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MÖBIUS CONE STRUCTURES ON 3-MANIFOLDS

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Abstract

We show that for any given angle $\alpha \in (0, 2\pi)$, any closed 3-manifold has Möbius cone structure with cone angle α.

1. Introduction

In this paper we prove

Theorem 1. *For any positive $r < 2$ any closed orientable 3-manifold M has a singular Riemannian metric ds of the following form: There are local coordinates (z, t) (z complex and t real) in M so that in the coordinate the metric ds is:*

(a) *conformally flat: $ds = u(z, t)(|dz|^2 + |dt|^2)$ or,*

(b) *conformally flat with cone singularity of angle $r\pi$,*

$$ds = u(z, t) \cdot (|dz|^2/|z|^{2-r} + |dt|^2),$$

where $u(z, t)$ is a smooth positive function of z, t. Furthermore, if $r = 2/n$ for some positive integer $n > 1$, then the monodromy group of the conformally flat structure is a discrete subgroup of $SO(4, 1)$.

Due to the solution of the Yamabe problem by Schoen [14] it seems highly possible that in each such conformal class, there exists a Riemannian metric having the same form as above so that the scalar curvature is constant. Furthermore, the metric is unique if the singular set is nonempty and the pair (M, singular set) is not (S^3, circle).

We may also state the result in terms of Möbius cone geometry as follows. Given $\alpha \in (0, 2\pi)$, a *Euclidean lens of angle* α is defined to be the intersection of two balls at an angle α if $\alpha < \pi$, to be a ball together with a circle in the boundary if $\alpha = \pi$, and to be the complement of the interior of a Euclidean lens of angle $2\pi - \alpha$ if $\alpha > \pi$. An α-cone 3-sphere S^3_α is the quotient of a Euclidean lens of angle α by the rotation about the edge of the lens which identifies the two boundary half-spheres of the lens. A 3-manifold M is said to have Möbius cone structure with cone

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angle α if it has a singular conformally flat structure so that each point in M has a neighborhood conformal to an open set in S_α. The above result is the same as

Theorem 2. *Given any positive $\alpha < 2\pi$, any closed orientable 3-manifolds M has a singular conformally flat structure so that each point in M has a neighborhood which is conformal to an open set in S^3_α. Furthermore, if the given cone angle is $2\pi/n$ for some $n \in Z_+$, then the monodromy group is a discrete subgroup of $SO(4, 1)$.*

The singularity forms a link in the manifold. As the cone angle α tends to 2π, the number of components of the singularity increases to infinity. We call the singular conformal structure a Möbius cone structure with cone angle α.

Essentially, we show that 3-dimensional Dehn surgery can be realized in Möbius cone geometry S^3_α. The basic idea of the proof comes from Gromov, Lawson, and Thurston's construction in [4]. The main geometric object that we will study in detail is the Möbius Polygon in S^3. These are solid tori in S^3 whose boundary is a union of finitely many annuli so that each annulus is in a 2-sphere and the interiors of these annuli are disjoint. Regular Möbius Polygons were first used by Gromov, et al. in their construction of hyperbolic structures on plan bundles over surfaces.

As another consequence of the study on Möbius Polygons, we show

Theorem 3. *Let $R^2 \to W_{e,g} \to \Sigma_g$ be the plan bundle over a surface of genus $g > 1$ so that the Euler number is e. If $|e| = g-1$, then there exists a complete hyperbolic metric on $W_{e,g}$. Furthermore, the conformal infinity of $W_{e,g}$ in the hyperbolic metric is a Möbius structure on the associated circle bundle over Σ_g.*

This answers a question of Kuiper [6] affirmatively. We were informed by Kapovich that Kuiper and Waterman have also discovered the similar metric.

In [10], we prove a stronger result that Theorem 3 is still true for all $|e| \leq (g-1)$ using different constructions.

The most interesting Möbius Polygons are the ideal ones. These are obtained as the complement of a necklace formed by tangent balls in R^3. We discovered the analog phenomena to Thurston's construction of hyperbolic metrics on compact surfaces by using hyperbolic ideal polygons in H^2. Similar to Thurston's completion of hyperbolic metric, the conformal completion to noncusp ends in this case is to add a torus to the end and the boundary of the ideal Möbius Polygon behaves like a leaf of Reeb foliation in the interior of a solid torus. We intend to carry out the study in [10].

The paper is organized as follows. We define Möbius Polygons and discuss their basic properties in §2. The most useful invariant introduced in §2 is the torsion of a Möbius n-gon. We show that torsion is addictive with respect to the gluing of two Möbius Polygons. In §3, we discuss regular convex Möbius Polygons and prove a result concerning the existence of regular Möbius Polygons with prescribed torsion. The main theorem and Theorem 3 are proved in §§4 and 5 respectively.

2. Möbius Polygons in S^3

The basic facts about Möbius geometry in $(S^3, \mathrm{Mob}(S^3))$ may be found in Beardon's book [1] or in [15]. We will use S^3 to denote the unit sphere in C^2. $\mathrm{Mob}(S^3)$ is the group of all diffeomorphisms of S^3 preserving angles. We will identify S^3 with $\bar{R}^3 = R^3 \cup \{\infty\}$ by a stereographic projection. We will abuse the use of language by saying that lines and planes in $\bar{R}^3$ are special types of circles and 2-spheres in S^3. Elements in $\mathrm{Mob}^+(S^3)$ (orientation-preserving Möbius transformations) are classified into three types: hyperbolic, parabolic, and elliptic. For each circle C in S^3, the *half-turn* about C, denoted by H_C, is the orientation-preserving Möbius involution leaving C pointwise fixed. Given a set X containing more than one point, the *span* of X, denoted by $\mathrm{sp}(X)$, is the sphere of the smallest dimension containing X. A pair of circles A, B is said to be *standard* if (A, B) is Möbius equivalent to the pair (z-axis, the unit circle in the xy-plan).

The basic geometric objects that we will study are in the following.

2.1. Definition. (a) A *Möbius annulus* A is a topological annulus in a 2-sphere so that A is bounded by two circles. The *middle circle* of a Möbius annulus A is the circle C in A so that inversion about C leaves A invariant. An *orthogonal arc* in a Möbius annulus A is a circular arc orthogonal to both boundary components of A. The *module* of A, denoted by $m(A)$, is the module of the open annulus $\mathrm{int}(A)$ in $\mathrm{sp}(A)$.

(b) A *Möbius n-gon* P is a topological solid torus in S^3 so that its boundary ∂P is a union of Möbius annuli $F_1, F_2, \ldots, F_n$, and their interiors are all disjoint. Each F_i is called a *face* of P, and each circle in ∂F_i is called an *edge* of P. A Möbius Polygon is said to be *convex* if the dihedral angle at each edge of P is less than π. Two Möbius Polygons are said to be *equivalent* if there is a Möbius transformation sending one to the other.

For each Möbius annulus A, there exists a unique circle C in S^3 so that C and each component of ∂A form a standard pair. Furthermore, $A \cap C = \phi$, and C is orthogonal to $\mathrm{sp}(A)$. We call C the *axis* of A. A Möbius n-gon is said to be of type $\mathrm{PSL}(2, R)$ if the axes of its faces are the same; and is said to be of type $\mathrm{PSL}(2, C)$ if the axes of its faces are in a fixed 2-sphere, i.e., all its faces are orthogonal to a fixed 2-sphere. Clearly a $\mathrm{PSL}(2, R)$ Möbius n-gon is also a $\mathrm{PSL}(2, C)$ Möbius n-gon.

All $\mathrm{PSL}(2, R)$ Möbius n-gons are constructed as follows. Take a circle C (to be the common axis) and a disc D with $\partial D = C$, and consider $\mathrm{int}(D)$ as a model for the hyperbolic plane H^2. Given any hyperbolic polygon Δ of n sides (not necessary convex) in $\mathrm{int}(D)$, the rotation of Δ about C, denoted by $\Delta \times S^1$, is a $\mathrm{PSL}(2, R)$ Möbius n-gon. Conversely, all $\mathrm{PSL}(2, R)$ Möbius n-gons which are disjoint from their axes are equivalent to a $\Delta \times S^1$. In particular, all convex $\mathrm{PSL}(2, R)$ Möbius n-gons are of the form $\Delta \times S^1$. If a $\mathrm{PSL}(2, R)$ Möbius n-gon intersects its axis, then it is equivalent to S^3-$\mathrm{int}(\Delta \times S^1)$ for a hyperbolic polygon Δ. Furthermore, dihedral angles of $\Delta \times S^1$ are the same as inner angles of Δ, and the modules of the faces are the same as the hyperbolic length of the edges of Δ.

2.2. Lemma. *Given a Möbius n-gon P, let* $\mathrm{Aut}(P) = \{g \in \mathrm{Mob}(S^3) | g(P) = P\}$. *Then the following hold*:

(a) *P is of type* $\mathrm{PSL}(2, R)$ *if and only if* $\mathrm{Aut}(P)$ *is infinite.*

(b) *P is of type* $\mathrm{PSL}(2, C)$ *if and only if there exists* $g \in \mathrm{Mob}(S^3) - \{\mathrm{id}\}$ *so that g leaves each face invariant.*

(c) *If P is not of type* $\mathrm{PSL}(2, C)$, *then* $\mathrm{Aut}(P)$ *is a subgroup of a dihedral group. Furthermore, if* $\mathrm{Aut}(P)$ *acts transitively on the set of all faces of P, then there is a cyclic subgroup of order n in* $\mathrm{Aut}(P)$ *which acts transitively on the set of all faces of P.*

(d) *There are no Möbius bi-gons in S^3; all Möbius* 3*-gons are of type* $\mathrm{PSL}(2, R)$ *and all Möbius* 4*-gons are of type* $\mathrm{PSL}(2, C)$.

Proof. The proof is based on the following fact about $\mathrm{Mob}(S^3)$. Namely, if $g \in \mathrm{Mob}(S^3)$ leaves three pairwise disjoint unlinked circles C_1, C_2, C_3 invariant, then g is either an inversion about a 2-sphere or a rotation about a circle, or an orientation-reversing involution with exactly two fixed points. In the first case, all C_i's are orthogonal to $\mathrm{Fix}(g)$, in the second case $(\mathrm{Fix}(g), C_i)$ is a standard pair for each i, and in the third case, we may assume after a Möbius conjugation that g sends x to $-x$ in $\bar{R}^3$. Thus all circles are centered at the origin. We now use this fact to prove the statements.

To prove (a), if P is of type $\mathrm{PSL}(2, R)$, then any rotation about the common axis is in $\mathrm{Aut}(P)$. To show the converse, we first note that there is an element g in $\mathrm{Aut}(P)$ having infinite order. Indeed, since $\mathrm{Aut}(P)$ is a closed Lie subgroup of $\mathrm{SO}(4, 1)$ by definition, it either contains a nontrivial element of infinite order or is discrete. If it is discrete, then exhausting $\mathrm{Aut}(P)$ by finitely generated subgroups and using Selberg's lemma, we conclude that $\mathrm{Aut}(P)$ has to be finite if it contains no element of infinite order. Now $h = g^{n!}$ is of infinite order and leaves each face of P invariant. Since there are no Möbius bi-gons in S^3, h leaves at least three pairwise unlinked circles invariant (namely the edges of P). By the fact above, h must be a rotation about a circle C, and C and each edge of P form a standard pair. Thus P is of type $\mathrm{PSL}(2, R)$.

To see (b), if P is of type $\mathrm{PSL}(2, C)$, we take g to be the inversion about the 2-sphere orthogonal to all edges (hence to all faces). Conversely, if $g \in \mathrm{Aut}(P) - \{\mathrm{id}\}$ leaves each edge invariant, then by the fact above, g is either a rotation about a circle or an inversion about a 2-sphere, or an orientation-reversing involution with exactly two fixed points, and P is of type $\mathrm{PSL}(2, C)$ in the first two cases. The last case does not occur. Indeed, we may assume after a Möbius transformation that g sends x to $-x$ in $\bar{R}^3$. Then all edges of P are circles whose Euclidean centers are the origin. This implies that all faces of P are planes passing through the origin. This is absurd.

To prove (c), we construct a natural homomorphism $\rho : \mathrm{Aut}(P) \to \mathrm{Dih}_n = \mathrm{Iso}(\text{regular Euclidean } n\text{-gon})$ by simply coding the action of $\mathrm{Aut}(P)$ on the set of faces of P. If P is not of type $\mathrm{PSL}(2, C)$, then $\mathrm{Ker}(\rho)$ consists of the identity element by (b). Thus the first and second statements follow.

To prove (d), we note first that the set of the spans of n faces of P has no common intersection points. Indeed, each common point must be in the intersection of all edges, which is the empty set. Now if $n = 3$, we apply the fact that any three 2-spheres with no common intersection point is orthogonal to a circle. If $n = 4$, we apply the fact that any four 2-spheres in S^3 with no common intersection point is orthogonal to a fifth 2-sphere (see [9] for a proof). This completes the proof. q.e.d.

The following lemma shows the special feature of convex Möbius Polygons.

2.3. Lemma. *Suppose P is a convex Möbius n-gon with faces F_1, $F_2, \dots F_n$ ordered cyclically. Take a point p in $\mathrm{int}(P)$, and let B_i be the ball in S^3 so that $\partial B_i = \mathrm{sp}(F_i)$ and p is not in B_i. Then $N = \cup_{i=1}^n B_i$ is* *a solid torus in* S^3 *with boundary* ∂P. *In particular,* $P \cap \mathrm{int}(B_i) = \phi$ *for each* i.

Proof. After a Möbius transformation we may assume that p is the infinity. Thus B_i is the 3-ball in R^3 bounded by $\mathrm{sp}(F_i)$. Each F_i has two sides: the concave and convex sides. By the dihedral angle assumption on P, the concave side of F_i is in P. Thus a small neighborhood of $F_i \cup F_{i+1}$ in $B_i \cup B_{i+1}$ is in P^c, the complement of P in $\overline{\mathbb{R}}^3$. Now construct an abstract solid torus N' which is a disjoint union of these B_i so that B_i and B_{i+1} are identified according to their configuration in R^3. To be more precise, a point $x \in B_i$ and a point $y \in B_{i+1}$ are identified if and only if $x = y$ in $B_i \cap B_{i+1}$ in R^3 (the index i is counted mod(n)). There exists a natural immersion $\Phi : N' \to S^3$ by sending each point $x \in N'$ to its real representative. Since $\Phi : \partial N' \to S^3$ is an embedding, Φ is also an embedding. Thus the image of N' under Φ is either P or $\mathrm{cl}(P^c)$. By the choice of the initial point p, the second case must occur, and the result follows. q.e.d.

We call $\mathrm{int}(P^c)$ (again a Möbius n-gon) a *necklace* if P is a convex Möbius n-gon.

An *oriented Möbius n-gon* P is a Möbius n-gon with an orientation in P together with orientations in all edges of P so that these oriented edges represent the same homology class in $H_1(\partial P, Z)$. Each face F of P has the induced orientation from P. We label the faces and their boundary components by F_i and E_i, E_{i-1} so that the induced orientation on E_i from F_i is the same as the given orientation on E_i. Thus $E_i = F_i \cap F_{i+1}$, i mod(n). We also orient each orthogonal arc of F_i so that it starts from E_{i-1} and ends at E_i. Let C_i be the middle circle of F_i. Then the *twist map* τ_{E_i} of E_i is the Möbius transformation $H_{C_i} \circ \ldots \circ H_{C_n} \circ H_{C_1} \circ \ldots \circ H_{C_{i+1}}|_{E_i}$: $E_i \to E_i$. Clearly τ_{E_i} is orientation preserving and is conjugated to τ_{E_j}. For PSL(2, R) Möbius n-gons, the twist map is always the identity map.

The twist map is closely related to a natural oriented foliation on ∂P. Consider the set of all oriented orthogonal arcs in the faces. The joining of these orthogonal arcs gives a foliation on ∂P. The leaves of the foliation are transverse to the edges and the middle circles. Indeed, the twist map is the holonomy map of the oriented foliation at the edge. There is a closed leave in the foliation if and only if τ_{E_i} has some periodic points.

2.4. Torsion of Möbius Polygons. We are mainly interested in those Möbius n-gons P so that (a) the twist map is elliptic and (b) a meridian curve of P intersects an edge at one point. For instance, a convex Möbius

Polygon satisfies the above condition (b) by Lemma 2.3. For these P, we will define the torsion of P as follows.

Suppose m is a meridian of P which intersects each edge in P at one point. We then orient m so that the intersection number of m with the oriented edge is $+1$.

Since τ_{E_i} is elliptic and preserves the orientation of E_i, τ_{E_i} is a rotation, say by an angle $2\pi\alpha$ where $\alpha \in [0, 1)$. Given $x \in E_i$ let $y = \tau_{E_i}(x)$. Let L be the oriented segment in the oriented leaf starting at x and ending at y; and let M be the arc in E_i starting at y and ending at x according to the orientation on E_i. Both L and M have the induced orientations. We call the oriented simple closed curve $L \cup M$ a *characteristic curve* of P. Suppose the intersection number between the oriented meridian m and the characteristic curve $L \cup M$ is k. Then *the torsion* of P is defined to be $k - \alpha$, denoted by $\tau(P)$. It is an integer if and only if the twist map is the identity map and it is a rational number if and only if the twist map is a rational rotation (periodic). For a convex PSL(2, R) Möbius n-gon, its torsion is always zero.

2.5. Lemma. *Given an oriented Möbius n-gon P, let $-P$ be the same Möbius n-gon with opposite orientation on P but the same edge orientation. Then*

$$\tau(-P) = -\tau(P).$$

Indeed, the twist map τ_{E_i} for $-P$ has rotation angle $2\pi(1-\alpha)$ and a characteristic curve for $-P$ is $-(L) \cup -(E_i - M)$. Thus the result follows.

We say that a twist map τ_{E_i} of P is *comparable* to a face F_i containing E_i if τ_{E_i} is the restriction of a Möbius transformation of F_i.

2.6. Lemma (Gluing lemma). *Suppose that P, P' are two oriented convex Möbius Polygons, and that F_1, F_1' are two faces of P, P' respectively so that their modules are the same. Let $\partial F_1 = E_n \cup E_1$ and $\partial F_1' = E_m' \cup E_1'$, and let $h : (F_1, E_n, E_1) \to (F_1', E_m', E_1')$ be an orientation-reversing Möbius transformation preserving the orientations in the edges. Then $Q = P \cup_h P'$ is an oriented Möbius $(n+m-2)$-gon in S^3. Furthermore, the following hold:*

(a) *The twist map of Q at $E_1 (= h^{-1}(E_m'))$ is $h^{-1} \circ \tau_{E_m'} \circ h \circ \tau_{E_1}$; in particular, if P' is a convex* PSL(2, R) *Möbius Polygon, then the twist map of Q is the same as the twist map of P, and $\tau(Q) = \tau(P)$.*

(b) *If τ_{E_1} and $\tau_{E_m'}$ are both comparable with the faces F_1 and F_1', then*

$$\tau(Q) = \tau(P) + \tau(P').$$

Proof. Suppose P has n faces $F_1, F_2, \ldots, F_n$, and P' has m faces $F'_1, \ldots, F'_m$ ordered cyclically according to the orientations. Let B_1 and B'_1 be the balls in the necklaces $\mathrm{cl}(P^c)$ and $\mathrm{cl}(P'^c)$ corresponding to F_1 and F'_1 respectively. Then by Lemma 2.3, P is in the complement of $\mathrm{int}(B_1)$, and P' is in the complement of $\mathrm{int}(B'_1)$. We find a copy of P inside B_1 after the inversion about ∂B_1. By composing with a Möbius transformation, we may assume that $F_1 = F'_1$ (thus $B_1 = B'_1$) and $\mathrm{int}(P)$ and $\mathrm{int}(P')$ lie in the different sides of ∂B_1. Thus the gluing $P \cup_h P'$ can be realized in S^3 and Q is still an oriented Möbius $(n+m-2)$-gon. A meridian curve m_Q of Q is the homological sum of two meridians of P and P' respectively. Thus m_Q intersects each edge of Q at one point.

The first statement (1) follows from the definition.

To see (2), since τ_{E_1} and τ'_{E_1} are comparable to the faces, $h^{-1} \circ \tau_{E'_m} \circ h$ and τ_{E_1} commute, and their composition is again an elliptic transformation of rotation angle $2\pi(\alpha+\beta)$ where $2\pi\alpha$ and $2\pi\beta$ are the rotation angles of τ_{E_1} and $\tau_{E'_m}$ respectively. Take a point $x \in E_1$, let L be the leaf of the oriented natural foliation on ∂P starting at x and ending at $\tau_{E_1}(x) = y$, and let M be the oriented arc of E_1 from y to x. Similarly let L' be the leaf of the natural foliation on $\partial P'$ starting at $h(y)$ and ending at $z = \tau_{E'_m}(h(y))$, and let M' be the oriented arc in E'_m from z to $h(y)$. To find a characteristic curve of Q, we need to distinguish two cases.

Case 1. $\alpha+\beta < 1$. Then $M \cup h^{-1}(M')$ is an oriented embedded arc in E_1 from $h^{-1}(z)$ to x. Thus, the curve $K = L \cup h^{-1}(L') \cup M \cup h^{-1}(M') - (L \cap \mathrm{int}(F_1))$ is a characteristic curve of Q. Since an oriented meridian of Q is the homological sum of oriented meridian curves of P and P', the intersection number adds. Therefore the intersection number between the oriented meridians of Q and K is the sum of the intersection numbers between the meridians of P and P' and the characteristic curves of P and P' respectively. This shows that $\tau(Q) = \tau(P) + \tau(P')$.

Case 2. $\alpha+\beta \geq 1$. Then $M \cup h^{-1}(M')$ goes around E_1 once from $h^{-1}(z)$ to x. The oriented arc $M \cup h^{-1}(M') - E_1$ is used in constructing a characteristic curve of Q. Thus the intersection number is one less than the sum of the intersection numbers between the meridians of P and P' with their characteristic curves. Since the rotation angle of the twist map for Q at E_1 is now $2\pi(\alpha+\beta-1)$, this shows again that $\tau(Q) = \tau(P) + \tau(P')$.

3. Regular Möbius n-gons

A Möbius n-gon P is called regular if $\mathrm{Aut}(P)$ acts transitively on the set of all faces of P. We will be interested in regular Möbius n-gons which are not of type $\mathrm{PSL}(2, C)$. Thus by Lemma 2.2, there exists $\phi \in \mathrm{Aut}(P)$ so that $\phi(F_i) = F_{i+1}$ for all i mod(n), where $F_1, \dots, F_n$ are cyclically ordered faces of P. We may assume after a conjugation that ϕ is an elliptic element in the maximal compact subgroup $O(4)$ of $\mathrm{Mob}(S^3)$. This motivates the following spherical geometric construction of regular Möbius n-gons. See Gromov et al. [4] for a reference.

3.1. Given an integer p and two complex numbers of norm less than one ϵ, ϵ' so that the sum of their norms is 1, let $\Gamma = \Gamma_{\epsilon, p} \subset S^3 = \{(z, w) \in C^2 | |z|^2 + |w|^2 = 1\}$ be given by $\{(\epsilon e^{it}, \epsilon' e^{ipt}) | t \in [0, 2\pi]\}$. Γ is unknotted in S^3 and oriented according to the natural order of t. Then the linking number $\mathrm{lk}(\Gamma_{\epsilon, p}, \Gamma_{\delta, p}) = p$ for $|\delta| \neq |\epsilon|, |\epsilon'|$. For each integer $n > 2$, construct a regular sphere polygon $\gamma = \gamma_{\epsilon, p, n}$ whose vertices are $v_k = (\epsilon \eta^k, \epsilon' \eta^{pk}) \in \Gamma_{\epsilon, p}$ where $\eta = e^{2\pi i/n}$ and $k = 0, 1, \dots, n-1$. Let $\phi : S^3 \to S^3$ be the periodic isometry defined by $\phi(z, w) = (z e^{2i\pi/n}, w e^{2pi\pi/n})$. Then $v_k = \phi^k(v_0)$, and ϕ leaves γ invariant.

The *local torsion* τ of an edge of γ is defined as follows. Assume an orientation is given on S^3, and γ is oriented according to the natural orders of v_k's. Along each edge e of γ, let $N_x(e)$ be the oriented normal plane to e. Now define at each vertex v of γ a distinguished unit normal vector $n_v \in N_v(e_+) \cap N_v(e_-)$ where e_- and e_+ are the edges $e_- \cap e_+ = v$, and e_+ follows e_- in the orientation of γ. We assume that $\langle n_v, e_- \times e_+ \rangle$ is positive. Suppose now that e is an edge of γ with ends v_- and v_+ (v_+ follows v_-). Then the torsion τ of e is the unique angle formed in passing from n_{v_-} to n_{v_+} in the normal plane to e. The following formula was obtained in [4] in the case that ϵ and ϵ' are real. It still holds for complex ϵ and ϵ'.

3.2. Lemma. *The local torsion τ can be calculated as*

$$\cos \tau = \frac{|\epsilon'|^2 \sin^2(2p\pi/n) \cos(2\pi/n) + |\epsilon|^2 \sin^2(2\pi/n) \cos(2p\pi/n)}{|\epsilon'|^2 \sin^2(2p\pi/n) + |\epsilon|^2 \sin^2(2\pi/n)}.$$

We now construct a necklace $N = N_{\epsilon, n, p} = B_1 \cup \dots \cup B_n$ based on γ by putting a ball of spherical radius r centered at v_k. The condition on n, r, and γ to guarantee that $P = \mathrm{int}(N)^c$ is a Möbius n-gon is complicated. However, there is a very easy sufficient condition: (1) each B_i intersects B_{i+1} nontangentially i mod(n), and (2) $B_i \cap B_j = \phi$ if the indexes i and j are not adjacent mod(n). To translate these in terms of distances, we use $d_E(x, y)$ to denote the Euclidean distance between $x, y \in C^2$; and use $d_S(x, y)$ to denote the spherical distance between $x, y \in S^3$. Clearly $d_E(x, y) = 2\sin(d_S(x, y)/2)$ if $x, y \in S^3$. Then the above two conditions become:

(C_1) $\min_{k=1,2,\dots,n-1} d_E(v_0, v_k)$ is $d_E(v_0, v_1)$ and $d_E(v_0, v_{n-1})$;

(C_2) Suppose $\min_{k=2,3,\dots,n-2} d_E(v_0, v_k) = d_E(v_0, v_m)$. Then the spherical radius r of ball B_i satisfies

$$d_S(v_0, v_1)/2 < r < d_S(v_0, v_m)/2.$$

One calculates that

$$\begin{aligned} d_E^2(v_0, v_k) &= |\epsilon|^2|1-\eta^k|^2 + |\epsilon'|^2|1-\eta^{pk}|^2 \\ &= 4|\epsilon|^2\sin^2(k\pi/n) + 4|\epsilon'|^2\sin^2(kp\pi/n). \end{aligned}$$

For fixed ϵ, p, n, P is parametrized by the radius r of B_i. Larger radius corresponds to larger inner angle. Under conditions (C_1), (C_2) the largest inner angle of P corresponding to $r = d_S(v_0, v_m)/2$ is given by $\pi - 2\sin^{-1}(d_E(v_0, v_1)/d_E(v_0, v_m))$. Thus, under (C_1), (C_2), the inner angle of P takes all values in $(0, \pi - 2\sin^{-1}(d_E(v_0, v_1)/d_E(v_0, v_m)))$.

The local torsion τ of γ is closely related to the twist map of the regular Möbius n-gon P. We orient P and edges of P as follows. P has the induced orientation from S^3, and edges are oriented so that the linking number between an edge and the oriented core curve γ of the necklace $\mathrm{int}(P)^c$ is 1.

3.3. Lemma. *Let P be an oriented regular convex Möbius n-gon invariant under the symmetry $\phi(z, w) = (ze^{2\pi i/n}, we^{2p\pi i/n})$.*

(a) *Then ϕ action on P is conjugated to $\mu(x, t) = (xe^{2\pi i/n}, te^{2p\pi i/n})$ action on the solid torus $\{x \in C||x| \le 1\} \times \{t \in C||t| = 1\}$.*

(b) *Suppose F_1 is a face of P with two boundary components E_0 and $E_1 = \phi(E_0)$ and middle circle C_1. Then, $H_{C_1} \circ \phi$ is an elliptic transformation of S^3 leaving E_0 invariant so that $H_{C_1} \circ \phi$ rotates E_0 by an angle τ and rotates oriented normal planes to E_0 by an angle $2\pi - \alpha$, where α is the dihedral angle of P at its edges. In particular, the twist map of an edge of P is a rotation by the angle τn.*

Proof. (a) By Lemma 2.2, P is a solid torus so that the edges of P are the longitude curves of P. Since ϕ acts transitively on the set of all edges, ϕ is conjugated to $\mu(x, t) = (xe^{2\pi i/n}, te^{2k\pi i/n})$ action on the solid torus $\{x \in C||x| \le 1\} \times \{t \in C||t| = 1\}$ for some integer k. We claim that

$k \equiv p \mod(n)$. Indeed, ϕ has two invariant circles $C_z = \{(z, 0)||z| = 1\}$ and $C_w = \{(0, w)||w| = 1\}$ in S^3 so that the rotation angle of ϕ on C_w is $2p\pi/n$. One calculates easily that the linking numbers $\mathrm{lk}(C_z, \gamma) = p$, and $\mathrm{lk}(C_w, \gamma) = 1$. Thus, if L is an oriented invariant core curve of P, then L is an unknot in S^3 so that $\mathrm{lk}(\gamma, L) = 1$. This implies that L and C_w are isotopic in $S^3 - \gamma$, and the rotation angles of ϕ on L and γ are the same. Thus the result follows. It seems highly possible that if condition (C_1) holds, then C_w is always in the interior of P.

(b) Let L be the great 2-sphere in S^3 intersecting F_1 orthogonally at the middle circle C_1, and let R be the spherical reflection about L. Then, $H_{C_1} = R \circ \mathrm{Inv}$ where Inv is the inversion about $\mathrm{sp}(F_1)$. In particular, $H_{C_1}|_{\mathrm{sp}(F_1)} = R|_{\mathrm{sp}(F_1)}$. Thus, $H_{C_1} \circ \phi| : E_0 \to E_0$ is an isometry with respect to the induced metric on E_0. To find the rotation angle, we mark the normal vector at v_i by n_i ($= n_{v_i}$). Let m_i be the (spherical) parallel translation of n_i from v_i to the middle point of the edge $v_i v_{i+1}$. Then $\phi(m_i) = m_{i+1}$ for $i \mod(n)$. $R(m_0)$ is obtained from n_0 by parallel translating it along the edge $v_0 v_1$ to its middle point. Thus $R\phi(m_{-1})$ ($= Rm_0$) and m_{-1} form an angle τ counted positively from m_{-1} to $R(m_{-1})$ in the oriented normal bundle to the edge. This shows that the rotation angle of $H_{C_1} \circ \phi$ on E_0 is τ. On the other hand, since $H_{C_1} \circ \phi(\mathrm{sp}(F_0)) = \mathrm{sp}(F_1)$, the rotation angle of $H_{C_1} \circ \phi$ in the oriented normal bundle of E_0 is then $2\pi - \alpha$ where α is the dihedral angle of P at E_0. q.e.d.

We summarize the observations about constructing a regular convex Möbius n-gon as follows.

3.4. Proposition. *Given positive integers p, n and $\tau \in [0, \pi)$, let*

$$\lambda = |\epsilon'|^2/|\epsilon|^2 = \frac{\sin^2(2\pi/n)(\cos\tau - \cos 2p\pi/n)}{\sin^2(2\pi p/n)(\cos 2\pi/n - \cos\tau)},$$

and let $a_k = \sin^2(k\pi/n) + \lambda \sin^2(pk\pi/n)$. Suppose $a_i > a_1$ for all $i = 2, 3, \ldots, [n/2] + 1$ and $a_m = \min_{1<i<[n/2]+1} a_i$. Then there is a regular convex Möbius n-gon P such that

(1) *P is invariant under $\phi(z, w) = (ze^{2\pi i/n}, we^{2p\pi i/n})$,*

(2) *the dihedral angle of P is any given number in $(0, \pi - 2\sin^{-1}\sqrt{\frac{a_1}{a_m}})$,*

(3) *the torsion of P is $p - n\tau/2\pi$.*

Proof. Since $a_i = a_{i-n}$, $a_i > a_1$ for all $i = 2, 3, \ldots, [n/2]+1$ is equivalent to condition (C_1). Thus, (1) and (2) follow. To see (3), we compute the intersection number in two steps. Take $x = x_0 \in E_0$ and let $x_i = \phi^i(x_0)$. Joint x_i to x_{i+1} by a circular arc c_i in F_{i+1}, and let $C' = \cup_{i=1}^n c_i$. C' is oriented so that x_1 follows x_0 in c_0. Then C' is ϕ-invariant, and the intersection number between the oriented meridian of P and C' is p. Indeed, C' is isotopic to a curve of the form $\Gamma_{\delta,p}$ for some δ ($0 < |\delta| < 1$) in P. Since the linking number $\mathrm{lk}(\Gamma_{\delta,p}, \Gamma_{\epsilon,p}) = p$, the assertion follows. Now let C be a characteristic curve on ∂P. Then in the homology group $H_1(P, Z)$, $[C] = [C'] - [n\tau/2\pi][E_0]$ due to the local twisting of degree τ in each face. This implies that the intersection number between the oriented meridian and the characteristic curve is $p - [n\tau/2\pi]$. Since the rotation angle of a twist map of P is $2\pi(n\tau/2\pi - [n\tau/2\pi])$, thus the torsion of P is $p - n\tau/2\pi$.

3.5. Corollary. *For any real number T, there exists a regular convex Möbius n-gon P with arbitrary small dihedral angle so that the torsion of P is T.*

Proof. We may assume that T is nonnegative since a change of the orientation of P will reverse the sign of the torsion. Take any integer $p > T + 2$, $\tau = 2\pi(p - T)/n$ we claim that for large n, the conditions in Proposition 3.4 hold. First note that

$$\lim_{n\to\infty} \lambda = (2pT - T^2)/p^2((p-T)^2 - 1)$$

is a positive number. The inequality $a_k > a_1$ is equivalent to

$$\sin^2(k\pi/n) - \sin^2(\pi/n) > \lambda(\sin^2 p\pi/n - \sin^2 k\pi p/n)$$

for $k = 2, 3, \ldots, [n/2] + 1$.

The left-hand side of the above inequality is strictly increasing in $k \in [1, n/2]$, and the right-hand side of it is strictly decreasing in $k \in [1, n/2p]$. Thus, the above inequality holds for all $k \in [1, n/2p]$. Now choose n so large that

$$\sin^2([n/2p]\pi/n) - \sin^2 \pi/n > \lambda \sin^2(p\pi/n).$$

Indeed, as n tends to infinity, the left-hand side above tends to the positive number $\sin^2 2\pi/p$ and the right-hand side tends to zero since λ is bounded.

Then for all $k \geq [n/2p]$ and $k \leq n/2$,

$$\begin{aligned}\sin^2(k\pi/n) - \sin^2(\pi/n) &\geq \sin^2([n/2p]\pi/n) - \sin^2 \pi/n \\ &> \lambda \sin^2(p\pi/n) > \lambda(\sin^2(p\pi/n) - \sin^2(k\pi p/n)).\end{aligned}$$

Thus the above inequality holds, and by Proposition 3.4, the dihedral angle of P can be arbitrary small.

4. Proof of the main theorem

Recall an α-cone sphere S^3_α is the quotient of a Euclidean lens of angle α by the rotation about the edge of the lens which identifies the two boundary half-spheres of the lens. Our goal is to prove the following.

4.1. Theorem. *Given any $\alpha \in (0, 2\pi)$, any closed orientable 3-manifold M has a singular conformally flat structure so that each point in M has a neighborhood which is conformal to an open set in S^3_α. Furthermore, if the cone angle is $2\pi/n$, $n \in Z_+$, then the monodromy group is a discrete subgroup of $SO(4, 1)$.*

Since the technical details of the proof are complicated, we will describe below the basic idea of the proof in the case that $\alpha = \pi$.

It is known from the work of Lickorish [7] (see Rolfsen's book [13]) that any closed orientable 3-manifold is obtained by doing 1 or -1 Dehn surgeries on the components of a closed pure braid in S^3.

Our goal is to realize this surgery construction in Möbius cone geometry. We first cover each component of the braid by small balls so that their union forms a necklace with small exterior angles. These necklaces are all disjoint and form a regular neighborhood of the braid. The edges of the Möbius Polygon are the meridian curves of the braids. We will start a sequence of modification in each necklace to achieve the Dehn surgery. Suppose N is such a necklace with cyclically ordered faces $F_1, F_2, \ldots, F_n$, and suppose H_i is the half-turn about the middle circle of F_i for each i. We then introduce an identification on ∂N by these half-turns, i.e., each side F_i is self-identified by H_i. The quotient space will be homeomorphic to a ± 1-Dehn surgeries on the component of the braid if we choose the necklace suitably. To see this, take the edge $E = F_1 \cap F_n$ of the Möbius n-gon $\mathrm{int}(N)^c$. The Möbius transformation $\phi = H_n \circ H_{n-1} \circ \ldots \circ H_1$ sends E to itself. This shows that points $x, \phi(x), \phi^2(x), \ldots$ in E are all identified in the quotient. Thus the quotient is a manifold if and only if ϕ is periodic in, i.e., $\phi^k = \mathrm{id}$ in E for some integer k. We require that $k = 1$.

(1) $\phi = \mathrm{id}$ in E.

Assume that (1) holds. Then the quotient is homeomorphic to an integer coefficient Dehn surgery on ∂N. To see this, consider a characteristic curve C in ∂N. Since the twist map is the identity map, C is invariant under the identification. The quotient of C is a wedge of closed intervals and is contractible. The quotient of N is homeomorphic to a Dehn surgery on ∂N killing C. The Dehn surgery coefficient is the intersection number between a meridian m of $P = \text{int}(N)^c$ and C which in turn is the torsion of the Möbius n-gon P. Thus we require that

(2) the torsion of $\text{int}(N)^c$ is $+1$ or -1 depending on the Dehn surgery coefficient.

Lastly, since all edges of N are identified to one edge in the quotient, we also need the following:

(3) The sum of the exterior angles of N is 2π, i.e., the sum of the interior angles of P is 2π.

Now the modification of N goes as follows. Start with N having small inner angles and torsion T. Use Corollary 3.5 to construct a regular Möbius n'-gon P' of torsion $-T \pm 1$ and small inner angles. We choose P' so that the module of a face of P' is the same as the module of a face of P. Glue P' to P along the face to obtain a new Möbius Polygon Q with torsion ± 1 using Lemma 2.6. Finally we attach a $\text{PSL}(2, R)$ Möbius Polygon to a face of Q to make the sum of the inner angles to be 2π. Thus, Q satisfies (1), (2), and (3).

To show the theorem for arbitrary angle α, we replace each Möbius annulus F_i which is a face of N by a union of two Möbius annuli which intersect along one boundary circle at an angle α.

4.2. Spherical polygons. Recall that S^3 is the unit sphere in C^2 with the standard induced metric.

4.3. Lemma. *Given any knot K and any neighborhood U of K in S^3, there is $\delta > 0$ so that for all $\epsilon \in (0, \delta)$, there exists a spherical polygon L_ϵ in U so that the following hold*:

(1) *L_ϵ is isotopic to K in U*;

(2) *the length of each edge of L_ϵ is ϵ*,

(3) *the exterior angle of L_ϵ at each vertex is at most $C\sqrt{\epsilon}$ for some constant C depending on K*,

(4) *two vertices of L_ϵ are at most $1.5\,\epsilon$ apart if and only if they are adjacent.*

Proof. We may assume that K is C^∞ smooth and contains a small geodesic segment K' of length δ_0. By a standard approximation argument, there exists $\delta_1 > 0$ so that for all $\epsilon \in (0, \delta_1)$, if A_ϵ is a spherical polygon satisfies (a) each vertex of A_ϵ is in K and (b) the length of each edge of A_ϵ is in $(\epsilon/2, 2\epsilon)$, then A_ϵ is isotopic to K in U, and the exterior angle of A_ϵ is at most ϵ. Furthermore, for each x in K, the sphere of radius ϵ centered at x intersects K (and A_ϵ) at two points. To construct L_ϵ, we fix an orientation on K and take

$\delta = \min(\delta_1, \delta_0/10, \pi/100)$, and let $K' = [v_-, v_+]$ where v_+ follows v_- in the orientation of K. In general, if $x, y \in S^3$ so that their spherical distance is less than π, then we use $[x, y]$ to denote the oriented geodesic segment from x to y. Now for $\epsilon \in (0, \delta)$, take $p_1 \in [v_-, v_+]$ so that $d_S(p_1, v_+) = \epsilon$. Inductively, suppose p_i is chosen in K. Then p_{i+1} is the point in K following p_i so that $d_S(p_i, p_{i+1}) = \epsilon$. Let p_m be the first point in $p_1, p_2, \ldots$ so that $p_m \in [v_-, p_1]$ and $d_S(p_m, v_-) \in (0, \epsilon]$. Now the length l of $[p_m, p_1]$ satisfies $\delta_0 - 2\epsilon \le l \le \delta_0 - \epsilon$. We divide $[p_m, p_1]$ into $[l/2\epsilon]+1$ equal segments, say by the points $q_1, \ldots, q_{[l/2\epsilon]}$. On each small segment $[q_i, q_{i+1}]$, construct a spherical triangle $\Delta q_i q_{i+1} r_i$ of side lengths ϵ, ϵ, $l/([l/2\epsilon]+1)$. Since $2\epsilon/(1+2\epsilon/l) < l/([l/2\epsilon]+1]) \le 2\epsilon$, an easy calculation shows that the inner angles of triangle $\Delta q_i q_{i+1} r_i$ at q_i and q_{i+1} and the exterior angle at r_i are at most $C\sqrt{\epsilon}$ for small ϵ. We take the spherical polygon L_ϵ to be the one with vertices $p_1, \ldots, p_m, q_1, r_1, q_2, \ldots, q_{[l/2\epsilon]}$. By the construction, L_ϵ satisfies all the conditions. q.e.d.

4.4. Spherical regular necklaces. Suppose now that $L = L_\epsilon$ is a spherical n-gon so that each edge has length ϵ, and exterior angle at each vertex is less than $C\sqrt{\epsilon}$. By choosing ϵ very small, we may assume that the exterior angle of L is very small. We label the vertices of L_ϵ to be $v_1, \ldots, v_n$ cyclically. Construct a spherical necklace $N = N_{\epsilon, r, n}$ by putting spherical ball B_i of radius $r > \epsilon/2$ centered at v_i. The existence of such a necklace is guaranteed by Lemma 4.3 (4). We call N and $P = P_{\epsilon, r, n} = \mathrm{int}(N)^c$ Möbius Polygons *based on* the spherical polygon L. We also call r the *radius* of the necklace N. Let $F_i = P \cap \partial B_i$ be the ith face of P, $E_i = F_i \cap F_{i-1}$ be the ith edge of P, and C_i be the middle circle of F_i.

4.5. Lemma. *The twist map $\tau_{E_i} : E_i \to E_i$ is an isometry with respect to the induced metric on E_i. Furthermore, the torsion of $P = P_{\epsilon, r, n}$ is independent of the radius r.*

We also call τ the *torsion* of the spherical equal-sided polygon L.

Proof. Let L_i be the great 2-sphere which bisects the angle $\angle v_{i-1} v_i v_{i+1}$ at v_i, and let R_i be the spherical reflection about L_i. Since L has equal edge lengths, L_i intersects ∂B_i orthogonally at the middle circle C_i of F_i. In particular, $H_{C_i} = R_i \circ \mathrm{Inv}_i = \mathrm{Inv}_i \circ R_i$ where Inv_i is the inversion about ∂B_i. Thus $H_{C_i}|_{E_{i-1}} : E_{i-1} \to E_i$ is the same as $R_i|_{E_{i-1}}$ (which preserves the natural orientations). This implies $\tau_{E_i} = R_i \circ R_{i-1} \circ \ldots \circ R_n \circ R_{n-1} \circ \ldots R_{i+1}|_{E_i}$ is an isometry of $E_i \to E_i$ with respect to the induced metric. The rotation angle of τ_{E_i} is independent of the radius r since these R_i's are independent of the radius. On the other hand the torsion

$\tau(r)$ of $P_{\epsilon,r,n}$ is continuous in r (for fixed ϵ, n) and $\tau(r) = -na \bmod(Z)$ where a is the rotation angle of τ_{E_i}. It follows that the torsion of $P_{\epsilon,r,n}$ is independent of the radius r.

4.6. Given an equal-sided spherical n-gon L and a positive integer $k \geq 2$, we may divide each edge of L into k equal parts to obtain an equal-sided spherical kn-gon L'. Suppose $N = N_{\epsilon,r,n}$ and $N' = N'_{\epsilon/k,r',nk}$ are two spherical necklaces of radii r and r' based on L and L' respectively.

4.7. Lemma. (a) *The torsion of* $P = \mathrm{int}(N)^c$ *and of* $P' = \mathrm{int}(N')^c$ *are the same, i.e., the torsions of* L *and* L' *are the same.*

(b) *Suppose* B'_i *is a ball in the necklace* N' *centered at a partition point, and* F'_i *is the corresponding face with an edge* E'_i. *Then the twist map* $\tau_{E'_i}$ *of* E'_i *in* N' *is comparable with* F'_i.

Proof. (a) By Lemma 4.5, the torsion of $N'_{\epsilon/k,r',nk}$ is independent of the radius. We compare the two Möbius Polygons $P = P_{\epsilon,r,n}$ and $P' = P'_{\epsilon/k,r,nk}$. P is obtained from P' by attaching a PSL(2, R) Möbius Polygon (each of them has a common axis $\mathrm{sp}([v_i, v_{i+1}])$) along faces of P. Indeed, suppose $[v_i, v_{i+1}]$ is an edge of L_ϵ, and $[v_i, v_{i+1}]$ is covered by the same radius balls $B_i, B_{i_1}, B_{i_2}, \dots, B_{i_{k-1}}, B_{i+1}$. Then, $\cup_{j=1}^{k-1} B_{i_j} - \mathrm{int}(B_i \cup B_{i+1})$ is a PSL(2, R) Möbius Polygon with axis $\mathrm{sp}([v_i, v_{i+1}])$. These are the attaching Möbius Polygons to P to obtain P'. By Lemma 2.6, the result follows.

(b) Consider three adjacent balls B'_{i-1}, B'_i, B'_{i+1}. They all have the same spherical radii and they are centered at v'_{i-1}, v'_i and v'_{i+1} so that these three points lie on a great circle. This shows that any spherical rotation about $\mathrm{sp}([v_i v_{i+1}])$ leaves the face F'_i invariant. In particular, it rotates E'_i with respect to the induced metric. By the proof of Lemma 4.5, $\tau_{E'_i} : E'_i \to E'_i$ is a spherical rotation, and we obtain the result.

4.8. *Proof of Theorem* 4.1. By the work of Lickorish [7] (see also [13]), M is obtained by doing $+1$ or -1 Dehn surgery on the components of a pure closed braid in S^3. Fix a tubular neighborhood U of the braid in S^3. There are now three cases according to the given angle $\alpha = \pi$, $\alpha < \pi$ or $\alpha > \pi$.

Case 1. $\alpha = \pi$. Since all geometric construction to achieve the Dehn surgery will be within any given regular neighborhood U of the braid, we will simply focus on one component K of the braid. By Lemmas 4.3 and 4.6, we may assume that K is isotopic to a spherical polygon L of equal side length in U. Let τ be the torsion of L. By Corollary 3.4

and Lemma 4.3, we construct a regular spherical polygon Γ so that the torsion of Γ is $-\tau - 1$ or $-\tau + 1$ where $+1$ or -1 depends on the Dehn surgery coefficient. Divide each edge of L into k (k very large and to be determined) equal parts to obtain a new equal-sided spherical polygon L_ϵ of edge length ϵ. Let N_r be the spherical necklace based on L_ϵ of radius r. Choose k very large, so that there exists N_{r_0} satisfying

(*) the sum of the exterior angles of N_{r_0} is at least 2π, and each exterior angle of N_{r_0} is less than any given number, say less than $\pi/4$ in our case (k depends on this number).

The exterior angles of N_r are estimated as follows. Fix two positive numbers $\delta_1 < \delta_2 < 2$, and consider the necklace N_r so that the radius $r \in (\epsilon/\delta_2 k, \epsilon/\delta_1 k)$. Then the exterior angle of N_r at each edge is given by $2\cos^{-1}(\frac{\sin(\epsilon/2k)}{\sin(r)})$ which is at most $2\cos^{-1}(\frac{\sin(\epsilon/2k)}{\sin(\epsilon/\delta_1 k)})$ and is at least $2\cos^{-1}(\frac{\sin(\epsilon/2k)}{\sin(\epsilon/\delta_2 k)})$. Thus, for δ_1 sufficiently near 2, the exterior angle is arbitrary small. On the other hand, the sum of the exterior angles of N_r is $2nk\cos^{-1}(\frac{\sin(\epsilon/2k)}{\sin(\epsilon/\delta_2 k)})$ which tends to infinity as k goes to infinity.

Perform the same subdivision procedure to Γ to obtain a new spherical polygon $\Gamma_{\epsilon'}$ of edge length ϵ'. Let $N'_{r'}$ be the spherical necklace based on $\Gamma'_{\epsilon'}$ of radius r'. Then, there exists a radius r'_0 so that the sum of the exterior angles of $N'_{r'_0}$ is $> 2\pi$ and each exterior angle of $N'_{r'_0} < \pi/4$. Take two faces F_r and $F'_{r'}$ of the Möbius Polygons $\operatorname{int}(N_r)^c$ and $\operatorname{int}(N'_{r'})^c$ so that both faces correspond to balls centered at division points. The module of F_r, $m(F_r)$ ($m(F'_{r'})$ respectively) depends strictly monotonically and continuously on the radius r (and r' respectively). Furthermore, the module $m(F_r)$ tends to ∞ as $r \to \epsilon/2$. We may assume without loss of generality that $m(F_{r_0}) \geq m(F'_{r'_0})$. Then there exists a strictly monotonic continuous function $\phi(r')$ of r' sending the open interval $(0, r'_0)$ to $(0, r_0)$ so that $m(F_{\phi(r')}) = m(F'_{r'})$ for all r'. We now glue the Möbius polygon $P_{\phi(r')} = \operatorname{int}(N_{\phi(r')})^c$ to $P'_{r'} = \operatorname{int}(N'_{r'})^c$ along the face $F_{\phi(r')}$ and $F'_{r'}$. By Lemma 4.7, the twist maps of $P_{\phi(r')}$ (or $P'_{r'}$) at the edges of the face $F_{\phi(r')}$ (or $F_{r'}$) are compatible with the faces F_r (or $F'_{r'}$). Thus, the glued Möbius polygon $Q_{r'} = P_{\phi(r')} \cup P'_{r'}$ has torsion $+1$ or -1 depending on the Dehn surgery coefficient by Lemma 2.6. $Q_{r'}$ is still convex by the assumption on the dihedral angles on P_r and $P'_{r'}$. The sum of the inner angles of $Q_{r'}$ is arbitrary small as r' tends to $\epsilon'/2$ and is at least 2π for the initial $r' = r'_0$ by the construction. Thus, there exists a radius r' so that the sum of the inner angles of $Q_{r'}$ is 2π. By our previous argument, the identification on $\partial Q_{r'}$ induced by the half-turns about the middle circles in faces gives rise to the $+1$ or -1 Dehn surgery. By the construction, the quotient space has a Möbius cone structure with cone angle π at the set corresponding to the middle circles of faces of $Q_{r'}$. Furthermore by Poincaré polyhedron theorem, the monodromy is a discrete subgroup of SO(4,1).

Case 2. $\alpha < \pi$. Given any positive number l, let $\Delta_{l,\alpha}$ be the unique isosceles hyperbolic triangle so that the base has length l, and the angle at the top vertex is α. Let $\Delta_{l,\alpha} \times S^1$ be the corresponding convex PSL$(2, R)$ Möbius 3-gon. For each $r' \in (\epsilon'/2, r_0')$ and each face F of $Q_{r'}$ constructed above, we attach the Möbius 3-gon $\Delta_{m(F),\alpha} \times S^1$ to $Q_{r'}$ along the face F to obtain $Q_{r',\alpha} = Q_{r'} \cup_{faces} \Delta_{m(F),\alpha} \times S^1$. We claim that $Q_{r',\alpha}$ is still a convex Möbius Polygon in S^3 if we choose $Q_{r'}$ appropriately. Indeed, both the dihedral angles of $Q_{r'}$ and the dihedral angles of $\Delta_{m(F),\alpha} \times S^1$ at the $\{$bottom vertices$\} \times S^1$ can be made arbitrary small if both k and the modules of the faces of the necklaces are large. Thus the attaching procedure can be realized in S^3 as in the proof of Lemma 2.6.

The torsion of $Q_{r',\alpha}$ is still the same as $Q_{r'}$ which is ± 1 by Lemma 2.6(a). The sum of the dihedral angles of $Q_{r',\alpha}$ at the edges of $Q_{r'}$ tends to zero as $r' \to \epsilon'/2$ and is larger than 2π for the initial radius $r' = r_0'$. Thus, we find one radius r' so that the sum is exactly 2π. Now, introduce an identification on $\partial Q_{r',\alpha}$ as follows: for each adjacent face F, F' of $Q_{r',\alpha}$ corresponding to $\Delta_{m(B),\alpha} \times S^1$, let $C = F \cap F'$ be the edge corresponding to $\{$top vertex$\} \times S^1$. The degree α rotation about C, denoted by $H_{C,\alpha}$, identifies F with F'. These $H_{C,\alpha}$ generate the identification on $\partial Q_{r',\alpha}$. By the previous argument, the quotient space is the same as performing $+1$ or -1 Dehn surgery on K.

Again by the construction, the quotient space has a Möbius cone structure with cone angle α. Furthermore, the monodromy group is discrete (due to the convexity of $Q_{r',\alpha}$) if $\alpha = 2\pi/n$ for some positive integer n by Poincaré polyhedral theorem.

Case 3. $\alpha > \pi$. This is the most difficult case since we now need to "dig" out Möbius 3-gon from $Q_{r'}$. To achieve this, we now attach Möbius 3-gons $\Delta_{l,\alpha} \times S^1$ to $Q_{r'}$ inside $Q_{r'}$. The only problem that may occur is that the result may not be a Möbius Polygon in S^3. To guarantee the embeddedness, we first choose the spherical polygons L and Γ with very small exterior angle. We again subdivide both L and Γ enough times so that the condition $(*)$ in Case 1 still holds. Thus we obtain two subdivided spherical equal-sided polygons L_ϵ and $\Gamma_{\epsilon'}$. We also have two spherical necklaces N_r and $N'_{r'}$ of radii r and r' based on them respectively.

To show that attaching $\Delta_{m(F),\beta} \times S^1$ to all faces of N_r (respectively $N'_{r'}$) still produces a Möbius Polygon in S^3, we first consider the very special case that the spherical polygon L is a regular polygon whose vertices are in a circle. Thus, the corresponding spherical necklace is of type $\mathrm{PSL}(2, R)$ whose axis is the given circle. In this case, if we attach $\Delta_{m(F),\beta} \times S^1$ to all its faces, the resulting Möbius Polygon is still embedded in S^3. Indeed, the attaching procedure is actually achieved in the hyperbolic plan H^2 in this case. It is implied by the following lemma.

4.9. Lemma. *Given $0 < \beta < \pi$, let $N = [2\pi/\beta]$. If $n > N$, and P is a regular hyperbolic n-gon whose edge length is l in H^2, then isometrically attaching $\Delta_{l,\beta}$ to each edge of P inside P will produce an embedded hyperbolic $2n$-gon.*

Indeed, suppose O is the center of P, and the vertices of P are v_1, ..., v_n ordered cyclically. The isosceles triangle $\Delta v_i v_{i+1} O$ has top angle $2\pi/n < \beta$. Moving O toward the middle point of $v_i v_{i+1}$ will then produce an isosceles hyperbolic triangle of top angle β based on $v_i v_{i+1}$ inside P. Performing this procedure at every such isosceles triangle $v_i v_{i+1} O$, we obtain the embedded hyperbolic $2n$-gon.

Now the general case of attaching $\Delta_{m(F),\beta} \times S^1$ follows from the special case since we can always choose the spherical polygon L to have extremely small exterior angles, and we can subdivide L enough time so that locally the attaching procedure is almost the same as the special case above. Thus, one obtains an embedded Möbius Polygon after attaching these $\mathrm{PSL}(2, R)$ 3-gons.

Now suppose subdivisions are fine enough for both L_ϵ and $\Gamma_{\epsilon'}$. Then the corresponding spherical necklace N_r and $N'_{r'}$ based on them are enlarged by attaching these Möbius 3-gon $\Delta_{m(F),\beta}$ to all faces except for two faces F_r and $F'_{r'}$ respectively. Both of these faces are centered at division points. By the discussion above, we have two Möbius Polygons $A_r = \mathrm{int}(N_r \cup_{F \neq F_r} \Delta_{m(F),\beta} \times S^1)^c$ and $A'_{r'} = \mathrm{int}(N'_{r'} \cup_{F' \neq F'_{r'}} \Delta_{m(F'),\beta} \times S^1)^c$ in S^3.

The dihedral angles of A_r (respectively $A'_{r'}$) are estimated in the same way as before. Fix a positive number $\delta < 2$, and consider necklaces N_r so that the radius $r \in (\epsilon/\delta k, \epsilon/k)$. Then the dihedral angle of A_r at each edge corresponding to N_r is at least a constant angle depending on ϵ, δ. The sum of the dihedral angles of A_r at the edges corresponding to N_r can be arbitrary large if k is large. Thus, we may assume that there are radii r_0 and r'_0 so that the sum of the dihedral angles of A_{r_0} and $A'_{r'_0}$ at the edges corresponding to N_{r_0} and $N'_{r'_0}$ are both larger than 2π.

We assume without loss of generality that $m(F_{r_0}) > m(F'_{r'_0})$. Thus, for each $r' \leq r'_0$, there exists a unique $\phi(r')$ so that $m(F_{\phi(r')}) = m(F'_{r'})$. We now glue $A'_{r'}$ to $A_{\phi(r')}$ along $F_{\phi(r')}$ to obtain a Möbius Polygon $Q_{r',\alpha}$ in S^3. That $Q_{r',\alpha}$ is still embedded in S^3 follows from the fact that $A'_{r'}$ and $A_{\phi(r')}$ lie in balls bounded by $\mathrm{sp}(F_{r'})$ and $\mathrm{sp}(F_{\phi(r')})$ respectively. Thus the gluing process can always be realized in S^3 (see the proof of Lemma 2.6).

The torsion of $Q_{r'}$ is $+1$ or -1 according to the Dehn surgery coefficient. Again, as $r' = r'_0$, the sum of the inner angles of $Q_{r',\alpha}$ at the edges corresponding to the edges of $N'_{r'}$ and $N_{\phi(r')}$ is $\geq 2\pi$. Thus, we find an intermediate radius r' so that the sum of the inner angle is 2π. Now, introduce an identification on $\partial\, Q_{r',\alpha}$ as in the second case. The quotient is the same as performing a $+1$ or -1 Dehn surgery on the corresponding link. Furthermore, by Poincaré polyhedron theorem, the quotient has a Möbius cone structure of cone angle α.

4.10. Remark. The restriction to Dehn surgery on trivial knots in the proof is not necessary. The proof works on Dehn surgery on any conformally flat 3-manifold, and the Dehn surgery coefficient can be any given rational number, i.e., Dehn surgery can always be realized in Möbius cone geometry. In particular, the result holds for nonorientable closed 3-manifolds as well.

5. A solution of a problem of Kuiper

Our construction of Möbius structures on circle bundles over surface is based on a simple topological identification. In dimension two, if the opposite sides of a planar $2n$-gon are identified by homeomorphisms reversing the induced orientations of the sides, then the quotient is a closed surface of genus $[n/2] - 1$. There are two cycles of vertices if n is odd, and only one cycle of vertices if n is even.

5.1. Proposition. *Suppose P is a regular convex Möbius $2n$-gon invariant under $\phi(z, w) = (ze^{2\pi i/n}, we^{2p\pi i/n})$ so that*

(1) *the inner angle of* P *is* $2\pi/m$ *where* $m = 2n$ *if* n *is even, and* $m = n$ *if* n *is odd*; *and*

(2) *the local torsion of* P *is* $2\pi T/m$ *for some nonnegative integer* T.

Identify the opposite sides of P *by* $H_{C_i} \circ \phi^n$ *where* C_i *is the middle circle of the* ith *face of* P. *Then the quotient is homeomorphic to a circle bundle over surface* Σ_g *of genus* $g = [n/2] - 1$ *and has a Möbius structure with discrete monodromy group isomorphic to* $\pi_1(\Sigma_g)$. *Furthermore, the Euler number of the fibration is the torsion* $p - nT/m$ *of* P.

Proof. Suppose $E_{i_1}, \dots, E_{i_m}$ form a cycle of edges under the identification. Then one calculates easily that the cycle transformation of E_{i_1} is $(H_{C_{i_1}} \circ \phi)^m$. Under conditions (1) and (2), by Lemma 3.3, the cycle transformation is the identity map. Thus, by Poincaré polyhedron theorem, the side pairing generates a discrete group isomorphic to $\pi_1(\Sigma_g)$ where Σ_g denotes the surface of genus g. Furthermore, the quotient space is homeomorphic to a circle bundle over surface of genus $[n/2] - 1$ (see [5], [6], or [8] for more details).

To find the Euler number of the fibration of the quotient over Σ_g, we consider a characteristic curve C in ∂P. Since the twist map of each edge is the identity map, C is invariant under the identification, and intersects each edge transversely at one point. The Euler number of the fibration is the intersection number of meridian curve of P with the characteristic curve. Thus the proposition is proved. q.e.d.

We now apply Proposition 5.1 to show

5.2. Theorem. *There exists a Möbius structure with discrete monodromy group on the circle bundle over surface of genus* 2 *so that the Euler number of the bundle is one.*

Proof. Take a regular convex Möbius 10-gon P, with local torsion $\tau = 2\pi/5$ so that P is invariant under periodic map

$$\phi(z, w) = (ze^{2i\pi/10}, we^{6i\pi/10}).$$

By Proposition 3.7, it suffices to show that among these regular 10-gons, there is one with inner angle $2\pi/5$. We now calculate the range of the dihedral angles. $\lambda = \frac{\sin^2(2\pi/10)(\cos 2\pi/5 - \cos 6\pi/10)}{\sin^2(6\pi/10)(\cos 2\pi/10 - \cos 2\pi/5)} \approx 0.472135954$, $a_k = \sin^2(k\pi/5) + \lambda \sin^2(3k\pi/5)$ are found to be

$$a_1 \approx 0.40458495$$

$$a_2 \approx 0.772542473$$

$$a_3 \approx 0.699593468$$

$$a_4 \approx 1.06762757$$

$$a_5 \approx 1.472135954$$

and $a_k = a_{10-k}$. Thus, the smallest a_k is $a_1 = a_9$ and the next smallest one is $a_3 = a_7$. $\beta = \pi - 2\sin^{-1}\sqrt{a_1/a_3} \approx 81.00141029^\circ > 72^\circ$. By Proposition 3.3, the dihedral angle of these P take all values in $(0, \beta)$. Hence, there is one with inner angle $2\pi/5$, and the theorem follows from Proposition 3.6. q.e.d.

Note that the construction actually exists in H^4. See Gromov et al. [4] or Kuiper [6] for detailed discussion concerning Möbius Polygons in S^3 and their convex hulls in H^4. We have actually produced a complete hyperbolic metric on a nontrivial plane bundle over a surface of genus 2. Since all closed orientable surfaces are covering spaces of Σ_2, the above theorem implies that all plane bundles over Σ_g ($g > 1$) with Euler number g-1 have complete hyperbolic structures.

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