J. DIFFERENTIAL GEOMETRY
41 (1995) 397–448

ON THE GLUING PROBLEM FOR THE η-INVARIANT

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Abstract

We solve the gluing problem for the η-invariant. Consider a generalized Dirac operator D over a compact Riemannian manifold M that is partitioned by a compact hypersurface N such that $M := M_1 \cup_N M_2$. We assume that the Riemannian metric of M and D have a product structure near N, i.e., $D = I(\partial/\partial\tau + D_N)$ with some Dirac operator D_N on N. Using boundary conditions of Atiyah-Patodi-Singer type parametrized by Lagrangian subspaces L_i of $\ker D_N$ we define selfadjoint extensions D_i, $i = 1, 2$, over M_i. We express the η-invariant of D in terms of the η-invariants of D_i, an invariant $m(L_1, L_2)$ of the pair of the Lagrangian subspaces L_1, L_2, which is related to the Maslov index and an integer-valued term J. In the adiabatic limit, i.e., if a tubular neighborhood of N is long enough, the vanishing of J is shown under certain regularity conditions. We apply this result in order to prove cutting and pasting formulas for the η-invariant, a Wall non-additivity result for the index of Atiyah-Patodi-Singer boundary value problems and a splitting formula for the spectral flow.

1. The gluing problem and applications

1.1. **Introduction.** We solve the gluing problem for the η-invariant of generalized Dirac operators. Consider a closed, compact Riemannian manifold carrying a Dirac bundle with a generalized Dirac operator. Assume that this manifold is separated into two pieces by a compact hypersurface. The gluing problem for the η-invariant consists in expressing the η-invariant of the original Dirac operator in terms of the η-invariants of the Dirac operators living on the pieces.

The selfadjoint operators on the two components with boundary depend on a boundary condition given by Lagrangian subspaces L_1, L_2 of a certain symplectic vector space. The gluing formula also contains an additional real-valued term $m(L_1, L_2)$, which is nicely related to the symplectic geometry and the Maslov index.

Received July 3, 1993 and, in revised form, February 3, 1994. This work has been supported by the Sonderforschungsbereich 288, which was founded by the Deutsche Forschungsgemeinschaft.

Moreover, there will be an integer-valued term which is not very manageable. Hence it is interesting to have conditions under which this term vanishes. In fact, we will give such conditions under certain regularity assumptions. This leads to a very explicit version of the gluing formula for the η-invariant.

We apply our gluing formula to derive a cutting and pasting formula for the η-invariant. Thus, we compute how the η-invariant changes if the manifold and the Dirac bundle are cut into two pieces by a compact hypersurface and glued together again using a different identification.

As another application we derive a Wall nonadditivity result for Atiyah-Patodi-Singer index problems. This generalizes the special case of the signature operator considered by Wall in a topological context.

A gluing problem for the η-invariant appeared first in Cheeger [20] in the context of manifolds with conical singularities. By now the main idea of studying the gluing problem has been to use the adiabatic limit. Let D_N be the Dirac operator on the induced Dirac bundle over the separating hypersurface. The easiest case is the one where $\dim \ker D_N = 0$. Under this assumption the additivity of the η-invariant $\bmod \mathbf{Z}$ follows easily from the results of Douglas-Wojciechowski [24]. Moreover, the author was informed by K. Wojciechowski about a real-valued version of the sum formula [40]. Without the regularity assumption it contains a nonexplicit integer depending on the exponentially small (in the adiabatic limit) eigenvalues of D.

The case where $\ker D_N \neq \{0\}$ was studied by Mazzeo-Melrose and Müller. Using the b-pseudodifferential calculus, Mazzeo-Melrose [33] defined a $b-\eta$-invariant and proved a gluing formula. Without the regularity assumption it involves the signature for the exponentially small eigenvalues. The author was informed by W. Müller that he has proved the gluing formula in full generality [36].

In the present section we give a detailed introduction to the subject and present the main results and applications. The remainder of the paper consists of three sections. In §2 we discuss the Maslov index, the signature defect and the function $m(L_1, L_2)$. In §3 we provide the details of the proof of the gluing formula, Theorem 1.9. We first study the relative index of projections, and then review the finite propagation speed method, which is applied to compare the heat kernels of different Dirac operators. We carry out the program sketched in §1.6. In §4 we deal with the behavior of the small eigenvalues in the adiabatic limit and prove Theorem 1.17. The results of this paper have been announced in [13], [14], [15].

I want to thank W. Müller for introducing me to the gluing problem for

the η-invariant, for showing me related literature and for many interesting discussions. I am also grateful to M. Lesch for showing me his computation on the cylinder, which was very helpful for guessing the right formula.

1.2. The η-invariant. The η-invariant first appeared in the study of index problems for generalized Dirac operators on compact manifolds with boundary (Atiyah-Patodi-Singer [1]). Let (X, M) be a compact Riemannian manifold X of even dimension with boundary M. We assume that there is a metric product collar $(-\varepsilon, 0] \times M \to X$ with a metric $g^X = (dr)^2 + g^M$, where g^M is a Riemannian metric on M independent of the normal coordinate r. Let $D_X \colon \Gamma(E_X) \to \Gamma(F_X)$ be an elliptic first order differential operator. We assume that, on the product collar, there exist identifications $E_X \cong \mathrm{pr}_M^* E$ and $F_X \cong \mathrm{pr}_M^* F$ for certain real or hermitian vector bundles E, F over M, and that $D_X = \sigma(\partial/\partial r + D)$ for a bundle isomorphism $\sigma \in C^\infty(M, \mathrm{Hom}(E, F))$ and a selfadjoint first order differential operator $D\colon \Gamma(E) \to \Gamma(E)$. Here, $\mathrm{pr}_M\colon (-\varepsilon, 0] \times M \subset X \to M$ is the projection of the collar $(-\varepsilon, 0] \times M$ to M. Note that X is considered to be on the left-hand side of M. Thus, M is a right boundary, and ∂r points in the outward normal direction. Let $P_{\leq}$ be the projection onto the subspace of $L^2(M, E)$ spanned by the eigenvectors corresponding to nonpositive eigenvalues of D. Let $C^\infty(X, E_X, P_{\leq})$ be the space of smooth sections ψ of E_X such that $P_{\leq}\psi_{|M} = 0$.

Theorem 1.1 (Atiyah-Patodi-Singer). *The operator*

$$D_X\colon C^\infty(X, E_X, P_{\leq}) \to C^\infty(X, F_X)$$

has a finite-dimensional kernel and cokernel and its index is given by

$$\operatorname{index} D_X = \int_X \Omega(D_X) + \frac{\eta(D) - \dim\ker D}{2}, \tag{1}$$

where $\Omega(D_X)$ is the local index density (defined with the asymptotic expansion of the heat kernel associated to D_X), and $\eta(D)$ is the η-invariant of D defined below.

We will only consider η-invariants of generalized Dirac operators. A generalized Dirac operator D on a hermitian or real vector bundle $E \to M$ with compatible connection ∇^E is a first order elliptic formally selfadjoint differential operator satisfying a Weizenboeck formula of the type $D^2 = (\nabla^E)^*\nabla^E + \mathscr{R}$, where $\mathscr{R}$ is a bundle endomorphism of E. We say briefly that D is the Dirac operator associated with the real or complex Dirac bundle E. Examples of generalized Dirac operators are $d + \delta$ on $E = \bigwedge^* T^*M$, the spin Dirac operator on the spinor bundle associated with a spin structure on M, and the odd signature operator on $\bigwedge^{\mathrm{ev}} T^*M$

given by $(-1)^{k+p}(*d - d*)\omega$ for $\omega \in \Gamma(\Lambda^{2p}T^*M)$ if M is oriented and $\dim M = 4k-1$ (some of these examples are in fact real operators). More examples can be obtained by twisting a generalized Dirac operator with an auxiliary real or hermitian vector bundle with compatible connection. A detailed exposition on generalized Dirac operators can be found in Lawson-Michelsohn [30] and Berline-Getzler-Vergne [4]. Readers who are not so familiar with Dirac operators should keep in mind the example of the odd signature operator.

Let M be a closed Riemannian manifold, $E \to M$ be a complex or real Dirac bundle and D be the associated generalized Dirac operator on E. The η-invariant $\eta(D)$ is a global spectral invariant of D in the sense that it is not the integral of a density locally determined by D, but is defined as follows. D has eigenvalues $\cdots\lambda_{-1} < 0 \leq \lambda_0 \leq \lambda_1 \cdots$ which are counted with multiplicity, and its η-function is defined by

$$\eta(D)(s) := \sum_{i\in\mathbf{Z}} \operatorname{sign}(\lambda_i)|\lambda_i|^{-s},$$

where $\operatorname{sign}(x)$ is the sign of x for $x \neq 0$ and $\operatorname{sign}(0) = 0$. This sum converges for $\operatorname{Re}(s) > n$, where $n := \dim M$. Using the heat-equation method one can show that $\eta(M)(s)$ has a meromorphic extension to the whole complex plane and is regular at $s = 0$ (see, e.g., Gilkey [25]).

Definition 1.2. The *η-invariant of the Dirac operator* is defined as the value of its η-function at $s = 0$:

$$\eta(D) := \eta(D)(0).$$

It is a regularized version of the signature of the quadratic form defined by D, i.e., formally $\eta(D) = \sharp\{\lambda_i > 0\} - \sharp\{\lambda_i < 0\}$.

1.3. The gluing problem. The gluing problem for the η-invariant fits well into the class of gluing problems for spectral invariants of elliptic differential operators. Let M be a closed Riemannian manifold, $E \to M$ be a real or complex Dirac bundle and D be the generalized Dirac operator on E. Let $N \subset M$ be a compact hypersurface cutting M into two components M_i, $i = 1, 2$. We assume that there is a product collar $(-\varepsilon, \varepsilon) \times N \to M$ such that $\{0\} \times N \to N$, carrying a product metric $(dr)^2 + g^N$, where g^N is a Riemannian metric on N, which is independent of the normal coordinate. We assume that ∂r points to M_2. Thus, N is the right boundary of M_1 and the left boundary of M_2. Concerning the generalized Dirac operator D over $(-\varepsilon, \varepsilon)\times N \to M$ we assume that there is a hermitian vector bundle E_N over N with a compatible connection, a parallel automorphism I (i.e., $[\nabla^{E_N}, I] = 0$), and a generalized Dirac operator D_N such that $E_{|(-\varepsilon,\varepsilon)\times N} = \mathrm{pr}_N^* E_N$ and $D = I(\partial/\partial r + D_N)$. It is easy to see that I satisfies

$$I^* = -I, \qquad I^2 = -1, \qquad ID_N + D_N I = 0.$$

The gluing problem for a spectral invariant consists in defining the corresponding spectral invariant for the restrictions D_i of D to M_i (using a suitable boundary condition) and in computing the invariant of D in terms of the invariants of D_i and possibly some invariant depending on the boundary condition and D_N. On even-dimensional manifolds the interesting spectral invariant is the index (associated to a $\mathbf{Z}_2$-grading of the Dirac bundle E), and the corresponding gluing problem was solved by Atiyah-Patodi-Singer [1] and, more generally, by Booss-Wojciechowski [7]. The gluing problem for the η-invariant is interesting for odd-dimensional manifolds M.

The first step is to define selfadjoint extensions of D_i using suitable boundary conditions. In this paper we will consider global boundary conditions of Atiyah-Patodi-Singer type. The conditions are very natural and can be defined in any case (Lesch-Wojciechowski [31]), while there are topological obstructions against the existence of selfadjoint local boundary conditions (e.g., for the odd signature operator). The η-invariant of Dirac operators with global boundary conditions was first considered by Cheeger [19, p. 612]. Cheeger attached cones over N to the boundaries. The choice of an ideal boundary condition there corresponds to the choice of the Lagrangian subspaces in our case (see below).

Let $P_>$ denote the positive spectral projection of D_N, i.e., the orthogonal projection in $L^2(N, E_N)$ onto the subspace spanned by all eigenvectors corresponding to positive eigenvalues, let $P_<$ be the negative spectral projection and $P_0 = 1 - P_< - P_>$. Let $V := \ker D_N = \operatorname{im} P_0$. Note that V is a ("hermitian" in the complex case) symplectic vector space with a symplectic structure given by $\Phi(x, y) := \langle Ix, y\rangle$, where $\langle x, y\rangle$ is the L^2-scalar product. In fact, the multiplication by I anticommutes with D_N and, thus, acts on $\ker D_N$. A Lagrangian subspace $L \subset V$ is a subspace satisfying $L \oplus IL = V$, $\Phi(L, L) = 0$. Let pr_L be the orthogonal projection onto L. Set $\Pi_L^r = P_< + P_0 - \mathrm{pr}_L$, $\Pi_L^l = P_> + P_0 - \mathrm{pr}_L$. Here the index l stands for left while r stands for right. The projection $\Pi_L^{r(l)}$ satisfies $-I\Pi_L^{r(l)} I \oplus \Pi_L^{r(l)} = 1$.

Let us choose two Lagrangian subspaces L_1, L_2. We define essentially selfadjoint operators D_i depending on the choice by setting

$$\operatorname{dom} D_1 := \{\psi \in C^\infty(M_1, E),\ \Pi^r_{L_1} \psi_{|N} = 0\},$$
$$\operatorname{dom} D_2 := \{\psi \in C^\infty(M_2, E),\ \Pi^l_{L_2} \psi_{|N} = 0\}.$$

For this kind of boundary conditions one has the usual elliptic regularity (see Atiyah-Patodi-Singer [1], Booss-Wojciechowski [7], and also the calculation in [10]).

The Dirac operators D_i, $i = 1, 2$, have again a pure point spectrum, and the η-function $\eta(D_i, L_i)(s)$ can be defined as above. It is regular in $s = 0$ (Douglas-Wojciechowski [24]), and we set $\eta(D_i, L_i) := \eta(D_i, L_i)(0)$. We include the Lagrangian subspace L into the notation in order to indicate that $\eta(D, L)$ depends on L in a very definite way (Lesch-Wojciechowski [31] and Corollary 1.12).

A more precise version of the gluing problem for the η-invariant is to compute

$$\eta(D) - \eta(D_1, L_1) - \eta(D_2, L_2) =: \tilde{d}(D_N, L_1, L_2).$$

For a real number x let $[x]$ be its class in $\mathbf{R}/\mathbf{Z}$. Let δD be a deformation of generalized Dirac operators which is trivial in a neighborhood of the hypersurface N. Then there exists a locally defined density $\Omega(D, \delta D)$ such that

$$\delta[\eta(D)] = \int_M \Omega(D, \delta D), \tag{2}$$

$$\delta[\eta(D_i, L_i)] = \int_{M_i} \Omega(D, \delta D). \tag{3}$$

Hence $[\tilde{d}(D_N, L_1, L_2)]$ is a deformation invariant with respect to deformations δD in the interior of M_i.

We conclude this subsection with some remarks on our sign conventions made for gluing. Note that, above, M_1 is considered to be on the left-hand side of the boundary N, while M_2 is on the right-hand side and ∂r points into M_2. To make our considerations more symmetric we will sometimes consider two manifolds M_i with right boundaries N_i and Dirac bundles $E_i \to M_i$ with Dirac operators D_i. We assume product structures as above. Let E_{N_i} be the corresponding Dirac bundles over N_i with Dirac operators D_{N_i}. Let $N_1 \to N_2$ be an isometry denoted by a and assume an isomorphism of tuples $(E_{N_1}, D_{N_1}, I_1) \to (E_{N_2}, D_{N_2}, I_2)$ over a denoted by A. We can glue the manifolds M_i using a and obtain $M := M_1 \cup_a M_2$. Moreover we glue the bundles E_1, E_2 using the composition of A with

I_2 and obtain $E := E_1 \cup_{I_2 \circ A} E_2$. It is easy to check that D_1 and $-D_2$ glue nicely.

We will also use the convention that the symbol M stands for all structures over M. Thus, we write, e.g., $\eta(M_1, L_1)$ for $\eta(D_1, L_1)$. If the left manifold M_2 is considered as a right manifold with Dirac operator $-D_2$, we write $-M_2$, and vice versa. In particular we have

$$\eta(M_2, L_2) = -\eta(-M_2, IL_2). \tag{4}$$

The bundle E_{N_2} is identified with the restriction of E_2 to N_2 using I. This is the reason for the appearance of IL_2 on the right-hand side of (4). We will always identify $V_i = \ker D_{N_i}$, $i = 1, 2$, using A.

1.4. The η-invariant of cylinders. A simple manifold with boundary consisting of two copies of N is the cylinder $Z = [-1, 1] \times N$. Let $E_Z := \mathrm{pr}_N^* E_N$. Then E_Z is a real or complex Dirac bundle with the generalized Dirac operator $D_Z := I(\partial/\partial r + D_N)$. Of course, Z has a left boundary $\{-1\} \times N$ and a right boundary $\{1\} \times N$. Hence, we have to fix two Lagrangian subspaces $L_i \subset V$, $i = 1, 2$, in order to define the boundary conditions, where L_2 is employed at the right boundary $\{1\} \times N$, and L_1 at the left boundary $\{-1\} \times N$ of Z.

The η-invariant $\eta(Z, L_1, L_2)$ was first computed by Lesch-Wojciechowski [31] using separation of variables and the spectral symmetry of D_N. In order to state this result we introduce the involution $\sigma_L := 1 - 2\mathrm{pr}_L$ on V. Note that with I σ_L anticommutes while the unitary $\sigma_{L_1}\sigma_{L_2}$ commutes.

We define the real-valued function $m(L_1, L_2)$ on pairs of Lagrangian subspaces by

Definition 1.3.

$$m(L_1, L_2) := -\frac{1}{\pi} \sum_{\substack{e^{\imath\lambda} \in \mathrm{spec}(-[(\imath+I)/2\imath]\sigma_{L_1}\sigma_{L_2}) \\ \lambda \in (-\pi, \pi)}} \lambda$$

in the complex case and by

Definition 1.4.

$$m(L_1, L_2) := -\frac{1}{\pi} \sum_{\substack{e^{\imath\lambda} \in \mathrm{spec}(-\sigma_{L_1}\sigma_{L_2}) \\ \lambda \in (-\pi, \pi)}} \lambda$$

in the real case, where we consider V as a complex vector space with the complex structure given by I.

Theorem 1.5 (Lesch-Wojciechowski). $\eta(Z, L_1, L_2) = m(L_1, L_2)$.

Note that $\eta(Z, L_1, L_2)$ is independent of the length of Z (which is 1 in [31]). There is a simple relation of the functions $m(L_1, L_2)$ in the real and the complex cases. Consider V to be complex. Assume that V has a real structure $V := V^{\mathbf{R}} \otimes_{\mathbf{R}} \mathbf{C}$ and I respects this real structure, i.e., acts as $I^{\mathbf{R}} \otimes 1_{\mathbf{C}}$. Then $V^{\mathbf{R}}$ has a euclidean structure given by $\langle x, y \rangle = \mathrm{Re}\langle x \otimes 1, y \otimes 1 \rangle$ and a symplectic structure $\Phi^{\mathbf{R}}(x, y) = \langle I^{\mathbf{R}} x, y \rangle$. We make the following simple observation:

Lemma 1.6. *If $L_i = l_i \otimes_{\mathbf{R}} \mathbf{C}$ are complexifications of real Lagrangian subspaces l_i, then $m(L_1, L_2) = m(l_1, l_2)$.*

Proof. V decomposes as $V = V^+ \oplus V^-$, where $V^{\pm}$ are the $\pm\imath$-eigenspaces of I. There is a canonical identification of complex vector spaces $W: V^{\mathbf{R}} \to V^+$ given by $W(x) := x \otimes 1 - I^{\mathbf{R}} x \otimes \imath$. The extension $A \otimes 1$ to V of any operator $A \in \mathrm{End}(V^{\mathbf{R}})^{I^{\mathbf{R}}}$ respects the splitting of V and decomposes as $A \otimes 1 = A^+ \oplus A^-$. Moreover, $A^+ = WAW^{-1}$. Thus, Lemma 1.6 follows. q.e.d.

We give now a relation of m to the Maslov index. Let (V, Φ) be a real symplectic vector space with a compatible complex structure (see Definition 2.5). The Maslov index is an integer $\tau(L_1, L_2, L_3)$ defined for a triple of Lagrangian subspaces (see Lion-Vergne [32] and §2, Definition 2.3). The complex structure I determines a maximal compact subgroup K of the symplectic group G of (V, Φ) as its stabilizer. We have (see Propositions 2.6 and 2.8)

Proposition 1.7. *If $L_i \subset V$, $i = 1, 2$, is a pair of Lagrangian subspaces, then*

$$m(L_1, L_2) = \int_K \tau(kL, L_1, L_2)\, dk,$$

where L is an arbitrary Lagrangian subspace.

The function m is the unique function on pairs of Lagrangian subspaces, which is invariant under K (with respect to the diagonal action $k(L_1, L_2) = (kL_1, kL_2)$, $k \in K$) such that

$$m(L_1, L_2) + m(L_2, L_3) + m(L_3, L_1) = \tau(L_1, L_2, L_3).$$

1.5. The gluing formula. We start with stating the gluing formula $\mathrm{mod}\, \mathbf{Z}$. Recall the geometric situation described in §1.3. In particular we consider M_2 to be on the right-hand side of N. Then

Theorem 1.8 (Gluing formula $\mathrm{mod}\ \mathbf{Z}$). *The class*

$$[\eta(D) - \eta(D_1, L_1) - \eta(D_2, L_2)] = [\tilde{d}(D_N, L_1, L_2)] \in \mathbf{R}/\mathbf{Z}$$

depends only on the relative position of the Lagrangian subspaces L_1, L_2 *with respect to the automorphism group* K *of* $(V, <, >, I)$ *and*

$$[\tilde{d}(D_N, L_1, L_2)] = [m(L_1, L_2)].$$

In order to state the real-valued version of the gluing formula we modify the situation slightly. We change the notation for the manifolds M_i and M which now are denoted by $\widetilde{M}_i$ and $\widetilde{M}$ respectively. Let M be the manifold obtained by gluing $\widetilde{M}_1 \cup_N Z \cup_N \widetilde{M}_2$. Then, M is derived from $\widetilde{M}$ by stretching the product collar of N such that a copy of Z is embedded into M. There is a natural generalized Dirac operator D over M. Let δD be the infinitesimal deformation of D corresponding to the stretching. It is supported in a tubular neighborhood of N, which can also be considered to be a subset of Z. Applying the local variation formula first to Z we get $\int_Z \Omega(D_Z, \delta D_Z) = 0$ by Theorem 1.5 since $\eta(D_Z, L, L) = 0$ and hence $\eta(D_Z, L, L)$ does not depend on the length of the cylinder. Consequently

$$\delta[\eta(D)] = \int_M \Omega(D, \delta D) = 0.$$

Thus, varying the length of the cylindrical part of M changes the η-invariant at most by an integer. Alternatively, one could show $\Omega(D, \delta D) = 0$ by direct computation.

We set $M_1 := \widetilde{M}_1 \cup_N Z$ and $M_2 := Z \cup_N \widetilde{M}_2$. M_i can be considered as overlapping subsets of M with the overlap $M_1 \cap M_2 = Z$. There are generalized Dirac operators D_i over M_i in a natural way. By the same discussion as above the η-invariants of M_i differ from that of $\widetilde{M}_i$ at most by an integer. The gluing problem for this modified geometric situation consists in computing

$$d(D_N, L_1, L_2) := \eta(M) - \eta(M_1, L_1) - \eta(M_2, L_2).$$

Recall that D is the Dirac operator on M, while D_Z lives on Z. The latter is essentially selfadjoint with boundary conditions given by the Lagrangian subspaces L_2, L_1. Here L_1 is employed at $\{1\} \times N$, and L_2 at $\{-1\} \times N$. Let $H := L^2(M, E) \oplus L^2(Z, E_Z)$. There we have the Dirac operator $D_+ = D \oplus D_Z$. Analogously we define $H_0 := L^2(M_1, E_1) \oplus L^2(M_2, E_2)$ and $D_0 := D_1 \oplus D_2$. Again, D_i are essentially selfadjoint with the boundary condition given by the L_i. The η-invariant is additive under the direct sum. Thus, $\eta(D_+) = \eta(M) + \eta(Z, L_2, L_1)$ and $\eta(D_0) = \eta(M_1, L_1) + \eta(M_2, L_2)$. However $\eta(Z, L_2, L_1) = m(L_2, L_1)$ is known by Theorem 1.5. We want to compare $\eta(D_+)$ and $\eta(D_0)$. This is done by comparing suitable functions of D_+ and D_-. For this, both operators must act on the same Hilbert space, say H. The key idea is the definition of a unitary operator $U: H \to H_0$. Let $D_- := U^* D_0 U$. U is defined using the canonical identifications of the parts of $M \cup Z$ on the one hand and of $M_1 \cup M_2$ on the other hand. We employ smooth cut-off functions. The unitary U identifies the corresponding boundary pieces such that $\operatorname{dom} D_+ = \operatorname{dom} D_-$. Moreover, $G := D_+ - D_-$ is a bounded but nonlocal operator on H.

Let $P_\pm$ be the *positive* spectral projections of $D_\pm$. Then it turns out that $P_+ - P_-$ is compact (see §3.3); therefore the relative index $I(P_+, P_-) \in \mathbf{Z}$ of these projections is well defined (see Kato [28], Avron-Seiler-Simon [2], [3], [11], and §3.1, Proposition 3.3).

The gluing formula for the η-invariant is given by

Theorem 1.9 (Gluing formula).

$$\begin{aligned}\eta(M) - \eta(M_1, L_1) - \eta(M_2, L_2) = m(L_1, L_2) &- 2I(P_+, P_-) \\ &+ \dim\ker D_+ - \dim\ker D_- .\end{aligned}$$

The mod $\mathbf{Z}$-gluing formula (Theorem 1.8) follows immediately from Theorem 1.9 and the discussion on how stretching the cylinder influences the η-invariant. The proof of Theorem 1.9 is outlined in §1.6 and completed in §3.

Note that

$$\begin{aligned}\dim\ker D_+ &= \dim\ker D_M + \dim\ker D_Z , \\ \dim\ker D_Z &= \dim(L_1 \cap L_2) , \\ \dim\ker D_- &= \dim\ker D_{M_1} + \dim\ker D_{M_2} .\end{aligned}$$

In general, the index term as well as the dimensions of the kernels cannot be computed. Hence, we would prefer to have a simpler formula not involving those nonexplicit terms. Later in this paper (see Corollary 1.18) we will show that, under a certain regularity condition, the index term and the kernels vanish if one uses a specially adapted boundary condition (given by the limiting values of the extended L_2-solutions) and the cylindrical part of M is long enough.

In [20] Cheeger explains the use of a sort of a gluing formula for computing the η-invariant of mapping tori. He employs manifolds with cones attached. In order to untwist the mapping torus he pinches a cross-section to obtain a manifold with two conical singularities, and then deforms this manifold back to a product. The change of the η-invariant during the latter deformation is calculated by his formula for the variation of the η-invariant for manifolds with conical singularities; see also [5]. The pinching process leaves the η-invariant unchanged (mod $\mathbf{Z}$). The idea of pinching provides an alternative approach to the gluing problem. The η-invariant of a Dirac operator on a manifold with a cone attached differs from the η-invariant of the Dirac operator with Atiyah-Patodi-Singer boundary conditions by at most an integer.

1.6. Outline of the proof of the gluing formula. We compare suitable functions of $D_{\pm}$. First we represent the η-function of a generalized Dirac operator using the heat operator

$$\eta(D)(s) = \frac{1}{\Gamma((s+1)/2)} \int_0^\infty r^{(s-1)/2} \operatorname{Tr} D e^{-rD^2}\, dr.$$

This integral converges absolutely for $\operatorname{Re}(s) > -2$ (see Bismut-Freed [6], Branson-Gilkey [8]). Hence, it also provides the required analytic extension of $\eta(D)(s)$ up to $s = 0$.

Define $\operatorname{sign}(D) := E_D(0, \infty) - E_D(-\infty, 0)$, where $E_D(\ )$ is the family of spectral measures of D. Set

$$R(s, t) := \frac{1}{\Gamma((s+1)/2)} \int_0^\infty r^{(s-1)/2} D e^{-(t+r)D^2}\, dr,$$

where the integral converges in the trace norm for $\operatorname{Re}(t) > 0$ and $\operatorname{Re}(s) > -1$ or $\operatorname{Re}(t) \geq 0$ and $\operatorname{Re}(s) > n$. Then

$$R(s, t) = e^{-tD^2} \operatorname{sign}(D) |D + E_D\{0\}|^{-s}.$$

For $\operatorname{Re}(s) > n$ we have

$$\eta(D)(s) = \operatorname{Tr} R(s, 0).$$

The idea of introducing the new parameter t is that we get an holomorphic extension of $\operatorname{Tr} R(s, t)$ up to the interesting point $(s, t) = (0, 0)$. Thus, we can replace analytic continuation by the limit $(0, t) \overset{t\to 0}{\to} (0, 0)$ in the discussion below.

Let $R_i(s, t)$ be defined as above for D_i with $i = \pm$. Set $\Delta(s, t) := \operatorname{Tr}(R_+(s, t) - R_-(s, t))$ and $A(t) := R_+(0, t) - R_-(0, t)$. The function $\Delta(s, t)$ is holomorphic for $\operatorname{Re}(s) > -1/2$, $\operatorname{Re}(t) \geq 0$, and

$$\lim_{t\to 0} \Delta(s, t) = \eta(D_+)(s) - \eta(D_-)(s)$$

exists uniformly in s on a compact set for $\operatorname{Re}(s) \geq -1/2$. This follows from the estimate

$$|\operatorname{Tr}(D_+ e^{-tD_+^2}) - \operatorname{Tr}(D_- e^{-tD_-^2})| \leq C e^{-ct} e^{-c/t} \qquad \forall t > 0$$

with $C < \infty$, $c > 0$, which is proved by the finite propagation speed method.

We want to compute

$$\delta := \lim_{t\to 0}\Delta(0, t) = \eta(D_+) - \eta(D_-). \tag{5}$$

Formally, δ is the trace of the strong limit

$$s-\lim_{t\to 0} A(t) = \operatorname{sign}(D_+) - \operatorname{sign}(D_-) = 2P_+ - 2P_- + E_{D_+}\{0\} - E_{D_-}\{0\}.$$

Unfortunately the limit on the left-hand side of this equation does not exist in the trace norm, and the right-hand side is not of trace class.

The first idea is that one can add a suitable commutator K to $P_+ - P_-$ (see Lemma 3.2) such that $P_+ - P_- + K$ is of trace class and such that

$$\operatorname{Tr}(P_+ - P_- + K) = -I(P_+, P_-).$$

The second idea is to get control over the convergence of $A(t)$ in the trace norm as follows. We can extend K to a holomorphic family of commutators $K(t)$ that are of trace class for $\operatorname{Re}(t) > 0$ (see Definition 3.4). We find another holomorphic family of trace class operators $B(t)$ with $\operatorname{Tr} B(t) = 0$ and commutators $\widetilde{K}(t)$ such that the limits $B := s-\lim_{t\to 0} B(t)$, $\widetilde{K} := s-\lim_{t\to 0}\widetilde{K}(t)$ exist. Moreover, $B + 2\widetilde{K}$ is of trace class and $\operatorname{Tr}(B + 2\widetilde{K}) = 0$. These families are constructed such that

$$A(t)+2K(t)-B(t)-2\widetilde{K}(t) \overset{t\to 0}{\rightrightarrows} 2P_+-2P_-+2K-B-2\widetilde{K}+E_{D_+}\{0\}-E_{D_-}\{0\}$$

in the trace norm. It follows that

$$\begin{aligned}\delta &= \lim_{t\to 0}\operatorname{Tr}(A(t) + 2K(t) - B(t) - 2\widetilde{K}(t))\\ &= \operatorname{Tr}(2P_+ - 2P_- + 2K - B - 2\widetilde{K}) + \operatorname{Tr}(E_{D_+}\{0\} - E_{D_-}\{0\})\\ &= -2I(P_+, P_-) + \dim\ker D_+ - \dim\ker D_-.\end{aligned}$$

The gluing formula follows immediately.

The families $B(t)$, $\widetilde{K}(t)$ will be constructed analogously to $A(t)$, $K(t)$. Note that $M \cup Z$ and also $M_1 \cup M_2$ contain two copies of the cylinder Z. Thus, we can consider the corresponding Hilbert space $h := L^2(Z \cup Z, E_Z \cup E_Z)$ as a subspace of H. On h we consider the unbounded operators $\widetilde{D}_+ := D_Z \oplus D_Z$ and $\widetilde{D}_- := U^*(D_Z \oplus D_Z)U$. We extend these operators by zero in order to obtain operators on H, and we define $\widetilde{R}(s, t)_\pm$ and $\widetilde{K}(t)$ by the same procedure as for $D_\pm$. Then $B(t) := \widetilde{R}_+(0, t) - \widetilde{R}_-(0, t)$. The main point behind this construction is that $D_\pm$ and $\widetilde{D}_\pm$ coincide locally on the set where the cut-off functions used for defining U are not constant. Note that $D_\pm$ and $\widetilde{D}_\pm$ should not be compared in a neighborhood of the boundary of the copies of Z.

1.7. Applications. Let (M_i, N_i), $i = 1, 2, 3$, be a triple of compact Riemannian manifolds with boundary, and $E_i \to M_i$ be real or complex Dirac bundles with Dirac operators D_i over M_i. All manifolds are considered as left manifolds. We assume that N_i are pairwise isometric (we denote the isometry by a) and $(E_{i|N_i}, I_i, D_{N_i})$ are isomorphic (with the isomorphism A over the given isometry a) as explained in §1.3 such that we can glue arbitrary pairs, thus obtaining $M_{12} := M_1 \cup_N -M_2$, $M_{23} := M_2 \cup_N -M_3$, and $M_{31} := M_3 \cup_N -M_1$ together with the corresponding Dirac operators D_{12}, D_{23}, D_{31}. From the gluing formula (Theorem 1.9) and (see the proof of Lemma 2.9)

$$(6)\qquad \tau_I(L_1, L_2, L_3) := m(L_1, IL_2) + m(L_2, IL_3) + m(L_3, IL_1) \in \mathbf{Z}$$

we obtain

Corollary 1.10 (Cocycle). $[\eta(M_{12}) + \eta(M_{23}) + \eta(M_{31})] = 0 \in \mathbf{R}/\mathbf{Z}$.

Later we will study the integer-valued cocycle $\eta(M_{12})+\eta(M_{23})+\eta(M_{31})$ in the adiabatic limit. In the special case of the odd signature operator this cocycle can be computed from the nonadditivity theorem for the signature of Wall [39]. For the odd signature operator on oriented manifolds (M, N) of dimension $n = 4k - 1$ we consider the real symplectic vector space $V := H^{2k-1}(N, \mathbf{R})$ with the symplectic structure given by the intersection form. Let $l_i := \operatorname{im}(H^{2k-1}(M_i, \mathbf{R}) \to H^{2k-1}(N, \mathbf{R}))$, $i = 1, 2, 3$. Then the l_i are Lagrangian subspaces. In the signature operator case we have (see Lemma 2.12)

$$(7)\qquad \eta(M_{12}) + \eta(M_{23}) + \eta(M_{31}) = \tau(l_1, l_2, l_3),$$

where $\tau(l_1, l_2, l_3)$ is the Maslov index of l_1, l_2, l_3. Note that the identifications of the bundles of differential forms are different from those used in the case of general Dirac operators. This is the reason for the appearance of the Maslov index instead of the I-twisted Maslov index (6) in (7).

Let $(\widetilde{M}_i, N_i)$, $i = 1, 2$, be a pair of compact Riemannian manifolds with boundary, and $E_i \to \widetilde{M}_i$ be real or complex Dirac bundles with the Dirac operator D_i being again isomorphic over N_i. If $\widetilde{M}_1 \supset N_1 \xrightarrow{f} N_2 \subset \widetilde{M}_2$ is another isometry, and f is covered by some isomorphism $F: E_{N_1} \to E_{N_2}$ which intertwines (I_1, D_{N_1}) with (I_2, D_{N_2}), then we can also use (f, F) instead of (a, A) for gluing. We denote the resulting manifold by $M^f = \widetilde{M}_1 \cup Z \cup_f -\widetilde{M}_2$. It carries the Dirac bundle $E^F \to M^f$ and the Dirac operator D^F. We use the boundary conditions defined by the Lagrangian subspace L at N_1 and the subspace IL at N_2.

Let $P_\pm^F$ be the positive spectral projections for $D_\pm^F$ (of course D_- and D_-^F are unitary equivalent).

Corollary 1.11 (Cutting and pasting formula).

$$\begin{aligned}\eta(M^f) - \eta(M) = m(F(L), L) + \dim(L \cap F(L)) - \tfrac{1}{2}\dim\ker D_N \\ + \dim\ker D^F - \dim\ker D - 2I(P_+^F, P_-^F) + 2I(P_+, P_-),\end{aligned}$$

where $F(L)$ is the image of L under the action of F on $\ker D_N$. *In particular,*

$$[\eta(M^f) - \eta(M)] = [m(F(L), L)] \in \mathbf{R}/\mathbf{Z}.$$

One can simplify

$$-I(P_+^F, P_-^F) + I(P_+, P_-) = I(UP_+U^*, U^F P_+^F (U^F)^*).$$

One can also reproduce and refine the result of Lesch-Wojciechowski about the dependence of the η-invariant on the boundary condition. Let $(\widetilde{M}_0, N)$ be a compact Riemannian manifold with boundary, and $E_0 \to \widetilde{M}_0$ be a real or complex Dirac bundle with the Dirac operator D. If we are given two Lagrangian subspaces L_i, $i = 1, 2$, we can take $(\widetilde{M}_1, N) := (\widetilde{M}_0, N)$ with the boundary condition L_1 and $(\widetilde{M}_2, N) := (\widetilde{M}_0, N)$ with the boundary condition L_2. By gluing we obtain $M = \widetilde{M}_1 \cup_N Z \cup_N -\widetilde{M}_2$ and $\eta(M) = 0$ for symmetry reasons. The gluing formula gives

Corollary 1.12 (Dependence of η on the Lagrangian subspace).

$$\begin{aligned}\eta(M_0, L_1) - \eta(M_0, L_2) = m(IL_2, L_1) + 2I(P_+, P_-) - \dim\ker D_M \\ - \dim(L_1 \cap IL_2) + \dim\ker D_{M_1} + \dim\ker D_{M_2}.\end{aligned}$$

Taking this modulo $\mathbf{Z}$ we obtain Theorem 3.1 of Lesch-Wojciechowski [31]. Note that $m(L_2, IL_2) + m(IL_2, L_1) + m(L_2, L_2) = \tau(L_2, IL_2, L_1)$, $m(L_2, IL_2) = 0$, and, hence,

$$m(IL_2, L_1) = m(L_2, L_1), \qquad \bmod \mathbf{Z}.$$

1.8. The adiabatic limit. In this subsection we discuss the vanishing of the index term $I(P_+, P_-)$ and of the kernels of $D_\pm$ in the adiabatic limit. Recall the modification of the geometric situation made in §1.5. There we have defined $M := \widetilde{M}_1 \cup_N Z \cup_N \widetilde{M}_2$, $M_1 := \widetilde{M}_1 \cup_N Z$, and $M_2 := Z \cup_N \widetilde{M}_2$. For $r \geq 0$ we could glue a cylinder $Z_r := [-r, r] \times N$ into $\widetilde{M}$ and obtain $M_\tau := \widetilde{M}_1 \cup_N Z_r \cup_N \widetilde{M}_2$. Let $M_{1,r} = \widetilde{M}_1 \cup_N Z_r$ and $M_{2,r} = Z_r \cup_N \widetilde{M}_2$. There are generalized Dirac operators D_r, $D_{i,r}$ over M_r, $M_{i,r}$ in a natural way. We will often omit the parameter r to make the notation more compact. The gluing formula Theorem 1.9 is valid for any $r \geq 1$. We will study the index term $I(P_+, P_-)$ and $\dim\ker D_\pm$ for large r. Note that both terms depend very sensitively on the data D_i, L_i, $i = 1, 2$, as, e.g., the dimension of kernels or eigenspaces in general. Thus, there is no hope to compute $I(P_+, P_-)$ in general. Our way out is to make a special choice of the Lagrangian subspaces L_i depending on D_i (and not on r) such that, under certain regularity conditions, $I(P_+, P_-) = 0$ and $\ker D_\pm = \{0\}$ for all $r \geq r_0$.

We will also need more general boundary conditions. Recall that $V := \ker D_N$. To a general subspace $L \subset V$ we associate the boundary conditions $B_L := \{(P_< + \mathrm{pr}_{L^\perp})\phi_{|N} = 0\}$ at the right boundaries and $B_L := \{(P_> + \mathrm{pr}_{L^\perp})\phi_{|N} = 0\}$ at the left boundaries. Of course, B_L gives rise to a selfadjoint extension of the corresponding Dirac operator iff L is Lagrangian.

Let B_i be the closure of D_i subject to the boundary condition B_V. Then $\ker B_i$ is finite dimensional. Let us have, for $i = 1, 2$,

Definition 1.13.

$$L_{M_i} := \{\mathrm{pr}_V \phi_{|N} | \phi \in \ker B_i\}. \tag{8}$$

In other words, L_{M_i} is the space of the limiting values of the extended L^2-solutions of the Dirac operator if one glues a complete half-cylinder on M_i. The following fact was proved by Yoshida [41] in a special case (with a different method) and also by Nicolaescu [37].

Proposition 1.14. *The $L_{M_i} \subset V$, $i = 1, 2$, are Lagrangian subspaces.*

See Proposition 4.1 for a proof. Of course, the position of the L_{M_i} inside V is highly sensitive with respect to D_i.

Next we discuss the regularity assumptions. Note that B_i^* is the closure of D_i subject to the boundary conditions $B_{\{0\}}$.

Assumption 1.15. $\ker B_i^* = \{0\}$.

Assumption 1.16. $L_{M_1} \cap L_{M_2} = \{0\}$.

Set $L_1 := L_{M_1}$ and $L_2 := L_{M_2}$, and define the selfadjoint extensions D_i and D_Z as above using the Lagrangian subspaces L_i. We employ L_1 at $\{-r\} \times N$ and L_2 at $\{r\} \times N$. Then the assumptions will imply

$\ker D_r = \{0\}$ for $r \geq r_0$,

$\ker D_Z = 0$,

$\ker D_i = 0$, $i = 1, 2$.

In fact, $\ker D_Z \cong L_1 \cap L_2$, and with Assumption 1.15 also $\ker D_i \cong L_1 \cap L_2$. The claim for D_r follows from the discussion of the small eigenvalues when r becomes large. Thus, for $r \geq r_0$, also $\ker D_+ = \ker D_- = \{0\}$.

Unfortunately the regularity condition excludes a lot of interesting cases.

For the odd signature operator this condition is valid iff the image of $H^*(M_i, N)$ in $H^*(M)$ vanishes. This follows from the discussion of the limiting L^2-solutions given in Atiyah-Patodi-Singer [1]. On the other hand, in Yoshida [41] it is shown by example that regularity holds in other cases.

Theorem 1.17 [Vanishing of the relative index term]. *Assume Assumptions* 1.15 *and* 1.16. *Then there is an* $r_0 \geq 1$ *such that for all* $r \geq r_0$

$$I(P_+, P_-) = 0.$$

This vanishing is stable against small perturbations of the L_i. We prove Theorem 1.17 in §4 by first interpreting the relative index as the spectral flow of the family $D(u) := D_- + uG$, $u \in [0, 1]$ (see [11]) and then proving a splitting formula showing that this spectral flow vanishes indeed for $r \geq r_0$. The argument is a variation of Yoshida's proof of the splitting formula [41].

Using Theorem 1.17 we can restate our gluing formula for the η-invariant as follows.

Corollary 1.18 (Simplified gluing formula). *Assume Assumptions* 1.15 *and* 1.16. *Then there is an* $r_0 \geq 1$ *such that for all* $r \geq r_0$

$$\eta(M) = \eta(M_1, L_2) + \eta(M_2, L_1) + m(L_2, L_1).$$

Note that the L_i are not arbitrary.

1.9. Applications in the adiabatic limit. As in §1.7 we consider here all manifolds (M_i, N_i) with boundary to be left manifolds. Moreover, we assume isometries (called a in §1.3) of the N_i and identifications (denoted by A) of the bundles E_{N_i}. We will denote L_{M_i} by L_i. Assumption 1.16 has to be modified to

Assumption 1.19. $L_1 \cap IL_2 = \{0\}$.

If M and D are obtained by gluing M_1 and $-M_2$ using the given identifications under the regularity conditions 1.15 and 1.19, the gluing formula for the η-invariant reads

$$\eta(M) = \eta(M_1, IL_2) - \eta(M_2, IL_1) + m(IL_2, L_1) \tag{9}$$

if $r \geq r_0$.

Let $(\widetilde{M}_i, N_i)$, $i = 1, 2, 3$, be a triple of compact Riemannian manifolds with boundary, and $E_i \to \widetilde{M}_i$ be real or complex Dirac bundles with Dirac operators D_i over M_i. We assume the product structures and that N_i and (E_{N_i}, I_i, D_{N_i}) are pairwise isomorphic such that by gluing arbitrary pairs we can obtain $M_{12}+ := \widetilde{M}_1 \cup_N Z_r \cup_N -\widetilde{M}_2$, $M_{23} :=$

$\widetilde{M}_2 \cup_N Z_r \cup_N -\widetilde{M}_3$, and $M_{31} := \widetilde{M}_3 \cup_N Z_r \cup_N -\widetilde{M}_1$, together with the corresponding Dirac operators D_{12}, D_{23} and D_{31}. Moreover, recall that L_i, $i = 1, 2, 3$, are given by

$$L_i := \{\mathrm{pr}_V \phi_{|N_i} | \phi \in \ker B_i\}, \tag{10}$$

where B_i is the closure of D_i with respect to the boundary condition B_V.

Recall the I-twisted Maslov index of the triple of Lagrangian subspaces (L_1, L_2, L_3):

$$\tau_I(L_1, L_2, L_3) := m(L_1, IL_2) + m(L_2, IL_3) + m(L_3, IL_1). \tag{11}$$

Corollary 1.20 (Cocycle). *Assume the regularity conditions* 1.15 *and* 1.19, *i.e.*, $\ker B_i^* = \{0\}$ *for* $i = 1, 2, 3$, *and* $L_i \cap IL_j = \{0\}$ *for* $i \neq j$, $i, j = 1, 2, 3$. *Then there is an* $r_0 > 1$ *such that for all* $r \geq r_0$

$$\eta(M_{12}) + \eta(M_{23}) + \eta(M_{31}) = \tau_I(L_1, L_2, L_3) \in 2\mathbf{Z}.$$

Proof. Recall that we consider all M_i to be left manifolds. We apply equation (9). Let r_0 be large enough. Then

$$\begin{aligned} \eta(M_{12}) + \eta(M_{23}) + \eta(M_{31}) & \\ &= \eta(M_1, IL_2) - \eta(M_2, IL_1) - m(L_1, IL_2) \\ &+ \eta(M_2, IL_3) - \eta(M_3, IL_2) - m(L_2, IL_3) \\ &+ \eta(M_3, IL_1) - \eta(M_1, IL_3) - m(L_3, IL_1). \end{aligned} \tag{12}$$

We also have

$$\begin{aligned} \eta(M_1, IL_2) - \eta(M_1, IL_3) &= m(L_1, IL_2) - m(L_1, IL_3), \\ \eta(M_2, IL_3) - \eta(M_2, IL_1) &= m(L_2, IL_3) - m(L_2, IL_1), \\ \eta(M_3, IL_1) - \eta(M_3, IL_2) &= m(L_3, IL_1) - m(L_3, IL_2). \end{aligned} \tag{13}$$

We explain (13). Note that there is a path of Lagrangian subspaces $L(t)$ from IL_3 to IL_2 inside of the affine set $\Lambda_{L_1} := \{L \in V | L \cap L_1 = \{0\}\}$. Consider $D_Z(t)$ on Z_r subject to the boundary conditions B_{L_1} at $\{-r\} \times N$ and $B_{L(t)}$ at $\{r\} \times N$. Then, by the method of Lesch-Wojciechowski [31],

$$\frac{\partial}{\partial t}\eta(D_Z(t)) = \frac{\partial}{\partial t}\eta(D_1, L(t)).$$

Both families of η-invariants are smooth by the definition of the path $L(t)$. Equation (13) follows from Theorem 1.5. Substituting this in (12) we get the claim. It will be shown in Lemma 2.9 that the I-twisted Maslov index is even. q.e.d.

Via the index theorem of Atiyah-Patodi-Singer [1] (see Theorem 1.1) Corollary 1.20 provides a generalization of Wall's nonadditivity of the index for the signature operator [39] (which is discussed extensively in §2) to general Atiyah-Patodi-Singer boundary value problems. We recall the geometric situation described in §1.2 adapted to our current problem. Assume that $X_\pm$ are $(n+1)$-dimensional Riemannian manifolds with $\mathbf{Z}_2$-graded Dirac bundles $F_\pm = F_\pm^+ \oplus F_\pm^-$ and Dirac operators $D_\pm$. Moreover, assume that $\partial X_+ = M_{12}$, $\partial X_- = M_{23}$ (metrically) and that the metric and the bundles respect a product structure near the boundary. Finally, we assume that

$$D_+^+ = \sigma(\tfrac{\partial}{\partial s} + D_{12}), \qquad D_-^+ = \sigma(\tfrac{\partial}{\partial s} + D_{23})$$

near the boundary, where $D_\pm^+ \colon \Gamma(F_\pm^+) \to \Gamma(F_\pm^-)$ is 'half' of $D_\pm$ with respect to the grading of $F_\pm$. Hence, we assume that

$$F^+_{+|M_{12}} = E_{12}, \qquad F^+_{-|M_{23}} = E_{23}.$$

Recall that, e.g., $M_{12} = \widetilde{M}_1 \cup_N Z_r \cup_N -\widetilde{M}_2$. One can now glue $X_+ \cup_{\widetilde{M}_2 \cup_N [0,r]\times N} X_-$, obtaining the Riemannian manifold X and a Dirac bundle $F \to X$ with the Dirac operator D (here one has to smooth the corner in some nonunique way). X has the boundary $\partial X = \widetilde{M}_1 \cup_N Z_r \cup_N -\widetilde{M}_3$. Consider the Dirac operators as maps

$$\begin{aligned}
D_+^+ &\colon \{\psi \in C^\infty(X_+, F_+^+) | E_{D_{12}}(-\infty, 0]\psi_{|M_{12}} = 0\} \to C^\infty(X_+, F_+^-),\\
D_-^+ &\colon \{\psi \in C^\infty(X_-, F_-^+) | E_{D_{23}}(-\infty, 0]\psi_{|M_{23}} = 0\} \to C^\infty(X_-, F_-^-),\\
D^+ &\colon \{\psi \in C^\infty(X, F^+) | E_{D_{13}}(-\infty, 0]\psi_{|M_{13}} = 0\} \to C^\infty(X, F^-),
\end{aligned}$$

where $E_D(-\infty, 0]$ denotes the spectral projection onto the nonpositive subspace of D. Then the index of these maps is well defined, and by the theorem of Atiyah-Patodi-Singer

$$\begin{aligned}
\operatorname{index}(D_+^+) &= \int_{X_+} \Omega(D_+^+) + \frac{\eta(D_{12}) - \dim\ker D_{12}}{2},\\
\operatorname{index}(D_-^+) &= \int_{X_-} \Omega(D_-^+) + \frac{\eta(D_{23}) - \dim\ker D_{23}}{2},\\
\operatorname{index}(D^+) &= \int_{X} \Omega(D^+) + \frac{\eta(D_{13}) - \dim\ker D_{13}}{2}.
\end{aligned}$$

Applying Corollary 1.20 we obtain

Corollary 1.21. *Assume the regularity condition* 1.15, *i.e.,* $\ker B_i = \{0\}$ *for* $i = 1, 2, 3$, *as well as* 1.19, *i.e.,* $L_i \cap IL_j = \{0\}$ *for* $i \neq j$, $i, j = 1, 2, 3$. *Then there is an* $r_0 > 1$ *such that for all* $r \geq r_0$

$$\operatorname{index}(D_+^+) + \operatorname{index}(D_-^+) - \operatorname{index}(D^+) = \tfrac{1}{2}\tau_I(L_1, L_2, L_3).$$

In fact, the integrals of the index densities cancel out each other. Using the product structure one can smooth the objects near the corner such that the index density vanishes in that region.

Let $(\widetilde{M}_i, N_i)$, $i = 1, 2$, be a pair of compact Riemannian manifolds with boundary, and $E_i \to \widetilde{M}_i$ be real or complex Dirac bundles with Dirac operators D_i. We assume product structures near the boundary. Moreover, we assume that $a: N_1 \to N_2$ is an isometry, and there is an isomorphism $A: E_{N_1} \to E_{N_2}$ of bundles intertwining with I_i and D_{N_i} living over a.

If $\widetilde{M}_1 \supset N_1 \xrightarrow{f} N_2 \subset \widetilde{M}_2$ is another isometry, and f is covered by some isomorphism $F: E_{N_1} \to E_{N_2}$ of bundles intertwining with I_i and D_{N_i}, then we can also use (f, F) for gluing. We denote the resulting manifold by $M^f = \widetilde{M}_1 \cup Z_r \cup_f -\widetilde{M}_2$. It carries the Dirac bundle $E^F \to M^f$ and the Dirac operator D^F. Again this construction contains a parameter r being the half of the length of the cylinder glued in. Recall that we identify $\ker D_{N_i}$ using A.

Corollary 1.22. *Assume* 1.15 *for* $i = 1, 2$ *and*

$$L_1 \cap IL_2 = F(L_1) \cap IL_2 = \{0\}.$$

Then there is an $r_0 > 1$ *such that for all* $r \geq r_0$

$$\eta(D_r^F) - \eta(D_r) = m(F(L_1), IL_2) - m(L_1, IL_2).$$

Proof. Both D_i, $i = 1, 2$, are considered to live on the left-hand side of the boundary N_i. We apply equation (9).

$$\begin{aligned}\eta(M^f) - \eta(M) = \eta(M_1, F^{-1}(IL_2)) &- \eta(M_2, IF(L_1)) - m(F(L_1), IL_2)\\ &- \eta(M_1, IL_2) + \eta(M_2, IL_1) + m(L_1, IL_2).\end{aligned}$$

Moreover, similarly as for equation (13)

$$\begin{aligned}\eta(M_1, IF^{-1}(L_2)) - \eta(M_1, IL_2) &= m(F(L_1), IL_2) - m(L_1, IL_2),\\ \eta(M_2, IF(L_1)) - \eta(M_2, IL_1) &= m(IL_2, F(L_1)) - m(IL_2, L_1).\end{aligned}$$

The claim now follows.

1.10. A splitting formula for the spectral flow. Another application of Theorem 1.17 provides a splitting formula for the spectral flow. Let

$(\widetilde{M}_i, N_i)$ be a pair of compact Riemannian manifolds with boundary and $E_i \to \widetilde{M}_i$ be Dirac bundles. We assume a product structure of the metrics and the Dirac bundles near the boundary N_i. This provides $E_{N_i} \to N_i$ and $I_i \in \Gamma(\mathrm{End}(E_{N_i}))$ as explained in §1.3. Moreover, we assume that N_i are isometric with an isometry denoted by a and that there is an isomorphism $A: (E_{N_1}, I_1) \to (E_{N_2}, I_2)$. Let $D_i(t)$, $i = 1, 2$, $t \in [0, 1]$, be a family of Dirac operators on E_i such that $D_i(t)$ have a constant principal symbol. Equivalently, $D_i(t) - D_i(s)$, $s, t \in [0, 1]$, is a bundle endomorphism. Such a family, for example, comes from a family of connections on the Clifford bundles E_i. We assume that the Dirac operators $D_i(t)$ have a product structure near N for all $t \in [0, 1]$. We obtain a family $D_{N_i}(t)$ on E_{N_i}. Moreover, we assume that $(E_{N_i}, I_i, D_{N_i}(t))$ are isomorphic for $i = 1, 2$ and all t via the identification A. Then we can glue the operators for each $t \in [0, 1]$ and obtain a family $D(t)$ of Dirac operators on M (recall that we glue in a cylinder of length r). Let $V(t) := \ker D_{N_1}(t)$. Note that the dimension of $V(t)$ may be discontinuous. Let $L_i(t) \subset V(t)$, $i = 1, 2$, be families of Lagrangian subspaces and define families of selfadjoint extensions $D_i(t)$, $i = 1, 2$, and $D_Z(t)$ as above using the boundary conditions given by $L_i(t)$. Recall the construction of $D_\pm$ in §1.5. Thus, we can also form the families $D_\pm(t)$ and make the following assumptions.

Assumption 1.23. *$D_\pm(t)$ are 'continuous' such that the spectral flow is well defined.*

Assumption 1.24. $\ker D_+(t) = \ker D_-(t) = \{0\}$ *for* $t = 0, 1$.

Assumption 1.23 is formulated informally, but it is, what we really need. At the points t where $\dim V(t)$ is continuous, it is enough to have a continuous family of Lagrangian subspaces $L_i(t)$ (the union of all $V(t)$, $t \in [0, 1]$, has the structure of a continuous bundle of symplectic vector spaces at those points as explained in Yoshida [41]). The continuity condition for $L_i(t)$ at points where $\dim V(t)$ jumps has to take into account which vectors of $V(t)$ come from or move into the positive or negative spectral subspace of $D_N(t)$ (see Nicolaescu [37] for a detailed account).

Assuming 1.23 and 1.24 it can be shown (compare [11] or Proposition 3.3) that $P_\pm(1) - P_\pm(0)$ is compact and that

$$sf\{D_\pm(t)\} = I(P_\pm(1), P_\pm(0)),$$

where $P_\pm(t)$ are the *positive* spectral projections of $D_\pm(t)$. By the algebraic properties of the relative index we have

$$0 = I(P_+(1), P_+(0)) + I(P_+(0), P_-(0)) + I(P_-(0), P_-(1)) + I(P_-(1), P_+(1))$$

and, thus,

(14) $$sf\{D_+(t)\} - sf\{D_-(t)\} = -I(P_+(0), P_-(0)) + I(P_+(1), P_-(1)).$$

Hence we obtain

Corollary 1.25 (Splitting Formula for the Spectral Flow). *Assume* 1.15, 1.19 (1.24 *is implied by* 1.15 *and* 1.19 *if* $r \geq r_0$) *for* $D_i(0)$ *and* $D_i(1)$, $i = 1, 2$, *and* 1.23. *If* $L_i(0)$ *and* $L_i(1)$ *are given by* (10), *then there is an* $r_0 \geq 1$ *such that for all* $r \geq r_0$

$$sf\{D(t)\} = sf\{D_1(t)\} + sf\{D_2(t)\} - sf\{D_Z(t)\}.$$

The spectral flow of $\{D_Z(t)\}$ is related to a symplectic invariant of the family of pairs of Lagrangian subspaces $\{L_1(t), IL_2(t)\}$ as explained in Yoshida [41] (see also [18]). Such a splitting formula was proved also by Cappell-Lee-Miller in [17] as cited in Kirk-Klassen [29]. Moreover there is a very interesting paper by Nicolaescu [37], which the author received after this work was finished. There, a more general splitting theorem is proved without the assumption of a constant principal symbol and regularity. Moreover, one finds also a nice discussion of the Maslov index for families of pairs of Lagrangian subspaces in a infinite-dimensional framework.

2. The Maslov index

2.1. Cochains. We recall the definition of the complex of (measurable) cochains $C^*_{(\lambda)}(X)$ associated with a (measurable) space X. If there is a group G acting on X, we consider the cohomology of the complex of G-invariant cochains.

Let X be a measurable space. Set

$$C^q(X) := \{\text{bounded measurable functions on } \underbrace{X \times \cdots \times X}_{q+1}\}.$$

Without further notice we will consider real-valued functions. The differential $d\colon C^q(X) \to C^{q+1}(X)$ is given by the formula

$$df(x_0, \cdots, x_{q+1}) := \sum_{i=0}^{q+1} (-1)^i f(x_0, \cdots, \hat{x}_i, \cdots, x_{q+1})$$

for $f \in C^q(X)$. Let $C^*_\lambda(X) \subset C^*(X)$ be the subcomplex of completely antisymmetric functions.

Lemma 2.1. *The complexes* $(C^*(X), d)$ *and* $(C^*_\lambda(X), d)$ *are acyclic.*

Proof. Fix a point $o \in X$. Then there is a contraction $s_o\colon C^q_{(\lambda)}(X) \to C^{q-1}_{(\lambda)}(X)$ given by

$$(s_o f)(x_0, \cdots, x_{q-1}) := f(o, x_0, \cdots, x_{q-1}).$$

In fact

$$ds_o + s_o d = \mathrm{id}. \tag{15}$$

See also Spanier [38]. q.e.d.

Let G be a group acting on X (preserving the measurable structure). Then G acts on $X \times \cdots \times X$ ($q+1$ factors) diagonally. Hence, it acts on the cochains $C^*_{(\lambda)}(X)$ by

$$gf(x_0, \cdots, x_q) := f(g^{-1}x_0, \cdots, g^{-1}x_q),$$

where $g \in G$, and $gf, f \in C^q_{(\lambda)}(X)$. Let $C^*_{(\lambda)}(X)^G \subset C^*_{(\lambda)}(X)$ denote the G-invariant functions.

The complex $(C^*_{(\lambda)}(X)^G, d)$ need not be acyclic in general. If, e.g., G is a countable group and $X = G$, then

$$H^*(C^{\cdot}(G)^G) = H^*(G, \mathbf{R})$$

(compare Brown [9]).

Let now G be a compact topological group with normalized Haar measure dg. Assume that $G \times X \to X$ is measurable. Then there is a projection $m\colon C^*_{(\lambda)}(X) \to C^*_{(\lambda)}(X)^G$ given by

$$mf := \int_G gf\, dg.$$

It is easy to check that

$$dm = md. \tag{16}$$

It follows that

Lemma 2.2. *If G is a compact group acting on X such that $G \times X \to X$ is measurable, then $H^*(C_{(\lambda)}(X)^G) = 0$.*

In fact, for finite groups $H^*(G, \mathbf{Z})$ is a torsion group and thus $H^*(G, \mathbf{R}) = 0$.

2.2. The Maslov index. We recall the definition of the Maslov index $\tau(l_1, l_2, l_3)$ associated to a triple (l_1, l_2, l_3) of Lagrangian subspaces of a real symplectic vector space (V, Φ). τ gives a 2-cocycle over Λ, the space of Lagrangian subspaces of V (see Lion-Vergne [32] for most of that material). We will find 1-cochains μ such that $d\mu = \tau$.

Let (V, Φ) be a symplectic vector space over $\mathbf{R}$, and G be the group of linear symplectic automorphisms $G = \mathrm{Sp}(V)$. Let Λ be the space of Lagrangian subspaces of V. Λ is a compact manifold, and there is an obvious G-action on Λ.

A triple $l_1, l_2, l_3 \in \Lambda$ determines a quadratic form Q on $l_1 \oplus l_2 \oplus l_3$:

$$Q(x_1, x_2, x_3) = \Phi(x_1, x_2) + \Phi(x_2, x_3) + \Phi(x_3, x_1),$$

where $x_i \in l_i$, $i = 1, 2, 3$.

Definition 2.3. The *Maslov index* of l_1, l_2, l_3 is given by the integer

$$\tau(l_1, l_2, l_3) := \operatorname{sign} Q.$$

Here $\operatorname{sign} Q := p - q$, where p is the number of 1's, while q is the number of -1's on the diagonal of a matrix representing Q with respect to a suitable diagonalizing basis.

The following is proved in Lion-Vergne [32].

Theorem 2.4 (Vergne). *The Maslov index τ satisfies*

1. $\tau \in C^2_\lambda(\Lambda)^G$ *and*
2. $d\tau = 0$.

Note that there is no G-invariant $\mu \in C^1_\lambda(\Lambda)$ with $d\mu = \tau$. In fact,

$$(g_1, g_2) \to c_l(g_1, g_2) := \tau(l, g_1 l, g_1 g_2 l)$$

is a nontrivial (over $\mathbf{R}$) group cochain, and thus $0 \neq [\tau] \in H^2(C_\lambda(\Lambda)^G)$.

If $K \subset G$ is a maximal compact subgroup, we can find of course a K-invariant cochain μ with $d\mu = \tau$. Such a cochain is then naturally associated to a compatible complex structure on V.

Definition 2.5. A complex structure I in V is said to be compatible if $\langle \cdot, \cdot \rangle := \Phi(\cdot, I\cdot)$ is a positive-definite scalar product.

Let us fix a compatible complex structure I on V. Then $K := \mathrm{Stab}_G(I) = Gl(V, I) \cap \mathrm{Sp}(V) \cong U(V)$ is a maximal compact subgroup of G. Note that K acts transitively on Λ.

Proposition 2.6. *There is a unique $\mu \in C^1_\lambda(\Lambda)^K$ with $d\mu = \tau$.*

Proof. Fix some $l \in \Lambda$ and set $\mu := m s_l(\tau)$, where m is the averaging with respect to K. Thus, by the G-invariance of τ

$$\mu(l_1, l_2) = \int_K \tau(kl, l_1, l_2)\, dk.$$

By (15), (16) and Theorem 2.4(2) we have $dms_l(\tau) = mds_l(\tau) = m(\tau) = \tau$. Obviously, μ is K-invariant.

We have to show the uniqueness. Let $\mu_i \in C^1_\lambda(\Lambda)^K$, $i = 1, 2$, with $d\mu_1 = d\mu_2 = \tau$. Then $d(\mu_1 - \mu_2) = 0$ and there is a $\xi \in C^0_\lambda(\Lambda)^K$ with

$d\xi = \mu_1 - \mu_2$ by Lemma 2.2. However K acts transitively on Λ and thus ξ is constant. Hence, $d\xi = 0$. q.e.d.

We compute μ in the two-dimensional case $V := \mathbf{R}^2$. Let $V := \mathbf{R}P \oplus \mathbf{R}Q$, where (P, Q) is a basis, with the symplectic structure given by $\Phi(P, Q) = 1$, $\Phi(P, P) = \Phi(Q, Q) = 0$. Fix the complex structure I by $I(P) = Q$, $I(Q) = -P$. Then $\langle P, P\rangle = \langle Q, Q\rangle = 1$, $\langle P, Q\rangle = 0$. We have $G = \mathrm{Aut}(V, \Phi) = SL(2, \mathbf{R})$ and $K = \mathrm{Stab}_G(I) = SO(2)$. Any one-dimensional subspace l parametrized by the angle $\alpha(l)$ between $\mathbf{R}P$ and l, with $\alpha \in [0, \pi)$, is a Lagrangian subspace.

Lemma 2.7.

$$\mu(l_1, l_2) = \begin{cases} 1 - \frac{2(\alpha(l_2) - \alpha(l_1))}{\pi}, & \alpha(l_2) > \alpha(l_1), \\ -1 - \frac{2(\alpha(l_2) - \alpha(l_1))}{\pi}, & \alpha(l_2) < \alpha(l_1), \\ 0, & \alpha(l_1) = \alpha(l_2). \end{cases}$$

Proof. This is an easy computation if one uses that $\tau(l_1, l_2, l_3) = 0$ if any two of the subspaces coincide and that $\tau(l_1, l_2, l_3) = 1$ if $\alpha(l_1) < \alpha(l_2) < \alpha(l_3)$. q.e.d.

2.3. The equality $\mu = m$. Let V be a real vector space with an euclidean structure $\langle\ ,\ \rangle$ and a compatible complex structure I, i.e., $I = -I^*$ and $I^2 = -1$. I defines a symplectic structure $\Phi(x, y) := \langle Ix, y\rangle$. Let $\Lambda := \Lambda(V)$ be the space of Lagrangian subspaces of V, and K be the automorphism group of $(V, \langle,\rangle, I)$. For $l \in \Lambda$ we define the involution $\sigma_l := \mathrm{pr}_l - (1 - \mathrm{pr}_l)$ that acts as the identity on l and by -1 on the orthogonal complement $l^\perp$. The following properties are easy to verify:

$\sigma_l^2 = 1$, $\sigma_l^* = \sigma_l$,
$\sigma_l I + I\sigma_l = 0$,
$\sigma_{ul} = u\sigma_l u^*$, $u \in K$.

For a pair $l_1, l_2 \in \Lambda$ we form $M(l_1, l_2) := \sigma_{l_1}\sigma_{l_2}$. Then

$M(l_1, l_2) = M(l_2, l_1)^{-1}$,
$M(l_1, l_2) \in K$,
$M(l_1, l_2)M(l_2, l_3)M(l_3, l_1) = 1$,
$M(ul_1, ul_2) = uM(l_1, l_2)u^*$,
$M(Al, l) = A\overline{A}^{-1}$ for $A \in \mathrm{Sp}(V)$ with $\overline{A} := \sigma_l A_l \sigma_2$.

Recall that from Definition 1.4 and Proposition 2.6 we have, respectively,

$$m(l_1, l_2) := -\frac{1}{\pi} \sum_{\substack{e^{\imath\lambda} \in \mathrm{spec}(-M(l_1, l_2)) \\ \lambda \in (-\pi, \pi)}} \lambda,$$

$$\mu(l_1, l_2) = \int_K \tau(kl, l_1, l_2)\, dk. \tag{17}$$

Proposition 2.8. $m(l_1, l_2) = \mu(l_1, l_2)$.

Proof. If $l_1, l_2, l_3 \in \Lambda$ are pairwise transverse, then (see Guillemin-Sternberg [27, 2.2.26])

$$m(l_1, l_2) + m(l_2, l_3) + m(l_3, l_1) = \tau(l_1, l_2, l_3). \tag{18}$$

By the properties of $M(l_1, l_2)$ listed above, $m(ul_1, ul_2) = m(l_1, l_2)$ for all $u \in K$ and $m(l_1, l_2) = -m(l_2, l_1)$. Let $l_1, l_2 \in \Lambda$ be transverse, and define

$$\dot{K} := \{u \in K | ul \cap l_1 = \{0\}, ul \cap l_2 = \{0\}\}.$$

Then $K \backslash \dot{K}$ has measure zero. Substituting (18) in (17) yields

$$\begin{aligned} \mu(l_1, l_2) &= \int_{\dot{K}} \tau(ul, l_1, l_2)\, du \\ &= \int_{\dot{K}} (m(ul, l_1) + m(l_2, ul))\, du + m(l_1, l_2) \\ &= m(l_1, l_2). \end{aligned}$$

Here the integral vanishes for symmetry reasons. It requires a little bit more effort to check the equality in the case if l_1, l_2 are not transverse. This can be done by studying the jumps of $m(l_1, l(t))$ and $\mu(l_1, l(t))$ if $l(t)$ is a family with $l(0) = l_2$ that goes transversely through $\{l \in \Lambda | l \cap l_1 \neq \{0\}\}$. Then

$$\begin{aligned} 2m(l_1, l_2) &= m(l_1, l(+0)) + m(l_1, l(-0)), \\ 2\mu(l_1, l_2) &= \mu(l_1, l(+0)) + \mu(l - 1, l(-0)). \end{aligned}$$

We omit the details.

2.4. The twisted Maslov index. Recall the definition of the twisted Maslov index (11)

$$\tau_I(L_1, L_2, L_3) = m(L_1, IL_2) + m(L_2, IL_3) + m(L_3, IL_1).$$

Lemma 2.9. *If* $L_i \cap IL_j = \{0\}$ *for* $j, i = 1, 2, 3$, $i \neq j$, *then* $\tau_I(L_1, L_2, L_3) \in 2\mathbf{Z}$.

Proof. Let L be any Lagrangian subspace. By the antisymmetry of m, $m(L, L) = 0$. Since $I \in K$ and $I^2 = -1$ we have $m(L, IL) = m(IL, L) = -m(L, IL)$. Hence, $m(L, IL) = 0$. A simple computation shows

$$\begin{aligned} \tau_I(L_1, L_2, L_3) = \tau(L_1, IL_2, L_2) + \tau(L_2, IL_3, L_3) \\ + \tau(L_3, IL_1, L_1) + \tau(L_1, L_2, L_3). \end{aligned}$$

By Proposition 1.9.3 of Lion-Vergne [32] for any triple of Lagrangian subspaces L_1, L_2, L_3,

$$\tau(L_1, L_2, L_3) = n+\dim(L_1\cap L_2)+\dim(L_2\cap L_3)+\dim(L_3\cap L_1) \qquad (\mathrm{mod}\ 2),$$

where $n = \dim(V)/2$. Thus we obtain

$$\begin{aligned}\tau_1(L_1, L_2, L_3) &= 2(\dim(L_1 \cap L_2) + \dim(L_2 \cap L_3) + \dim(L_3 \cap L_1)) + 4n \\ &= 0 \ (\mathrm{mod}\ 2).\end{aligned}$$

2.5. The signature defect. We recall the theorem of Atiyah-Patodi-Singer [1] on the signature of a compact manifold with boundary and the result of Wall [39] on the signature defect.

Let (X, M) be a compact oriented Riemannian manifold with boundary of dimension $\dim X = 4k$. Assume that the metric is product near the boundary. Let $\eta(M)$ be the η-invariant associated with the odd signature operator introduced by Atiyah-Patodi-Singer [1]. Then

Theorem 2.10 (APS).

$$\mathrm{sign}(X) = \int_X \mathscr{L} - \eta(M),$$

where $\mathscr{L}$ is the Hirzebruch-$\mathscr{L}$ polynomial in the Pontrjagin forms of X.

Let $(X_+, M_1, -M_2, N)$, $(X_-, M_2, -M_3, N)$ be oriented manifold 4-tuples (the sign indicates the orientation) with $M_1 \cap M_2 = N$, $M_2 \cap M_3 = N$, $M_1 \cup_N -M_2 = \partial X_+$, $M_2 \cup_N -M_3 = \partial X_-$. Give N the orientation as the boundary of M_2. We form $X := X_+ \cup_{M_2} X_-$. Let $V := H_{2k-1}(N, \mathbf{R})$. V is a symplectic vector space, where the symplectic structure Φ is given by the intersection number. Let

$$l_i := \ker(V \to H_{2k-1}(M_i)), \qquad i = 1, 2, 3.$$

It can be shown that the l_i are Lagrangian subspaces. The result of Wall [39] is

Theorem 2.11.

$$\mathrm{sign}(X) = \mathrm{sign}(X_+) + \mathrm{sign}(X_-) + \tau(l_1, l_2, l_3). \tag{19}$$

By Theorem 2.10 we have

$$\begin{aligned}\mathrm{sign}(X_+) &= \int_{X_+} \mathscr{L} - \eta(M_1 \cup_N -M_2), \\ \mathrm{sign}(X_-) &= \int_{X_-} \mathscr{L} - \eta(M_2 \cup_N -M_3), \\ \mathrm{sign}(X) &= \int_X \mathscr{L} - \eta(M_1 \cup_N -M_3).\end{aligned}$$

Substituting these equations into (19) we obtain

(20) $\eta(M_1 \cup_N -M_2) + \eta(M_2 \cup_N -M_3) + \eta(M_3 \cup_N -M_1) = \tau(l_1, l_2, l_3)$.

Lemma 2.12. *Equation* (20) *holds for any three pairs* (M_i, N), $i = 1, 2, 3$, *of compact, oriented Riemannian manifolds of dimension* $4k-1$ *with a boundary isometric to* N *and a product metric near the boundary.*

Proof. We form the compact oriented Riemannian manifolds

$$Y_1 := M_1 \cup_N -M_2, \qquad Y_2 := M_2 \cup_N -M_3.$$

Since the oriented cobordism vanishes rationally $\Omega^{SO}_{4k-1} \otimes \mathbf{Q} = \{0\}$ in dimension $4k-1$, there are oriented manifolds with boundary (X_+, nY_1), (X_-, mY_2), $n, m \in \mathbf{N}$. In fact, we can assume $n = m$. Applying now Theorem 2.10 to the 4-tuples $(X_+, nM_1, -nM_2, nN)$, $(X_-, nM_2, -nM_3, nN)$, we see that n-times equation (20) holds. The lemma follows.

3. The proof of the gluing formula

3.1. The relative index of projections. Let H be a separable Hilbert space. The space of bounded operators $L(H)$ contains the ideals K of compact operators and $L^1(H) \subset K$ of trace class operators. Moreover, there is the family $L^1(H) \subset L^N(H) \subset K$ of Schatten classes for $N \geq 1$ (see Connes [23, Appendix]). The Nth Schatten class consists of all operators $A \in L(H)$ such that $|A|^N$ is of trace class. A projection $P \in L(H)$ is a selfadjoint idempotent, i.e., $P^* = P$ and $P^2 = P$.

Let P, Q be infinite dimensional projections on H such that $P - Q \in K$. Then the operator $PQ\colon \mathrm{Im}(Q) \to \mathrm{Im}(P)$ is Fredholm, and its index is the relative index of P and Q denoted by $I(P, Q)$. We consider the space $h := \mathrm{Im}(P) \oplus \mathrm{Im}(Q)$ and the operator

$$A := \begin{pmatrix} 0 & PQ \\ W & 0 \end{pmatrix} \in L(h),$$

where $W\colon \mathrm{Im}(P) \to \mathrm{Im}(Q)$ is some isometric isomorphism which exists by the assumption on the dimensions of P, Q. Let

$$B := \begin{pmatrix} 0 & W^* \\ QP & 0 \end{pmatrix} \in L(h).$$

Then

$$S := 1 - BA = \begin{pmatrix} 0 & 0 \\ 0 & Q(Q-P)Q \end{pmatrix}.$$

and

$$T := 1 - AB = \begin{pmatrix} P(P-Q)P & 0 \\ 0 & 0 \end{pmatrix}$$

are compact operators. Thus, A is Fredholm with parametrix B and

$$\operatorname{index}(A) = I(P, Q).$$

Assume now that $S, T \in L^N(H)$, i.e., they are in the Nth Schatten class for some $N \geq 1$. Then we have the following proposition (proved, e.g., in Connes [23])

Proposition 3.1. $I(P, Q) := \operatorname{Tr} S^N - \operatorname{Tr} T^N$.

In terms of the projections P, Q this formula is equivalent to

$$I(P, Q) = \operatorname{Tr}((Q - QPQ)^N - (P - PQP)^N).$$

The following lemma provides us with the commutator K announced in the outline of the proof in §1.6.

Lemma 3.2. *If P, Q are projections on H, then*

$$(Q - QPQ)^N - (P - PQP)^N = Q - P + K,$$

where K is a commutator given by

$$K = \left[(QP - PQ), \frac{1}{2}\sum_{i=0}^{N-1} \binom{N}{i} (-1)^{N-i}(Q(PQ)^{N-i-1} + P(QP)^{N-i-1})\right].$$

Proof. A simple computation shows that for $l \geq 0$

$$[[Q, P], (Q(PQ)^l + P(QP)^l)] = 2(Q(PQ)^{l+1} - P(QP)^{l+1}). \tag{21}$$

Now we expand the Nth powers $(Q - QPQ)^N$ and $(P - PQP)^N$ and use the fact that Q commutes with QPQ and P commutes with PQP. Substituting (21) in these expansions and collecting all commutator terms prove the lemma.

3.2. Construction of the isometry U. Recall the modified geometric situation introduced in §1.5 and the Hilbert spaces $H := L^2(M, E) \oplus L^2(Z, E_Z)$ and $H_0 := L^2(M_1, E_1) \oplus L^2(M_2, E_2)$. We define an isometry $U: H \to H_0$.

We start with choosing a function $\chi \in C^\infty[-1, 1]$ with $\chi = 1$ on $[-1, -1/2]$, $\chi(r) \in [0, 1]$ if $r \in [-1/2, 1/2]$ and $\chi = 0$ on $[1/2, 1]$ such that $\gamma := \sqrt{1 - \chi^2}$ is also smooth. χ gives rise to a function on Z depending only on the radial variable. We extend χ by 1 to the left on M_1 and by 0 to the right on M_2. The corresponding extension of γ is $\gamma := \sqrt{1 - \chi^2}$. Thus, we obtain functions on M, M_1, M_2 also denoted by χ, γ.

We define operators $a, b, c, d: H \to H_0$ as follows (compare[12]):

a is the multiplication by χ on M followed by the transfer to M_1.

b is the multiplication by χ on Z followed by the transfer to M_2.
c is the multiplication by γ on M followed by the transfer to M_2.
d is the multiplication by γ on Z followed by the transfer to M_1.
Here we use the canonical identifications of the different parts of M, M_1, M_2, and Z and of the Dirac bundles over these parts.

Note that a^*a is the multiplication by χ^2 on M. Analogously, c^*c is the multiplication by γ^2 on M and similarly for b, d. Thus, $a^*a + b^*b + c^*c + d^*d = 1_H$. We define

$$U := a + b + c - d.$$

Then

$$\begin{aligned} U^*U &= a^*a + b^*b + c^*c + d^*d + a^*(b + c - d) \\ &\quad + b^*(a + c - d) + c^*(a + b - d) - d^*(a + b + c) \\ &= 1_H - a^*d + b^*c + c^*b - d^*a \\ &= 1_H, \end{aligned}$$

where we have employed that $a^*d = c^*b$, $b^*c = d^*a$, $a^*b = 0$, $a^*c = 0$, $b^*a = 0$, $b^*d = 0$, $c^*a = 0$, $c^*d = 0$, $d^*b = 0$, and $d^*c = 0$. Analogously $UU^* = 1_{H_0}$.

3.3. The difference of the positive spectral projections. We consider now the Dirac operators D_+ on H and D_0 on H_0. Their domains are specified by the boundary conditions defined by Lagrangian subspaces L_1 and L_2 at $\{1\} \times N$ and $\{-1\} \times N$, respectively. Since the transfer used above identifies the boundary components, where we have put the same condition, and U is defined by smooth cut-off functions, we have $U^* \operatorname{dom} D_0 = \operatorname{dom} D_+$. Let $D_- := U^* D_0 U$. Then we can consider D as a bounded perturbation of D_+. The operator $G := D_+ - D_-$ is the sum of nonlocal bundle endomorphisms over the flip interchanging the cylindrical parts of $M \cup Z$. It is supported on $(-1/2, 1/2) \times N \cup (-1/2, 1/2) \times N$ in the cylindrical parts, and is the sum of products of functions depending only on the r-coordinate with the composition of the flip and the multiplication by I.

We define Sobolev spaces H^k, $k \in \mathbf{Z}$, using the operator D_+. We set $H^k = \operatorname{dom} D_+^k$ with the norm $\|\psi\|_k := \|(1 + |D_+|^k)\psi\|$. Using D_- yields equivalent spaces. Let $L^N(H)$ denote the Nth Schatten class of H. Let $P_\pm$ be the positive spectral projections of $D_\pm$.

Proposition 3.3. *The difference $P_+ - P_-$ is continuous from H to $H^{1/2}$ and, hence, $(P_+ - P_-) \in L^N(H)$ for $N > 2n$.*

Proof. We use the technique of [11, §3]. We represent $P_\pm$ by strongly convergent integrals

$$P_\pm = 1/2 + \lim_{S\to\infty} \frac{1}{2\pi} \int_{-S}^{S} (D_\pm - \varepsilon + \imath\lambda)^{-1}\, d\lambda,$$

where $\varepsilon > 0$ is smaller than the smallest positive eigenvalue of $D_\pm$. Then

$$P_+ - P_- = \frac{1}{2\pi} \int_{-\infty}^{\infty} [(D_+ - \varepsilon + \imath\lambda)^{-1} - (D_- - \varepsilon + \imath\lambda)^{-1}]\, d\lambda,$$

and this integral converges in the norm of $L(H, H^{1/2})$. In fact, using the resolvent identity we can estimate the norm of the integrand by

$$\|(D_+ - \varepsilon + \imath\lambda)^{-1} G (D_- - \varepsilon + \imath\lambda)^{-1}\|_{L(H, H^{1/2})} \le C(1 + |\lambda|)^{-3/2}.$$

The proposition now follows. q.e.d.

Applying now Proposition 3.1 we see that the relative index of P_+ and P_- is well defined and can be computed by the formula given in Lemma 3.2 as follows:

$$I(P_-, P_+) = \mathrm{Tr}(P_+ - P_- + K),$$

where

$$K := [[P_+, P_-], Q],$$

$$Q := \frac{1}{2} \sum_{i=0}^{N-1} \binom{N}{i} (-1)^{N-i} (P_+(P_-P_+)^{N-i-1} + P_-(P_+P_-)^{N-i-1}).$$

For $\mathrm{Re}(t) \ge 0$ we give now the extension of K to a holomorphic family of commutators $K(t)$ announced in §1.6. We introduce the regularized 'projections' $P_\pm(t) = e^{-tD_\pm^2} P_\pm$.

Definition 3.4. $K(t) := [[P_+(t), P_-(t)], Q]$.

Then $K(t)$ is of trace class and obviously $\mathrm{Tr}\, K(t) = 0$ if $\mathrm{Re}(t) > 0$.

3.4. Finite propagation speed and comparison of heat kernels. In this section we compare the heat kernels of two Dirac operators in places where they coincide locally. We explain the general method (introduced by Cheeger-Gromov-Taylor [21] and employed for comparison in [10]) that will be frequently applied in the sequel.

For $i = 0$, let (M_i, g_i) be Riemannian manifolds of dimension n, and $E_i \to M_i$ be Dirac bundles with Dirac operators D_i. We assume that M_i, $i = 0, 1$, decompose as $K_i \cup U_i$ with U_i open and precompact, and there exists an isometry between U_0 and U_1 covered by a bundle isomorphism $E_{0|U_0} \to E_{1|U_1}$ intertwining with the Dirac operators. Let

$$\mathscr{H} = L^2(K_0, E_0) \oplus L^2(K_1, E_1) \oplus L^2(U_0, E_0).$$

Identifying the sections over U_0 with those over U_1 there are natural embeddings $L^2(M_i, E_i) \to \mathscr{H}$. Let P_i be the projections onto these subspaces. We extend the Dirac operators to $\mathscr{H}$ by zero where they have not been defined before. Let $H_i := D_i^2$. Let $W_i(t, x, y)$, $t \geq 0$, be the heat kernel of D_i, i.e., the integral kernel of e^{-tH_i}. If M_i has a nonempty boundary, we assume a selfadjoint elliptic boundary condition. We represent the heat kernels by the Fourier transform in terms of the wave operator:

$$e^{-tH} = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-\lambda^2/4t} e^{\imath\lambda D}\, d\lambda,$$

where the integral converges in the operator norm for $t > 0$. The operators $D_i^l e^{-tH_i}$, $l \geq 0$, are smoothing and we represent the kernels $D_i^l W_i$ by

$$D_i^l W_i(t, x, y) = \langle \delta(x), D_i^l e^{-tH_i} \delta(y)\rangle \in E_{i,x} \otimes E_{i,y}^*.$$

The following lemma states the finite propagation speed property of the wave operator (Chernoff [22]).

Lemma 3.5 (Finite propagation speed). *The operators $e^{\imath\lambda D_i}$ extend to all Sobolev spaces $H^k(M_i, E_i)$ and, for $\psi \in H^k(M_i, E_i)$, $\operatorname{supp} e^{\imath\lambda D_i}\psi$ is contained in a $|\lambda|$-neighborhood of $\operatorname{supp}\psi$.*

Let $x, y \in U_i$ such that $\max(\operatorname{dist}(x, K_1), \operatorname{dist}(y, K_1)) =: u > 0$. We can choose a family of smooth cut-off functions $\chi_u(r)$ such that $\chi_u(r) = 1$ if $|r| \geq u/2$ and $\chi_u(r) = 0$ if $|r| \leq u/4$ such that the C^k-norms of χ_u depend polynomially on u^{-1}. Then

$$D_0^l W_0(t, x, y) - D_1^l W_1(t, x, y) \tag{22}$$

$$= \left\langle \delta(x), \frac{1}{\sqrt{4\pi t}} \int_{\mathbf{R}} e^{-\lambda^2/4t} [D_0^l e^{\imath\lambda D_0} - D_1^l e^{\imath\lambda D_1}]\, d\lambda\, \delta(y)\right\rangle. \tag{23}$$

We can exclude $[-u, u]$ from the integration since by Lemma 3.5 the integrand vanishes in that interval, if it is applied to the delta distributions. Thus,

$$\begin{aligned} &D_0^l W_0(t, x, y) - D_1^l W_1(t, x, y) \\ &= \left\langle \delta(x), \frac{1}{\sqrt{4\pi t}} \int_{\mathbf{R}} \chi_u(\lambda) e^{-\lambda^2/4t} [D_0^l e^{\imath\lambda D_0} - D_1^l e^{\imath\lambda D_1}]\, d\lambda\, \delta(y)\right\rangle. \end{aligned} \tag{24}$$

Using partial integration in order to regularize the difference, (24) can be estimated by

$$Ce^{-cu^2/t} \|\delta(x)\|_{H^{-k}(B_\epsilon(x), E)^* \otimes E_x} \|\delta(y)\|_{H^{-k}(B_\epsilon(y), E)^* \otimes E_y}.$$

Here $B_\varepsilon(x)$ is the ball of radius $\varepsilon > 0$ centered at x and $c > 0$. If $k > n/2$, the norms of the δ-distributions are bounded.

Lemma 3.6 (Estimate of the difference of heat kernels).

$$\|D_0^l W_0(t, x, y) - D_1^l W_1(t, x, y)\| < Ce^{-cu^2/t}. \tag{25}$$

By essentially the same technique we also have the off-diagonal estimate:

Lemma 3.7 (Off-diagonal estimate of a heat kernel).

$$\|D_i^l W_i(t, x, y)\| < Ce^{-c\,\mathrm{dist}(x, y)^2/t} \tag{26}$$

for all $x, y \in M_i$.

The finite propagation speed property fails to be true along the boundary ∂M_i, where a global boundary condition is posed. Let $x \in M_0$ and consider the distribution $\phi_\lambda := e^{\imath\lambda D_0}\delta(x)$. As long as $|\lambda| < \mathrm{dist}(\partial M_0, x)$ we have $\mathrm{supp}(\phi_\lambda) \subset B(x, |\lambda|)$. If $\mathrm{supp}(\phi_\lambda)$ hits ∂M_0, then it instantaneously spreads out along ∂M_0. Thus,

$$\mathrm{supp}(\phi_\lambda) \subset B(x, |\lambda|) \cup \{y \in M_0 | \mathrm{dist}(y, \partial M_0) \le |\lambda| - \mathrm{dist}(x, \partial M_0)\}.$$

The same holds on M_1. Consider now the integrand of (22). It vanishes as long as the supports of $e^{\imath\lambda D_i}\delta(z)$, $i = 0, 1$, $z = x, y$ are contained in the set where D_0 and D_1 coincide. Let us redefine the distance of points of U_i from K_i. For $x \in U_i$ let

$$\widetilde{\mathrm{dist}}(x, K_i) := \min(\mathrm{dist}(x, K_i), \mathrm{dist}(x, \partial M_i) + \mathrm{dist}(\partial M_i, K_i)).$$

With this definition of the distance the estimate (25) holds. A similar modification works for (26).

3.5. Comparison away from the cylinder. Recall the definition of $R(s, t)_\pm$:

$$R(s, t)_\pm = e^{-tD_\pm^2}\mathrm{sign}(D_\pm)|D_\pm + E_{D_\pm}\{0\}|^{-s}.$$

We use the finite propagation speed method in order to compare these functions away from the set $(-5/8, 5/8) \times N \cup (-5/8, 5/8) \times N \subset M \cup Z$, where D_+ and D_- differ. Let v, l, r be smooth cut-off functions with $l + v + r = 1$ and such that the following hold:

$\mathrm{supp}\, v \subset [-6/8, 6/8] \times N \cup [-6/8, 6/8] \times N$,
$\mathrm{supp}\, l \subset \widetilde{M}_1 \cup_N Z \cup Z$,
$l = 0$ on $[-5/8, 1] \times N \cup [-5/8, 1] \times N$,
$\mathrm{supp}\, r \subset Z \cup Z \cup_N \widetilde{M}_2$,
$r = 0$ on $[-1, 5/8] \times N \cup [-1, 5/8] \times N$.

Let $W_\pm(t, x, y)$ be the integral kernels of $e^{-tD_\pm^2}$ and $M_+ := M \cup Z$. The kernels $W_\pm(t, x, y)$ are smooth on $(t, x, y) \in (0, \infty) \times M_+ \times M_+$ up to the boundary of M_+. They diverge on the diagonal if $t \to 0$. However, these divergences cancel each other if one forms differences. This can by quantified by the finite propagation speed method which provides the following estimates.

Corollary 3.8. *There are constants $c > 0$ and $C < \infty$ such that for all $t > 0$, $(x, y) \in M_+ \times M_+$*

$$\begin{aligned}
|l(x)(W_+(t, x, y) - W_-(t, x, y))| &\le Ce^{-c/t}, \\
|l(x)(D_+W_+(t, x, y) - D_-W_-(t, x, y))| &\le Ce^{-c/t}e^{-ct}, \\
|(W_+(t, x, y) - W_-(t, x, y))l(x)| &\le Ce^{-c/t}, \\
|(D_+W_+(t, x, y) - D_-W_-(t, x, y))l(x)| &\le Ce^{-c/t}e^{-ct}, \\
|r(x)(W_+(t, x, y) - W_-(t, x, y))| &\le Ce^{-c/t}, \\
|r(x)(D_+W_+(t, x, y) - D_-W_-(t, x, y))| &\le Ce^{-c/t}e^{-ct}, \\
|(W_+(t, x, y) - W_-(t, x, y))r(x)| &\le Ce^{-c/t}, \\
|(D_+W_+(t, x, y) - D_-W_-(t, x, y))r(x)| &\le Ce^{-c/t}e^{-ct}.
\end{aligned}$$

Similar estimates hold for higher derivatives in the (x, y)-coordinates.

Lemma 3.9. *The integral kernels of the restricted differences $l(R(s, t)_+ - R(s, t)_-)$, $r(R(s, t)_+ - R(s, t)_-)$, $(R(s, t)_+ - R(s, t)_-)l$, $(R(s, t)_+ - R(s, t)_-)r$ are smooth and uniformly bounded on $(s, t, x, y) \in (-1/3, C] \times [0, \infty] \times M_+ \times M_+$. In particular, the following limits exist in the trace norm:*

$$\begin{aligned}
&\lim_{t\to 0} l(R(0, t)_+ - R(0, t)_-) = l(R(0, 0)_+ - R(0, 0)_-), \\
&\lim_{t\to 0} r(R(0, t)_+ - R(0, t)_-) = r(R(0, 0)_+ - R(0, 0)_-), \\
&\lim_{t\to 0} (R(0, t)_+ - R(0, t)_-)l = (R(0, 0)_+ - R(0, 0)_-)l, \\
&\lim_{t\to 0} (R(0, t)_+ - R(0, t)_-)r = (R(0, 0)_+ - R(0, 0)_-)r.
\end{aligned}$$

Proof. The operator-valued function $R(s, t)$ has the following integral representation:

$$R(s, t) = \frac{1}{\Gamma((s+1)/2)} \int_0^\infty u^{(s-1)/2} De^{-(u+t)D^2}\, du. \tag{27}$$

The integral kernel of difference $l(R(s, t)_+ - R(s, t)_-)$ is given by the integral

$$\frac{1}{\Gamma((s+1)/2)} \int_0^\infty u^{(s-1)/2} l(x)(D_+W_+(t+u, x, y) - D_-W_-(t+u, x, y))\, du.$$

By Corollary 3.8 this integral absolutely converges in C^k, $k \geq 0$, in the region $(s, t, x, y) \in (-1/3, C] \times [0, \infty] \times M_+ \times M_+$. The claim for the kernel of $l(R(s, t)_+ - R(s, t)_-)$ follows. The other differences are treated similarly. q.e.d.

It remains to consider the behavior of $v(R(0, t)_+ - R(0, t)_-)v$ as $t \to 0$. Unfortunately it does not converge in the trace norm. This is exactly the reason for introducing other families $\widetilde{R}_\pm(s, t)$ to compare with.

3.6. Comparison on the cylinder. In this section we carry out the construction of the operators $B(t)$, $\widetilde{K}(t)$ used for comparison on the cylinder. Note that $M \cup Z$ contains two copies of the cylinder Z, namely Z itself and the middle part of M. Thus, we have a canonical embedding $L^2(Z, E_Z) \oplus L^2(Z, E_Z) \to H$, where the first summand is the subspace of $L^2(M, E)$. Moreover, $M_1 \cup M_2$ contains two copies of the cylinder Z (recall the modified geometry in §1.5) and we have the embedding $L^2(Z, E_Z) \oplus L^2(Z, E_Z) \to H_0$. Then U restricts to a unitary isomorphism of these subspaces. Let $\widetilde{D}_+$ be the sum of two copies of D_Z with boundary conditions given by the Lagrangian subspaces L_1, L_2 as before, and let $\widetilde{D}_0$ be the sum of two copies of D_Z now considered on H_0. Define $\widetilde{D}_- := U^* \widetilde{D}_0 U$. We extend these operators by zero to the complement of the cylinders. Repeating all the above constructions for $\widetilde{D}_\pm$ we obtain $\widetilde{R}(s, t)_\pm$, $\widetilde{P}_\pm$, $\widetilde{P}_\pm(t)$ and $\widetilde{K}(t)$. Let $\widetilde{W}_\pm(t, x, y)$ be the integral kernels of $e^{-t\widetilde{D}^2_\pm}$, which are smooth on $(t, x, y) \in (0, \infty) \times [Z \cup Z] \times [Z \cup Z]$. By the finite propagation speed method we have

Corollary 3.10. *There are constants $c > 0$ and $C < \infty$ such that for all $t > 0$, $(x, y) \in M_+ \times M_+$*

$$|l(x)(\widetilde{W}_+(t, x, y) - \widetilde{W}_-(t, x, y))| \leq Ce^{-c/t},$$
$$|l(x)(\widetilde{D}_+\widetilde{W}_+(t, x, y) - \widetilde{D}_-\widetilde{W}_-(t, x, y))| \leq Ce^{-c/t}e^{-ct},$$
$$|(\widetilde{W}_+(t, x, y) - \widetilde{W}_-(t, x, y))l(x)| \leq Ce^{-c/t},$$
$$|(\widetilde{D}_+\widetilde{W}_+(t, x, y) - \widetilde{D}_-\widetilde{W}_-(t, x, y))l(x)| \leq Ce^{-c/t}e^{-ct},$$
$$|r(x)(\widetilde{W}_+(t, x, y) - \widetilde{W}_-(t, x, y))| \leq Ce^{-c/t},$$
$$|r(x)(\widetilde{D}_+\widetilde{W}_+(t, x, y) - \widetilde{D}_-\widetilde{W}_-(t, x, y))| \leq Ce^{-c/t}e^{-ct},$$
$$|(\widetilde{W}_+(t, x, y) - \widetilde{W}_-(t, x, y))r(x)| \leq Ce^{-c/t},$$
$$|(\widetilde{D}_+\widetilde{W}_+(t, x, y) - \widetilde{D}_-\widetilde{W}_-(t, x, y))r(x)| \leq Ce^{-c/t}e^{-ct}.$$

Similar estimates hold for higher derivatives in the (x, y)-coordinates.

Lemma 3.11. *The integral kernels of the restricted differences* $l(\widetilde{R}(s,t)_+ - \widetilde{R}(s,t)_-)$, $r(\widetilde{R}(s,t)_+ - \widetilde{R}(s,t)_-)$, $(\widetilde{R}(s,t)_+ - \widetilde{R}(s,t)_-)l$, $(\widetilde{R}(s,t)_+ - \widetilde{R}(s,t)_-)_r$ *are smooth and uniformly bounded on* $(s,t,x,y) \in (-1/3, C] \times [0,\infty] \times [Z \cup Z] \times [Z \cup Z]$. *In particular the following limits exist in the trace norm:*

$$\lim_{t\to 0} l(\widetilde{R}(0,t)_+ - \widetilde{R}(0,t)_-) = l(\widetilde{R}(0,0)_+ - \widetilde{R}(0,0)_-),$$
$$\lim_{t\to 0} r(\widetilde{R}(0,t)_+ - \widetilde{R}(0,t)_-) = r(\widetilde{R}(0,0)_+ - \widetilde{R}(0,0)_-),$$
$$\lim_{t\to 0}(\widetilde{R}(0,t)_+ - \widetilde{R}(0,t)_-)l = (\widetilde{R}(0,0)_+ - \widetilde{R}(0,0)_-)l,$$
$$\lim_{t\to 0}(\widetilde{R}(0,t)_+ - \widetilde{R}(0,t)_-)r = (\widetilde{R}(0,0)_+ - \widetilde{R}(0,0)_-)r.$$

Lemma 3.12.

$$\operatorname{Tr}(\widetilde{R}(s,t)_+ - \widetilde{R}(s,t)_-) = 0, \qquad \operatorname{Tr}(\widetilde{P}_+ - \widetilde{P}_- + \widetilde{K}) = 0.$$

Proof. The first equation is obvious since $\widetilde{R}(s,t)_+$ and $\widetilde{R}(s,t)_-$ are unitary equivalent. The second equation is true since one can interchange the roles of $\widetilde{D}_\pm$ by applying U. q.e.d.

Note that the domains of $D_\pm$ and $\widetilde{D}_\pm$ are not compatible, but $D_\pm$ and $\widetilde{D}_\pm$ coincide on $([-7/8, 7/8] \times N) \cup ([-7/8, 7/8] \times N)$ locally.

Lemma 3.13. *The integral kernels of* $v(R(s,t)_\pm - \widetilde{R}(s,t)_\pm)v$ *are uniformly bounded and smooth on* $(s,t,x,y) \in [-1/3, C] \times [0,\infty] \times M_+ \times M_+$. *In particular, the limits*

$$\lim_{t\to 0} v(R(0,t)_\pm - \widetilde{R}(0,t)_\pm)v = v(R(0,0)_\pm - \widetilde{R}(0,0_\pm)v$$

exist in the trace norm.

Proof. We employ again the integral representation (27). The finite propagation speed method provides constants $C < \infty$, $c > 0$ such that for all $t > 0$ and $(x,y) \in M_+ \times M_+$

$$|v(x)(D_\pm W_\pm(t,x,y) - \widetilde{D}_\pm \widetilde{W}_\pm(t,x,y))v(y)| \le Ce^{-ct}e^{-c/t}$$

holds. A similar estimate holds for higher derivatives in the (x,y)-coordinates. q.e.d.

Define $B(t) = \widetilde{R}(s,t)_+ - \widetilde{R}(s,t)_-$. Then

$$s - \lim_{t\to 0} B(t) = 2\widetilde{P}_+ - 2\widetilde{P}_- + E_{\widetilde{D}_+}\{0\} - E_{\widetilde{D}_+}\{0\} - E_{\widetilde{D}_-}\{0\}.$$

3.7. The proof of the gluing formula. In the preceding sections we have constructed families of trace class operators $A(t)$, $K(t)$, $B(t)$ and $\widetilde{K}(t)$.

We already know that

$$\lim_{t\to 0} \operatorname{Tr} A(t) = \lim_{t\to 0} \operatorname{Tr}(A(t) + 2K(t)) = \delta$$

(see equation (5)). Moreover, we know that

$$A(0) + 2K(0) = 2(P_+ - P_-) + 2K + E_{D_+}\{0\} - E_{D_-}\{0\}$$

is of trace class. Since the limit $\lim_{t\to 0}(A(t) + 2K(t))$ does not exist in the trace norm, we use the family $B(t) + 2K(t)$ for comparison. In fact,

$$\operatorname{Tr}(A(t) + 2K(t)) = \operatorname{Tr}(A(t) + 2K(t) - B(t) - 2\widetilde{K}(t))$$

and

$$\operatorname{Tr}(A(0) + 2K(0)) = \operatorname{Tr}(A(0) + 2K(0) - B(0) - 2\widetilde{K}(0))$$

by Lemma 3.12.

Theorem 3.14. *The limit*

$$\lim_{t\to 0}(A(t) + 2K(t) - B(t) - 2\widetilde{K}(t)) = A(0) + 2K(0) - B(0) - 2\widetilde{K}(0)$$

exists in the trace norm. Hence,

$$\delta = \operatorname{Tr}(A(0) + 2K(0)) = -2I(P_+, P_-) + \dim\ker D_+ - \dim\ker D_- .$$

Proof. It is sufficient to show that the following families converge in the trace norm:

$$\begin{gathered} l(A(t) + 2K(t) - B(t) - 2\widetilde{K}(t)), \\ r(A(t) + 2K(t) - B(t) - 2\widetilde{K}(t)), \\ v(A(t) + 2K(t) - B(t) - 2\widetilde{K}(t))(l + r), \\ v(A(t) + 2K(t) - B(t) - 2\widetilde{K}(t))v . \end{gathered}$$

According to Lemmas 3.9, 3.11 and 3.13 it is enough to show the convergence in the trace norm of the terms $lK(t)$, $l\widetilde{K}(t)$, $rK(t)$, $r\widetilde{K}(t)$, $vK(t)(l + r)$, $v\widetilde{K}(t)(l + r)$, $v(K(t) - \widetilde{K}(t))v$. The arguments for all these terms are very similar. Thus, we consider only the first and the last ones. Let us start with

$$\begin{aligned} lK(t) &= l[[P_+(t), P_-(t)], Q] \\ &= l[P_+(t), P_-(t)]Q - lQ[P_+(t), P_-(t)]. \end{aligned}$$

Let l_1 be a smooth cut-off function with $ll_1 = l$ and $l_1 = 0$ in a 1/100-neighborhood of $([-6/8, 6/8] \times N) \cup ([-6/8, 6/8] \times N)$. Inserting l_1 between Q and the commutator provides

$$lK(t) = l[P_+(t), P_-(t)]Q - lQl_1[P_+(t), P_-(t)] - lQ(1 - l_1)[P_+(t), P_-(t)].$$

It is enough to show that

$$l[P_+(t), P_-(t)] \overset{t\to 0}{\rightrightarrows} l[P_+, P_-], \tag{28}$$

$$l_1[P_+(t), P_-(t)] \overset{t\to 0}{\rightrightarrows} l[P_+, P_-], \tag{29}$$

$$lQ(1-l_1)[P_+(t), P_-(t)] \overset{t\to 0}{\rightrightarrows} [P_+, P_-] \tag{30}$$

exist in the trace norm. First we consider the term (28). For this, we need the following lemmas.

Lemma 3.15. *The integral kernel of $l(P_+(t) - P_-(t))$ is uniformly bounded and smooth on $(t, x, y) \in [0, \infty) \times M_+ \times M_+$.*

Proof. We represent $P_\pm(t)$ as follows:

$$2P_\pm(t) = R(0, t)_\pm + e^{-tD_\pm^2} - e^{-tD_\pm^2} E_{D_\pm}\{0\}.$$

Hence, the claim follows immediately from Corollary 3.8 and Lemma 3.9. q.e.d.

The same is true for l replaced by l_1. Applying the finite propagation speed method in order to obtain an off-diagonal estimate we have

Lemma 3.16. *The integral kernels of $lP_\pm(t)(1-l_1)$, $(1-l_1)P_\pm(t)l$ are uniformly bounded and smooth on $(t, x, y) \in [0, \infty) \times M_+ \times M_+$.*

We have

$$l[P_+(t), P_-(t)] = lP_+(t)(P_-(t) - P_+(t)) - l(P_-(t) - P_+(t))P_+(t). \tag{31}$$

We discuss the first term of the right-hand side of (31)

$$\begin{aligned} &lP_+(t)(P_-(t) - P_+(t)) \\ &\quad = lP_+(t)l_1(P_-(t) - P_+(t)) + lP_+(t)(1-l_1)(P_-(t) - P_+(t)). \end{aligned} \tag{32}$$

Let us again consider the first term in more detail. By Lemma 3.15 one can find a compact operator T and a family $S(t)$ converging in the trace norm as $t \to 0$ such that $l_1(P_-(t) - P_+(t)) = TS(t)$. Since $lP_+(t) \overset{t\to 0}{\longrightarrow} lP_+$ strongly, the family $P_+(t)T$ converges uniformly. Hence, $P_+(t)TS(t) = lP_+(t)l_1(P_-(t) - P_+(t))$ converges in the trace norm. The second term in (32) is handled in a similar way using Lemma 3.16. Analogously, we show the convergence in the trace norm of the second term in (31). The same method applies to (29). The term (30) converges in the trace norm since $lQ(1-l_1)$ has a bounded smooth kernel. This can also be shown with the finite propagation speed method without appealing for the pseudodifferential calculus and using the fact that $l(1-l_1) = 0$.

Since $[P_+(t), P_-(t)]$ converges strongly as $t \to 0$, we have proved the convergence in the trace norm of $lK(t)$ as $t \to 0$.

Now we show how to deal with $v(K(t)-\tilde{K}(t))v$. Let v_1 be a smooth cut-off function with $v_1 v = v$ and

$$\operatorname{supp} v_1 \subset ([-6/8, 6/8]\times N)\cup([-6/8, 6/8]\times N).$$

Set

$$\tilde{Q} := \frac{1}{2}\sum_{i=0}^{N-1}\binom{N}{i}(-1)^{N-i}(\tilde{P}_+(\tilde{P}_-\tilde{P}_+)^{N-i-1}+\tilde{P}_-(\tilde{P}_+\tilde{P}_-)^{N-i-1}).$$

We have to consider

$$\begin{aligned} v(K(t)-\tilde{K}(t))v &= v[[P_+(t), P_-(t)], Q]v - v[[\tilde{P}_+(t), \tilde{P}_-(t)], \tilde{Q}]v \\ &= v[[P_+(t), P_-(t)], (Q-\tilde{Q})]v && (33)\\ &\quad + v[[P_+(t), P_-(t)] - [\tilde{P}_+(t), \tilde{P}_-(t)], \tilde{Q}]v. && (34)\end{aligned}$$

We write the term (33) as

$$\begin{aligned} v[[P_+(t), P_-(t)], (Q-\tilde{Q})]v &= v[P_+(t), P_-(t)](Q-\tilde{Q})v \\ &\quad - v(Q-\tilde{Q})[P_+(t), P_-(t)]v \\ &= v[P_+(t), P_-(t)]v_1(Q-\tilde{Q})v \\ &\quad - v(Q-\tilde{Q})v_1[P_+(t), P_-(t)]v \\ &\quad + v[P_+(t), P_-(t)](1-v_1)(Q-\tilde{Q})v \\ &\quad - v(Q-\tilde{Q})(1-v_1)[P_+(t), P_-(t)]v. \end{aligned}$$

All these terms converge in the trace norm as $t\to 0$. To see this we use the following facts:

$v_1(Q-\tilde{Q})v$ has a smooth bounded integral kernel.
$v[P_+(t), P_-(t)]v_1$ converges strongly.
$v[P_+(t), P_-(t)](1-v_1)$ converges in the trace norm due to the off-diagonal estimates.

The term (34) is handled in a similar way. Thus, we have proved the theorem. q.e.d.

This also finishes the proof of the gluing formula for the η-invariant (Theorem 1.9):

$$\begin{aligned}\eta(M)-\eta(M_1, L_1)-\eta(M_2, L_2) &= m(L_1, L_2) - 2I(P_+, P_-) \\ &\quad + \dim\ker D_+ - \dim\ker D_-.\end{aligned}$$

Probably this formula could also be proved using a variation formula. In fact $D_+ - D_- = G$ is a compactly supported bundle endomorphism (nonlocal!). It seems not to be too complicated to show that the variation of the η-invariant for the family $D_t = D_+ - tG$ vanishes. Then one has only to count the jumps of the η-invariant of D_t. The resulting integer is exactly $-2I(P_+, P_-) + \dim\ker D_+ - \dim\ker D_-$. A related application of the variation formula was discussed by Cheeger [20].

4. Vanishing of the relative index term

4.1. The space of limiting values is Lagrangian. We start with proving Proposition 1.14. Let (M, N) be a compact odd-dimensional Riemannian manifold with right boundary N, and $E \to M$ be a Dirac bundle with Dirac operator D. We assume a product structure for the Riemannian metric and for E near N as explained in §1.3. Moreover, let B be the closure of D with boundary condition $B_{\{V\}}$ (§1.8), i.e., for $\phi \in \operatorname{dom} B$ we require that $P_-\phi_{|N} = 0$, where P_- is the negative spectral projection of D_N. Recall the definition of

$$L := L_M := \{\mathrm{pr}_V \phi_{|N} | \phi \in \ker B\}.$$

L is a subspace of the symplectic vector space $V := \ker D_N$ with the symplectic structure given by $\Phi(u, v) = \langle Iu, v\rangle_{L^2}$.

Proposition 4.1. *L is a Lagrangian subspace of V, i.e., $L \perp IL$ and $L \oplus IL = V$.*

Proof. We show that the restriction of Φ to L vanishes. Let $u, v \in L$ and $\phi, \psi \in \ker B$ with $\mathrm{pr}_V \phi_{|N} = u$, $\mathrm{pr}_V \psi_{|N} = v$. Then, because of $IP_-I = -P_+$, by the partial integration formula for the Dirac operator

$$0 = \langle B\phi, \psi\rangle_{L^2} - \langle \phi, B\psi\rangle_{L^2} = \langle Iu, v\rangle_{L^2(N, E_N)} = \Phi(u, v).$$

Since this is true for all pairs $u, v \in L$, the claim follows. Note that B^* is the closure of D with the initial domain given by the boundary condition $B_{\{0\}}$. Thus, a vector $\phi \in \ker B$ satisfying $\mathrm{pr}_V \phi_{|N} = 0$ belongs already to $\ker B^*$. It follows that

$$\dim L = \dim\ker B - \dim\ker B^* = \operatorname{index} B.$$

It remains to show that $\operatorname{index} B = \dim V/2$. We consider $\overline{M} := [-1, 0] \times N$ and let $\overline{E} \to \overline{M}$ be the Dirac bundle induced from that of M over the product collar with Dirac operator $\overline{D}$. We define $\overline{B}$ to be the closure of $\overline{D}$ subject to the boundary condition B_V at $\{0\} \times N$ and to $B_{\overline{L}}$ at $\{-1\} \times N$, where $\overline{L}$ is an arbitrary Lagrangian subspace of V. By separation of variables one can compute $\ker \overline{B} \cong \overline{L}$ and $\operatorname{coker} \overline{B} = \{0\}$. Hence, $\operatorname{index} \overline{B} = \dim \overline{L} = \dim V/2$. We can apply a variant of the relative index theorem of Gromov-Lawson [26] to infer $\operatorname{index} B = \dim V/2$. In fact, $\overline{B}$ and B coincide on the cocompact set $(-\varepsilon, 0] \times N$ and are both Fredholm. Thus, the difference of their indices is the difference of integrals of index densities (the left boundary of $\overline{M}$ does not contribute since we have chosen a selfadjoint boundary condition). Since the dimension of M is odd, this integral vanishes. This proves the claim and finishes also the proof of the proposition.

4.2. Adjusting U in the adiabatic limit. Recall that M, M_i, D, D_i depend on a parameter $r > 1$, that is half of the length of the cylinder glued in at the hypersurface N as explained in §1.8. We will now make a more convenient r-dependent choice of the unitary U. Let $F_s: [-s, s] \times (N \cup N) \to [-1, 1] \times (N \cup N)$ be the stretching map and F^* be the induced pullback of sections of the Dirac bundle. Let U_1 denote the isometry U defined above on the cylinder of length 2. For $r \geq 1$ we define

$U_r := U_1$ outside $[-r/2, r/2] \times (N \cup N)$,

$U_r := F_r^* U_1 F_r^{-1*}$ on $[-r/2, r/2] \times (N \cup N)$.

In what follows we will again omit the index r, assuming this new choice of U. Then also G becomes r-dependent and supported in $[-r/2, r/2] \times (N \cup N)$. Since G involves derivatives of cut-off functions, there is a $C < \infty$ such that

$$\|G\| \leq C/r. \tag{35}$$

Since in Theorem 1.9 the relative index term $I(P_+, P_-)$ (note that $P_\pm$ are the positive spectral projections of $D_\pm$) is the only one depending on the unitary U, it turns out that $I(P_+, P_-)$ is independent of the choice of U.

4.3. The operator $B(u)$. Recall the definition of $D(u) := D_- + uG$, $u \in [0, 1]$, with $G := D_+ - D_-$. As explained below the statement of Theorem 1.17 we have to show the vanishing of the spectral flow of this family of Dirac operators with constant principal symbol if r is large. These operators live on $M \cup Z$. Note that $M \cup Z$ contains cylindrical parts $Z \cup Z$, $Z^+ := [r/2, r] \times (N \cup N)$ and $Z^- := [-r, r/2] \times (N \cup N)$, where the operator $D(u)$ has a product structure for all u. We refer to the component of $Z \cup Z$, which is not glued to $\widetilde{M}_i$ as the second component, while the first component is a part of M. On $Z \cup Z$ we define another family $B(u)$ as an extension of $(D_Z \oplus D_Z) + (u-1)G$ subject to the following boundary conditions:

We require B_{L_1} at the left boundary of the second component $\{-r\} \times N$, and B_{L_2} at the right boundary of the second component $\{r\} \times N$.

On the first component we require that $\psi \in \operatorname{dom} B(u)$ satisfies $\psi \in (L_1 \oplus N_+)$ at $\{-r\} \times N$ and $\psi \in (L_2 \oplus N_-)$ at $\{r\} \times N$, where $N_\pm$ are the positive and negative spectral subspaces of D_N.

On the first component the boundary conditions is *nonelliptic.* However $B(u)$ decomposes into a sum of one-dimensional operators labeled by $\operatorname{spec}(D_N)$. These one-dimensional operators are essentially selfadjoint and elliptic. This gives the selfadjoint extension of $B(u)$. Note that $D(u)$ and $B(u)$ coincide locally over $Z^{\pm}$. We will compare the spectral flow of $B(u)$ with that of $D(u)$ for large r. Since this concerns the small eigenvalues, actually only a part of $B(u)$ is interesting. The next lemma takes the special structure of G into account.

Lemma 4.2. *Let $h_0 := L^2([-r, r], V \oplus V) \subset L^2(Z \cup Z, E_Z \cup E_Z)$ be the subspace of ψ with $\psi(s) \in V \oplus V$ for all $s \in [-r, r]$. Then $B(u)$ leaves h_0 invariant.*

Proof. This holds for $B(1) = I(\partial/\partial r + (D_N \oplus D_N))$. However G consists of compositions of multiplication operators ρI with functions ρ only depending on the r-variable with transportation using the flip interchanging the two copies of Z. Hence, G also leaves h_0 invariant. q.e.d.

We define $B_0(u)$ to be the restriction of $B(u)$ to h_0. Let

$$d := \inf(|\operatorname{spec}(D_N)| \setminus \{0\}).$$

Note that $|\operatorname{spec}(B(1)_{|h_0^\perp})| \geq d$. There is an r_0 such that if $r \geq r_0$, then by (35), $\|G\| \leq d/2$ and also $|\operatorname{spec}(B(u)_{|h_0^\perp})| \geq d/2$ for all $u \in [0, 1]$. In this case all eigenvectors of $B(u)$ corresponding to eigenvalues λ with $|\lambda| \leq d/2$ are in fact eigenvectors of $B_0(u)$. Hence, for $r > r_0$ we have $sf\{B(u)\} = sf\{B_0(u)\}$.

4.4. The structure of the eigenvectors on the cylinder. We identify $L^2(Z \cup Z, E_Z \cup E_Z)$ with $L^2([-r, r], L^2(N, E_N) \oplus L^2(N, E_N))$. Let $N_\pm \subset L^2(N, E_Z)$ be the positive and negative spectral subspaces of D_N, respectively. If ψ is an eigenvector of $D(u)$ or $B(u)$ corresponding to the eigenvalue λ, then we can split it over $Z \cup Z$ as $\psi = \psi_0 + \psi_+ + \psi_-$, where $\psi_0(s) \in (V \oplus V)$, $\psi_+(s) \in N_+ \oplus N_+$, and $\psi_-(s) \in (N_- \oplus N_-)$.

We give now expressions for the eigenfunctions of $I(\partial/\partial r + D_N)$. Let $\{h_\mu\}$ and $\{Ih_\mu\}$ be orthonormal bases of N_+ and N_- such that $D_N h_\mu = \mu h_\mu$ with $\mu > 0$ and $D_N I h_\mu = -\mu I h_\mu$. If $|\lambda| \leq d/2$, we can define $\alpha_\mu \in (-\pi/2, \pi/2)$ by $\sin\alpha_\mu = \lambda/\mu$. Let

$$H_\mu^+(s) := [(1 - \cos\alpha_\mu)h_\mu + \sin\alpha_\mu I h_\mu]e^{s\mu\cos\alpha_\mu}, \tag{36}$$

$$H_\mu^-(s) := [\sin\alpha_\mu h_\mu + (1 - \cos\alpha_\mu)I h_\mu]e^{-s\mu\cos\alpha_\mu}. \tag{37}$$

Then one can check $I(\partial/\partial r + D_N)H_\mu^\pm = \lambda H_\mu^\pm$.

4.5. Definition of the map K. We start now with the comparison of the spectral flow of the families $D(u)$ and $B(u)$ by defining a linear, almost unitary map K mapping eigenvectors of $B(u)$ corresponding to small eigenvalues to approximate eigenvectors of $D(u)$ of the same eigenvalue. The approximation becomes better when r becomes larger and $|\lambda|$ smaller. Starting from now on we define $D(u)$ and $B(u)$ using the Lagrangian subspaces $L_1 := L_{M_1}$ and $L_2 := L_{M_2}$ given by the limiting values. Note that M_2 is considered as a right manifold. L_1 is employed at $\{-r\} \times (N \cup N)$, while L_2 at $\{r\} \times (N \cup N)$.

We fix linear maps $x_i \colon L_i \to \ker \widetilde{B}_i$ ($\widetilde{B}_i$ is the extension of $\widetilde{D}_i$ subject to the boundary condition B_V living on $\widetilde{M}_i$, i.e., without the cylinder glued on (see §1.8)), such that $\mathrm{pr}_V x_1(v)(-r) = v$, $v \in L_1$ and $\mathrm{pr}_V x_2(u)(r) = u$, $u \in L_2$. Note that every element $\phi \in \ker \widetilde{B}_1$ has an obvious extension to the first component of Z^- of the form $\phi = \phi_0 + \phi_+$, where

$$\|\phi_+(s)\|_N \le Ce^{c(-r-s)} \|\phi_0(-r)\|_N \tag{38}$$

and $\|\cdot\|_N$ is the norm on $L^2(N, E_N)$. Every $\chi \in \ker \widetilde{B}_2$ has an extension $\chi_0 + \chi_-$ to the first component of Z_r^+ with

$$\|\chi_-(s)\|_N \le Ce^{c(s-r)} \|\chi_0(r)\|_N . \tag{39}$$

We choose smooth cut-off functions $\alpha_\pm$ with
$\alpha_-(s) = 1$ for $s < -3r/4 - 1$,
$\alpha_-(s) = 0$ for $s > -3r/4 + 1$,
$\alpha_+(s) = 1$ for $s > 3r/4 + 1$,
$\alpha_+(s) = 0$ for $s < 3r/4 - 1$
such that the C^1-norm is uniformly bounded with respect to r. Let ψ be an eigenvector of $B(u)$ corresponding to an eigenvalue λ with $|\lambda| \le d/2$. Then we have $\psi = \psi_0 + \psi_- + \psi_+$ on $Z \cup Z$. Since λ is small, actually $\psi = \psi_0$. Moreover, $\psi_0(-r) \in L_1$ and $\psi_0(r) \in L_2$ on both components.

Let $\phi := x_1(\psi_0(-r))$ and $\chi := x_2(\psi_0(r))$ (we take the boundary values on the first component). We define $K(\psi)$ to be ψ on the second component of $Z \cup Z$ and by
ϕ on $\widetilde{M}_1$,
$\psi + \phi_+$ on $[-r, -3r/4 - 1] \times N$,
$\psi + \alpha_- \phi_+$ on $[-3r/4 - 1, -3r/4 + 1] \times N$,
ψ on $[-3r/4 + 1, 3r/4 - 1] \times N$,
$\psi + \alpha_+ \chi_-$ on $[3r/4 - 1, 3r/4 + 1] \times N$,
$\psi + \chi_-$ on $[3r/4 + 1, r] \times N$,
χ on $\widetilde{M}_2$.

Then $K(\psi)$ is continuous, piecewise smooth, and satisfies the boundary conditions for $D(u)$. We extend K linearly to linear combinations of eigenvectors.

Lemma 4.3 (K maps eigenvectors to approximate eigenvectors). *There are constants $C < 0$, $c > 0$, and $r_0 \geq 2$ such that the following holds for all $r \geq r_0$, $u \in [0, 1]$: Let ψ be a normed eigenvector of $B(u)$ to the eigenvalue λ with $|\lambda| \leq d/2$. Then*

$$\|D(u)K(\psi) - \lambda K(\psi)\| \leq C(|\lambda|/\sqrt{r} + e^{-cr}).$$

If ψ_1, ψ_2 are two such eigenvectors, then

$$|\langle \psi_1, \psi_2\rangle - \langle K(\psi_1), K(\psi_2)\rangle| \leq C/\sqrt{r}.$$

Proof. We compute $D(u)K(\psi) - \lambda K(\psi)$ obtaining 0 on the second component, and on the first component the result is

$-\lambda\phi$ on $\widetilde{M}_1$,
$-\lambda\phi_+$ on $[-r, -3r/4 - 1] \times N$,
$\operatorname{grad}\alpha_-\phi_+ - \lambda\alpha_-\phi_+$ on $[-3r/4 - 1, -3r/4 + 1] \times N$,
0 on $[-3r/4 + 1, 3r/4 - 1] \times N$,
$\operatorname{grad}\alpha_+\chi_- - \lambda\alpha_+\chi_-$ on $[3r/4 - 1, 3r/4 + 1] \times N$,
$-\lambda\chi_-$ on $[3r/4 + 1, r] \times N$,
$-\lambda\chi$ on $\widetilde{M}_2$.

Note that ψ_0 is oscillating. Hence,

$$1 = \|\psi\|_{L^2} \geq \|\psi_0\|_{L^2(Z^\pm, E_Z)} = \sqrt{r/2}\|\psi_0(\pm r)\|_N. \tag{40}$$

Applying this and (38), (39) we obtain

$\|\lambda\phi\|_{L^2} \leq C|\lambda|/\sqrt{r}$ on $\widetilde{M}_1$,
$\|\lambda\phi_+\|_{L^2} \leq C|\lambda|/\sqrt{r}$ on $[-r, -3r/4 - 1] \times N$,
$\|\operatorname{grad}\alpha_-\phi_+ - \lambda\alpha_-\phi_+\|_{L^2} \leq C(1/\sqrt{r})e^{-cr}$ on $[-3r/4-1, -3r/4+1]\times N$,
$\|\operatorname{grad}\alpha_+\chi_- - \lambda\alpha_+\chi_-\|_{L^2} \leq C(1/\sqrt{r})e^{-cr}$ on $[3r/4 - 1, 3r/4 + 1] \times N$,
$\|\lambda\chi_-\|_{L^2} \leq C|\lambda|/\sqrt{r}$ on $[3r/4 + 1, r] \times N$,
$\|\lambda\chi\|_{L^2} \leq C|\lambda|/\sqrt{r}$ on $\widetilde{M}_2$.

Summing up the terms we obtain the first inequality of the lemma. In order to see the second inequality note that most of the mass of ψ_i is concentrated in $(\psi_i)_0$ on the cylinder. q.e.d.

Thus for large r we know by the variational principle, that for every small eigenvalue λ of multiplicity m of $B(u)$, there are eigenvalues of $D(u)$ of total multiplicity at least m in a neighborhood of λ of size $C(|\lambda|/\sqrt{r}+e^{-cr})$. In order to check that we cover all small eigenvalues with the right multiplicity we will construct a map A from the eigenvectors of $D(u)$ to the approximate eigenvectors of $B(u)$.

4.6. The small eigenvectors of $D(u)$. We first study the structure of the eigenvectors of $D(u)$ corresponding to small eigenvalues.

Lemma 4.4. *For every J_0 there are constants $C < \infty$, $c > 0$, and $r_0 > 1$ such that for all $r \geq r_0$ the following holds: Let ψ be a normed eigenvector of $I(\partial/\partial r + D_N)$ on $[0, r] \times N$ corresponding to the eigenvalue λ with $|\lambda| \leq c$ and assume*

$$\|\psi(0)\|_N \leq J \leq J_0. \tag{41}$$

Then

$$\|\psi_-(0)\|_N \leq C(J|\lambda| + e^{-cr}).$$

Proof. Since

$$1 = \|\psi\|^2_{L^2} \geq \int_0^r \|\psi(s)_+\|^2_N \, ds + \int_0^r \|\psi_-(s)\|^2_N \, ds,$$

there is a point $v \in [3r/4, r]$ with

$$\|\psi_-(v)\|^2_N \leq 4/r. \tag{42}$$

Without loss of generality we can assume that $\psi = c^+ H^+_\mu + c^- H^-_\mu$ (see (36), (37)). Writing (41) and (42) out provides

$$|(1 - \cos\alpha_\mu)c^+ + \sin\alpha_\mu c^-| \leq J, \tag{43}$$

$$|\sin\alpha_\mu c^+ + (1 - \cos\alpha_\mu)c^-| \leq J, \tag{44}$$

$$|\sin\alpha_\mu e^{v\mu\cos\alpha_\mu} c^+ + (1 - \cos\alpha_\mu)e^{-v\mu\cos\alpha_\mu} c^-| \leq 2/\sqrt{r}. \tag{45}$$

From (43) we derive

$$|c^-| \leq \frac{1}{|\sin\alpha_\mu|}(J + (1 - \cos\alpha_\mu)|c^+|). \tag{46}$$

Substituting (46) in (45) yields

$$\begin{aligned} &|c^+| \left(|\sin\alpha_\mu| e^{v\mu\cos\alpha_\mu} - \frac{(1 - \cos\alpha_\mu)^2 e^{-v\mu\cos\alpha_\mu}}{|\sin\alpha_\mu|} \right) \\ &\quad \leq \frac{2}{\sqrt{r}} + \frac{J(1 - \cos\alpha_\mu)e^{-v\mu\cos\alpha_\mu}}{|\sin\alpha_\mu|}. \end{aligned} \tag{47}$$

If $c > 0$ is small enough, the factor at $|c^+|$ is positive. Choosing $c > 0$ even smaller (depending on J_0) we can simplify (47) to

$$|c^+| \leq 8e^{-v\mu\cos\alpha_\mu}/\sqrt{r}|\sin\alpha_\mu|. \tag{48}$$

Substituting (48) in (46) we obtain

$$|c^-| \le \frac{J}{|\sin\alpha_\mu|} + \frac{8(1-\cos\alpha_\mu)e^{-v\mu\cos\alpha_\mu}}{\sqrt{r}(\sin\alpha_\mu)^2} \tag{49}$$

$$\le \frac{J}{|\sin\alpha_\mu|} + \frac{100e^{-v\mu\cos\alpha_\mu}}{\sqrt{r}}. \tag{50}$$

From (44), (48), and (49) we get

$$\|\psi_-(0)\|_N \le \frac{8e^{-v\mu\cos\alpha_\mu}}{\sqrt{r}} + \frac{J(1-\cos\alpha_\mu)}{|\sin\alpha_\mu|} + \frac{100(1-\cos\alpha_\mu)e^{-v\mu\cos\alpha_\mu}}{\sqrt{r}}$$
$$\le C(J|\lambda| + e^{-cr}),$$

since $(1-\cos\alpha_\mu)/|\sin\alpha_\mu| \le C|\lambda|$ and $v \ge 3r/4$. Here C and c are independent of μ.

Lemma 4.5. *There are constants $C < \infty$, $c > 0$, and $r_0 > 1$ such that the following holds for all $r \ge r_0$ and $u \in [0, 1]$: Let ψ be a normed eigenvector of $D(u)$ corresponding to the eigenvalue λ with $|\lambda| \le c$. Then, on the first component*

$$\|\psi_-(-r)\|_N \le C(|\lambda| + e^{-cr}), \tag{51}$$

$$\|\psi_+(r)\|_N \le C(|\lambda| + e^{-cr}). \tag{52}$$

Proof. In order to get the first estimate we apply Lemma 4.4 to ψ on the first component of Z^-, noting that a priori there is a $J_0 < \infty$ with $\|\psi(-r)\|_N \le J_0\|\psi\|_{L^2} \le J_0$. The second estimate is obtained by a reflection-symmetric argument.

Lemma 4.6. *Assume* 1.15. *Then there are constants $C < \infty$, $c > 0$, and $r_0 > 1$ such that for all $r \ge r_0$ and $u \in [0, 1]$ the following holds: Let ψ be a normed eigenvector of $D(u)$ corresponding to an eigenvalue λ with $|\lambda| < c$. Then*

$$\|\psi_{|\widetilde{M}_i}\|_{L^2} \le C(|\lambda| + 1/\sqrt{r})$$

for $i = 1, 2$.

Proof. We consider the assertion for $\widetilde{M}_1$. Let F be the closure of $\widetilde{D}_1$ with the boundary condition $B_{L_1^\perp}$. Then F is selfadjoint and $\inf|\mathrm{spec}(F)| \ge 2c > 0$ for some $c > 0$ by the regularity assumption 1.15. Hence, for any $\phi \in \mathrm{dom}\, F$ if $|\lambda| \le c$ we have

$$\|(F-\lambda)\phi\|_{L^2} \ge c\|\phi\|_{L^2}. \tag{53}$$

Let γ be a smooth cut-off function on $\widetilde{M}_1$ with $\gamma = 0$ on $\{-r\} \times N$ and $\gamma = 1$ outside of the collar $(-r-\varepsilon, -r] \times N$.

Now let ψ be a normed eigenvector of $D(u)$ corresponding to the eigenvalue λ with $|\lambda| \le c$ and $\psi = \psi_0 + \psi_- + \psi_+$ near $\{-r\} \times N$ of the first component. We define $\phi \in \operatorname{dom} F$ to be

ψ on $M_1 \backslash (-r-\varepsilon, -r] \times N$,

$\psi_+ + \gamma(\psi_0 + \psi_-)$ on $(-r-\varepsilon, -r] \times N$.

Since ψ_0 is oscillating, we have $\psi_0(s) \le C/\sqrt{r}$ for all $s \in (-\varepsilon, 0] \times N$. By Lemma 4.5 we also have

$$\|\psi_-(s)\|_N \le C(|\lambda| + e^{-cr})$$

for $s \in (-\varepsilon, 0] \times N$. This shows

$$\|\psi_{|\widetilde{M}_1} - \phi\|_{L^2} \le C(|\lambda| + 1/\sqrt{r}). \tag{54}$$

On $s \in (-r-\varepsilon, -r] \times N$ we compute

$$(F-\lambda)\phi = [\lambda(1-\gamma) + \operatorname{grad}\gamma]\psi_- + \lambda(\gamma - 1)\psi_+ + \operatorname{grad}\gamma\psi_0 .$$

We give the L^2-norm estimates of the various terms using Lemma 4.5:

$$\|[\lambda(1-\gamma) + \operatorname{grad}\gamma]\psi_-\|_{L^2} \le C(|\lambda| + e^{-cr}), \tag{55}$$

$$\|\lambda(\gamma - 1)\psi_+\|_{L^2} \le C|\lambda|, \tag{56}$$

$$\|\operatorname{grad}\gamma\psi_0\|_{L^2} \le C/\sqrt{r}.$$

Hence,

$$\|(F-\lambda)\phi\|_{L^2} \le C(|\lambda| + 1/\sqrt{r}).$$

Now (53) also implies $\|\phi\|_{L^2} \le C(|\lambda| + 1/\sqrt{r})$ and the claim follows from (54).

Lemma 4.7. *Assume* 1.15. *Then there are constants* $C < \infty$, $c > 0$, *and* $r_0 > 1$ *such that for all* $r \ge r_0$ *and* $u \in [0, 1]$ *the following holds: Let* ψ *be a normed eigenvector of* $D(u)$ *corresponding to an eigenvalue* λ *with* $|\lambda| < c$. *Then for the boundary values on the first component the following hold*:

$$\|\psi(-r)\|_N \le C(|\lambda| + 1/\sqrt{r}), \tag{57}$$

$$\|\psi(r)\|_N \le C(|\lambda| + 1/\sqrt{r}), \tag{58}$$

$$\|\psi_-(-r)\|_N \le C(\lambda^2 + |\lambda|/\sqrt{r} + e^{-cr}), \tag{59}$$

$$\|\psi_+(r)\|_N \le C(\lambda^2 + |\lambda|/\sqrt{r} + e^{-cr}). \tag{60}$$

Proof. The first two estimates (57) and (58) follow from Lemma 4.6 since there is an a priori estimate of the C^1-norm. In order to get (59) and (60) substitute (57) and (58) in Lemma 4.4, i.e., use $J = C(|\lambda| + 1/\sqrt{r})$.

Lemma 4.8. *Assume* 1.15. *Then there are constants* $C < \infty$, $c > 0$, *and* $r_0 > 1$ *such that for all* $r \geq r_0$ *and* $u \in [0, 1]$ *the following holds: Let* ψ *be a normed eigenvector of* $D(u)$ *corresponding to an eigenvalue* λ *with* $|\lambda| < c$. *Then*

$$\|\psi_{|\widetilde{M}_i}\|_{L^2} \leq C(\lambda^2 + 1/\sqrt{r})$$

for $i = 1, 2$.

Proof. We proceed as in the proof of Lemma 4.6, but we substitute the better estimates (59) for ψ_- in (55) and (57) for ψ_+ into (56).

Lemma 4.9. *Assume* 1.15. *Then there are constants* $C < \infty$, $c \geq 0$, *and* $r_0 > 1$ *such that for all* $r \geq r_0$ *and* $u \in [0, 1]$ *the following holds: Let* ψ *be a normed eigenvector of* $D(u)$ *corresponding to the eigenvalue* λ *with* $|\lambda| \leq c$. *Then, on the first component, the following hold:*

$\|\mathrm{pr}_{L_1^\perp} \psi_0(-r)\|_N \leq C(\lambda^2 + |\lambda|/\sqrt{r} + e^{-cr})$ *and*

$\|\mathrm{pr}_{L_2^\perp} \psi_0(r)\|_N \leq C(\lambda^2 + |\lambda|/r + e^{-cr})$.

Proof. We consider the first estimate. Let F_L be the closure of $\widetilde{D}_1$ with the boundary condition B_L, where L is Lagrangian. By the method of Lesch-Wojciechowski [31] it can be shown that the spectrum of F_L depends continuously on L. Since the parameter space of all Lagrangian subspaces of V is compact, we have

$$(61) \qquad \inf_{x \in V} \sup_{x \in L \subset V} \inf(|\mathrm{spec}(F_L)| \backslash \{0\}) = 2c_1 > 0.$$

Let ψ be a normed eigenvector of $D(u)$ corresponding to the eigenvalue λ with $|\lambda| \leq \min(c, c_1)$ (the c from Lemma 4.8) and $\psi = \psi_0 + \psi_- + \psi_+$ its decomposition on $Z \cup Z$. We define ϕ on $\widetilde{M}_1$ by

ψ on $\widetilde{M}_1 \backslash (-r - \varepsilon, -r] \times N$,

$\psi_0 + \psi_+ + \gamma\psi_-$ on $(-r - \varepsilon, -r] \times N$.

Let $L \subset V$ realize the maximum of $\inf|(\mathrm{spec}(F_L) \backslash \{0\}|)$ subject to $\psi_0(-r) \in L$. Then $\phi \in \mathrm{dom}\, F_L$. We compute

$$(F_L - \lambda)\phi = [\lambda(1 - \gamma) + \mathrm{grad}\, \gamma]\psi_- + \lambda(\gamma - 1)\psi_+ .$$

Applying now Lemma 4.7 to ψ_+ and ψ_- yields

$$\|(F_L - \lambda)\phi\|_{L^2} \leq C(\lambda^2 + |\lambda|/\sqrt{r} + e^{-cr}).$$

Let $\phi = \phi_n + \phi_o$ with $F_L \phi_n = 0$ and $\phi_o \perp \ker F_L$. Then

$$\|(F_L - \lambda)\phi_o\|_{L^2} \leq C(\lambda^2 + |\lambda|/\sqrt{r} + e^{-cr})$$

and, hence, $\|\phi_o\|_{L^2} \leq C(\lambda^2 + |\lambda|/\sqrt{r} + e^{-cr})$. Since the C^1-norm of the ϕ is a priori bounded, we have $\phi_o(-r) \leq C(\lambda^2 + |\lambda|/\sqrt{r} + e^{-cr})$ too. From

$\mathrm{pr}_V\phi_n(-r) \in L \cap L_1 \subset L_1$, it follows that $\mathrm{pr}_{L_1^\perp}\psi_0(-r) = \mathrm{pr}_{L_1^\perp}\phi_o(-r)$. Hence the lemma follows.

4.7. The construction of the map A**.** Now we construct a linear, almost unitary map A mapping eigenvectors of $D(u)$ corresponding to small eigenvalues λ to approximate eigenvectors of $B(u)$. The approximation becomes better when r increases and $|\lambda|$ is small.

Choose smooth cut-off functions $\rho_\pm$ with

$\rho_- = 0$ near $\{-r\}$,
$\rho_- = 1$ on $[-r+1, r]$,
$\rho_+ = 0$ near $\{r\}$,
$\rho_+ = 1$ on $[-r, r-1]$

such that the C^1-norms are uniformly bounded with respect to r. Let ψ be an eigenvector of $D(u)$ corresponding to λ, and let $\psi = \psi_0 + \psi_- + \psi_+$ be the decomposition on $Z \cup Z$. We define $A(\psi)$ to be ψ on the second component, and on the first component to be

$\mathrm{pr}_{L^1}\psi_0 + \psi_+ + \rho_-(\mathrm{pr}_{L_1^\perp}\psi_0 + \psi_-)$ on $[-r, -r+1] \times N$,
ψ on $[-r+1, r-1] \times (N \cup N)$,
$\mathrm{pr}_{L_2}\psi_0 + \psi_- + \rho_+(\mathrm{pr}_{L_2^\perp}\psi_0 + \psi_+)$ on $[r-1, r] \times N$.

Then $A(\psi) \in \mathrm{dom}\, B(u)$. A will be extended linearly to linear combinations of eigenvectors.

Lemma 4.10 (A maps eigenvectors to approximate eigenvectors). *There are constants $C < \infty$, $c > 0$, and $r_0 > 1$ such that for all $r \geq r_0$, $u \in [0, 1]$, the following holds: Let ψ be a normed eigenvector of $D(u)$ corresponding to an eigenvalue λ with $|\lambda| < c$. Then*

$$\|B(u)A(\psi) - \lambda A(\psi)\|_{L^2} \leq C(\lambda^2 + |\lambda|/\sqrt{r} + e^{-cr}).$$

If ψ_1, ψ_2 are two such eigenvectors, then

$$|\langle\psi_1, \psi_2\rangle - \langle A(\psi_1), A(\psi_2)\rangle| \leq C(1/\sqrt{r} + \lambda^2). \tag{62}$$

Proof. We compute $B(u)A(\psi) - \lambda A(\psi)$ and obtain 0 on the second component, compute

$$([\lambda(1-\rho_-) + \mathrm{grad}\,\rho_-]\mathrm{pr}_{L_1^\perp} + \lambda(\rho_- - 1)\mathrm{pr}_{L_1})\psi_0 + (\mathrm{grad}\,\rho_- + \lambda(1-\rho_-))\psi_- \\ + \lambda(\rho_- - 1)\psi_+$$

on $[-r, -r+1] \times N$, and obtain 0 on $[-r+1, r-1] \times N$, and compute

$$([\lambda(1-\rho_+) + \mathrm{grad}\,\rho_+]\mathrm{pr}_{L_2^\perp} + \lambda(\rho_+ - 1)\mathrm{pr}_{L^2})\psi_0 + (\mathrm{grad}\,\rho_+ + \lambda(1-\rho_+))\psi_+ \\ + \lambda(\rho_+ - 1)\psi_-$$

on $[r-1, r] \times N$. Lemmas 4.9 and 4.7 provide

$$\|B(u)A(\psi) - \lambda A(\psi)\|_{L^2} \le C(\lambda^2 + |\lambda|/\sqrt{r} + e^{-cr}).$$

Using Lemmas 4.7 and 4.8 we further obtain (62). q.e.d.

Thus, for large r by the variational principle we know that for every small eigenvalue λ of multiplicity m of $D(u)$, there are eigenvalues of $B(u)$ of total multiplicity at least m in a neighborhood of λ of size $C(\lambda^2 + |\lambda|/\sqrt{r} + e^{-cr})$.

4.8. Comparison of the spectral flows of $D(u)$ and $B(u)$. We fix now some large r_0. The small eigenvalues λ of $B(u)$ with $|\lambda| \le d/2$ are eigenvalues of $B_0(u)$. For $r \ge r_0$ and an eigenvector ψ of $B_0(u)$ let $\psi_r(s) := \psi(r_0 s/r)$. Then ψ_r is an eigenvector of $B_0(u)_r$ (it is more convenient not to omit the r in this subsection) to the eigenvalue $r_0\lambda/r$. Here we employ the r-dependent choice of U.

Lemma 4.11. *Assume* 1.15 *and* 1.16. *There is an $r_1 \ge r_0$ such that, for all $r \ge r_1$, we have equality of the spectral flows*

$$sf\{B(u)_r\} = sf\{D(u)\}.$$

Proof. We label the eigenvalues of $B_0(1)_{r_0}$ as follows (note that $\ker B_0(1)_{r_0} = \{0\}$ by Assumption 1.16, i.e., $L_1 \cap L_2 = \{0\}$) $\cdots \le \lambda_{-2} \le \lambda_{-1} < 0 < \lambda_0 \le \lambda_1 \le \cdots$. Then we extend this labeling by continuity to all $u \in [0, 1]$ such that the order is maintained, i.e., $\lambda_i(u) \le \lambda_{i+1}(u)$. By rescaling this induces a labeling of the small eigenvalues (i.e., with $|\lambda| \le d/2$) of $B(u)_r$ for all $r \ge r_0$.

There exists a small $c_1 > 0$ such that for $r = r_0$ and every $u \in [0, 1]$ there is an $N(u) \in \mathbf{Z}$ such that $|\lambda_{N(u)}(v) - \lambda_{N(u)+1}(v)| \ge c_1$ with $v \in [u^-, u^+]$, where $[u^-, u^+]$ is a small neighborhood of u. We choose finitely many u_i such that $\bigcup_i [u_i^-, u_i^+] = [0, 1]$.

Now we choose the r_1 so large that the following hold for all $r \ge r_1$:

(i) $\lambda_{N(u_i)}(v) \le c$ for all i and $v \in [u_i^-, u_i^+]$, where c is the smallest c of the preceding lemmas.

(ii)

$$\frac{r_0 c_1}{r} \ge 100C(\lambda_{N(u_i)}(v)^2 + \frac{|\lambda_{N(u_i)}(v)|}{\sqrt{r}} + e^{-cr}) \tag{63}$$

for all $v \in [u_i^-, u_i^+]$ and i, where C is the maximum of the C's given in the preceding lemmas and c is again the minimum of the c's given there. Note that $\lambda_{N(u_i)}(v) \sim 1/r$. Hence, the inequality above can be fulfilled if r_1 is large enough.

(iii) Let $q := \inf|\operatorname{spec}(B_0(1)_{r_0})| > 0$ (by (1.16!). Then

$$(64)\qquad qr_0/r - 100C((qr_0/r)^2 + qr_0/r^{3/2} + e^{-cr}) > 0.$$

To every eigenvalue λ of $D(u)$ with $|\lambda| \le c$, there corresponds an eigenvalue of $B(u)_r$ with a neighborhood of λ of size $C(\lambda^2+\lambda/\sqrt{r}+e^{cr})$, and vice versa. Here all eigenvalues are counted with multiplicity. We will use this in order to construct an induced labeling of the small eigenvalues of $D(u)$. Let us consider some i. For $v \in [u_i^-, u_i^+]$ define $\rho_{N(u_i)}(v)$ to be the largest eigenvalue of $D(v)$ smaller than

$$\lambda_{N(u_i)} + C(\lambda_{N(u_i)}^2 + |\lambda_{N(u_i)}|/\sqrt{r} + e^{-cr}).$$

It is easy to see that $\rho_{N(u_i)}(v)$ is continuous since no other branch of eigenvalues of $D(v)$ may enter the neighborhood of $\lambda_{N(u_i)}$ from above because we have chosen $N(u_i)$ such that there is a 'large' gap according to (63). Now we can label all the other eigenvalues of $D(v)$ using their order. Then all branches $\rho_i(v)$ become continuous. Moreover, one can check that the labeling is compatible over the intersections $[u_i^-, u_i^+] \cap [u_j^-, u_j^+]$. In fact, there are only finitely many eigenvalues between $\lambda_{N(u_i)}(v)$ and $\lambda_{N(u_j)}(v)$, and the same is true for $\rho_{N(u_i)}(v)$ and $\rho_{N(u_j)}(v)$. Furthermore we have for all $u \in [0, 1]$, $i \in \mathbf{Z}$,

$$(65)\qquad |\lambda_i(u) - \rho_i(u)| \le C(\lambda_i(u)^2 + |\lambda_i(u)|/\sqrt{r} + e^{-cr})$$

if $|\lambda_i(u)| \le c$. Due to (64) the choice of r_1 assures that

$$\cdots \le \rho_{-2}(1) \le \rho_{-1}(1) < 0 < \rho_0(1) \le \rho_1(1) \le \cdots.$$

Let $f := -sf\{B(u)_r\}$ be the spectral flow of $\{B(u)_r\}$. Then

$$\cdots \le \lambda_{-2+f}(0) \le \lambda_{-1+f}(0) < 0 < \lambda_f(0) \le \lambda_{1+f}(0) \le \cdots.$$

Hence, again by (64) and (65)

$$\cdots \le \rho_{-2+f}(0) \le \rho_{-1+f}(0) < 0 < \rho_f(0) \le \rho_{1+f}(0) \le \cdots.$$

Thus $sf\{D(u)\} = sf\{B(u)_r\}$.

The following lemma finishes the proof of our main Theorem 1.17.

Lemma 4.12. $sf\{B(u)_r\} = 0$.

Proof. Let V be the restriction of the unitary U to h_0. Then, by the definition of G, we have $VB_0(u)_rV^* = B_0(1-u)_r$ and the claim follows since $sf\{B(u)_r\} = sf\{B_0(u)_r\}$.

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