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ON SURFACES OF FINITE TOTAL CURVATURE

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Abstract

We consider surfaces M immersed into $\mathbf{R}^n$ and we prove that the quantity $\int_M |A|^2$ (where A is the second fundamental form) controls in many ways the behaviour of conformal parametrizations of M. If M is complete, connected, noncompact and $\int_M |A|^2 < \infty$ we obtain a more or less complete picture of the behaviour of the immersions. In particular we prove that under these assumptions the immersions are proper. Moreover, if $\int_M |A|^2 \leq 4\pi$ or if $n = 3$ and $\int_M |A|^2 < 8\pi$, then M is embedded. We also prove that conformal parametrizations of graphs of $W^{2,2}$ functions on $\mathbf{R}^2$ exist, are bilipschitz and the conformal metric is continuous. The paper was inspired by recent results of T.Toro.

1. Introduction

Let M be a complete , connected, noncompact, oriented two-dimensional manifold immersed in $\mathbf{R}^n$. If the second fundamental form A satisfies $\int_M |A|^2 < +\infty$ then a well-known result of Huber implies that there exists a conformal parametrization $f : S \setminus \{a_1, \ldots, a_q\} \to M \hookrightarrow \mathbf{R}^n$, where S is a compact Riemannian surface. One of our aims in this paper is to study f (viewed as a map into $\mathbf{R}^n$) in a neighbourhood of the "ends" a_i. We shall see that f resembles (in a rather weak sense, cf. Proposition 4.2.10) the function $(z - a_i)^{-m_i}$ in that neighbourhood. We can call the integer m_i the multiplicity of the end at a_i. One con-

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sequence of our analysis is that M is properly immersed which resolves a conjecture of White [39] (see Corollary 4.2.5). Using results of Shiohama [30] (or Li and Tam [19]) we obtain in addition the Gauss-Bonnet formula (see Corollary 4.2.5)

$$\int_M K = 2\pi(\chi_M - m),$$

where χ_M is the Euler characteristic and m is the total number of ends (counted with their multiplicity). In particular if $\int_M K = 0$ then M is conformally equivalent to $\mathbf{C}$. Moreover, we show that if $\int_M |A|^2 < 4\pi$ (or 8π for surfaces in $\mathbf{R}^3$) then the conformal parametrizations $f : \mathbf{C} \to M \hookrightarrow \mathbf{R}^n$ satisfy (after a suitable normalization)

$$e^{-C}|z_1 - z_2| \leq |f_0(z_1) - f_0(z_2)| \leq e^C|z_1 - z_2|.$$

where C depends only on $\int_M |A|^2$ and thus M is embedded. This can be considered as a generalization of a result from [20] to the noncompact case.

Our proofs rely mainly on PDE techniques. In particular we use the fact that for a conformal parametrization with metric $e^{2u}\delta_{ij}$ one has the identity

$$-\Delta u = Ke^{2u},$$

where K denotes the Gauss curvature. At first glance this does not seem to be of much use as the assumption $\int_M |A|^2 < +\infty$ only implies that the right hand side is in L^1 while there is no L^1 theory for the Laplace operator. Using (and in fact generalizing) recent results of Coifman, Meyer, Lions and Semmes [7] (see also [23]) one can show, however, that Ke^{2u} is in fact bounded in the Hardy space $\mathcal{H}^1$. Then one can apply classical results of Fefferman and Stein [11] to obtain good estimates for u. We refer to section 3 for the details. For Hardy space estimates in other problems with critical growth we also refer to [2, 8, 9] and [12].

We remark that for most of our purposes here the use of Hardy spaces is not stricly necessary. Instead of using the $\mathcal{H}^1$-estimates for the Laplace operator, one can apply the results of Wente [38] about the equation $\Delta u = \det D\varphi$. See also [3] and [35].

Our work was inspired by the remarkable results of Toro [36]. She showed, among other things, that the graph Γ of any $W^{2,2}$ function

$w : \mathbf{R}^2 \to \mathbf{R}$ admits a bilipschitz parametrization $f : \mathbf{R}^2 \to \Gamma \subset \mathbf{R}^3$. Using our methods we obtain the following variant of her result. Let $w : \mathbf{R}^2 \to \mathbf{R}$ be a function of $W^{2,2}_{\text{loc}}$ and assume that $D^2 w \in L^2(\mathbf{R}^2)$. Then the graph Γ of w can be parametrized by a bilipschitz map $F : \mathbf{R}^2 \to \Gamma \subset \mathbf{R}^3$ such that

$$\begin{aligned}(1 + c||D^2w||^2_{L^2})^{-1/2}|x-y| &\le |F(x) - F(y)| \\ &\le (1 + c||D^2w||^2_{L^2})^{1/2}|x-y|\,, \\ ||D^2F||_{L^2} &\le c||D^2w||_{L^2}\end{aligned}$$

and the metric $(DF)^t(DF)$ is continuous.

The well-known example (see e.g. [16]) $w(x_1, x_2) = x_1 \sin(\log|\log\sqrt{x_1^2 + x_2^2}|)$ (considered in a neighbourhood of 0) shows that in general the normal to the graph of w may not be continuous.

Global properties of complete minimal surfaces of finite total curvature have been studied by Osserman [24] in the case $n = 3$ and by Chern and Osserman [5] in general. In the paper of White [39] some of their results are proved without assuming that the surface is a minimal surface. We also refer the reader to a recent note by Cheung [6]. See also the nice expositions by Lawson [18] and Rosenberg [25]. In the fundamental paper of Huber [15] complete surfaces with finite total (Gauss) curvature are studied from the intrinsic point of view. For further "intrinsic" results see Li and Tam [19]. The extrinsic geometry of surfaces whose Gauss map is merely small in the space BMO has recently been studied by Semmes [26, 27, 28].

2. Preliminaries

2.1. We shall identify $\mathbf{R}^2$ and $\mathbf{C}$ in the obvious way: $(x_1, x_2) \sim z = x_1 + ix_2$. The Lebesgue spaces L^p and the Sobolev spaces $W^{k,p}$ are defined in the usual way. To avoid any misunderstanding, we recall some facts concerning the spaces $W^{1,2}_0(\mathbf{C})$ and $W^{-1,2}(\mathbf{C})$ which will be frequently used throughout the paper. As usual, we denote by $W^{1,2}_0(\mathbf{C})$ the space of all distributions u with $Du \in L^2(\mathbf{C})$. It is well-known (and easily verified) that smooth functions with compact support are dense in $W^{1,2}_0(\mathbf{C})$ (with respect to the semi-norm given by $\int_{\mathbf{C}} |Du|^2$). The dual of $W^{1,2}_0(\mathbf{C})$ (or more precisely, the space of all distributions on $\mathbf{C}$

which are continuous with respect to the seminorm $\int_{\mathbf{C}} |Du|^2)$ will be denoted by $W^{-1,2}(\mathbf{C})$. For a locally integrable function v on $\mathbf{C}$ we let

$$||v||_{W^{-1,2}} = \sup\{\int_{\mathbf{C}} vu\,;\ u : \mathbf{C} \to \mathbf{R} \text{ is smooth,}$$
$$\text{compactly supported, and } \int_{\mathbf{C}} |Du|^2 \le 1\}.$$

The Sobolev spaces of differential forms on manifolds are defined in the usual way, for example by using charts. See [22] for details.

2.2. We denote by $\mathbf{G}_{n,2}$ the Grassmannian manifold of oriented, two-dimensional subspaces of $\mathbf{R}^n$. We recall that $\mathbf{G}_{n,2}$ embeds naturally as the quadric $\{z_0^2 + \dots z_{n-1}^2 = 0\}$ into $\mathbf{P}^{n-1}(\mathbf{C})$ and can therefore be considered as a Kähler manifold. In particular, the standard Kähler two-form ω on $\mathbf{P}^{n-1}(\mathbf{C})$ gives a two-form on $\mathbf{G}_{n,2}$. We recall that ω can be defined as follows: if $\pi : \mathbf{S}^{2n-1} \to \mathbf{P}^{n-1}(\mathbf{C})$ is the canonical fibration, then $\pi^*\omega = \sum_{k=0}^{n-1} i dz_k \wedge d\bar{z}_k$.

2.3. Let Σ be a surface (i.e. a two-dimensional oriented manifold) immersed into $\mathbf{R}^n$. We use the letter Σ when dealing with surfaces which are possibly not complete. Basically these will be open parts of the surface M from the introduction. We use the notation $\Sigma \hookrightarrow \mathbf{R}^n$ to denote that Σ is immersed into $\mathbf{R}^n$. (Hence $\Sigma \hookrightarrow \mathbf{R}^n$ and $\Sigma \subset \mathbf{R}^n$ do not have the same meaning.) We consider Σ as a Riemannian manifold, the metric being induced by the immersion. Since the dimension of Σ is two, the metric defines also an integrable complex structure on Σ, and Σ can thus be also considered as a one-dimensional complex manifold.

We denote by $G : \Sigma \to \mathbf{G}_{n,2} \subset \mathbf{P}^{n-1}(\mathbf{C})$ the Gauss map which assigns to each $x \in \Sigma$ the oriented tangent plane to Σ at x. (Recall that Σ is assumed to be oriented.) The second fundamental form of Σ is denoted by A. Up to a suitable normalization, A can be identified with the derivative DG of the Gauss map. With our choice of the metric on $\mathbf{G}_{n,2}$, we have $\frac{1}{2}|A|^2 = |DG|^2$. (Here $|A|^2 = \sum_{i,j} |A(e_i, e_j)|^2$, where (e_1, e_2) is any (locally defined) orthogonal frame.)

The Gauss curvature of Σ is denoted by K. We have $K\sigma = G^*\omega$, where σ is the volume form on Σ.

2.4. Let Σ be as in 2.3 and assume that there exists a conformal parametrization $f : \Omega \to \Sigma \hookrightarrow \mathbf{R}^n$, where $\Omega \subset \mathbf{C}$ is an open domain. If we consider f as a mapping $\Omega \to \mathbf{R}^n$, the Cauchy-Riemann conditions imply $|f_{x_1}| = |f_{x_2}|$ and $f_{x_1} \cdot f_{x_2} = 0$.

Let u be given by $e^u = |f_{x_1}|$. Let us also define $\varphi : \Omega \to \mathbf{G}_{n,2} \subset \mathbf{P}^{n-1}(\mathbf{C})$ by $\varphi = G \circ f$. We recall that

$$-\Delta u = Ke^{2u}, \tag{2.4.1}$$

where $\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$. This can be rewritten as

$$-d * du = \varphi^* \omega. \tag{2.4.2}$$

The conformal invariance of the Dirichlet integral gives

$$\int_\Omega |D\varphi|^2 = \int_\Sigma |DG|^2 = \int_\Sigma \frac{1}{2}|A|^2. \tag{2.4.3}$$

2.5. Assume that M is a complete surface (i.e. complete oriented two-dimensional manifold) immersed in $\mathbf{R}^n$ such that $\int_M |A|^2 < \infty$. An obvious consequence of the well-known results of Huber (see [15] and also [19]) is

2.5.1. Theorem. *M is conformally equivalent to a compact Riemannian surface with finitely many points deleted. If M is simply connected, then it is conformally equivalent to $\mathbf{C}$.*

3. $\mathcal{H}^1$-estimates

3.1. We first recall the definition of the Hardy space $\mathcal{H}^1(\mathbf{R}^n)$. See [11] for details. Let ψ be a smooth compactly supported function on $\mathbf{R}^n$ satisfying $\int_{\mathbf{R}^n} \psi = 1$. For $\varepsilon > 0$ we let $\psi_\varepsilon(x) = \varepsilon^{-n}\psi(\frac{x}{\varepsilon})$. Let $v \in L^1(\mathbf{R}^n)$. We set $v^*(x) = \sup_{\varepsilon>0} |(\psi_\varepsilon * v)(x)|$. The Hardy space $\mathcal{H}^1(\mathbf{R}^n)$ consists of all $v \in L^1(\mathbf{R}^n)$ for which v^* is integrable. The norm $||\cdot||_{\mathcal{H}^1}$ is given by $||v||_{\mathcal{H}^1} = \int_{\mathbf{R}^n} v^*$. This definition is independent of ψ, modulo equivalence of norms.

3.2. The following result (which follows from [11]) will play an important role throughout this paper. In what follows we addopt the usual convention and denote by c or C generic constants (whose values may change from line to line).

3.2.1. Theorem. *Let $\nu \in \mathcal{H}^1(\mathbf{C})$. Then the equation $\Delta u = \nu$ (considered in $\mathbf{C}$) admits a solution $u_0 : \mathbf{C} \to \mathbf{R}$ which is continuous,* *belongs to* $W^{2,1}_{loc}$*, and satisfies:*

$$\lim_{z\to\infty} u_0(z) = 0,$$

$$\int_{\mathbf{C}} |D^2 u_0| \le c\|\nu\|_{\mathcal{H}^1},$$

$$\left\{\int_{\mathbf{C}} |Du_0|^2\right\}^{\frac{1}{2}} \le c\|\nu\|_{\mathcal{H}^1}, \quad \textit{and}$$

$$|u_0| \le c\|\nu\|_{\mathcal{H}^1} \quad \textit{in } \mathbf{C}.$$

Proof. Let $\mathcal{H}^1_{00} \subset \mathcal{H}^1$ be the space of functions whose Fourier transform is a compactly supported smooth function with the support away from zero. Since $\mathcal{H}^1_{00} \subset \mathcal{H}^1$ is dense in $\mathcal{H}^1$, see [32], p. 231, it is enough to consider the case $\nu \in \mathcal{H}^1_{00}$. For $\nu \in \mathcal{H}^1_{00}$ we define u_0 by $\hat{u}_0(\xi) = -\frac{\hat{\nu}(\xi)}{|\xi|^2}$ (where $\hat{}$ denotes the Fourier transform) and we note that u_0 belongs to the class $\mathcal{S}$ of rapidly decreasing smooth functions (see [29], Chap. VII). We can use the results in [11], section 3, to obtain the first inequality. The second inequality follows from the standard Sobolev inequality $\{\int_{\mathbf{C}} |Dv|^2\}^{\frac{1}{2}} \le \tilde{c}\int_{\mathbf{C}} |D^2 v|$ which is valid for all $v \in \mathcal{S}$. Finally, writing $u(x) = \int_{-\infty}^{x_1} dy_1 \int_{-\infty}^{x_2} dy_2\, \frac{\partial^2 u(y_1,y_2)}{\partial y_1 \partial y_2}$ we obtain the last estimate. (See also Adams [1], Lemma 5.8. for the imbedding of $W^{n,1}_{\mathrm{loc}}(\mathbf{R}^n)$ into continuous functions.)

3.3. We shall be using the following theorem:

3.3.1. Theorem. ([7], *see also* [23].) *Let* $\varphi : \mathbf{R}^n \to \mathbf{R}^n$ *be a function belonging to* $W^{1,n}(\mathbf{R}^n)$*. Then* $\det D\varphi$ *belongs to* $\mathcal{H}^1(\mathbf{R}^n)$ *and*

$$\|\det D\varphi\|_{\mathcal{H}^1} \le c\|D\varphi\|^n_{L^n}.$$

In this article we shall need $\mathcal{H}^1$-estimates for v given by $v\,dx_1 \wedge dx_2 = \varphi^*\omega$, where $\varphi : \mathbf{C} \to \mathbf{P}^n(\mathbf{C})$ is a $W^{1,2}_0$-function and ω the Kähler form on $\mathbf{P}^n(\mathbf{C})$. In what follows we shall (with some inaccuracy) identify $\varphi^*\omega$ with the function v given by $v\,dx_1 \wedge dx_2 = \varphi^*\omega$. The $\mathcal{H}^1$-estimates of $\varphi^*\omega$ do not seem to be an obvious consequence of (3.3.1). There is, however, one situation, where (3.3.1) can be directly applied:

3.3.2. Corollary. *Let* $\varphi : \mathbf{C} \to \mathbf{S}^2 \subset \mathbf{R}^3$ *be a function belonging to* $W^{1,2}_0(\mathbf{C})$*. Assume that there is* $a \in \mathbf{S}^2$ *and* $\delta > 0$ *such that* $|\varphi - a| \ge \delta$ *a.e. in* $\mathbf{C}$*. Let* ω *be the canonical volume form on* $\mathbf{S}^2$*. Then* $\varphi^*\omega \in \mathcal{H}^1(\mathbf{C})$ *and*

$$\|\varphi^*\omega\|_{\mathcal{H}^1} \le \frac{c}{\delta^2}\|D\varphi\|^2_{L^2}.$$

Proof. We can assume that $a = (0,0,-1)$. We consider the polar coordinates (ρ, ϑ) on $\mathbf{S}^2$ (given by $(\rho, \vartheta) \to (\sin\rho\cos\vartheta, \sin\rho\sin\vartheta, \cos\rho)$). Let (r, θ) be the polar coordinates in $\mathbf{R}^2$ and let $T : \mathbf{S}^2 \setminus \{a\} \to \mathbf{R}^2$ be defined by $r = \sqrt{2(1-\cos\rho)}$, $\theta = \vartheta$. Since T is volume-preserving, our statement follows from 3.3.1 by considering the mapping $T \circ \varphi$.

Theorems 3.2.1 and 3.3.1 imply that the solution of the equation $\Delta u = \det D\varphi$, where $\varphi \in W_0^{1,2}(\mathbf{R}^2, \mathbf{R}^2)$ are in fact more regular than standard estimates suggest. This fact was according to our knowledge first recognized by H.Wente in [38], where essentially the following theorem was proved. Our formulation incorporates a result from [3] regarding optimal constants. (See the appendix of [3].)

3.3.3. Theorem. *Let $\varphi \in W_0^{1,2}(\mathbf{R}^2, \mathbf{R}^2)$. Then the equation $\Delta u = \det D\varphi$ (considered in $\mathbf{R}^2$) admits a solution which is continuous and satisfies:*

$$\lim_{z\to\infty} u_0(z) = 0\,,$$

$$||Du_0||_{L^2(\mathbf{R}^2)} \le \frac{1}{8}\sqrt{\frac{3}{2\pi}}||D\varphi||^2_{L^2(\mathbf{R}^2)}\,, \quad \text{and}$$

$$|u_0| \le \frac{1}{4\pi}||D\varphi||^2_{L^2(\mathbf{R}^2)} \quad \text{in } \mathbf{R}^2.$$

Proof. As we have mentioned above, this is a consequence of results in [38] and [3]. Appart from the numerical values of the constants in the estimates this also obviously follows from 3.2.1 and 3.3.1. The constant $\frac{1}{8}\sqrt{\frac{3}{2\pi}}$ in the estimate of $||Du_0||_{L^2}$ was obtained in [38], the constant $\frac{1}{4\pi}$ in the estimate of $|u_0|$ follows trivially from estimates obtained in [3].

3.4. Let us consider $\varphi : \mathbf{C} \to \mathbf{P}^n(\mathbf{C})$ belonging to $W_0^{1,2}(\mathbf{C}, \mathbf{P}^n(\mathbf{C}))$. We note that $\varphi^*\omega$ does not necessarily belong to $\mathcal{H}^1$, since a necessary condition for $\varphi^*\omega \in \mathcal{H}^1$ is $\int_{\mathbf{C}} \varphi^*\omega = 0$. Assuming this and trying to prove $\varphi^*\omega \in \mathcal{H}^1$ following the method in [7], one finds that difficulties arise from the fact that ω is not exact. We can try to remove these difficulties by lifting φ to $F : \mathbf{C} \to \mathbf{S}^{2n+1} \subset \mathbf{C}^{n+1}$ (i.e. $\varphi = \pi \circ F$, where $\pi : \mathbf{S}^{2n+1} \to \mathbf{P}^n(\mathbf{C})$ is the canonical fibration) in such a way that we control the $W_0^{1,2}$-norm of F. We shall see that this is possible.

We introduce the following notation: for $\varphi : \mathbf{C} \to \mathbf{P}^n(\mathbf{C})$ (belonging to $W_0^{1,2}(\mathbf{C}, \mathbf{P}^n(\mathbf{C}))$) we denote (with some inaccuracy) by $|D\varphi \wedge D\varphi|$ the area element induced on $\mathbf{C}$ by φ. For $n = 1$ we clearly have $2|D\varphi \wedge D\varphi| = |\varphi^*\omega|$. (Note that the volume form on $\mathbf{P}^1(\mathbf{C})$ is $\frac{1}{2}\omega$, where ω is the Kähler form on $\mathbf{P}^1(\mathbf{C})$.)

The Hermitian product on $\mathbf{C}^k$ is denoted by $\langle\cdot\rangle$, i.e. $\langle z,w\rangle = \sum_{j=1}^{j=k} z_j\overline{w}_j$.

3.4.1. Proposition. *Let $\varphi \in W_0^{1,2}(\mathbf{C},\mathbf{P}^n(\mathbf{C}))$ and let $\varepsilon > 0$. Then there exists a smooth $\tilde\varphi : \mathbf{C} \to \mathbf{P}^n(\mathbf{C})$ which is constant outside a compact subset of $\mathbf{C}$ and $\int_{\mathbf{C}} |D\varphi - D\tilde\varphi|^2 \le \varepsilon$. (To make sense of the last integral, we consider $\mathbf{P}^n(\mathbf{C})$ as a submanifold of some $\mathbf{R}^N$.)*

Proof. We recall that smooth functions are dense in $W^{1,2}(\mathbf{S}^2,\mathbf{P}^n(\mathbf{C}))$ by [33], Section 4. From this we see easily that also smooth functions which are constant in a neighbourhood of a given point (the neighbourhood can depend on the function) are dense in $W^{1,2}(\mathbf{S}^2,\mathbf{P}^n(\mathbf{C}))$. The proof is finished easily by using the stereographic projection $\mathbf{S}^2 \to \mathbf{C}$.

3.4.2. Proposition. *Let $\varphi : \mathbf{C} \to \mathbf{P}^n(\mathbf{C})$ be smooth and constant outside a compact subset of $\mathbf{C}$. A necessary and sufficient condition for the existence of a smooth $F : \mathbf{C} \to \mathbf{S}^{2n+1}$ with $\varphi = \pi \circ F$ is that $\int_{\mathbf{C}} \varphi^*\omega = 0$.*

Proof. This is well-known and follows easily for example from [31], Chapter 8.

3.4.3. Proposition. *Let $\varphi : \mathbf{C} \to \mathbf{P}^n(\mathbf{C})$ be smooth and constant outside a compact subset of $\mathbf{C}$. Assume that $\int_{\mathbf{C}} \varphi^*\omega = 0$. Then there is a smooth lifting $F : \mathbf{C} \to \mathbf{S}^{2n+1}$ of φ, which minimizes $\int_{\mathbf{C}} |D\tilde F|^2$ among all liftings $\tilde F \in W_0^{1,2}(\mathbf{C},\mathbf{S}^{2n+1})$ of φ. Moreover,*

$$\int_{\mathbf{C}} |DF|^2 = \int_{\mathbf{C}} |D\varphi|^2 + ||\varphi^*\omega||^2_{W^{-1,2}},$$

and F is unique up to the multiplication by a complex unity.

Proof. Let α be the 1-form on $\mathbf{S}^{2n+1}$ defined at $z = (z_0,\dots,z_n) \in \mathbf{S}^{2n+1}$ by $\alpha(\xi) = \operatorname{Re}\langle\xi, iz\rangle$ for each vector ξ from the tangent space $T_z\mathbf{S}^{2n+1}$. Since $\sum z_k\overline{z}_k = 1$, we can also write $\alpha = \sum -i\overline{z}_k\,dz_k$. Clearly $d\alpha = \pi^*\omega$. Let $\tilde F$ be any smooth lifting of φ. We decompose $\beta = \tilde F^*\alpha$ as $\beta = d\theta + *d\psi$, where θ and ψ are smooth functions of $W_0^{1,2}(\mathbf{C})$, which are uniquely determined up to constants. We note that $\int_{\mathbf{C}} |\beta|^2 = \int_{\mathbf{C}} |d\theta|^2 + \int_{\mathbf{C}} |d\psi|^2$. Since $d * d\psi = d\beta = \varphi^*\omega$, the function ψ depends only on φ (modulo a constant) and $\int_{\mathbf{C}} |d\psi|^2 = ||\varphi^*\omega||^2_{W^{-1,2}}$. We have $\int_{\mathbf{C}} |D\tilde F|^2 = \int_{\mathbf{C}} |D\varphi|^2 + \int_{\mathbf{C}} |\beta|^2$ and we see that $F = e^{-i\theta}\tilde F$ is the required minimizer. Now $\tilde F$ determines θ uniquely up to a constant and hence F is unique up to the multiplication by a complex unity.

Remark. Proposition 3.4.3 is related to a well-known result of Uhlenbeck (see [37]) regarding the existence of good gauges. In our situation the gauge group is $SO(2)$ and our liftings correspond to the potentials A in [37]. The commutativity of $SO(2)$ accounts for the fact that we can find the required lifting by solving the linear equation $d * d\psi = \varphi^*\omega$ above.

3.5. We next aim to obtain estimates of $\varphi^*\omega$ in $W^{-1,2}$. Let $(z_0, \dots, z_n)$ be the homogeneous coordinates in $\mathbf{P}^n(\mathbf{C})$. Let $a \in \mathbf{P}^n(\mathbf{C})$ be a point with homogeneous coordinates $(a_0, \dots, a_n)$. We shall denote by H_a the hyperplane in $\mathbf{P}^n(\mathbf{C})$ which is determined by the equation $\sum_{k=0}^{k=n} a_k \bar{z}_k = 0$. We denote by $\hat{\mathbf{P}}^n(\mathbf{C})$ the manifold of all hyperplanes in $\mathbf{P}^n(\mathbf{C})$. We consider the standard metric on $\hat{\mathbf{P}}^n(\mathbf{C})$ which is defined so that the 1-1 correspondence $a \to H_a$ between $\mathbf{P}^n(\mathbf{C})$ and $\hat{\mathbf{P}}^n(\mathbf{C})$ is an isometry. We denote by μ the multiple of the standard $2n$-dimensional measure on $\hat{\mathbf{P}}^n(\mathbf{C})$ for which $\mu(\hat{\mathbf{P}}^n(\mathbf{C})) = 1$.

3.5.1. For each hyperplane $H \in \hat{\mathbf{P}}^n(\mathbf{C})$ we define a one form α_H on $\mathbf{P}^n(\mathbf{C}) \setminus H$ in the following way. We consider the point $a \in \mathbf{P}^n(\mathbf{C})$ for which $H = H_a$ and we choose $A \in \pi^{-1}(a)$, where $\pi : \mathbf{S}^{2n+1} \to \mathbf{P}^n(\mathbf{C})$ is the canonical fibration. Let $s_H : \mathbf{P}^n(\mathbf{C}) \setminus H \to \mathbf{S}^{2n+1}$ be the section which is determined by $s_H(a) = A$ and by the condition that for each geodesics in $\mathbf{P}^n(\mathbf{C}) \setminus H$ passing through the point a its image under s_H is perpendicular to the fibres. (If H is given by the equation $z_0 = 0$, $A = (1, 0, \dots, 0)$ and $z \in \mathbf{P}^n(\mathbf{C}) \setminus H$ has homogeneous coordinates $(z_0, \dots, z_n) \in \mathbf{S}^{2n+1}$, then $s_H(z) = \left(|z_0|, \frac{|z_0|}{z_0} z_1, \dots, , \frac{|z_0|}{z_0} z_n\right)$. We recall that the one-form α on $\mathbf{S}^{2n+1}$ is defined as $\alpha = -i \sum_{k=0}^{k=n} \bar{z}_k dz_k$ and we set $\alpha_H = s_H^* \alpha$. (This definition is clearly independent of the choice of A in the fibre $\pi^{-1}(a)$.) Clearly $d\alpha_H = \omega$ in $\mathbf{P}^n(\mathbf{C}) \setminus H$ and it is easy to check that for each $z \in \mathbf{P}^n(\mathbf{C}) \setminus H$ we have $|\alpha_H(z)| = \operatorname{cotan} \operatorname{dist}(z, H)$.

3.5.2. We recall that for $0 < r < \pi/2$ the volume of the ball $B_{a,r} = \{z \in \mathbf{P}^n(\mathbf{C}), \operatorname{dist}(a, z) < r\}$ is given by $\operatorname{Vol}(B_{a,r}) = \frac{\alpha(2n-1)}{2n} \sin^{2n} r$, where $\alpha(m)$ denotes the standard m-dimensional measure of $\mathbf{S}^m$. (See, for example [13], Chapter I.4, p. 168.) For a hyperplane $H \subset \mathbf{P}^n(\mathbf{C})$ and $0 < r < \pi/2$ we let $H_{(r)} = \{z \in \mathbf{P}^n(\mathbf{C}), \operatorname{dist}(z, H) < r\}$. Since $\mathbf{P}^n(\mathbf{C}) \setminus H_{(r)}$ is a closed ball of radius $\pi/2 - r$ (see, for example [13], Chapter I.4.2) we have $\operatorname{Vol}(H_{(r)}) = \frac{\alpha(2n-1)}{2n}(1 - \cos^{2n} r)$.

In what follows we shall use the notation $\#S$ for the number of elements of a set S.

3.5.3. Lemma. *Let $\varphi : \mathbf{C} \to \mathbf{P}^n(\mathbf{C})$ be a smooth mapping belonging* *to* $W_0^{1,2}(\mathbf{C}, \mathbf{P}^n(\mathbf{C}))$. *Then*

$$\pi \int_{\hat{\mathbf{P}}^n(\mathbf{C})} \#\varphi^{-1}(H)\, d\mu(H) \le \int_{\mathbf{C}} |D\varphi \wedge D\varphi|$$

Proof. This is an easy consequence of the general integral-geometric formula in [4], Theorem 5.5. (The formula from [4] can be directly applied to images of balls $B \subset \mathbf{C}$ for which the restriction of φ to B is an embedding. The general case follows by the standard application of Sard's theorem and an easy covering argument. See, for example [10], the proof of Theorem 3.2.3.)

3.5.4. Lemma. *Let* $\varphi : \mathbf{C} \to \mathbf{P}^n(\mathbf{C})$ *be smooth and constant outside a compact subset of* $\mathbf{C}$. *If* $\int \varphi^*\omega = 0$, *then* $\#\varphi^{-1}(H)$ *is even for a.e.* $H \in \hat{\mathbf{P}}^n(\mathbf{C})$.

Proof. Since $\int_{\mathbf{C}} \varphi^*\omega = 0$, the mapping φ is homotopic to a constant mapping. (See, for example, [31], Chapter 8.) If H is such that φ is transversal to H, then $\#\varphi^{-1}(H)$ is even by the standard intersection theory. (See, for example, [14].) A standard application of Sard's theorem and easy dimension arguments show that φ is transversal to a.e. $H \in \hat{\mathbf{P}}^n(\mathbf{C})$.

3.5.5. Proposition. *Let* $0 < \varepsilon < 1$. *Let* $\varphi \in W_0^{1,2}(\mathbf{C}, \mathbf{P}^n(\mathbf{C}))$ *and assume that* $\int_{\mathbf{C}} \varphi^*\omega = 0$ *and that* $\int_{\mathbf{C}} |D\varphi \wedge D\varphi| \le 2\pi\varepsilon$. *Then* $||\varphi^*\omega||_{W^{-1,2}} \le \frac{2n(1-\varepsilon^{\frac{1}{n}})^{\frac{1}{2}}}{1-\varepsilon} ||D\varphi||_{L^2}$.

Proof. It is clearly enough to prove the estimate under the assumption $\int_{\mathbf{C}} |D\varphi \wedge D\varphi| < 2\pi\varepsilon$. Using 3.4.1 we see that we can also assume that φ is smooth and constant outside a compact subset of $\mathbf{C}$. From 3.5.3 and 3.5.4 we see that there is a closed set $E \subset \hat{\mathbf{P}}^n(\mathbf{C})$ with $\mu(E) = 1 - \varepsilon$ such that $\varphi^{-1}(H) = \emptyset$ for each $H \in E$. Let $\tilde{E} \subset \mathbf{P}^n(\mathbf{C})$ be the union of all hyperplanes of E. We note that $\tilde{E}$ is closed and that $\varphi(\mathbf{C}) \subset \mathbf{P}^n(\mathbf{C}) \setminus \tilde{E}$. We define a one form α_E on $\mathbf{P}^n(\mathbf{C}) \setminus \tilde{E}$ by $\alpha_E = \frac{1}{\mu(E)} \int_E \alpha_H d\mu(H)$, where α_H is defined in 3.5.1. (In fact this formula defines α_E well also on $\tilde{E}$, but we will not need this.) Clearly α_E is smooth in $\mathbf{P}^n(\mathbf{C}) \setminus \tilde{E}$ and satisfies $|\alpha_E(z)| \le \frac{1}{\mu(E)} \int_E \operatorname{cotan} \operatorname{dist}(z, H)\, d\mu(H)$ for each $z \in \mathbf{P}^n(\mathbf{C}) \setminus \tilde{E}$. Since $d\alpha_H = \omega$ in $\mathbf{P}^n(\mathbf{C}) \setminus H$, we have $d\alpha_E = \omega$ in $\mathbf{P}^n(\mathbf{C}) \setminus \tilde{E}$. Let $0 < \delta < \pi/2$ be such that $\mu(E) = 1 - \varepsilon = 1 - \cos^{2n}\delta$. For $z \in \mathbf{P}^n(\mathbf{C})$ let $E_{z,\delta} = \{H \in \hat{\mathbf{P}}^n(\mathbf{C});\ \operatorname{dist}(z, H) < \delta\}$ and let $\hat{z}$ be the hyperplane in

$\hat{\mathbf{P}}^n(\mathbf{C})$ consisting of all hyperplanes in $\mathbf{P}^n(\mathbf{C})$ passing through z. Since clearly $\operatorname{dist}(z, H) = \operatorname{dist}_{\hat{\mathbf{P}}^n}(H, \hat{z})$, where $\operatorname{dist}_{\hat{\mathbf{P}}^n}$ denotes the distance in $\hat{\mathbf{P}}^n(\mathbf{C})$, we see from 3.5.2 that $\mu(E_{z,\delta}) = 1 - \cos^{2n}\delta = \mu(E)$. It is not difficult to see that for each $z \in \mathbf{P}^n(\mathbf{C})$ we have

$$\frac{1}{\mu(E)} \int_E \operatorname{cotan} \operatorname{dist}(z, H)\, d\mu(H) \leq$$

$$\frac{1}{\mu(E_{z,\delta})} \int_{E_{z,\delta}} \operatorname{cotan} \operatorname{dist}_{\hat{\mathbf{P}}^n}(H, \hat{z})\, d\mu(H).$$

Using the formulae in 3.5.2 and the isometry of $\mathbf{P}^n(\mathbf{C})$ and $\hat{\mathbf{P}}^n(\mathbf{C})$ we see that the last integral is equal to $2n \int_0^\delta \cos^{2n} t\, dt \leq 2n \sin\delta = 2n(1 - \varepsilon^{\frac{1}{n}})^{\frac{1}{2}}$. Hence $|\alpha_E| \leq \frac{2n(1-\varepsilon^{\frac{1}{n}})^{\frac{1}{2}}}{1-\varepsilon}$ in $\mathbf{P}^n(\mathbf{C}) \setminus \tilde{E}$. We have $\varphi(\mathbf{C}) \subset \mathbf{P}^n(\mathbf{C}) \setminus \tilde{E}$ and $\varphi^*\omega = \varphi^* d\alpha_E = d\varphi^*\alpha_E$. Since $|\varphi^*\alpha_E| \leq |\alpha_E|\,|D\varphi|$, the result follows.

3.5.6. Theorem. *Let $0 < \varepsilon < 1$. Let $\varphi \in W_0^{1,2}(\mathbf{C}, \mathbf{P}^n(\mathbf{C}))$ and assume that $\int_{\mathbf{C}} \varphi^*\omega = 0$ and that $\int_{\mathbf{C}} |D\varphi \wedge D\varphi| \leq 2\pi\varepsilon$. Let $\pi : \mathbf{S}^{2n+1} \to \mathbf{P}^n(\mathbf{C})$ be the canonical fibration. Then there exists $F \in W_0^{1,2}(\mathbf{C}, \mathbf{S}^{2n+1})$ so that $\pi \circ F = \varphi$ and $||DF||_{L^2}^2 \leq C(n, \varepsilon)||D\varphi||_{L^2}^2$, where $C(n, \varepsilon) = 1 + \frac{4n^2(1-\varepsilon^{\frac{1}{n}})}{(1-\varepsilon)^2}$.*

Proof. This follows directly from 3.4.3 and 3.5.5.

3.5.7. Corollary. *Under the assumptions of 3.5.6 we have $\varphi^*\omega \in \mathcal{H}^1$ with $||\varphi^*\omega||_{\mathcal{H}^1} \leq c_1 C(n, \varepsilon)||D\varphi||_{L^2}^2$, where $C(n, \varepsilon) = 1 + \frac{4n^2(1-\varepsilon^{\frac{1}{n}})}{(1-\varepsilon)^2}$ and c_1 is independent of n and ε. Moreover, the equation $\Delta u = \varphi^*\omega$ (considered in $\mathbf{C}$) admits a solution $u_0 : \mathbf{C} \to \mathbf{R}$ which is continuous and satisfies:*

$$\lim_{z \to \infty} u_0(z) = 0\,,$$

$$\int_{\mathbf{C}} |D^2 u_0| \leq c_2 C(n, \varepsilon)||D\varphi||_{L^2}^2\,,$$

$$\left\{\int_{\mathbf{C}} |Du_0|^2\right\}^{\frac{1}{2}} \leq \frac{1}{4}\sqrt{\frac{3}{2\pi}}C(n, \varepsilon)||D\varphi||_{L^2}^2\,, \quad \textit{and}$$

$$|u_0| \leq \frac{1}{2\pi}C(n, \varepsilon)||D\varphi||_{L^2}^2 \qquad \textit{in } \mathbf{C},$$

where $C(n, \varepsilon)$ is as in Theorem 3.5.6 and c_2 is independent of n and ε.

Proof. Let $F : \mathbf{C} \to \mathbf{S}^{2n+1} \subset \mathbf{C}^{n+1}$ be the lifting from (3.5.6). We can write $F = (F_0, \dots, F_n)$ where F_k are $\mathbf{C}$-valued functions on $\mathbf{C}$. We have $\varphi^*\omega = F^*\pi^*\omega = i\sum_{k=0}^{k=n} dF_k \wedge d\overline{F}_k$. We can now use (3.2.1), (3.3.1) and (3.3.3) and the results follow.

Easy examples show that without the assumption $\int_{\mathbf{C}} |D\varphi \wedge D\varphi| \leq 2\pi\varepsilon$ in 3.5.5 we cannot expect a bound for $||\varphi^*\omega||_{W^{-1,2}}$ which would depend only on $||D\varphi||_{L^2}$. In view of this, the following result is more or less optimal.

3.5.8. Proposition. *Let $\varphi \in W_0^{1,2}(\mathbf{C}, \mathbf{P}^n(\mathbf{C}))$ with $\int_{\mathbf{C}} \varphi^*\omega = 0$. Then $\varphi^*\omega \in W^{-1,2}(\mathbf{C})$ and the norm $||\varphi^*\omega||_{W^{-1,2}}$ can be estimated in terms of $||D\varphi||_{L^2}$ and the modulus of continuity of the measure $|D\varphi \wedge D\varphi|$. More precisely, let $\tau : \mathbf{C} \to \mathbf{S}^2$ be the stereographic projection and let $0 < \varepsilon < 1$ and $r > 0$ be such that $\int_{\tau^{-1}(B)} |D\varphi \wedge D\varphi| \leq \pi\varepsilon$ for each ball $B \subset \mathbf{S}^2$ of radius $\leq r$. Then $||\varphi^*\omega||_{W^{-1,2}} \leq c(n, r, \varepsilon)||D\varphi||_{L^2}$.*

Proof. Let us fix $0 < \varepsilon < 1$ and $r > 0$. Using 3.4.1 we see that it is enough to prove the estimate $||\varphi^*\omega||_{W^{-1,2}} \leq c(n, r, \varepsilon)||D\varphi||_{L^2}$ under the assumption that φ is smooth, constant outside a compact subset of $\mathbf{C}$, and satisfies $\int_{\tau^{-1}(B)} |D\varphi \wedge D\varphi| \leq \pi\varepsilon$ for each ball $B \subset \mathbf{S}^2$ of radius $\leq r$. Let $1 = \tilde{\eta}_1 + \cdots + \tilde{\eta}_m$ be a partition of unity on $\mathbf{S}^2$ such that $\operatorname{diam}(\operatorname{supp}\tilde{\eta}_k) \leq r$ for each k and that $\tilde{\eta}_2 = \cdots = \tilde{\eta}_m = 0$ in a neighbourhood of the north pole $(=\infty)$. Let $\eta_k = \tilde{\eta}_k \circ \tau$ and let $E_k = \operatorname{supp}\eta_k$. We can clearly assume $\int_{E_k} |D\varphi \wedge D\varphi| < \pi\varepsilon$ and, proceeding in a similar way as in the proof of 3.5.5 (the main difference is that we no longer know that $\#\varphi^{-1}(H) \cap E_k$ is even), we can find for each $k = 1, \dots, m$ a one-form α_k with $|\alpha_k| \leq \frac{2n(1-\varepsilon^{\frac{1}{n}})^{\frac{1}{2}}}{1-\varepsilon}$ and $d\alpha_k = \omega$ in a neighbourhood of $\varphi(E_k)$. We have

$$\varphi^*\omega = \Sigma\eta_k\varphi^*\omega = \Sigma\eta_k\varphi^*(d\alpha_k) = \Sigma d(\eta_k\varphi^*\alpha_k) - \Sigma d\eta_k \wedge \varphi^*\alpha_k\,.$$

Since $\Sigma|(\eta_k\varphi^*\alpha_k)| \leq \frac{2n(1-\varepsilon^{\frac{1}{n}})^{\frac{1}{2}}}{1-\varepsilon}|D\varphi|$ in $\mathbf{C}$, we see that the $W^{-1,2}$-norm of $\Sigma d(\eta_k\varphi^*\alpha_k)$ can be controled in the required way. We can choose our partition of unity so that we controll the L^2-norm of $\Sigma d\eta_k \wedge \varphi^*\alpha_k$ and also diameter of the support of this function by quantities depending only on r, ε, and n. Since we also have $\int_{\mathbf{C}} \Sigma d\eta_k \wedge \varphi^*\alpha_k = 0$, we see that the $W^{-1,2}$-norm of $\Sigma d\eta_k \wedge \varphi^*\alpha_k$ can be controlled in the required way. The proof is finished.

Remark. Let X be a compact Riemannian manifold and let ω be a closed n-form on X. Let $\varphi : \mathbf{R}^n \to X$ be a function belonging to $W_0^{1,n}(\mathbf{R}^n, X)$ such that $\int_{\mathbf{R}^n} \varphi^*\omega = 0$. It is natural to ask whether one can obtain estimates of $\varphi^*\omega$ in $\mathcal{H}^1(\mathbf{R}^n)$ in this general situation. Results in this direction have been recently obtained by Malý [21].

4. Estimates for conformal maps of punctured discs

Throughout this section M denotes a complete, connected, noncompact, oriented two-dimensional manifold immersed in $\mathbf{R}^n$ with second fundamental form A satisfying $\int_M |A|^2 < +\infty$. By Huber's result (see 2.5.1), M can be parametrized by a conformal mapping $\tilde{f}$: $S \setminus \{a_1, \dots, a_q\} \to M \hookrightarrow \mathbf{R}^n$, where S is a compact Riemannian surface. Our goal in this section is to study the behaviour of $\tilde{f}$ near the "ends" a_i. Passing to local charts, this reduces to the study of conformal maps into M which are defined on punctured discs. In fact we find it more convenient to move the singularity from 0 to ∞ and thus to deal with maps,

$$f : \Omega^* \to \Sigma \subset M \hookrightarrow \mathbf{R}^n,$$

where

$$\Omega^* = \{z \in \mathbf{C} : |z| > 1\}.$$

We shall also use the notation $\Omega^*_r = r\Omega^*$.

4.1.1. Definition. We say that a metric g on Ω^*_r *is complete at* ∞ if $\operatorname{dist}_g(z_0, z) \to \infty$ when $z \to \infty$ for some (and hence all) $z_0 \in \Omega^*_r$.

The results of this section may be summarized as follows. The map f behaves, in a sense to be made precise, like z^m (where $m \in \mathbf{N}$) as $z \to \infty$ and hence we can associate to each end a_i its multiplicity m_i. One has $\tilde{f}(z) \to \infty$ (in $\mathbf{R}^n$) as $z \to a_i$ which proves White's [39] conjecture that M is properly immersed. Moreover we show that there exists constants c_i such that the induced metric $(Df)^t(Df)(z)$ on Ω^* is of the form $c_i|z|^{2m_i-2}\delta_{kl} + o(|z|^{2m_i-2})$ as $z \to \infty$. In view of a result of Shiohama [30] (or Li and Tam [19]) this implies that

$$\int_M K = 2\pi(\chi_M - \sum_{i=1}^{q} m_i),$$

where χ_M denotes the Euler characteristic. Note that $\sum m_i$ can be thought of as the total number of ends (counted with multiplicity). We finally show that if $\int_M |A|^2 \leq 4\pi$ or if $n = 3$ and $\int_M |A|^2 < 8\pi$ then the conformal type of M is $\mathbf{C}$ and M is embedded.

4.1.2. Lemma. *Let $H : \Omega^* \to \mathbf{R}$ be a harmonic function. The metric $e^{2H}\delta_{ij}$ is complete at ∞ if and only if*

$$H(z) = \alpha \log |z| + h(z),$$

where $h : \Omega^* \to \mathbf{R}$ is harmonic and bounded at ∞ and where $\alpha \geq -1$.

Proof. Only the "only if" part of the statement is nontrivial. Assume that $e^{2H}\delta_{ij}$ is complete at ∞ and let

$$\alpha = \frac{1}{2\pi} \int_\Gamma *dH, \quad \text{where} \quad \Gamma = \{z \in \mathbf{C} : |z| = 2\}.$$

Set $h(z) = H(z) - \alpha \log |z|$. Let $\Phi : \Omega^* \to \mathbf{C}$ be a holomorphic function with $\operatorname{Re} \Phi = h$ (such a function exists since h is harmonic and $\int_\Gamma *dh = 0$). Let P be a non-zero polynomial of degree $\geq \alpha$ such that $\int_\Gamma P(z)e^{\Phi(z)}dz = 0$. Since degree $P \geq \alpha$ and $e^{2H}\delta_{ij}$ is complete at ∞, the metric $|Pe^\Phi|^2\delta_{ij}$ is complete at ∞.

Let $F : \Omega^* \to \mathbf{C}$ be a holomorphic function with $F' = Pe^\Phi$ (which exists since $\int_\Gamma P(z)e^{\Phi(z)}dz = 0$). We prove that F cannot have an essential singularity at ∞. Arguing by contradiction, assume that F has an essential singularity at ∞. Let us consider a sufficiently large $r > 0$ such that $F' \neq 0$ in Ω_r^*. A standard application of the monodromy theorem and of the implicit function theorem shows that we cannot have $F(\Omega_r^*) = \mathbf{C}$. Since F has an essential singularity at ∞ Picard's theorem then implies that there is a $w_0 \in \mathbf{C}$ such that $F(\Omega_s^*) = \mathbf{C}\backslash\{w_0\}$ for each $s > r$.

Let $w_1 \neq w_0$ be such that the segment $[w_0, w_1]$ does not intersect the (compact) set $F(\partial\Omega_s^*)$ for some $s > r$. Considering a connected component of $F^{-1}([w_0, w_1])$ which is contained in Ω_s^*, we see that $|F'|^2\delta_{ij}$ is not complete at ∞, a contradiction. Since F thus cannot have an essential singularity at ∞, Φ must be bounded at ∞ and the proof is easily finished.

4.2.1. Theorem. *Let $\Sigma \hookrightarrow \mathbf{R}^n$ be a surface immersed into $\mathbf{R}^n$. Assume that Σ is conformally equivalent to Ω^* and let $f : \Omega^* \to \Sigma$ be a conformal parametrization for Σ, with $|f_{x_1}| = |f_{x_2}| = e^u$. Let $\varphi : \Omega^* \to \mathbf{G}_{n,2} \subset \mathbf{P}^{n-1}(\mathbf{C})$ be defined by $\varphi = G \circ f$ where G is the*

Gauss map. Assume that

$$\int_{\Omega^*} |D\varphi \wedge D\varphi| \leq \pi/2, \quad \textit{and} \quad \int_{\Omega^*} |D\varphi|^2 = \frac{1}{2}\int_\Sigma |A|^2 < \infty.$$

Then

$$u(z) = u_0(z) + H(z),$$

where $H : \Omega^* \to \mathbf{R}$ is harmonic and $u_0 : \Omega^* \to \mathbf{R}$ is a (smooth) function which satisfies:

$$\lim_{z\to\infty} u_0(z) = 0, \quad \textit{and} \quad |u_0| \leq c\int_\Sigma |A|^2 \quad \textit{in} \quad \Omega^*, \tag{4.2.2}$$

$$\left\{\int_{\Omega^*} |Du_0|^2\right\}^{1/2} \leq c\int_\Sigma |A|^2, \tag{4.2.3}$$

$$\int_{\Omega^*} |D^2 u_0| \leq c\int_\Sigma |A|^2.$$

If moreover the metric $e^{2u}\delta_{ij}$ is complete at ∞, then

$$H(z) = (m-1)\log|z| + h(z),$$

where $m \geq 1$ is an integer and h is a harmonic function bounded at ∞, and we also have

$$\lim_{z\to\infty} \frac{|f(z)|}{|z|^m} = \frac{e^\lambda}{m},$$

where $\lambda = \lim_{z\to\infty} h(z)$.

4.2.4. Definition. In the situation of 4.2.1, when $e^{2u}\delta_{ij}$ is complete, Σ is a surface (with boundary) which has one end and we shall refer to the number m as the *multiplicity* of the end. For a manifold M as in the beginning of this section with conformal parametrization $\tilde{f} : S \setminus \{a_1, \dots, a_n\} \to M$ one assigns a multiplicity m_i to each end a_i by passing to local charts. (Clearly m_i does not depent on the particular choice of the conformal parametrization or the local chart.)

4.2.5. Corollary. *Let $M \hookrightarrow \mathbf{R}^n$ be a two-dimensional manifold as in the beginning of this section. Then for each $x_0 \in M$*

$$\lim_{\mathrm{dist}_M(x_0,x)\to\infty} \frac{\mathrm{dist}_M(x_0,x)}{|x_0 - x|} = 1\,.$$

Moreover, one has the Gauss-Bonnet formula

$$\int_M K = 2\pi(\chi_M - \sum_{i=1}^{q} m_i),$$

where m_i is the multiplicity of the end a_i. If $\int_M K = 0$, then M is conformally equivalent to $\mathbf{C}$.

Proof of Corollary 4.2.5. Let us consider a conformal parametrization $\tilde{f} : S \setminus \{a_1, \ldots, a_q\} \to M \hookrightarrow \mathbf{R}^n$, where S is a compact Riemannian surface. The existence of such parametrizations follows from Theorem 2.5.1. Let us choose punctured neighbourhoods U_i of the points a_i which are conformaly equivalent to Ω^*. Let us fix points $b_i \in U_i$. Using Theorem 4.2.1 (more specifically, we use (4.2.2), the formula for H in the case when $e^{2u}\delta_{ij}$ is complete at ∞ and the formula for $\lim_{z\to\infty} \frac{|f(z)|}{|z|^m}$) we see that, for each $i = 1, \ldots, q$

$$\lim_{z\to a_i} \frac{\operatorname{dist}_M(\tilde{f}(b_i), \tilde{f}(z))}{|\tilde{f}(b_i) - \tilde{f}(z)|} = 1.$$

Since $S \setminus \cup_{j=1}^{j=q} U_j$ is compact, the first statement follows easily.

To prove the Gauss-Bonnet formula we fix $x_0 \in M$ and we denote by $A(r)$ the area of the geodesic ball of radius r centered at x_0. In [30] it has been shown (see also [19]) that

$$2 \lim_{r\to\infty} \frac{A(r)}{r^2} = 2\pi\chi_M - \int_M K.$$

Applying Theorem 4.2.1 (in a similar way as in the proof of the first statement) at each end a_i we obtain

$$\lim_{r\to\infty} \frac{A(r)}{r^2} = \pi \sum_{i=1}^{q} m_i,$$

and the formula follows.

If $\int_M K = 0$, then $\chi(M) = \sum_{i=1}^q m_i \geq 1$. On the other hand, we have $\chi(M) \leq 1$ since M is noncompact and connected. Hence $\chi(M) = \sum_{i=1}^q m_i = 1$ and we see that M is homeomorphic to $\mathbf{C}$. By Huber's result (see 2.5.1) the proof is finished.

Proof of Theorem 4.2.1. We extend φ to $\mathbf{C}$ by $\varphi(z) = \varphi(\frac{1}{\bar{z}})$. Since

$$\int_{\mathbf{C}} \varphi^*\omega = 0, \quad \int_{\mathbf{C}} |D\varphi \wedge D\varphi| = 2\int_{\Omega^*} |D\varphi \wedge D\varphi| \quad \text{and}$$

$$\int_{\mathbf{C}} |D\varphi|^2 = 2\int_{\Omega^*} |D\varphi|^2,$$

we see from Corollary 3.5.7 that we can find u_0 with $-\Delta u_0 = \varphi^*\omega$ in $\mathbf{C}$ which satisfies (4.2.2) and (4.2.3). The function $H = u - u_0$ is clearly harmonic in Ω^*. If $e^{2u}\delta_{ij}$ is complete at ∞, then in view of (4.2.2) $e^{2H}\delta_{ij}$ is also complete at ∞ and from Lemma 4.1.2 we see that

$$H(z) = \alpha \log|z| + h(z), \quad \text{where} \quad \alpha \geq -1, \tag{4.2.6}$$

and where h is harmonic and bounded at ∞.

We now prove that α is a nonnegative integer. For $\varepsilon > 0$ and $z \in \Omega_\varepsilon^*$ we let

$$f_\varepsilon(z) = \varepsilon^{\alpha+1}[f(z/\varepsilon) - f(2/\varepsilon)].$$

We also let $\varphi_\varepsilon(z) = \varphi(z/\varepsilon)$, and $u_\varepsilon(z) = \log|f_{\varepsilon x_1}(z)| = \log|f_{\varepsilon x_2}(z)| = u_0(z/\varepsilon) + \alpha\log|z| + h(z/\varepsilon)$. For each $0 < \varepsilon < r$ we have $\int_{\Omega_r^*} |D\varphi_\varepsilon|^2 = \int_{\Omega_{r/\varepsilon}^*} |D\varphi|^2 \to 0$ as $\varepsilon \to 0$. We also have for each compact set $K \subset \mathbf{C} \setminus \{0\}$ and ε sufficiently small (so that $K \subset \Omega_\varepsilon^*$) the equality $\int_K |Du_\varepsilon|^2 = \int_{\frac{1}{\varepsilon}K} |Du|^2$. These equalities and Lemma 4.2.7 below also show that, for small ε the integral $\int_K |D^2 f_\varepsilon|^2$ is bounded independently of ε. Using these estimates we infer that there exists a sequence $\varepsilon_k \to 0$ with the following properties:

(i) There exists $L \in \mathbf{G}_{n,2}$ such that $\varphi_{\varepsilon_k} \to L$ in $W^{1,2}_{\mathrm{loc}}(\mathbf{C} \setminus \{0\})$.

(ii) the maps f_{ε_k} converge uniformly on compact subsets of $\mathbf{C} \setminus \{0\}$ to a conformal mapping $f_0 : \mathbf{C} \setminus \{0\} \to L \subset \mathbf{R}^n$ which satifies $|f_{0x_i}(z)| = e^\lambda |z|^\alpha$ with $\lambda = \lim_{z\to\infty} h(z)$. (This limit exists as h is harmonic and bounded at ∞.)

Identifying L (as an oriented subspace) with $\mathbf{C}$, we can consider f_0 as a holomorphic function on $\mathbf{C} \setminus \{0\}$. We have $|f_0'(z)| = e^\lambda|z|^\alpha$ and therefore α is an integer $\neq -1$. Since $\alpha \geq -1$, we see that α is a nonnegative integer.

To prove the last statement of the theorem, we note that the function f_0 is of the form $f_0(z) = a\frac{e^\lambda}{\alpha+1}z^{\alpha+1} + c$, where $|a| = 1$. Using the locally uniform convergence in (ii), we see that $\lim_{z\to\infty} \frac{|f(z)|}{|z|^{\alpha+1}} = \frac{e^\lambda}{\alpha+1}$ will follow if we show that $\varepsilon^{\alpha+1}|f(\frac{2}{\varepsilon})|$ is bounded as $\varepsilon \to 0$. We have $f(\frac{2}{\varepsilon}) = f(2) + \int_2^{2/\varepsilon} f_{x_1}(s)\,ds$. Since $|f_{x_1}(z)|$ is bounded by $\tilde{c}e^\lambda|z|^\alpha$ for some $\tilde{c} > 0$ and we know that $\alpha \geq 0$, the result follows easily.

4.2.7. Lemma. *The conformal parametrization $f : \Omega^* \to \Sigma \hookrightarrow \mathbf{R}^n$ of Σ satisfies*

$$|D^2 f|^2 = e^{2u}(4|Du|^2 + 2|D\varphi|^2)$$

(pointwise) in Ω^*.

Proof. This is obtained easily by taking derivatives of $e^{2u}\delta_{ij} = f_{x_i} \cdot f_{x_j}$ and of $\varphi = \frac{1}{\sqrt{2}} e^{-2u}(f_{x_i} \wedge f_{x_j})$.

The following lemma relates the intrinsic and the extrinsic geometry of Σ.

4.2.8. Lemma. *Let Q be a square and assume that $f : Q \to \Sigma$ is a conformal parametrization of a surface Σ immersed into $\mathbf{R}^n$. As before let $|f_{x_1}| = |f_{x_2}| = e^u$ and $\varphi = G \circ f$, where G is the Gauss map of Σ. Assume that*

$$\beta - \varepsilon_1 \le u \le \beta + \varepsilon_1 \quad \textit{on} \quad Q \quad \textit{for some } \beta \in \mathbf{R} \quad \textit{and } \varepsilon_1 > 0,$$

and that

$$\int_Q (2|D\varphi|^2 + 4|Du|^2) < \varepsilon_2^2 e^{-2\varepsilon_1} \textit{ for some } 0 < \varepsilon_2 < \sqrt{\frac{\pi \tanh \pi}{2}} e^{-\varepsilon_1}.$$

Let z_1, z_2 be two neighbouring vertexes of Q and denote by d_Σ the intrinsic distance on Σ. Then

$$d_\Sigma(f(z_1), f(z_2)) \le \sqrt{1 + \frac{2\varepsilon_2^2 e^{2\varepsilon_1}}{\pi \tanh \pi}} |f(z_1) - f(z_2)|.$$

Remark. In [26] Semmes proved (for the codimension one case) the deeper result that a similar estimate for d_Σ holds under the weaker assumption that the Gauss map is small in BMO.

Proof. Changing $f(z)$ to $\frac{e^{-\beta}}{|a|} f(az + b)$ if necessary, we may assume that $\beta = 0$, $Q = [0, 1] \times [0, 1]$, $z_1 = 0$ and $z_2 = 1$. Let

$$g(t) = f_{x_1}(t), \quad \mathbf{A} = \int_0^1 g, \quad A = |\mathbf{A}|, \quad B = \int_0^1 |g|.$$

Then

$$B^2 \le \int_0^1 |g|^2 = A^2 + \int_0^1 |g - \mathbf{A}|^2$$

By standard results about traces (applied to the function $h(x,y) = g(x,y) - \int_0^1 g(x,y)dx$) one has

$$(4.2.9)\quad \int_0^1 |g - \mathbf{A}|^2 \le \frac{1}{\pi \tanh \pi} \int_Q |Dg|^2 \le \frac{1}{\pi \tanh \pi} \int_Q |D^2 f|^2.$$

(This can be seen, for example from the fact that $||h||_{L^2(Q)} \le \pi^{-1}||g_x||_{L^2(Q)}$ and $||h(\cdot,0)||_{L^2}^2 \le \frac{\pi}{\tanh \pi}(||h||_{L^2(Q)}^2 + \pi^{-2}||h_y||_{L^2(Q)}^2)$.) We infer that

$$B^2 - A^2 \le \frac{1}{\pi \tanh \pi}||D^2 f||_{L^2(Q)}^2.$$

Now $B^2 \ge e^{-2\varepsilon_1}$ and $||D^2 f||_{L^2(Q)} \le \varepsilon_2$ (by Lemma 4.2.7) and hence $A^2 \ge e^{-2\varepsilon_1} - \frac{1}{\pi \tanh \pi}\varepsilon_2^2 \ge e^{-2\varepsilon_1}/2$. Thus

$$\frac{B}{A} \le \sqrt{1 + \frac{2\varepsilon_2^2 e^{2\varepsilon_1}}{\pi \tanh \pi}}$$

and the result follows.

To end this subsection we discuss in what sense the end of Σ behaves like $z \to z^m$. Let $\pi : X_m \to \Omega^*$ be the m-fold covering of Ω^* by a connected surface X_m. As usual, we consider X_m with the metric $\pi^*\delta_{ij}$ (where δ_{ij} denotes the standard metric on Ω^*).

Let $f : \Omega^* \to \Sigma$ be the conformal parametrization considered above. We have seen that the constant α in (4.2.6) is a nonnegative integer. We let $m = \alpha + 1$ and for $\xi \in X_m$ we set $\tilde{f}(\xi) = f(\xi^{1/m})$, where the $\mathbf{C}$-valued function $\xi \to \xi^{1/m}$ on X_m is defined in the usual way. Clearly $\tilde{f}$ is a conformal parametrization of Σ and $\tilde{u}$ defined by $|D\tilde{f}|^2 = 2e^{2\tilde{u}}$ satisfies

$$\tilde{u}(\xi) = \tilde{u}_0(\xi) + \tilde{h}(\xi) - \log m,$$

where $\tilde{u}_0(\xi) = u_0(\xi^{1/m})$ and $\tilde{h}(\xi) = h(\xi^{1/m})$.

4.2.10. Proposition. *For each $\varepsilon > 0$ there exists $R > 0$ such that the following statement holds. If $Q \subset \Omega^*$ is a square such that $Q \cap \{z \in \mathbf{C} : |z| \le R\} = \emptyset$ and if $\xi_1, \xi_2 \in X_m$ are neighbouring vertices of a connected component of $\pi^{-1}(Q)$, then*

$$\frac{e^\lambda}{m}(1-\varepsilon)|\xi_1 - \xi_2| \le |\tilde{f}(\xi_1) - \tilde{f}(\xi_2)| \le \frac{e^\lambda}{m}(1+\varepsilon)|\xi_1 - \xi_2|,$$

where $\lambda = \lim_{z\to\infty} h(z)$.

Proof. The function $\tilde{u}_0$ above clearly satifies (4.2.2) and (4.2.3) with Ω^* replaced by X_m. We see that we can apply Lemmas 4.2.7 and 4.2.8. The proof is finished.

4.3. In this subsection we look in more detail at conformal parametrization of surfaces with one simple end which are conformally equivalent to $\mathbf{C}$.

4.3.1. Theorem. *Let $0 < \varepsilon < 1$. Let $M \hookrightarrow \mathbf{R}^n$ be a complete, connected, noncompact surface with $\int_M |A|^2 = 8\pi\varepsilon$. If $n \geq 4$, assume in addition that $\int_M K = 0$. Then M is embedded and admits a conformal parametrization $f : \mathbf{C} \to M \hookrightarrow \mathbf{R}^n$ such that*

$$e^{-2c(n,\varepsilon)}|z_1 - z_2| \leq |f(z_1) - f(z_2)| \leq e^{c(n,\varepsilon)}|z_1 - z_2|$$

for each $z_1, z_2 \in \mathbf{C}$ and

$$\int_{\mathbf{C}} |D^2 f|^2 \leq 2\pi e^{2c(n,\varepsilon)}(4\varepsilon + \frac{3}{4}(c(n,\varepsilon))^2),$$

where $c(n,\varepsilon) = 2(1 + \frac{4n^2(1-\varepsilon^{\frac{1}{n}})}{(1-\varepsilon)^2})\varepsilon$.

4.3.2. Corollary. *Let $M \hookrightarrow \mathbf{R}^n$ be a complete, connected, noncompact surface immersed into $\mathbf{R}^n$. Assume that either*

$$\int_M |A|^2 < 8\pi \quad \textit{and} \quad n = 3$$

or

$$\int_M |A|^2 \leq 4\pi \quad \textit{and} \quad n \geq 4.$$

Then M is embedded.

Remarks. 1. We refer the reader to the paper of Li and Yau [20] for a similar statement concerning compact surfaces immersed into $\mathbf{R}^n$. In fact it is not difficult to deduce 4.3.2 from results in [20] and our results regarding the behaviour of f near ∞ (see Theorem 4.2.1 and its proof).

2. The constants 8π and 4π are optimal. Indeed for $n = 3$. Enneper's surface is not embedded and satisfies $\int_M |A|^2 = -2\int_M K = 8\pi$. For $n = 4$ one can consider surfaces $M_R \hookrightarrow \mathbf{R}^4 \simeq \mathbf{C}^2$ given by the immersions $f_R(z) = (\eta(|z|/R)z, z^2)$, where $\eta \in C_0^\infty([0,2))$ and $\eta_{|[0,1]} \equiv 1$, $R > 0$. As $R \to \infty$ one has $\int_{M_R} |A|^2 \to 4\pi$, $\int_{M_R} K \to -2\pi$.

Proof of Theorem 4.3.1. We first note that also for $n = 3$ our assumptions imply that $\int K = 0$, since, on one hand, for $n = 3$ the value of the integral $\int_M K$ is an integral multiple of 4π by [39] and on the other hand $|\int_M K| \leq \frac{1}{2}\int_M |A|^2 < 4\pi$. We can use the Gauss-Bonnet formula (see 4.2.5) together with 2.5.1 to infer that M is conformally equivalent to $\mathbf{C}$ and has one end of multiplicity one. Let $f : \mathbf{C} \to M \hookrightarrow \mathbf{R}^n$ be a conformal parametrization of M. As above let $\varphi = G \circ f$, where G is the Gauss map of M and let $u = \log|f_{x_1}| = \log|f_{x_2}|$. The function u satisfies $-\Delta u = \varphi^*\omega$ in $\mathbf{C}$ and using 4.2.1 and the fact that the multiplicity of the end is one we see that $u(z)$ has a finite limit as $z \to \infty$. Multiplying f by a suitable constant, if necessary, we can assume that $\lim_{z\to\infty} u(z) = 0$. This "boundary condition" and the equation $-\Delta u = \varphi^*\omega$ determine u uniquely, and hence we can apply Corollary 3.5.7 to obtain $|u| \leq c(n,\varepsilon)$. The estimates in 3.5.7 and 4.2.7 give also the required bound for $\int_{\mathbf{C}} |D^2 f|^2$.

To show that M is embedded, let us consider a point $w \in \mathbf{C}$ such that $f(w)$ is not a point of selfintersection of M. (Such points exist by 4.2.8.) We prove that

$$|f(z) - f(w)| \geq e^{-2c(n,\varepsilon)}|z - w| \tag{4.3.3}$$

for each $z \in \mathbf{C}$. To prove this, let us assume (without loss of generality) that $w = 0$ and $f(w) = 0$. For $z \neq 0$ we set

$$\tilde{f}(z) = \frac{f(\frac{1}{z})}{|f(\frac{1}{z})|^2}$$

Clearly $\tilde{f}$ is a conformal parametrization of a surface $\tilde{\Sigma} \hookrightarrow \mathbf{R}^n$ which is the image of $\Sigma = M \setminus \{0\}$ by the inversion $\phi : x \to \frac{x}{|x|^2}$ of $\mathbf{R}^n$. We let $\tilde{\varphi} = \tilde{G} \circ \tilde{f}$ and $\tilde{u} = \log|\tilde{f}_{x_1}| = \log|\tilde{f}_{x_2}|$, where $\tilde{G}$ is the Gauss mapping of $\tilde{\Sigma}$. Let $\tilde{A}$ be the second form of $\tilde{\Sigma}$. The basic fact here is that

$$\int_M |A|^2 = \int_{\tilde{\Sigma}} |\tilde{A}|^2,$$

see Lemma 4.3.4 below. This implies that $\int_{\mathbf{C}\setminus\{0\}} |D\tilde{\varphi}|^2 = \int_{\mathbf{C}} |D\varphi|^2 = 4\pi\varepsilon$ and hence $\tilde{\varphi}$ can be considered as an element of $W_0^{1,2}(\mathbf{C}, \mathbf{P}^n(\mathbf{C}))$. Moreover, from the proof of Lemma 4.3.4 below we see that $\int_{\mathbf{C}} \tilde{\varphi}^*\omega = 0$. The function $\tilde{u}$ is smooth in $\mathbf{C} \setminus \{0\}$, since $0 = f(0)$ is not a point of selfintersection of M. An easy calculation shows that $\lim_{z\to\infty} \tilde{u}(z) = -u(0)$ and using the fact that $\lim_{z\to\infty} \frac{|f(z)|}{|z|} = 1$ (see 4.2.1) we easily verify that $\lim_{z\to 0}\tilde{u}(z) = 0$. We also have $-\Delta\tilde{u} = \tilde{\varphi}^*\omega$ in $\mathbf{C}\setminus\{0\}$. Let $\tilde{v} = v - u(0)$, where v is the solution of $-\Delta v = \tilde{\varphi}^*\omega$ in $\mathbf{C}$ given in 3.5.7. The function $\tilde{u} - \tilde{v}$ is continuous in $\mathbf{C}\setminus\{0\}$, has a finite limit as $z \to 0$, tends to 0 as $z \to \infty$, and is harmonic in $\mathbf{C}\setminus\{0\}$. Hence $\tilde{u} = \tilde{v}$ and we see from 3.5.7 that $|\tilde{u} + u(0)| \le c(\varepsilon, n)$. From this and the above estimate of u we obtain $|\tilde{u}| \le 2c(\varepsilon, n)$ and (4.3.3) follows easily.

We note that what we have proved implies that the set $\mathcal{U}$ of points of M which are not points of selfintersection is closed. By 4.2.8 it is nonempty and since M is properly immersed by 4.2.5, it is also open. Hence $\mathcal{U} = M$ and we see that 4.3.3 in fact holds for each $z, w \in \mathbf{C}$. The proof is finished.

Proof of Corollary 4.3.2. In view of the obvious inequality $|\int_M K| \le \int_M \frac{1}{2}|A|^2$ and the Gauss-Bonnet formula in 4.2.5, the only case which does not directly follow from Theorem 4.3.1 is the case when $n \ge 4$, $\int_M |A|^2 = 4\pi$ and $\int_M K = -2\pi$. It is easy to describe explicitly the surfaces M satisfying these conditions. First we note that under these conditions we have $|A|^2 = -2K$ (pointwise) and hence M is a minimal surface. From the Gauss-Bonnet formula in 4.2.5 we see that M has one end of multiplicity two and is conformally equivalent to $\mathbf{C}$. This means that M admits a conformal parametrization $f : \mathbf{C} \to M \hookrightarrow \mathbf{R}^n$ such that $\frac{|f(z)|}{|z|^2}$ has a finite and nonvanishing limit as $z \to \infty$. Moreover, since M is minimal, f is harmonic. Hence f has to be a quadratic polynomial and an easy calculation shows that M is contained in a four-dimensional subspace of $\mathbf{R}^n$ and that, using suitable coordinates, we can identify this subspace with $\mathbf{C}^2$ so that f becomes $f(z) = (z + az^2, bz^2)$ for some $a, b \in \mathbf{C}$, $b \ne 0$. These surfaces are clearly embedded. The proof is finished.

4.3.4. Lemma. *In the notation introduced in the proof of Theorem 4.3.1 we have*

$$\int_{\tilde{\Sigma}} |\tilde{A}|^2 = \int_M |A|^2.$$

Proof. Let $\tilde{K}$ and K denote respectively the Gauss curvature of $\tilde{\Sigma}$ and M and let $d\sigma$ and $d\tilde{\sigma}$ be respectively the area elements on $M\setminus\{0\}$ and $\tilde{\Sigma}$. We have (pointwise) $(|\tilde{A}|^2 - 2\tilde{K})\,d\tilde{\sigma} = (|A|^2 - 2K)\,d\sigma$. See, for example, [40], Chapter 5. We know that K is integrable on M and we have seen in the proof of 4.3.1 that $\int_M K = 0$. It remains to check that we have some control over $\int_{\tilde{\Sigma}} \tilde{K}$. For $r > 0$ we let

$v(r) = \frac{1}{2\pi}\int_0^{2\pi} \log|f(re^{i\theta})|\,d\theta$, where $f : \mathbf{C} \to M \hookrightarrow \mathbf{R}^n$ is the conformal parametrization of M introduced in the proof of 4.3.1. (We recall that $f(0) = 0$ and $f(z) \neq 0$ for $z \neq 0$.) Since $\lim_{z\to\infty} \frac{|f(z)|}{|z|} = 1$ by 4.2.1, we have $\lim_{r\to\infty}(v(r) - \log r) = 0$. Using this and applying the mean value theorem on intervals of the form $(2^j, 2^{j+1})$ we see that there exists a sequence $r_j \to \infty$ such that $r_j v'(r_j) \to 1$. An elementary calculation shows that $\lim_{r\to 0} rv'(r) = 1$. Let $\rho_j > 0$ be a sequence converging to 0 and let $\Omega_j = \{z \in \mathbf{C},\, \rho_j < |z| < r_j\}$. Since $\tilde{\Sigma}$ is the image of Σ under the inversion $\phi : x \to \frac{x}{|x|^2}$ of $\mathbf{R}^n$, the metric induced on $\mathbf{C}\setminus\{0\}$ by $\phi\circ f : \mathbf{C}\setminus\{0\} \to \tilde{\Sigma} \hookrightarrow \mathbf{R}^n$ is $\frac{1}{|f(z)|^4}e^{2u}\delta_{kl} = e^{2u_1}\delta_{kl}$. Hence

$$\begin{aligned} -\int_{\phi\circ f(\Omega_j)} \tilde{K} &= \int_{\Omega_j} \Delta u_1 = \int_{\Omega_j} \Delta(-2\log|f(z)| + u) \\ &= 4\pi\rho_j v'(\rho_j) - 4\pi r_j v'(r_j) + \int_{f(\Omega_j)} K \to 0 \end{aligned}$$

as $j \to \infty$ and thus

$$\int_{\phi\circ f(\Omega_j)} |\tilde{A}|^2 - \int_{f(\Omega_j)} |A|^2 \to 0$$

as $j \to \infty$. The proof is finished easily.

5. Lipschitz parametrization of $W^{2,2}$ graphs

Let $w : \mathbf{R}^2 \to \mathbf{R}$ be a function belonging to $W^{2,2}_{\mathrm{loc}}$ such that $\int_{\mathbf{R}^2} |D^2 w|^2 < +\infty$ and let $\Gamma \subset \mathbf{R}^3$ be its graph. We aim to prove Γ can be parametrized by a bilipshitz map $F : \mathbf{R}^2 \to \Gamma \subset \mathbf{R}^3$ which belongs to $W^{2,2}_{\mathrm{loc}}$ and for which the induced metric is continuous. This can be considered as an extension of a result of T.Toro [36], which in fact inspired the current work.

5.1. Let w be as above and assume moreover that it is smooth. Let Γ be the graph of w. Let $N : \Gamma \to \mathbf{S}^2$ be the classical Gauss map, i.e. for $X = (x_1, x_2, w(x_1, x_2)) \in \Gamma$ we have

$$N(X) = \frac{1}{\sqrt{1 + |Dw(x)|^2}}(-w_{x_1}(x), -w_{x_2}(x), 1).$$

The orientation on Γ is defined, as usual, by the requirement that the diffeomorphism $x \to (x, w(x))$ between $\mathbf{R}^2$ and Γ is orientation preserving. (The Grassmannian $G_{3,2}$ introduced in 2.2 can be of course identified with $\mathbf{S}^2$. With this identification the Kähler metric on $G_{3,2}$ introduced in 2.2 is the $\frac{1}{2}$-multiple of the canonical metric on $\mathbf{S}^2$ and the Kähler form ω is exactly the volume form given by the canonical metric.)

The second fundamental form A of Γ clearly satisfies $\int_\Gamma |A|^2 \leq \int_{\mathbf{R}^2} |D^2 w|^2$. Let $f : \mathbf{C} \to \Gamma \subset \mathbf{R}^3$ be a conformal parametrization, the existence of which follows from Theorem 2.5.1. We let $u = \log|f_{x_1}| = \log|f_{x_2}|$ and $\varphi = N \circ f$. We recall that $-\Delta u = \varphi^*\omega$, where ω is the canonical volume form on $\mathbf{S}^2$ and that $\int_{\mathbf{C}} |D\varphi|^2 = \int_\Gamma |A|^2$. (The previous remark concerning the identification of $G_{3,2}$ and $\mathbf{S}^2$ accounts for the fact that the factor $\frac{1}{2}$ appearing in 2.4.3 has become 1 .)

Let $\mathbf{S}^2_+ = \{X = (x_1, x_2, x_3) \in \mathbf{S}^2,\ x_3 \geq 0\}$ be the closed upper half sphere and let $T : \mathbf{S}^2_+ \to \mathbf{R}^2$ be the volume preserving map constructed in the proof of 3.3.2. We let $\tilde{\varphi} = T \circ \varphi$. Clearly $|D\tilde{\varphi}| \leq 2|D\varphi|$ and $\varphi^*\omega = \det D\tilde{\varphi}$. (As above, we slightly abuse the notation by identifying the form $\varphi^*\omega$ with the function $*\varphi^*\omega$.)

We have $-\Delta u = \det D\tilde{\varphi}$ and $||D\tilde{\varphi}||_{L^2} \leq 2||D\varphi||_{L^2}$. Let u_1 be the solution of $-\Delta v = \det D\tilde{\varphi}$ given in 3.3.3. (Thus, in particular, $\lim_{z\to\infty} u_1(z) = 0$.) Let $H = u - u_1$. In view of Lemma 4.1.2 the harmonic function H must be constant. Replacing f by $z \to f(az+b)$ if necessary, we see that we can choose the conformal parametrization of Γ so that $H \equiv 0$, $f(0) = (0, w(0))$, $f_{x_2}(0) \cdot (1,0,0) = 0$, and $f_{x_2}(0) \cdot (0,1,0) > 0$.

These conditions determine f uniquely and in what follows we denote the unique conformal parametrization of Γ satisfying the above conditions by f_0.

Let $u_0 = \log|f_{0x_1}| = \log|f_{0x_2}|$. The function u_0 satisfies the same estimates as u_1, i.e.

$$\int_{\mathbf{C}} |D^2 u_0| \leq c \int_\Gamma |A|^2, \tag{5.1.1}$$

$$\left\{\int_{\mathbf{C}} |Du_0|^2\right\}^{1/2} \leq \frac{1}{2}\sqrt{\frac{3}{2\pi}} \int_\Gamma |A|^2, \tag{5.1.2}$$

$$|u_0| \leq \frac{1}{\pi} \int_\Gamma |A|^2, \tag{5.1.3}$$

and

$$\lim_{z\to\infty} u_0(z) = 0.$$

From these estimates we see that we can control $|Df_0|$ and $|D(f_0^{-1})|$. To control the bilipschitz constant of f_0 we still need to compare the instrinsic metric of Γ with the distance in $\mathbf{R}^3$. This is done in the following lemma. The result can be deduced (with different constants) from the assertions in section 4, but we prefer to give a simple direct proof.

5.1.5 Lemma. *Let* dist_Γ *denote the intrinsic distance on* Γ. *For each* $X, Y \in \Gamma$ *we have*

$$\mathrm{dist}_\Gamma(X,Y) \le \sqrt{1 + \frac{1}{\pi \tanh \pi}||D^2 w||_{L^2}^2}\,|X-Y|.$$

Proof. Let $X = (x, w(x))$ and $Y = (y, w(y))$, where $x, y \in \mathbf{R}^2$. Since our statement is invariant under changing w to $\frac{1}{\lambda} w \circ \lambda R$, where $\lambda > 0$ and R is an isometry, we can assume $x = (0,0)$ and $y = (1,0)$. Let $v(t) = w_{x_1}(t,0)$, $a = \int_0^1 v$, and $\Phi(s) = \sqrt{1+|s|^2}$. We have

$$\mathrm{dist}_\Gamma^2(X,Y) - |X-Y|^2 \le \int_0^1 \Phi^2(v) - |X-Y|^2 = \int_0^1 |v|^2 - |a|^2$$

and as in (4.2.4) we see that

$$\begin{aligned}\int_0^1 |v|^2 - |a|^2 = \int_0^1 |v-a|^2 &\le \frac{1}{\pi \tanh \pi}\int_{(0,1)^2} |Dv|^2 \\ &\le \frac{1}{\pi \tanh \pi}||D^2 w||_{L^2}^2.\end{aligned}$$

Since $|X-Y| \ge 1$, we obtain

$$\left(\frac{\mathrm{dist}_\Gamma(X,Y)}{|X-Y|}\right)^2 \le 1 + \frac{1}{\pi \tanh \pi}||D^2 w||_{L^2}^2$$

and the result follows.

5.2. Theorem. *Let* $w : \mathbf{R}^2 \to \mathbf{R}$ *belong to* $W^{2,2}_{loc}$ *and assume that* $\int_{\mathbf{R}^2} |D^2 w|^2 < +\infty$. *Let* $\Gamma \subset \mathbf{R}^3$ *be the graph of* w. *Then there is a conformal parametrization* $f : \mathbf{C} \to \Gamma$ *which belongs to* $W^{2,2}_{loc}$ *and satisfies*

(i) $(1 + \frac{1}{\pi \tanh \pi}||D^2 w||_{L^2}^2)^{-\frac{1}{2}} e^{-\frac{1}{\pi}||D^2 w||_{L^2}^2}|x-y| \le$

$$|f(x) - f(y)| \le e^{\frac{1}{\pi}||D^2 w||_{L^2}^2}|x-y|,$$

(ii) *the metric* $(Df)^t(Df)$ *is continuous, and*

(iii) $\int_{\mathbf{C}} |D^2 f|^2 \le e^{\frac{2}{\pi}||D^2w||^2_{L^2}}(||D^2w||^2_{L^2} + \frac{3}{2\pi}||D^2w||^4_{L^2})$

Moreover, if $\tilde{f} : \mathbf{C} \to \Gamma \subset \mathbf{R}^3$ *belongs to* $W^{1,1}_{loc}$ *and satisfies* $|\tilde{f}_{x_1}| = |\tilde{f}_{x_2}|$, $\tilde{f}_{x_1} \cdot \tilde{f}_{x_2} = 0$, *and* $(\tilde{f}_{x_1} \wedge \tilde{f}_{x_2}) \cdot (0,0,1) \ge 0$ *a.e. in* $\mathbf{C}$, *then* $\tilde{f} = f \circ h$ *for a holomorphic function* $h : \mathbf{C} \to \mathbf{C}$.

Proof. For $\varepsilon > 0$ let $w_\varepsilon = w * \rho_\varepsilon$, where ρ_ε is the standard mollifying function, let Γ_ε be the graph of w_ε and let $f_0^\varepsilon : \mathbf{C} \to \Gamma_\varepsilon$ be the unique conformal parametrization of Γ_ε which we obtained in section 5.1. Using the estimates in that section, we see that there is a sequence $\varepsilon_k \to 0$ such that $f_0^{\varepsilon_k}$ converges uniformly on compact subsets to $f : \mathbf{C} \to \mathbf{R}^3$, which has the required properties (see 4.2.7 for the estimates of $D^2 f$).

As for the proof of the last statement, let us consider $\tilde{f} : \mathbf{C} \to \Gamma$ which belongs to $W^{1,1}_{\mathrm{loc}}(\mathbf{C})$ and satisfies the conformality conditions above. We have to prove that $h = f^{-1} \circ \tilde{f}$ is holomorphic. Using the fact that f satisfies (i) we see that $h \in W^{1,1}_{\mathrm{loc}}(\mathbf{C})$. Hence h is approximately differentiable a.e. in $\mathbf{C}$, see [10], Theorem 3.1.4 and [22], Lemma 3.1.1. We aim to prove that the approximate differential $D_{\mathrm{ap}} h$ satisfies the Cauchy-Riemann conditions a.e. in $\mathbf{C}$. This would be clear if we knew that we can apply the chain rule when taking the derivatives of $f^{-1} \circ \tilde{f}$. We prove that the chain rule can indeed be applied. Let us say that a linear map $L : \mathbf{R}^2 \to \mathbf{R}^3$ satisfies the condition (C) if it is conformal (i.e. $L^t L = \lambda \mathrm{Id}$ for some $\lambda \ge 0$) and satisfies $(Le_1 \wedge Le_2) \cdot (0,0,1) > 0$, where e_1, e_2 is the cannonical basis of $\mathbf{R}^2$. Let

$$\begin{aligned}\Gamma_{\mathrm{reg}} = \{y \in \Gamma,\ & f \text{ is differentiable} \\ & \text{at } a = f^{-1}(y) \text{ and } Df(a) \text{ satisfies (C)}\}.\end{aligned}$$

From the properties of f we see that $H^2(\Gamma \backslash \Gamma_{\mathrm{reg}}) = 0$, where H^2 denotes the two-dimensional Hausdorff measure. Let A be the set of all points of $\mathbf{C}$ at which $\tilde{f}$ is approximately differentiable and $D_{\mathrm{ap}}\tilde{f}(z)$ satisfies the conformality conditions above. Let also $A_1 = \{z \in A,\ D_{\mathrm{ap}}\tilde{f} \ne 0\}$. Since f satisfies (i), we see that h is approximately differentiable (with $D_{\mathrm{ap}} h = 0$) on $A \setminus A_1$. Let $z \in A_1$ with $\tilde{f}(z) \in \Gamma_{\mathrm{reg}}$. In this case it is not difficult to verify that h is approximately differentiable at z and that $D_{\mathrm{ap}} h(z)$ satisfies the Cauchy-Riemann conditions.

Under our assumptions the area formula ([10], Theorem 3.2.5) implies

$$\int_E \frac{1}{2}|D_{\mathrm{ap}}\tilde{f}|^2 = \int_E |\mathrm{det} D_{\mathrm{ap}}\tilde{f}| = \int_{\tilde{f}(E)} N(y, \tilde{f}, E)\, dH^2(y),$$

where E is any measurable subset of A_1 and $N(y, \tilde{f}, E)$ denotes the number of elements of the set $\{x \in E, \tilde{f}(x) = y\}$. See the proof of Theorem 2 from [34] to check that Theorem 3.2.5 from [10] can be applied in our situation. We infer that $H^2(\tilde{f}(E)) > 0$ whenever the measure of $E \subset A_1$ is positive. This shows that $\tilde{f}(z) \in \Gamma_{\mathrm{reg}}$ for a.e. $z \in A_1$. Since the measure of $\mathbf{C} \setminus A$ is zero (use Theorem 4.5.9 from [10] or Lemma 3.1.1 from [22] and our assumptions), we see that $D_{\mathrm{ap}}h$ exists and satisfies the Cauchy-Riemann conditions a.e. in $\mathbf{C}$. Under our assumption the distributional derivative of h and $D_{\mathrm{ap}}h$ coincide a.e. in $\mathbf{C}$ (see [10], Theorem 4.5.9 or [22], Lemma 3.1.1). Using the Weyl's lemma, we see that h is holomorphic. The proof is finished.

The estimates in 5.2 involve exponential dependence on $||D^2w||^2_{L^2}$. One can deduce from 5.2 also the existence of bilipschitz parametrizations of Γ with Lipschitz constants that depend on $||D^2w||_{L^2}$ only linearly:

5.3. Corollary. *There exist constants $c, \tilde{c} > 0$ such that the following holds. Let $w : \mathbf{R}^2 \to \mathbf{R}$ belong to $W^{2,2}_{loc}$ and let $\int_{\mathbf{R}^2} |D^2w|^2 < +\infty$. Let Γ be the graph of w. Then Γ admits a parametrization by a bilipschitz map $F : \mathbf{C} \to \Gamma$ which satisfies*

(i) $\max(\tilde{c}, 1 - c||D^2w||^2_{L^2})|x - y| \le |F(x) - F(y)|$

$$\le (1 + c||D^2w||^2_{L^2})^{1/2}|x - y|,$$

(ii) the metric $(DF)^t(DF)$ *is continuous, and*

(iii) $\int_{\mathbf{C}} |D^2F|^2 \le c||D^2w||^2_{L^2}$.

Proof. We can use a trick from [36]. Apply Theorem 5.2 to εw for a suitable small $\varepsilon > 0$ and then scale back. The details are left to the reader.

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