) = \partial_0 E(T) \cup \partial_1 E(T)$.

Lemma 3.1. *Let (B, T) be an atoroidal tangle. Let A be an incompressible annulus in $E(T)$ so that $\partial A \subset \partial E(T)$ can be isotoped to be disjoint from $\partial_1 E(T)$. Then A is parallel to an annulus on $\partial E(T)$.*

Proof. After an isotopy if necessary we may assume that ∂A is on $\partial_0 E(T)$. For homological reasons, ∂A bounds either an annulus on $\partial_0 E(T)$ or an annulus on $\partial E(T)$ containing a component U_i of $\partial_1 E(T)$. In the second case, after isotoping a component of ∂A through U_i, we get an annulus with boundary a pair of parallel curves on $\partial_0 E(T)$. Therefore, we may assume that this is already true for A. Let A' be the annulus on $\partial_0 E(T)$ bounded by ∂A.

Since $E(T)$ is atoroidal, the torus $A \cup A'$ bounds a solid torus V. Since A is incompressible in $E(T)$, it cannot be meridional on V. Note that a component of ∂A bounds a disk D on ∂B, so if A is not longitudinal on V, then $V \cup N(D)$ would be a punctured lens space in the 3-ball B, which is absurd. Therefore, A is longitudinal, and is parallel to the annulus A' on $\partial E(T)$. q.e.d.

Note that the annulus A in the lemma is not assumed essential. The condition that ∂A can be isotoped into $\partial_0 E(T)$ cannot be omitted, otherwise there would be many counter examples.

A disk D in $E(T)$ is called a *monogon* if $\partial D \cap \partial_i E(T)$ is an essential arc for $i = 0, 1$. It is called a *bigon* if $\partial D \cap \partial_i E(T)$ consists of two essential arcs.

Lemma 3.2. *If T is a nontrivial atoroidal tangle, then $\partial_0 E(T)$ is incompressible, and the tangle space $E(T)$ has no monogons or bigons.*

Proof. A compressing disk of $\partial_0 E(T)$ cuts B into two 3-balls B_1, B_2, each B_i containing a string t_i of T. Since $E(T)$ is atoroidal, t_i is a trivial arc in B_i, so T is a trivial tangle.

A monogon of $E(T)$ can be extended to a disk D in B with $\partial D = t_i \cup \alpha$, where $\alpha \subset \partial B$. The frontier of $N(D)$ is then a compressing disk of $\partial_0 E(T)$.

Now consider a bigon D. If the two components of $\partial D \cap \partial_1 E(T)$ are on different components of $\partial_1 E(T)$, then D extends to a band D' in B connecting the two strings. The frontier of $N(D')$ is an incompressible annulus in $E(T)$ with boundary on $\partial_0 E(T)$, so by Lemma 3.1 it is ∂-parallel. Since T can be isotoped into $\partial N(D')$, T is trivial. If the two components of $\partial D \cap \partial_1 E(T)$ are on the same component of $\partial_1 E(T)$, then D extends to an annulus A in B containing a string t_1 of T. Since $\partial D \cap \partial_0 E(T)$ are essential arcs, the other string of T is in the ball component of $B - A$. Pushing A off this component, we get an incompressible annulus in $E(T)$. By Lemma 3.1 this annulus is ∂-parallel, therefore t_1 is also ∂-parallel, which implies that $\partial_0 E(T)$ is compressible, so T is trivial by the above.

Theorem 3.3. *Suppose (S^3, K) is a union of nontrivial atoroidal tangles (B_1, T_1) and (B_2, T_2). If T_1 is ∂-irreducible, then all nontrivial surgeries on K produce hyperbolic Haken manifolds.*

Proof. Decompose S^3 as the union of the tangle space $E(T_1)$ and the handlebody $H = B_2 \cup N(T_1)$. Then K is a knot in H intersecting each meridian disk of $N(T_1)$ once. Denote by M the manifold $H - \operatorname{Int} N(K)$. Let D_1, D_2 be the two disks in H which are meridian disks of T_1, so that $H - \operatorname{Int} N(D_1 \cup D_2) = B_2$. Let U_i be the annulus $M \cap D_i$. Clearly, U_i is essential in M. So ∂H is compressible in M if and only if after cutting along U_i, the surface $\partial_0 E(T)$ is compressible in $E(T)$, which is the closure of $M - \operatorname{Int} N(U_1 \cup U_2)$. Hence by Lemma 3.2, ∂H is incompressible in M. By Menasco's result [21] it remains incompressible after all nontrivial surgeries.

Let γ be a nontrivial slope on $\partial N(K)$. Clearly, both $E(T_1)$ and $H - \operatorname{Int} N(K)$ are irreducible. Since K intersects a disk of H just once, it cannot be a cabled knot in H, so by Scharlemann's theorem [26] $(H, K; \gamma)$ is irreducible. Thus $(S^3, K; \gamma) = E(T_1) \cup (H, K; \gamma)$ is a Haken manifold. Moreover, the incompressible surface ∂H in $(S^3, K; \gamma)$ is separating, so $(S^3, K; \gamma)$ is not a small Seifert fiber space, i.e, a Seifert fiber space with orbifold a 2-sphere having at most 3 singular points.

In the following we will show that $(S^3, K; \gamma)$ is atoroidal. It will then follow from Thurston's hyperbolization theorem [30] that $(S^3, K; \gamma)$ is a hyperbolic manifold. In general both $E(T_1)$ and $(H, K; \gamma)$ may contain some essential annuli. What we will show below is that the boundaries of these annuli will never match up to produce an essential torus.

Lemma 3.4. *The manifold $(H, K; \gamma)$ is atoroidal.*

Proof. Let T be an essential torus in $(H, K; \gamma)$, isotoped to have least intersection with $\partial N(K)$. Then $P = T \cap M$ is a punctured torus such that ∂P is a set of curves on $\partial N(K)$ parallel to γ. Since T is an essential torus, such P is an essential surface in M. Isotop P so that it has least intersection with U_i. By an innermost circle – outermost arc argument one can show that $P \cap U_i$ has no trivial circles or ∂-parallel arcs. Since P has no intersection with the component of ∂A that lies on ∂H, this implies that $P \cap U_i$ is a set of essential circles. In particular, $P = T$, so T lies in M.

If $T \cap (U_1 \cup U_2) = \emptyset$, T would be an essential torus in the tangle space $E(T_2) = M - \operatorname{Int} N(U_1 \cup U_2)$, contradicting the assumption that T_2 is atoroidal. So assume $T \cap E(T_2)$ is a set of annuli. One can show that an inessential component of $T \cap E(T_2)$ is parallel to one of the annuli in $\partial N(T_2)$. Thus if none of the annuli in $T \cap E(T_2)$ is essential, then T is isotopic to $\partial N(K)$ in M, so it would not be an essential torus. If some component of $T \cap E(T_2)$ is an essential annulus, by Lemma 3.1 $E(T_2)$ would be toroidal. q.e.d.

Now consider $(H, K; \gamma)$. Let $M_1 = N(D_1 \cup D_2 \cup K)$, and let $M_2 = \overline{H - M_1}$. It is clear that M_2 is homeomorphic to the tangle space $E(T_2)$, and the homeomorphism can be chosen so that the surface $F = M_1 \cap M_2$ is mapped to $\partial_1 E(T_2)$. Use $\partial_0 M_i$ to denote the surface $\partial M_i - \operatorname{Int} F$.

Lemma 3.5. *An essential annulus A in $(H, K; \gamma)$ can be isotoped to be disjoint from F.*

Proof. We may assume that A has minimal intersection with F. Then by an innermost circle outermost arc argument we may assume that each of $A \cap F$, $A \cap \partial_0 M_1$ and $A \cap \partial_0 M_2$ consists of essential circles or essential arcs in F, $\partial_0 M_1$ and $\partial_0 M_2$, respectively. If $A \cap F$ consists of essential circles, then $A \cap M_2$ is a union of essential annuli which can be isotoped to be disjoint from $\partial_1 E(T_2)$, contradicting Lemma 3.1. If $A \cap F$ are essential arcs, then these arcs cut A into bigons, half of which lie in $M_2 = E(T_2)$, so by Lemma 3.2 T_2 would be either trivial or toroidal, contradicting the assumption of the theorem.

We remark that in general $(H, K; \gamma)$ may contain some essential annuli, but the above lemma says that the annuli can be pushed off F.

Now suppose T is an essential torus in $(S^3, K; \gamma)$. Since $(S^3, K; \gamma)$ is Haken, we may isotop T so that $T \cap (H, K; \gamma)$ and $T \cap E(T_1)$ consist of essential annuli. By Lemma 3.5 we can choose T to be disjoint from the surface F. Note that $\partial F = \partial(\partial_1 E(T_1))$, so a component of $T \cap E(T_1)$ can be isotoped off $\partial_1 E(T_1)$. But this contradicts Lemma 3.1, completing the proof of Theorem 3.3.

Theorem 3.6. *If K is a type III arborescent knot, then $K(\gamma)$ is a hyperbolic Haken manifold for all nontrivial γ. In particular, this is true for all Montesinos knots $K = K(p_1/q_1, \dots, p_n/q_n)$ with $q_i \geq 2$ and $n \geq 4$.*

Proof. By the proof of Theorem 2.4, if K is not of type I, then it is a union of two nontrivial algebraic tangles T_1 and T_2. By Lemmas 3.2 and 3.3 of [31], T_i is atoroidal, and it is ∂-reducible if and only if it is a sum of $R(1/2)$ and $R(p_i/q_i)$ with $|q_i| \geq 2$. Therefore, if both T_i are ∂-reducible, then K is a type II knot. The first part of the theorem now follows from Theorem 3.3. As for the second part, notice that if there are two i's such that $p_i/q_i = 1/2$, then $K(p_1/q_1, \dots, p_n/q_n)$ is a link of at least two components. Therefore a type II knot cannot be a Montesinos knot.

4. Surgery on type II knots

Let $(S^3, K) = (B_1, T_1) \cup (B_2, T_2)$ be a type II knot, where each T_i is a sum of a $(1/2)$ rational tangle and a (p_i/q_i) rational tangle, as shown in Figure 2.1, where the 4-string braid determines the gluing map $\partial B_1 \to \partial B_2$. Let P be the planar surface $\partial B_i \cap E(T_i)$. It cuts $E(K)$ into the two tangle spaces $E(T_i)$. As in Section 2, ∂P cuts the torus $\partial N(K)$ into four annuli U_1, U_2, V_1, V_2, where $U_i = \partial N(t_i) \cap \partial E(T_1)$, and $V_i = \partial N(s_i) \cap \partial E(T_2)$, t_i, s_i being the strings of T_1 and T_2 respectively. We choose the indices so that t_1 and s_1 are the unknotted strings in T_1 and T_2. The following are some basic facts about the tangles T_i and K.

Lemma 4.1. *(a) T_i is a nontrivial atoroidal tangle;*
(b) $E(T_i)$ is a handlebody;
(c) $\partial E(T_1) - U_1$ (resp. $\partial E(T_2) - V_1$) is incompressible, and $\partial E(T_1) -$

U_2 (resp. $\partial E(T_2) - V_2$) is compressible;

(d) $E(K)$ is atoroidal.

Proof. One of the strings of T_i has exterior the same as that of a (p_i/q_i) 2-bridge knot in S^3, so T_i is nontrivial. Since T_i is a nontrivial sum of two atoroidal tangles, it is also atoroidal; see for example Lemma 3.2 of [31].

As t_1 is a trivial string, $E(t_1) = B_1 - \operatorname{Int} N(t_1)$ is a solid torus. One can untangle T_1 by sliding t_2 over t_1, which means that the string t_2 is isotopic to a trivial arc in the solid torus $E(t_1)$. Hence $E(T_1)$ is a handlebody of genus 2 (this is also proved in Lemma 3.3 of [31]), and $\partial E(T_1) - U_2$ is compressible. By Lemma 2.2, $\partial E(T_1) - U_1$ is incompressible.

Let S be an essential torus in $E(K)$. Since the T_i are atoroidal, we may assume that P cuts S into incompressible annuli A_i, none of which is parallel to an annulus on P. By Lemma 3.1 each A_i is parallel to U_j or V_j. Hence S is parallel to $\partial N(K)$. q.e.d.

There are 6 surfaces obtained by tubing P along $\partial N(K)$. Two of them are isotopic to $\partial E(T_1)$ and $\partial E(T_2)$, and are compressible. Now take a union $P \cup U_1$, and push the U_1 part into $E(K)$; then take the union of this surface with $V_1 \cup U_1 \cup V_2$ and push it into $E(K)$. We thus obtain a surface, denoted by F_{U_1}. Similarly, we have F_{U_2}, F_{V_1} and F_{V_2}. Two of these surfaces, F_{U_2} and F_{V_2}, are actually compressible in $E(K)$. We will see that F_{U_1} (similarly F_{V_1}) remains incompressible after all non-integral surgeries.

Let V_1', V_2' be two longitudinal annuli on the boundary of a solid torus W. Construct a manifold $X = E(T_2) \cup W$ by gluing V_i to V_i'.

Lemma 4.2. *The manifold X is irreducible, ∂-irreducible, and atoroidal. Any essential annulus in X is isotopic to V_i.*

Proof. Consider the surface $S = V_1 \cup V_2$ in X. Clearly, it is incompressible and ∂-incompressible in the solid torus W. By Lemma 4.1(a) and 3.2, it is also incompressible and ∂-incompressible in $E(T_2)$. Therefore, S is an essential surface in X. It is well known and easy to prove by an innermost circle outermost arc argument, that if X is reducible or ∂-reducible, then after cutting along an essential surface, either one of the components is reducible, or the surface $F = \partial X - N(S)$ is compressible in one of the components. Now as a tangle space, $E(T_2)$ is irreducible. Since W is a solid torus, it is also irreducible. $\partial X \cap W$ is a pair of longitudinal annuli, and $\partial X \cap E(T_2)$ is the surface P, which is already known to be incompressible. Therefore, X is irreducible and ∂-irreducible.

If X has an essential torus Q, by minimizing its intersection with S, we may assume that $Q \cap E(T_2)$ is a set of incompressible annuli. Let A be a component of $Q \cap E(T_2)$. Since S is a pair of annuli, ∂A can be isotoped into P, so by Lemma 3.1, A is parallel to an annulus on $\partial E(T_2)$. Thus we can isotope the torus Q to reduce $|Q \cap S|$. Since both $E(T_2)$ and W are atoroidal, this would eventually lead to a contradiction.

Now suppose Q is an essential annulus in X, isotoped so that $|Q \cap S|$ is minimal. Then $Q \cap S$ is a set of essential arcs or circles in Q. If they are arcs, a component of $Q \cap E(T_2)$ would be a bigon of $E(T_2)$, contradicting Lemma 3.2. If $Q \cap S$ are circles, one can reduce $|Q \cap S|$ by the same argument as above for an essential torus. So assume Q is disjoint from S. If Q is in W, one can see that it is parallel to V_i. If Q is in $E(T_2)$, by Lemma 3.1 it is parallel to an annulus Q' on $\partial E(T_2)$. Since Q is essential in X, Q' must contain one of the V_i. Thus Q is isotopic to V_i.

Consider a solid torus W. Let U_1' be an annulus on ∂W running at least twice along the longitude of W. Construct a manifold $Y = E(T_1) \cup W$ by gluing U_1 to U_1'.

Lemma 4.3. *The manifold Y is irreducible, ∂-irreducible, and atoroidal. There is no essential annulus in Y with at least one boundary parallel to ∂U_2.*

Proof. The proof is essentially the same as that of Lemma 4.2. When proving the ∂-irreducibility of Y, use the fact that $\partial E(T_1) - U_1$ is incompressible (Lemma 4.1). For the proof about the annulus, notice that if Q is an essential annulus with one boundary parallel to ∂U_2, then a component of $Q \cap E(T_1)$ can still be isotoped off $U_1 \cup U_2$, so the argument in the proof of Lemma 4.2 applies, and one would finally conclude that Q is isotopic to U_1. (It cannot be isotopic to U_2 because it then would not be essential in Y.) But since the curves ∂U_2 are not isotopic to ∂U_1 on ∂Y, this is impossible.

Theorem 4.4. *If K is a type II arborescent knot, then $K(\gamma)$ is a hyperbolic Haken manifold for all non-integral slopes γ.*

Proof. Let $F = F_{U_1}$ be the surface constructed above by tubing P with some annuli on $\partial N(K)$. It cuts $E(K)$ into two components. From the construction we can see that the component Y' containing $\partial N(K)$ is homeomorphic to $E(T_1) \cup_{U_1} (\partial N(K) \times I)$ with U_1 glued to a meridional annulus on $\partial N(K) \times I$, and the other component X is homeomorphic to $E(T_2) \cup (U_1 \times I)$, with $V_1 \cup V_2$ glued to the two annuli $(\partial U_1) \times I$. Thus X is the manifold constructed prior to Lemma 4.2.

Now attach a solid torus W' to Y' along the slope γ. The resulting manifold $Y = Y' \cup W'$ can be written as $E(T_1) \cup_{U_1} ((\partial N(K) \times I) \cup W')$. Let W be the solid torus $(\partial N(K) \times I) \cup W'$. Then $Y = E(T_1) \cup_{U_1} W$. Moreover, since γ is a non-integral slope, U_1 runs at least twice along the longitude of W. Hence Y is a manifold as constructed prior to Lemma 4.3.

The surgered manifold $K(\gamma)$ is the union of X and Y, with $\partial X = \partial Y = F$. Therefore, by Lemma 4.2 and Lemma 4.3, F is incompressible in $K(\gamma)$, and $K(\gamma)$ is irreducible. So $K(\gamma)$ is a Haken manifold. Since F is separating, $K(\gamma)$ is not a small Seifert fiber space.

It remains to show that $K(\gamma)$ is atoroidal. Assume Q is an essential torus in $K(\gamma)$. Since both X and Y are atoroidal, $Q \cap X$ and $Q \cap Y$ consist of essential annuli. By Lemma 4.2, all components of $Q \cap X$ are parallel to V_i. As each V_i has one boundary on U_1 and the other on U_2, it follows that at least one of the essential annuli in $Q \cap Y$ has a boundary curve parallel to the curves ∂U_2. But this is impossible by Lemma 4.3. q.e.d.

Let α be a 1-manifold properly embedded in a 3-manifold M, and F a properly embedded surface in M. By an isotopy of F in (M, α) we mean an isotopy $\varphi : F \times I \to M$ of F in M such that $\varphi((F \cap \alpha) \times I) \subset \alpha$. A disk D in M is called a *peripheral compressing disk* of F if $D \cap F = \partial D$, D intersects α just once, and on F there is no disk D', which intersects α at most once such that $\partial D = \partial D'$. If such a disk exists, F is *peripheral compressible*, otherwise it is peripheral incompressible. F is *α-essential* if $F - \alpha$ is essential in $M - \alpha$, and F is peripheral incompressible.

Lemma 4.5. *Let $\widetilde{M} \to M$ be a double cover with branch set α. Let $\widetilde{F}$ be the lift of F. If F is α-essential, then $\widetilde{F}$ is incompressible and ∂-incompressible in $\widetilde{M}$.*

Proof. If $\widetilde{F}$ is compressible, by the $\mathbb{Z}_2$-equivariant Dehn's Lemma [19], there is a compressing disk $\widetilde{D}$ of $\widetilde{F}$ such that either $\eta(\widetilde{D}) = \widetilde{D}$, or $\eta(\widetilde{D}) \cap \widetilde{D} = \emptyset$, where η is the covering transformation map. In the first case the image D of $\widetilde{D}$ in M is a peripheral compressing disk, and in the second case it is a compressing disk of F.

If $\widetilde{F}$ is ∂-compressible, consider the double $2\widetilde{M}$ of $\widetilde{M}$, i.e., take two copies of $\widetilde{M}$ and glue their boundaries together by the identity map.

The double $2\widetilde{F}$ of $\widetilde{F}$ is compressible in $2\widetilde{M}$, so by the above $2F$ is compressible in $2M$, implying that F is compressible or ∂-compressible.

Lemma 4.6. *Suppose $(B,T) = R(1/2) + R(p/q)$, $|q| \geq 2$. Let F be a T-essential surface in B such that F is not a disk intersecting T at most once or is a sphere intersecting T at most twice. Then the following hold:*

(a) The ends of each string t_i of T are in the same component of $\partial B - \partial F$.

(b) If $F \subset \operatorname{Int} B$, then F is isotopic to ∂B in (B,T).

Proof. (a) Let $\widetilde{M}$ be the double cover of B branched over T, and let $\widetilde{F}$ be the lift of F. The conditions of the lemma guarantee that $\widetilde{F}$ is not a disk or sphere. By Lemma 4.5, $\widetilde{F}$ is an essential surface in $\widetilde{M}$.

Let D be the gluing disk between $R(1/2)$ and $R(p/q)$. By our definition the lift of the rational tangles are solid tori, and the lift of D is an annulus $\widetilde{D}$ representing $m_1 + 2l_1$ and $pm_2 + ql_2$ in $H_1(\partial W_i)$ with respect to some meridian-longitude pairs (m_i, l_i). Thus $\widetilde{M}$ is a Seifert fiber space with two singular fibers of type $(1,2)$ and (p,q), and its orbifold is a disk with 2 singular points. So $\widetilde{D}$ is the only vertical essential annulus, and $\partial\widetilde{M}$ is the only closed incompressible surface in $\widetilde{M}$.

CLAIM. *Each component of $\partial\widetilde{F}$ intersects each component of $\partial\widetilde{D}$ an even number of times.*

By Theorem VI.34 of [18], $\widetilde{F}$ is either vertical (i.e., a union of fibers) or horizontal (i.e., transverse to all fibers). If it is vertical, it is isotopic to $\widetilde{D}$, so the claim is true. Now assume $\widetilde{F}$ is horizontal. Glue a solid torus V to $\widetilde{M}$ to get a new manifold X, so that $\partial\widetilde{F}$ bounds meridians of V. The Seifert fibration of $\widetilde{M}$ extends to a Seifert fibration of X in a unique way. Furthermore, if s is the intersection number of a component of $\partial\widetilde{F}$ with a component of $\partial\widetilde{D}$, then the center of V is a singular fiber of type (r,s) for some r relatively prime to s, because $\partial\widetilde{D}$ are fibers and $\partial\widetilde{F}$ are meridians of V. Now the union of $\widetilde{F}$ and some meridians of V is a horizontal surface. It is well known that if a Seifert fiber space has a horizontal surface, then its Euler number is zero. By the formula on p.437 of [27] the Euler number of X is $(1/2) + (p/q) + (r/s) \bmod \mathbb{Z}$. Therefore, $(r/s) = -(q+2p)/(2q)$, mod $\mathbb{Z}$. Since q is odd (otherwise T would have a closed component), we have $s = 2q$. This proves the claim.

Now consider a component β of ∂F on ∂B. Since F is peripheral incompressible, there must be two points of ∂T on each component of

$\partial B - \beta$. Let α be an arc in a component of $\partial B - \beta$ connecting the two points of ∂T in that component. The conclusion (a) of the lemma is true if and only if $\partial\alpha = \partial t_i$ for some string t_i of T. Let u be the intersection number of α with ∂D. Since the two points on each component of $\partial B - \partial D$ belong to the same string of T, we see that $\partial\alpha = \partial t_i$ for some i if and only if u is even. Notice that β is the boundary of a regular neighborhood of α on ∂B, so $|\beta \cap \partial D| = 2|\alpha \cap \partial D| = 2u$.

Let $\widetilde{\beta}$ be the lift of β. The $2u$ points in $\beta \cap \partial D$ lift to $4u$ points of intersection $\widetilde{\beta} \cap \partial\widetilde{D}$. Each of $\widetilde{\beta}$ and $\partial\widetilde{D}$ has two components. Therefore, each component of $\widetilde{\beta}$ intersects each component of $\partial\widetilde{D}$ at u points. By the above claim, u is even. We have just shown that this implies (a).

(b) The only closed incompressible surfaces in $\widetilde{M}$ are tori parallel to $\partial\widetilde{M}$. By calculating the Euler number of F, we see that F is either a torus disjoint from T, or a 2-sphere intersecting T at four points. But the first case cannot happen, since $E(T)$ is atoroidal.

Let D be the disk between $R(1/2)$ and $R(p/q)$ as before. Since F is peripheral incompressible, by an isotopy in (B,T) we may assume that $F \cap D$ consists of circles parallel to ∂D on $D - T$, and each component of $F - D$ is an annulus or a disk intersecting T twice. By Lemma 3.1, each annuli is parallel to one on D. Thus after an isotopy in (B,T) we may assume that F intersects D in a single circle. Let (B_i, T_i), $i = 1, 2$, be the tangles $R(1/2)$ and $R(p/q)$. The disk $F \cap B_1$ cuts (B_1, T_1) into two tangles T' and T''. Since a rational tangle cannot be a nontrivial sum, one of T' and T'' is trivial, so $F \cap B_1$ is isotopic in (B_1, T_1) to D or $\partial B_1 - \operatorname{Int} D$. Similar arguments hold for $F \cap B_2$. If one of the $F \cap B_i$ is isotopic to D, we can see that $F - T$ would be compressible in $B - T$, which is impossible because F is T-essential. Therefore, both $F \cap B_i$ are isotopic to $\partial B_i - \operatorname{Int} D$ in (B_i, T_i), and F is isotopic to ∂B in (B,T).

Lemma 4.7. *Suppose $(S^3, K) = (B_1, T_1) \cup (B_2, T_2)$ is a type II arborescent knot, where $(B_i, T_i) = R(1/2) + R(p_i/q_i)$, as in the definition. Let F be a K-essential connected surface in S^3, and assume that F is not a sphere intersecting K at most twice. Then F is isotopic in (S^3, K) to the sphere $S = \partial B_1 \cap \partial B_2$.*

Proof. Isotop F to minimize $|F \cap S|$. Clearly, no component of $S - F$ is a disk disjoint from K. If D is a closed up component of $S - F$ intersecting K just once, then by the peripheral incompressibility of F, the circle ∂D bounds a disk D' on F intersecting K once. $D \cup D'$ cuts S^3

into two 3-balls W_1, W_2. Let W_1 be the one with interior disjoint from F. Let $K_i = W_i \cap K$. If K_2 is a trivial arc in W_2, then $F - \operatorname{Int} N(K_2)$ is an incompressible surface in the solid torus $W_2 - \operatorname{Int} N(K_2)$. Thus $F - \operatorname{Int} N(K_2)$ is an annulus. But then F is a sphere intersecting K twice, contradicting our assumption. Therefore K_2 is knotted. Since $E(K)$ is atoroidal (Lemma 4.1(d)), K is not a composite knot, so K_1 is a trivial arc in W_1. We can then isotop F through W_1 to reduce $|F \cap S|$. Hence $F \cap S$ is a set of parallel circles on S, such that each disk component of $S - F$ contains two points of K.

Let $F_i = F \cap B_i$. We want to show that F_i is T_i-essential in B_i. Suppose D is a compressing disk of $F_1 - T_1$ in $B_1 - T_1$. Since F is incompressible, ∂D bounds a disk D' on F. D' cannot be in B_1, otherwise D would not be a compressing disk. Thus $D' \cap S \neq \emptyset$. A disk component of $D' - S$ would then be a compressing disk of $S - K$, contradicting the fact that $S - K$ is incompressible in $S^3 - K$. Similarly, one can show that $F \cap (B_i - T_i)$ is peripheral incompressible in (B_i, T_i). It remains to show that $F_i - T_i$ is ∂-incompressible in $B_i - T_i$.

Let D be a ∂-compressing disk. If the arc $D \cap S$ connects two different components of $F \cap S$, then after isotoping F through D we would get a surface with less components of intersection with S, contradicting the choice of F. If $D \cap S$ connects the same component of $F \cap S$, then after isotoping F through D, we get a surface F' such that $|F' \cap S| = |F \cap S| + 1$. But there are two components α_1, α_2 of $F' \cap S$ which bound disks on S intersecting K just once. Since F is peripheral incompressible, such components can be removed by an isotopy, so we will get a surface with less intersection to S than F, again a contradiction to the minimality of $|F \cap S|$. Therefore F_i is T_i-essential in B_i.

It now follows that F is disjoint from S, for otherwise by Lemma 4.6(a) the two points of K in a disk component of $S - F$ would belong to the same string in each (B_i, T_i), which means that K would be a link of two components. Finally, if F is in a B_i, by Lemma 4.6(b) it is isotopic to S.

If D is a peripheral compressing disk of F, let $D \times I$ be a product neighborhood of D with $(D \times I) \cap F = \partial D \times I$. Then the surface $F' = (F - \partial D \times I) \cup (D \times \partial I)$ is said to be obtained from F by 2-surgery along D. The reverse process of getting F from F' is called *tubing* along K, or more precisely, along the arc $K \cap (D \times I)$. The annulus $\partial D \times I$ is called a tube.

Theorem 4.8. *If K is a type II arborescent knot, then all closed incompressible surfaces in $E(K)$ are compressible in $K(\gamma)$ for all integral slopes γ.*

Proof. Let F be a connected essential surface in $E(K)$. By 2-surgery of F along peripheral compressing disks, we will get a K-essential surface F'. Since K is atoroidal (Lemma 4.1(d)), F is not a torus, so no component of F' is a 2-sphere intersecting K at most twice. By Lemma 4.7, F' is a union of parallel copies of the sphere S. F can be obtained by tubing F' along K.

Suppose F' has n components $F'_1, \dots, F'_n$. Let $F_n = F'_n - \operatorname{Int} N(K)$. Let A be an annulus in $N(K)$ with one boundary on K, and the other on $\partial N(K)$ representing the slope γ. We may assume that the tubes are all inside of $N(K)$, with boundary on $\partial N(K)$. Label a point of $\partial A \cap F$ by i if it is in F_i. Thus, when traveling around ∂A, the labels of $\partial A \cap F$ are $1, 2, \dots, n, n, \dots, 1, 1, \dots, n, n, \dots, 1$, as shown in Figure 4.1, where $n = 4$. Each tube intersects A in an arc connecting two points of $F \cap \partial A$. Figure 4.1 shows a possible diagram of $A \cap F$. (It can be shown that the surface corresponding to this diagram is a connected essential surface in $E(K)$; see for example [24]. One can construct infinitely many connected essential surfaces in this way.)

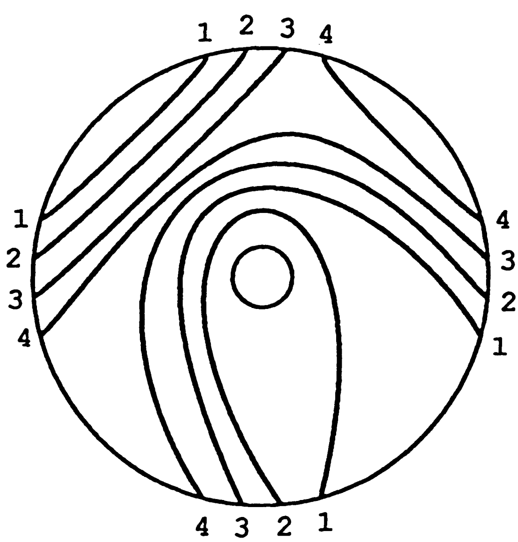

FIGURE 4.1

CLAIM. *An outermost arc α of $A \cap F$ has the same label at its two ends.*

Otherwise, the tube corresponding to α would lie between some F_i' and F_{i+1}', so the union F_i, F_{i+1} and the tube would be a compressible surface, and its compressing disks would become compressing disks of F after all the other tubings, so F would be compressible. This completes the proof of the claim.

Now let N' be a smaller regular neighborhood of K. Perform the γ surgery on this neighborhood. When removing $\operatorname{Int} N'$, the effect on A is to remove a small neighborhood of the inner circle. By our choice of A, the inner circle represents the slope γ on $\partial N'$. Thus after surgery A can be extended to a disk D. By the same reason as in the claim, one can see that if an outermost arc of $A \cap D$ has different labels on its ends, then F is compressible in $K(\gamma)$ and we are done. So assume that each outermost arc has the same label on its two ends. Notice that this label must be either 1 or n.

If $A \cap D$ has 3 or more outermost arcs, then two of them have the same label at their 4 ends. If $A \cap D$ has only two outermost arcs, then all the arcs are parallel, so by the way $\partial A \cap F$ are labeled, one can see that the two outermost arcs again have the same label on its 4 ends. This means that the union of the two corresponding tubes and a component of F' would make a closed surface. Since F is assumed connected, this is impossible unless $n = 1$. When $n = 1$, the two arcs $F \cap D$ are isotopic (in $K(\gamma)$) to arcs on ∂D, which implies that after surgery, the two tubes are isotopic to the two annuli $\partial N(K) \cap E(T_i)$ for some i, so F is isotopic to $\partial E(T_i)$. Since $E(T_i)$ are handlebodies, F is compressible in $K(\gamma)$. This completes the proof of Theorem 4.8.

In [13] Hatcher showed that, given a knot K, there are at most finitely many slopes on $\partial N(K)$ that are the boundary of essential surfaces in $E(K)$. Combining Theorem 2.4, Theorem 4.8 with Hatcher's theorem, we have the following corollary.

Corollary 4.9. *All but finitely many integral surgeries on a type II arborescent knot produce non-Haken laminar manifolds.*

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