J. DIFFERENTIAL GEOMETRY
41 (1995) 53–157

# EQUIVARIANT IMMERSIONS AND QUILLEN METRICS

JEAN-MICHEL BISMUT

**Abstract**

The purpose of this paper is to construct Quillen metrics on the equivariant determinant of the cohomology of a holomorphic vector bundle with respect to the action of a compact group $G$. We calculate the behaviour of the equivariant Quillen metric by immersions, and thus extend a formula of Bismut-Lebeau to the equivariant case.

Let $i: Y \to X$ be an embedding of compact complex manifolds. Let $\eta$ be a holomorphic vector bundle on $X$, and let

$$(0.1)\qquad (\xi, v): 0 \to \xi_m \underset{v}{\to} \xi_{m-1} \to \cdots \to \xi_0 \to 0$$

be a holomorphic chain complex of vector bundles on $X$, which, together with a restriction map $r: \xi_{0|Y} \to \eta$, provides a resolution of the sheaf $i_*\mathscr{O}_Y(\eta)$.

Let $\lambda(\xi)$, $\lambda(\eta)$ be the complex lines which are the inverses of the determinants of the cohomology of $\xi$, $\eta$, i.e.,

$$(0.2)\qquad \lambda(\xi) = (\det H(X, \xi))^{-1}, \qquad \lambda(\eta) = (\det H(Y, \eta))^{-1}.$$

Let $G$ be a compact Lie group acting holomorphically on $X$ and preserving $Y$, whose action lifts holomorphically to $(\xi, v)$ and $\eta$. Let $\widehat{G}$ be the set of equivalence classes of complex irreducible representations of $G$. Then we have the isotypical splittings

$$(0.3)\qquad \begin{aligned} H(X, \xi) &= \bigoplus_{W \in \widehat{G}} \operatorname{Hom}_G(W, H(X, \xi)) \otimes W, \\ H(Y, \eta) &= \bigoplus_{W \in \widehat{G}} \operatorname{Hom}_G(W, H(Y, \eta)) \otimes W. \end{aligned}$$

Received October 13, 1993.

Set

$$(0.4)\qquad \begin{aligned} \lambda_G(\xi) &= \bigoplus_{W\in\widehat{G}} \det(\mathrm{Hom}_G(W, H(X, \xi)) \otimes W), \\ \lambda_G(\eta) &= \bigoplus_{W\in\widehat{G}} \det(\mathrm{Hom}_G(W, H(Y, \eta)) \otimes W). \end{aligned}$$

An obvious extension of [22] shows that we have a canonical isomorphism of direct sums of complex lines

$$(0.5)\qquad \lambda_G(\eta) \simeq \lambda_G(\xi).$$

Let $\sigma = \bigoplus_{W\in\widehat{G}} \sigma_W \in \lambda_G^{-1}(\eta)\otimes\lambda_G(\xi)$ be the direct sum of nonzero sections, which defines the canonical isomorphism (0.5).

Let $h^{TX}$, $h^{\xi}$, $h^{TY}$, $h^{\eta}$ be $G$-invariant Hermitian metrics on $TX$, $\xi$, $TY$, $\eta$, respectively. By Hodge theory, one can construct corresponding $L_2$ metrics on the lines $\det(\mathrm{Hom}_G(W, H(X, \xi)) \otimes W)$, $\det(\mathrm{Hom}_G(W, H(Y, \eta))\otimes W)$, which we denote $|\ |_{\det(\mathrm{Hom}_G(W, H(X, \xi))\otimes W)}$, $|\ |_{\det(\mathrm{Hom}_G(W, H(Y, \eta))\otimes W)}$.

If $W \in \widehat{G}$, let $\chi(W)$ be the corresponding character. Set

$$(0.6)\qquad \begin{aligned} \log(|\ |^2_{\lambda_G(\xi)}) &= \sum_{W\in\widehat{G}} \log(|\ |^2_{\det(\mathrm{Hom}_G(W, H(X, \xi))\otimes W)})\frac{\chi(W)}{\mathrm{rk}(W)}, \\ \log(|\ |^2_{\lambda_G(\eta)}) &= \sum_{W\in\widehat{G}} \log(|\ |^2_{\det(\mathrm{Hom}_G(W, H(Y, \eta))\otimes W)})\frac{\chi(W)}{\mathrm{rk}(W)}. \end{aligned}$$

By imitating the construction by Quillen of the Quillen metric [27], [11], [13] on $\lambda(\xi)$, $\lambda(\eta)$, one can modify the symbols $\log(|\ |^2_{\lambda^G(\xi)})$, $\log(|\ |^2_{\lambda^G(\eta)})$ into new symbols $\log(\|\ \|^2_{\lambda^G(\xi)})$, $\log(\|\ \|^2_{\lambda^G(\eta)})$, which we call equivariant Quillen metrics on $\lambda^G(\xi)$, $\lambda^G(\eta)$. The modification involves an obvious extension of the Ray-Singer analytic torsion [28] to the equivariant case.

Then the function $g \in G \to \log(\|\sigma\|^2_{\lambda_G^{-1}(\eta)\otimes\lambda_G(\xi)})(g)$ is a central function on $G$. The purpose of this paper is to give a formula for this function in terms of local secondary invariants of the holomorphic Hermitian vector bundles considered above, under natural assumptions on the metrics. This generalizes earlier work by Bismut-Lebeau [15], where the case where $G = \{1\}$ was considered, to the equivariant setting.

Our assumptions are essentially the same as in [15]. Namely we suppose that $h^{TX}$ is Kähler, and that $h^{TY}$ is the restriction of $h^{TX}$ to $TY$. Let $N_{Y/X}$ be the normal bundle to $Y$ in $X$, and let $h^{N_{Y/X}}$ be the metric induced by $h^{TX|_Y}$ on $N_{Y/X}$. Then we assume that the metrics $h^{\xi_0}, \cdots, h^{\xi_m}$

on $\xi_0, \cdots, \xi_m$ verify assumption (A) of [5, Definition 1.5] with respect to $h^{N_{Y/X}}, h^\eta$.

If $g \in G$, set

$$X_g = \{x \in X, gx = x\}, \qquad Y_g = \{y \in Y, gy = y\}. \tag{0.7}$$

Let $i_g$ be the embedding $Y_g \to X_g$.

Let $\mathrm{Td}_g(TX, h^{TX})$ be the Chern-Weil Todd form on $X_g$ associated to the holomorphic Hermitian connection on $(TX, h^{TX})$, which appears in the Lefschetz formulas of Atiyah-Bott [1]. Other Chern-Weil forms will be denoted in a similar way. In particular the form $\mathrm{ch}_g(\xi, h^\xi)$ on $X_g$ is the Chern-Weil representative of the $g$-Chern character form of $\xi$ associated to $h^{\xi_0}, \cdots, h^{\xi_m}$. Also we denote by $\mathrm{Td}_g(TX), \mathrm{ch}_g(\xi) \cdots$ the cohomology classes of $\mathrm{Td}_g(TX, h^{TX}), \mathrm{ch}_g(\xi, h^\xi), \cdots$.

In this paper, by an extension of [14], we first construct a current $T_g(\xi, h^\xi)$ on $X_g$, such that

$$\frac{\bar\partial\partial}{2i\pi} T_g(\xi, h^\xi) = \mathrm{Td}_g^{-1}(N_{Y/X}, h^{N_{Y/X}})\,\mathrm{ch}_g(\eta, h^\eta)\delta_{Y_g} - \mathrm{ch}_g(\xi, h^\xi). \tag{0.8}$$

Let $\zeta(\theta, s)$, $\eta(\theta, s)$ be the real and imaginary parts of the Lerch series, i.e.,

$$\zeta(\theta, s) = \sum_{n\geq 1} \frac{\cos(n\theta)}{n^s}, \qquad \eta(\theta, s) = \sum_{n\geq 1} \frac{\sin(n\theta)}{n^s}. \tag{0.9}$$

Let $R(\theta, x)$ be the power series introduced in [7]

$$\begin{aligned} R(\theta, x) = &\sum_{\substack{n\geq 1\\ n \text{ odd}}} \left(\sum_1^n \frac{1}{j}\zeta(\theta, -n) + \frac{2\partial\zeta}{\partial s}(\theta, -n)\right)\frac{x^n}{n!} \\ &+ \sum_{\substack{n\geq 0\\ n \text{ even}}} i\left(\sum_1^n \frac{1}{j}\eta(\theta, -n) + \frac{2\partial\eta}{\partial s}(\theta, -n)\right)\frac{x^n}{n!}. \end{aligned} \tag{0.10}$$

Let $\zeta(s)$ be the Riemann zeta function. Let $R(x)$ be the Gillet-Soulé power series [20]

$$R(x) = \sum_{\substack{n\geq 1\\ n \text{ odd}}} \left(\sum_1^n \frac{1}{j}\zeta(-n) + \frac{2\partial\zeta}{\partial s}(-n)\right)\frac{x^n}{n!}. \tag{0.11}$$

Clearly

$$R(0, x) = R(x). \tag{0.12}$$

Let $\widetilde{\mathrm{Td}}_g(TY|_{Y_g}, TX|_{Y_g}, h^{TX|_{Y_g}})$ be the Bott-Chern class of forms on $Y_g$ associated to the exact sequence of holomorphic Hermitian vector bundles on $Y_g$, $0 \to TY|_{Y_g} \to TX|_{Y_g} \to N_{Y/X|_{Y_g}} \to 0$, constructed in [11, §1f)], such that

$$\begin{aligned}(0.13)\quad &\frac{\bar{\partial}\partial}{2i\pi}\widetilde{\mathrm{Td}}_g(TY|_{Y_g}, TX|_{Y_g}, h^{TX|_{Y_g}}) \\ &= i_g^* \mathrm{Td}_g(TX, h^{TX}) - \mathrm{Td}_g(TY, h^{TY})\,\mathrm{Td}_g(N_{Y/X}, h^{N_{Y/X}}).\end{aligned}$$

Over $X_g$, $TX|_{X_g}$ splits as a direct sum $TX|_{X_g} = \bigoplus TX|_{X_g}^{\theta}$, where the $\theta \in [0, 2\pi[$ are distinct and locally constant, and $g$ acts on $TX|_{X_g}^{\theta}$ by multiplication by $e^{i\theta}$. Set

$$(0.14)\qquad R_g(TX) = \sum R(\theta, TX|_{X_g}^{\theta}).$$

We use a similar notation for $R_g(TY)$.

The main result of this paper is the following extension of [15, Theorem 0.1].

**Theorem 0.1.** *For $g \in G$, the following identity holds:*

(0.15)

$$\begin{aligned}\log(\|\sigma\|^2_{\lambda_G^{-1}(\eta)\otimes\lambda_G(\xi)})(g) = &-\int_{X_g} \mathrm{Td}_g(TX, h^{TX}) T_g(\xi, h^{\xi}) \\ &+ \int_{Y_g} \mathrm{Td}_g^{-1}(N_{Y/X}, h^{N_{Y/X}})\widetilde{\mathrm{Td}}_g(TY|_{Y_g}, TX|_{Y_g}, h^{TX|_{Y_g}})\,\mathrm{ch}_g(\eta, h^{\eta}) \\ &- \int_{X_g} \mathrm{Td}_g(TX) R_g(TX)\,\mathrm{ch}_g(\xi) + \int_{Y_g} \mathrm{Td}_g(TY) R_g(TY)\,\mathrm{ch}_g(\eta).\end{aligned}$$

In fact [15, Theorem 0.1] is exactly our Theorem 0.1 with $g = 1$. The formula of [15, Theorem 0.1] is an important step in the proof of Gillet and Soulé [21] of the Riemann-Roch Theorem in Arakelov geometry which they had conjectured in [20]. In particular the genus $R(x)$ was obtained by Gillet and Soulé [20] by a difficult calculation (with Zagier) of the Ray-Singer holomorphic torsion of $\mathbf{P}^n$ equipped with the Fubini-Study metric. The genus $R$ reappeared in an analytic construction of characteristic classes in [6]. The calculations of [6] were then a key ingredient to the proof of the final formula of [15].

In [23], Köhler has calculated the equivariant analytic torsion of $\mathbf{P}^n$ associated to an isometry of $\mathbf{P}^n$ having isolated fixed points. In [7] the calculations of [6] were extended to the equivariant case. For isometries with isolated fixed points, the calculation of [23] and [7] fit as well as [20] and [6].

Theorem 0.1 should be considered as a new block in the construction of an Arakelov theory to be in an equivariant context.

Most of our techniques and arguments are taken from [15]. However, there are certain complications, which we now describe.

## 1. Anomaly formulas for equivariant Quillen metrics

In [13], Bismut-Gillet-Soulé have established anomaly for usual Quillen metrics (here for $G = \{1\}$), which calculate the ratio of two such metrics associated to two couples of metrics on $TX$, $\xi$ and $TY$, $\eta$. In §1, we extend this result in an equivariant context. We express the ratio in terms of Bott-Chern classes evaluated on $X_g$ or $Y_g$. Then formula (0.15) is easily seen to be compatible with these anomaly formulas.

## 2. Localization on $X_g$, $Y_g$ and finite propagation speed

In [15], a key point was the study of the supertrace of certain heat kernel evaluated on the diagonal of $X \times X$ as a function of two parameters $u > 0$, $T > 0$. Here at a formal level the diagonal is replaced by the graph of $g$ in $X \times X$.

As for the classical heat equation proofs of the Atiyah-Bott Lefschetz formula [19], [9], [3], [2], this accounts for the localization of certain supertraces on the fixed point set $X_g$. Also in [15], certain supertraces localized on $Y$. Here the presence of $g$ forces the localization on $Y_g$.

In [15], the needed estimates were obtained by using a heavy functional analytic machinery, which was used to prove that certain rescaled kernels exhibit a decay faster than the polynomial decay on the diagonal in the directions normal to $Y$ in $X$. Here, there is the extra complication that not only we have to show that nonfixed points do not contribute to the asymptotics, but also that the rescaled kernels also exhibit the right Gaussian decay in the directions normal to $X_g$ in $X$. Ultimately, the combination of these two arguments explains the localization of the supertraces on $Y_g$.

In [15], finite propagation speed methods were used to prove that the calculation of certain asymptotics was effectively local on $X$, i.e., that one could replace $X$ by a small ball. Here finite propagation speed is also used to study certain heat kernels inside the considered small ball, to obtain the required Gaussian decay.

Otherwise, the general outlook of the proof of Theorem 0.1 is very similar to the proof of [15, Theorem 0.1]. We refer to the introduction of [15] for more details. As explained before, instead of [6], we use [7] to evaluate a mysterious current $\mathbf{B}_g(TY, TX|_Y, h^{TX|_Y})$ on $Y_g$, which is responsible for the appearance of $R_g$ in (0.15), instead of $R$ in [15].

Because many arguments in the proofs are taken from [15], to avoid duplicating the arguments of [15], we tried to refer as much as necessary to [15], including sometimes for notation. However we give as many details as needed, especially in the construction of local coordinate systems and of local trivializations of certain vector bundles, and also for the Gaussian estimates in directions which are normal to $X_g$ in $X$.

The organization of the paper, and even the organization of most of the sections are deliberately related to [15].

In §1, we give a few algebraic preliminaries. In §2, we construct the equivariant Quillen metrics, and prove the corresponding anomaly formulas. §3 describes the geometric setting of the $G$-equivariant immersion problem. Let $\tilde{\lambda}_G(\xi)$ be the equivariant determinant of the hypercohomology of $\xi$. Then $\lambda_G(\xi)$ and $\tilde{\lambda}_G(\xi)$ are canonically isomorphic. In §4, we extend a result of [15, §2], by comparing the Quillen metrics on $\lambda_G(\xi)$ and $\tilde{\lambda}_G(\xi)$. §5 contains a construction of a closed form $\beta$ on $\mathbf{R}_+^* \times \mathbf{R}_+^*$ and a contour $\Gamma$ by extending [15, §3]. As in [15], Theorem 0.1 will be obtained from the equality $\int_\Gamma \beta = 0$ by taking the boundary of $\Gamma$ to infinity.

In §6, by extending [14], we construct the Bott-Chern current $T_g(\xi, h^\xi)$. §7 summarizes the results of [7].

In §8, we state seven intermediary results, the proofs of six of which are deferred to §§9–13. We take advantage of the formal similarity of our intermediary results to corresponding results in [15, §6] to adapt formally the discussion of [15, §6] in our context, and we prove Theorem 0.1.

§§9–13 are devoted to the proofs of six of the intermediary results which were alluded to before. These sections are an obvious extension of [15, §§9–13] to the case of a nontrivial group $G$, but we still use the results of [15] very much to establish our own results.

As in [15], we use the superconnection formalism of Quillen [26]. In particular $\mathrm{Tr}_s$ is our notation for the supertrace, and $[A, B]$ denotes the supercommutator of $A$ and $B$.

The results contained in this paper were announced in [10].

## I. Algebraic preliminaries

Let

$$(V, \partial) : 0 \to V^0 \underset{\partial}{\to} V^1 \to \dots \underset{\partial}{\to} V^n \to 0 \tag{1.1}$$

be a chain complex of finite dimensional complex vector spaces. Here $V = \bigoplus_{i=0}^{n} V^i$.

Let $h^{V^0}, \cdots, h^{V^n}$ be Hermitian metrics on $V^0, \cdots, V^n$ respectively. We equip $V$ with the metric $h^V = \bigoplus_{i=0}^{n} h^{V^i}$.

Let $G$ be a compact group. Let $\rho : G \to \mathrm{End}(V)$ be a representation of $G$, with values in chain homomorphisms of $V$, which preserve the metric $h^V$. In particular, if $g \in G$, then $\rho(g)$ preserves the $V^i$'s.

Let $\widehat{G}$ be the set of equivalence classes of complex irreducible representations of $G$. An element of $\widehat{G}$ is specified by a complex finite dimensional vector space $W$ together with an irreducible representation $\rho_W : G \to \mathrm{End}(W)$.

For $W \in \widehat{G}$, set

$$V_W^i = \mathrm{Hom}_G(W, V^i) \otimes W, \qquad V_W = \mathrm{Hom}_G(W, V) \otimes W. \tag{1.2}$$

Then $V_W = \bigoplus_{i=0}^{n} V_W^i$. Let $\partial_W$ be the map induced by $\partial$ on $V_W$. Then

$$(V_W, \partial_W) : 0 \to V_W^0 \underset{\partial_W}{\to} V_W^1 \to \dots \underset{\partial_W}{\to} V_W^n \to 0 \tag{1.3}$$

is a chain complex. Finally we have the isotypical splitting

$$(V, \partial) = \bigoplus_{W \in \widehat{G}} (V_W, \partial_W), \tag{1.4}$$

and the decomposition (1.4) is orthogonal.

If $E$ is a complex finite dimensional representation space for $G$, let $\chi(E)$ be the character of the representation. Put

$$\begin{aligned} \chi(V) &= \sum_{i=0}^{n} (-1)^i \chi(V^i), \\ \mathrm{e}(V) &= \sum_{i=0}^{n} (-1)^i \dim V^i, \\ \mathrm{e}(V_W) &= \sum_{i=0}^{n} (-1)^i \dim(V_W^i). \end{aligned} \tag{1.5}$$

By (1.4), we get

$$\chi(V) = \sum_{W \in \widehat{G}} \mathrm{e}(V_W) \frac{\chi(W)}{\mathrm{rk}(W)}. \tag{1.6}$$

If $\lambda$ is a complex line, let $\lambda^{-1}$ be the dual line. If $E$ is a finite dimensional complex vector space, set

$$\det E = \Lambda^{\max}(E). \tag{1.7}$$

Put

$$(1.8)\qquad \begin{aligned} \det V &= \bigotimes_{i=0}^{n} (\det V^i)^{(-1)^i}, \\ \det V_W &= \bigotimes_{i=0}^{n} (\det V^i_W)^{(-1)^i}. \end{aligned}$$

By (1.4), we obtain

$$(1.9)\qquad \det V = \bigotimes_{W \in \widehat{G}} \det V_W.$$

For $0 \leq i \leq n$, $V^i_W$ is a vector subspace of $V^i$. Let $h^{V^i_W}$ be the induced metric on $V^i_W$. Let $\| \ \|_{\det V^i_W}$ be the metric on $\det V^i_W$ induced by $h^{V^i_W}$, and let $\| \ \|_{(\det V^i_W)^{-1}}$ be the dual metric on $(\det V^i_W)^{-1}$. Let $\| \ \|_{\det V_W}$ be the obvious tensor product metric on $\det V_W$. Similarly let $\| \ \|_{\det V}$ be the metric on $\det V$ induced by $h^V$. Then (1.9) is an isometry of line bundles.

Put

$$(1.10)\qquad \det(V, G) = \bigoplus_{W \in \widehat{G}} \det V_W.$$

**Definition 1.1.** We introduce the formal symbol

$$(1.11)\qquad \log(\| \ \|^2_{\det(V, G)}) = \sum_{W \in \widehat{G}} \log(\| \ \|^2_{\det V_W}) \frac{\chi(W)}{\operatorname{rk}(W)}.$$

For $W \in \widehat{G}$, let $\sigma_W \in \det V_W$, $\sigma_W \neq 0$. Set $\sigma = \bigoplus_{W \in \widehat{G}} \sigma_W \in \det(V, G)$. Then by definition,

$$(1.12)\qquad \log(\|\sigma\|^2_{\det(V, G)}) = \sum_{W \in \widehat{G}} \log(\|\sigma_W\|^2_{\det V_W}) \frac{\chi(W)}{\operatorname{rk}(W)}.$$

Tautologically, (1.12) is an identity of characters on $G$. In particular

$$(1.13)\qquad \log(\|\sigma\|^2_{\det(V, G)})(1) = \sum_{W \in \widehat{G}} \log(\|\sigma_W\|^2_{\det V_W}).$$

In fact (1.13) just implies that

$$(1.14)\qquad \log(\| \ \|^2_{\det(V, G)})(1) = \log(\| \ \|^2_{\det V}).$$

Of course, using the orthogonality of the $\chi_W$'s, knowing the formal symbol $\log(\| \ \|^2_{\det(V, G)})$ is equivalent to knowing the metrics $\| \ \|_{\det V_W}$.

Clearly

$$H(V_W, \partial_W) = \mathrm{Hom}_G(W, H(V, \partial)) \otimes W, \tag{1.15}$$
$$H(V, \partial) = \bigoplus_{W \in \widehat{G}} H(V_W, \partial_W).$$

For $W \in \widehat{G}$, we define $\det H(W, \partial_W)$ as in (1.8). Set

$$\det(H(V, \partial), G) = \bigoplus_{W \in \widehat{G}} \det H(V_W, \partial_W). \tag{1.16}$$

By [22], [11, §1a)], for $W \in \widehat{G}$, we have the canonical isomorphism of complex lines

$$\det V_W \simeq \det H(V_W, \partial_W). \tag{1.17}$$

From (1.17), we get

$$\det(V, G) \simeq \det(H(V, \partial), G). \tag{1.18}$$

Let $\| \ \|_{\det H(V_W, \partial_W)}$ be the metric on $\det H(V_W, \partial_W)$ corresponding to $\| \ \|_{\det V_W}$ via the canonical isomorphism (1.17).

**Definition 1.2.** We introduce the formal symbol

$$\log(\| \ \|^2_{\det(H(V, \partial), G)}) = \sum_{W \in \widehat{G}} \log(\| \ \|^2_{\det H(V_W, \partial_W)}) \frac{\chi(W)}{\mathrm{rk}(W)}. \tag{1.19}$$

Tautologically, under the identification (1.18),

$$\log(\| \ \|^2_{\det(V, G)}) = \log(\| \ \|^2_{\det H(V, \partial), G)}). \tag{1.20}$$

By an abuse of notation, we will call the formal symbol $\| \ \|_{\det(V, G)}$ a metric on $\det(V, G)$. In effect, it is a direct sum of metrics on $\det(V, G) = \bigoplus_{W \in \widehat{G}} \det V_W$.

Let $\partial^*$, $\partial_W^*$ be the adjoints of $\partial$, $\partial_W$. Put

$$D = \partial + \partial^*, \qquad D_W = \partial_W + \partial_W^*. \tag{1.21}$$

Under the identification (1.4), we have

$$D = \bigoplus_{W \in \widehat{G}} D_W.$$

Set

$$K = \ker D, \qquad K_W = \ker D_W. \tag{1.22}$$

Then $K$, $K_W$ inherit metrics $h^K$, $h^{K_W}$ from the metrics $h^V$, $h^{V_W}$. By Hodge theory,

$$K \simeq H(V, \partial), \qquad K_W \simeq H(V_W, \partial_W). \tag{1.23}$$

Let $|\ |_{\det H(V,\partial)}$, $|\ |_{\det H(V_W,\partial_W)}$ be the metrics on $\det H(V, \partial)$, $\det H(V_W, \partial_W)$ induced by the metrics $h^K$, $h^{K_W}$ via the canonical identifications (1.23).

Set

$$\log(|\ |^2_{\det(H(V,\partial),G)}) = \sum_{W\in\widehat{G}} \log(|\ |^2_{\det H(V_W,\partial_W)})\frac{\chi(W)}{\operatorname{rk}(W)}. \tag{1.24}$$

Tautologically, the symbol

$$\log\left(\frac{\|\ \|^2_{\det(H(V,\partial),G)}}{|\ |^2_{\det H(V,\partial),G)}}\right) = \sum_{W\in\widehat{G}} \log\left(\frac{\|\ \|^2_{\det H(V_W,\partial_W)}}{|\ |^2_{\det H(V_W,\partial_W)}}\right)\frac{\chi(W)}{\operatorname{rk}(W)} \tag{1.25}$$

is a character of $G$.

Let $K^\perp$ be the orthogonal space to $K$ in $V$. Let $P$ be the orthogonal projection operator from $V$ on $K$. Set $P^\perp = 1 - P$. Then $D^2$ acts as an invertible operator on $K^\perp$. Let $(D^2)^{-1}$ be the corresponding inverse.

Let $N$ be the number operator of $V$, i.e., $N$ acts on $V^i$ by multiplication by $i$.

In the sequel, if $A \in \operatorname{End}(V)$, $\operatorname{Tr}_s[A]$ denotes the supertrace of $A$ [26].

**Definition 1.3.** For $s \in \mathbb{C}$, $g \in G$, set

$$\theta(s)(g) = -\operatorname{Tr}_s[gN(D^2)^{-s}P^\perp]. \tag{1.26}$$

Then $g \to \theta(s)(g)$ is a character of $G$.

**Theorem 1.4.** *For $g \in G$, the following equality holds*:

$$\log\left(\frac{\|\ \|^2_{\det(H(V,\partial),G)}}{|\ |^2_{\det H(V,\partial),G)}}\right) = \frac{\partial\theta}{\partial s}(0)(g). \tag{1.27}$$

*Proof.* Clearly

$$\theta(s)(g) = -\sum_{W\in\widehat{G}} \operatorname{Tr}_s[N(D^2_W)^{-s}P^\perp_W]\frac{\chi(W)}{\operatorname{rk}(W)}(g). \tag{1.28}$$

Also when $G$ acts trivially on $V$, (1.27) is the equality of [11, Proposition 1.5]. Using (1.25), (1.28), we get (1.27). q.e.d.

Let $h'^{V^0}, \cdots, h'^{V^n}$ be other $G$-invariant metrics on $V^0, \cdots, V^n$ respectively, and let $h'^V = \bigoplus_{i=0}^n h'^{V^i}$ be the corresponding metric on $V$. Let $\|\ \|'_{\det(V,G)}$ be the associated metric on $\det(V, G)$.

**Proposition 1.5.** *For any* $g \in G$,

$$\log\left(\frac{\| \ \|'^2_{\det(H(V,\partial),G)}}{\| \ \|^2_{\det(H(V,\partial),G)}}\right)(g) = \mathrm{Tr}_s\left[g \log\left(\frac{h'^V}{h^V}\right)\right]. \tag{1.29}$$

*Proof.* For $g = 1$, (1.29) is obvious. For a general $g \in G$, (1.29) follows by summation.

## II. Equivariant Quillen metrics and their anomaly formulas

The purpose of this section is to construct the Quillen metrics on the equivariant determinant of the cohomology of a holomorphic vector bundle and to establish corresponding anomaly formulas. Thus we extend the anomaly formulas of [11, Theorem 0.2] and [13, Theorem 1.23] in an equivariant setting.

This section is organized as follows. In part a, we construct the equivariant Quillen metrics by a straightforward extension of [27], [13]. In part b, we state our anomaly formula, the remainder of the section being devoted to the proof of this formula. In part c, we establish a simple formula on Clifford algebras. Finally, part d contains our proof of the anomaly formula, along the lines of [4], [9], [13].

a. *Equivariant Quillen metrics.* Let $X$ be a compact complex manifold of complex dimensions $l$. Let $\xi$ be a holomorphic vector bundle on $X$.

Let $G$ be a compact Lie group. We assume that $G$ acts on $X$ by holomorphic diffeomorphisms, and that the action of $G$ lifts to a linear holomorphic action on $\xi$. Then $G$ acts naturally on $H(X, \xi)$.

Let $E = \bigoplus_0^{\dim X} E^i$ be the vector space of smooth sections of

$$\Lambda(T^{*(0,1)}X) \otimes \xi = \bigoplus_0^{\dim X} \Lambda^i(T^{*(0,1)}X) \otimes \xi$$

on $X$. Let $\bar\partial^X$ be the Dolbeault operator acting on $E$. Then

$$H(E, \bar\partial^X) \simeq H(X, \xi). \tag{2.1}$$

If $g \in G$, $s \in E$, let $gs \in E$ be given by

$$gs(x) = gs(g^{-1}x). \tag{2.2}$$

Then $G$ acts on $(E, \bar\partial^X)$ by chain homomorphisms, and (2.1) is an identification of $G$-spaces.

Let $h^{TX}$, $h^{\xi}$ be smooth $G$-invariant Hermitian metrics on $TX$, $\xi$ respectively. Let $dv_X$ be the volume element on $X$ associated to $h^{TX}$. Let $\langle\ ,\ \rangle_{\Lambda(T^{*(0,1)}X)\otimes\xi}$ be the Hermitian product on $\Lambda(T^{*(0,1)}X)\otimes\xi$ associated to $h^{TX}$, $h^{\xi}$.

If $s, s' \in E$, set

$$\langle s, s'\rangle_E = \left(\frac{1}{2\pi}\right)^{\dim X} \int_X \langle s, s'\rangle_{\Lambda(T^{*(0,1)}X)\otimes\xi}\, dv_X. \tag{2.3}$$

Then $\langle\ ,\ \rangle_E$ is a $G$-invariant Hermitian product on $E$.

Let $\bar\partial^{X*}$ be the formal adjoint of $\bar\partial^X$ with respect to $\langle\ ,\ \rangle_E$. Set

$$D^X = \bar\partial^X + \bar\partial^{X*}. \tag{2.4}$$

If $g \in G$, then

$$[g, D^X] = 0. \tag{2.5}$$

Set

$$K = \ker D^X. \tag{2.6}$$

By Hodge theory, we have

$$K \simeq H(X, \xi). \tag{2.7}$$

Also by (2.5), $G$ acts on $K$. Then (2.7) is an identification of $G$-spaces.

Clearly $K$ inherits a $G$-invariant metric from $\langle\ ,\ \rangle_E$. Let $h^{H(X,\xi)}$ be the corresponding $G$-invariant metric on $H(X, \xi)$.

As in (1.15), we have the isotypical decomposition

$$H(X, \xi) = \bigoplus_{W\in\widehat{G}} \operatorname{Hom}_G(W, H(X, \xi)) \otimes W, \tag{2.8}$$

which is orthogonal with respect to $h^{H(X,\xi)}$.

For $W \in \widehat{G}$, set

$$\lambda_W(\xi) = (\det(\operatorname{Hom}_G(W, H(X, \xi)) \otimes W))^{-1}. \tag{2.9}$$

Put

$$\lambda_G(\xi) = \bigoplus_{W\in\widehat{G}} \lambda_W(\xi). \tag{2.10}$$

In the sequel $\lambda_G(\xi)$ will be called the inverse of the equivariant determinant of the cohomology of $\xi$. Then $\lambda_G(\xi)$ is a direct sum of complex lines.

Let $|\ |_{\lambda_W(\xi)}$ be the metric induced by $h^{H(X,\xi)}$ on $\lambda_W(\xi)$, and set

$$\log(|\ |^2_{\lambda_G(\xi)}) = \sum_{W\in\widehat{G}} \log(|\ |^2_{\lambda_W(\xi)})\frac{\chi(W)}{\mathrm{rk}(W)}. \tag{2.11}$$

The symbol $|\ |^2_{\lambda_W(\xi)}$ will be called the (equivariant) $L_2$ metric on $\lambda_G(\xi)$.

Let $K^{\perp}$ be the orthogonal space to $K$ in $E$. Then $D^{X,2}$ acts as an invertible operator on $K^{\perp}$. Let $(D^{X,2})^{-1}$ be the corresponding inverse.

Take $g\in G$, and set

$$X_g = \{x\in X,\ gx=x\}. \tag{2.12}$$

Then $X_g$ is a compact complex totally geodesic submanifold of $X$.

Let $N$ be the number operator of $E$, i.e., $N$ acts by multiplication by $i$ on $E^i$. Then by the standard heat equation methods, we know that as $t\to 0$, for any $k\in\mathbb{N}$,

$$\mathrm{Tr}_s[gN\exp(-tD^{X,2}] = \frac{a_{-l}}{t^l} + \cdots + a_0 + a_1 t + \cdots + a_k t^k + o(t^k). \tag{2.13}$$

**Definition 2.1.** For $g\in G$, $s\in\mathbb{C}$, $\mathrm{Re}(s)>l$, set

$$\theta^X(g)(s) = -\mathrm{Tr}_s[gN(D^{X,2})^{-s}]. \tag{2.14}$$

By (2.13), $\theta^X(g)(s)$ extends to a meromorphic function of $s\in\mathbb{C}$, which is holomorphic at $s=0$. In particular $g\to(\partial\theta^X/\partial s)(g)(0)$ is a central function on $G$.

**Definition 2.2.** For $g\in G$, set

$$\log(\|\ \|^2_{\lambda_G(\xi)})(g) = \log(|\ |^2_{\lambda_G(\xi)})(g) - \frac{\partial\theta^X(g)(s)}{\partial s}(0). \tag{2.15}$$

The quantity $\exp\{(-\partial\theta^X/\partial s)(g)(0)\}$ is an extension of the Ray-Singer analytic torsion [28] to the equivariant setting.

The symbol $\|\ \|_{\lambda_G(\xi)}$ will be called a Quillen metric on the equivariant determinant $\lambda_G(\xi)$. In effect the case where $G=\{1\}$ was already considered in [27], [11], [13].

b. *Anomaly formulas for equivariant Quillen metrics.* Let $\nabla^{TX}$, $\nabla^{\xi}$ be the holomorphic Hermitian connections on $(TX, h^{TX})$, $(\xi, h^{\xi})$ respectively, and let $R^{TX}$, $R^{\xi}$ be their curvatures.

Take $g \in G$. Then

$$TX_g = \{U \in TX|_{X_g},\ gU = U\}. \tag{2.16}$$

Let $N_{X_g/X}$ be the normal bundle to $X_g$ in $X$. Then $g$ acts on $N_{X_g/X}$. Let $e^{i\theta_1}, \cdots, e^{i\theta_q}$ $(0 < \theta_j < 2\pi)$ be the locally constant distinct eigenvalues of $g$ acting on $N_{X_g/X}$, and let $N^{\theta_1}_{X_g/X}, \cdots, N^{\theta_q}_{N_g/X}$ be the corresponding eigenbundles. Then $N_{X_g/X}$ splits holomorphically as

$$N_{X_g/X} = N^{\theta_1}_{X_g/X} \oplus \cdots \oplus N^{\theta_q}_{N_g/X}. \tag{2.17}$$

Also, we have the holomorphic splitting

$$TX|_{X_g} = TX_g \oplus N_{X_g/X}. \tag{2.18}$$

Moreover the splitting (2.18) of $TX|_{X_g}$ is orthogonal with respect to $h^{TX|_{X_g}}$. Let $h^{TX_g}$, $h^{N^{\theta_1}_{X_g/N}}, \cdots$ be the Hermitian metrics induced by $h^{TX|_{X_g}}$ on $TX_g$, $N^{\theta_1}_{X_g/X}, \cdots$. Then $\nabla^{TX|_{X_g}}$ induces the holomorphic Hermitian connections $\nabla^{TX_g}$, $\nabla^{N^{\theta_1}}_{X_g/X}, \cdots$ on $(TX_g, h^{TX_g})$, $(N^{\theta_1}_{X_g/X}, h^{N^{\theta_1}_{X_g/X}}) \cdots$. Let $R^{TX_g}$, $R^{N^{\theta_1}_{X_g/X}}, \cdots$ be their curvatures.

**Definition 2.3.** Let $P^{X_g}$ be the vector space of smooth forms on $X_g$, which are the sums of forms of type $(p, p)$. Let $P^{X_g, 0}$ be the subspace of the $\omega \in P^{X_g}$ such that there exist smooth forms $\alpha, \beta$ on $X_g$ with $\omega = \partial\alpha + \bar{\partial}\beta$.

If $A$ is a $(q, q)$ matrix, set

$$\mathrm{Td}(A) = \det\left(\frac{A}{1 - e^{-A}}\right), \qquad \mathrm{e}(A) = \det(A). \tag{2.19}$$

The genera associated to Td and e are called the Todd genus and the Euler genus.

**Definition 2.4.** Set

(2.20)

$$\mathrm{Td}_g(TX, h^{TX}) = \mathrm{Td}\left(\frac{-R^{TX_g}}{2i\pi}\right)\prod_{j=1}^{q}\left(\frac{\mathrm{Td}}{\mathrm{e}}\right)\left(\frac{-R^{N^{\theta_j}_{X_g/X}}}{2i\pi} + i\theta_j\right),$$

$$\mathrm{Td}'_g(TX, h^{TX}) = \frac{\partial}{\partial b}\left[\mathrm{Td}\left(\frac{-R^{TX_g}}{2i\pi} + b\right)\times\prod_{j=1}^{q}\left(\frac{\mathrm{Td}}{\mathrm{e}}\right)\left(\frac{-R^{N^{\theta_j}_{X_g/X}}}{2i\pi} + i\theta_j + b\right)\right]_{b=0},$$

$$(\mathrm{Td}_g^{-1})'(TX, h^{TX}) = \frac{\partial}{\partial b}\left[\mathrm{Td}^{-1}\left(\frac{-R^{TX_g}}{2i\pi} + b\right)\times\prod_{j=1}^{q}\left(\frac{\mathrm{Td}}{\mathrm{e}}\right)^{-1}\left(\frac{-R^{N^{\theta_j}_{X_g/X}}}{2i\pi} + i\theta_j + b\right)\right]_{b=0},$$

$$\mathrm{ch}_g(\xi, h^{\xi}) = \mathrm{Tr}\left[g\exp\left(\frac{-R^{\xi|X_g}}{2i\pi}\right)\right].$$

Then the forms in (2.20) are closed forms on $X_g$, which lie in $P^{X_g}$, and their cohomology class does not depend on the $G$-invariant metrics $h^{TX|X_g}$, $h^{\xi}$. We denote by $\mathrm{Td}_g(TX)$, $\mathrm{Td}'_g(TX)$, $\cdots$, $\mathrm{ch}_g(\xi)$ these cohomology classes, which appear in the Lefschetz formulas of Atiyah-Bott [1].

Let $h'^{TX}$, $h'^{\xi}$ be another couple of $G$-invariant metrics on $TX$, $\xi$. We denote by a prime the objects which are just considered and attached to $h'^{TX}$, $h'^{\xi}$.

By [11, §1f)], there are uniquely defined classes $\widetilde{\mathrm{Td}}_g(TX, h^{TX}, h'^{TX})$ and $\widetilde{\mathrm{ch}}_g(\xi, h^{\xi}, h'^{\xi})$ in $P^{X_g}/P^{X_g,0}$ such that

$$(2.21)\quad \begin{aligned}&\frac{\bar{\partial}\partial}{2i\pi}\widetilde{\mathrm{Td}}_g(TX, h^{TX}, h'^{TX}) = \mathrm{Td}_g(TX, h'^{TX}) - \mathrm{Td}_g(TX, h^{TX}),\\ &\frac{\bar{\partial}\partial}{2i\pi}\widetilde{\mathrm{ch}}_g(\xi, h^{\xi}, h'^{\xi}) = \mathrm{ch}_g(\xi, h'^{\xi}) - \mathrm{ch}_g(\xi, h^{\xi}).\end{aligned}$$

The main result of this section extends the anomaly formulas of [11, Theorem 0.2], [13, Theorem 1.23] to equivariant Quillen metrics.

**Theorem 2.5.** *Assume that the metrics $h^{TX}$ and $h'^{TX}$ are Kähler. Then for any $g \in G$,*

$$\log\left(\frac{\| \|'^2_{\lambda_G(\xi)}}{\| \|^2_{\lambda_G(\xi)}}\right)(g) = \int_{X_g} \widetilde{\mathrm{Td}}_g(TX, h^{TX}, h'^{TX})\,\mathrm{ch}_g(\xi, h^\xi) + \int_{X_g} \mathrm{Td}_g(TX, h'^{TX})\widetilde{\mathrm{ch}}_g(\xi, h^\xi, h'^\xi). \tag{2.22}$$

*Proof.* The rest of the section is devoted to the proof of Theorem 2.5.

c. *Supertraces and Clifford algebras.* Let $E$ be a complex vector space of dimension $l$, and $h^E$ be a Hermitian product on $E$.

Let $c(E_{\mathbf{R}})$ be the Clifford algebra of $(E_{\mathbf{R}}, h^{E_{\mathbf{R}}})$. Recall that $\Lambda(\overline{E}^*)$ is a $c(E_{\mathbf{R}})$-Clifford module. In fact if $X \in E$, let $X^* \in \overline{E}^*$ correspond to $X$ by the metric $h^E$. If $X \in E$, set

$$c(X) = \sqrt{2}X^*\wedge, \qquad c(\overline{X}) = -\sqrt{2}i_{\overline{X}}. \tag{2.23}$$

We extend the map $Y \to c(Y)$ by $\mathbf{C}$-linearity.

If $A \in \mathrm{End}(E_{\mathbf{R}})$ is antisymmetric, we identify $A$ with the 2-form $X, Y \in E_{\mathbf{R}} \to \langle X, AY\rangle$.

Let $g$ be a linear isometry of $(E, h^E)$. If $g-1$ is invertible, $(g+1)/(g-1) \in \mathrm{End}(E)$ is skew-adjoint, and so it defines a 2-form in $\Lambda(E^*_{\mathbf{R}})$ which is of type $(1,1)$. Also $g$ acts naturally on $\Lambda(\overline{E}^*)$. Moreover

$$\mathrm{Tr}_s^{\Lambda(\overline{E}^*)}[g] = \det{}^E(1-g), \tag{2.24}$$

and so $\mathrm{Tr}_s^{\Lambda(E^*)}[g]$ vanishes if and only if $g-1$ is noninvertible.

In the sequel, if $a \in \Lambda^{\mathrm{even}}(E^*_{\mathbf{R}})$, $\exp(a)$ denotes the exponential of $a$ in $\Lambda^{\mathrm{even}}(E^*_{\mathbf{R}})$.

Assume first that $g-1$ is invertible. Then

$$\mathrm{Tr}_s^{\Lambda(\overline{E}^*)}[g]e^{(g+1)/(g-1)} \in \Lambda^{\mathrm{even}}(E^*_{\mathbf{R}}). \tag{2.25}$$

In view of (2.24), one verifies easily that the expression (2.25) extends by continuity to an arbitrary unitary $g$.

Let $F$ be a finite dimensional complex vector space. Let $\alpha \in \Lambda^{\mathrm{odd}}(F^*_{\mathbf{R}}) \hat{\otimes} E_{\mathbf{R}}$. If $g-1$ is invertible, then

$$\mathrm{Tr}_s^{\Lambda(\overline{E}^*)}[g]\exp\{-\frac{1}{2}\left\langle \alpha, \frac{g+1}{g-1}\alpha\right\rangle \in \Lambda^{\mathrm{even}}(F^*_{\mathbf{R}}). \tag{2.26}$$

By the argument we just gave, this expression extends by continuity to an arbitrary unitary $g$.

Clearly $\alpha$ can be written in the form

$$\alpha = \sum_1^l \alpha^i \otimes e_i, \qquad \alpha^i \in \Lambda^{\mathrm{odd}}(F_{\mathbf{R}}^*), \ e_i \in E_{\mathbf{R}}. \tag{2.27}$$

Let $c(\alpha) \in (\Lambda(F^*)\hat{\otimes} c(E))^{\mathrm{even}}$ be given by

$$c(\alpha) = -\sum_1^n \alpha^i c(e_i). \tag{2.28}$$

Then $g\exp(c(\alpha)) \in (\Lambda(F_{\mathbf{R}}^*)\hat{\otimes}\,\mathrm{End}(\Lambda(\overline{E}^*)))^{\mathrm{even}}$.

**Theorem 2.6.** *The following equality holds:*

$$\mathrm{Tr}_s^{\Lambda(\overline{E}^*)}[g\exp(c(\alpha))] = \mathrm{Tr}_s^{\Lambda(\overline{E}^*)}[g]\exp\left\{-\frac{1}{2}\left\langle \alpha, \frac{g+1}{g-1}\alpha\right\rangle\right\}. \tag{2.29}$$

*Proof.* Let $e_1, \cdots, e_{2l}$ be an orthonormal basis of $E_{\mathbf{R}}$. We may and will assume that on this basis, the matrix of $g$ has diagonal blocks

$$\begin{bmatrix} \cos(\theta_j) & -\sin(\theta_j) \\ \sin(\theta_j) & \cos(\theta_j) \end{bmatrix}, \qquad 0 \le \theta_j < 2\pi.$$

Then one verifies easily that the action of $g$ on $\Lambda(\overline{E}^*)$ is given by

$$g = \prod_{1\le j\le l}\left(\cos\left(\frac{\theta_j}{2}\right) + \sin\left(\frac{\theta_j}{2}\right)c(e_{2j-1})c(e_{2j})\right)e^{(i/2)\sum_1^l \theta_j}. \tag{2.30}$$

Also since in (2.28), the $\alpha^i$'s are odd, we have

$$\exp(c(\alpha)) = \prod_{1\le j\le 2l}(1-\alpha^j c(e_j)). \tag{2.31}$$

Also $c(e_1)\cdots c(e_{2n})$ is the only monomial in the $c(e_j)$'s whose supertrace on $\Lambda(\overline{E}^*)$ is nonzero. Using (2.30), (2.31), we get

$$\begin{aligned} &\mathrm{Tr}_s^{\Lambda(\overline{E}^*)}[g\exp(c(\alpha))] \\ &= \prod_{1\le j\le l}\left(\sin\left(\frac{\theta_j}{2}\right) - \cos\left(\frac{\theta_j}{2}\right)\alpha^{2j-1}\wedge\alpha^{2j}\right)e^{(i/2)\sum_1^l \theta_j} \\ &\quad \times \mathrm{Tr}_s^{\Lambda(\overline{E}^*)}[e_1\cdots e_{2l}]. \end{aligned} \tag{2.32}$$

If $g-1$ is invertible, i.e., if no $\theta_j$ is equal to $0$, from (2.32) we deduce

$$\begin{aligned} \mathrm{Tr}_s^{\Lambda(\overline{E}^*)}[g\exp(c(\alpha))] &= \mathrm{Tr}_s^{\Lambda(\overline{E}^*)}[g] \\ &\quad \times \prod_{1\le j\le l}\left(1 - \frac{\cos(\theta_j/2)}{\sin(\theta_j/2)}\alpha^{2j-1}\wedge\alpha^{2j}\right). \end{aligned} \tag{2.33}$$

Also

$$\prod_{1\le j\le l}\left(1-\frac{\cos(\theta_j/2)}{\sin(\theta_j/2)}\alpha^{2j-1}\wedge\alpha^{2j}\right)\tag{2.34}$$
$$=\exp\left(-\sum_{1\le j\le l}\frac{\cos(\theta_j/2)}{\sin(\theta_j/2)}\alpha^{2j-1}\wedge\alpha^{2j}\right).$$

Moreover the matrix of $(g+1)/(g-1)$ has diagonal blocks given by

$$\begin{bmatrix} 0 & \frac{\cos(\theta_j/2)}{\sin(\theta_j/2)} \\ \frac{-\cos(\theta_j/2)}{\sin(\theta_j/2)} & 0 \end{bmatrix}.$$

Therefore

$$\frac{1}{2}\left\langle\alpha,\frac{g+1}{g-1}\alpha\right\rangle=\sum_{1\le j\le l}\frac{\cos(\theta_j/2)}{\sin(\theta_j/2)}\alpha^{2j-1}\wedge\alpha^{2j}.\tag{2.35}$$

From (2.33)–(2.35), we get (2.29) when $g-1$ is invertible. The general case follows by continuity. Hence the proof of Theorem 2.6 is completed.

d. *Proof of Theorem* 2.5. Let $c\in[0,1]\to(h_c^{TX},h_c^{\xi})$ be a smooth family of $G$-invariant Hermitian metrics on $TX$, $\xi$ such that for any $c$, $h_c^{TX}$ is Kähler and also $(h_0^{TX},h_0^{\xi})=(h^{TX},h^{\xi})$, $(h_1^{TX},h_1^{\xi})=(h'^{TX},h'^{\xi})$. Let $\|\ \|_{\lambda_G(\xi),c}$ be the corresponding equivariant Quillen metric on $\lambda_G(\xi)$.

Let $\bar\partial_c^{X*}$ be the adjoint of $\bar\partial^X$ with respect to $(h_c^{TX},h_c^{\xi})$. Then

$$[\bar\partial^X,g]=0,\qquad[\bar\partial_c^{X*},g]=0.\tag{2.36}$$

Set

$$D_c^X=\bar\partial^X+\bar\partial_c^{X*}.\tag{2.37}$$

Let $*_c$ be the Hodge star operator attached to $h_c^{TX}$. Set

$$Q_c=-*_c^{-1}\frac{\partial *_c}{\partial c}-(h_c^{\xi})^{-1}\frac{\partial h_c^{\xi}}{\partial c}.\tag{2.38}$$

By an obvious analogue of (2.13), as $t\to 0$, we have an asymptotic expansion

$$\operatorname{Tr}_s[gQ_c\exp(-tD_c^{X,2})]=\sum_{j=-l}^{0}M_{j,c}t^j+\mathscr{O}(t).\tag{2.39}$$

Using Proposition 1.5, (2.39) and proceeding formally as in [13, Theorem 1.18], we get

$$\frac{\partial}{\partial c}\log\left(\frac{\|\ \|^2_{\lambda_G(\xi),c}}{\|\ \|^2_{\lambda_G(\xi)}}\right)(g) = M_{0,c}. \tag{2.40}$$

Assume first that $h_c^{TX} = h^{TX} = h'^{TX}$. Then since $h^{TX}$ is Kähler, we may use the local index techniques of [9], [3], [2, Chapter 6] to find that

$$M_{0,c} = -\int_{X_g} \mathrm{Td}_g(TX, h^{TX})\,\mathrm{Tr}\left[g(h_c^{\xi})^{-1}\frac{\partial h^{\xi}}{\partial c}\exp\left(\frac{-R_c^{\xi}}{2i\pi}\right)\right]. \tag{2.41}$$

Moreover by [11, §1f)], we obtain

$$\begin{aligned} -\int_0^1 \mathrm{Tr}&\left[g(h_c^{\xi})^{-1}\frac{\partial h^{\xi}}{\partial c}\exp\left(\frac{-R_c^{\xi}}{2i\pi}\right)\right] \\ &= \widetilde{\mathrm{ch}}_g(\xi, h^{\xi}, h'^{\xi}) \quad \text{in } P^{X_g}/P^{X_g,0}, \end{aligned} \tag{2.42}$$

which together with (2.40) yields (2.22).

Assume now that $h_c^{\xi} = h^{\xi} = h^{\xi'}$. Let $\omega_c$ be the Kähler form of $h_c^{TX}$. Let $J^{TX}$ be the complex structure of $T_{\mathbf{R}}X$. If $U, V \in T_{\mathbf{R}}X$, then $\omega_c(U, V) = \langle U, J^{TX}V\rangle_{T_{\mathbf{R}}X,c}$. Set

$$\dot{\omega}_c = \partial\omega_c/\partial c. \tag{2.43}$$

To simplify our notation, in the sequel, we will not always write the subscript $c$ explicitly. Then by proceeding as in the above references and in [12, proof of Theorem 2.16], we find that

$$\begin{aligned} &M_j = 0, \qquad j \le -2, \\ &M_{-1} = \int_{X_g} \frac{\dot{\omega}}{2\pi}\mathrm{Td}_g(TX, h^{TX})\,\mathrm{ch}_g(\xi, h^{\xi}). \end{aligned} \tag{2.44}$$

Let $da, d\bar{a}$ be two odd Grassmann variables. In particular, $da, d\bar{a}$ anticommute with the operator $D^X$. Set

$$L_u = -uD^{X,2} - \sqrt{\frac{u}{2}}\,da D^X - \sqrt{\frac{u}{2}}\,d\bar{a}[D^X, Q] + da\,d\bar{a}Q. \tag{2.45}$$

If $\alpha \in \mathbb{C}(da, d\bar{a})$, let $[\alpha]^{dad\bar{a}} \in \mathbb{C}$ be the coefficient of $dad\bar{a}$ in the expansion of $\alpha$. Using (2.36), and proceeding formally as in [13, Theorem 1.20], we get

$$\frac{\partial}{\partial u} u \operatorname{Tr}_s[gQ\exp(-uD^{X,2})] = \operatorname{Tr}_s[g\exp(L_u)]^{dad\bar{a}}, \tag{2.46}$$

which together with (2.40) leads to that as $u \to 0$,

$$\operatorname{Tr}_s[g\exp(L_u)]^{dad\bar{a}} = M_0 + \mathscr{O}(u). \tag{2.47}$$

As we saw in (2.23), $\Lambda(T^{*(0,1)}X)\hat{\otimes}\xi$ is a $c(T_{\mathbf{R}}X, h^{T_{\mathbf{R}}X})$ Clifford module. If $U \in T_{\mathbf{R}}X$, let $c(U)$ denote the corresponding Clifford action.

Let $\nabla^{\Lambda(T^{*(0,1)}X)\otimes\xi}$ be the connection induced by $\nabla^{TX}$ and $\nabla^{\xi}$ on $\Lambda(T^{*(0,1)}X)\otimes\xi$. Set

$$R'^{\xi} = R^{\xi} + \tfrac{1}{2}\operatorname{Tr}[R^{TX}]. \tag{2.48}$$

Let $K$ be the scalar curvature of $(X, h^{T_{\mathbf{R}}X})$.

Let $e_1, \cdots, e_{2n}$ be a locally defined smooth orthonormal basis of $(T_{\mathbf{R}}X, h^{T_{\mathbf{R}}X})$. Recall that $J^{TX}$ is the complex structure of $T_{\mathbf{R}}X$. By [13, Theorem 1.21] (and keeping in mind that the operator $D$ in [13] coincides with $\sqrt{2}D^X$), we get

$$\begin{aligned} L_u = {} & \frac{u}{2}\left(\nabla_{e_i}^{\Lambda(T^{*(0,1)}X)\hat{\otimes}\xi} - \frac{dac(e_i)}{2\sqrt{u}} - \frac{d\bar{a}\sqrt{-1}}{2\sqrt{u}}\dot{\omega}(e_k, e_i)c(e_k)\right)^2 \\ & + \frac{dad\bar{a}}{4}\dot{\omega}(e_j, J^X e_j) - \frac{\sqrt{u}}{8}d\bar{a}c(e_i)\nabla_{e_i}\dot{\omega}(e_j, J^{TX}e_j) \\ & - \frac{uK}{8} - \frac{u}{4}c(e_i)e(e_j)\otimes R'^{\xi}(e_i, e_j). \end{aligned} \tag{2.49}$$

Let $T_u(x, x')$ $(x, x' \in X)$ be the smooth kernel associated to $\exp(L_u)$ with respect to $dv_X(x')/(2\pi)^{\dim X}$. Then

$$\operatorname{Tr}_s[g\exp(L_u)] = \int_X \operatorname{Tr}_s[gT_u(g^{-1}x, x)]\frac{dv_X(x)}{(2\pi)^{\dim X}}. \tag{2.50}$$

Using normal geodesic coordinates to $X_g$ in $X$, we may and will identify an $\varepsilon$-neighborhood of $X_g$ in $N_{X_g/X,\mathbf{R}}$ to an open neighborhood $\mathscr{U}_\varepsilon$ of $X_g$ in $X$.

By standard estimates on heat kernels, one finds easily that there exist $c \geq 0$, $C > 0$ such that for $x \in X \backslash \mathscr{U}_\varepsilon$, $0 < u \leq 1$,

$$|T_u(g^{-1}x, x)| \leq c \exp(-C/u). \tag{2.51}$$

Let $dv_{N_{X_g/X}}$ be the natural volume element along the fibers of $N_{X_g/X, \mathbf{R}}$. Let $k(x, Z)$ $(x \in X, \ Z \in N_{X_g/X, \mathbf{R}}, \ |Z| < \varepsilon)$ be defined by

$$dv_X(x, Z) = k(x, Z)\, dv_{X_g}(x)\, dv_{N_{X_g/X}}(Z). \tag{2.52}$$

Then $k(x, 0) = 1$.

Clearly

$$\begin{aligned} &\int_{\mathscr{U}_\varepsilon} \mathrm{Tr}_s[gT_u(g^{-1}x, x)] \frac{dv_X(x)}{(2\pi)^{\dim X}} \\ &= \int_{\substack{(x, Z) \in N_{X_g/X} \\ |Z| \leq \varepsilon/\sqrt{u}}} u^{\dim N_{X_g/X}} \mathrm{Tr}_s[gT_u(g^{-1}(x, \sqrt{u}Z), (x, \sqrt{u}Z))] \\ &\qquad \times k(x, \sqrt{u}Z) \frac{dv_{X_g}(x) dv_{N_{X_g/X}}(z)}{(2\pi)^{\dim X}}. \end{aligned} \tag{2.53}$$

Of course, since we have used normal geodesic coordinates to $X_g$ in $X$, if $(x, Z) \in N_{X_g/X}$, then

$$g^{-1}(x, Z) = (x, g^{-1}Z). \tag{2.54}$$

Now we calculate the asymptotic behavior of

$$u^{\dim N_{X_g/X}} \mathrm{Tr}_s[gT_u(g^{-1}(x, \sqrt{u}Z), (x, \sqrt{u}Z))] \text{ as } u \to 0.$$

For this, we combine the methods of [9], where we gave a proof of the Lefschetz formulas of Atiyah-Bott [1], with the methods of [4], where we proved the local family index theorem for Dirac operators.

Take $x \in X_g$. We assume that $e_1, \cdots, e_{2p}$ form an orthonormal basis of $T_{\mathbf{R}}X_{g,x}$, and that $e_{2p+1}, \cdots, e_{2l}$ form an orthonormal basis of $N_{X_g/X, \mathbf{R}, x}$. If $Z \in N_{X_g/X, \mathbf{R}, x}$, let $Q_{(g^{-1}Z, Z)}$ be the probability law on $\mathscr{C}([0, 1], N_{X_g/X, \mathbf{R}, x})$ of the Brownian bridge $s \to w_s$, with $w_0 = g^{-1}Z$, $w_1 = Z$ [8, Definition 2.6]. Then by proceeding as in [9, proof of Theorems 4.9, 4.11], we get

$$\lim_{u\to 0}\int_{|Z|\le \varepsilon/\sqrt{u}} u^{\dim N_{X_g/X}} \operatorname{Tr}_s[gT_u(g^{-1}(x,\sqrt{u}Z),(x,\sqrt{u}Z))]$$

$$\times k(x,\sqrt{u}Z)\frac{dv_{N_{X_g/X}}(Z)}{(2\pi)^{\dim X}}$$

$$= \left(\frac{1}{2\pi i}\right)^{\dim X_g}\left\{\int_{N_{X_g/X,\mathbf{R},x}} E^{Q_{(g^{-1}Z,Z)}}\exp\left[\frac{1}{2}\int_0^1\langle R^{TX}w,dw\rangle\right.\right.$$

$$+\frac{1}{2}idad\bar{a}\left(\int_0^1\dot{\omega}(w,dw)-\dot{\omega}(g^{-1}Z,Z)\right)$$

$$-\frac{1}{2}dad\bar{a}\operatorname{Tr}\left((h^{TX})^{-1}\frac{\partial h^{TX}}{\partial c}\right)$$

(2.55)

$$-\frac{d\bar{a}}{\sqrt{2}}\sqrt{-1}\int_0^1\left(\sum_1^{2l}\nabla_{w_s}\dot{\omega}(e_k,dw_s)e^k\wedge\right)$$

$$\left.+\frac{d\bar{a}}{2\sqrt{2}}d\operatorname{Tr}\left((h^{TX})^{-1}\frac{\partial h^{TX}}{\partial c}\right)\right]$$

$$\times\operatorname{Tr}_s^{\Lambda(N^{*(0,1)}_{X_g/X})}\left[g\exp\left(\frac{1}{2}c(da(1-g^{-1})Z\right.\right.$$

$$\left.\left.+d\bar{a}\sqrt{-1}\sum_{2p+1}^{2l}\dot{\omega}(e_k,(1-g^{-1})Z)e_k)\right)\right]$$

$$\times\exp\left(-\frac{1}{2}|(1-g^{-1})Z|^2\right)\frac{dv_{N_{X_g/X}}(Z)}{(2\pi)^{\dim N_{X_g/X}}}$$

$$\left.\times\exp\left(-\frac{1}{2}\operatorname{Tr}[R^{TX}]\right)\operatorname{Tr}[g\exp(-R^{\xi})]\right\}^{\max},$$

uniformly on $X_g$.

From Theorem 2.6 it follows that

$$\operatorname{Tr}_s \Lambda(N^{*(0,1)}_{X_g/X})\left\{ g \exp\left[\frac{1}{2}c\left(da(1-g^{-1})Z + d\bar{a}\sqrt{-1}\sum_{2p+1}^{2l}\dot{\omega}(e_k, (1-g^{-1})Z)e_k\right)\right]\right\}$$

$$(2.56)\quad = \operatorname{Tr}_s^{\Lambda(N^{*(0,1)}_{X_g/X})}[g]\exp\left[-\frac{1}{8}\left\langle da(1-g^{-1})Z + d\bar{a}\sqrt{-1}\sum_{2p+1}^{2l}\dot{\omega}(e_k, (1-g^{-1}Z)e_k, \frac{g+1}{g-1}\left(da(1-g^{-1})Z + d\bar{a}\sqrt{-1}\sum_{2p+1}^{2l}\dot{\omega}(e_k, (1-g^{-1})Z)e_k\right)\right\rangle\right].$$

Now the form $\dot{\omega}$ is $g$-invariant. Therefore

$$(2.57)\qquad \sum_{2p+1}^{2l}\dot{\omega}(e_k, g^{-1}Z)e_k = g^{-1}\sum_{2p+1}^{2l}\dot{\omega}(e_k, Z)e_k,$$

which together with (2.56) gives

$$(2.58)\quad \operatorname{Tr}_s^{\Lambda(N^{*(0,1)}_{X_g/X})}\left\{ g \exp\left[\frac{1}{2}c\left(da(1-g^{-1})Z + d\bar{a}\sqrt{-1}\sum_{2p+1}^{2l}\dot{\omega}(e_k, (1-g^{-1})Z)e_k\right)\right]\right\}^{da d\bar{a}} = \operatorname{Tr}_s^{\Lambda(N^{*(0,1)}_{X_g/X})}[g]\exp\left(\frac{i}{2}\dot{\omega}(g^{-1}Z, Z)da d\bar{a}\right).$$

So by (2.55), (2.58), we obtain

$$\lim_{u\to 0}\int_{|Z|\le\varepsilon/\sqrt{u}} u^{\dim N_{X_g/X}} \operatorname{Tr}_s[gT_u(g^{-1}(x,\sqrt{u}Z),(x,\sqrt{u}Z))]^{da d\bar a} \times k(x,\sqrt{u}Z)\frac{dv_{N_{X_g/X}}(Z)}{(2\pi)^{\dim X}}$$

$$\begin{aligned} = \left(\frac{1}{2\pi i}\right)^{\dim X_g}\Bigg\{\int_{N_{X_g/X}} E^{Q_{(g^{-1}Z,Z)}}\Bigg[\exp\Bigg(\frac{1}{2}\int_0^1\langle R^{TX}w,dw\rangle \\ + \frac{1}{2} i\,da d\bar a\int_0^1 \dot\omega(w,dw) \\ - \frac{1}{2}da d\bar a\operatorname{Tr}\left((h^{TX})^{-1}\frac{\partial h^{TX}}{\partial c}\right) \\ -\frac{1}{2}\operatorname{Tr}[R^{TX}]\Bigg)\Bigg] \\ \times\exp\left(-\frac{|(g-1)Z|^2}{2}\right)\frac{dv_{N_{X_g/X}}(Z)}{(2\pi)^{\dim N_{X_g/X}}} \\ \times \operatorname{Tr}_s^{\Lambda(N^{*(0,1)}_{X_g/X})}[g]\operatorname{Tr}[g\exp(-R^{\xi})]\Bigg\}^{\max} \end{aligned} \tag{2.59}$$

uniformly on $X_g$. Clearly

$$\int_0^1 \dot\omega(w,dw)=\int_0^1\left\langle w, J^{TX}(h^{TX})^{-1}\frac{\partial h^{TX}}{\partial c}dw\right\rangle. \tag{2.60}$$

Using (2.60), we find that the right-hand side of (2.59) is given by

(2.61)

$$\begin{aligned} \Bigg\{\left(\frac{1}{2\pi i}\right)^{\dim X_g}\frac{\partial}{\partial b}\Bigg[\int_{N_{X_g/X}} E^{Q_{(Z,gZ)}} \\ \times\left(\frac{1}{2}\int_0^1\left\langle\left(R^{TX}-ibJ^{TX}(h^{TX})^{-1}\frac{\partial h^{TX}}{\partial c}\right)w,dw\right\rangle\right. \\ \left.\times\exp\left(-\frac{1}{2}b\operatorname{Tr}\left((h^{TX})^{-1}\frac{\partial h^{TX}}{\partial c}\right)-\frac{1}{2}\operatorname{Tr}[R^{TX}]\right)\right) \\ \times\exp\left(-\frac{1}{2}|(g-1)Z|^2\right)\frac{dv_{N_{X_g/X}}(Z)}{(2\pi)^{\dim N_{X_g/X}}}\det{}^{N_{X_g/X}}(1-g)\Bigg]_{b=0}\operatorname{ch}_g(\xi,\nabla^{\xi})\Bigg\}^{\max}. \end{aligned}$$

By proceeding as in [9, Theorem 4.13], it is easily seen that (2.61) is equal to

(2.62)

$$\left\{ \frac{\partial}{\partial b} \left[ \mathrm{Td}\left( -\frac{R^{TX_g}}{2i\pi} - b(h^{TX_g})^{-1} \frac{\partial h^{TX_g}}{\partial c} \right) \times \prod_{j=1}^{q} \frac{\mathrm{Td}}{\mathrm{e}} \left( -\frac{R^{N^{\theta_j}_{X_g/X}}}{2i\pi} - b(h^{N^{\theta_j}_{X_g/X}})^{-1} \frac{\partial h^{N^{\theta_j}_{X_g/X}}}{\partial c} + i\theta_j \right) \right]_{b=0} \mathrm{ch}_g(\xi, \nabla^{\xi}) \right\}^{\max}.$$

So using (2.47), (2.50), (2.51), (2.53), (2.59)–(2.62), we get

$$\begin{aligned} M_0 &= \lim_{u \to 0} \mathrm{Tr}_s[g \exp(L_u)] \\ &= \int_{X_g} \frac{\partial}{\partial b} \left[ \mathrm{Td}\left( -\frac{R^{TX_g}}{2i\pi} - b(h^{TX_g})^{-1} \frac{\partial h^{TX_g}}{\partial c} \right) \right. \\ &\quad \left. \times \prod_{j=1}^{q} \frac{\mathrm{Td}}{\mathrm{e}} \left( -\frac{R^{N^{\theta_j}_{X_g/X}}}{2i\pi} - b(h^{N^{\theta_j}_{X_g/X}})^{-1} \frac{\partial h^{N^{\theta_j}_{X_g/X}}}{\partial c} + i\theta_j \right) \right]_{b=0} \mathrm{ch}_g(\xi, \nabla^{\xi}). \end{aligned}$$

By (2.40), we have

$$\log \left( \frac{\| \; \|'^2_{\lambda_G(\xi)}}{\| \; \|^2_{\lambda_G(\xi)}} \right)(g) = \int_0^1 M_{0,c}\, dc, \tag{2.64}$$

which together with [11, §1f)] and (2.63) gives (2.22).

Hence the proof of Theorem 2.5 is completed.

## III. Complex immersions, equivariant resolutions, and Quillen metrics

Let $i : Y \to X$ be an embedding of a compact complex manifold, let $\eta$ be a holomorphic vector bundle on $Y$, and let $(\xi, v)$ be a complex of holomorphic vector bundles on $X$, which provides a resolution of $i_* \eta$. Let $G$ be a compact group acting holomorphically on the objects which we just introduced. Let $\lambda_G(\xi)$ and $\lambda_G(\eta)$ be the equivariant determinants of the cohomology of $\xi$ and $\eta$.

An obvious extension of [22] shows that $\lambda_G(\xi) \simeq \lambda_G(\eta)$. Let $\sigma \in \lambda_G^{-1}(\eta) \otimes \lambda_G(\xi)$ be the canonical section inducing this identification. The purpose of this paper is to calculate the Quillen norm of $\sigma$.

In this section we describe in more detail the objects considered above, and make various simplifying assumptions on the considered metrics on $TX$, $\xi$, $TY$, $\eta$, along the lines of assumption (A) of [5].

This section is organized as follows. Part a contains an introduction of our basic geometric setting. In part b, we describe the canonical section $\sigma$. In part c, we construct an equivariant Quillen metric on an intermediary object $\tilde{\lambda}_G(\xi)$. Finally, part d gives various assumptions on the metrics on $TX$, $\xi$, $TY$, $\eta$.

This section extends [15, §1] to an equivariant setting.

a. *Complex immersions and resolutions.* Let $X$ be a compact connected complex manifold of complex dimension $l$. Let $Y = \bigcup_1^d Y_j$ be a finite union of compact connected submanifolds of $X$ such that $Y_j \cap Y_{j'} = \varnothing$ for $1 \leq j < j' \leq d$. Let $i$ be the embedding $Y \to X$. For $1 \leq j \leq d$, let $l'_j$ be the complex dimension of $Y_j$.

Let $\eta$ be a holomorphic vector bundle on $Y$. Let

$$(\xi, v) : 0 \to \xi_m \underset{v}{\to} \cdots \underset{v}{\to} \xi_0 \to 0 \tag{3.1}$$

be a holomorphic chain complex of vector bundles on $X$. In the sequel, we identify $\xi$ with $\bigoplus_{k=0}^m \xi_k$. Let $r$ be a holomorphic restriction map: $\xi_0|_Y \to \eta$.

We assume that $(\xi, v)$ provides a resolution of the sheaf $i_*\mathscr{O}_Y(\eta)$, i.e., we have the exact sequence of sheaves

$$0 \to \mathscr{O}_X(\xi_m) \underset{v}{\to} \mathscr{O}_X(\xi_{m-1}) \to \cdots \underset{v}{\to} \mathscr{O}_X(\xi_0) \underset{r}{\to} i_*\mathscr{O}_Y(\eta) \to 0. \tag{3.2}$$

Let $N_H \in \operatorname{End}(\xi)$ be the number operator of $\xi$, i.e., $N_H$ acts on $\xi_k$ by multiplication by $k$.

Let $\delta_X$, $\delta_Y$ be the Čech coboundary operators on $X$, $Y$. By definition the cohomology groups $H^*(X, \xi_i)$ $(0 \leq i \leq m)$, $H^*(Y, \eta)$ are the cohomology groups of the complexes $(\mathscr{O}_X(\xi_i), \delta_X)$, $(\mathscr{O}_Y(\eta), \delta_Y)$. Of course $H^*(Y, \eta) = \bigoplus_{j=1}^d H^*(Y_j, \eta|_{Y_j})$.

Let $N_\delta$ be the operator acting on $q$ cochains by multiplication by $q$.

We choose sign conventions, so that $\delta_X v + v\delta_X = 0$, i.e., $(\mathscr{O}_X(\xi), \delta_X + v)$ is a complex. We define the $\mathbb{Z}$-grading on $\mathscr{O}_X(\xi)$ by $N_\delta - N_H$, so that

$\delta_X + v$ increases the total degree by 1. Similarly, we define the $\mathbb{Z}$-grading on $\mathscr{O}_Y(\eta)$ by $N_\delta$.

We extend $r$ to a map from $\mathscr{O}_X(\xi)$ into $i_*\mathscr{O}_Y(\eta)$, with the convention that it vanishes on $\mathscr{O}_X(\xi_i)$ for $i > 0$, and coincides with the given $r$ for $i = 0$.

Tautologically, $r$ is a quasi-isomorphism of $\mathbf{Z}$-graded complexes, which induces the canonical identification

$$H^*(\mathscr{O}_X(\xi), \delta_X + v) \simeq H^*(Y, \eta) = \bigoplus_{j=1}^{d} H^*(Y_j, \eta|_{Y_j}). \tag{3.3}$$

Clearly

$$\Lambda(T^{*(0,1)}X) = \bigoplus_{p=0}^{\dim X} \Lambda^p(T^{*(0,1)}X).$$

Let $N_V^X$ be the operator defining the $\mathbf{Z}$-grading of $\Lambda(T^{*(0,1)}X)$.

We can form the $\mathbf{Z}$-graded tensor product $\Lambda(T^{*(0,1)}X)\hat{\otimes}\xi$. We define the $\mathbf{Z}$-grading on $\Lambda(T^{*(0,1)}X)\hat{\otimes}\xi$ by the operator $N_V^X\hat{\otimes}1 - 1\hat{\otimes}N_H$, which we also note $N_V^X - N_H$.

**Definition 3.1.** For $0 \le p \le l$, $0 \le i \le m$, let $E_i^p$ be the vector space of smooth sections of $\Lambda^p(T^{*(0,1)}X)\hat{\otimes}\xi_i$ on $X$. Set

$$E_+ = \bigoplus_{p-i \text{ even}} E_i^p, \quad E_- = \bigoplus_{p-i \text{ odd}} E_i^p, \quad E = E_+ \oplus E_-. \tag{3.4}$$

Then $E$ is exactly the set of smooth sections of $\Lambda(T^{*(0,1)}X)\hat{\otimes}\xi$ on $X$. It is $\mathbf{Z}$-graded by the operator $N_V^X - N_H$. Also $E = E_+ \oplus E_-$ describes the corresponding $\mathbf{Z}_2$-grading of $E$.

The Dolbeault operator $\bar{\partial}^X$ acts as an odd operator on $E$. Also $v$ acts on $\xi$ as an odd operator. We extend $v$ to an odd operator acting on $\Lambda(T^{*(0,1)}X)\hat{\otimes}\xi$, with the convention that if $\alpha \in \Lambda(T^{*(0,1)}X)$, $f \in \xi$, then

$$v(\alpha\hat{\otimes}f) = (-1)^{\deg\alpha}\alpha\hat{\otimes}vf. \tag{3.5}$$

Then

$$(\bar{\partial}^X + v)^2 = 0, \tag{3.6}$$

i.e., $\bar{\partial}^X + v$ is a chain map on $E$.

By [15, Proposition 1.5], there is a canonical identification of **Z**-graded vector spaces

$$\begin{aligned} &H^*(E_i, \bar\partial^X) \simeq H^*(X, \xi_i), \\ &H^*(E, \bar\partial^X + v) \simeq H^*(\mathscr{O}_X(\xi), \delta^X + v). \end{aligned} \tag{3.7}$$

Let $N_V^Y$ be the operator defining the **Z**-grading on $\Lambda(T^{*(0,1)}Y)$. For $1 \le j \le d$, $1 \le q \le l'_j$, let $F_j^q$ be the set of smooth sections of $\Lambda^q(T^{*(0,1)}Y_j)\hat{\otimes}\eta|_{Y_j}$ on $Y_j$. Set

$$\begin{aligned} &F_{j,+} = \bigoplus_{q \text{ even}} F_j^q, \quad F_{j,-} = \bigoplus_{q \text{ odd}} F_j^q, \quad F_j = F_{j,+} \oplus F_{j,-}, \\ &F_\pm = \bigoplus_{j=1}^d F_{j,\pm}, \qquad F = F_+ \oplus F_-. \end{aligned} \tag{3.8}$$

The operator $N_V^Y$ defines the **Z**-grading on $F_j$ and $F$.

Let $\bar\partial^Y$ be the Dolbeault operator acting on $F$. Then

$$H^*(F, \bar\partial^Y) \simeq H^*(Y, \eta). \tag{3.9}$$

By (3.3), (3.7), (3.9), we find that there is a canonical isomorphism of **Z**-graded vector spaces

$$H^*(E, \bar\partial^X + v) \simeq H^*(F, \bar\partial^Y). \tag{3.10}$$

If $\alpha \in \Lambda(T^{*(0,1)}X)|_Y$, $f \in \xi_{k|Y}$, set

$$\begin{aligned} r(\alpha\hat{\otimes}f) &= 0 \quad \text{if } k \neq 0, \\ &\phantom{=}\, i^*\alpha \otimes rf \quad \text{if } k = 0. \end{aligned} \tag{3.11}$$

Now we recall a result in [15, Theorem 1.7].

**Theorem 3.2.** *The map* $r\colon (E, \bar\partial^X + v) \to (F, \bar\partial^Y)$ *is a quasi-isomorphism of* **Z***-graded complexes, and induces the canonical identification* $H^*(E, \bar\partial^X + v) \simeq H^*(Y, \eta)$.

b. *Resolutions and group actions.* Let $G$ be a compact Lie group. We assume that $G$ acts on $X$ by holomorphic diffeomorphisms, which preserve $Y$. Also we assume that the action of $G$ on $X$ and $Y$ lifts to a holomorphic action on the chain complex $(\xi, v)$ and on $\eta$, and that the restriction map $r: \xi_0|_Y \to \eta$ is $G$-equivariant.

Then $G$ acts naturally by chain maps on $(\mathscr{O}_X(\xi_i), \delta^X)$ $(0 \le i \le m)$, $(\mathscr{O}_X(\xi), \delta^X + v)$, $(\mathscr{O}_Y(\eta), \delta^Y)$ and on $(E_i, \bar\partial^X)$, $(E, \bar\partial^X + v)$, $(F, \bar\partial^Y)$. Also the quasi-isomorphisms $r : (\mathscr{O}_X(\xi), \bar\partial^X + v) \to (\mathscr{O}_{Y(\eta)}, \delta^Y)$ and $r: (E, \bar\partial^X + v) \to (F, \bar\partial^Y)$ are $G$-equivariant. Therefore $G$ acts on $H^*(X, \xi_i)$ $(0 \le i \le m)$, $H^*(\mathscr{O}_X(\xi), \delta^X + v) \simeq H^*(E, \bar\partial^X + v)$, and $H^*(Y, \eta)$. Finally the canonical identification

$$H^*(E, \bar\partial^X + v) \simeq H^*(F, \bar\partial^Y) \tag{3.12}$$

is an identification of finite dimensional $G$-spaces.

For given $W \in \widehat{G}$, if $\lambda_W$, $\mu_W$ are complex lines, $\lambda = \bigoplus_{W\in\widehat{G}} \lambda_W$, and $\mu = \bigoplus_{W\in\widehat{G}} \mu_W$, then set

$$\lambda^{-1} = \bigoplus_{W\in\widehat{G}} \lambda_W^{-1}, \qquad \lambda\otimes\mu = \bigoplus_{W\in\widehat{G}} \lambda_W \otimes \mu_W. \tag{3.13}$$

Now use the notation of §1. Set

$$\begin{aligned} \lambda_G(\xi_i) &= (\det(H^*(X, \xi_i), G))^{-1}, \\ \lambda_G(\xi) &= \bigotimes_{i=0}^{m} (\lambda_G(\xi_i))^{(-1)^i}, \\ \tilde\lambda_G(\xi) &= (\det(H^*(E, \bar\partial^X + v), G))^{-1}, \\ \lambda_G(\eta) &= (\det(H^*(Y, \eta), G))^{-1}. \end{aligned} \tag{3.14}$$

Then (3.12) induces the identification

$$\tilde\lambda_G(\xi) \simeq \lambda_G(\eta). \tag{3.15}$$

For $0 \le i \le m$, consider the exact sequences of complexes

$$\begin{aligned} 0 &\to \left(\bigoplus_{j\le i-1} \mathscr{O}_X(\xi_j), \delta_v^X\right) \to \left(\bigoplus_{j\le i} \mathscr{O}_X(\xi_j), \delta^X + v\right) \\ &\qquad \to (\mathscr{O}_X(\xi_i), \delta^X) \to 0, \\ 0 &\to \left(\bigoplus_{j\le i-1} E_j, \bar\partial^X + v\right) \to \left(\bigoplus_{j\le i} E_j, \bar\partial^X + v\right) \\ &\qquad \to (E_i, \bar\partial^X) \to 0. \end{aligned} \tag{3.16}$$

The objects appearing in (3.16) are $\mathbf{Z}$-graded by the operators $N_\delta - N_H$, and $N_V^X - H_H$, so that the arrows in (3.16) are morphisms of $\mathbf{Z}$-graded complexes. Then by [15, Proposition 1.8], the corresponding exact sequences in cohomology are isomorphic. So we write the second one in the form

$$\begin{aligned}(3.17)\qquad &\to H^p\left(\bigoplus_{j\le i-1} E_j, \bar\partial^X + v\right) \to H^p\left(\bigoplus_{j\le i} E_j, \bar\partial^X + v\right)\\ &\to H^{p+i}(E_i, \bar\partial^X) \to H^{p+1}(E_i, \bar\partial^X + v) \to .\end{aligned}$$

Clearly, $G$ acts on the exact sequences (3.16), (3.17) by chain maps. From (3.17), we get the canonical isomorphism

$$\begin{aligned}(3.18)\qquad \lambda_G(\xi_i) \simeq \bigotimes_{p=-m}^{l} &\left(\det\left(H^p\left(\bigoplus_{j\le i-1} E_j, \bar\partial^X + v\right), G\right)^{-1}\right.\\ &\left.\otimes \det\left(H^p\left(\bigoplus_{j\le i} E_j, \bar\partial^X + v\right), G\right)\right)^{(-1)^{p+i+1}}\end{aligned}$$

Using (3.18), we obtain the canonical isomorphism

$$(3.19)\qquad \tilde\lambda_G(\xi) \simeq \lambda_G(\xi),$$

which together with (3.15) gives

$$(3.20)\qquad \lambda_G(\xi) \simeq \tilde\lambda_G(\xi) \simeq \lambda_G(\eta).$$

By (2.10), we have

$$(3.21)\qquad \lambda_G(\xi) = \bigoplus_{W\in\widehat G} \lambda_W(\xi).$$

Similarly, we write

$$(3.22)\qquad \tilde\lambda_G(\xi) = \bigoplus_{W\in\widehat G} \tilde\lambda_W(\xi), \qquad \lambda_G(\eta) = \bigoplus_{W\in\widehat G} \lambda_W(\eta).$$

By (3.20), for $W \in \widehat G$, the lines $\lambda_W^{-1}(\eta) \otimes \lambda_W(\xi)$, $\lambda_W^{-1}(\eta) \otimes \tilde\lambda_W(\xi)$, $\tilde\lambda_W^{-1}(\xi) \otimes \lambda_W(\xi)$ have canonical nonzero section $\sigma_W$, $\rho_W$, $\tau_W$. Clearly

$$(3.23)\qquad \sigma_W = \rho_W \otimes \tau_W.$$

Set

$$(3.24)\qquad \begin{aligned} \sigma &= \bigoplus_{W\in\widehat{G}} \sigma_W \in \lambda_G^{-1}(\eta)\otimes\lambda_G(\xi),\\ \rho &= \bigoplus_{W\in\widehat{G}} \rho_W \in \lambda_G^{-1}(\eta)\otimes\tilde{\lambda}_G(\xi),\\ \tau &= \bigoplus_{W\in\widehat{G}} \tau_W \in \tilde{\lambda}_G^{-1}(\xi)\otimes\lambda_G(\xi). \end{aligned}$$

Then from (3.21), (3.22), and using an obvious notation we obtain

$$(3.25)\qquad \sigma = \rho\otimes\tau.$$

c. *A Quillen metric on* $\tilde{\lambda}_G(\xi)$. Let $h^{TX}$, $h^\xi = \bigoplus_{i=0}^m h^{\xi^i}$ be $G$-invariant smooth Hermitian metrics on $TX$, $\xi = \bigoplus_{i=0}^m \xi^i$, and $h^{TY}$, $h^\eta$ be $G$-invariant metrics on $TY$, $\eta$, respectively. By §2a, these metrics induce equivariant Quillen metrics $\| \ \|_{\lambda_G(\xi_i)}$, $\| \ \|_{\lambda_G(\eta)}$ on $\lambda_G(\xi_i)$ $(0\le i\le m)$, $\lambda_G(\eta)$.

We now briefly explain how to construct an equivariant Quillen metric $\| \ \|_{\tilde{\lambda}_G(\xi)}$ on $\tilde{\lambda}_G(\xi)$. Let $dv_X$ be the volume element on $X$ associated to $h^{TX}$. Let $\langle\ ,\ \rangle_{\Lambda(T^{*(0,1)})\hat{\otimes}\xi}$ be the Hermitian product induced by $h^{TX}$, $h^\xi$ on $\Lambda(T^{*(0,1)}X)\hat{\otimes}\xi$. If $s, s'\in E$, set

$$(3.26)\qquad \langle s, s'\rangle = \left(\frac{1}{2\pi}\right)^{\dim X}\int_X \langle s, s'\rangle_{\Lambda(T^{*(0,1)})\hat{\otimes}\xi}\, dv_X.$$

This Hermitian product is $G$-invariant.

Let $v^*$ be the adjoint of $v$ with respect to $h^\xi$. Then $v^*$ acts as an odd operator on $\Lambda(T^{*(0,1)}X)\hat{\otimes}\xi$. Let $\bar{\partial}^{X*}$ be the formal adjoint of $\bar{\partial}^X$ with respect to $\langle\ ,\ \rangle$. Set

$$(3.27)\qquad K = \{e\in E,\ (\bar{\partial}^X + v)s = 0,\ (\bar{\partial}^{X*}+v^*)s = 0\}.$$

By Hodge theory, we have a canonical identification of $\mathbf{Z}$-graded $G$-spaces $H^*(E, \bar{\partial}^X + v)\simeq K$. Let $|\ |_{\tilde{\lambda}_G(\xi)}$ be the equivariant metric on $\tilde{\lambda}_G(\xi)$ induced on $\tilde{\lambda}_G(\xi) = (\det(H(E, \bar{\partial}^X + v), G))^{-1}$ by the restriction of the $L_2$ metric $\langle\ ,\ \rangle$ on $K$.

Let $K^\perp$ be the orthogonal space to $K$ in $E$. Then $(\bar{\partial}^X + v + \bar{\partial}^{X*} + v^*)^2$ acts as an invertible operator on $K^\perp$. Let $P$, $P^\perp$ be the orthogonal projection operators from $E$ on $K$, $K^\perp$.

For $g\in G$, $s\in\mathbb{C}$, $\mathrm{Re}(s) > \dim X$, set

$$(3.28)\qquad \tilde{\theta}_\xi^X(g)(s) = -\mathrm{Tr}_s[g(N_V^X - N_H)[(\bar{\partial}^X + v + \bar{\partial}^{X*} + v^*)^2]^{-s}P^\perp].$$

The same arguments as in (2.14), (2.15) show that $s \to \tilde{\theta}_\xi^X(g)(s)$ extends to a meromorphic function which is holomorphic at $s = 0$.

**Definition 3.3.** For $g \in G$, set

$$\log(\| \|^2_{\tilde{\lambda}_G(\xi)})(g) = \log(| |^2_{\tilde{\lambda}_G(\xi)})(g) - \frac{\partial \tilde{\theta}_\xi^X}{\partial s}(g)(0). \tag{3.29}$$

Then $\lambda_G(\xi)$, $\tilde{\lambda}_G(\xi)$, $\lambda_G(\eta)$ are equipped with Quillen metrics. We equip the inverses or the tensor products of such sums of lines with the inverses or the tensor products of the corresponding Quillen metrics.

Tautologically, by (3.25), we get

$$\|\sigma\|_{\lambda_G^{-1}(\eta)\otimes\lambda_G(\xi)} = \|\rho\|_{\lambda_G^{-1}(\eta)\otimes\tilde{\lambda}_G(\xi)}\|\tau\|_{\tilde{\lambda}_G^{-1}(\xi)\otimes\lambda_G(\xi)}. \tag{3.30}$$

The purpose of this paper is to calculate the central functions of $g \in G$ which appear in (3.30).

By Theorem 2.5, once we know how to calculate (3.30) for one choice of metrics $h^{TX}$, $h^{\xi_0}, \cdots, h^{\xi_m}$, $h^{TY}$, $h^\eta$, with $h^{TX}$, $h^{TY}$ Kähler, we also get a formula for (3.30) for arbitrary metrics $h'^{TX}$, $h'^{\xi_0}, \cdots, h'^{\xi_m}$, $h'^{TY}$, $h^\eta$, with $h'^{TX}$, $h'^{TY}$ Kähler. This is why we are free to impose as many restrictions as needed on the choice of these metrics.

d. *Assumptions on the metric on $TX$, $TY$, $\xi$, $\eta$.* Our first basic assumption is that the $G$-invariant metric $h^{TX}$ is Kähler. Also we assume that $h^{TY}$ is the metric induced by $h^{TX}$ on $TY$.

Let $\omega^{TX}$, $\omega^{TY}$ be the Kähler forms of $(X, h^{TX})$, $(Y, h^{TY})$. Then $\omega^{TX}$, $\omega^{TY}$ are $G$-invariant $(1, 1)$ closed forms, and moreover

$$\omega^{TY} = i^* \omega^{TX}. \tag{3.31}$$

Let $N_{Y/X}$ be the normal bundle to $Y$ in $X$. On $Y$, we have the exact sequence of holomorphic vector bundles

$$0 \to TY \to TX|_Y \to N_{Y/X} \to 0. \tag{3.32}$$

Then, $TY$ and $TX|_Y$ are $G$-bundles. Therefore $N_{Y/X}$ is also a $G$-bundle.

We identify $N_{Y/X}$ with the orthogonal bundle to $TY$ in $TX|_Y$. Let $h^{N_{Y/X}}$ be the metric induced by $h^{TX|_Y}$ on $N_{Y/X}$. Then $h^{N_{Y/X}}$ is $G$-invariant.

For $y \in Y$, let $H_y(\xi, v)$ be the homology of the complex $(\xi, v)_y$. If $y \in Y$, $u \in TX_y$, let $\partial_u v(y)$ be the derivative of $v$ at $y$ in the direction $u$ in any given holomorphic trivialization of $(\xi, v)$ near $y$.

Then using the local uniqueness of resolutions [29, Chapter IV], [17, Theorem 8], the following results were proved in [5, §1b)].

• $H_y(\xi, v)$ are the fibers of a holomorphic $\mathbf{Z}$-graded vector bundle $H(\xi, v)$ on $Y$. The map $\partial_u v(y)$ acts on $H(\xi, v)_y$ as a chain map, and this action does not depend on the trivialization of $(\xi, v)$ near $y$, and only depends on the image $z$ of $U$ in $N_{Y/X, y}$. From now on, we will write $\partial_z v(y)$ instead of $\partial_u v(y)$.

• Let $\pi$ be the projection $N_{Y/X} \to Y$. Then over $N_{Y/X}$, we have a canonical identification of $\mathbf{Z}$-graded chain complexes

$$(3.33) \qquad (\pi^* H(\xi, v), \partial_z v) \simeq (\pi^*(\Lambda N^*_{Y/X} \otimes \eta), \sqrt{-1} i_z).$$

Clearly, $G$ acts on both complexes in (3.33) by holomorphic chain maps. It is then easy to verify that the canonical identification (3.33) is an identification of $G$-bundles.

By finite dimensional Hodge theory, we know that for any $y \in Y$, there is a canonical isomorphism of $\mathbf{Z}$-graded vector spaces

$$(3.34) \qquad H(\xi, v)_y \simeq \{f \in \xi_y, \ vf = 0, \ v^* f = 0\}.$$

The identification (3.34) induces an identification of smooth $G$-vector bundles on $Y$. The vector bundle $H(\xi, v)$ can then be considered as a smooth $\mathbf{Z}$-graded $G$-vector subbundle of $\xi$. Let $h^{H(\xi, v)}$ be the induced metric on $H(\xi, v)$. This metric is $G$-invariant.

Let $h^{\Lambda(N^*_{Y/X}) \otimes \eta}$ be the metric induced by $h^{N_{Y/X}}$ and $h^\eta$ on $\Lambda(N^*_{Y/X}) \otimes \eta$. This metric is again $G$-invariant.

**Definition 3.4.** We say that the metrics $h^{\xi_0}, \cdots, h^{\xi_m}$ verify assumption (A) with respect to $h^{N_{Y/X}}$, $h^\eta$ if the identification (3.33) also identifies the metrics.

**Proposition 3.5.** *There exist G-invariant metrics* $h^{\xi_0}, \cdots, h^{\xi_m}$ *on* $\xi_0, \cdots, \xi_m$ *which verify assumption* (A) *with respect to* $h^{N_{Y/X}}$, $h^\eta$.

*Proof.* By [5, Proposition 1.6], there exist metrics $h'^{\xi_0}, \cdots, h'^{\xi_m}$ on $\xi_0, \cdots, \xi_m$ which verify assumption (A) with respect to the $G$-invariant metrics $h^{N_{Y/X}}$, $h^\eta$. By averaging $h'^{\xi_0}, \cdots, h'^{\xi_m}$ with respect to the Haar measure on $G$, we obtain $G$-invariant metrics $h^{\xi_0}, \cdots, h^{\xi_m}$ which have the required property. q.e.d.

In the sequel, we assume that $h^{\xi_0}, \cdots, h^{\xi_m}$ verify assumption (A) with respect to $h^{N_{Y/X}}$, $h^\eta$.

## IV. The equivariant norm of the section $\tau$

Here we give an extension of [15, Theorem 2.1].

**Theorem 4.1.** *For any $g \in G$, the following identity holds*:

$$\log(\|\tau\|^2_{\lambda_G^{-1}(\xi)\otimes\lambda_G(\xi)})(g) = 0. \tag{4.1}$$

*Proof.* The proof of (4.1) follows the same lines as that of [15, Theorem 2.1], except an essentially new argument which is an extension of the curvature theorem of [11, Theorem 0.1] to the equivariant case for the metrics $\| \ \|_{\lambda_G(\xi)}$, in a trivial situation. For an arbitrary $g \in G$, the same local index techniques as in §2 show that as in [15], the "curvature" (which here depends on $g$) vanishes.

Also one needs an extension of a result in [13, equation (2.23)] in an equivariant context. However by splitting the considered finite dimensional complex in its irreducible components as in §1, the result of [13, equation (2.23)] can be used verbatim as in [15, §2].

Details, which are easy to fill, are left to the reader.

## V. A contour integral

This section is the obvious extension of [15, §3].

Set

$$D^X = \bar{\partial}^X + \bar{\partial}^{X*}, \qquad V = v + v^*. \tag{5.1}$$

For $u > 0$, $T \geq 0$, set

$$B_{u,T} = u(D^X + TV). \tag{5.2}$$

Then the operators in (5.1), (5.2) act on the **Z**-graded vector space $E$. As explained in the introduction, $\mathrm{Tr}_s$ is our notation for the supertrace.

In the sequel, $g \in G$ is fixed once and for all.

**Theorem 5.1.** *Let $\beta_{u,T}$ be the* 1*-form on $\mathbf{R}_+^* \times \mathbf{R}_+^*$*:

$$\begin{aligned}\beta_{u,T} = {} & \frac{du}{u}\mathrm{Tr}_s[(N_V^X - N_H)g\exp(-B^2_{u,T})] \\ & - \frac{dT}{T}\mathrm{Tr}_s[N_H g\exp(-B^2_{u,T})].\end{aligned} \tag{5.3}$$

*Then $\beta_{u,T}$ is closed.*

*Proof.* Clearly $g$ is an even operator which commutes with the operators $\bar{\partial}^X$, $v$, $\bar{\partial}^{X*}$, $v^*$, $N_V^X$, $N_H$. The proof of Theorem 5.1 is then formally identical to that of [15, Theorem 3.5].

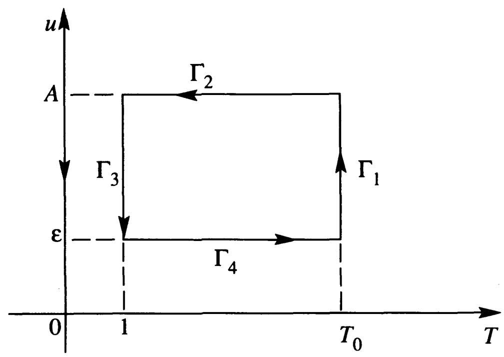


FIGURE 1

Take $\varepsilon$, $A$, $T$, $0 < \varepsilon \leq 1 \leq A < +\infty$, $1 \leq T_0 < +\infty$. Let $\Gamma = \Gamma_{\varepsilon, A, T_0}$ be the oriented contour in $\mathbf{R}_+^* \times \mathbf{R}_+^*$ indicated on Figure 1.

The contour $\Gamma$ is made of the four oriented pieces $\Gamma_1, \cdots, \Gamma_4$ indicated above. For $1 \leq k \leq 4$, set

$$I_k^0 = \int_{\Gamma_k} \beta. \tag{5.4}$$

**Theorem 5.2.** *The following identity holds*:

$$\sum_1^4 I_k^0 = 0. \tag{5.5}$$

*Proof.* This follows from Theorem 5.1.

**Remark 5.3.** As in [15], we will prove Theorem 0.1 by making in succession $A \to +\infty$, $T_0 \to +\infty$, $\varepsilon \to 0$ in (5.5).

## VI. A singular equivariant Bott-Chern current

In this section, we construct equivariant Bott-Chern currents associated to the Hermitian chain complex $((\xi, v), h^\xi)$. Namely if $g \in G$, $X_g = \{x \in X, \ gx = x\}$, and $Y_g = Y \cap X_g$, then we construct a current $T_g(\xi, h^\xi)$ on $X_g$, which verifies equation (0.8). Our construction uses results of [5]. Thus we extend [14] to the equivariant setting.

This section is organized as follows. Part a contains various short exact sequences of holomorphic Hermitian vector bundles naturally associated to the equivariant immersion problem. In part b, by the superconnection formalism of Quillen [26], we construct equivariant Chern character superconnection forms on $X_g$. In part c we use the results of [5] to establish convergence results for these currents as a parameter $u$ tends to $+\infty$. Finally in part d, by extending [14], we construct the Bott-Chern current $T_g(\xi, h^\xi)$.

This section is the obvious extension of [15, §4] to the equivariant context.

a. *Equivariant short exact sequences.* Take $g \in G$ and set

$$X_g = \{x \in X,\ gx = x\}, \qquad Y_g = \{y \in Y,\ gy = y\}. \tag{6.1}$$

Then $X_g$ and $Y_g$ are compact complex manifolds and

$$Y_g = Y \cap X_g. \tag{6.2}$$

Also since $g$ is an isometry,

$$\begin{aligned} TX_g &= \{U \in TX|_{X_g},\ gU = U\}, \\ TY_g &= \{U \in TY|_{Y_g},\ gU = U\}, \end{aligned} \tag{6.3}$$

and so

$$TY_g = TY|_{Y_g} \cap TX_g|_{Y_g}. \tag{6.4}$$

In particular $TX_g|_{Y_g} + TY|_{Y_g}$ is a subbundle of $TX|_{Y_g}$. Let $\widetilde{N}$ be the excess normal bundle

$$\widetilde{N} = \frac{TX|_{Y_g}}{TX_g|_{Y_g} + TY|_{Y_g}}. \tag{6.5}$$

We have the exact sequence of holomorphic Hermitian vector bundles on $Y_g$

$$E : 0 \to TY|_{Y_g} \to TX|_{Y_g} \to N_{Y/X}|_{Y_g} \to 0. \tag{6.6}$$

Of course $g$ acts on $E$ as a holomorphic parallel isometry.

Let $E^0$ be the subcomplex of $E$ associated to the eigenvalue 1 of $g$, and let $E^{0,\perp}$ be the direct sum of the subcomplexes of $E$ associated to eigenvalues of $g$ distinct of 1. Then $E$ splits holomorphically as

$$E = E^0 \oplus E^{0,\perp}. \tag{6.7}$$

Let $N_{Y_g/X_g}$ be the normal bundle to $Y_g$ in $X_g$. Then $E^0$, $E^{0,\perp}$ are given by

$$\begin{aligned} E^0 &: 0 \to TY_g \to TX_g|_{Y_g} \to N_{Y_g/X_g} \to 0, \\ E^{0,\perp} &: 0 \to N_{Y_g/Y} \to N_{X_g/X}|_{Y_g} \to \widetilde{N} \to 0. \end{aligned} \tag{6.8}$$

As explained before, the metric $h^{TX|_{Y_g}}$ induces metrics on all the vector bundles appearing in (6.8). In particular, we see that $X_g$ and $Y$ intersect orthogonally along $Y_g$, i.e., $N_{Y_g/Y}$ and $TX_g|_{Y_g}$ are orthogonal in $TX|_{Y_g}$.

Also we have the exact sequence

$$F : 0 \to N_{Y_g/X_g} \oplus N_{Y_g/Y} \to N_{Y_g/X} \to \widetilde{N} \to 0. \tag{6.9}$$

Again $g$ acts on $F$. Using the same conventions as in (6.8), we get

$$\begin{aligned} F^0 &: 0 \to N_{Y_g/X_g} \to N_{Y_g/X_g} \to 0 \to 0, \\ F^{0,\perp} &: 0 \to N_{Y_g/Y} \to N_{X_g/X}|_{Y_g} \to \widetilde{N} \to 0, \end{aligned} \tag{6.10}$$

and also

$$F = F^0 \oplus F^{0,\perp}. \tag{6.11}$$

Observe that

$$E^{0,\perp} = F^{0,\perp}. \tag{6.12}$$

Over $X_g$, $g$ acts on $(\xi, v)|X_g$ as a holomorphic unitary chain map. Let $e^{i\theta'_1}, \cdots, e^{i\theta'_{q'}}$ $(0 \le \theta'_j < 2\pi)$ be the distinct locally constant eigenvalues of $g$. Then $(\xi, v)|_{X_g}$ splits holomorphically and metrically as a direct sum of complexes $(\xi^{\theta'_j}, v)$ on which $g$ acts by multiplication by $e^{i\theta'_j}$.

b. *The equivariant Chern character superconnection forms.*

**Definition 6.1.** Let $\mathrm{ch}_g(\xi, h^\xi)$, $\mathrm{ch}'_g(\xi, h^\xi)$ be the closed differential forms on $X_g$, i.e.,

$$\begin{aligned} \mathrm{ch}_g(\xi, h^\xi) &= \sum_1^m (-1)^i \,\mathrm{ch}_g(\xi_i, h^{\xi_i}), \\ \mathrm{ch}'_g(\xi, h^\xi) &= \sum_1^m (-1)^i i \,\mathrm{ch}_g(\xi_i, h^{\xi_i}). \end{aligned} \tag{6.13}$$

Set

$$\xi_+ = \bigoplus_{i \text{ even}} \xi_i, \qquad \xi_- = \bigoplus_{i \text{ odd}} \xi_i. \tag{6.14}$$

Then $\xi = \xi_+ \oplus \xi_-$ is a $\mathbf{Z}_2$-graded vector bundle.

Let $\nabla^\xi = \bigoplus_{i=0}^m \nabla^{\xi_i}$ be the holomorphic Hermitian connection on $(\xi, h^\xi) = \bigoplus_{i=0}^m (\xi^i, h^{\xi_i})$. Clearly $V = v + v^*$ is a selfadjoint section of $\mathrm{End}^{\mathrm{odd}}(\xi)$.

For $u > 0$, set

$$C_u = \nabla^\xi + \sqrt{u}V. \tag{6.15}$$

Then $C_u$ is a $G$-invariant superconnection [26] on the $\mathbf{Z}_2$-graded vector bundle $\xi$. Since $C_u$ is $G$-invariant, over $X_g$ we have

$$[C_u, g] = 0. \tag{6.16}$$

By definition [26], $C_u^2$ is the curvature of $C_u$. It is a smooth section of $(\Lambda(T_{\mathbf{R}}^* X) \hat{\otimes} \operatorname{End}(\xi))^{\text{even}}$. Let $\Phi$ be the map: $\alpha \in \Lambda(T_{\mathbf{R}}^* X_g) \to (2i\pi)^{-\deg\alpha/2}\alpha \in \Lambda(T_{\mathbf{R}}^* X_g)$.

By the same arguments as in [26], the forms $\Phi \operatorname{Tr}_s[g \exp(-C_u^2)]$ over $X_g$ are closed, and their cohomology class does not depend on $u$. For $u = 0$, the forms $\Phi \operatorname{Tr}_s[g \exp(-C_u^2)]$ are standard equivariant Chern character forms. As in [11, Theorem 1.9], we find that the forms $\Phi \operatorname{Tr}_s[g \exp(-C_u^2)]$ and $\Phi \operatorname{Tr}_s[N_H g \exp(-C_u^2)]$ lie in $P^{X_g}$.

**Theorem 6.2.** *For $u > 0$, the following equality of forms holds on $X_g$:*

$$\frac{\partial}{\partial u} \Phi \operatorname{Tr}_s[g \exp(-C_u^2)] = \frac{\bar{\partial}\partial}{2i\pi} \Phi \operatorname{Tr}_s[N_H g \exp(-C_u^2)]. \tag{6.17}$$

*Proof.* By (6.16), the proof of (6.17) is the same as that of [11, Theorem 1.15].

c. *Convergence of equivariant superconnection forms.* Let $C^1(X_g)$ be the vector space of forms on $X_g$, which are continuous with continuous first derivatives, and let $\| \, \|_{C^1(X_g)}$ be a natural norm on $C^1(X_g)$. Here we use the notation of Definition 2.4.

**Theorem 6.3.** *There exists a constant $C > 0$ such that for any $\mu \in C^1(X_g)$, $u \geq 1$,*

(6.18)

$$\left| \int_{X_g} \mu \Phi \operatorname{Tr}_s[g \exp(-C_u^2)] - \int_{Y_g} \mu \operatorname{Td}_g^{-1}(N_{Y/X}, h^{N_{Y/X}}) \operatorname{ch}_g(\eta, h^\eta) \right| \leq \frac{C}{\sqrt{u}} \|\mu\|_{C^1(X_g)},$$

$$\left| \int_{X_g} \mu \Phi \operatorname{Tr}_s[N_H g \exp(-C_u^2)] + \int_{Y_g} \mu (\operatorname{Td}_g^{-1})'(N_{Y/X}, h^{N_{Y/X}}) \operatorname{ch}_g(\eta, h^\eta) \right| \leq \frac{C}{\sqrt{u}} \|\mu\|_{C^1(X_g)}.$$

*Proof.* The proof of this result is essentially identical to that of [5, Theorems 5.1 and 5.4]. The only difference is that $\exp(-C_u^2)$ is replaced by $g\exp(-C_u^2)$, but this does not introduce any difference in the analysis.

In effect, let $\nabla^{H(\xi,v)} = \bigoplus_{i=0}^m \nabla^{H^i(\xi,v)}$ be the holomorphic Hermitian connection on $(H(\xi,v), h^{H(\xi,v)}) = \bigoplus_{i=0}^m (H^i(\xi,v), h^{H^i(\xi,v)})$. As we saw in §3d, if $z \in N_{Y/X}$, then $\partial_z v$ acts on $H(\xi,v)$. Let $\partial_z^* v$ be the adjoint of $\partial_z v$. If $Z = z + \bar{z} \in N_{Y/X,\mathbf{R}}$, set

$$\partial_Z V = \partial_z v + \partial_z v^*. \tag{6.19}$$

By (3.33), there is $C > 0$ such that if $f \in H(\xi,v)$, then

$$|\partial_Z V f|^2 \geq C|Z|^2_{N_{Y/X,\mathbf{R}}} |f|^2. \tag{6.20}$$

Of course (3.33) gives an explicit description of $H(\xi,v)$, $\partial_z v = \sqrt{-1}i_z$, and by assumption (A), $\partial_z v^* = -\sqrt{-1}i_z^*$.

Let $B$ be the superconnection on $\pi^* H(\xi,v)$, i.e.,

$$B = \pi^* \nabla^{H(\xi,v)} + \partial_Z V. \tag{6.21}$$

Recall that $N_{Y_g/X_g}$ is a subbundle of $N_{Y/X}|_{X_g}$. Then by [5, Theorems 5.1 and 5.4], there exists $C > 0$ such that for $\mu \in C^1(X_g)$, $u \geq 1$,

$$\begin{aligned} &\left| \int_{X_g} \mu \Phi \operatorname{Tr}_s[g\exp(-C_u^2)] - \int_{Y_g} \mu \int_{N_{Y_g/X_g}} \Phi \operatorname{Tr}_s[g\exp(-B^2)] \right| \\ &\qquad \leq \frac{C}{\sqrt{u}} \|\mu\|_{C^1(X_g)}, \\ &\left| \int_{X_g} \mu \Phi \operatorname{Tr}_s[N_H g\exp(-C_u^2)] - \int_{Y_g} \mu \int_{N_{Y_g/X_g}} \Phi \operatorname{Tr}_s[N_H g\exp(-B^2)] \right| \\ &\qquad \leq \frac{C}{\sqrt{u}} \|\mu\|_{C^1(X_g)}. \end{aligned} \tag{6.22}$$

Let $\nabla^{\Lambda(N^*_{Y_g/X_g})}$, $\nabla^{\Lambda(\widetilde{N}^*)}$ be the holomorphic Hermitian connections on $\Lambda(N^*_{Y_g/X_g})$, $\Lambda(\widetilde{N}^*)$. We still denote by $N_H$ the number operators of $\Lambda(N^*_{Y_g/X_g})$, $\Lambda(\widetilde{N}^*)$. If $Z = z + \bar{z} \in N_{Y_g/X_g,\mathbf{R}}$, $z \in N_{Y_g/X_g}$, set

$$V'(Z) = \sqrt{-1}(i_z - i_z^*). \tag{6.23}$$

Let $\pi'$ be the projection $N_{Y_g/X_g} \to Y_g$. Let $B'$ be the superconnection on $\pi'^* \Lambda(N^*_{Y_g/X_g})$, i.e.,

$$B' = \pi'^* \nabla^{\Lambda(N^*_{Y_g/X_g})} + V'(Z). \tag{6.24}$$

Then using assumption (A), which guarantees that the canonical identification (3.33) identifies the metrics, and also (6.8), we get

(6.25)

$$\begin{aligned}
&\int_{N_{Y_g/X_g}} \Phi \operatorname{Tr}_s[g\exp(-B^2)] \\
&\quad = \Phi \operatorname{Tr}_s[g\exp(-\nabla^{\Lambda(\widetilde{N}^*)})^2] \int_{N_{Y_g/X_g}} \Phi \operatorname{Tr}_s[g\exp(-B'^2)]\operatorname{ch}_g(\eta, h^\eta), \\
&\int_{N_{Y_g/X_g}} \Phi \operatorname{Tr}_s[N_H g\exp(-B^2)] \\
&\quad = \Big\{ \Phi \operatorname{Tr}_s[N_H g\exp(-(\nabla^{\Lambda(\widetilde{N}^*)})^2)] \\
&\qquad \times \int_{N_{Y_g/X_g}} \Phi \operatorname{Tr}_s[g\exp(-B'^2)] + \Phi \operatorname{Tr}_s[g\exp(-(\nabla^{\Lambda(\widetilde{N}^*)})^2)] \\
&\qquad\qquad \times \int_{N_{Y_g/X_g}} \Phi \operatorname{Tr}_s[N_H \exp(-B'^2)] \Big\} \operatorname{ch}_g(\eta, h^\eta).
\end{aligned}$$

We have the trivial equalities:

(6.26)

$$\begin{aligned}
&\Phi \operatorname{Tr}_s[g\exp(-(\nabla^{\Lambda(\widetilde{N}^*)})^2)] = \prod_{\theta_j \neq 0} \left(\frac{\mathrm{Td}}{\mathrm{e}}\right)^{-1} \left(\frac{-R^{N^{\theta_j}_{Y/X}}}{2i\pi} + i\theta_j\right), \\
&\Phi \operatorname{Tr}_s[N_H g\exp(-(\nabla^{\Lambda(\widetilde{N}^*)})^2)] \\
&\quad = -\frac{\partial}{\partial b}\left[\prod_{\theta_j \neq 0} \left(\frac{\mathrm{Td}}{\mathrm{e}}\right)^{-1} \left(\frac{-R^{N^{\theta_j}_{Y/X}}}{2i\pi} + i\theta_j + b\right)\right]_{b=0},
\end{aligned}$$

and also

(6.27)

$$\begin{aligned}
&\int_{N_{Y_g/X_g}} \Phi \operatorname{Tr}_s[g\exp(-B'^2)] = \mathrm{Td}^{-1}\left(\frac{-R^{N_{Y_g/X_g}}}{2i\pi}\right), \\
&\int_{N_{Y_g/X_g}} \Phi \operatorname{Tr}_s[N_H \exp(-B'^2)] = -\frac{\partial}{\partial b}\mathrm{Td}^{-1}\left(\frac{-R^{N_{Y_g/X_g}}}{2i\pi} + b\right)_{b=0}
\end{aligned}$$

by [25, Theorem 4.5], [5, Theorem 3.2]. From (6.22)–(6.27), we thus arrive at (6.18).

**Remark 6.4.** As in [5, Theorem 3.2], one can also show that the convergence in (6.18) takes place microlocally, and obtain corresponding estimates for corresponding microlocal semi-norms.

d. *Equivariant Bott-Chern currents.* Now we imitate [14]. Let $\delta_{Y_g}$ be the current of integration of $Y_g$.

**Definition 6.5.** For $s \in \mathbf{C}$, $0 < \mathrm{Re}(s) < 1/2$, let $R_g(\xi, h^\xi)(s)$ be the current on $X_g$:

$$\begin{aligned} &R_g(\xi, h^\xi)(s) \\ &= \frac{1}{\Gamma(s)} \int_0^{+\infty} u^{s-1} \Big[\Phi \operatorname{Tr}_s[N_H g \exp(-C_u^2)] \\ &\qquad + (\mathrm{Td}_g^{-1})'(N_{Y/X}, h^{N_{Y/X}}) \operatorname{ch}_g(\eta, h^\eta) \delta_{Y_g}\Big]\, du. \end{aligned} \tag{6.28}$$

By Theorem 6.3, the current $R_g(\xi, h^\xi)(s)$ is well defined.

**Definition 6.6.** Let $T_g(\xi, h^\xi)$ be the current on $X_g$, i.e.,

$$T_g(\xi, h^\xi) = \frac{\partial}{\partial s} R_g(\xi, h^\xi)(0). \tag{6.29}$$

One finds easily that $T_g(\xi, h^\xi)$ is given by the formula

$$\begin{aligned} T_g(\xi, h^\xi) &= \int_0^1 \Phi \operatorname{Tr}_s[N_H g(\exp(-C_u^2) - \exp(-C_0^2))] \frac{du}{u} \\ &+ \int_1^{+\infty} \Big(\Phi \operatorname{Tr}_s[N_H \exp(-C_u^2)] \\ &\qquad + (\mathrm{Td}_g^{-1})'(N_{Y/X}, h^{N_{Y/X}}) \operatorname{ch}_g(\eta, h^\eta)\delta_{Y_g}\Big) \frac{du}{u} \\ &- \Gamma'(1)\{\operatorname{ch}_g'(\xi, h^\xi) + (\mathrm{Td}_g^{-1})'(N_{Y/X}, h^{N_{Y/X}})\delta_{Y_g}\}. \end{aligned} \tag{6.30}$$

**Theorem 6.7.** *The current $T_g(\xi, h^\xi)$ is a sum of currents of type $(p, p)$. Also*

$$\frac{\bar\partial \partial}{2i\pi} T_g(\xi, h^\xi) = \mathrm{Td}_g^{-1}(N_{Y/X}, h^{N_{Y/X}}) \operatorname{ch}_g(\eta, h^\eta)\delta_{Y_g} - \operatorname{ch}_g(\xi, h^\xi). \tag{6.31}$$

*Proof.* It is clear that $T_g(\xi, h^\xi)$ is a sum of currents of type $(p, p)$. Equation (6.31) follows from Theorems 6.2 and 6.3.

**Remark 6.8.** As in [14], one can show that the wave front set $\mathrm{WF}(T_g(\xi, h^\xi))$ of $T_g(\xi, h^\xi)$ is included in $N^*_{Y_g/X_g, \mathbf{R}}$.

## VII. The analytic torsion forms of an equivariant short exact sequence

In this section, we describe the construction in [7] of analytic torsion forms associated to a short exact sequence $0 \to L \to M \to N \to 0$ of holomorphic Hermitian vector bundles equipped with a parallel isometry

$g$. The construction of [7] extends the construction of [6] to an equivariant situation. Also we describe the main result of [7], which is the evaluation of these analytic torsion forms in terms of Bott-Chern classes and an additive genus $D(\theta, x)$.

This section is organized as follows. In part a, we recall various results on Clifford algebras. Part b gives a formula for the curvature $\mathscr{B}_u^2$ of the superconnection $\mathscr{B}_u$ considered in [6], [7]. In part c, we construct the generalized supertrace of $g \exp(-\mathscr{B}_u^2)$, which is a smooth differential form on the considered manifold. In part d, we recall the results of [7] on the asymptotics of these forms as $u \to 0$ and $u \to +\infty$. Part e reviews the construction given in [7] of analytic torsion forms. In part f, following [7] we evaluate these forms in terms of Bott-Chern classes and the genus $D(\theta, x)$. Finally part g contains a formula for $D(\theta, x)$ as a power series in $x$, introduces the genus $R(\theta, x)$ already given in (0.10), and reviews formulas of [7] for $\mathbf{R}(\theta, x)$.

This section is self-contained. It is the extension of [15, §5] to the equivariant setting. In the sequel, its results will be applied to the exact sequence $0 \to TY_g \to TX|_{Y_g} \to N_{Y/X}|_{Y_g} \to 0$ on $Y_g$.

a. *Clifford algebras and complex vector spaces.* Let $V$ be a complex Hermitian vector space of complex dimension $k$. Let $\overline{V}$ be the conjugate vector space. If $z \in V$, then $z$ represents $Z = z + \bar{z} \in V_{\mathbf{R}}$, so that $|Z|^2 = 2|z|^2$. Let $J \in \operatorname{End}(V_{\mathbf{R}})$ be the complex structure of $V_{\mathbf{R}}$.

Let $c(V_{\mathbf{R}})$ be the Clifford algebra of $V_{\mathbf{R}}$, i.e., the algebra generated by 1, $U \in V_{\mathbf{R}}$, and the commutation relations $UU' + U'U = -2\langle U, U'\rangle$. Then $\Lambda(\overline{V}^*)$ and $\Lambda(V^*)$ and Clifford modules. Namely if $X \in V$, $X' \in \overline{V}$, let $X^* \in \overline{V}^*$, $X'^* \in V^*$ correspond to $X$, $X'$ by the Hermitian product of $V$. Set

$$(7.1)\qquad \begin{aligned} c(X) &= \sqrt{2}X^*\wedge, & c(X') &= -\sqrt{2}i_{X'}, \\ \hat{c}(X) &= \sqrt{2}i_X, & \hat{c}(X') &= -\sqrt{2}X'^*\wedge. \end{aligned}$$

Note that our conventions in (7.1) for $\hat{c}$ from the conventions in [15, §5a)] (where our $\hat{c}(U)$ is $\hat{c}(JU)$ in [15]) and fit with the conventions in [7].

If $U, U' \in V_{\mathbf{R}} \otimes_{\mathbf{R}} \mathbf{C}$, then

$$(7.2)\qquad \begin{aligned} c(U)c(U') + c(U')c(U) &= -2\langle U, U'\rangle, \\ \hat{c}(U)\hat{c}(U') + \hat{c}(U')\hat{c}(U) &= -2\langle U, U'\rangle. \end{aligned}$$

Also $c(U)$, $\hat{c}(U)$ act as odd operators on $\Lambda(\overline{V}^*)\hat{\otimes}\Lambda(V^*)$. If $U, U' \in V_{\mathbf{R}} \otimes_{\mathbf{R}} \mathbf{C}$, then

$$c(U)\hat{c}(U') + \hat{c}(U')c(U) = 0. \tag{7.3}$$

b. *A formula for* $\mathscr{B}_u^2$. Let $B$ be a compact complex manifold. Let

$$E : 0 \to L \to M \to N \to 0 \tag{7.4}$$

be a short exact sequence of holomorphic vector bundles on $B$. Let $l$, $m$, $n$ be the dimensions of $L$, $M$, $N$ respectively. We identify $L$ with a subbundle of $M$, and $N$ with $M/L$. Let $\pi$ be the projection $M \to B$.

Let $h^M$ be a Hermitian metric on $M$. Let $h^L$ be the induced metric on $L$. By identifying $N$ to the orthogonal bundle to $L$ in $M$, $N$ inherits a Hermitian metric $h^N$. Let $P^L$, $P^N$ be the orthogonal projection operators from $M$ on $L$, $N$. Let $\nabla^L$, $\nabla^M$, $\nabla^N$ be the holomorphic Hermitian connections on $L$, $M$, $N$, and let $R^L$, $R^M$, $R^N$ be their curvatures. Then classically,

$$\nabla^L = P^L \nabla^M, \qquad \nabla^N = P^N \nabla^M. \tag{7.5}$$

Let ${}^0\nabla^M$ be the connection on $M$, i.e.,

$${}^0\nabla^M = \nabla^L \oplus \nabla^N. \tag{7.6}$$

Set

$$A = \nabla^M - {}^0\nabla^M. \tag{7.7}$$

Then $A$ is 1-form on $B$ with values in skew-adjoint endomorphisms of $M$, which exchange $L$ and $N$.

Let $e_1, \cdots, e_{2n}$ be an orthonormal base of $N_{\mathbf{R}}$.

**Definition 7.1.** Let $S \in \mathrm{End}^{\mathrm{even}}(\Lambda(\overline{N}^*) \otimes \Lambda(N^*))$ be given by

$$S = \frac{\sqrt{-1}}{2} \sum_1^{2n} c(e_i)\hat{c}(e_i). \tag{7.8}$$

Let $f_1, \cdots, f_{2k}$ be a basis of $T_{\mathbf{R}}B$, and let $f^1, \cdots, f^{2k}$ be the dual basis of $T^*_{\mathbf{R}}B$.

**Definition 7.2.** If $Z \in M_{\mathbf{R}}$, set

$$\hat{c}(AP^L Z) = -\sum_1^{2k} f^j \wedge \hat{c}(A(f_j)P^L Z). \tag{7.9}$$

**Definition 7.3.** If $y \in B$, then $I_y$ (resp. $J_y$) denotes the vector space of smooth section of $(\Lambda(\overline{M}^*)\hat{\otimes}\Lambda(N^*))_y$ (resp. $(\Lambda(\overline{N}^*)\hat{\otimes}(N^*))_y$), over the fibre $M_{\mathbf{R},y}$.

Let $R^{\Lambda(N^*)}$ be the obvious action of $R^N$ of $\Lambda(N^*)$. Then $R^{\Lambda(N^*)}$ acts on $\Lambda(\overline{M}^*)\hat{\otimes}\Lambda(N^*)$ like $1\hat{\otimes}R^{\Lambda(N^*)}$.

Let $e_1, \cdots, e_{2m}$ be an orthonormal basis of $M_{\mathbf{R}}$.

**Definition 7.4.** For $u > 0$, let $\mathscr{B}_u^2 \in (\Lambda(T_{\mathbf{R}}^*B)\hat{\otimes}\operatorname{End}(J))^{\text{even}}$ be given by

$$\mathscr{B}_u^2 = -\frac{1}{2}\sum_1^{2m}\left(\nabla_{e_i} + \frac{1}{2}\langle R^M Y, e_i\rangle\right)^2 + \frac{u|P^N Z|^2}{2} + \sqrt{u}S + \frac{\sqrt{-u}}{\sqrt{2}}\hat{c}(AP^L Z) + \frac{1}{2}\operatorname{Tr}[R^M] + R^{\Lambda(N^*)}. \tag{7.10}$$

c. *Generalized supertraces.* Let $dv_M$, $dv_N$ be the volume forms on the fibers $M_{\mathbf{R}}$, $N_{\mathbf{R}}$. The smooth kernels on the fibers of $M_{\mathbf{R}}$ will be calculated with respect to the volume form $dv_M/(2\pi)^{\dim M}$.

For $y \in B$, $u > 0$, let $Q_u^y(Z, Z')$ $(Z, Z' \in M_{\mathbf{R},y})$ be the smooth kernel associated to $\exp(-\mathscr{B}_u^{2,y})$. For the existence and uniqueness of $Q_u^y(Z, Z')$, we refer to [6, §4 a)]. Then

$$Q_u^y(Z, Z') \in (\Lambda(T_{\mathbf{R}}^*B)\hat{\otimes}\operatorname{End}(\Lambda(\overline{N}^*)\hat{\otimes}\Lambda(N^*)))_y^{\text{even}}.$$

By [7, Theorem 1.6], for $u > 0$, there exist $C > 0$, $C' > 0$, $C'' > 0$ such that for $y \in B$, $Z, Z' \in M_{\mathbf{R},y}$, then

$$|Q_u^y(Z, Z')| \leq C\exp\left(-\frac{|P^L(Z - Z')|^2}{2} + C'(|P^L Z| + |P^L Z'|) - C''(|P^N Z|^2 + |P^N Z'|^2)\right). \tag{7.11}$$

Let $g$ be a smooth section of $\operatorname{End}(M)$, which preserves $L$. Then $g$ acts naturally on $L$ and $N$.

We assume that $g$ is an isometry of $M$, which is parallel with respect to $\nabla^M$. Then $g$ also acts as an isometry of $L$, $N$, which is parallel with respect to $\nabla^L$, $\nabla^N$. So $g$ acts on the complex $E$.

Let $e^{i\theta_1}, \cdots, e^{i\theta_q}$ $(0 \leq \theta_j < 2\pi)$ be the locally constant distinct eigenvalues of $g$ acting on $L$, $M$, $N$. Then $E$ splits holomorphically as an orthogonal sum of complexes

$$E^{\theta_j} : 0 \to L^{\theta_j} \to M^{\theta_j} \to N^{\theta_j} \to 0, \tag{7.12}$$

and $g$ acts on $E^{\theta_j}$ by multiplication by $e^{i\theta_j}$. Moreover $M^{\theta_j}$ inherits a metric $h^{M^{\theta_j}}$ from the metric $h^M$ on $M$.

Set

$$E^{0,\perp} = \bigoplus_{\theta_j \neq 0} E^{\theta_j}. \tag{7.13}$$

Then $E$ splits holomorphically and metrically as

$$E = E^0 \oplus E^{0,\perp}. \tag{7.14}$$

Take $y \in Y$. If $s \in I_y$, let $gs \in I_y$ be given by

$$gs(Z) = gs(g^{-1}Z). \tag{7.15}$$

Then $g\exp(-\mathscr{B}_u^2)$ acts on $\Lambda(T_{\mathbf{R}}^*B)\hat{\otimes}I$, and the corresponding kernel is given by $gP_u(g^{-1}Z, Z')$.

Clearly $g$ acts on $\Lambda(\overline{L}^{0,\perp,*})\hat{\otimes}\Lambda(\overline{N}^*)\hat{\otimes}\Lambda(N^*)$ and so on $\Lambda(T_{\mathbf{R}}^*B)\,\hat{\otimes}\Lambda(\overline{L}^{0,\perp,*})\hat{\otimes}\Lambda(\overline{N}^*)\hat{\otimes}\Lambda(N^*)$. Also $Q_u(Z, Z')$ acts on the same bundle (it acts trivially on $\Lambda(\overline{L}^{0,\perp,*})$). Therefore

$$gQ_u(g^{-1}Z, Z) \in (\Lambda(T_{\mathbf{R}}^*B)\hat{\otimes}\operatorname{End}(\Lambda(\overline{L}^{0,\perp,*})\hat{\otimes}\Lambda(\overline{N}^*)\hat{\otimes}\Lambda(N^*)))^{\text{even}}.$$

Let $\operatorname{Tr}_s[gQ_u(g^{-1}Z, Z')] \in \Lambda(T_{\mathbf{R}}^*B)$ denote the corresponding supertrace.

By [7, equation (2.10)], which itself follows from (7.11), we find that given $u > 0$, there exist $C > 0$, $C' > 0$ such that for $y \in B$, $Z \in M_{\mathbf{R},y}^{0,\perp} \oplus N_{\mathbf{R},y}^0$,

$$|Q_u^y(g^{-1}Z, Z)| \le C\exp(-C'|Z|^2). \tag{7.16}$$

Let $N_H$ be the number operator of $\Lambda(N^*)$. Then $N_H$ acts like $1\hat{\otimes}N_H$ on $\Lambda(T_{\mathbf{R}}^*B)\hat{\otimes}\Lambda(\overline{L}^{0,\perp,*})\hat{\otimes}\Lambda(\overline{N}^*)\hat{\otimes}\Lambda(N^*)$.

In view of (7.16) and following [7, Definition 2.1], we now set the following definition.

**Definition 7.5.** For $u > 0$, set

$$\begin{aligned} &\operatorname{Tr}_s[g\exp(-\mathscr{B}_u^2)] \\ &\quad = \int_{M_{\mathbf{R}}^{0,\perp}\oplus N_{\mathbf{R}}^0} \operatorname{Tr}_s[gQ_u(g^{-1}Z, Z)]\frac{dv_{M^{0,\perp}\oplus N^0}(Z)}{(2\pi)^{\dim M^{0,\perp}+\dim N^0}}, \\ &\operatorname{Tr}_s[N_H g\exp(-B_U^2)] \\ &\quad = \int_{M_{\mathbf{R}}^{0,\perp}\oplus N_{\mathbf{R}}^0} \operatorname{Tr}_s[N_H gQ_u(g^{-1}Z, Z)]\frac{dv_{M^{0,\perp}\oplus N^0}(Z)}{(2\pi)^{\dim M^{0,\perp}+\dim N^0}}. \end{aligned} \tag{7.17}$$

The objects constructed in (7.17) are smooth forms on $B$.

d. *Convergence of generalized supertraces.* If $u \in \mathbf{R}_+ \to \omega_u$ is a family of smooth forms on $B$, we write that as $u \to 0$,

$$\omega_u = \omega_0 + \mathscr{O}(u) \tag{7.18}$$

if for any $k \in \mathbf{N}$, there is $C_k > 0$ such that the norm of $\omega_u - \omega_0$ in $C^k(B)$ is dominated by $C_k u$.

We define $P^B$, $P^{B,0}$ as in Definition 2.3, by simply replacing $X_g$ by $B$.

Now we recall a result of [7].

**Theorem 7.6.** *For $u > 0$, the form $\Phi \operatorname{Tr}_s[g \exp(-B_u^2)]$ is closed, lies in $P^B$, and its cohomology class does not depend on $u > 0$. The forms $\Phi \operatorname{Tr}_s[N_H g \exp(-B_u^2)]$ also lie in $P^B$. Moreover*

$$\frac{\partial}{\partial u} \Phi \operatorname{Tr}_s[g \exp(-\mathscr{B}_u^2)] = \frac{\bar{\partial}\partial}{2i\pi} \Phi \operatorname{Tr}_s \left[ \frac{N_H}{u} g \exp(-\mathscr{B}_u^2) \right] ; \tag{7.19}$$

*as $u \to 0$,*

$$\begin{aligned} \Phi \operatorname{Tr}_s[g \exp(-\mathscr{B}_u^2)] &= \operatorname{Td}_g(M, h^M) \operatorname{Td}_g^{-1}(N, h^N) + \mathscr{O}(u), \\ \Phi \operatorname{Tr}_s[N_H g \exp(-\mathscr{B}_u^2)] &= -(\operatorname{Td}_g^{-1})'(N, h^N) \operatorname{Td}_g(M, h^M) + \mathscr{O}(u); \end{aligned} \tag{7.20}$$

*as $u \to +\infty$,*

$$\begin{aligned} \Phi \operatorname{Tr}_s[g \exp(-\mathscr{B}_u^2)] &= \operatorname{Td}_g(L, h^L) + \mathscr{O}\left(\frac{1}{\sqrt{u}}\right), \\ \Phi \operatorname{Tr}_s[N_H g \exp(-\mathscr{B}_u^2)] &= \frac{\dim N}{2} \operatorname{Td}_g(L, h^L) + \mathscr{O}\left(\frac{1}{\sqrt{u}}\right). \end{aligned} \tag{7.21}$$

*Proof.* The results stated in our theorem are proved in [7, Theorems 2.5, 3.2, and 5.3].

e. *Generalized analytic torsion forms.* Now we reproduce the construction given in [7, §6] of analytic torsion forms.

**Remark 7.7.** By using the techniques of §13 and proceeding as in [15, §14], one can give a new proof of (7.21).

**Definition 7.8.** For $s \in \mathbf{C}$, $0 < \operatorname{Re}(s) < \frac{1}{2}$, let $A(s)$ be the form on $B$:

$$\begin{aligned} A(s) = \frac{1}{\Gamma(s)} \int_0^{+\infty} u^{s-1} \{ &\Phi \operatorname{Tr}_s[N_H g \exp(-\mathscr{B}_u^2)] \\ &- \frac{\dim N}{2} \operatorname{Td}_g(L, h^L) \} \, du. \end{aligned} \tag{7.22}$$

By (7.20), (7.21), one verifies that $s \to A(s)$ extends to a function which is holomorphic near $s = 0$.

**Definition 7.9.** Set

$$\mathbf{B}_g(L, H, h^M) = \frac{\partial A}{\partial s}(0). \tag{7.23}$$

By [7, equation (6.3)], we have

$$\begin{aligned}\mathbf{B}_g(L, M, h^M) = &\int_0^1 \{\Phi \operatorname{Tr}_s[N_H g \exp(-\mathscr{B}_u^2)] \\ &\qquad + \operatorname{Td}_g(M, h^M)(\operatorname{Td}_g^{-1})'(N, h^N)\} \frac{du}{u} \\ &+ \int_1^{+\infty} \left\{\Phi \operatorname{Tr}_s[N_H g \exp(-B_u^2)] \right. \\ &\qquad \left. - \frac{\dim N}{2} \operatorname{Td}_g(L, h^L)\right\} \frac{du}{u} \\ &+ \Gamma'(1) \left\{\operatorname{Td}_g(M, h^M)(\operatorname{Td}_g^{-1})'(N, h^N) \right. \\ &\qquad \left. + \frac{\dim N}{2} \operatorname{Td}_g(L, h^L)\right\}.\end{aligned} \tag{7.24}$$

The following result is proved in [7, Theorem 6.3].

**Theorem 7.10.** *The form* $\mathbf{B}_g(L, M, h^M)$ *lies in* $P^B$. *Moreover,*

$$\frac{\bar{\partial}\partial}{2i\pi}\mathbf{B}_g(L, M, h^M) = \operatorname{Td}_g(L, h^L) - \frac{\operatorname{Td}_g(M, h^M)}{\operatorname{Td}_g(N, h^N)}. \tag{7.25}$$

f. *Evaluation of the generalized analytic torsion forms.* For $u \in \mathbf{C}$, $\eta \in \mathbf{C}$, $x \in \mathbf{C}$, set

(7.26)

$$\sigma(u, \eta, x) = 4 \sinh\left(\frac{x - 2\eta + \sqrt{x^2 + 4u}}{4}\right) \sinh\left(\frac{-x + 2\eta + \sqrt{x^2 + 4u}}{4}\right).$$

In the sequel, $\theta \in \mathbf{R}$, and $x \in \mathbf{C}$ are such that $|x| < 2\pi$ if $\theta \in 2\pi\mathbf{Z}$, and $|x| < \inf_{k \in \mathbf{Z}} |\theta + 2k\pi|$ if $\theta \notin 2\pi\mathbf{Z}$. Then by [7, equation (6.6)], as $u \to +\infty$

$$\frac{\partial\sigma/\partial x}{\sigma}(u, i\theta, x) = \mathscr{O}\left(\frac{1}{\sqrt{u}}\right). \tag{7.27}$$

**Definition 7.11.** For $s \in \mathbf{C}$, $0 < \operatorname{Re}(s) < \frac{1}{2}$, set

$$C(s, \theta, x) = \frac{1}{\Gamma(s)} \int_0^{+\infty} u^{s-1} \frac{\partial\sigma/\partial x}{\sigma}(u, i\theta, -x)\, du. \tag{7.28}$$

Then $s \to C(s, \theta, x)$ extends to a holomorphic function near $s = 0$.

**Definition 7.12.** For $\theta \in \mathbf{R}$, $x \in \mathbf{C}$, $|x| < 2\pi$ if $\theta \in 2\pi\mathbf{Z}$, and $|x| < \inf_{k\in\mathbf{Z}} |\theta + 2k\pi|$ if $\theta \notin 2\pi\mathbf{Z}$, put

$$D(\theta, x) = \partial C(0, \theta, x)/\partial s. \tag{7.29}$$

Then $D(\theta, x)$ is a periodic function of $\theta$ with period $2\pi$. Also $D(\theta, x)$ is holomorphic in $x$ on its domain of definitions.

For $\theta \in \mathbf{R}$, we identify $D(\theta, x)$ with the corresponding additive genus. Set

$$D(\theta_j, N^{\theta_j}, h^{N^{\theta_j}}) = \operatorname{Tr}[D(\theta_j, -R^{N^{\theta_j}}/2i\pi)]. \tag{7.30}$$

Then $D(\theta_j, N^{\theta_j}, h^{N^{\theta_j}})$ lies in $P^B$ and is closed.

Now we follow [7, Definition 6.7].

**Definition 7.13.** Set

$$D_g(N, h^N) = \sum D(\theta_j, N^{\theta_j}, h^{N^{\theta_j}}). \tag{7.31}$$

The class of $\operatorname{Td}_g(L, h^L)D_g(N, h^N)$ in $P^B/P^{B,0}$ does not depend on the metric $h^L = \bigoplus h^{L^{\theta_j}}$, $h^N = \bigoplus h^{N^{\theta_j}}$. We denote this class by $\operatorname{Td}_g(L)D_g(N)$.

Let $\widetilde{\operatorname{Td}}_g(L, M, h^M)$ be the Bott-Chern class in $P^B/P^{B,0}$ constructed in [11, Theorem 1.29], such that

$$\frac{\bar\partial\partial}{2i\pi}\widetilde{\operatorname{Td}}_g(L, M, h^M) = \operatorname{Td}_g(M, h^M) - \operatorname{Td}_g(L, h^L)\operatorname{Td}_g(N, h^N). \tag{7.32}$$

The following result is proved in [7, Theorem 6.8].

**Theorem 7.14.** *The following equality holds*:

$$\begin{aligned}\mathbf{B}_g(L, M, h^M) = &-\operatorname{Td}_g^{-1}(N, h^N)\widetilde{\operatorname{Td}}_g(L, M, h^M)\\ &+\operatorname{Td}_g(L)D_g(N) \quad \textit{in } P^B/P^{B,0}.\end{aligned} \tag{7.33}$$

g. *Evaluation of the function* $D(\theta, x)$.

**Definition 7.15.** For $y \in \mathbf{R}$, $s \in \mathbf{C}$, $\operatorname{Re}(s) > 1$, set

$$\zeta(y, s) = \sum_{n=1}^{+\infty} \frac{\cos(ny)}{n^s}, \qquad \eta(y, s) = \sum_{n=1}^{+\infty} \frac{\sin(ny)}{n^s}. \tag{7.34}$$

Then for a fixed $y \in \mathbf{R}$, both functions in (7.34) extend to a holomorphic function of $s$ for $\operatorname{Re}(s) < 1$.

We now recall a result of [7, Theorem 7.2].

**Theorem 7.16.** *For* $\theta \in \mathbf{R}$, $x \in \mathbf{C}$, $|x| < 2\pi$ *if* $\theta \in 2\pi\mathbf{Z}$, *and* $|x| < \inf_{k\in\mathbf{Z}} |\theta + 2k\pi|$ *if* $\theta \notin 2\pi\mathbf{Z}$, $D(\theta, x)$ *is given by the convergent power* *series*

$$D(\theta, x) = \sum_{\substack{n\geq 0 \\ n \text{ even}}} i\left(\left(\Gamma'(1) + \sum_1^n \frac{1}{j}\right)\eta(\theta, -n) + \frac{2\partial\eta}{\partial s}(\theta, -n)\right)\frac{x^n}{n!} + \sum_{\substack{n\geq 1 \\ n \text{ odd}}} \left(\left(\Gamma'(1) + \sum_1^n \frac{1}{j}\right)\zeta(\theta, -n) + \frac{2\partial\zeta}{\partial s}(\theta, -n)\right)\frac{x^n}{n!}. \tag{7.35}$$

Recall that the Hirzebruch polynomial $\widehat{A}(x)$ is given by

$$\widehat{A}(x) = \frac{x/2}{\sinh(x/2)}. \tag{7.36}$$

Set

$$\begin{aligned} \alpha(x, \theta) &= \widehat{A}(x) && \text{if } \theta \in 2\pi\mathbf{Z}, \\ &= \widehat{A}(x + i\theta)/(x + i\theta) && \text{if } \theta \notin 2\pi\mathbf{Z}. \end{aligned} \tag{7.37}$$

**Definition 7.17.** For $\theta \in \mathbf{R}$, $x \in \mathbf{C}$, $|x| < 2\pi$ if $\theta \in 2\pi\mathbf{Z}$, and $|x| < \inf_{k\in\mathbf{Z}} |\theta + 2k\pi|$ if $\theta \notin 2\pi\mathbf{Z}$, set

$$R(\theta, x) = D(\theta, x) - \Gamma'(1)\frac{\partial\widehat{\alpha}/\partial x}{\widehat{\alpha}}(\theta, x). \tag{7.38}$$

Again $R(\theta, x)$ is a periodic function of $\theta$ with period $2\pi$.

The following result was established in [7, Theorem 7.7].

**Theorem 7.18.** *For $\theta \in \mathbf{R}$, $x \in \mathbf{C}$, $|x| < 2\pi$ if $\theta \in 2\pi\mathbf{Z}$, and $|x| < \inf_{k\in\mathbf{Z}} |\theta + 2k\pi|$ if $\theta \notin 2\pi\mathbf{Z}$, we have*

$$R(\theta, x) = \sum_{\substack{n\geq 0 \\ n \text{ even}}} i\left(\sum_1^n \frac{1}{j}\eta(\theta, -n) + \frac{2\partial\eta}{\partial s}(\theta, -n)\right)\frac{x^n}{n!} + \sum_{\substack{n\geq 1 \\ n \text{ odd}}} \left(\sum_1^n \frac{1}{j}\zeta(\theta, -n) + \frac{2\partial\zeta}{\partial s}(\theta, -n)\right)\frac{x^n}{n!}. \tag{7.39}$$

Now we recall the definition of the function $R(x)$ by Gillet and Soulé [20]. Let $\zeta(s) = \sum_{n=1}^{+\infty} 1/n^s$ be the Riemann zeta function.

**Definition 7.19.** For $x \in \mathbf{C}$, $|x| < 2\pi$, set

$$R(x) = \sum_{\substack{n\geq 1 \\ n \text{ odd}}} \left(\sum_1^n \frac{1}{j}\zeta(-n) + \frac{2\partial\zeta}{\partial s}(-n)\right)\frac{x^n}{n!}. \tag{7.40}$$

Clearly

$$R(0, x) = R(x). \tag{7.41}$$

**Definition 7.20.** For $x \in \mathbf{C}$, $|x| < 2\pi$, set

$$\rho(x) = \sum_{\substack{n \geq 1 \\ n \text{ odd}}} 2\zeta'(-n)\frac{x^n}{n!}. \tag{7.42}$$

Finally, we recall results in [7, Theorems 7.8 and 7.11].

**Theorem 7.21.** *If* $\theta \in ]-2\pi, 2\pi[ \setminus \{0\}$, *if* $x \in \mathbf{C}$, $|x| < \inf_{k \in \mathbf{Z}} |\theta + 2k\pi|$, *then*

$$\begin{aligned} R(\theta, x) = R(x + i\theta) &+ i \sum_{k \in \mathbf{Z}^*} \frac{\log(1 + \theta/2k\pi)}{2k\pi - i(x + i\theta)} \\ &+ \frac{2\Gamma'(1) - \log(\theta^2) - \log(1 - ix/\theta)}{x + i\theta}. \end{aligned} \tag{7.43}$$

*Also for* $\theta \in \mathbf{R}$,

$$\begin{aligned} R(\theta, 0) &= \frac{2i\partial\eta}{\partial s}(\theta, 0), \\ R(\theta, 0) &= \rho(i\theta) + \frac{2\Gamma'(1) - \log(\theta^2)}{i\theta}. \end{aligned} \tag{7.44}$$

## VIII. A formula for $\log(\|\rho\|^2_{\lambda_G^{-1}(\eta) \otimes \tilde{\lambda}_G(\xi)})(g)$

This section extends [15, §6] to the equivariant situation. Namely, we give a formula for $\log(\|\rho\|^2_{\lambda_G^{-1}(\eta) \otimes \tilde{\lambda}_G(\xi)})(g)$, which combined with Theorem 4.1 implies Theorem 0.1. This formula is the main result of the paper.

To establish our main result, we proceed as in [15]. Namely we start from the identity $\sum_{k=1}^4 I_k^0 = 0$ of Theorem 5.2. Then we state seven intermediate results, the proofs of six of which are delayed to the next sections. These results have a strong formal resemblance with the corresponding results in [15]. We can thus formally import from [15] most of the discussion on the asymptotics of the $I_0^k$'s as $A \to +\infty$, $T_0 \to +\infty$, $\varepsilon \to 0$. This leads us very quickly to the proof of our main result.

This section is organized as follows. In part a, we state our main result. In part b, we introduce a rescaled metric on $E$. In part c, we state our seven intermediate results. In part d, following [15, §6 d], we discuss very briefly the asymptotics of the $I_0^k$'s. Part e gives a local formula for $\log(\|\rho\|^2_{\lambda_G^{-1}(\eta) \otimes \tilde{\lambda}_G(\xi)})(g)$ in terms of $T_g(\xi, h^\xi)$ and $\mathbf{B}_g(TY|_{Y_g}, TX|_{Y_g}, h^{TX|_{Y_g}})$. In part f, using the results of [7] which were recalled in §7, we establish our main result.

a. *The main theorem.* We now state the main result of this paper, whose proof occupies §§8–13. It extends [15, Theorem 6.1].

**Theorem 8.1.** *For $g \in G$, the following equality holds*:

$$\log(\|\rho\|^2_{\lambda_G^{-1}(\eta)\otimes\tilde\lambda_G(\xi)})(g) = -\int_{X_g} \mathrm{Td}_g(TX, h^{TX})T_g(\xi, h^\xi)$$

$$(8.1)\qquad + \int_{Y_g} \mathrm{Td}_g^{-1}(N_{Y/X}, h^{N_{Y/X}})\widetilde{\mathrm{Td}}_g(TY|_{Y_g}, TX|_{Y_g}, h^{TX|_{Y_g}})\,\mathrm{ch}_g(\eta, h^\eta)$$

$$- \int_{Y_g} \mathrm{Td}_g(TX)R_g(N_{Y/X})\,\mathrm{ch}_g(\eta),$$

$$\log(\|\rho\|^2_{\lambda_G^{-1}(\eta)\otimes\tilde\lambda_G(\xi)})(g) = -\int_{X_g} \mathrm{Td}_g(TX, h^{TX})T_g(\xi, h^\xi)$$

$$(8.1')\qquad + \int_{Y_g} \mathrm{Td}_g^{-1}(N_{Y/X}, h^{N_{Y/X}})\widetilde{\mathrm{Td}}_g(TY|_{Y_g}, TX|_{Y_g}, h^{TX|_{Y_g}})\,\mathrm{ch}_g(\eta, h^\eta)$$

$$- \int_{X_g} \mathrm{Td}_g(TX)R_g(TX)\,\mathrm{ch}_g(\xi) + \int_{Y_g} \mathrm{Td}_g(TY)R_g(TY)\,\mathrm{ch}_g(\eta).$$

*Proof.* The remainder of the paper is devoted to the proof of Theorem 8.1.

b. *A rescaled metric on $E$.*

**Definition 8.2.** For $T > 0$, we denote by $\langle\ ,\ \rangle_T$ the Hermitian product on $E$ associated with the metrics $h^{TX}$, $h^{\xi_0}$, $h^{\xi_1}/T^2, \cdots, h^{\xi_m}/T^{2m}$ on $TX, \xi_0, \cdots, \xi_m$ respectively. Set

$$(8.2)\qquad K_T = \{s \in E;\ (\bar\partial^X + v)s = 0;\ (\bar\partial^{X*} + T^2 v^*)s = 0\}.$$

Let $P_T$ be the orthogonal projection operator from $E$ on $K_T$ with respect to the Hermitian product $\langle\ ,\ \rangle_T$.

In (3.27), we saw that for any $T > 0$, there is a canonical isomorphism of **Z**-graded $G$-spaces,

$$(8.3)\qquad K_T \cong H^*(E, \bar\partial^X + v).$$

Let $|\ |_{\tilde\lambda_G(\xi), T}$ be the equivariant metric on $\tilde\lambda_G(\xi)$ inherited from the Hermitian product $\langle\ ,\ \rangle_T$ restricted to $K_T$. Clearly, with the notation of §3b, we have

$$K_1 = K;\quad P_1 = P;\quad |\ |_{\tilde\lambda_G(\xi),1} = |\ |_{\tilde\lambda_G(\xi)}.$$

For $T > 0$, set

$$(8.4)\qquad \widetilde K_T = \{s \in E;\ (D^X + TV)s = 0\}.$$

Let $\widetilde{P}_T$ be the orthogonal projection operator from $E$ on $\widetilde{K}_T$ with respect to the Hermitian product $\langle\ ,\ \rangle = \langle\ ,\ \rangle_1$ on $E$. Then we have the simple formula of [15, equation (6.5)]

$$T^{-N_H}(\bar{\partial}^X + v + \bar{\partial}^{X^*} + T^2 v^*)T^{N_H} = D^X + TV. \tag{8.5}$$

From (8.5), we deduce that

$$\widetilde{P}_T = T^{-N_H} P_T T^{N_H}. \tag{8.6}$$

The map $s \in K_T \to T^{-N_H}s \in \widetilde{K}_T$ is an isomorphism of $\mathbf{Z}$-graded $G$-spaces. We thus find that as a $\mathbf{Z}$-graded $G$-space, $\widetilde{K}_T$ is also isomorphic to $H^*(E, \bar{\partial}^X + v)$.

Set

$$D^Y = \bar{\partial}^Y + \bar{\partial}^{Y^*}. \tag{8.7}$$

For $1 \leq j \leq d$, let $D^{Y_j}$ be the restriction of $D^Y$ to $Y_j$.

Let $Q$ be the orthogonal projection operator from $F$ on $K' = \ker(D^Y)$ with respect to the given Hermitian product on $F$.

c. *Seven intermediate results.* Recall that $\omega^{TX}$, $\omega^{TY}$ are the Kähler forms of $X$, $Y$. Since these forms are closed, they can be paired with characteristic classes of vector bundles on $X_g$, $Y_g$ respectively.

For $0 \leq i \leq m$, $1 \leq j \leq d$, $g \in G$, set

$$\chi_g(\xi_i) = \mathrm{Tr}_s^{H(X,\xi_i)}[g], \qquad \chi_g(\eta|_{Y_j}) = \mathrm{Tr}_s^{H(X_j,\eta|_{Y_j})}[g]. \tag{8.8}$$

Then by the Lefschetz fixed point formula of Atiyah-Bott [1],

$$\begin{aligned} \chi_g(\xi_i) &= \int_{X_g} \mathrm{Td}_g(TX)\,\mathrm{ch}_g(\xi_i), \\ \chi_g(\eta|_{Y_j}) &= \int_{Y_{j,g}} \mathrm{Td}_g(TY)\,\mathrm{ch}_g(\eta|_{Y_j}). \end{aligned} \tag{8.9}$$

In the sequel, we will often use the notation

$$\begin{aligned} \int_{Y_g} \dim Y\,\mathrm{Td}_g(TY)\,\mathrm{ch}_g(\eta) &= \sum_{j=1}^{d} \dim Y_j \int_{Y_{j,g}} \mathrm{Td}_g(TY_j)\,\mathrm{ch}_g(\eta|_{Y_j}), \\ \dim N_{Y/X}(\eta) &= \sum_1^d \dim N_{Y_j/X}\chi_g(\eta|_{Y_j}). \end{aligned} \tag{8.10}$$

In the sequel, $g \in G$ is fixed once and for all.

We now state in Theorems 8.3 to 8.9 seven intermediate results which play an essential role in the proof of Theorem 8.1. The proofs of Theorems 8.4–8.9 are deferred to §§9–13.

**Theorem 8.3.** *As* $u \to 0$,

$$\begin{aligned}
&\operatorname{Tr}_s[(N_V^X - N_H) g \exp(-u(D^X + V)^2)] \\
&\quad = \frac{1}{u}\int_{X_g} \frac{\omega^{TX}}{2\pi} \operatorname{Td}_g(TX)\operatorname{ch}_g(\xi) \\
&\quad + \int_{X_g} [\dim X \operatorname{Td}_g(TX)\operatorname{ch}_g(\xi) \\
&\qquad - \operatorname{Td}'_g(TX)\operatorname{ch}_g(\xi) - \operatorname{Td}_g(TX)\operatorname{ch}'_g(\xi)] \\
&\quad + \mathscr{O}(u), \\
&\operatorname{Tr}_s[N_V^Y g \exp(-uD^{Y,2})] \\
&\quad = \frac{1}{u}\int_{Y_g} \frac{\omega^{TY}}{2\pi} \operatorname{Td}_g(TY)\operatorname{ch}_g(\eta) \\
&\quad + \int_{Y_g} [\dim Y \operatorname{Td}_g(TY) - \operatorname{Td}'_g(TY)]\operatorname{ch}_g(\eta) \\
&\quad + \mathscr{O}(u).
\end{aligned} \tag{8.11}$$

*Proof.* First we prove the second equality in (8.11). For $t > 0$, set $h_t^{TY} = h^{TY}/t$. Let $*^{TY}$, $*_t^{TY}$ be the star operators associated to the metrics $h^{TY}$, $h_t^{TY}$. Clearly, when acting on $\Lambda(T^{*(0,1)}Y) \otimes \eta$,

$$*_t^{TY} = t^{-\dim Y} *^{TY} t^{N_V^Y}, \tag{8.12}$$

and so

$$(*_t^{TY})^{-1}\frac{\partial *_t^{TY}}{\partial t} = \frac{N_V^Y - \dim Y}{t}. \tag{8.13}$$

By (2.38), (2.39), (2.44), (2.63), we see that as $u \to 0$

$$\begin{aligned}
\operatorname{Tr}_s[(N_V - \dim Y) g \exp(-uD^{Y,2})] &= \frac{1}{u}\int_{Y_g} \frac{\omega^{TY}}{2\pi}\operatorname{Td}_g(TY)\operatorname{ch}_g(\eta) \\
&\quad - \int_{Y_g} \operatorname{Td}'_g(TY)\operatorname{ch}_g(\eta) + \mathscr{O}(u).
\end{aligned} \tag{8.14}$$

Also using the McKean-Singer formula [24] and the Lefschetz formula of Atiyah-Bott [1], we get

$$\operatorname{Tr}_s[g\exp(-uD^{Y,2})] = \int_{Y_g} \operatorname{Td}_g(TY)\operatorname{ch}_g(\eta), \tag{8.15}$$

which together with (8.14) gives the second identity in (8.11).

The proof of the first equality follows from the same line. It is left to the reader.

**Theorem 8.4.** *For any* $\alpha_0 > 0$, *there exists* $C > 0$ *such that for* $\alpha \geq \alpha_0$, $T \geq 1$

$$|\operatorname{Tr}_s[N_H g \exp(-\alpha(D^X + TV)^2)] - \frac{1}{2}\dim N_{Y/X}\chi_g(\eta)| \leq C/\sqrt{T},$$

$$|\operatorname{Tr}_s[(N_V^X - N_H) g \exp(-\alpha(D^X + TV)^2)] - \operatorname{Tr}_s[N_V^Y g \exp(-\alpha D^{Y,2})]| \leq C/\sqrt{T}. \tag{8.16}$$

**Theorem 8.5.** *There exist* $c > 0$, $C > 0$ *such that for* $\alpha \geq 1$, $T \geq 1$,

$$|\operatorname{Tr}_s[(N_V^X - N_H) g \exp(-\alpha(D^X + TV)^2)] - \operatorname{Tr}_s[(N_V^X - N_H) g \widetilde{P}_T]| \leq \exp(-C\alpha). \tag{8.17}$$

**Theorem 8.6.** *There exist* $C > 0$, $\gamma \in ]0, 1]$, *such that for* $u \in ]0, 1]$, $0 \leq T \leq 1/u$,

$$|\operatorname{Tr}_s[N_H g \exp(-(uD^X + TV)^2)] - \int_{X_g} \operatorname{Td}_g(TX, h^{TX}) \times \Phi \operatorname{Tr}_s[N_H g \exp(-C_{T^2}^2)]| \leq C(u(1+T)^\gamma). \tag{8.18}$$

*Moreover, there exists a constant* $C' > 0$ *such that for* $u \in ]0, 1]$, $0 \leq T \leq 1$,

$$|\operatorname{Tr}_s[N_H g \exp(-(uD^X + TV)^2)] - \operatorname{Tr}_s[N_H g \exp(-(uD^X)^2)]| \leq C'T. \tag{8.19}$$

Consider the exact sequence of holomorphic Hermitian vector bundles on $Y_g$

$$E : 0 \to TY|_{Y_g} \to TX|_{Y_g} \to N_{Y/X}|_{Y_g} \to 0. \tag{8.20}$$

Clearly $g \in G$ acts as a parallel isometry on $E$. In the sequel, we use the notation of §7 applied to this exact sequence. In particular for $u > 0$, we will consider the operator $\mathscr{B}_u^2$ of Definition 7.4.

**Theorem 8.7.** *For any* $T > 0$, *the following equality holds*:

$$\lim_{u \to 0} \operatorname{Tr}_s\left[N_H g \exp\left(-\left(uD^X + \frac{T}{u}V\right)^2\right)\right] = \int_{Y_g} \Phi \operatorname{Tr}_s[N_H g \exp(-\mathscr{B}_{T^2}^2)] \operatorname{ch}_g(\eta, h^\eta). \tag{8.21}$$

**Theorem 8.8.** *There exist* $C > 0$, $\delta \in ]0, 1]$ *such that for* $u \in ]0, 1]$, $T \geq 1$,

$$\left|\operatorname{Tr}_s\left[N_H g \exp\left(-\left(uD^X + \frac{T}{u}V\right)^2\right)\right] - \frac{1}{2}\dim N_{Y/X}\chi_g(\eta)\right| \leq \frac{C}{T^\delta}. \tag{8.22}$$

**Theorem 8.9.** *As* $T \to +\infty$,

$$\log\left(\frac{|\ |_{\tilde{\lambda}_G(\xi),T}}{|\ |_{\tilde{\lambda}_G(\xi)}}\right)^2(g) = \dim N_{Y/X}\chi_g(\eta)\log(T) - \log(|\rho|^2_{\lambda_G^{-1}(\eta)\otimes\tilde{\lambda}_G(\xi)})(g) + \varepsilon(1/T). \tag{8.23}$$

**Remark 8.10.** As in [15, Remark 6.10], one immediately verifies that Theorems 8.4–8.8 are compatible with each other.

Besides, at a formal level, Theorems 8.3–8.9 can be obtained from [15, Theorems 6.3–6.9] by introducing the operator $g$ in the right place. This will permit us to transfer formally the discussion in [15, §6] to our situation.

d. *The asymptotics of the* $I_k^0$*'s.* We start from the equality in (5.5):

$$\sum_{k=1}^{4} I_k^0 = 0. \tag{8.24}$$

Because of the formal analogies with [15, §6] which were indicated before, the discussion of the asymptotics of the $I_k^0$'s as $A \to +\infty$, $T_0 \to +\infty$, $\varepsilon \to 0$ can be formally transferred from [15, §§6d and 6e]. Of course, we use here the more general Theorems 8.3–8.9, instead of [15, Theorems 6.3–6.9].

Consider the forms on $Y_g$

$$\begin{aligned} A_g(N_{Y/X}, h^{N_{Y/X}}) &= \widehat{A}\left(\frac{-R^{N^0_{Y/X}}}{2i\pi}\right)\prod_{\theta_j\neq 0}\frac{\widehat{A}}{\mathrm{e}}\left(\frac{-R^{N^{\theta_j}_{Y/X}}}{2i\pi}+i\theta_j\right), \\ \widehat{A}'_g(N_{Y/X}, h^{N_{Y/X}}) &= \frac{\partial}{\partial b}\left[\widehat{A}\left(\frac{-R^{N^0_{Y/X}}}{2i\pi}+b\right)\prod_{\theta_j\neq 0}\frac{\widehat{A}}{\mathrm{e}}\left(\frac{-R^{N^{\theta_j}_{Y/X}}}{2i\pi}+i\theta_j+b\right)\right]_{b=0}. \end{aligned} \tag{8.25}$$

Let $\widehat{A}_g(N_{Y/X})$, $\widehat{A}'_g(N_{Y/X})$ be the corresponding classes in $P^{Y_g}/P^{Y_g,0}$. Then

$$\frac{\mathrm{Td}'_g}{\mathrm{Td}_g}(N_{Y/X}, h^{N_{Y/X}}) = \frac{\widehat{A}'_g}{\widehat{A}_g}(N_{Y/X}) + \frac{1}{2}\dim N_{Y/X}. \tag{8.26}$$

Ultimately, by proceeding as indicated, we obtain an extension of [15, Theorem 6.16].

**Theorem 8.11.** *The following equality holds*:

$$\begin{aligned}\log(\| \ \|^2_{\lambda_G^{-1}(\eta)\otimes\tilde{\lambda}_G(\xi)})(g) = &-\int_{X_g} \mathrm{Td}_g(TX, h^{TX})T_g(\xi, h^\xi)\\ &-\int_{Y_g} \mathbf{B}_g(TY|_{Y_g}, TX|_{Y_g}, h^{TX|_{Y_g}})\,\mathrm{ch}_g(\eta, h^\eta)\\ &+\Gamma'(1)\int_{Y_g} \mathrm{Td}_g(TY)\frac{\widehat{A}'_g}{\widehat{A}_g}(N_{Y/X})\,\mathrm{ch}_g(\eta).\end{aligned} \tag{8.27}$$

e. *A formula for* $\log(\|\rho^2_{\lambda_G^{-1}(\eta)\otimes\tilde{\lambda}_G(\xi)})(g)$.

**Theorem 8.12.** *For* $g \in G$, *the following equality holds*:

(8.28)

$$\begin{aligned}&\log(\|\rho\|^2_{\lambda_G^{-1}(\eta)\otimes\tilde{\lambda}_G(\xi)})(g)\\ &\quad= -\int_{X_g} \mathrm{Td}_g(TX, h^{TX})T_g(\xi, h^\xi)\\ &\quad+\int_{Y_g} \mathrm{Td}_g^{-1}(N_{Y/X}, h^{N_{Y/X}})\widetilde{\mathrm{Td}}_g(TY|_{Y_g}, TX|_{Y_g}, h^{TX|_{Y_g}})\,\mathrm{ch}_g(\eta, h^\eta)\\ &\quad-\int_{Y_g} \mathrm{Td}_g(TY)\left(D_g - \Gamma'(1)\frac{\widehat{A}'_g}{\widehat{A}_g}\right)(N_{Y/X})\,\mathrm{ch}_g(\eta).\end{aligned}$$

*Proof.* Since $\mathrm{ch}_g(\eta, h^\eta) \in P^{X_g}$ is closed, Theorem 8.12 follows from Theorems 7.14 and 8.11.

f. *Proof of Theorem* 8.1. Equality (8.1) follows from (7.38) and Theorem 8.12. Let $i_g : Y_g \to X_g$ be the obvious embedding. Then we have the equalities in $H^*(Y_g)$:

$$\begin{aligned}\mathrm{Td}_g(TY) &= i_g^*\,\mathrm{Td}_g(TX)/\mathrm{Td}_g(N_{Y/X}),\\ R_g(N_{Y/X}) &= i_g^* R_g(TX) - R_g(TY).\end{aligned} \tag{8.29}$$

Using (6.31) and (8.29), we get

$$\begin{aligned}\int_{Y_g} \mathrm{Td}_g(TY)R_g(N_{Y/X})\,\mathrm{ch}_g(\eta) = &\int_{X_g} \mathrm{Td}_g(TX)R_g(TX)\,\mathrm{ch}_g(\xi)\\ &-\int_{Y_g} \mathrm{Td}_g(TY)R_g(TY)\,\mathrm{ch}_g(\eta).\end{aligned} \tag{8.30}$$

Then equality (8.1′) follows from (8.1) and (8.30).

Hence the proof of Theorem 8.1 is completed.

## IX. Proofs of Theorems 8.4 and 8.5

In this section, we give a proof of Theorems 8.4 and 8.5. This proof relies essentially on the results of [15, §§8 and 9], where the corresponding results were established when $G$ is trivial.

Let $L$ be a smooth $G$-invariant section of $\mathrm{End}(\Lambda(T^{*(0,1)}X)\hat{\otimes}\xi)^{\text{even}}$. The section $L^\theta$ of $\mathrm{End}(\Lambda(T^{*(0,1)}Y)\hat{\otimes}\eta)$ was defined in [15, §8b]. Again, $L^\theta$ is $G$-invariant.

First, we state the obvious analogues of [15, Theorems 8.2 and 8.3].

**Theorem 9.1.** *For any $\alpha_0 > 0$, there is $C > 0$ such that for $\alpha \geq \alpha_0$, $T \geq 1$,*

$$(9.1)\qquad |\mathrm{Tr}[Lg\exp(-\alpha(D^X + TV)^2)] - \mathrm{Tr}[L^\theta g\exp(-\alpha D^{Y,2})]| \leq C/\sqrt{T}.$$

**Theorem 9.2.** *There exist $c > 0$, $C > 0$ such that for $\alpha \geq 1$, $T \geq 1$,*

$$(9.2)\qquad |\mathrm{Tr}[Lg\exp(-\alpha(D^X + TV)^2)] - \mathrm{Tr}[Lg\widetilde{P}_T]| \leq c\exp(-C\alpha).$$

**Remark 9.3.** As in [15], Theorems 8.4 and 8.5 follow easily from Theorems 9.1 and 9.2.

The proof of Theorems 9.1 and 9.2 is essentially the same as that of [15, Theorems 8.2 and 8.3] given in [15, §§8 and 9].

Let $\nabla^{\Lambda(T^{*(0,1)}X)}$ be the connection induced by $\nabla^{TX}$ on $\Lambda(T^{*(0,1)}X)$. The first fundamental observation is that all the constructions of [15, §8] are $G$-equivariant. In fact these constructions involve the following:

1. In [15, §8e], an identification of a neighborhood of $Y$ in $X$ with a neighborhood of $Y$ in $N_{Y/X,\mathbf{R}}$ where $Y$ is considered as the zero section of $N_{Y/X}$, by using geodesic coordinates normal to $Y$. Now since $g \in G$ is an isometry which preserves $Y$, $g$ preserves the geodesics which are normal to $Y$. Of course under this identification, $g$ acts linearly in the fibers of $N_{Y/X,\mathbf{R}}$.

2. In [15, §8f], an orthogonal splitting of $\mathbf{Z}$-graded vector bundles $\xi = \xi^+ \oplus \xi^-$ of $\xi$ near $Y$, already considered in [5, §1]. In effect for $y \in Y$, let $\mu(y)$ be the smallest nonzero eigenvalue of $V^2(y)$. Since $\ker V^2|_Y$ is a smooth vector bundle, $\mu$ has a positive lower bound $2b$ on $Y$. If $d^X(x, Y)$ is small enough, $b$ is not an eigenvalue of $V^2(x)$. Then $\xi_x^-$ (resp. $\xi_x^+$) is the direct sum of the eigenspaces of $V^2(x)$ associated to the eigenvalues which are smaller (resp. larger) than $b$. Since $G$ acts on $\xi$ as

an isometry which preserves $V^2$, $G$ also acts on $\xi^+$ and $\xi^-$. Let $\widetilde{\nabla}^{\xi^\pm}$ be the connection on $\xi^\pm$, which is the orthogonal projection of $\nabla^\xi$ on $\xi^\pm$. Set $\widetilde{\nabla}^\xi = \widetilde{\nabla}^{\xi^+} \oplus \widetilde{\nabla}^{\xi^-}$. Then $\widetilde{\nabla}^\xi$ is $G$-invariant.

3. In [15, §8g], a trivialization of $\Lambda(T^{*(0,1)}X)\hat{\otimes}\xi$ along geodesics normal to $Y$, with respect to the connection $\nabla^{\Lambda(T^{*(0,1)}X)}\hat{\otimes}1 + 1\hat{\otimes}\widetilde{\nabla}^\xi$, which is again $G$-invariant.

Now we use the notation of [15, §9]. Let $E^0$ (resp. $F^0$) be the vector space of square integrable sections of $\Lambda(T^{*(0,1)}X)\hat{\otimes}\xi$ over $X$ (resp. $\Lambda(T^{*(0,1)}Y)\hat{\otimes}\eta$ over $Y$). Then for any $T \geq 0$, the linear isometric embedding $J_T$ of $F^0$ in $E^0$ defined in [15, Definition 9.4] is $G$-equivariant. Set $E_T^0 = J_T(F^0)$. Let $E_T^{0,\perp}$ be the orthogonal space to $E_T^0$ in $E^0$ with respect to the Hermitian product (3.26) on $E$. It follows from the previous considerations that for any $T > 0$, the orthogonal splitting $E^0 = E_T^0 \oplus E_T^{0,\perp}$ of $E^0$ considered in [15, Definition 9.4] is $G$-invariant, i.e., $G$ acts on $E_T^0$ and $E_T^{0,\perp}$.

Therefore the matrix of the unitary operator $g$ with respect to the splitting $E^0 = E_T^0 \oplus E_T^{0,\perp}$ is diagonal, and so it can be written in the form

$$g = \begin{bmatrix} g_{0,T} & 0 \\ 0 & g_{1,T} \end{bmatrix}, \tag{9.3}$$

and moreover

$$g_{0,T}J_T = J_T g. \tag{9.4}$$

The proofs of Theorems 9.1 and 9.2 then proceed as in [15, §§9g and 9h].

## X. The $L_2$ metrics on $\tilde{\lambda}_G(\xi)$ and $\tilde{\lambda}_G(\eta)$

For $T \geq 1$, let $|\ |_{\lambda_G^{-1}(\eta)\otimes\tilde{\lambda}_G(\xi),T}$ be the metric on $\lambda_G^{-1}(\eta)\otimes\tilde{\lambda}_G(\xi)$ which is associated to the $L_2$ metrics $|\ |_{\lambda_G(\eta)}$, $|\ |_{\tilde{\lambda}_G(\xi),T}$.

In this section, we will prove the following result.

**Theorem 10.1.** *For $g \in G$, as $T \to +\infty$,*

$$\log(|\rho|^2_{\lambda_G^{-1}(\eta)\otimes\tilde{\lambda}_G(\xi),T})(g) = \dim N_{Y/X}\chi_g(\eta)\log(T) + \varepsilon(1/T). \tag{10.1}$$

**Remark 10.2.** Theorems 8.9 and 10.1 are equivalent.

Clearly

$$H(Y,\eta) = \bigoplus_{j=1}^{d} H(Y_j, \eta|_{Y_j}). \tag{10.3}$$

Let $h^{H(Y,\eta)}$ be the metric on $H(Y,\eta)$ induced by the Hermitian product of $F$ on $\ker(D^Y) \simeq H(Y,\eta)$. Then the $H(Y_j, \eta|_{Y_j})$'s are mutually orthogonal in $H(Y,\eta)$ with respect to $h^{H(Y,\eta)}$.

Let $s \in E$ be such that $(\bar\partial^X + v)s = 0$. Then $P_T s$ represents the cohomology class $[s] \in H(E, \bar\partial^X + v)$ of $s$ in $K_T \simeq H(E, \bar\partial^X + v)$. In the sequel, we will write $P_T[s]$ instead of $P_T s$.

Also, we use the canonical identification of Theorem 3.2,

$$H(E, \bar\partial^X + v) \simeq H(Y, \eta). \tag{10.4}$$

Then the splitting (10.3) of $H(Y,\eta)$ induces a corresponding splitting of $H(E, \bar\partial^X + v)$.

Take $s \in H(Y_j, \eta|_{Y_j})$, $s' \in H(Y_{j'}, \eta|_{Y_j})$. Then by [15, Theorem 10.9], as $T \to +\infty$,

$$\begin{aligned} \langle P_T[s], P_T[s']\rangle_T &= \mathscr{O}(T^{-\infty}) \quad \text{if } j \neq j', \\ &= T^{-\dim N_{Y_j/X}}(\langle [s], [s']\rangle_{h^{H(Y,\eta)}} + \mathscr{O}(1/\sqrt{T})), \\ &\qquad \text{if } j = j'. \end{aligned} \tag{10.5}$$

Let $\dim N_{Y/X}$ be the operator acting on $H(Y_j, \eta|_{Y_j})$ by multiplication by $\dim N_{Y_j/X}$. Let $h_T^{H(E,\bar\partial^X+v)}$ be the $L_2$ metric on $H(E, \bar\partial^X + v)$ associated to $\langle\ ,\ \rangle_T$, and let $h_T^{H(Y,\eta)}$ be the corresponding metric on $H(Y,\eta)$. From (10.5), we deduce that

$$\log(h_T^{H(Y,\eta)}/h^{H(Y,\eta)}) = -\dim N_{Y/X}\log(T) + \mathscr{O}(1/\sqrt{T}). \tag{10.6}$$

By definition,

$$\log(|\rho|^2_{\lambda_G^{-1}(\eta)\otimes\tilde\lambda_G(\xi),T})(g) = -\operatorname{Tr}_s[g\log(h_T^{H(Y,\eta)}/h^{H(Y,\eta)})]. \tag{10.7}$$

By (10.6), (10.7), we see that as $T \to +\infty$,

$$\log(|\rho|^2_{\lambda_G^{-1}(\eta)\otimes\tilde\lambda_G(\xi),T})(g) = \operatorname{Tr}_s^{H(Y,\eta)}[g\dim N_{Y/X}]\log(T) + \mathscr{O}\left(\frac{1}{\sqrt{T}}\right). \tag{10.8}$$

Also

$$\operatorname{Tr}_s^{H(Y,\eta)}[g\dim N_{Y/X}] = \sum_1^d \dim N_{Y_j/X}\chi_g(\eta|_{Y_j}) = \dim N_{Y/X}\chi_g(\eta), \tag{10.9}$$

which together with (10.8) gives (10.1). Hence the proof of Theorem 10.1 is completed.

## XI. The analysis of the two parameters operator $g\exp(-(uD^X + TV)^2)$ in the range $u \in ]0, 1]$, $T \in [0, 1/u]$

The purpose of this section is to prove Theorem 8.6. The main point of Theorem 8.6 is the existence of $C > 0$, $\gamma \in ]0, 1]$ such that if $u \in ]0, 1]$, $T \in [0, 1/u]$, then

$$\left| \operatorname{Tr}_s[N_H g \exp(-(uD^X + TV)^2)] - \int_{X_g} \operatorname{Td}_g(TX, h^{TX}) \Phi \operatorname{Tr}_s[N_H g \exp(-C_{T^2}^2)] \right| \leq C(u(1+T))^\gamma. \tag{11.1}$$

To establish Theorem 8.6, we essentially use the methods of [15, §11], where Theorem 8.6 was established when $G$ is trivial, combined with finite propagation speed techniques. In effect, we use four main ideas, some of which are taken from [15].

- A first simple idea is that the proof of (11.1) is local on $X$, and also local near $X_g$.
- A second idea is to combine the rescaling techniques of Getzler [18] with the splitting $\xi = \xi^+ \oplus \xi^-$ of $\xi$ near $Y$, which was already considered in [5, §1], [15, §8f] and §9, together with the fixed point techniques of [9], [2].
- As in [15], functional analytic techniques play an important role in handling the difficulties related to the splitting $\xi = \xi^+ \oplus \xi^-$, in the concentration of the local supertrace on $Y$ as $T \to +\infty$, which follows from the invertibility of $V^2$ on $X \backslash Y$, and in the concentration of the local supertrace on the fixed point set $X_g$ as $u \to 0$, which ultimately forces the concentration of the local supertrace on $Y_g$.
- While the concentration of the local supertrace on $Y$ as $T \to +\infty$ is controlled by the methods of [15, §11], the concentration of the local supertrace on $X_g$, and ultimately on $Y_g$, is obtained by using finite propagation speed techniques.

Ultimately, once the considered heat kernels are adequately rescaled, we obtain a decay faster than the polynomial decay in the directions normal to $Y$, because of the presence of a harmonic oscillator in a direction normal to $Y$, and a Gaussian decay in the directions normal to $X_g$ by finite propagation speed. By slightly improving the estimates in [15], we also get a Gaussian decay in the directions normal to $Y$.

The organization and the content of this section are much related to [15, §11], to which the reader is referred when necessary. In part a, we calculate the limit as $u \to 0$ of $\operatorname{Tr}_s[N_H g \exp(-(uD^X + TV)^2)]$, and obtain the second easy half of Theorem 8.6. In part b, we show that the proof of (11.1) can be localized near $Y_g$. Parts c and d contain a construction of a coordinate system and a trivialization of $\Lambda(T^{*(0,1)}X)\hat{\otimes}\xi$ near $Y_g$. In part e, we perform a Getzler rescaling [18] on certain Clifford variables. In part f, following [15, §11j], we briefly discuss the matrix structure of the new rescaled operator $L^{3,Z_0/T}_{u,T}$ with respect to the splitting $\xi = \xi^+ \oplus \xi^-$. In part g, we obtain the key decay estimates on the rescaled kernel $P^{3,Z_0/T}_{u,T}$. Finally part h contains our proof of the estimate (11.1).

At many intermediary stages of the proofs, we rely on the results of [15, §11], which can be adapted without any change to the situation which is considered here.

a. *The limit as* $u \to 0$ *of* $\operatorname{Tr}_s[N_H g \exp(-(uD^X + TV)^2)]$.

**Proposition 11.1.** *Let* $T_0 \in [0, +\infty[$. *Then there exists* $C > 0$ *such that for* $u \in ]0, 1]$, $T \in [0, T_0]$,

$$\left| \operatorname{Tr}_s[N_H g \exp(-(uD^X + TV)^2)] - \int_{X_g} \operatorname{Td}_g(TX, h^{TX}) \Phi \operatorname{Tr}_s[N_H g \exp(-C^2_{T^2})] \right| \leq Cu, \tag{11.2}$$

$$| \operatorname{Tr}_s[N_H g \exp(-(uD^X + TV)^2)] - \operatorname{Tr}_s[N_H g \exp(-(uD^X)^2)]| \leq CT.$$

*Proof.* By proceeding as in [15, proof of Proposition 11.7] and §2, we find easily that for $T \geq 0$, as $u \to 0$

$$\operatorname{Tr}_s[N_H g \exp(-(uD^X + TV)^2)] = \int_{X_g} \operatorname{Td}_g(TX, h^{TX}) \Phi \operatorname{Tr}_s[N_H g \exp(-C^2_{T^2})] + \mathscr{O}(u).$$

Since $T$ only plays the role of a parameter, one obtains the existence of $C > 0$ such that the first identity in (11.2) holds.

Also,

$$\begin{aligned} \frac{\partial}{\partial T} & \operatorname{Tr}_s[N_H g \exp(-(uD^X + TV)^2)] \\ &= \frac{\partial}{\partial b} \operatorname{Tr}_s[N_H g \exp(-(uD^X + TV)^2 - b[uD^X + TV, V])]_{b=0} \\ &= T \frac{\partial}{\partial b} \operatorname{Tr}_s[[V, N_H] g \exp(-(uD^X + TV)^2 + bV)]_{b=0} \\ &= T \frac{\partial}{\partial b} \operatorname{Tr}_s[(v - v^*) g \exp(-(uD^X + TV)^2 + bV)]_{b=0}. \end{aligned} \tag{11.3}$$

By using again the techniques of §2, one finds that for $u \to 0$, the right-hand side of (11.3) converges boundedly for $T \leq T_0$. The second inequality in (11.2) follows.

b. *Localization of the problem near* $Y_g$.

**Definition 11.2.** For $u > 0$, $T \geq 0$, let $P_{u,T}(x, x')$ $(x, x' \in M)$ be the smooth kernel associated to the operator $\exp(-(uD^X + TV)^2)$, calculated with respect to $dv_X(x')/(2\pi)^{\dim X}$.

Then

$$\begin{aligned} &\operatorname{Tr}_s[N_H g \exp(-(uD^X + TV)^2)] \\ &\quad = \int_X \operatorname{Tr}_s[N_H g P_{u,T}(g^{-1}x, x)] \frac{dv_X(x)}{(2\pi)^{\dim X}}. \end{aligned} \tag{11.4}$$

Let $d^X$ be the Riemann distance on $(X, h^{TX})$.

**Proposition 11.3.** *For any* $\alpha > 0$, *there exist* $c > 0$, $C > 0$ *such that for* $u \in ]0, 1]$, $T \in [0, 1/u]$, $x, x' \in X$, $d^X(x, x') \geq \alpha$, *we have*

$$|P_{u,T}(x, x')| \leq c \exp(-C/u^2). \tag{11.5}$$

*Proof.* Let $\Delta$ be the Laplace-Beltrami operator associated to $h^{TX}$. For $t > 0$, let $p_t(x, x')$ be the smooth kernel associated to $e^{t\Delta}$.

Consider now the differential equation in [15, equation (11.18)]. By [15, equation (11.24)], for $u \in ]0, 1]$, $T \leq 1/u$, its solution is uniformly bounded. Making use of Ito's formula as in [15, equation (11.20)], we find that there exists $c > 0$ such that for $u \in ]0, 1]$, $T \in [0, 1/u]$, $x, x' \in X$, we have

$$|P_{u,T}(x, x')| \leq c p_{u^2}(x, x'). \tag{11.6}$$

Now classically, there exist $c' > 0$, $C' > 0$ such that for $u \in ]0, 1]$, $x, x' \in X$, $d^X(x, x') \geq \alpha$,

$$p_{u^2}(x, x') \leq c' \exp(-C'/u^2), \tag{11.7}$$

which together with (11.6) gives (11.5). q.e.d.

Let $a > 0$ be the injectivity radius of $(X, h^{TX})$. For $x \in X$, $b \in \mathbf{R}_+$, let $B^X(x, b)$ be the open ball of center $x$ and radius $b$ in $X$.

In the sequel, we take $b \in ]0, a/2]$.

**Definition 11.4.** For $x_0 \in X$, let $P^{x_0}_{u,T}(x, x')$ $(x, x' \in B^X(x_0, b))$ be the smooth kernel associated to the operator $\exp(-(uD^X + TV)^2)$ with Dirichlet conditions on $\partial B^X(x_0, b)$.

**Proposition 11.5.** *There exist* $c > 0$, $C > 0$ *such that for any* $x_0 \in X$, $u \in ]0, 1]$, $T \in [0, 1/u]$, $x, x' \in B^X(x_0, b/2)$, *we have*

$$\|(P_{u,T} - P^{x_0}_{u,T})(x, x')\| \le c \exp(-C/u^2). \tag{11.8}$$

*Proof.* In [15, Proposition 11.10], this result was proved for $x = x'$. Our proposition can be proved by exactly following the method of [15] in the general case. Details are left to the reader. q.e.d.

Let $\mathscr{V}$ be an open neighborhood of $X_g$ in $X$. Using Proposition 11.3, we find that there exist $c > 0$, $C > 0$ such that for $u \in ]0, 1]$, $T \in [0, 1/u]$,

$$\left| \int_{X \setminus \mathscr{V}} \operatorname{Tr}_s[N_H g P_{u,T}(g^{-1}x, x)] \frac{dv_X(x)}{(2\pi)^{\dim X}} \right| \le c \exp\left(\frac{-C}{u^2}\right). \tag{11.9}$$

From (11.4), (11.9), we see that the proof of (8.18) is local near $X_g$. Besides, by Proposition 11.5, the proof of (8.18) can be localized near any arbitrary $x \in X$, which by (11.9) can be taken in a neighborhood of $X_g$.

As in [15, §11], we will in fact work locally near $Y$, and so in our case, near $Y_g = Y \cap X_g$, and the estimates obtained by a formal argument as in [15, Remark 11.14] are near points close to $X_g$, but far from $Y$.

c. *A rescaling of the coordinate* $Z_0 \in N_{Y_g/X_g}$. In the sequel, if $x \in X$, $Z \in (T_{\mathbf{R}}X)_x$, then $t \in \mathbf{R} \to x_t = \exp^X_x(tZ) \in X$ denotes the geodesic in $X$ such that $x_0 = x$, $dx/dt|_{t=0} = Z$. Similar notation will be used on $Y$, $X_g$. Of course, we recall here that $X_g$ is totally geodesic in $X$.

First we identify a neighborhood of $Y_g$ in $X_g$ with a neighborhood of $Y_g$ in $N_{Y_g/X_g,\mathbf{R}}$ using geodesic coordinates normal to $Y_g$. Namely for $0 < \varepsilon \le a/2$, and $\varepsilon$ small enough, if $y \in Y_g$, $Z \in N_{Y_g/X_g,\mathbf{R},y}$, $|Z| \le \varepsilon$, then we identify $(y, Z)$ with $\exp^{X_g}_y(Z)$, where $y_t = \exp^{X_g}_y(tZ)$ is the geodesic in $X_g$ (which is also a geodesic in $X$) such that $y_0 = y$, $dy/dt|_{t=0} = Z$. Along the geodesic $y_t$, we trivialize $N_{X_g/X}$ by parallel transport with respect to $\nabla^{N_{X_g/X}}$.

Also we identify a neighborhood of $X_g$ in $X$ with a neighborhood of $X_g$ in $N_{X_g/X,\mathbf{R}}$ using geodesic coordinates normal to $X_g$ in $X$.

Let $y \in Y_g$, $Z \in N_{Y_g/X_g,\mathbf{R},y}$, $Z' \in N_{X_g/X,\mathbf{R},y}$. Recall that $Z' \in N_{X_g/X,\mathbf{R},y}$ is identified with an element of $N_{X_g/X,\mathbf{R},\exp^{X_g}_y(Z)}$. Then $(y, Z, Z') \to \exp^X_{\exp^{X_g}_y(Z)}(Z')$ identifies an open neighborhood of $Y_g$ in $N_{Y_g/X,\mathbf{R}} \simeq N_{Y_g/X_g,\mathbf{R}} \oplus (N_{X_g/X,\mathbf{R}})|_{Y_g}$ with an open neighborhood of $Y_g$ in $X$.

In particular, if $Z \in N_{Y_g/X_g,\mathbf{R},y}$, then $(y, Z)$ represents an element of $X_g$.

Let $dv_{X_g}$ be the volume form on $X_g$ with respect to $h^{TX_g}$. Let $dv_{N_{X_g/X}}$ be the volume form on the fibres of $(N_{X_g/X}, h^{N_{X_g/X}})$.

**Definition 11.6.** Let $\beta_T$ be the smooth function of $T \geq 0$, $x \in X_g$ such that

$$\beta_T(x)\frac{dv_{X_g}(x)}{(2\pi)^{\dim X_g}} = \{\mathrm{Td}_g(TX, h^{TX})\Phi \operatorname{Tr}_s[N_H g \exp(-C_{T^2}^2)]\}^{\max}. \tag{11.10}$$

For $\varepsilon > 0$ small enough, $y \in Y_g$, $Z \in N_{Y_g/X_g,\mathbf{R},y}$, $Z' \in N_{X_g/X,\mathbf{R},y}$, $|Z| < \varepsilon/2$, $|Z'| < \varepsilon/2$, let $k(y, Z, Z')$ be defined by

$$dv_X(y, Z, Z') = k(y, Z, Z')\, dv_{X_g}(y, Z)\, dv_{N_{X_g/X}}(Z'). \tag{11.11}$$

Then

$$k(y, Z, 0) = 1. \tag{11.12}$$

Similarly, let $k'(y, Z')$ be defined by

$$dv_{X_g}(y, Z') = k'(y, Z') dv_{Y_g}(y)\, dv_{N_{Y_g/X_g}}(Z'). \tag{11.13}$$

In the sequel, $\varepsilon \in ]0, a/2]$ is taken to be small enough so that the identifications considered above hold.

**Theorem 11.7.** *There exists* $\gamma \in ]0, 1]$ *such that for* $p \in \mathbf{N}$, *there is* $C_p > 0$ *such that if* $u \in ]0, 1]$, $T \in [1, 1/u]$, $y_0 \in Y_g$, $Z_0 \in N_{Y_g/X_g,\mathbf{R},y_0}$, $|Z_0| \leq \varepsilon T/2$, *then*

$$\begin{aligned}\Bigg| \frac{1}{T^{2\dim N_{Y_g/X_g}}} \Bigg\{ \int_{\substack{Z \in N_{X_g/X,\mathbf{R},(y_0, Z_0/T)} \\ |Z| \leq \varepsilon/8}} &\operatorname{Tr}_s \left[ N_H g P_{u,T}\left(g^{-1}\left(y_0, \frac{Z_0}{T}, Z\right), \left(y, \frac{Z_0}{T}, Z\right)\right)\right] \\ &\times k\left(y_0, \frac{Z_0}{T}, Z\right) \frac{dv_{N_{X_g/X}}(Z)}{(2\pi)^{\dim N_{X_g/X}}} - \beta_T\left(y_0, \frac{Z_0}{T}\right) \Bigg\} \Bigg| \\ &\leq C_p (1 + |Z_0|)^{-p} (u(1+T))^{\gamma}.\end{aligned} \tag{11.14}$$

**Remark 11.8.** Now we show how to derive (8.18) from Theorem 11.7. By Proposition 11.1, we may restrict ourselves to the case where $T$ is $\geq 1$.

For $\varepsilon > 0$, let $\mathscr{U}_\varepsilon(X_g)$ be the set of $x \in X$ such that $d^X(x, X_g) < \varepsilon$. Let $\mathscr{U}_\varepsilon(Y_g/X_g)$ denotes a corresponding open neighborhood of $Y_g$ in $X_g$. Clearly

(11.15)

$$\begin{aligned}&\int_{\mathscr{U}_{\varepsilon/8}(X_g)} \operatorname{Tr}_s[N_H g P_{u,T}(g^{-1}x, x)] \frac{dv_X(x)}{(2\pi)^{\dim X}} \\ &\qquad - \int_{X_g} \operatorname{Td}_g(TX, h^{TX}) \Phi \operatorname{Tr}_s[N_H g \exp(-C_{T^2}^2)] \\ &\qquad = \int_{X_g} \left\{ \int_{\substack{Z \in N_{X_g/X,\mathbf{R}} \\ |Z| \leq \varepsilon/8}} \operatorname{Tr}_s[N_H g P_{u,T}(g^{-1}(x, Z), (x, Z))] \right. \\ &\qquad\qquad \left. \times k(x, Z) \frac{dv_{N_{X_g/X}}}{(2\pi)^{\dim N_{X_g/X}}}(Z) - \beta_T(x) \right\} \frac{dv_{X_g}(x)}{(2\pi)^{\dim X_g}}.\end{aligned}$$

Also

(11.16)

$$\begin{aligned}&\int_{\mathscr{U}_{\varepsilon/2}(Y_g/X_g)} \left\{ \int_{\substack{Z \in N_{X_g/X,\mathbf{R}} \\ |Z| \leq \varepsilon/8}} \operatorname{Tr}_s[N_H g P_{u,T}(g^{-1}(x, Z), (x, Z))] \right. \\ &\qquad \left. \times k(x, Z) \frac{dv_{N_{X_g/X}}}{(2\pi)^{\dim N_{X_g/X}}}(Z) - \beta_T(x) \right\} \frac{dv_{X_g}(x)}{(2\pi)^{\dim X_g}} \\ &= \int_{Y_g} \frac{1}{T^{2\dim N_{Y_g/X_g}}} \left\{ \int_{\substack{Z_0 \in N_{Y_g/X_g,\mathbf{R}} \\ |Z_0| \leq \varepsilon T/2}} \left[ \int_{\substack{Z \in N_{X_g/X,\mathbf{R}} \\ |Z| \leq \varepsilon/8}} \right. \right. \\ &\qquad \times \operatorname{Tr}_s \left[ N_H g P_{u,T} \left( g^{-1} \left( y, \frac{Z_0}{T}, Z \right), \left( y, \frac{Z_0}{T}, Z \right) \right) \right] \\ &\qquad \left. \times k \left( y, \frac{Z_0}{T}, Z \right) \frac{dv_{N_{X_g/X}}(Z)}{(2\pi)^{\dim N_{X_g/X}}} - \beta_T \left( y, \frac{Z_0}{T} \right) \right] \\ &\qquad \left. \times k' \left( y, \frac{Z_0}{T} \right) \frac{dv_{N_{Y_g/X_g}}(Z_0)}{(2\pi)^{\dim N_{Y_g/X_g}}} \right\} \frac{dv_{Y_g}(y)}{(2\pi)^{\dim Y_g}}.\end{aligned}$$

Using Theorem 11.7, it is clear that the absolute value of the left-hand side of (11.16) is dominated by $C(u(1+T))^\gamma$.

Also the contribution of the complement of an open neighborhood of $Y_g$ in $\mathscr{U}_{\varepsilon/4}(X_g)$ to the integral on the right-hand side of (11.4) can be estimated by applying formally Theorem 11.7 with $Y = \phi$. From (11.9), (11.14), (11.16), we obtain (8.18). Hence the proof of Theorem 8.6 is completed.

d. *A local coordinate system near* $Y_g$ *and a trivialization of* $\Lambda(T^{*(0,1)}X) \hat{\otimes}\xi$. To establish Theorem 11.7, we will use an adequate coordinate system near $y_0 \in Y_g$.

If $Z \in (T_{\mathbf{R}}X_g)_{y_0}$, we identify $Z$ with $\exp_{y_0}^{X_g}(Z) \in X_g$. Again we trivialize $N_{X_g/X}$ along the geodesic $t \to tZ$ by parallel transport with respect to $\nabla^{N_{X_g/X}}$. Then we identify $(Z, Z') \in ((T_{\mathbf{R}}X_g) \times N_{X_g/X,\mathbf{R}})_{y_0} \cong (T_{\mathbf{R}}X)_{y_0}$ with $\exp^X_{\exp_{y_0}^{X_g}(Z)}(Z')$. Observe that since $X_g$ is totally geodesic in $X$ and $g$ preserves the geodesics in $X$, the action of $g$ near $y_0$ in the coordinates $(Z, Z')$ is given by

$$g(Z, Z') = (Z, gZ'). \tag{11.17}$$

Recall that the connection $\tilde{\nabla}^\xi$ on $\xi$ was already considered in §9.

We fix $Z_0 \in N_{Y_g/X_g,\mathbf{R},y_0}$, $|Z_0| < \varepsilon/2$, and take $Z \in (T_{\mathbf{R}}X)_{y_0}$, $|Z| < \varepsilon/2$. The curve $t \in [0, 1] \to Z_0 + tZ$ lies in $B_{y_0}^{TX}(0, \varepsilon)$. We identify $TX_{Z_0+Z}$, $\Lambda(T^{*(0,1)}X)_{Z_0+Z}$ with $TX_{Z_0}$, $\Lambda(T^{*(0,1)}X)_{Z_0}$ (resp. $\xi_{Z_0+Z}$ to $\xi_{Z_0}$) by parallel transport along the curve $t \in [0, 1] \to Z_0 + tZ$ with respect to the connections $\nabla^{TX}$, $\nabla^{\Lambda(T^{*(0,1)}X)}$ (resp. $\tilde{\nabla}^\xi$).

When $Z_0 \in N_{Y_g/X_g,\mathbf{R}}$ is allowed to vary, we identify $TX_{Z_0}$, $\Lambda(T^{*(0,1)}X)_{Z_0}$ with $TX_{y_0}$, $\Lambda(T^{*(0,1)}X)_{y_0}$ (resp. $\xi_{Z_0}$ to $\xi_{y_0}$) by parallel transport with respect to $\nabla^{TX}$, $\nabla^{\Lambda(T^{*(0,1)}X)}$ (resp. $\tilde{\nabla}^\xi$) along the curve $t \in [0, 1] \to tZ_0$.

Let $k''(Z)$ be the function defined on $B_{y_0}^{TX_g}(0, \varepsilon)$ such that for $Z \in (T_{\mathbf{R}}X_g)_{y_0}$, $|Z| < \varepsilon/2$,

$$dv_{X_g}(Z) = k''(Z)\, dv_{TX_g}(Z). \tag{11.18}$$

Let $\mathbf{H}_{y_0}$ be the vector space of smooth sections of $(\Lambda(T^{*(0,1)}X)\hat{\otimes}\xi)_{y_0}$ over $(T_{\mathbf{R}}X)_{y_0}$. Let $\Delta^{TX}$ be the ordinary Laplacian on $(T_{\mathbf{R}}X)_{y_0}$ with respect to the metric $h^{TX_{y_0}}$. Let $\gamma$ be a smooth function defined on $\mathbf{R}$ with values in $[0, 1]$ such that

$$\begin{aligned} \gamma(a) &= 1 \quad \text{for } a \leq 1/2, \\ &= 0 \quad \text{for } a \geq 1. \end{aligned} \tag{11.19}$$

If $Z \in (T_{\mathbf{R}}X)_{y_0}$, set

$$\rho(Z) = \gamma(2|Z|/\varepsilon). \tag{11.20}$$

Then

$$\begin{aligned} \rho(Z) &= 1 \quad \text{if } |Z| \le \varepsilon/4, \\ &= 0 \quad \text{if } |Z| \ge \varepsilon/2. \end{aligned} \tag{11.21}$$

We now fix $Z_0 \in N_{Y_g/X_g,\mathbf{R},y_0}$, $|Z_0| \le \varepsilon/2$. Recall that the considered trivialization of $\Lambda(T^{*(0,1)}X)\hat{\otimes}\xi$ depends explicitly on $Z_0$. Therefore the actions of $D^X$, and $V$ on $\mathbf{H}_{y_0}$ depends on $Z_0$. We will denote these actions by $D^X_{Z_0}$ and by $V(Z_0+.)$ respectively.

Recall that near $Y$, the vector bundles $\xi^\pm$ were defined in §9. Let $P^{\xi^\pm}$ be the orthogonal projection operator from $\xi$ on $\xi^\pm$.

Now we follow [15, Definition 11.18].

**Definition 11.9.** For $u > 0$, $T \ge 0$, let $L^{1,Z_0}_{u,T}$, $M^{2,Z_0}_u$ be the operators

$$\begin{aligned} L^{1,Z_0}_{u,T} &= (1-\rho^2(Z))\left(\frac{-u^2\Delta^{TX}}{2} + T^2P^{\xi^+}_{y_0}\right) \\ &\quad + \rho^2(Z)(uD^X_{Z_0} + TV(Z_0+Z))^2, \\ M^{1,Z_0}_u &= -u^2(1-\rho^2(Z))\frac{\Delta^{TX}}{2} + \rho^2(Z)(uD^X)^2. \end{aligned} \tag{11.22}$$

For $|Z'| < \varepsilon$, the volume element $dv_X(Z')$ is well defined. For $|Z_0| < \varepsilon/2$, let $P^{1,Z_0}_{u,T}(Z, Z')$ $(Z, Z' \in (T_{\mathbf{R}}X)_{y_0}$, $|Z'| < \varepsilon/2)$ be the smooth kernel associated to $\exp(-L^{1,Z_0}_{u,T})$ with respect to $dv_X(Z_0+Z')/(2\pi)^{\dim X}$.

Using (11.17) and proceeding as in Proposition 11.5, we see that there exist $c > 0$, $C > 0$ such that for $u \in ]0, 1]$, $T \in [1, 1/u]$, $y_0 \in Y_g$, $Z_0 \in N_{Y_g/X_g,\mathbf{R}}$, $|Z_0| \le \varepsilon/2$, $Z \in N_{X_g/X,\mathbf{R},y_0}$, $|Z| \le \varepsilon/2$,

$$\begin{aligned} |P_{u,T}(g^{-1}(y, Z_0, Z), (y, Z_0, Z)) - P^{1,Z_0}_{u,T}(g^{-1}Z, Z)| \\ \le c\exp(-C/u^2). \end{aligned} \tag{11.23}$$

Therefore, in the estimate (11.14), we may and will replace $P_{u,T}$ by $P^{1,Z_0}_{u,T}$.

e. *Rescaling of the variable $Z$ and the Clifford variables.* For $u > 0$, let $F_u$ be the linear map

$$(11.24) \qquad h \in \mathbf{H}_{y_0} \to F_u h \in \mathbf{H}_{y_0}, \qquad F_u h(Z) = h(Z/u).$$

For $u > 0$, $T \geq 0$, set

$$(11.25) \qquad L^{2,Z_0}_{u,T} = F_u^{-1} L^{1,Z_0}_{u,T} F_u, \qquad M^{2,Z_0}_u = F_u^{-1} M^{1,Z_0}_u F_u.$$

Let Op be the set of scalar differential operators acting on smooth functions on $(T_{\mathbf{R}}X)_{y_0}$. As in [15, equation (11.51)], we find that

$$L^{2,Z_0}_{u,T}, M^{2,Z_0}_{u,T} \in (c(T_{\mathbf{R}}X) \hat{\otimes} \operatorname{End} \xi)_{y_0} \otimes \operatorname{Op}.$$

Let $e_1, \cdots, e_{2l'}$, $(e_{2l'+1}, \cdots, e_{2l''})$, $e_{2l''+1}, \cdots, e_{2l}$ be orthonormal bases of $(T_{\mathbf{R}}Y_g)y_0, N_{Y_g/X_g,\mathbf{R},y_0}, N_{X_g/X,\mathbf{R},y_0}$ respectively. Then $e_1, \ldots, e_{2l}$ is an orthonormal basis of $(T_{\mathbf{R}}X)_{y_0}$. Let $e^1, \cdots, e^{2l}$ be the corresponding dual basis of $(T^*_{\mathbf{R}}X)_{y_0}$.

Because $X_g$ is totally geodesic, it is important to observe that under the considered identification of $(T_{\mathbf{R}}X)_{Z_0}$ with $(T_{\mathbf{R}}X)_{y_0}$, at $Z_0 \in N_{Y_g/X_g,\mathbf{R},y_0}$ which represents an element of $X_g$, $(e_1, \cdots, e_{2l''})$ and $(e_{2l''+1}, \cdots, e_{2l})$ are orthonormal bases of $(T_R X_g)_{Z_0}$ and $N_{X_g/X,\mathbf{R},y_0}$ respectively.

**Definition 11.10.** For $u > 0$, $T > 0$, set

$$(11.26) \qquad \begin{aligned} c_{u,T}(e_j) &= \frac{\sqrt{2}e^j}{u} \wedge - \frac{u}{\sqrt{2}} i_{e_j}, \qquad 1 \leq j \leq 2l', \\ c_{u,T}(e_j) &= \frac{\sqrt{2}e^j}{uT} \wedge - \frac{uT}{\sqrt{2}} i_{e_j}, \qquad 2l' + 1 \leq j \leq 2l''. \end{aligned}$$

The operators $c_{u,T}(e_j)$, $1 \leq j \leq 2l''$, act naturally on $(\Lambda(T^*_{\mathbf{R}}X_g) \hat{\otimes} \xi)_{y_0}$.

**Definition 11.11.** For $u > 0$, $T \geq 0$, let

$$L^{3,Z_0}_{u,T}, M^{3,Z_0}_{u,T} \in (\operatorname{End}(\Lambda(T^*_{\mathbf{R}}X_g) \hat{\otimes} \xi) \hat{\otimes} c(N_{X_g/X,\mathbf{R}}))_{y_0} \otimes \operatorname{Op}$$

be obtained from $L^{2,Z_0}_{u,T}$, $M^{2,Z_0}_u$ by replacing the Clifford variables $c(e_j)$, $1 \leq j \leq 2l''$, by the operators $c_{u,T}(e_j)$, while leaving unchanged the $c(e_j)$ $(2l'' + 1 \leq j \leq 2l)$.

The complicated fact with respect to [15, §11 i] is that the $c(e_j)$'s, $2l'' + 1 \leq j \leq 2l$, are not rescaled.

Let $P^{3,Z_0}_{u,T}(Z, Z')$ $(Z, Z' \in (T_{\mathbf{R}}X)_{y_0}, |Z'| < \varepsilon/2)$ be the smooth kernel associated to $\exp(-L^{3,Z_0}_{u,T})$ calculated with respect to $k''(y_0, Z_0) dv_{(TX)_{y_0}}(Z')/(2\pi)^{\dim X}$. Note that, at $Z' = 0$ (representing $(y_0, Z_0)$), this last density coincides with $dv_X/(2\pi)^{\dim X}$. Here $P^{3,Z_0}_{u,T}(Z, Z')$ lies in $(\operatorname{End}(\Lambda(T^*_{\mathbf{R}}X_g)) \hat{\otimes} c(N_{X_g/X,\mathbf{R}}) \hat{\otimes} \operatorname{End}(\xi))_{y_0}$. Moreover $g$

acts naturally on $(\Lambda(\overline{N}^*_{X_g/X})\hat{\otimes}\xi)_{y_0}$ as an element of $(c(N_{X_g/X,\mathbf{R}})\hat{\otimes}\operatorname{End}(\xi))_{y_0}$. So $gP^{3,Z_0}_{u,T}(Z, Z')$ lies in $(\operatorname{End}(\Lambda(T^*_{\mathbf{R}}X_g))\hat{\otimes}c(N_{X_g/X,\mathbf{R}})\hat{\otimes}\operatorname{End}(\xi))_{y_0}$.

Now we use the notation of [15, equation (11.53)]. Namely $P^{3,Z_0}_{u,T}(g^{-1}Z, Z)$ can be expanded in the form

$$\tag{11.27} \begin{aligned} &P^{3,Z_0}_{u,T}(g^{-1}Z, Z) \\ &\quad = \sum_{\substack{1\le i_1<\cdots<i_p\le 2l'' \\ 1\le j_1<\cdots<j_q\le 2l''}} e^{i_1}\wedge\cdots\wedge e^{i_p}\wedge i_{e_{j_1}}\cdots i_{e_{j_q}}\hat{\otimes}Q^{j_1\cdots j_q}_{i_1\cdots i_p}(g^{-1}Z, Z), \\ &\qquad\qquad Q^{j_1\cdots j_q}_{i_1\cdots i_p}(g^{-1}Z, Z)\in(c(N_{X_g/X,\mathbf{R}})\hat{\otimes}\operatorname{End}(\xi))_{y_0}. \end{aligned}$$

Set

$$\tag{11.28} \begin{aligned} [P^{3,Z_0}_{u,T}(g^{-1}Z, Z)]^{\max} &= Q_{1,\ldots,2l''}(g^{-1}Z, Z) \\ &\in (c(N_{N_g/X,\mathbf{R}})\hat{\otimes}\operatorname{End}(\xi))_{y_0}. \end{aligned}$$

Since $(c(N_{X_g/N,\mathbf{R}})\hat{\otimes}\operatorname{End}(\xi))_{y_0}$ acts on $(\Lambda(\overline{N}^*_{X_g/X})\hat{\otimes}\xi)_{y_0}$, the supertrace of the elements of this algebra is well defined.

The obvious extension of [15, Proposition 11.21] is as follows.

**Proposition 11.12.** *If* $Z\in N_{X_g/X,\mathbf{R},y_0}$, *the following equality holds*:

$$\tag{11.29} \begin{aligned} &\frac{1}{T^{2\dim N_{Y_g/X_g}}}\operatorname{Tr}_s[N_H g P^{1,Z_0}_{u,T}(g^{-1}Z, Z)]k(y_0, Z_0, Z) \\ &\quad = (-i)^{\dim X}\frac{1}{u^{2\dim N_{X_g/X}}}\operatorname{Tr}_s\left[N_H g\left[P^{3,Z_0}_{u,T}\left(\frac{g^{-1}Z}{u}, \frac{Z}{u}\right)\right]^{\max}\right]. \end{aligned}$$

*Proof.* Observe that since $g$ preserves the geodesics and the relevant connections on $\Lambda(T^{*(0,1)}X)\hat{\otimes}\xi \simeq (\Lambda(T^{*(0,1)}X)\hat{\otimes}\xi)_{y_0}$, $g$ just acts as the obvious constant linear map on $(\Lambda(T^{*(0,1)}X)\hat{\otimes}\xi)_{y_0}$. Since $g$ acts like the identity on $\Lambda(T^{*(0,1)}X_g)$, we have $g\in c(N_{X_g/X,\mathbf{R}})_{y_0}$. Therefore the rescaling of the Clifford variables in (11.26) has no effect on $g$. Equality (11.29) is now a trivial consequence of [18], [15, Proposition 11.2]. q.e.d.

By (11.29), we find that for $Z_0 \in N_{Y_g/X_g,\mathbf{R},y_0}$, $|Z_0| \le \varepsilon T/2$,

$$(11.30)\quad \begin{aligned} &\frac{1}{T^{2\dim N_{Y_g/X_g}}} \int_{\substack{Z\in N_{X_g/X,\mathbf{R},y_0}\\ |Z|\le\varepsilon/8}} \mathrm{Tr}_s\left[N_H g P^{1,Z_0/T}_{u,T}\left(g^{-1}\left(y_0, \frac{Z_0}{T}, Z\right),\right.\right. \\ &\qquad\qquad \left.\left.\left(y_0, \frac{Z_0}{T}, Z\right)\right)\right] \\ &\qquad \times k\left(y_0, \frac{Z_0}{T}, Z\right) \frac{dv_{N_{X_g/X}}(Z)}{(2\pi)^{\dim N_{X_g/X}}} \\ &= (-i)^{\dim X_g} \int_{\substack{Z\in N_{X_g/X,\mathbf{R},y_0}\\ |Z|\le\varepsilon/8u}} \\ &\qquad \times \mathrm{Tr}_s[N_H g[P^{3,Z_0/T}_{u,T}(g^{-1}Z, Z)]^{\max}] \frac{dv_{N_{X_g/X}}(Z)}{(2\pi)^{\dim N_{X_g/X}}}. \end{aligned}$$

Also by [9, Theorems 4.11–4.15], one finds easily that

$$(11.31)\quad \begin{aligned} &(-i)^{\dim X_g} \int_{Z\in N_{X_g/X,\mathbf{R}}} \mathrm{Tr}_s[N_H g[P^{3,Z_0/T}_{0,T}(g^{-1}Z, Z)]^{\max}] \frac{dv_{N_{X_g/X}}(Z)}{(2\pi)^{\dim N_{X_g/X}}} \\ &= \beta_T\left(y_0, \frac{Z_0}{T}\right). \end{aligned}$$

Moreover, observe that there is $C > 0$ such that if $Z \in N_{X_g/X,\mathbf{R}}$, then

$$(11.32)\quad |g^{-1}Z - Z|^2 \ge C|Z|^2.$$

In view of (11.16), (11.23), (11.29)–(11.32), Theorem 11.7 follows from the following result.

**Theorem 11.13.** *There exist $\gamma \in ]0, 1]$, $C > 0$ such that for any $m \in \mathbf{N}$, there is $C > 0$, $r \in \mathbf{N}$ such that for $u \in ]0, 1]$, $T \in [1, 1/u]$, $y_0 \in Y_g$, $Z_0 \in N_{Y_g/X_g,\mathbf{R},y_0}$, $|Z_0| \le \varepsilon T/2$, $Z, Z' \in (T_{\mathbf{R}}X)_{y_0}$, $|Z|$, $|Z'| \le \varepsilon/8u$, we have*

$$(11.33)\quad \begin{aligned} &|(P^{3,Z_0/T}_{u,T} - P^{3,Z_0/T}_{0,T})(Z, Z')| \\ &\le \frac{c(1+|Z|+|Z'|)^{2r}}{(1+|Z_0|)^m} \exp(-C|Z-Z'|^2)(u(1+T))^\gamma. \end{aligned}$$

*Proof.* The remainder of the section is devoted to the proof of Theorem 11.13, which is similar to that of [15, equation (11.58)]. The essential difference between these two proofs is that we need to obtain the bounding function $(1+|Z|+|Z'|)^{2r}\exp(-C|Z-Z'|^2)$ on the right-hand side of

(11.33), while in [15], only the case where $X_g = X$, $Z = Z' = 0$ was considered.

We briefly describe the main steps which are needed to obtain Theorem 11.13, referring to [15] when necessary.

f. *The matrix structure of the operator* $L_{u,T}^{3,Z_0/T}$. As in [15, §11j], we calculate the asymptotic expansion of the operator $L_{u,T}^{3,Z_0/T}$ as $u \to 0$. The basic difference is that here, the operators $c(e_j)$, $2l'' + 1 \leq j \leq 2l$, are not rescaled, but this does not create any new difficulty. Also as in [15], we must study the matrix structure of $L_{u,T}^{3,Z_0/T}$ with respect to the splitting $\xi = \xi^+ \oplus \xi^-$. In particular, instead of [15, equation (11.60)], the operator obtained by rescaling $uT[D^X, V]$ is now

$$
\begin{aligned}
&T\sum_{j=1}^{2l'}\left(e^j \wedge -\frac{u^2}{2}i_{e_j}\right)\nabla^\xi_{e_j}V\left(\frac{Z_0}{T}+uZ\right)\\
&\quad+\sum_{j=2l'+1}^{2l''}\left(e^j \wedge -\frac{u^2T^2}{2}i_{e_j}\right)\nabla^\xi_{e_j}V\left(\frac{Z_0}{T}+uZ\right)\\
&\quad+\sum_{j=2l''+1}^{2l} uTc(e_j)\nabla^\xi_{e_j}V\left(\frac{Z_0}{T}+uZ\right).
\end{aligned}
\tag{11.34}
$$

By [5, Proposition 3.5], as in [15, equation (11.64)], we know that if $U \in T_{\mathbf{R}}Y$, then

$$P^{\xi^-}\nabla^\xi_U V P^{\xi^-} = 0 \quad \text{on } Y. \tag{11.35}$$

Therefore, for $1 \leq j \leq 2l'$,

$$P^{\xi^-}\nabla^\xi_{e_j}V(Z_0/T+uZ)P^{\xi^-} = \mathscr{O}(|Z_0/T+uZ|). \tag{11.36}$$

So at least formally, the situation is the same as in [15]. Finally the extra term $\sum_{j=2l''+1}^{2l} uTc(e_j)\nabla^\xi_{e_j}V(Z_0/T+uZ)$ does not introduce any extra difficulty, because $uT \leq 1$.

g. *A family of Sobolev spaces with weights.* Set

$$\Lambda^{(p,q)}(T^*_{\mathbf{R}}X_g)_{y_0} = \Lambda^p(T^*_{\mathbf{R}}Y_g)_{y_0}\hat{\otimes}\Lambda^q(N^*_{Y_g/X_g,\mathbf{R}})_{y_0}. \tag{11.37}$$

Let $\mathbf{I}_{y_0}$ be the set of smooth sections of $\Lambda(T^*_{\mathbf{R}}X_g)_{y_0}\hat{\otimes}\Lambda(\overline{N}^*_{X_g/X})\hat{\otimes}\xi_{y_0}$ over $(T_{\mathbf{R}}X)_{y_0}$, and $\mathbf{I}_{(p,q),y_0}$ be the set smooth section of $\Lambda^{(p,q)}(T^*_{\mathbf{R}}X_g)$ $\hat{\otimes}\Lambda(\overline{N}^*_{X_g/X,y_0})\hat{\otimes}\xi_{y_0}$ over $(T_{\mathbf{R}}X)_{y_0}$.

As in [15, §11 k], we introduce a family of Sobolev spaces with weights, which are strictly similar to the corresponding weights in [15, Definition 11.23]. The results contained in [15, Proposition 11.24, Theorem 11.30] remain valid, essentially because the operator $L_{u,T}^{3,Z_0/T}$ considered here has the same structure as in [15].

h. *Uniform estimates on the kernel* $P_{u,T}^{3,Z_0/T}$. Now we refine [15, Theorem 11.31] in our context.

**Theorem 11.14.** *There is* $C > 0$ *such that for* $m \in \mathbf{N}$, $m' \in \mathbf{N}$, *there exist* $c > 0$, $r \in \mathbf{N}$ *such that for any* $u \in ]0, 1]$, $T \in [1, 1/u]$, $y_0 \in Y_g$, $Z_0 \in N_{Y_g/X_g,\mathbf{R},y_0}$, $|Z_0| \leq \varepsilon T/2$, $Z, Z' \in (T_{\mathbf{R}}X)_{y_0}$, $|Z|$, $|Z'| \leq \varepsilon T/6u$, *we have*

$$(11.38)\qquad \begin{aligned}(1+|Z_0|)^m \sup_{\substack{|\alpha|\leq m'\\ |\alpha'|\leq m'}} \left| \frac{\partial^{|\alpha|+|\alpha'|}}{\partial Z^{\alpha}\partial Z'^{\alpha'}} P_{u,T}^{3,Z_0/T}(Z, Z') \right| \\ \leq c(1+|Z|+|Z'|)^r \exp(-C|Z-Z'|^2).\end{aligned}$$

*Proof.* First we will prove (11.38) with $C = 0$. Then by using finite propagation speed, we will get (11.38) with $C > 0$.

The proof starts in the same way as the proof of [15, Theorem 11.31]. In particular the inequality [15, equation (11.125)] is still valid.

Take $p \in \mathbf{N}$. Let $J_{p,y_0}^0$ be the set of square integrable sections of $\Lambda(T_{\mathbf{R}}^*X_g)_{y_0}\hat{\otimes}\Lambda(\overline{N}_{Y_g/X_g}^*)_{y_0}\hat{\otimes}\xi_{y_0}$ over $\{Z \in (T_{\mathbf{R}}X_g)_{y_0},\ |Z| \leq p + \frac{1}{2}\}$. We equip $J_{p,y_0}^0$ with the Hermitian product

$$(11.39)\qquad s, s' \in J_{p,y_0}^0 \to \langle s, s'\rangle = \int_{|Z|\leq p+1/2} \langle s, s'\rangle\, dv_{TX}.$$

Let $|\ |$ denote the obvious norm on $J_{p,y_0}^0$. If $A \in \mathscr{L}(J_{p,y_0}^0)$, let $\|A\|_{p,\infty}$ be the corresponding norm of $A$ with respect to $|\ |$.

In the sequel, the constants $C > 0$ may vary from line to line. They are uniform in $u \in ]0, 1]$, $T \in [1, 1/u]$, $p \in \mathbf{N}$.

The obvious analogue of [15, equation (11.127)] (where only the case $|Z| \leq 3/2$ was considered) is that there is $C > 0$ such that for any $p \in \mathbf{N}$, $s \in J_{p,y_0}^0$,

$$(11.40)\qquad |s| \leq |s|_{u,T,Z_0,0}, \qquad |s|_{u,T,Z_0,0} \leq C(1+|Z_0|)^{2l'}(1+p)^{2l''}|s|.$$

Using the notation in [15, equation (11.128)], we obtain, for $k$, $k'$, $k''$, $k'''$

$\in \mathbf{N}$,

$$\|\Delta^k A^{k'}_{u,T,Z_0/T}\Delta^{k''}\exp(-L^{3,Z_0,T}_{u,T})\Delta^{k'''}\|_{p,\infty} \leq C''(1+|Z_0|)^{2l'}(1+p)^{2l''}, \tag{11.41}$$

which together with Sobolev's inequalities implies that for $k', k'', k''' \in \mathbf{N}$,

$$\sup_{\substack{|Z|\leq p+1/4\\ |Z'|\leq p+1/4}} |A^{k'}_{u,T,Z_0/T}\Delta^{k''}_Z\Delta^{k'''}_{Z'}P^{3,Z_0/T}_{u,T}(Z,Z')| \leq C(1+|Z_0|)^{2l}(1+p)^{2l}. \tag{11.42}$$

By (11.21), if $|Z| \leq \varepsilon/4u$, then $\rho(uZ) = 1$. Using [15, equation (11.130)] and (11.42), we find that if $u \in ]0,1]$, $T \in [1, 1/u]$, $|Z_0| \leq \varepsilon T/2$, then

$$\sup_{\substack{|Z|,|Z'|\leq\\ \inf(p+\frac{1}{4},\frac{\varepsilon}{4u})}} |(\mathrm{T}d^X(Z_0/T+uZ,Y))^{2k'}\Delta^{k''}_Z\Delta^{k'''}_{Z'}P^{3,Z_0/T}_{u,T}(Z,Z')| \leq C(1+|Z_0|)^{2l}(1+p)^{2l}. \tag{11.43}$$

Also if $|Z_0| \leq \varepsilon T/2$, $|Z| \leq \inf(p+1/4, \varepsilon/4u)$, then

$$\begin{gathered}\mathrm{T}d^X(Z_0/T,Y) \leq \mathrm{T}d^X(Z_0/T+uZ,Y)+Cp,\\ \mathrm{T}d^X(Z_0/T,Y) = |Z_0|.\end{gathered} \tag{11.44}$$

Clearly (11.43) is valid for $k'=0$. Using (11.43), (11.44), we see that given $k', k'', k''' \in \mathbf{N}$, there is $C > 0$, $r \in \mathbf{N}$ such that for $u \in ]0,1]$, $T \in [1, \frac{1}{u}]$, $y_0 \in Y_g$, $Z_0 \in N_{Y_g/X_g,\mathbf{R},y_0}$, $|Z_0| \leq \varepsilon T/2$, $p \in \mathbf{N}$, we have

$$\sup_{|Z|,|Z'|\leq\inf(p+1/4,\varepsilon/4u)} ||Z_0|^{k'}\Delta^{k''}_Z\Delta^{k'''}_{Z'}P^{3,Z_0/T}_{u,T}(Z,Z')| \leq C(1+|Z_0|)^{2l}(1+p)^r. \tag{11.45}$$

Using Sobolev's inequalities again, we deduce from (11.45) that given $m \in \mathbf{N}$, $m' \in \mathbf{N}$, there is $c > 0$, $r \in \mathbf{N}$ such that if $u \in ]0,1]$, $T \in [1, 1/u]$, $y_0 \in Y_g$, $Z_0 \in N_{Y_g/X_g,\mathbf{R},y_0}$, $|Z_0| \leq \varepsilon T/2$, $Z, Z' \in (T_{\mathbf{R}}X)_{y_0}$, $|Z|, |Z'| \leq \varepsilon/6u$, then

$$(1+|Z_0|)^m \sup_{\substack{|\alpha|\leq m'\\ |\alpha'|\leq m'}} \left|\frac{\partial^{|\alpha|+|\alpha'|}}{\partial Z^\alpha \partial Z'^{\alpha}}P^{3,Z_0/T}_{u,T}(Z,Z')\right| \leq c(1+|Z|+|Z'|)^r. \tag{11.46}$$

Now we will get the exponential factor $\exp(-C|Z-Z'|^2)$ on the right-hand side of (11.38) by using the finite propagation speed. Let $u \in \mathbf{R} \to k(u) \in [0,1]$ be a smooth even function such that

$$k(u) = 0 \quad \text{for } |u| \le \tfrac{1}{2}, \qquad = 1 \quad \text{for } |u| \ge 1. \tag{11.47}$$

For $q \in \mathbf{R}_+^*$, $a \in \mathbf{C}$, set

$$K_q(a) = 2\int_0^{+\infty} \cos(t\sqrt{2}a)\exp\left(\frac{-t^2}{2}\right) k\left(\frac{t}{q}\right)\frac{dt}{\sqrt{2\pi}}. \tag{11.48}$$

First we prove an analogue of [15, Proposition 13.8].

**Proposition 11.15.** *For any $c>0$, $m, m' \in \mathbf{N}$, there is $C>0$, $C'>0$ such that for $q \ge 1$,*

$$\sup_{\substack{a\in\mathbf{C}\\ |\operatorname{Im} a|\le c}} |a|^m |K_q^{(m')}(a)| \le C\exp(-C'q^2). \tag{11.49}$$

*Proof.* Clearly the function $t \in \mathbf{R} \to k(t/q)$ vanishes for $|t| \le q/2$. Also if $a \in \mathbf{C}$, $|\operatorname{Im} a| \le c$, then

$$|\cos(ta)| \le \exp(ct). \tag{11.50}$$

Using (11.48) and integration by parts, we get (11.49). q.e.d.

Clearly $K_q(a)$ is an even holomorphic function of $a$. Therefore, there is a holomorphic function $a \in \mathbf{C} \to \widetilde{K}_q(a)$ such that

$$K_q(a) = \widetilde{K}_q(a^2). \tag{11.51}$$

**Definition 11.16.** Given $c>0$, set

$$V_c = \{\lambda \in \mathbf{C},\ \operatorname{Re}(\lambda) \ge (\operatorname{Im}\lambda)^2/4c^2 - c^2\}. \tag{11.52}$$

Now we proceed as in [15, Proposition 13.10].

**Proposition 11.17.** *For any $c>0$, $m, m' \in \mathbf{N}$, there exist $C>0$, $C'>0$ such that for $q \ge 1$,*

$$\sup_{a\in V_c} |a|^m |\widetilde{K}_q^{(m')}(a)| \le C\exp(-C'q^2). \tag{11.53}$$

*Proof.* The set $V_c$ is exactly the image of $\{\lambda \in \mathbf{C},\ |\operatorname{Im}\lambda| \le c\}$ by the map $\lambda \to \lambda^2$. Our result now follows from Proposition 11.15. q.e.d.

For $c>0$, set

$$U_c = \{\lambda \in \mathbf{C},\ \operatorname{Re}(\lambda) \le (\operatorname{Im}\lambda)^2/4c^2 - c^2\}, \qquad \Gamma_c = \{\lambda \in \mathbf{C},\ \operatorname{Re}(\lambda) = (\operatorname{Im}\lambda)^2/4c^2 - c^2\}. \tag{11.54}$$

By using an analogue of [15, Theorem 11.27], we find that if $c > 0$ is large enough, then for $u \in ]0, 1]$, $T \in [1, 1/u]$, $y_0 \in Y_g$, $Z_0 \in N_{Y_g/X_g, \mathbf{R}, y_0}$, $|Z_0| \leq \varepsilon T/2$, $\lambda \in U_c$, the resolvent $(\lambda - L_{u,T}^{3, Z_0/T})^{-1}$ exists, and extends to a continuous linear operator from $\mathbf{I}_{y_0}^{-1}$ into $\mathbf{I}_{y_0}^{1}$, and moreover, with the notation of [15, §11 1],

$$\begin{aligned} &\|(\lambda - L_{u,T}^{3, Z_0/T})^{-1}\|_{u,T,Z_0}^{0,0} \leq C, \\ &\|(\lambda - L_{u,T}^{3, Z_0/T})^{-1}\|_{u,T,Z_0}^{-1,1} \leq C(1 + |\lambda|^2). \end{aligned} \tag{11.55}$$

From Proposition 11.17 and (11.55) it follows that

$$\widetilde{K}_q(L_{u,T}^{3, Z_0/T}) = \frac{1}{2\pi i} \int_{\Gamma_c} \frac{\widetilde{K}_q(\lambda)}{\lambda - L_{u,T}^{3, Z_0/T}} \, d\lambda. \tag{11.56}$$

By (11.53), we see that given $r \in \mathbf{N}$, there is a holomorphic function $\widetilde{K}_{q,r}(a)$ defined in a neighborhood of $V_c$ and verifying the same estimates as $\widetilde{K}_q(a)$ in (11.53) such that

$$\widetilde{K}_{q,r}^{(r-1)}(a)/(r-1)! = K_q(a). \tag{11.57}$$

Thus

$$\widetilde{K}_q(L_{u,T}^{3, Z_0/T}) = \frac{1}{2\pi i} \int_{\Gamma_c} \frac{\widetilde{K}_{q,r}(\lambda)}{(\lambda - L_{u,T}^{3, Z_0/T})^r} \, d\lambda. \tag{11.58}$$

Using (11.58) and proceeding as in [15, equations (13.244)–(13.247)] instead of [15, (11.117)–(11.125)], we deduce that the kernel $\widetilde{K}_q(L_{u,T}^{3, Z_0/T})(Z, Z')(Z, Z' \in (T_{\mathbf{R}}X)_{y_0})$ of $\widetilde{K}_q(L_{u,T}^{3, Z_0/T})$ satisfies estimates similar to (11.46), the essential difference being that because of (11.53), there is an extra factor $\exp(-Cq^2)$. More precisely, there is $C > 0$ such that for $m, m' \in \mathbf{N}$, there exist $c > 0$, $r \in \mathbf{N}$, for which given $q \in \mathbf{N}$, $u \in ]0, 1]$, $T \in [1, 1/u]$, $y_0 \in Y_g$, $Z_0 \in N_{Y_g/X_g, \mathbf{R}, y_0}$, $|Z_0| \leq \varepsilon T/2$, $Z, Z' \in (T_{\mathbf{R}}X)_{y_0}$, $|Z|, |Z'| \leq \varepsilon/6u$, we have

$$\begin{aligned} (1 + |Z_0|)^m \sup_{\substack{|\alpha| \leq m' \\ |\alpha'| \leq m'}} \left| \frac{\partial^{|\alpha| + |\alpha'|}}{\partial Z^\alpha \partial Z'^{\alpha'}} \widetilde{K}_q(L_{u,T}^{3, Z_0/T})(Z, Z') \right| \\ \leq c(1 + |Z| + |Z'|)^r \exp(-Cq^2). \end{aligned} \tag{11.59}$$

If $t \geq q$, then $k(t/q) = 1$. Also using the finite propagation speed for solutions of hyperbolic equations for $\cos(s\sqrt{L_{u,T}^{3, Z_0/T}})$ [16, §7.8], [30, §4.4], we find that there is a fixed constant $C' > 0$ such that

$$P^{3,Z_0/T}_{u,T}(Z, Z') = \widetilde{K}_q(L^{3,Z_0/T}_{u,T})(Z, Z') \quad \text{if } |Z - Z'| \geq C'q. \tag{11.60}$$

From (11.59), (11.60), it follows that there exist $C > 0$, $C' > 0$ for which, given $m, m' \in \mathbf{N}$, there is $c > 0$, $r \in \mathbf{N}$ such that for $q \in \mathbf{N}$, $u \in ]0, 1]$, $T \in [1, 1/u]$, $y_0 \in Y_g$, $Z_0 \in N_{Y_g/X_g, \mathbf{R}, y_0}$, $|Z_0| \leq \varepsilon T/2$, $Z, Z' \in (T_{\mathbf{R}}X)_{y_0}$, $|Z|, |Z'| \leq \varepsilon/6u$, we have

$$\begin{aligned}(1 + |Z_0|)^m \sup_{\substack{|\alpha| \leq m' \\ |\alpha'| \leq m'}} &\left| \frac{\partial^{|\alpha|+|\alpha'|}}{\partial Z^\alpha \partial Z'^{\alpha'}} P^{3,Z_0/T}_{u,T}(Z, Z') \right| \\ &\leq c(1 + |Z| + |Z'|)^{2r} \exp(-Cq^2) \quad \text{if } |Z - Z'| \geq C'q,\end{aligned} \tag{11.61}$$

which together with (11.46) thus gives (11.38).

The proof of Theorem 11.14 is completed.

i. *Proof of Theorem* 11.13. The analogues of [15, Proposition 11.34, Theorem 11.36] hold for exactly the same reasons as in [15].

We use the same notation as in (11.41). By the analogue of [15, Theorem 11.36] and by (11.40), we find that there exists $C > 0$ such that for $p \in \mathbf{N}$, $u \in ]0, 1]$, $T \in [1, 1/u]$,

$$\| \exp(-L^{3,Z_0/T}_{u,T}) - \exp(-L^{3,Z_0/T}_{0,T}) \|_{p,\infty} \leq CuT(1+|Z_0|)^{2l}(1+p)^{2l}. \tag{11.62}$$

To establish Theorem 11.13, we proceed as in [15, §11 p].

Let $\varphi$ be a smooth function defined on $(T_{\mathbf{R}}X)_{y_0}$ with values in $\mathbf{R}_+$ and support in $\{Z \in (T_{\mathbf{R}}X)_{y_0}, |Z| \leq 1\}$ such that

$$\int \varphi(Z)\, dv_{TX}(Z) = 1. \tag{11.63}$$

Take $\beta \in ]0, 1]$. By Theorem 11.14, there exists $c > 0$ such that for $y_0 \in Y_g$, $Z_0 \in N_{Y_g/X_g, \mathbf{R}, y_0}$, $|Z_0| \leq \varepsilon T/2$, $Z, Z' \in (T_{\mathbf{R}}X)_{y_0}$, $|Z|, |Z'| \leq \varepsilon/8u$, $U, U' \in (\Lambda(T^*_{\mathbf{R}}X_g)\hat{\otimes}\Lambda(\overline{N}^*_{X_g/X})\hat{\otimes}\xi)_{y_0}$, we have

$$\begin{aligned}&\Big| \langle (P^{3,Z_0/T}_{u,T} - P^{3,Z_0/T}_{0,T})(Z, Z')U, U' \rangle \\ &\quad - \int_{(T_{\mathbf{R}}X)_{y_0} \times (T_{\mathbf{R}}X)_{y_0}} \langle (P^{3,Z_0/T}_{u,T} - P^{3,Z_0/T}_{0,T})(Z - \widetilde{Z}, Z' - \widetilde{Z}')U, U' \rangle \\ &\qquad \frac{1}{\beta^{2l}} \varphi\left(\frac{\widetilde{Z}}{\beta}\right) \frac{1}{\beta^{2l}} \varphi\left(\frac{\widetilde{Z}'}{\beta}\right) dv_{TX}(\widetilde{Z})\, dv_{TX}(\widetilde{Z}') \Big| \\ &\leq c\beta(1 + |Z| + |Z'|)^{2l}.\end{aligned} \tag{11.64}$$

On the other hand, by (11.62), we get

$$\left|\int_{(T_{\mathbf{R}}X)_{y_0}\times(T_{\mathbf{R}}X)_{y_0}} \langle (P_{u,T}^{3,Z_0/T} - P_{0,T}^{3,Z_0/T})(Z-\widetilde{Z}, Z'-\widetilde{Z}')U, U'\rangle \times \frac{1}{\beta^{2l}}\varphi\left(\frac{\widetilde{Z}}{\beta}\right)\frac{1}{\beta^{2l}}\varphi\left(\frac{\widetilde{Z}'}{\beta}\right) dv_{TX}(\widetilde{Z})\, dv_{TX}(\widetilde{Z}')\right| \leq C\frac{uT}{\beta^{2l}}(1+|Z_0|)^{2l+1}(1+|Z|+|Z'|)^{2l}. \tag{11.65}$$

By choosing $\beta = (uT)^{1/(2l+1)}$, we deduce from (11.64), (11.65) that if $Z_0$, $Z$, $Z'$ are taken as before, then

$$|\langle (P_{u,T}^{3,Z_0/T} - P_{0,T}^{3,Z_0/T})(Z, Z')U, U'\rangle| \leq C'(uT)^{1/(2l+1)}(1+|Z_0|)^{2l+1}(1+|Z|+|Z'|)^{2l}. \tag{11.66}$$

From (11.38), (11.66), it follows that under the assumptions of Theorem 11.13,

$$|(P_{u,T}^{3,Z_0/T} - P_{0,T}^{3,Z_0/T})(Z, Z')| \leq \frac{c(1+|Z|+|Z'|)^r}{(1+|Z_0|)^m}\exp\left(-\frac{C}{2}|Z-Z'|^2\right)(uT)^{1/2(2l+1)}, \tag{11.67}$$

from which (11.33) follows. Hence the proof of Theorem 11.13 is completed.

This terminates the proof of Theorem 8.6.

**Remark 11.18.** It should be observed that in the estimates (11.33), (11.38), (11.46), 11.59), 11.61), (11.66), if $Z, Z' \in N_{X_g/X,\mathbf{R},y_0}$, then the bounding factor $(1+|Z_0|)^{-m}$ can be replaced by a factor $\exp(-C''|Z_0|^2)$, with $C'' > 0$. In fact, for $q \in \mathbf{R}_+^*$, we still define $\widetilde{K}_q(a)$ as in (11.48)–(11.51). Set

$$\widetilde{K}'_q(a) = \exp(-a) - \widetilde{K}_q(a). \tag{11.68}$$

Using Proposition 11.17 and (11.68), we find that given $c, C > 0$, there exist $c', C' > 0$ such that if for $q \geq 1$, $a \in V_c$, $\mathrm{Re}(a) \geq C|q|^2$, then

$$|\widetilde{K}'_q(a)| \leq c'\exp(-C'|q|^2). \tag{11.69}$$

By (11.68), we get

$$P_{u,T}^{3,Z_0/T}(Z, Z') = \widetilde{K}_{|Z_0|/2}(L_{u,T}^{3,Z_0/T})(Z, Z') + \widetilde{K}'_{|Z_0|/2}(L_{u,T}^{3,Z_0/T})(Z, Z'). \tag{11.70}$$

By Proposition 11.15, and by proceeding as in the proof of Theorem 11.14, $\widetilde{K}_{|Z_0|/2}(L^{3,Z_0/T}_{u,T})(Z, Z')$ can be estimated as in (11.32), with the extra bounding factor $\exp(-C|Z_0|^2)$ $(C > 0)$. Also by the finite propagation speed, and using the fact that $Z, Z' \in N_{X_g/X,\mathbf{R},y_0}$ and (11.53), the same type of estimates can be proved for $\widetilde{K}'_{|Z_0|/2}(Z, Z')$. From (11.70), we ultimately get the required estimate.

## XII. The analysis of the kernel of $g\exp(-(uD^X + TV/u)^2)$ for $T$ positive as $u$ tends to 0

The purpose of this section is to prove Theorem 8.7. Our method of proof follows closely [15, §12], where the same result was proved in the case where $G$ is trivial.

As in §11, part of the analysis is taken from [15]. However the geometry of the situation is more difficult than in [15]. Also we use estimates already established in §11.

This section is organized as follows. In part a, we introduce our assumptions and notation. In part b, we show that the proof of Theorem 8.7 can be localized near $Y_g$. In part c, we construct a coordinate system near $y_0 \in Y_g$ and a trivialization of $\Lambda(T^{*(0,1)}X)\hat{\otimes}\xi$ near $y_0$. In part d, we replace $X$ by $(T_{\mathbf{R}}X)_{y_0}$. In part e, we rescale the coordinate $Z \in (T_{\mathbf{R}}X)_{y_0}$ and also certain Clifford variables. In part f, we calculate the asymptotics as $u \to 0$ of the operator $L^{3,y_0}_{u,T/u}$ obtained from $(uD^X + TV/u)^2$ by such a rescaling. As in [15, §12], the operator $\mathscr{B}^2_{T^2}$ appears in this process. In part g, we establish uniform estimates on the rescaled heat kernels. Finally, in part h, we prove Theorem 8.7.

a. *Assumptions and notation.* Consider the exact sequence of holomorphic Hermitian vector bundles over $Y_g$

$$E : 0 \to TY|_{Y_g} \to TX|_{Y_g} \to N_{Y/X}|_{Y_g} \to 0. \tag{12.1}$$

Then $g$ acts naturally on each term of the exact sequence, by parallel isometries.

In this section, we use the notation of §7 with respect to the exact sequence (12.1). In particular, for $u > 0$, $\mathscr{B}^2_u$ denotes the operator constructed in Definition 7.4, and $Q^y_u(Z, Z')$ $(y \in Y_g,\ Z, Z' \in (T_{\mathbf{R}}X)_y)$ denotes the smooth kernel associated to $\exp(-\mathscr{B}^{2,y}_u)$ calculated with respect to $dv_{TX}(Z')/(2\pi)^{\dim X}$. Also we use the notation of §11.

We have the identification of holomorphic Hermitian vector bundles

(12.2) $$N_{Y_g/X} = N_{Y_g/X_g} \oplus N_{X_g/X}|_{Y_g}.$$

Let $P^{N_{Y_g/X_g}}$, $P^{N_{X_g/X}|_{Y_g}}$ be the projection operators from $N_{Y_g/X}$ on $N_{Y_g/X_g}$, $N_{X_g/X|Y_g}$. By (6.6)–(6.11) we have

(12.3) $$\begin{aligned} &TX^{0,\perp}_{|X_0} = N_{X_g/X}, \qquad N^0_{Y/X}|_{Y_g} = N_{Y_g/X_g}, \\ &N^0_{Y/X}|_{Y_g} \oplus TX^{0,\perp}_{|Y_g} = N_{Y_g/X}. \end{aligned}$$

Clearly, for $u > 0$, $T > 0$,

(12.4) $$\begin{aligned} &\operatorname{Tr}_s\left[N_H g \exp\left(-\left(uD^X + \frac{T}{u}V\right)^2\right)\right] \\ &\quad = \int_X \operatorname{Tr}_s[N_H g P_{u,T/u}(g^{-1}x, x)] \frac{dv_X(x)}{(2\pi)^{\dim X}}. \end{aligned}$$

In the whole section, the constant $T > 0$ will be fixed.

b. *The problem is localizable near* $Y_g$.

**Proposition 12.1.** *For any* $\alpha > 0$, *there is* $c > 0$, $C > 0$ *such that for* $u \in ]0, 1]$, $x, x' \in X$, $d^X(x, x') \geq \alpha$, *we have*

(12.5) $$|P_{u,T/u}(x, x')| \leq c \exp(-C/u^2).$$

*Proof.* For $0 < T \leq 1$, this result is proved in Proposition 11.3. For a general $T$, our proposition follows by a simple scaling argument.

**Proposition 12.2.** *For any* $\alpha > 0$, *there exist* $c > 0$, $C > 0$ *such that for* $x, x' \in X$, $d^X(x, Y) \geq \alpha$ *or* $d^X(x', Y) \geq \alpha$, $u \in ]0, 1]$,

(12.6) $$|P_{u,T/u}(x, x')| \leq c \exp(-C/u^2).$$

*Proof.* Since the operator $\exp(-(uD^X + TV/u)^2)$ is selfadjoint and positive, $P_{u,T/u}(x, x')$ is also selfadjoint, and moreover

(12.7) $$|P_{u,T/u}(x, x')| \leq |P_{u,T/u}(x, x)|^{1/2} |P_{u,T/u}(x', x')|^{1/2}.$$

Also by [15, Proposition 12.1], there is $C > 0$ such that if $d^X(x, Y) \geq \alpha$, for $u \in ]0, 1]$, then

(12.8) $$|P_{u,T/u}(x, x)| \leq c \exp(-C/u^2).$$

Moreover, using Ito's formula as in [15, equations (12.11)–(12.15)], we find that there exists $C' > 0$ such that for $x' \in X$ and $u \in ]0, 1]$,

(12.9) $$|P_{u,T/u}(x', x')| \leq C'/u^{2\dim X}.$$

From (12.7), (12.9), we get (12.6). Hence the proof of our proposition is completed. q.e.d.

For $\varepsilon > 0$, let $\mathscr{U}_\varepsilon(Y_g)$ be the open neighborhood of $Y_g$ in $X$, which we define as in Remark 11.8. By Propositions 12.1 and 12.2, there exist $c > 0$, $C > 0$ such that for $u \in ]0, 1]$,

$$(12.10)\quad \left| \int_{X \backslash \mathscr{U}_{\varepsilon/8}(Y_g)} \mathrm{Tr}_s[N_H g P_{u,T/u}(g^{-1}x, x)] \frac{dv_X(x)}{(2\pi)^{\dim X}} \right| \leq c \exp\left(\frac{-C}{u^2}\right).$$

By (12.4), (12.10) indicates that the proof of Theorem 8.7 can be localized near $Y_g$.

For $b > 0$ small enough, the map $(y_0, Z_0) \in N_{Y_g/X,\mathbf{R}}$, $|Z_0| < b \to \exp^X_{y_0}(Z_0) \in X$ is a diffeomorphism of an open neighborhood of $Y_g$ in $N_{Y_g/X,\mathbf{R}}$ into a tubular neighborhood of $Y_g$ in $X$. In the sequel we assume that $\varepsilon \in ]0, \frac{b}{2}]$.

If $y_0 \in Y_g$, $Z_0 \in N_{Y_g/X,\mathbf{R},y_0}$, $|Z_0| \leq \varepsilon$, we identify $(y_0, Z_0)$ with $\exp^X_{y_0}(Z_0) \in X$. In particular, in the coordinates $(y_0, Z_0)$,

$$(12.11)\quad g^{-1}(y_0, Z_0) = (y_0, g^{-1}Z_0).$$

Let $k(y_0, Z_0)$ be such that for $|Z_0| < \varepsilon$,

$$(12.12)\quad dv_X(y_0, Z_0) = k(y_0, Z_0)dv_{Y_g}(y_0)dv_{N_{Y_g/X}}(Z_0).$$

Then

$$(12.13)\quad k(y_0, 0) = 1.$$

The main technical result of this section is as follows.

**Theorem 12.3.** *For $y_0 \in Y_g$, $Z_0 \in N_{Y_g/X,\mathbf{R},y_0}$, we have*

$$(12.14)\quad \begin{aligned} &\lim_{u\to 0} \frac{u^{2\dim N_{Y_g/X}}}{(2\pi)^{\dim X}} \mathrm{Tr}_s[N_H g P_{u,T/u}(g^{-1}(y_0, uZ_0), (y_0, uZ_0))] \\ &= \left\{ \frac{1}{(2\pi)^{\dim N_{Y_g/X}}} \Phi \mathrm{Tr}_s[N_H g Q^{y_0}_{T^2}(g^{-1}Z_0, Z_0)] \mathrm{ch}_g(\eta, h^\eta) \right\}^{\max}. \end{aligned}$$

*For any $p \in \mathbf{N}$, there is $c > 0$, $C > 0$ such that for $u \in ]0, 1]$, $y_0 \in Y_g$, $Z_0 \in N_{Y_g/X,\mathbf{R},y_0}$, $|Z_0| \leq \varepsilon/8u$, we have*

$$(12.15)\quad \begin{aligned} &u^{2\dim N_{Y_g/X}} |\mathrm{Tr}_s[N_H g P_{u,T/u}(g^{-1}(y_0, uZ_0), (y_0, uZ_0))]| \\ &\leq \frac{c}{(1 + |P^{N_{Y_g/X_g}} Z_0|)^p} \exp(-C|P^{N_{X_g/X}} Z_0|^2). \end{aligned}$$

*Proof.* The remainder of the section is devoted to the proof of Theorem 12.3.

**Remark 12.4.** Clearly

$$\begin{aligned}
&\int_{\mathscr{U}_{\varepsilon/8}(Y_g)} \operatorname{Tr}_s[N_H g P_{u,T/u}(g^{-1}x, x)] \frac{dv_X(x)}{(2\pi)^{\dim X}} \\
(12.16)\quad &= \frac{1}{(2\pi)^{\dim X}} \int_{Y_g} \left\{ \int_{\substack{Z_0 \in N_{Y_g/X,\mathbf{R},y_0} \\ |Z_0| \le \varepsilon/8u}} u^{2\dim N_{Y_g/X}} \right. \\
&\quad \times \operatorname{Tr}_s[N_H g P_{u,T/u}(g^{-1}(y_0, uZ_0), (y_0, uZ_0))] \\
&\quad \left. \times k(y_0, uZ_0)\, dv_{N_{Y_g/X}}(Z_0) \right\} dv_{Y_g}(y_0).
\end{aligned}$$

Using Theorem 12.3 and dominated convergence, we find that as $u \to 0$,

(12.17)

$$\begin{aligned}
&\int_{\mathscr{U}_{\varepsilon/8}(Y_g)} \operatorname{Tr}_s[N_H g P_{u,T/u}(g^{-1}x, x)] \frac{dv_X(x)}{(2\pi)^{\dim X}} \\
&\to \int_{Y_g} \left\{ \int_{N_{Y_g/X}} \Phi \operatorname{Tr}_s[N_H g Q_{T^2}(g^{-1}Z_0, Z_0)] \frac{dv_{N_{Y_g/X}}(Z_0)}{(2\pi)^{\dim N_{Y_g/X}}} \right\} \operatorname{ch}_g(\eta, h^\eta).
\end{aligned}$$

By (12.4), (12.10), (12.17), we get

$$\begin{aligned}
(12.18)\quad &\lim_{u\to 0} \operatorname{Tr}_s\left[ N_H g \exp\left( - \left( uD^X + \frac{T}{u} V \right)^2 \right) \right] \\
&= \int_{Y_g} \Phi \operatorname{Tr}_s[N_H g \exp(-\mathscr{B}_{T^2}^2)] \operatorname{ch}_g(\eta, h^\eta),
\end{aligned}$$

which is exactly Theorem 8.7.

c. *A local coordinate system and a trivialization of* $\Lambda(T^{*(0,1)}X)\hat{\otimes}\xi$. Let $a > 0$ be the injectivity radius of $(X, h^{TX})$. In the sequel, we take $\varepsilon \in ]0, \inf(a/2, b/2)]$. We fix $y_0 \in Y_g$. For $Z \in (T_{\mathbf{R}}X)_{y_0}$, $|Z| < \varepsilon$, we identify $Z$ with $\exp^X_{y_0}(Z) \in X$. Let $k'(Z)$, $|Z| < a$ be the function defined by

$$(12.19)\qquad dv_X(Z) = k'(Z)\, dv_{TX}(Z).$$

Then $k'(0) = 1$. If $|Z| < \varepsilon$, we identify $(TX)_Z$, $\Lambda(T^{*(0,1)}X)_Z$ with $(TX)_{y_0}$, $\Lambda(T^{*(0,1)}X)_{y_0}$ (resp. $\xi_Z$ with $\xi_{y_0}$) by parallel transport with respect to the connections $\nabla^{TX}$, $\nabla^{\Lambda(T^{*(0,1)}X)}$ (resp. $\widetilde{\nabla}^\xi$) along the geodesic $t \in [0, 1] \to tZ$.

With respect to §11 d, the main difference is that we do not need the intermediate $Z_0 \in N_{Y_g/X_g,\mathbf{R},y_0}$, which is here identically $0$.

Let $\Gamma_Z^{TX}$, $\Gamma_Z^{\xi}$ be the connection forms of the connections $\nabla^{TX}$, $\nabla^{\xi}$ in the considered trivialization of $TX$, $\xi$. As in [15, equation (12.23)], we have

$$\Gamma_0^{\xi} = B_{y_0}, \qquad \Gamma_Z^{TX} = \frac{1}{2}(\nabla^{TX})^2_{y_0}(Z, \cdot) + \mathscr{O}(|Z|^2). \tag{12.20}$$

If $|Z| \le \varepsilon$, $U \in (T_{\mathbf{R}}X)_Z$, then $\nabla_U$ denotes the standard differentiation operator in the direction $U$ acting on smooth sections of $(\Lambda(T^{*(0,1)}X)\hat{\otimes}\xi)_{y_0}$ over $(T_{\mathbf{R}}X)_{y_0}$.

Of course since $g$ is an isometry, $g$ acts linearly in the coordinates $Z$.

d. *Replacing the manifold $X$ by $(T_{\mathbf{R}}X)_{y_0}$.* We use the notation of Definition 11.9.

**Definition 12.5.** Set

$$L^{1,y_0}_{u,T/u} = L^{1,0}_{u,T/u}, \qquad M^{1,y_0}_u = M^{1,0}_u. \tag{12.21}$$

Let $P^{1,y_0}_{u,T/u}(Z, Z')$ $(Z, Z' \in (T_{\mathbf{R}}X)_{y_0})$ be the smooth kernel associated to $\exp(-L^{1,y_0}_{u,T/u})$, which is calculated with respect to $dv_{TX}(Z')/(2\pi)^{\dim X}$.

The same argument as in the proof of Proposition 11.5 shows that there exists $c > 0$ such that for $u \in ]0, 1]$, $y_0 \in Y_g$, $Z_0 \in N_{Y_g/X,\mathbf{R},y_0}$, $|Z_0| \le \varepsilon/8$, we have

$$\begin{aligned}|P_{u,T/u}(g^{-1}(y_0, Z_0), (y_0, Z_0))k'(Z_0) - P^{1,y_0}_{u,T/u}(g^{-1}Z_0, Z_0)| \\ \le c\exp(-C/u^2).\end{aligned} \tag{12.22}$$

From (12.22), it is clear that to prove Theorem 12.3, we only need to show that for $y_0 \in Y_g$, $Z_0 \in N_{Y_g/X,\mathbf{R},y_0}$,

$$\begin{aligned}&\lim_{u\to 0} \frac{1}{(2\pi)^{\dim X}} u^{2\dim N_{Y_g/X}} \operatorname{Tr}_s[N_H g P^{1,y_0}_{u,T/u}(g^{-1}uZ_0, uZ_0)] \\ &= \frac{1}{(2\pi)^{\dim N_{Y_g/X}}} \{\Phi \operatorname{Tr}_s[N_H g Q^{y_0}_{T^2}(g^{-1}Z_0, Z_0)] \operatorname{ch}_g(\eta, h^{\eta})\}^{\max},\end{aligned} \tag{12.23}$$

and that given $p \in \mathbf{N}$, there is $c > 0$, $C > 0$ such that for $u \in ]0, 1]$, $y_0 \in Y_g$, $Z_0 \in N_{Y_g/X,\mathbf{R},y_0}$, $|Z_0| \le \varepsilon/8u$, we have

$$\begin{aligned}u^{2\dim N_{Y_g/X}} |\operatorname{Tr}_s[N_H g P^{1,y_0}_{u,T/u}(g^{-1}uZ_0, Z_0)]| \\ \le c\exp(-C|P^{N_{X_g/X}}Z_0|^2)/(1 + |P^{N_{Y_g/X_g}}Z_0|)^p.\end{aligned} \tag{12.24}$$

e. *Rescaling of the variable* $Z$ *and the Clifford variables.* We still define $F_u$ as in (11.24). Set

$$L^{2,u_0}_{u,T/u} = F_u^{-1} L^{1,y_0}_{u,T/u} F_u, \qquad M_u^{2,y_0} = F_u^{-1} M_u^{1,y_0} F_u. \tag{12.25}$$

Let $(e_1, \cdots, e_{2l'})$ and let $(e_{2l'+1}, \cdots, e_{2l})$ be orthonormal bases of $(T_{\mathbf{R}} Y_g)_{y_0}$ and $N_{Y_g/X,\mathbf{R},y_0}$ respectively. Then $(e_1, \cdots, e_{2l})$ is an orthonormal basis of $(T_{\mathbf{R}}X)_{y_0}$.

**Definition 12.6.** For $u > 0$, $1 \le i \le 2l'$, set

$$c_u(e_i) = \sqrt{2} e^i \wedge /u - u i_{e_i} / \sqrt{2}. \tag{12.26}$$

For $u > 0$, $T > 0$, let $L^{3,y_0}_{u,T/u}$, $M_u^{3,y_0}$ be the operators obtained from $L^{2,y_0}_{u,T/u}$, $M_u^{2,y_0}$ by replacing the Clifford variables $c(e_i)$ by $c_u(e_i)$ for $1 \le i \le 2l'$, while leaving unchanged the $c(e_i)$'s $(2l'+1 \le i \le 2l)$.

Let $P^{3,y_0}_{u,T/u}(Z, Z')$ $(Z, Z' \in (T_{\mathbf{R}}X)_{y_0})$ be the smooth kernel associated to the operator $\exp(-L^{3,y_0}_{u,T/u})$, which is calculated with respect to $dv_{TX}(Z')/(2\pi)^{\dim X}$. We can still expand $P^{3,y_0}_{u,T/u}(g^{-1}Z, Z)$ as in (11.27), the difference being that on the right-hand side of (11.27), $l''$ is replaced by $l'$, and $N_{X_g/X}$ by $N_{Y_g/X}$. We define $[(P^{3,y_0}_{u,T/u})(g^{-1}Z, Z)]^{\max} \in (c(N_{Y_g/X,\mathbf{R}}) \hat{\otimes} \operatorname{End}(\xi))_{y_0}$ as in (11.28), $l''$ being replaced by $l'$.

Also $c(N_{Y_g/X,\mathbf{R}}) \hat{\otimes} \operatorname{End}(\xi)$ acts on $(\Lambda(\overline{N}^*_{Y_g/X}) \hat{\otimes} \operatorname{End}(\xi))_{y_0}$, and so the supertrace of elements in this algebra is well defined.

Now we extend [15, Proposition 12.9].

**Proposition 12.7.** *The following equality holds*:

$$\begin{aligned} &u^{2\dim N_{Y_g/X}} \operatorname{Tr}_s[N_H g P^{1,y_0}_{u,T/u}(g^{-1}uZ_0, uZ_0)] \\ &\qquad = (-i)^{\dim Y_g} \operatorname{Tr}_s[N_H g [P^{3,y_0}_{u,T/u}(g^{-1}Z_0, Z_0)]^{\max}]. \end{aligned} \tag{12.27}$$

*Proof.* Observe that since $g$ acts as the identity on $TY_g$, applying the Clifford rescaling on $g$ does not change $g$. Our proposition is now a trivial consequence of Getzler [18], [15, Proposition 11.2].

f. *The asymptotics of the operator* $L^{3,y_0}_{u,T/u}$ *as* $u \to 0$. If $U \in (T_{\mathbf{R}}X)_{y_0}$, let $(\tau U)_Z \in (T_{\mathbf{R}}X)_Z \simeq (T_{\mathbf{R}}X)_{y_0}$ be the parallel transport of $U$ along $t \in [0, 1] \to tZ$ with respect to $\nabla^{TX}$.

As in [15, equation (12.34)],

$$
\begin{aligned}
L^{3,y_0}_{u,T/u} &= M^{3,y_0}_u \\
&+ \rho^2(uZ)\left\{\frac{T}{u}\sum_1^{2l'}\left(e^i\wedge -\frac{u^2}{2}i_{e_i}\right)(\nabla^{\xi}_{\tau e_i}V)(uZ)\right. \\
&\left. +T\sum_{2l'+1}^{2l}\frac{c(e_i)}{\sqrt{2}}(\nabla^{\xi}_{\tau e_i}V)(uZ)+\frac{T^2}{u^2}V^2(uZ)\right\} \\
&+\frac{T^2}{u^2}(1-\rho^2(uZ))P^{\xi^+}_{y_0}.
\end{aligned}
\tag{12.28}
$$

Let $i_g$ be the embedding $Y_g \to X$. We will write that as $u \to 0$, $M^{3,y_0}_u$ converges to the differential operator $M^{3,y_0}_0$ if the smooth coefficients of $M^{3,y_0}_u$ converge to the coefficients of $M^{3,y_0}_0$ together with their derivatives uniformly over the compact sets of $(T_{\mathbf{R}}X)_{y_0}$.

**Theorem 12.8.** *Let $M^{3,y_0}_0$ be the operator*

$$
\begin{aligned}
M^{3,y_0}_0 = &-\frac{1}{2}\sum_1^{2l}\left(\nabla_{e_i}+\frac{1}{2}\langle i_g^*(\nabla^{TX})^2_{y_0}Z, e_i\rangle\right)^2 \\
&+ i_g^*\left((\nabla^{\xi})^2_{y_0}+\frac{1}{2}\operatorname{Tr}[(\nabla^{TX})^2]_{y_0}\right).
\end{aligned}
\tag{12.29}
$$

*Then as $u \to 0$,*

$$
M^{3,y_0}_u \to M^{3,y_0}_0. \tag{12.30}
$$

*Proof.* The proof of our theorem is the same as that of [15, Theorem 12.10]. q.e.d.

In the sequel, we may and will assume that $(e_{2l'+1}, \cdots, e_{2m})$ and $(e_{2m+1}, \cdots, e_{2l})$ are orthonormal bases of $N_{Y_g/Y,\mathbf{R},y_0}$ and $N_{Y/X,\mathbf{R},y_0}$ respectively.

Recall that we use the convention of (7.1) instead of [15, §5 a] for the definition of $\hat{c}$.

**Definition 12.9.** Let $S \in \operatorname{End}(\Lambda^{\text{even}}(\overline{N}^*_{Y/X})\hat{\otimes}\Lambda(N^*_{Y/X}))_{y_0}$ be given by

$$
S = \frac{\sqrt{-1}}{2}\sum_{2m+1}^{2l} c(e_i)\hat{c}(e_i). \tag{12.31}
$$

Clearly $S$ extends to an operator acting on $(\Lambda(T^*_{\mathbf{R}}Y_g)\hat{\otimes}\Lambda(\overline{N}^*_{Y/X})\hat{\otimes}\Lambda(N^*_{Y/X})\otimes\eta)_{y_0}$. Also recall that by [15, equation (8.31)], $\xi^-|_Y = \Lambda(N^*_{Y/X})\otimes\eta$. Therefore $S$ acts on $(\Lambda(T^*_{\mathbf{R}}Y_g)\hat{\otimes}\Lambda(\overline{N}^*_{Y/X})\otimes\xi^-)_{y_0}$.

Let $V^{\pm}$ be the restriction of $V$ to $\xi^{\pm}$, and $P^{TY}$, $P^{N_{Y/X}}$ be the orthogonal projection operators $TX|_Y \to TY$, $TX|_Y \to N_{Y/X}$. Now we prove an extension of [15, Theorem 12.12].

**Theorem 12.10.** *For $y_0 \in Y_g$, $Z \in (T_{\mathbf{R}}X)_{y_0}$, as $u \to 0$,*

$$\begin{aligned}
&\frac{1}{u}\sum_1^{2l'}\left(e^i\wedge -\frac{u^2}{2}i_{e_i}\right)(\nabla^{\xi}_{\tau e_i}V)(uZ)\\
&\qquad = \frac{1}{u}i^*_g\nabla^{\xi}V(y_0)+\sum_1^{2l'}e^i\wedge\tilde{\nabla}^{\xi}_Z\nabla^{\xi}_{\tau e_i}V(y_0)\\
&\qquad\quad + \frac{1}{u}\mathscr{O}(|uZ|^2+u^2),\\
&\sum_{2l'+1}^{2l}\frac{c(e_i)}{\sqrt{2}}\nabla^{\xi}_{\tau e_i}V(uZ) = \sum_{2l'+1}^{2l}\frac{c(e_i)}{\sqrt{2}}(\nabla^{\xi}_{e_i}V)(y_0)+\mathscr{O}(|uZ|),\\
&\frac{1}{u^2}(V^+(uZ))^2 = \frac{1}{u^2}V^+(y_0)^2+\frac{1}{u^2}\mathscr{O}(|uZ|),\\
&\frac{1}{u^2}V^-(uZ)^2 = \left((\tilde{\nabla}^{\xi}_Z V^-)(y_0)+\frac{1}{u}\mathscr{O}(|uZ|)^2\right)^2.
\end{aligned}\tag{12.32}$$

*Moreover, the following equalities hold*:

$$\begin{aligned}
&P^{\xi^-}i^*\nabla^{\xi}VP^{\xi^-} = 0,\\
&P^{\xi^-}\sum_{i=1}^{2l'}e^i\wedge\tilde{\nabla}^{\xi}_Z\nabla^{\xi}_{\tau e_i}V(y_0)P^{\xi^-} = P^{\xi^-}\frac{\sqrt{-1}}{\sqrt{2}}\hat{c}(AP^{TY}Z)P^{\xi^-},\\
&P^{\xi^-}\left(\sum_{i=2l'+1}^{2l}\frac{c(e_i)}{\sqrt{2}}\nabla^{\xi}_{e_i}V(y_0)\right)P^{\xi^-} = P^{\xi^-}SP^{\xi^-},\\
&(\tilde{\nabla}^{\xi}_Z V^-(y_0))^2 = \frac{|P^{N_{Y/X}}Z|^2}{2}P^{\xi^-},\\
&i^*(\nabla^{\xi^-})^2 = i^*(P^{\xi^-}(\nabla^{\xi})^2P^{\xi^-}\\
&\qquad - P^{\xi^-}(\nabla^{\xi}V)P^{\xi^+}[(V^+)^2]^{-1}P^{\xi^+}(\nabla^{\xi}V)P^{\xi^-}).
\end{aligned}\tag{12.33}$$

*Proof.* As in [15, Theorem 12.12], we get (12.32) by Taylor expansion. The first two equalities in (12.33) were already established in [15, Theorem 12.12]. By [5, Proposition 3.5], [15, equation (11.64)], we know that if $U \in (T_{\mathbf{R}}Y)_{y_0}$, then $\nabla^{\xi}_{U}V(y_0)$ maps $\xi^{-}_{y_0}$ into $\xi^{+}_{y_0}$. Also by [5, §§1c and 3j] or [15, Proposition 8.13 and equation (12.45)], if $U \in N_{Y/X,\mathbf{R},y_0}$, then

$$P^{\xi^-}\nabla^{\xi}_{U}V(y_0)P^{\xi^-} = \sqrt{-1}\hat{c}(U)/\sqrt{2}.$$

The third equality in (12.33) follows from these considerations, and the last two equalities in (12.33) were already established in [15, Theorem 12.12].

g. *Uniform estimates on* $P^{3,y_0}_{u,T/u}$. Here, we extend [15, Theorem 12.14].

**Theorem 12.11.** *There is* $C > 0$ *such that for* $m \in \mathbf{N}$, *there exist* $C' > 0$, $r \in \mathbf{N}$ *for which if* $u \in ]0, 1]$, $y_0 \in Y_g$, $Z, Z' \in N_{Y_g/X,\mathbf{R},y_0}$, $|Z|, |Z'| \le \varepsilon/8u$, *then*

$$(12.34)\quad \begin{aligned}&|P^{3,y_0}_{u,T/u}(Z, Z')|\\ &\quad\le C'(1+|P^{N_{Y/X}}Z|)^{-m}(1+|P^{N_{Y_g/Y}}Z|)^{r}\exp(-C|Z-Z'|^2).\end{aligned}$$

*Moreover, for* $M > 0$, $m' \in \mathbf{N}$, *there exists* $C'' > 0$ *such that for* $u \in ]0, 1]$, $y_0 \in Y_g$, *we have*

$$(12.35)\quad \sup_{\substack{Z,Z' \in (T_{\mathbf{R}}X)_{y_0}\\ |Z|,|Z'|\le M\\ |\alpha|,|\alpha'|\le m'}} \left|\frac{\partial^{|\alpha|+|\alpha'|}}{\partial Z^{\alpha}\partial Z'^{\alpha'}}P^{3,y_0}_{u,T/u}(Z, Z')\right| \le C''.$$

*Proof.* We will assume that $T \in ]0, 1]$. The general case will follow by a scaling argument. Given $u_0 \in ]0, 1]$, the inequality (12.34) is trivial for $u \ge u_0$. Then (12.35) follows from Theorem 11.14 for $Z_0 = 0$.

Also we use inequalities (11.43) with $Z_0 = 0$ and $T$ replaced by $T/u$. Clearly if $Z \in N_{Y_g/X,\mathbf{R},y_0}$, then

$$(12.36)\quad \frac{T}{u}d^X(uP^{N_{Y/X}}Y, Z) \le \frac{T}{u}d^X(uZ, Y) + C|P^{N_{Y_g/Y}}Z|.$$

Moreover, by construction,

$$(12.37)\quad (T/u)d^X(uP^{N_{Y/X}}Z, Y) = T|P^{N_{Y/X}}Z|.$$

From (11.43), (12.36), (12.37), we deduce that given $m \in \mathbf{N}$, there exists $C' > 0$ such that if $u \in ]0, 1]$, $y_0 \in Y_g$, $Z, Z' \in N_{Y_g/X,\mathbf{R},y_0}$, $|Z|$, $|Z'| \le \varepsilon/8u$, then

$$(12.38)\quad \begin{aligned}&(1+|P^{N_{Y/X}}Z|)^m|P^{3,y_0}_{u,T/u}(Z, Z')|\\ &\quad\le C'(1+|P^{N_{Y_g/Y}}Z|)^m(1+|Z|+|Z'|)^{2l},\end{aligned}$$

which together with (11.38) gives (12.34). Hence the proof of our theorem is completed.

h. *Proof of Theorem* 12.3. Clearly $g^{-1}(y_0, Z) = (y_0, g^{-1}Z)$. Also if $Z \in N_{Y_g/X}$, then

$$|g^{-1}Z - Z|^2 = |(g^{-1} - 1)P^{N_{X_g/X}}Z|^2 \geq C|P^{N_{X_g/X}}Z|^2 \geq C|P^{N_{Y_g/Y}}Z|^2. \tag{12.39}$$

Using Proposition 12.7, Theorem 12.11, and (12.39), we get (12.24).

The obvious analogue of [15, Theorem 12.16] still holds, for the same reasons as in [15]. As in [15, equation (12.120)], we find that as $u \to 0$

$$P^{3,y_0}_{u,T/u} \to Q^{y_0}_{T^2} \exp(-(\nabla^\eta)^2_{y_0}) \quad \text{in the sense of distributions.} \tag{12.40}$$

By the uniform bounds of Theorem 12.11, we deduce from (12.40) that as $u \to 0$,

$$P^{3,y_0}_{u,T/u}(Z, Z') \to Q^{y_0}_{T^2}(Z, Z') \exp(-(\nabla^\eta)^2_{y_0}) \quad \text{uniformly over compact set on } (T_{\mathbf{R}}X)_{y_0} \times (T_{\mathbf{R}}X)_{y_0}. \tag{12.41}$$

From Proposition 12.7 and (12.41), it follows that as $u \to 0$, for $Z_0 \in N_{Y_g/X, \mathbf{R}, y_0}$,

$$u^{2\dim N_{Y_g/X}} \operatorname{Tr}_s[N_H g P^{1,y_0}_{u,T/u}(g^{-1}uZ_0, uZ_0)] \to (-i)^{\dim Y_g} \{\operatorname{Tr}_s[N_H g Q^{y_0}_{T^2}(g^{-1}Z_0, Z_0)] \times \operatorname{Tr}[g \exp(-(\nabla^\eta)^2_{y_0})]\}^{\max}, \tag{12.42}$$

which is equivalent to (12.23).

The proof of Theorem 12.3 is completed.

**Remark 12.12.** If $Z, Z' \in N_{Y_g/X, \mathbf{R}, y_0}$, then

$$1 + |P^{N_{Y/X}}Z'| \leq (1 + |P^{N_{Y/X}}Z|)(1 + |Z - Z'|). \tag{12.43}$$

So (12.34) can be made symmetric in $Z$, $Z'$. Also the same arguments as in Remark 11.18 show that in Theorem 12.11, the weighting factor $(1 + |P^{N_{Y/X}}Z|)^{-m}$ can be replaced by $\exp(-C''|P^{N_{Y/X}}Z|^2)$, with $C'' > 0$.

## XIII. The analysis of the kernel of $g\exp(-(uD^X + TV)^2)$ in the range $u \in ]0, 1]$, $T \geq 1/u$

The purpose of this section is to prove Theorem 8.8. Note that for a fixed $u \in ]0, 1]$, inequality (8.22) follows from Theorem 8.4. So the whole point of Theorem 8.8 is to get uniformity for $u \in ]0, 1]$. This section is the obvious extension of [15, §13] to the case of a nontrivial group $G$.

As in [15], we first show that the proof of Theorem 8.8 can be localized on a tubular neighborhood of $Y$. Then as in [15], using the finite propagation speed, we show that the proof of Theorem 8.8 is also local on $X$. In our context, this allows us to localize the proof on an arbitrary open neighborhood of a point in $X_g$, and ultimately to localize the proof on an arbitrary open neighborhood of a point in $Y_g$. Once this reduction is done, we use the techniques of [15, §13] together with the finite propagation speed, which allows us to establish the Gaussian decay of the rescaled heat kernel in directions normal to $X_g$. This argument of finite propagation speed is essentially related to which we did in §§11 and 12.

This section is organized as follows. In part a, we show that our problem is localized globally near $Y$. Part b contains a reduction of the proof of Theorem 8.8 to a local problem on $X$. We construct a holomorphic function $\lambda \in \mathbf{C} \to \widetilde{F}_u(\lambda) \in \mathbf{C}$, and replace $\exp(-(uD^X + TV/u)^2)$ by $\widetilde{F}_u((uD^X + TV/u)^2)$. Part c describes various properties of $\widetilde{F}_u(\lambda)$ as $|\lambda| \to +\infty$.

Parts d and e construct a coordinate system and a trivialization of $\Lambda(T^{*(0,1)}X)\hat{\otimes}\xi$ near a given $y_0 \in Y_g$, and replace $X$ by $(T_{\mathbf{R}}X)_{y_0}$, and $(uD^X + TV/u)^2$ by an operator $\mathscr{L}_{u,T}^{1,y_0}$ acting on $(T_{\mathbf{R}}X)_{y_0}$.

In part f, we rescale the coordinate $Z_0 \in (T_{\mathbf{R}}X)_{y_0}$, and also use Getzler's rescaling on certain Clifford variables. The operator $\mathscr{L}_{u,T}^{1,y_0}$ is then changed to $\mathscr{L}_{u,T}^{3,y_0}$.

Parts g, h, i summarize very briefly the content of key subsections of [15, §13], whose results can be used here almost without any change. Part j establishes estimates on the kernel of $\widetilde{F}_u(\mathscr{L}_{u,T}^{3,y_0})$. Finally in part k we prove Theorem 8.8.

Let us again insist on the fact that we use many results of [15, §13], and that in particular part of the algebra and the functional analytic machine were already developed in [15].

We use the notation of §§11 and 12.

a. *The problem is localizable globally near* $Y$. As in (12.4), we have the formula

$$\operatorname{Tr}_s\left[N_h g \exp\left(-\left(uD^X + \frac{T}{u}V\right)^2\right)\right] = \int_X \operatorname{Tr}_s[N_H g P_{u,T/u}(g^{-1}x, x)] \frac{dv_X(x)}{(2\pi)^{\dim X}}. \tag{13.1}$$

**Proposition 13.1.** *For any $\alpha > 0$, there exist $c > 0$, $C > 0$ such that for $x, x' \in X$, $d^X(x, Y) \geq \alpha$ or $d^X(x', Y) \geq \alpha$, and $u \in ]0, 1]$, $T \geq 1$, the following inequality holds*:

$$|P_{u,T/u}(x, x')| \leq c\exp(-CT). \tag{13.2}$$

*Proof.* As in (12.7), we have the inequalities

$$|P_{u,T/u}(x, x')| \leq |P_{u,T/u}(x, x)|^{1/2}|P_{u,T/u}(x', x')|^{1/2}. \tag{13.3}$$

Assume that $d^X(x, Y) \geq \alpha$. By [15, equations (13.3), (13.4)], there is $c > 0$, $C > 0$ such that for $u \in ]0, 1]$, $T \geq 1$,

$$|P_{u,T/u}(x, x)| \leq c\exp(-CT/u^2). \tag{13.4}$$

Also, since $T \geq 1$,

$$|P_{u,T/u}(x', x')| \leq |P_{u/\sqrt{T},\sqrt{T}/u}(x', x')|. \tag{13.5}$$

Moreover by (12.9), there is $C > 0$ such that for any $s \in ]0, 1]$, $x' \in X$,

$$|P_{s,1/s}(x', x')| \leq C/s^{2\dim X}. \tag{13.6}$$

From (13.5), (13.6), we deduce that

$$|P_{u,T/u}(x', x')| \leq C(T/u^2)^{\dim X}, \tag{13.7}$$

which together with (13.3), (13.6) gives (13.2). Hence the proof of our proposition is completed.

**Remark 13.2.** For $\varepsilon > 0$ small enough, we define the tubular neighborhood $\mathscr{U}_\varepsilon(Y)$ of $Y$ in $X$ as in Remark 11.8. By (13.2), we find that

$$\left|\int_{X\setminus\mathscr{U}_\varepsilon(Y)} \mathrm{Tr}_s[N_H g P_{u,T/u}(g^{-1}x, x)]\frac{dv_X(x)}{(2\pi)^{\dim X}}\right| \leq c\exp(-CT). \tag{13.8}$$

It is now clear that to prove Theorem 8.8, we only need to show that there exist $C > 0$, $\delta \in ]0, 1]$ such that for $u \in ]0, 1]$, $T \geq 1$, we have

$$\left|\int_{\mathscr{U}_\varepsilon(Y)} \mathrm{Tr}_s[N_H g P_{u,T/u}(g^{-1}x, x)]\frac{dv_X(x)}{(2\pi)^{\dim X}} - \frac{1}{2}\dim N_{Y/X}\chi_g(\eta)\right| \leq \frac{C}{T^\delta}. \tag{13.9}$$

b. *Finite propagation speed and localization.* Take $\varepsilon_0 > 0$ small enough so that $\mathscr{U}_{\varepsilon_0}(Y)$ is a tubular neighborhood of $Y$ in $X$. Recall that $a > 0$ is the injectivity radius of $(X, h^{TX})$. Let $b$ be the injectivity radius of $(Y, h^{TY})$.

We fix $\varepsilon \in ]0, \inf(\varepsilon_0/2, a/2, b/2)]$. Let $\alpha$ be a positive constant, whose precise value will be determined in §13 e.

Let $f$ be a smooth even function defined on $\mathbf{R}$ with values in $[0, 1]$, such that

$$f(t) = 1 \quad \text{for } |t| \leq \alpha/2, \qquad = 0 \quad \text{if } |t| \geq \alpha. \tag{13.10}$$

Set

$$g(t) = 1 - f(t). \tag{13.11}$$

**Definition 13.3.** If $u \in ]0, 1]$, $a \in \mathbf{C}$, set

$$\begin{aligned} F_u(a) &= \int_{-\infty}^{+\infty} \exp\left(it\sqrt{2}a\right) \exp\left(\frac{-t^2}{2}\right) f(ut) \frac{dt}{\sqrt{2\pi}}, \\ G_u(a) &= \int_{-\infty}^{+\infty} \exp\left(it\sqrt{2}a\right) \exp\left(\frac{-t^2}{2}\right) g(ut) \frac{dt}{\sqrt{2\pi}}. \end{aligned} \tag{13.12}$$

Then

$$\exp(-a^2) = F_u(a) + G_u(a). \tag{13.13}$$

Since $f$ is even, as in [15, §13 b], we see that $F_u(a)$, $G_u(a)$ are even functions, which take real values on $\mathbf{R}$. Moreover $F_u$ and $G_u$ lie in the Schwartz space $S(\mathbf{R})$, and so, as in [15], $F_u(uD^X + TV/u)$ and $G_u(uD^X + TV/u)$ are trace class operators.

First we extend [15, Theorem 13.4].

**Theorem 13.4.** *There exist $c > 0$, $C > 0$ such that for $u \in ]0, 1]$, $T \geq 1$,*

$$\begin{aligned} &\left| \operatorname{Tr}_s \left[ N_H g G_u \left( uD^X + \frac{T}{u} V \right) \right] - \frac{1}{2} \dim N_{Y/X} \chi_g(\eta) G_u(0) \right| \\ &\qquad \leq \frac{c}{\sqrt{T}} \exp\left( -\frac{C}{u^2} \right). \end{aligned} \tag{13.14}$$

*Proof.* The proof of our theorem is essentially the same as that of [15, Theorem 13.4]. Of course here we use the arguments of §9 instead of [15, §§8 and 9]. In effect, we find that

$$\left|\operatorname{Tr}_s\left[N_H g G_u\left(uD^X + \frac{T}{u}V\right)\right] - \operatorname{Tr}_s[N_H^\theta g G_u(uD^Y)]\right| \leq \frac{cu}{\sqrt{T}}\exp\left(\frac{-C}{u^2}\right). \tag{13.15}$$

By [15, Proposition 8.4], $N_H^\theta = \frac{1}{2}\dim N_{Y/X}$. Since $G_u(a)$ is an even function, it is a holomorphic function of $a^2$. By an analogue of the McKean-Singer formula [25], we obtain that for $1 \leq j \leq d$,

$$\operatorname{Tr}_s[gG_u(uD^{Y_j})] = \chi_g(\eta|_{Y_j})G_u(0), \tag{13.16}$$

which together with (13.15) thus gives (13.14).

**Remark 13.5.** By (13.13), we get

$$F_u(0) + G_u(0) = 1. \tag{13.17}$$

In view of Theorem 13.4 and (13.17), we see that to prove Theorem 8.8, we only need to show that there exist $C > 0$, $\delta \in ]0, 1]$ such that for $u \in ]0, 1]$, $T \geq 1$,

$$\left|\operatorname{Tr}_s\left[N_H g F_u\left(uD^X + \frac{T}{u}V\right)\right] - \frac{1}{2}\dim N_{Y/X}\chi_g(\eta)F_u(0)\right| \leq \frac{C}{T^\delta}. \tag{13.18}$$

Since $f(t)$ vanishes for $|t| \geq \alpha$,

$$F_u(a) = \int_{-\alpha/u}^{\alpha/u}\exp\left(it\sqrt{2}a\right)\exp\left(\frac{-t^2}{2}\right)f(ut)\frac{dt}{\sqrt{2\pi}}. \tag{13.19}$$

In particular

$$F_u\left(uD^X + \frac{T}{u}V\right) = \int_{-\alpha/u}^{+\alpha/u}\exp\left(it\sqrt{2}\left(uD^X + \frac{T}{u}V\right)\right)\exp\left(\frac{-t^2}{2}\right)f(ut)\frac{dt}{\sqrt{2\pi}}. \tag{13.20}$$

Since $f$ is even, (13.20) can be rewritten in the equivalent form

$$F_u\left(uD^X + \frac{T}{u}V\right) = \int_{-\alpha/u}^{\alpha/u}\cos\left(t\sqrt{2}\left|uD^X + \frac{T}{u}V\right|\right)\exp\left(\frac{-t^2}{2}\right)f(ut)\frac{dt}{\sqrt{2\pi}}. \tag{13.21}$$

Let $F_u(uD^X + TV/u)(x, x')$ $(x, x' \in X)$ be the smooth kernel associated to $F_u(uD^X + TV/u)$ calculated with respect to $dv_X(x')/(2\pi)^{\dim X}$.

By using general results on hyperbolic equations [16, §7.8], [30, §4.4], we find that for $t \in \mathbf{R}$, $x \in X$, $h \in (\Lambda(T^{*(0,1)}X)\hat{\otimes}\xi)_x$,

$$\operatorname{supp}\cos\left(t\sqrt{2}\left|uD^X + \frac{T}{u}V\right|\right) h\delta_{\{x\}} \in B^X(x, ut). \tag{13.22}$$

From (13.22), we conclude that if $x \in X$, $x' \in X$, and $d^X(x, x') \geq \alpha$, then

$$F_u(uD^X + TV/u)(x, x') = 0, \tag{13.23}$$

and moreover, given $x \in X$, $F_u(uD^X + TV/u)(x, \cdot)$ only depends on the restriction of $uD^X + TV/u$ to $B^X(x, \alpha)$.

Clearly

$$\begin{aligned}&\operatorname{Tr}_s\left[N_H g F_u\left(uD^X + \frac{T}{u}V\right)\right] \\ &\quad = \int_X \operatorname{Tr}_s\left[N_H g F_u\left(uD^X + \frac{T}{u}V\right)(g^{-1}x, x)\right] \frac{dv_X(x)}{(2\pi)^{\dim X}}.\end{aligned} \tag{13.24}$$

In view of (13.23), (13.24), we obtain

$$\begin{aligned}\operatorname{Tr}_s\left[N_H g F_u\left(uD^X + \frac{T}{u}V\right)\right] &= \int_{x\in X,\, d^X(g^{-1}x, x)\leq\alpha} \operatorname{Tr}_s\left[N_H g F_u\right. \\ &\quad \times \left.\left(uD^X + \frac{T}{u}V\right)(g^{-1}x, x)\right] \frac{dv_X(x)}{(2\pi)^{\dim X}},\end{aligned} \tag{13.25}$$

which thus indicates that the analysis needed in the proof of (13.18) localizes near $X_g$.

Recall that by (6.9), we have the exact sequence of holomorphic Hermitian vector bundles on $Y_g$

$$F : 0 \to N_{Y_g/X_g} \oplus N_{Y_g/Y} \to N_{Y_g/X} \to \widetilde{N} \to 0. \tag{13.26}$$

Moreover $N_{Y_g/X_g}$ and $N_{Y_g/Y}$ are mutually orthogonal in $N_{Y_g/X}$. As usual, we identify $\widetilde{N}$ (as a smooth vector bundle) with the orthogonal bundle to $N_{Y_g/X_g} \oplus N_{Y_g/Y}$ in $N_{Y_g/X}$. So now we have an identification of smooth vector bundles

$$N_{Y_g/X} = N_{Y_g/X_g} \oplus N_{Y_g/Y} \oplus \widetilde{N}. \tag{13.27}$$

Take $y_0 \in Y_g$. If $Z_0 \in (T_{\mathbf{R}}X)_{y_0}$, we write $Z_0$ in the form

$$\begin{aligned}&Z_0 = Z'' + Z + Z' + \widetilde{Z},\ Z'' \in (T_{\mathbf{R}}Y_g)_{y_0}, \\ &Z \in (N_{Y_g/X_g,\mathbf{R}})_{y_0},\ Z' \in N(_{Y_g/Y,\mathbf{R}})_{y_0},\ \widetilde{Z} \in (\widetilde{N}_{\mathbf{R}})_{y_0}.\end{aligned} \tag{13.28}$$

If $y \in Y$, $U \in (T_{\mathbf{R}}Y)_y$, recall that $t \in \mathbf{R} \to y_t = \exp_y^U(tU) \in Y$ is the geodesic in $Y$ such that $y_0 = y$, $dy/dt|_{t=0} = U$. If $V \in N_{Y/X,\mathbf{R},y}$, we still denote by $V \in N_{Y/X,\mathbf{R},\exp_y^Y(U)}$ the parallel transport of $V$ with respect to $\nabla^{N_{Y/X}}$ along $t \in [0,1] \to y_t \in Y$.

For $\varepsilon > 0$ small enough, set

$$\mathscr{U}_\varepsilon = \{(y_0, Z_0) \in N_{Y_g/X,\mathbf{R}},\ |P^{N_{Y_g/Y}} Z_0| < \varepsilon,\ |P^{N_{Y/X}} Z_0| < \varepsilon\}. \tag{13.29}$$

We identify $(y_0, Z_0) \in \mathscr{U}_\varepsilon$ with $\exp^X_{\exp^Y_{y_0}(P^{N_{Y_g/Y}}Z)}(P^{N_{Y/X}}Z)$.

Let $k(y_0, Z_0)$ $((y_0, Z_0) \in N_{Y_g/X,\mathbf{R}})$, $k'(y_0, Z_0')$ $((y_0, Z_0') \in N_{Y_g/Y,\mathbf{R}})$ be the smooth functions defined by

$$\begin{aligned} dv_X &= k(y_0, Z_0)dv_{Y_g}(y_0)dv_{N_{Y_g/X}}(Z_0), \\ dv_Y &= k'(y_0, Z_0')dv_{Y_g}(y_0)dv_{N_{Y_g/Y}}(Z_0'). \end{aligned} \tag{13.30}$$

Then $k(y_0, 0) = 1$, $k'(y_0, 0) = 1$, and $k'$ is the restriction of $k$ to $N_{Y_g/Y,\mathbf{R}}$. Clearly,

(13.31)

$$\begin{aligned} &\int_{\mathscr{U}_{\varepsilon/8}(Y_g)} \operatorname{Tr}_s\left[N_H g F_u\left(uD^X + \frac{T}{u}V\right)(g^{-1}x, x)\right]\frac{dv_X(x)}{(2\pi)^{\dim X}} \\ &= \frac{1}{(2\pi)^{\dim X}}\int_{Y_g}\int_{\substack{Z_0 \in N_{Y_g/X,\mathbf{R}} \\ |P^{N_{Y_g/Y}}Z_0| \le \varepsilon/8u \\ |P^{N_{Y/X}}Z_0| \le \varepsilon\sqrt{T}/8u}} \frac{u^{2\dim N_{Y_g/X}}}{T^{\dim N_{Y/X}}} \\ &\quad \times \operatorname{Tr}_s\left[N_H g F_u\left(uD^X + \frac{T}{u}V\right)\left(g^{-1}\left(y_0, uP^{N_{Y_g/Y}}Z_0 + \frac{uP^{N_{Y/X}}Z_0}{\sqrt{T}}\right),\right.\right. \\ &\qquad\qquad \left.\left.\left(y_0, uP^{N_{Y_g/Y}}Z_0 + \frac{uP^{N_{Y/X}}Z_0}{\sqrt{T}}\right)\right)\right] \\ &\quad \times k\left(y_0, uP^{N_{Y_g/Y}}Z_0 + \frac{uP^{N_{Y/X}}Z_0}{\sqrt{T}}\right) dv_{N_{Y_g/X}}(Z_0). \end{aligned}$$

In the sequel, we always assume that given $\varepsilon > 0$, $\alpha > 0$ is chosen small enough so that if $x \in X$, $d^X(g^{-1}x, x) \le \alpha$, then $d^X(x, X_g) < \varepsilon/16$, and if $y \in Y$, $d^Y(g^{-1}y, y) \le \alpha$, then $d^Y(y, Y_g) < \varepsilon/16$.

We state the obvious extension of [15, Theorem 13.6].

Let $F_u(uD^Y)(y, y')$ be the smooth kernel associated to $F_u(uD^Y)$, with respect to the volume element $dv_Y(y)/(2\pi)^{\dim Y}$.

**Theorem 13.6.** *If* $\varepsilon \in ]0, \inf(\varepsilon_0/2, a/2, b/2)]$, $\alpha \in ]0, \varepsilon/8]$ *are small enough, there is* $C > 0$ *such that for any* $m \in \mathbf{N}$, *there is* $C' > 0$ *for which if* $u \in ]0, 1]$, $T \geq 1$, $y_0 \in Y_g$, $Z_0 = (Z, Z', \widetilde{Z}) \in N_{Y_g/X, \mathbf{R}, y_0}$, $|Z| \leq \varepsilon\sqrt{T}/8u$, $|Z'| \leq \varepsilon/8u$, $|\widetilde{Z}| \leq \varepsilon\sqrt{T}/8u$, *then*

(13.32)

$$\frac{u^{2\dim N_{Y_g/X}}}{T^{\dim N_{Y/X}}} \left| \operatorname{Tr}_s \left[ N_H g F_u\left(uD^X + \frac{T}{u}V\right) \left(g^{-1}\left(y_0, uP^{N_{Y_g/Y}}Z_0 + \frac{uP^{N_{Y/X}}Z_0}{\sqrt{T}}\right), \left(y_0, uP^{N_{Y_g/Y}}Z_0 + \frac{uP^{N_{Y/X}}Z_0}{\sqrt{T}}\right)\right]\right| \leq C'(1 + |P^{N_{Y_g/X_g}}Z_0|)^{-m} \exp(-C|P^{N_{X_g/X}}Z_0|^2).$$

*There exist* $C'' > 0$, $\delta' \in ]0, \frac{1}{2}]$ *such that under the same conditions as before, we have*

(13.33)

$$\left| \frac{1}{(2\pi)^{\dim X}} \frac{u^{2\dim N_{Y_g/X}}}{T^{\dim N_{Y/X}}} \times \operatorname{Tr}_s\left[N_H g F_u\left(uD^X + \frac{T}{u}V\right)\left(g^{-1}\left(y_0, uP^{N_{Y_g/Y}}Z_0 + \frac{uP^{N_{Y/X}}Z_0}{\sqrt{T}}\right), \left(y_0, uP^{N_{Y_g/Y}}Z_0 + \frac{uP^{N_{Y/X}}Z_0}{\sqrt{T}}\right)\right)\right] \times k\left(y_0, uP^{N_{Y_g/Y}}Z_0 + \frac{u}{\sqrt{T}}P^{N_{Y/X}}Z_0\right) - u^{\dim N_{Y_g/Y}} \frac{\exp(-|P^{N_{Y/X}}Z_0|^2)}{\pi^{\dim N_{Y/X}}} \frac{\dim N_{Y/X}}{2} \frac{1}{(2\pi)^{\dim Y}} \times \operatorname{Tr}_s[gF_u(uD^Y)(g^{-1}(y_0, uP^{N_{Y_g/Y}}Z_0), (y_0, uP^{N_{Y_g/Y}}Z_0))] \times k'(y_0, uP^{N_{Y_g/Y}}Z_0)\right| \leq C''/T^{\delta'}.$$

*Proof.* The remainder of this section is devoted to the proof of Theorem 13.6.

**Remark 13.7.** We define the open tubular neighborhood $\mathscr{U}_\varepsilon(Y_g)$ in $X$ as in Remark 11.8. From (13.30), (13.31) and Theorem 13.6, it is clear that there exists $C > 0$ such that for $u \in ]0, 1]$, $T \geq 1$,

$$\left| \int_{\mathscr{U}_{\varepsilon/8}(Y_g)} \operatorname{Tr}\left[N_H g F_u\left(uD^X + \frac{T}{u}V\right)(g^{-1}x, x)\right] \frac{dv_X(y)}{(2\pi)^{\dim X}} - \frac{\dim N_{Y/X}}{2} \int_{\mathscr{U}_{\varepsilon/8}(Y_g)\cap Y} \operatorname{Tr}_s[gF_u(uD^Y)(g^{-1}y, y)] \frac{dv_Y(y)}{(2\pi)^{\dim Y}} \right| \leq \frac{C}{T^{\delta'/2}}. \tag{13.34}$$

Applying Theorem 13.6 to the case where $Y = \phi$ yields

$$\left| \int_{X \setminus \mathscr{U}_{\varepsilon/8}(Y_g)} \operatorname{Tr}_s\left[N_H g F_u\left(uD^X + \frac{T}{u}V\right)(g^{-1}x, x)\right] \frac{dv_X(x)}{(2\pi)^{\dim X}} \right| \leq \frac{C}{T^{\delta'/2}}. \tag{13.35}$$

Using the finite propagation speed again, we obviously see that

$$gF_u(uD^Y)(g^{-1}y, y) = 0 \quad \text{if } d^Y(g^{-1}y, y) \geq \alpha. \tag{13.36}$$

If $d^Y(g^{-1}y, y) \leq \alpha$, then $d^Y(y, Y_g) \leq \varepsilon/16$ and so $y \in \mathscr{U}_{\varepsilon/8} \cap Y$. Therefore for $1 \leq j \leq d$,

$$\int_{\mathscr{U}_{\varepsilon/8}\cap Y} \operatorname{Tr}_s[gF_u(uD^{Y_j})(g^{-1}y, y)] \frac{dv_Y(y)}{(2\pi)^{\dim Y}} = \operatorname{Tr}_s[gF_u(uD^{Y_j})]. \tag{13.37}$$

Finally, the same arguments as in (13.16) show that for $1 \leq j \leq d$,

$$\operatorname{Tr}_s[gF_u(uD^{Y_j})] = \chi_g(\eta|_{Y_j})F_u(0). \tag{13.38}$$

By (13.34)–(13.38), we arrive at (13.18), and the proof of Theorem 8.8 is completed.

c. *The function* $F_u(a)$ *as a function of* $a^2$. The following result is elementary and was proved in [15, Proposition 13.8].

**Proposition 13.8.** *For* $c > 0$, $m \in \mathbf{N}$, $m' \in \mathbf{N}$, *there is* $C > 0$ *such that for* $u \in ]0, 1]$,

$$\sup_{\substack{a \in \mathbf{C} \\ |\operatorname{Im} a| \leq c}} |a|^m |F_u^{(m')}(a)| \leq C. \tag{13.39}$$

Since $F_u(a)$ is an even function of $a$, there exists a unique holomorphic function $\widetilde{F}_u(a)$ such that

$$F_u(a) = \widetilde{F}_u(a^2).$$

Recall that for $c > 0$, $V_c \subset \mathbf{C}$ was defined in Definition 11.16. The next result was proved in [15, Proposition 13.10].

**Proposition 13.9.** *For any $c > 0$, $m \in \mathbf{N}$, $m' \in \mathbf{N}$, there exists $C > 0$ such that for $u \in ]0, 1]$.*

$$\sup_{a \in V_c} |a|^m |\widetilde{F}_u^{(m')}(a)| \leq C. \tag{13.40}$$

d. *An orthogonal splitting of $TX$ and a connection on $TX$.* Now we follow [15, §13d]. In [15], near $Y$, a smooth orthogonal splitting

$$TX = TX^1 \oplus TX^2 \tag{13.41}$$

is defined, which, on $Y$, restricts to the smooth splitting

$$TX|_Y = TY \oplus N_{Y/X}. \tag{13.42}$$

By [15, Definition 13.11], the splitting (13.41) is obtained by parallel transport along the geodesics normal to $Y$ of the splitting (13.42), with respect to the connection $\nabla^{TX}$.

Also a connection ${}^0\nabla^{TX} = {}^0\nabla^{TX_1} \oplus {}^0\nabla^{TX_2}$ on $TX = TX^1 \oplus TX^2$ is constructed in [15, §13d] by projecting orthogonally $\nabla^{TX}$ on $TX^1$, $TX^2$. For details, we refer to [15].

Let ${}^0\nabla^{\Lambda(T^{*(0,1)}X)}$ be the connection induced by ${}^0\nabla^{TX}$ on $\Lambda(T^{*(0,1)}X)$.

e. *A local coordinate system near $y_0 \in Y_g$ and a trivialization of $\Lambda(T^{*(0,1)}X)\hat{\otimes}\xi$.* Take $y_0 \in Y_g$, and recall that $Y_g$ is totally geodesic in $Y$. So if $Z'' \in (T_{\mathbf{R}}Y_g)_{y_0}$, then $t \to y_t = \exp^Y_{y_0}(tZ'') \in Y_g$ is the geodesic in $Y_g$ such that $y|_{t=0} = y_0$, $dy/dt|_{t=0} = Z''$.

If $Z'' \in (T_{\mathbf{R}}Y_g)_{y_0}$, $Z'_0 \in N_{Y_g/X,\mathbf{R},y_0}$, we still denote by $Z'_0 \in N_{Y_g/X,\mathbf{R},\exp^Y_{y_0}(Z'')}$ the parallel transport of $Z'_0$ along the curve $t \in [0, 1] \to \exp^Y_{y_0}(tZ'')$ with respect to the connection $\nabla^{N_{Y_g/X_g}} \oplus \nabla^{N_{Y_g/Y}} \oplus \nabla^{\widetilde{N}}$.

If $y \in Y$, $Z \in (T_{\mathbf{R}}Y)_y$, $Z' \in N_{Y/X,\mathbf{R},y}$, we still denote by $Z' \in N_{Y/X,\mathbf{R},\exp^Y_y(Z)}$ the parallel transport of $Z'$ with respect to $\nabla^{N_{Y/X}}$ along the curve $t \in [0, 1] \to \exp^Y_y(tZ)$.

Ultimately, if $Z_0 \in (T_{\mathbf{R}}X)_{y_0}$, $|Z_0| < \varepsilon$, we identify $Z_0$ with $\exp^X_{\exp^Y_{\exp^Y_{y_0}(P^{TY_g}Z)}(P^{N_{Y_g/Y}}Z)}(P^{N_{Y/X}}Z) \in X$.

Let $k''(Z_0)$, $Z_0 \in (T_{\mathbf{R}}X)_{y_0}$, $|Z_0| < \varepsilon$, $k'''(Z_0')$, $Z_0' \in (T\mathbf{R}Y)_{y_0}$, $|Z_0'| < \varepsilon$ be the functions defined by

$$(13.43)\quad dv_X(Z_0) = k''(Z_0)dv_{TX}(Z_0), \qquad dv_Y(Z_0') = k'''(Z_0')dv_{TY}(Z_0').$$

Then one easily verifies that if $Z_0 \in N_{Y_g/X,\mathbf{R},y_0}$, $Z_0' \in N_{Y_g/Y,\mathbf{R},y_0}$, then

$$(13.44)\qquad k''(Z_0) = k(y_0, Z_0), \qquad k'''(Z_0') = k'(y_0, Z_0').$$

Take $Z_0 \in (T_{\mathbf{R}}X)_{y_0}$. We identify $(TX)_{Z_0}$, $\Lambda(T^{*(0,1)}X)_{Z_0}$ (resp. $\xi_{Z_0}$) with $TX_{y_0}$, $\Lambda(T^{*(0,1)}X)_{y_0}$ (resp. $\xi_{y_0}$) by parallel transport with respect to the connection ${}^0\nabla^{TX}$, ${}^0\nabla^{\Lambda(T^{*(0,1)}X)}$ (resp. $\widetilde{\nabla}^{\xi}$) along the path

$$(13.45)\qquad \begin{aligned} t \in [0, 3] \to tP^{TY_g}Z_0, &\qquad 0 \le t \le 1;\\ P^{TY_g}Z_0 + (t-1)P^{N_{Y_g/Y}}Z_0, &\qquad 1 \le t \le 2;\\ P^{TY}Z_0 + (t-2)P^{N_{Y/X}}Z_0, &\qquad 2 \le t \le 3.\end{aligned}$$

As in [15, §13e], we observe that for $2 \le t \le 3$, the parallel transport with respect to ${}^0\nabla^{TX}$ coincides with the parallel transport with respect to $\nabla^{TX}$. Also note that for $2 \le t \le 3$, this trivialization is essentially the one we considered in §9.

If $U \in (T_{\mathbf{R}}X)_{y_0}$, $Z_0 \in (T_{\mathbf{R}}X)_{y_0}$, let ${}^0\tau U(Z_0)$ be the parallel transport of $U$ along the curve (13.45) with respect to ${}^0\nabla^{TX}$.

Let $a \in \mathbf{R} \to \gamma(a) \in [0, 1]$ be a smooth function such that

$$(13.46)\qquad \begin{aligned}\gamma(a) &= 1 \quad \text{for } a \le \tfrac{1}{2},\\ &= 0 \quad \text{for } a \ge 1.\end{aligned}$$

Recall that $b$ is the injectivity radius of $(Y, h^{TY})$. Set

$$(13.47)\qquad \mu(U) = \gamma(4|U|/3b).$$

Then

$$(13.48)\qquad \begin{aligned}\mu(U) &= 1 \quad \text{if } |U| \le 3b/8,\\ &= 0 \quad \text{if } |U| \ge 3b/4.\end{aligned}$$

Let $e_1, \cdots, e_{2m}$ be an orthonormal basis of $(T_{\mathbf{R}}Y)_{y_0}$, and $\Delta^{TY}$ be the Euclidean Laplacian on $(T_{\mathbf{R}}Y)_{y_0}$.

**Definition 13.10.** Let $L$ be the differential operator on $(T_{\mathbf{R}}X)_{y_0}$,

$$L = (1-\mu^2(P^{TY}Z_0))\Delta^{TY} + \mu^2(P^{TY}Z_0)\sum_1^{2m}\nabla^2_{{}^0\tau e_i}(P^{TY}Z_0). \tag{13.49}$$

Recall that by [15, equation (8.31)] $\xi^-_{y_0} = (\Lambda N^*_{Y/X}\otimes\eta)_{y_0}$. Let $e_{2l'+1}, \cdots, e_{2l}$ be an orthonormal basis of $N_{Y/X,\mathbf{R},y_0}$, and $S\in \mathrm{End}(\Lambda(T^{*(0,1)}X)\hat{\otimes}\xi^-)_{y_0}$ be given by

$$S = \frac{\sqrt{-1}}{2}\sum_{2m+1}^{2l} c(e_i)\hat{c}(e_i). \tag{13.50}$$

Let $(a, b)\in\mathbf{R}^2\to\kappa(a, b)\in[0, 1]$ be a smooth function such that

$$\begin{aligned}\kappa(a, b) &= 1 \quad \text{if } |a|\le \tfrac{1}{2},\ |b|\le\tfrac{1}{2},\\ &= 0 \quad \text{if } |a|\ge\tfrac{3}{4} \text{ or } |b|\ge\tfrac{3}{4}.\end{aligned} \tag{13.51}$$

If $Z_0\in(T_{\mathbf{R}}X)_{y_0}$, set

$$\varphi(Z_0) = \kappa(|P^{TY}Z_0|/\varepsilon, |P^{N_{Y/X}}Z_0|/\varepsilon). \tag{13.52}$$

Then

$$\begin{aligned}\varphi(Z_0) &= 1 \quad \text{if } |P^{TY}Z_0|\le\varepsilon/2,\ |P^{N_{Y/X}}Z_0|\le\varepsilon/2,\\ &= 0 \quad \text{if } |P^{TY}Z_0|\ge 3\varepsilon/4 \text{ or } |P^{N_{Y/X}}Z_0|\ge 3\varepsilon/4.\end{aligned} \tag{13.53}$$

Let $\mathscr{W}_\varepsilon(y_0)$ be the open neighborhood of $y_0$ in $X$, given by

$$\mathscr{W}_\varepsilon(y_0) = \{Z\in(T_{\mathbf{R}}X)_{y_0},\ |P^{TY}Z|<\varepsilon/2,\ |P^{N_{Y/X}}Z|<\varepsilon/2\}.$$

Clearly, there exists $\alpha_0(\varepsilon)>0$ such that for $y_0\in Y_g$, $Z_0\in N_{Y/X,\mathbf{R},y_0}$, $|Z_0|<\varepsilon/8$, the open Riemannian ball $B^X(Z_0, \alpha_0(\varepsilon))$ in $X$ is contained in $\mathscr{W}_\varepsilon(y_0)$. In particular $0<\alpha_0(\varepsilon)\le\varepsilon/2\le b/4$.

Now we fix $\alpha\in]0, \alpha_0(\varepsilon)]$ small enough so that the conditions stated after (13.31) are satisfied. Let $\Delta^{N_{Y/X}}$ be the Laplacian on $N_{Y/X,\mathbf{R},y_0}$, and recall that $\mathbf{H}_{y_0}$ is the vector space of smooth sections of $(\Lambda(T^{*(0,1)}X)\hat{\otimes}\xi)_{y_0}$ over $(T_{\mathbf{R}}X)_{y_0}$.

**Definition 13.11.** Let $\mathscr{L}_{u,T}^{1,y_0}$, $\mathscr{M}_{u,T}^{1,y_0}$ be the operators acting on $\mathbf{H}_{y_0}$,

(13.54)
$$\mathscr{L}_{u,T}^{1,y_0} = (1-\varphi^2(Z_0))\left(\frac{-u^2}{2}(L+\Delta^{N_{Y/X}}) + TP^{\xi^-}SP^{\xi^-} + \frac{T^2}{u^2}\left(P^{\xi^+} + \frac{|P^N Z_0|^2}{2}P^{\xi^-}\right)\right) + \varphi^2(Z_0)\left(uD^X + \frac{T}{u}V(Z_0)\right)^2,$$

$$\mathscr{M}_u^{1,y_0} = -(1-\varphi^2(Z_0))\frac{u^2}{2}(L+\Delta^{N_{Y/X}}) + \varphi^2(Z_0)(uD^X)^2.$$

Let $\widetilde{F}_u(\mathscr{L}_{u,T}^{1,y_0})(Z_0, Z_0')$ $(Z_0, Z_0' \in (T_{\mathbf{R}}X)_{y_0})$ be the smooth kernel associated to $\widetilde{F}_u(\mathscr{L}_{u,T}^{1,y_0})$, calculated with respect to $dv_{TX}(Z_0')/(2\pi)^{\dim X}$. By construction $\varphi^2$ is equal to 1 on $B^X(Z_0, \alpha)$. Using the finite propagation speed, for $Z_0, Z_0' \in N_{Y_g/X,\mathbf{R},y_0}$, $|Z_0| \le \varepsilon/8$, we have

$$\begin{aligned}(13.55)\qquad &\widetilde{F}_u((uD^X + TV/u)^2)((y_0, Z_0), (y_0, Z_0'))k''(Z_0') \\ &= \widetilde{F}_u(\mathscr{L}_{u,T}^{1,y_0})((y_0, Z_0), (y_0, Z_0')).\end{aligned}$$

f. *Rescaling of the variable $Z_0$ and the Clifford variables.* Clearly if $Z_0 = (Z'', Z, Z', \widetilde{Z}) \in (T_{\mathbf{R}}X)_{y_0}$, then

$$(13.56)\qquad \frac{P^{TY}Z_0}{u} + \frac{\sqrt{T}}{u}P^{N_{Y/X}}Z_0 = \frac{Z'+Z''}{u} + \frac{\sqrt{T}}{u}(Z+\widetilde{Z}).$$

For $u>0$, $T>0$, let $G_{u,T}$ be the linear map $h \in \mathbf{H}_{y_0} \to G_{u,T}h \in \mathbf{H}_{y_0}$ such that if $Z_0 \in (T_{\mathbf{R}}X)_{y_0}$, then

$$(13.57)\qquad G_{u,T}h(Z_0) = h\left(\frac{P^{TY}Z_0}{u} + \frac{\sqrt{T}}{u}P^{N_{Y/X}}Z_0\right).$$

Set

$$(13.58)\qquad \mathscr{L}_{u,T}^{2,y_0} = G_{u,T}^{-1}\mathscr{L}_{u,T}^{1,y_0}G_{u,T}, \qquad \mathscr{M}_{u,T}^{2,y_0} = G_{u,T}^{-1}\mathscr{M}_u^{1,y_0}G_{u,T}.$$

Let $e_1, \cdots, e_{2l'}$ and $e_{2l'+1}, \cdots, e_{2m}$ be orthonormal bases of $(T_{\mathbf{R}}Y_g)_{y_0}$ and $N_{Y_g/Y,\mathbf{R},y_0}$ respectively. Then, as in §12 f, $e_1, \dots, e_{2m}$ is an orthonormal basis of $(T_{\mathbf{R}}Y)_{y_0}$. Also $e_{2m+1}, \cdots, e_{2l}$ still denote an orthonormal basis of $N_{Y/X,\mathbf{R},y_0}$. For $1 \leq i \leq 2l'$, we define $c_u(e_i)$ as in Definition 12.6.

**Definition 13.12.** Let $\mathscr{L}^{3,y_0}_{u,T}$, $\mathscr{M}^{3,y_0}_{u,T}$ be the operators obtained from $\mathscr{L}^{2,y_0}_{u,T}$, $\mathscr{M}^{2,y_0}_{u,T}$ by replacing the Clifford variables $c(e_i)$ by $c_u(e_i)$ for $1 \leq i \leq 2l'$, while leaving unchanged the $c(e_i)$'s for $2l'+1 \leq i \leq 2l$.

Let $\widetilde{F}_u(\mathscr{L}^{3,y_0}_{u,T})(Z_0, Z_0')$ $(Z_0, Z_0' \in (T_{\mathbf{R}}X)_{y_0})$ be the smooth kernel associated to $\widetilde{F}_u(\mathscr{L}^{3,y_0}_{u,T})$ with respect to $dv_{TX}(Z_0')/(2\pi)^{\dim X}$.

**Proposition 13.13.** *For $u > 0$, $T > 0$, $y_0 \in Y_g$, $Z_0 \in N_{Y_g/X,\mathbf{R},y_0}$, $|P^{TY}Z_0| \leq \frac{\varepsilon}{8u}$, $|P^{N_{Y/X}}Z_0| \leq \frac{\varepsilon\sqrt{T}}{8u}$, we have*

$$\begin{aligned}
&\frac{u^{2\dim N_{Y_g/X}}}{T^{\dim N_{Y/X}}} \operatorname{Tr}_s \Big[ N_H g F_u\left(uD^X + \frac{T}{u}V\right) \\
&\qquad \times \left(g^{-1}\left(y_0, uP^{N_{Y_g/Y}}Z_0 + \frac{u}{\sqrt{T}}P^{N_{Y/X}}Z_0\right),\right. \\
&\qquad\qquad \left.\left(y_0, uP^{N_{Y_g/Y}}Z_0 + \frac{u}{\sqrt{T}}P^{N_{Y/X}}Z_0\right)\right)\Big] \\
&\qquad \times k''\left(uP^{TY}Z_0 + \frac{u}{\sqrt{T}}P^{N_{Y/X}}Z_0\right) \\
&= (-i)^{\dim Y_g} \operatorname{Tr}_s[N_H g[\widetilde{F}_u(\mathscr{L}^{3,y_0}_{u,T})(g^{-1}Z_0, Z_0)]^{\max}].
\end{aligned} \tag{13.59}$$

*Proof.* Since $g$ preserves the geodesics in $X$ and $Y$ and also the connections on the vector bundles considered before, it is clear that $g$ acts linearly in the coordinate $Z_0$. Thus the proof of our proposition is the same as the proof of [15, Proposition 13.17].

g. *A formula for $\mathscr{L}^{3,y_0}_{u,T}$.* The discussion in [15, §13h] applies with minor modifications. The main difference is that the Clifford variables $c(e_i)$, $2l'+1 \leq i \leq 2m$, are not rescaled, while they are rescaled in [15]. However this just introduces fewer diverging terms than in [15]. In particular, the analogues of [15, Theorems 13.18 and 13.19] still hold.

h. *The algebraic structure of $\mathscr{L}^{3,y_0}_{u,T}$ as $u \to 0$.* The analogue of [15, §13i] still holds. It leads to another proof of our results in §12.

i. *The matrix structure of the operator $\mathscr{L}^{3,y_0}_{u,T}$ as $T \to +\infty$.* For a fixed $u > 0$, the analysis of the matrix structure of $\mathscr{L}^{3,y_0}_{u,T}$ as $T \to +\infty$ is the same as in [15, §13j]. Of course the rescaling on the Clifford variables, which depends on $u > 0$, is different, but this does not introduce any extra difficulty.

We still define the function $g_{u,T}(Z)$, $\tilde{g}_u(U)$ as in [15, Definition 13.24].

The algebra $\Lambda(T^*_{\mathbf{R}}Y_g)$ splits into

$$\Lambda(T^*_{\mathbf{R}}Y_g) = \bigoplus_0^{\dim Y_g} \Lambda^r(T^*_{\mathbf{R}}Y_g). \tag{13.60}$$

Then we introduce the obvious modifications of the system of norms $|\ |_{u,T,y_0,j}$, $j=-1,0,1$ of [15, Definitions 13.25 and 13.26], with respect to the splitting (13.60).

Thus [15, Theorem 13.27] still holds for essentially the same reasons as in [15]. The same is true for [15, Theorems 13.30 and 13.31]. In particular we choose $T_0 \geq 1$ as in [15, Theorem 13.27].

j. *Uniform estimate on the kernel of* $\widetilde{F}_u(\mathscr{L}^{3,y_0}_{u,T})$. We now establish an extension of [15, Theorem 13.32].

**Theorem 13.14.** *There exists* $C>0$ *such that for any* $m \in \mathbf{N}$, *there exists* $C'>0$ *such that if* $u \in ]0,1]$, $T \geq T_0$, $y_0 \in Y_g$, $Z_0, Z'_0 \in (T_{\mathbf{R}}X)_{y_0}$, $|P^{TY}Z_0|$, $|P^{TY}Z'_0| \leq \varepsilon/4u$, $|P^{N_{Y/X}}Z_0| \leq \frac{\varepsilon\sqrt{T}}{4u}$, $|P^{N_{Y/X}}Z'_0| \leq \varepsilon\sqrt{T}/4u$, *then*

$$\begin{aligned} &|\widetilde{F}_u(\mathscr{L}^{3,y_0}_{u,T})(Z_0,Z'_0)| \\ &\quad \leq C'(1+|P^{N_{Y/X}}Z_0|)^{-m}(1+|P^{TY}Z_0|)^{2l}\exp(-C|Z_0-Z'_0|^2). \end{aligned} \tag{13.61}$$

*There exists* $C>0$ *for which if* $m' \in \mathbf{N}$, *there exists* $C'>0$ *such that if* $|\alpha|$, $|\alpha'| \leq m'$, $u \in ]0,1]$, $T \geq T_0$, $y_0 \in Y_g$, $Z_0, Z'_0 \in (T_{\mathbf{R}}X)_{y_0}$, *then*

$$\begin{aligned} &\left|\frac{\partial^{|\alpha|+|\alpha'|}}{\partial Z^{\alpha}\partial Z'^{\alpha'}}\widetilde{F}_u(\mathscr{L}^{3,y_0}_{u,T})(Z_0,Z'_0)\right| \\ &\quad \leq C'(1+|Z_0|)^{2l}\exp(-C|Z_0-Z'_0|^2). \end{aligned} \tag{13.62}$$

*Proof.* We briefly indicate the principle of the proof of Theorem 13.14. The bounds in (13.61), (13.62) with $C=0$ are easily obtained by proceeding as in [15, proof of Theorem 13.32]. To get the required $C>0$, we proceed as in the proof of Theorem 11.14.

Take $q \in \mathbf{N}$, and recall that $f$ and $g$ were defined in (13.10), (13.11). Set

$$F_{u,q}(a) = \int_{-\infty}^{+\infty} \exp\left(it\sqrt{2}a\right)\exp\left(\frac{-t^2}{2}\right) f(ut)g\left(\frac{t}{q}\right)dt. \tag{13.63}$$

Then there is a holomorphic function $\widetilde{F}_{u,q}(a)$ such that

$$F_{u,q}(a) = \widetilde{F}_{u,q}(a^2). \tag{13.64}$$

As in Proposition 11.17,

$$\sup_{a \in V_c} |a^m \widetilde{F}_{u,q}^{(m')}(a)| \leq C \exp(-C' q^2). \tag{13.65}$$

Using the finite propagation speed, we see that there is $C'' > 0$ such that if $|Z_0 - Z_0'| \geq C'' q$, then

$$\widetilde{F}_u(\mathscr{L}_{u,T}^{3,y_0})(Z_0, Z_0') = \widetilde{F}_{u,q}(\mathscr{L}_{u,T}^{3,y_0})(Z_0, Z_0'), \tag{13.66}$$

which together with (13.65) and the same bounds as before yields (13.61), (13.62).

The proof of Theorem 13.14 is completed.

**Remark 13.15.** By proceeding as in Remark 12.12, we find that (13.61) and (13.62) are indeed symmetrical in $Z_0, Z_0'$.

Clearly, if $Z_0 \in N_{Y_g/X, \mathbf{R}, y_0}$, then

$$g^{-1} Z_0 - Z_0 = g^{-1} P^{N_{X_g/N}} Z_0 - P^{N_{X_g/X}} Z_0. \tag{13.67}$$

So there is $C > 0$ such that

$$|g^{-1} Z_0 - Z_0|^2 \geq C |P^{N_{X_g/X}} Z_0|^2. \tag{13.68}$$

Also if $Z_0 \in N_{Y_g/X, \mathbf{R}, y_0}$, then

$$P^{TY} Z_0 = P^{TY} P^{N_{X_g/X}} Z_0, \tag{13.69}$$

and so

$$|P^{TY} Z_0| \leq |P^{N_{X_g/X}} Z_0|. \tag{13.70}$$

Finally if $Z_0 \in N_{Y_g/X, \mathbf{R}, y_0}$, then

$$P^{N_{Y_g/X_g}} Z_0 = P^{N_{Y_g/X_g}} P^{N_{Y/X}} Z_0, \tag{13.71}$$

and so

$$|P^{N_{Y_g/X_g}} Z_0| \leq |P^{N_{Y/X}} Z_0|. \tag{13.72}$$

Equation (13.32) follows from Proposition 13.13, Theorem 13.14, and (13.68)–(13.72).

Let $\Xi_u^{y_0}$ be the analogue of the elliptic second order differential operator considered in [15, Definition 13.40]. The minor difference from [15] is that here, only the Clifford variables $c(e_i)$, $1 \le i \le 2l'$, are rescaled, while in [15], the Clifford variables $c(e_i)$, $1 \le i \le 2m$, were rescaled. Because our Clifford rescaling introduces fewer diverging terms as in [15, §13], the analogue of [15, Theorem 13.42] still holds, i.e., for $u \in ]0, 1]$, $T \ge T_0$,

$$\|\widetilde{F}_u(\mathscr{L}_{u,T}^{3,y_0}) - p\psi\widetilde{F}_u(\Xi_u^{y_0})\psi^{-1}p\|_{u,T,y_0}^{0,0} \le C/T^{1/4}. \tag{13.73}$$

The obvious analogue of [15, Theorem 13.43] still holds.

Let $\widetilde{F}_u(\Xi_u^{y_0})(U, U')$ $(U, U' \in (T_{\mathbf{R}}Y)_{y_0}$ be the smooth kernel associated to the operator $\widetilde{F}_u(\Xi_u^{y_0})$, calculated with respect to $dv_{TY}(U')/(2\pi)^{\dim Y}$.

Using Theorem 13.14, (13.73), and proceeding as in [15, §11p] or as in §11 h, we find that there exists $\delta' \in ]0, \frac{1}{2}]$ such that if $Z_0 \in N_{Y_g/X,\mathbf{R},y_0}$ is taken as in Theorem 13.16, then

$$\begin{aligned} &\left| \frac{1}{(2\pi)^{\dim X}} \widetilde{F}_u(\mathscr{L}_{u,T}^{3,y_0})(g^{-1}Z_0, Z_0) \right. \\ &\qquad \left. - \frac{\exp(-|P^{N_{Y/X}}Z_0|^2)}{\pi^{\dim Y/X}(2\pi)^{\dim Y}} \widetilde{F}_u(\Xi_u^{y_0})(g^{-1}P^{N_{Y_g/Y}}Z_0, P^{N_{Y_g/X}}Z_0)q \right| \\ &\qquad \le \frac{C(1+|Z_0|)^{2l+1}}{T^{\delta'}}. \end{aligned} \tag{13.74}$$

By (13.61), (13.68)–(13.72), (13.74), we get (13.75). The left-hand side of inequality (13.74) $\le C/T^{\delta'/2}$, which together with (13.43), (13.61) gives the proof of (13.33) in Theorem 13.6 as in [15, §13q]. Hence the proof of Theorem 13.6 is completed.

This concludes the proof of Theorem 8.8, and terminates the paper.

## Acknowledgment

The author acknowledges the support of the Institut Universitaire de France (IUF), and thanks the referees for their careful reading of the manuscript.

# References

[1] M. F. Atiyah & R. Bott, *A Lefschetz fixed point formula for elliptic complexes*, I, Ann. of Math. (2) **86** (1968) 546–604.

[2] N. Berline, E. Getzler & M. Vergne, *Heat kernels and the Dirac operator*, Grundlhren Math. Wiss., Band 298, Springer, Berlin, 1992.

[3] N. Berline & M. Vergne, *A computation of the equivariant index of the Dirac operator*, Bull. Soc. Math. France **113** (1985) 305–345.

[4] J.-M. Bismut, *The index theorem for families of Dirac operators: two heat equation proofs*, Invent. Math. **83** (1986) 91–151.

[5] ——, *Superconnection currents and complex immersions*, Invent. Math. **99** (1990) 59–113.

[6] ——, *Koszul complexes, harmonic oscillators and the Todd class*, J. Amer. Math. Soc. **3** (1990) 159–256.

[7] ——, *Equivariant short exact sequences of vector bundles and their analytic torsion forms*, preprint, Orsay, 92-79, Compositio Math., to appear.

[8] ——, *Large deviations and the Malliavin calculus*, Progr. Math., No. 45, Birkhaüser, Boston, 1984.

[9] ——, *The Atiyah-Singer index theorems: a probabilistic approach.* II, *The Lefschetz fixed point formulas*, J. Functional Analysis **57** (1984) 329–348.

[10] ——, *Métriques de Quillen équivariantes et plongements complexes*, C. R. Acad. Sci. Paris Sér. I Math. **316** (1993) 827–832.

[11] J.-M. Bismut, H. Gillet & C. Soulé, *Analytic torsion and holomorphic determinant bundles.* I, Comm. Math. Phys. **115** (1988) 49–78.

[12] ——, *Analytic torsion and holomorphic determinant bundles.* II, Comm. Math. Phys. **115** (1988) 79–126.

[13] ——, *Analytic torsion and holomorphic determinant bundles.* III, Comm. Math. Phys. **115** (1988) 301–351.

[14] ——, *Bott-Chern currents and complex immersions*, Duke Math. J. **60** (1990) 255–284.

[15] J.-M. Bismut & G. Lebeau, *Complex immersions and Quillen metrics*, Inst. Hautes Etudes Sci. Publ. Math. **74** (1991) 1–298.

[16] J. Charazain & A. Piriou, *Introduction à la théorie des équations aux dérivées partielles*, Gauthier-Villars, Paris, 1981.

[17] S. Eilenberg, *Homological dimension and local syzygies*, Ann. of Math. (2) **64** (1956) 328–336.

[18] E. Getzler, *A short proof of the Atiyah-Singer index theorem*, Topology **25** (1986) 111–117.

[19] P. Gilkey, *Lefschetz fixed point formulas and the heat equation*, Partial Differential Equations and Geometry (Proc. Clark City Conf., 1977) (C. Byrnes, ed.), Lecture Notes in Pure. and Appl. Math., Vol. 48, Dekker, New York, 1979, 91–147.

[20] H. Gillet & C. Soulé, *Analytic torsion and the arithmetic Todd genus*, Topology **30** (1991) 21–54.

[21] ——, *An arithmetic Riemann-Roch theorem*, Invent. Math. **110** (1992) 473–543.

[22] F. F. Knudsen & D. Mumford, *The projectivity of the moduli space of stable curves*, I. *Preliminaries on "det" and "div"*, Math. Scand. **39** (1976) 19–55.

[23] K. Köhler, Math. Annalen, to appear.

[24] H. McKean & I. M. Singer, *Curvature and the eigenvalues of the Laplacian*, J. Differential Geometry **1** (1967) 43–69.

[25] V. Mathai & D. Quillen, *Superconnections, Thom classes, and equivariant differential forms*, Topology **25** (1986) 85–110.

[26] D. Quillen, *Superconnections and the Chern character*, Topology **24** (1985) 89–95.

[27] ——, *Determinants of Cauchy-Riemann operators over a Riemann surface*, Functional Anal. Appl. **14** (1985) 31–34.

[28] D. B. Ray & I. Singer, *Analytic torsion for complex manifolds*, Ann. of Math. **98** (1973) 154–177.

[29] J.-P. Serre, *Algébre locale, Multiplicités*, Lecture Notes in Math., Vol. 11, Springer, Berlin, 1965.

[30] M. Taylor, *Pseudodifferential operators*, Princeton, Princeton Univ. Press, Princeton, NJ, 1981.

UNIVERSITÉ PARIS-SUD, ORSAY