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ON THE EXISTENCE OF CONVEX HYPERSURFACES OF CONSTANT GAUSS CURVATURE IN HYPERBOLIC SPACE

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Introduction

In this paper we shall prove that a codimension-one embedded submanifold Γ of $\partial_\infty(\mathbf{H}^{n+1})$ is the asymptotic boundary of a complete embedded K-hypersurface M of a hyperbolic $(n+1)$-space $\mathbf{H}^{n+1}$ for any $K \in (-1, 0)$. By a K-hypersurface M, we mean the Gauss-Kronecker curvature of M is the constant K (recall that $K = K_{\text{ext.}} - 1$, where $K_{\text{ext.}}$ is the extrinsic curvature of M, i.e., the determinant of the second fundamental form). Our approach is to construct the desired M as the limit of K-graphs over a fixed compact domain in a horosphere for appropriate boundary data. Thus an important part of our study is an existence theory for K-hypersurfaces which are graphs over a bounded domain in a horosphere. This is accomplished by solving a Monge-Ampere equation for the Gauss curvature using the recent work of [6].

In general, a codimension-two closed submanifold Γ of $\mathbf{H}^{n+1}$ does not bound a K-hypersurface with $K > -1$. There are topological obstructions for Γ to bound a hypersurface with $K > -1$ (cf. [13]). For example, let Γ be a smooth Jordan curve in $\mathbf{H}^3$, and assume Γ bounds a surface with $K > -1$. Then the curvature of Γ never vanishes, so let $n(x)$, $x \in \Gamma$, be the unit principal normal to Γ. For $x \in \Gamma$, let $\Gamma_\epsilon(x)$ be the endpoint of the geodesic starting at x, of length ϵ, and with $n(x)$ as tangent at x. For ϵ small, Γ_ϵ is embedded and disjoint from Γ. Then the linking number (mod 2) of Γ and Γ_ϵ is zero [13]; so it is easy to construct Γ which bound no surface with $K > -1$.

We will see that for Γ an embedded codimension-one submanifold of a horosphere $\subset \mathbf{H}^{n+1}$, and $K \in (-1, 0)$, there exists a K-hypersurface M with boundary $\partial M = \Gamma$.

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Let $\mathbf{H}^{n+1}$ be represented by the upper half-space model:

$$\mathbf{H}^{n+1} = \{(x, x_{n+1}) \in \mathbf{R}^{n+1} \mid x \in \mathbf{R}^n, \ x_{n+1} > 0\},$$

with the metric $ds^2 = (1/x_{n+1}^2)(dx_1^2 + \cdots + dx_n^2)$. Let P_∞ denote the extended plane $x_{n+1} = 0$, and denote by $P(c)$, $c > 0$ the horosphere $x_{n+1} = c$.

Let $P = P(1)$, and let $\Omega \subset P$ be a compact domain with $\partial\Omega = \Gamma$ a C^∞ submanifold. Here are our main results.

Theorem 1. *Let $\phi \in C^\infty(\Gamma)$, and suppose the graph of ϕ extends to a smooth graph $M = \{(x, x_{n+1}) : x_{n+1} = \underline{f}(x),\ \underline{f} \in C^\infty(\overline{\Omega}),\ \underline{f} = \phi$ on $\partial\Omega\}$ with*

$$K_M = \inf_{x \in M} K(x) > -1.$$

Then for any K, $-1 < K < K_M$, there exists an extension $f \in C^\infty(\overline{\Omega})$ of ϕ to Ω whose graph is a K-hypersurface.

Corollary 1. *For any $K \in (-1, 0)$, there exists a smooth K-hypersurface M with $\partial M = \Gamma$; M can be chosen a graph over $\overline{\Omega}$.*

To prove this corollary, one applies Theorem 1 with $\phi = 0$ on Ω; the horosphere P has curvature zero (extrinsic curvature one).

We remark that when $\partial\Omega$ is strictly convex, then the graph of ϕ over $\phi\Omega$ has an extension to a smooth graph over $\overline{\Omega}$ with curvature greater than -1. Thus we have

Corollary 2. *Let Ω strictly convex. Then the graph of ϕ extends to a smooth graph over $\overline{\Omega}$ with K constant, K sufficiently near -1.*

The technique of the proof of Theorem 1 is the continuity method applied to the equation for the curvature of a graph over Ω. This is a fully nonlinear equation of Monge-Ampère type, and the difficult part of the proof is to obtain *a priori* $C^{2+\alpha}$ bounds for solutions of the equation. An interesting point here is the absence of any convexity hypothesis on $\partial\Omega$. The recent work in [8] and [6] is used here to deal with domains of arbitrary geometry.

Using Theorem 1 as a tool, we construct, for any $K \in (-1, 0)$, a K-graph with given smooth asymptotic boundary. More precisely, we have

Theorem 2. *Let $\Gamma = \partial\Omega \subset \partial_\infty(\mathbf{H}^{n+1})$ be smooth. Then for any $K \in (-1, 0)$, $\Gamma = \partial_\infty(M)$ for M an embedded K-hypersurface of $\mathbf{H}^{n+1}$. Moreover, M can be represented as a graph $x_{n+1} = f(x)$ over Ω with $u(x) = \exp 2f(x) \in C^{1,1}(\overline{\Omega})$ and $u = 0$ on $\partial\Omega$.*

It is also of interest to consider the case of nonsmooth asymptotic boundaries.

Theorem 3. *Let $\Gamma = \partial\Omega \subset \partial_\infty(\mathbf{H}^{n+1})$. Assume the following:*

a. *For $n = 2$, Γ consists of a finite number of Jordan curves.*

b. *For* $n > 2$, *every point of* $\partial\Omega$ *is a regular point for Laplaces equation.*

Then the conclusions of Theorem 2 hold with $u(x) \in C^\infty(\Omega) \cup C^0(\overline{\Omega})$.

Finally, it is a remarkable property of $\mathbf{H}^3$ that for Γ a Jordan curve in $P(c)$ or P_∞, all of the K-surfaces which we can construct are in fact unique. To make this precise, we say that a Jordan curve Γ in P_∞ is the *asymptotic homological boundary* of a surface M in $\mathbf{H}^3$ if for $c > 0$ sufficiently small, $M \cap P(c)$ contains a connected component $\Gamma(c)$ such that $\Gamma(c)$ converges to Γ as $c \to 0$ and $\Gamma(c)$ is homologous to zero on M, i.e., there exists a compact submanifold $M(c)$ of M and $\Gamma(c) = \partial M(c)$. We write $\Gamma = \partial_\infty M$ to mean that Γ is the asymptotic homological boundary of M.

Theorem 4. *Let* Ω *be a bounded simply connected domain in* $P(c)$, *respectively* P_∞ *with boundary a Jordan curve* Γ. *Then there are exactly two embedded* K*-surfaces* M *in* $\mathbf{H}^3$ *with* $\partial M = \Gamma$, *respectively* $\partial_\infty M = \Gamma$ (*in the ball model of* H^3). *Each surface is a graph over one of the two components of* $P_\infty - \Gamma$. *Moreover, if* M *is any immersed* K*-surface with* $\partial M = \Gamma$, *respectively* $\partial_\infty M = \Gamma$, *then* M *is embedded and thus is one of the two graphical disks.*

An outline of the paper is as follows. In §1 and §2, we derive, respectively, the equation for the curvature of a graph over a domain in a horosphere, and $C^{2+\alpha}$ bounds for smooth admissible solutions to this equation. These estimates provide strong compactness estimates for K-graphs and the basis for our subsequent arguments. §3 contains a sketch of the proof of Theorem 1 by using these estimates. In §4 we construct appropriate approximating graphs $x_{n=1} = f(x; c)$ with boundaries in $P(c)$, and obtain sharp $C^{1,1}$ estimates independent of c for $u(x; c) = \exp 2f(x; c)$ as c tends to zero. We then pass to the limit to get Theorem 2. In §5 we prove Theorem 3 by an approximation process. Section 6 contains the proof of Theorem 4 using foliation and comparison arguments. These are based on the formula for the linearized operator associated to a K-surface in $\mathbf{H}^3$, and this formula and its consequent implications for stability of K-surfaces is explained in the Appendix.

1. The equation for K

The hyperbolic distance from a point $(x_1, \cdots, x_{n+1})$ to the horosphere $P = \{x_{n+1} = 1\}$ is $y = \ln x_{n+1}$. Now suppose M is a graph $h = f(x)$, over a domain $\Omega \subset P$, $x = (x_1, \cdots, x_n, 1)$. We parametrize M by the coordinates $x_1, \cdots, x_n$ and let f_i, f_{ij} denote the usual partial derivatives of f.

Proposition 1.1. *The equation for* K *is*:

$$\det(f_{ij} + 2f_if_j + e^{-2f}\delta_{ij}) = (K+1)e^{-2nf}(1 + e^{2f}|\nabla f|^2)^{(n+2)/2}. \tag{1.1}$$

Formula (1.1) is well known (see for example [1]).

Proof of Proposition 1.1. The proof is a long and tedious calculation. We list the principal steps and let the courageous reader verify the statements.

Let $e_1, \cdots, e_{n+1}$ be the standard basis of $\mathbf{R}^{n+1}$ and let $e^y = x_{n+1}$, $\partial_y = x_{n+1}e_{n+1}$. One has the coordinate vector fields on M: $X_i = e_i + f_i\partial_y$, and the induced metric on M is given by

$$g_{ij} = \delta_{ij}e^{-2f} + f_if_j.$$

The Christoffel symbols of the hyperbolic metric are (let $m = n+1$):

$$\Gamma^i_{m,m} = \begin{cases} 0 & \text{if } i < m, \\ -1/x_m & \text{if } i = m, \end{cases}$$

$$\Gamma^i_{j,k} = \begin{cases} 0 & \text{if } i \neq m,\ j \neq m,\ k \neq m, \\ -\delta_{jk}/x_m & \text{if } i = m, \end{cases}$$

$$\Gamma^i_{j,m} = \begin{cases} 0 & \text{if } j < m,\ i \neq j, \\ -1/x_m & \text{if } i = j < m. \end{cases}$$

Let ∇ be the Riemannian connection of $\mathbf{H}^{n+1}$. Then

$$\nabla_{X_i}X_j = -f_ie_j - f_je_i + (f_{ij} + e^{-2f}\delta_{ij})\partial_y.$$

The upward pointing unit normal ν to M is:

$$\nu = \frac{1}{\tau}\left(\partial_y - e^{2y}\sum_{i=1}^n f_ie_i\right),$$

$$\tau^2 = \left(1 + e^{2y}\sum_{i=1}^n f_i^2\right).$$

One then calculates the coefficients of the second fundamental form:

$$b_{ij} = \langle \nabla_{X_i}X_j, \nu\rangle = \frac{1}{\tau}(f_{ij} + 2f_if_j + e6{-}2f\delta_{ij}).$$

Then (1.1) results from

$$(K+1)\det g_{ij} = \det b_{ij}.$$

2. $C^{2+\alpha}$ A Priori Bounds

We shall consider an equation slightly more general than (1.1). Let Ω be a smooth domain in $\mathbf{R}^n$ and consider the equation:

$$\tag{2.1} \begin{aligned} \det(f_{ij} + 2f_i f_j + e^{-2f}\delta_{ij}) &= \psi(x, f, \nabla f) \quad \text{in } \Omega, \\ f &= \phi \qquad\qquad\quad \text{on } \partial\Omega, \end{aligned}$$

where ϕ, ψ are smooth, and $\psi_0 = \inf_\Omega \psi > 0$ for $f \in \mathscr{A}$ (see (2.5)). The choice $\psi = (K+1)e^{-2nf}(1+e^{2f}|\nabla f|^2)^{(n+2)/2}$ with $K+1 = K(x, f)+1 \geq \epsilon_0 > 0$, corresponds to prescribed Gauss curvature $K = K(x, f)$.

We assume ψ satisfies

$$\tag{2.2} g(x, f, p) = \psi(x, f, p)^{1/n} \text{ is convex in } p.$$

In order for (2.1) to be elliptic, f must be "hyperbolic strictly locally convex"; that is,

$$\tag{2.3} \{f_{ij} + 2f_i f_j + e^{-2f}\delta_{ij}\} > 0 \quad \text{in } \overline{\Omega}.$$

We assume for the boundary data ϕ, the existence of a strict subsolution $\underline{f}$ of (2.1) (satisfying (2.3)):

$$\tag{2.4} \begin{aligned} \det(\underline{f}_{ij} + 2\underline{f}_i\underline{f}_j + e^{-2\underline{f}}\delta_{ij}) &\geq \psi(x, \underline{f}, \nabla\underline{f}) + \delta_0 \quad \text{in } \Omega, \\ \underline{f} &= \phi \qquad\qquad\qquad\qquad \text{on } \partial\Omega \end{aligned}$$

for some $\delta > 0$, and define the class of admissible functions

$$\tag{2.5} \begin{aligned} \mathscr{A} = \{f \in C^\infty(\overline{\Omega}) &\text{ satisfying (2.3)}, \ f = \phi \text{ on } \partial\Omega, \\ &\det(f_{ij} + 2f_i f_j + e^{-2f}) \geq \psi_0 \text{ and } f \geq \underline{f}\}. \end{aligned}$$

In deriving our estimates, it is much more convenient to work with $u = e^{2f}$. We observe that f satisfies (2.3) if and only if u satisfies

$$\tag{2.6} \{u_{ij} + 2\delta_{ij}\} > 0.$$

Set $\varphi = e^{2\phi}$, $\underline{u} = e^{2\underline{f}} > 0$ and define

$$\tag{2.7} \begin{aligned} \widetilde{\mathscr{A}} = \{u \in C^\infty(\overline{\Omega}) &\text{ satisfying (2.6)}, \ u = \varphi \text{ on } \partial\Omega, \\ &\det(u_{ij} + 2\delta_{ij}) \geq (2\inf_\Omega \underline{u})^n \psi_0 \equiv \tilde{\psi}_0 \text{ and } u \geq \underline{u} > 0 \text{ in } \overline{\Omega}\}. \end{aligned}$$

Note that $f \in \mathscr{A}$ satisfies (2.1) if and only if $u \in \widetilde{\mathscr{A}}$ satisfies

$$\tag{2.8} \det(u_{ij} + 2\delta_{ij}) = 2^n u^n \psi(x, \tfrac{1}{2}\ln u, \tfrac{1}{2}(\nabla u/u)) \equiv \tilde{\psi}(x, u, \nabla u).$$

Lemma 2.1. *Let $u \in \widetilde{\mathscr{A}}$. Then $\underline{u} \leq u \leq h - |x|^2$ where h is harmonic in Ω, $h = \varphi + |x|^2$ on $\partial\Omega$. Also, $|\nabla u| \leq C$ in Ω for a controlled constant C.*

Proof. Observe that $u \in \widetilde{\mathscr{A}}$ implies that $\tilde{u} \equiv u + |x|^2$ is convex, since $\tilde{u}_{ij} = u_{ij} + 2\delta_{ij}$. In particular $\tilde{u}$ is subharmonic and thus $\tilde{u} \le h$. This shows that

$$\underline{u} + |x|^2 \le \tilde{u} \le h \quad \text{in } \Omega.$$

Consequently, $|\nabla \tilde{u}| \le C$ on $\partial\Omega$. But for a convex function $|\nabla \tilde{u}|$ achieves its maximum on $\partial\Omega$ and so $|\nabla \tilde{u}| \le C$ in Ω. The lemma follows. q.e.d.

We turn next to second derivative estimates for u on $\partial\Omega$. Consider a point $0 \in \partial\Omega$ and choose coordinates so that the positive x_n-axis is the interior normal to $\partial\Omega$ at 0. Near 0, we can represent $\partial\Omega$ as a graph

$$x_n = \rho(x') = \frac{1}{2} \sum_{\alpha, \beta < n} B_{\alpha\beta} x_\alpha x_\beta + \mathscr{O}(|x'|^3), \tag{2.9}$$

where $x' = (x_1, \cdots, x_{n-1})$. If $u \in \widetilde{\mathscr{A}}$, then $(u - \underline{u})(x', \rho(x')) = 0$; thus

$$(u - \underline{u})_{\alpha\beta}(0) = -(u - \underline{u})_n(0) B_{\alpha\beta}; \qquad \alpha, \beta < n. \tag{2.10}$$

In particular for $u \in \widetilde{\mathscr{A}}$,

$$|u_{\alpha\beta}(0)| \le C, \qquad \alpha, \beta < n. \tag{2.11}$$

We need to establish, in addition, the strict tangential (hyperbolic) convexity of u, i.e.,

$$\sum_{\alpha, \beta} (u_{\alpha\beta} + 2\delta_{\alpha\beta}) \xi_\alpha \xi_\beta \ge c_0 > 0. \tag{2.12}$$

By rotating coordinates, it suffices to show that

$$u_{11} + 2 \ge c_0 \tag{2.13}$$

for a controlled constant $c_0 > 0$. This is easily proven directly as in [6], and in fact we can transform to the case studied there. To see this, note that $u \in \widetilde{\mathscr{A}}$ implies that $\tilde{u} = u + |x|^2$ is locally (Euclidean) convex and satisfies (recall Lemma 2.1)

$$\begin{aligned} \det \tilde{u}_{ij} &\ge \tilde{\psi}_0 > 0 \quad \text{in } \Omega, \\ \tilde{u} &= \varphi + |x|^2 \quad \text{on } \partial\Omega. \end{aligned}$$

Moreover $\tilde{\underline{u}} \equiv \underline{u} + |x|^2$ is strictly locally convex, $\tilde{\underline{u}} \le \tilde{u}$ in Ω, $\tilde{\underline{u}} = \tilde{u}$ on $\partial\Omega$. This is exactly the classical Monge-Ampère case as studied in [6, Proposition 2.1]. This gives

Proposition 2.2. *There exists* $c_0 = c_0(\Omega, \varphi, \underline{u})$ *so that* (2.12) *holds for any* $u \in \widetilde{\mathscr{A}}$.

Remark. Thus far we have not made use of condition (2.2), nor the strictness of the subsolution, i.e. $\delta_0 > 0$ in (2.4). These conditions will be utilized below to obtain an estimate for $|u_{\alpha n}(0)|$.

Set $F(D^2u) = (\det(u_{ij} + 2\delta_{ij}))^{1/n}$, and let $L = F^{ij}\partial_i\partial_j$ denote the linearized operator, i.e., $F^{ij} = \partial F/\partial u_{ij}$. If $u \in \widetilde{\mathscr{A}}$ is a solution of (2.8), then $F^{ij} = \tilde{g}(x, u, \nabla u)\tilde{b}^{ij}/n$, where $\{\tilde{b}^{ij}\}$ is the inverse of the positive matrix $\{\tilde{b}_{ij}\}$ given in (2.6), and from (2.8) we have

$$\tilde{g} = \tilde{\psi}^{1/n} = 2u\psi^{1/n}\left(x, \frac{1}{2}\ln u, \frac{1}{2}\frac{\nabla u}{u}\right) = 2ug\left(x, \frac{1}{2}\ln u, \frac{1}{2}\frac{\nabla u}{u}\right).$$

Since $\tilde{g}_{p_ip_j} = g_{p_ip_j}/2u$, we see that $\tilde{g}$ is convex in ∇u. Set

$$\mathscr{L} = L - \tilde{g}_{p_i}\partial_i - C_0 \tag{2.14}$$

with $C_0 = \max|\partial\tilde{g}/\partial u| \geq 0$, the maximum taken over the compact set (see Lemma (2.1)) $x \in \overline{\Omega}$, $|u| + |\nabla u| \leq C$ so that C_0 is a controlled constant.

Lemma 2.3. *Let $u \in \widetilde{\mathscr{A}}$ be a solution of (2.8). Then there is a controlled positive constant $\epsilon 1$ so that*

$$\mathscr{L}(u - \underline{u}) \leq -\epsilon_1\left(1 + \sum_i F^{ii}\right) \quad \text{in } \Omega. \tag{2.15}$$

Proof. Consider $\underline{w} = \underline{u} - \epsilon|x|^2/2$; for $\epsilon > 0$ small enough, $\{\underline{w}_{ij} + 2\delta_{ij}\} > 0$ and

$$\begin{aligned}\det(\underline{w}_{ij} + 2\delta_{ij}) &= \det(\underline{u}_{ij} + (2-\epsilon)\delta_{ij})\\ &\geq \det(\underline{u}_{ij} + 2\delta_{ij}) - C\epsilon \quad \text{in } \Omega\\ &\geq \tilde{\psi}(x, \underline{u}, \nabla\underline{u}) + (2^n\underline{u}^n\delta_0 - C\epsilon)\end{aligned}$$

for a uniform constant C. Hence for ϵ small enough,

$$(\det(\underline{w}_{ij} + 2\delta ij))^{1/n} \geq \tilde{g}(x, \underline{u}, \nabla u) + \epsilon_0 \tag{2.16}$$

for a controlled constant $\epsilon_0 > 0$.

Since $F(D^2u)$ is concave in D^2u (see [3]),

$$F(D^2w) \leq F(D^2u) + L(w - u),$$

and hence

$$L(u - \underline{u}) \leq -\epsilon_0 - \epsilon\sum F^{ii} + \tilde{g}(x, u, \nabla u) - \tilde{g}(x, \underline{u}, \nabla\underline{u}). \tag{2.17}$$

Using the convexity of $\tilde{g}(\cdot, \cdot, p)$ in p, we obtain

(2.18) $\tilde{g}(x, u, \nabla u) - \tilde{g}(x, \underline{u}, \nabla \underline{u}) \leq C_0(u - \underline{u}) + \tilde{g}_{p_i}(x, u, \nabla u)(u - \underline{u})_i.$

Combining (2.17) and (2.18) gives (2.15) (recall (2.14)).

Lemma 2.4. *Let $u \in \widetilde{\mathscr{A}}$ be a solution of (2.8). Then $|u_{\alpha n}(0)| \leq C$, $\alpha < n$, for a controlled constant C.*

Proof. In $\Omega \cap B_\sigma(0)$, consider the barrier

$$w = A(u - \underline{u}) + B|x|^2 \geq 0. \tag{2.19}$$

With $T = \partial_\alpha + \rho_\alpha \partial_n$,the tangential boundary operator corresponding to $\partial/\partial x_\alpha$, we have

$$\begin{aligned} T(u - \underline{u}) &= 0 \quad \text{on } \partial\Omega \cap B_\sigma(0), \\ |T(u - \underline{u})| &\leq C \quad \text{on } \Omega \cap \partial B_\sigma(0), \end{aligned}$$

and

$$|\mathscr{L} T(u - \underline{u})| \leq C\left(1 + \sum F^{ii}\right) \quad \text{in } \Omega \cap B_\sigma(0).$$

(To see this last inequality we use the formulas $\mathscr{L} u_i = \mathscr{O}(1)$,

$$\mathscr{L} T(u - \underline{u}) = \mathscr{L} u_\alpha + \rho_\alpha \mathscr{L} u_n + u_n(L\rho_\alpha + \tilde{g}_{pi}\rho_{\alpha i}) + 2F^{ij}\rho_{\alpha i}u_{nj}$$

and

$$\sum_j F^{ij}u_{nj} = \sum_j (F^{ij}\tilde{b}_{nj} - 2F^{in}) = \frac{1}{n}\tilde{g}\sum_j \tilde{b}^{ij}\tilde{b}_{nj} - 2F^{in} = \frac{1}{n}\tilde{g}\delta_{ij} - 2F^{in}.)$$

Choosing $A \gg B \gg 1$ in (2.19), by Lemma 2.3 we find that

$$\mathscr{L}(w \pm T(u - \underline{u})) \leq 0 \quad \text{in } \Omega \cap B_\sigma(0)$$

and

$$w \geq |T(u - \underline{u})| \quad \text{on } \partial(\Omega \cap B_\sigma(0)).$$

Thus by the maximum principle,

$$w \geq \pm T(u - \underline{u}) \quad \text{in } \Omega \cap B_\sigma(0)$$

and thus, in consequence of $w(0) = T(u - \underline{u})(0) = 0$,

$$|(u - \underline{u})_{\alpha n}(0)| \leq w_n(0) = A(u - \underline{u})_n(0)$$

or

$$|u_{\alpha n}(0)| \leq C_1 A + C_2. \quad \text{q.e.d.}$$

Corollary 2.5. *Let $u \in \widetilde{\mathscr{A}}$ be a solution of (2.8). Then $|u_{nn}(0)| \leq C$, for a controlled constant C.*

Proof. Expanding the left-hand side of (2.8) in cofactors and using Lemmas 2.4 and 2.1 (and 2.11) give $A^{nn}(u_{nn}+2) \le C$ at 0. By Proposition 2.2, $A^{nn} \ge c_0^{n-1}$ and so $-2 \le u_{nn} \le C/c_0^{n-1}$. q.e.d.

We now have completed the proof of the *a priori* estimate

$$\sum_{i,j} |u_{ij}| \le C \quad \text{on } \partial\Omega \tag{2.20}$$

and we now complete the proof of the global second derivative bounds

$$\sum_{i,j} |u_{ij}| \le C \quad \text{in } \overline{\Omega}. \tag{2.21}$$

Instead of carrying out the well-known argument we directly appeal to the classical result by again utilizing $\tilde{u} = u + |x|^2$, which satisfies

$$\det \tilde{u}_{ij} = \tilde{\psi}(x, \tilde{u} - |x|^2, \nabla(\tilde{u} - |x|^2)) \equiv \eta(x, \tilde{u}, \nabla, \tilde{u}) \text{ in } \Omega.$$

We note that η is a smooth function of its arguments and that $\sum |\tilde{u}_{ij}| \le C + 2n$ on $\partial\Omega$ by (2.20). Appealing to [3], we obtain a global bound for $\sum |\tilde{u}_{ij}|$ and thus (2.21) is proven.

From (2.21) and the elliptic regularity theory for concave fully nonlinear elliptic equations (see[2]), we finally obtain

Theorem 2.6. *Let $u \in \widetilde{\mathscr{A}}$ be a solution of* (2.8). *Then $\|u\|_{C^{2+\alpha}(\overline{\Omega})} \le C$ for controlled constants $\alpha \in (0, 1)$ and $C > 0$, depending only on $\Omega, \psi, \underline{u}$.*

3. Existence

In this section we sketch a proof of the existence of a smooth admissible solution $f \in \mathscr{A}$ to (2.1). As explained in §2, we study the equivalent problem of finding a smooth solution $u \in \widetilde{\mathscr{A}}$ to (2.8).

Recall that from Lemma 2.1,

$$|u| + |\nabla u| \le C \quad \text{for } u \in \widetilde{\mathscr{A}}. \tag{3.1}$$

Set

$$M = \sup\left\{\frac{\partial\tilde{\psi}}{\partial u}(x, u, \nabla u) : x \in \overline{\Omega},\ |u| + |\nabla u| \le C\right\} \tag{3.2}$$

with C as in (3.1) and $\tilde{\psi}$ as in (2.8). Consider the iterative increasing sequence $\{u^k\}_{k\ge 1}$ defined by the problems

$$(3.3)_k \qquad \begin{aligned} \det(u^k_{ij} + 2\delta_{ij}) &= \tilde{\psi}(x, u^{k-1}, \nabla u^k) + M(u^k - u^{k-1}) \quad \text{in } \Omega, \\ u^k &= \varphi \quad \text{on } \partial\Omega, \end{aligned}$$

where $u^k \in \widetilde{\mathscr{A}}$ and $u^0 \equiv \underline{u}$. Observe that if $u^{k-1} \in \widetilde{\mathscr{A}}$, then by (3.1) and (3.2),

$$\tilde{\psi}(x, u^{k-1}, \nabla \underline{u}) - Mu^{k-1} \leq \tilde{\psi}(x, \underline{u}, \nabla \underline{u}) - M\underline{u},$$

and so $\underline{u}$ is a strict subsolution to (3.3_k). Note also that u^{k-1} is a subsolution to $(3.3)_k$. In fact,

$$\det(u^r_{ij} + 2\delta_{ij}) \geq \tilde{\psi}(x, u^r, \nabla u^r), \qquad r = 1, \cdots, k-1. \tag{3.4}$$

The existence of a unique solution $u^k \in \widetilde{\mathscr{A}}$ to $(3.3)_k$ follows in a straightforward way from the continuity method and the estimates (Theorem 2.6) of §2. We briefly sketch the argument. Set

$$\eta^k(x, w, \nabla w) \equiv \tilde{\psi}(x, u^{k-1}, \nabla w) + M(w - u^{k-1}),$$

and consider the family of problems for w^t, $t \in [0, 1]$:

$$\det(w^t_{ij} + 2\delta_{ij}) \equiv \eta^t(x, w^t, \nabla w^t), \qquad w^t \in \widetilde{\mathscr{A}}, \ w^0 \equiv u^{k-1}, \tag*{$(3.5)_t$}$$

where (recall $u^0 \equiv \underline{u}$)

$$\eta^t(x, w, \nabla w) = \begin{cases} (1-t)\eta^{k-1}(x, w, \nabla w) + t\eta^k(x, w, \nabla w), & k \geq 2 \\ (1-t)\det(\underline{u}_{ij} + 2\delta_{ij}) + t\eta^1, & k = 1. \end{cases}$$

By our choice of M,

$$\eta^t(x, w, \nabla w) \leq \psi(x, \underline{u}, \nabla w) + M(w - \underline{u}), \qquad k \geq 2.$$

Thus $\underline{u}$ is a strict subsolution of $(3.5)_t$ $\forall t \in [0, 1]$ for $k \geq 2$ and $\forall t \in (0, 1]$ for $k = 1$. Starting from $w^0 \equiv \underline{u}$ at $t = 0$ we solve $(3.5)_t$ using the Implicit Function Theorem for $0 \leq t \leq 2t_0$ (with t_0 small enough to insure $\{w^t + 2\delta_{ij}\} > 0$ in $\overline{\Omega}$). Then by the maximum principle, $w^t \geq \underline{u}$ so that $w^t \in \widetilde{\mathscr{A}}$. Apply Theorem 2.6 for $t \geq t_0$ to obtain $\|w^t\|_{C^{2+\alpha}} \leq C$ independent of t. Therefore we can repeat the process and reach $t = 1$ in a finite number of steps. Thus we arrive at a sequence of solutions to $(3.3)_k$: $u^0 \leq u^1 \leq \cdots \leq u^k$.

It follows that $\{u^k\}_{k \geq 1}$ converges to some $u \in C^{0,1}(\overline{\Omega})$ with $u \geq \underline{u}$. We will in fact show that u is a smooth solution to (2.8) by establishing the *a priori* estimates

$$\|u^k\|_{C^{2\alpha}(\overline{\Omega})} \leq C, \quad \text{independent of } k. \tag{3.6}$$

It suffices, as remarked earlier to derive an *a priori* C^2 estimate

$$\|u^k\|_{C^2(\overline{\Omega})} \leq C, \quad \text{independent of } k \tag{3.7}$$

from which (3.6) follows. In view of (3.1), we need only estimate $|D^2u^k|$ on $\overline{\Omega}$. We first estimate $|D^2u^k|$ on $\partial\Omega$. Since each u^k satisfies (3.4), Proposition 2.2 applies. Therefore from the discussion of §2, it suffices to estimate $|u^k_{\alpha n}| \le C$, independent of k at any point $0 \in \partial\Omega$. But each $u = u^k$ satisfies $(3.3)_k$, and $\underline{u}$ is a strict subsolution so that Lemma 2.4 and the discussion preceding it apply (one easily checks that the nth root of the right-hand side of (3.3) is convex in ∇u). This yields bounds independent of k for $|u^k_{\alpha n}(0)|$ since we already have obtained uniform C^1 estimates.

Thus the essential point is to obtain uniform bounds for $|D^2u^k|$ in $\overline{\Omega}$, knowing that such an estimate holds on $\partial\Omega$. Set

$$M_k = \max_{x\in\overline{\Omega},\ \xi\in s^{n-1}} e^{\mu(|\nabla u^k|^2+4u^k)/2}(u^k_{\xi\xi}+2)$$

with

$$\mu = 1 + \sup_k \sup_{|\xi|=1,\ x\in\overline{\Omega}} f_{p_ip_j}\xi_i\xi_j(x, u^{k-1}, \nabla u^k),$$

where $f = \log\{\tilde{\psi}(x, u^{k-1}, \nabla u^k) + M(u^k - u^{k-1})\}$. By (3.1) μ is well-defined. As in the proof of (2.21) (see [3] or §4.3 of this paper) we easily derive

$$M_k^2 \le C_1M_k + C_2M_{k-1} + C_3$$

with C_i, $i = 1, \cdots, 3$ independent of k. Hence

$$M_k^2 \le M_{k-1}^2/2 + (C_1^2 + 2C_2^2 + 2C_3), \qquad k = 1, 2, \cdots,$$

and so

$$M_k^2 \le M_0^2/2^k + (2C_1^2 + 4C_2^2 + 4C_3) \le C.$$

This completes the proof of the smooth convergence.

Remark. 1. For the case of Gauss curvature, we can take $M = 0$ in $(3.3)_k$, and the proof is somewhat simpler.

2. It is easy to see that we have found among the admissible solutions $u \in \widetilde{\mathscr{A}}$, the "smallest", that is the one closest to $\underline{u}$.

4. Proof of Theorem 2

Let $\Gamma \subset \partial_\infty(\mathbf{H}^{n+1})$ be a smooth embedded codimension-one submanifold. We think of $\Gamma \subset \{x_{n+1} = 0\} \subset \mathbf{R}^{n+1}$ as $\Gamma = \partial\Omega$, with Ω a smooth domain in $\{x_{n+1} = 0\}$. Denote by $P(c)$ the hyperplanes $x_{n+1} = c$, so that $P(c)$ is a horosphere of $\mathbf{H}^{n+1}$ for $c > 0$. Let $\Gamma(c)$, $\Omega(c)$ be the vertical translations of Γ and Ω to $P(c)$. By Corollary 1, we know that $\Gamma(c)$ bounds a locally strictly convex graph $y = f(x; c)$ of constant Gauss curvature $K \in (-1, 0)$, where $y = \ln x_{n+1}$ is the signed distance to the horosphere $P(1)$. Thus f satisfies

$$\begin{aligned} &\det(f_{ij} + 2f_i f_j + e^{-2f}\delta_{ij}) \\ &\qquad = (K+1)e^{-2nf}(1 + e^{2f}|\nabla f|^2)^{(n+2)/2} \quad \text{in } \Omega(1), \\ &f = \ln c \quad \text{on } \partial\Omega(1). \end{aligned} \tag{4.1}$$

By setting as before $u(x; c) = e^{2f}$, then $u = u(x; c)$ is a solution of

$$\begin{aligned} \det(u_{ij} + 2\delta_{ij}) = 2^n(K+1)(1 + |\nabla u|^2/4u)^{(n+2)/2} \quad \text{in } \Omega(1), \\ u = c^2 \quad \text{on } \Gamma(1) = \partial(\Omega(1)). \end{aligned} \tag{4.2}$$

Our goal is to pass to the limit in (4.2) by obtaining sufficiently strong *a priori* estimates for the family $\{u(x; c)\}_{0<c\leq 1}$, which are independent of c. In fact we will show that $\|u\|_{C^2(\overline{\Omega}(1))} \leq C$ for a constant C independent of c. Moreover for any compact subdomain Ω' of $\Omega(1)$, it then follows from Evans' theorem [5] that

$$\|u(x, c)\|_{C^{2+\alpha}(\Omega')} \leq C', \tag{4.3}$$

where α, C' are again independent of c. These estimates are strong enough to pass to the limit as $c \to 0$ and obtain a solution $u = u(x, 0) \in C^{2+\alpha}(\Omega(1)) \cap C^{1,1}(\overline{\Omega}(1))$. With a little more effort, one could find the precise asymptotic behavior for u as in Lee-Melrose [10], but these estimates are not essential here.

4.1. Comparison surfaces. In this section we construct lower and upper radial comparison surfaces that will enable us to obtain estimates that are uniform in c as c tends to zero.

Consider a radial function $w(x) = w(r)$, $r = |x|$. A simple computation gives

$$w_{ij} + 2\delta ij = \left(\frac{w'}{r} + 2\right)\delta_{ij} + \left(w'' - \frac{w'}{r}\right)\frac{x_i x_j}{r^2}. \tag{4.4}$$

We will choose comparison functions w satisfying $w'' - w'/r \geq 0$. This implies the eigenvalues of $\{w_{ij} + 2\delta_{ij}\}$ are $w'/r + 2$ with multiplicity $n-1$ and $w'' + 2$ with multiplicity 1. Thus

$$\det(w_{ij} + 2\delta_{ij}) = (w'/r + 2)^{n-1}(w'' + 2). \tag{4.5}$$

Given $\delta > 0$ set $w(r, \delta) = (-a + \sqrt{R^2 - r^2})^2$, where

$$c = -a + \sqrt{R^2 - \delta^2}, \qquad R > a > 0. \tag{4.6}$$

Then

$$\frac{w'}{r} + 2 = \frac{2a}{\sqrt{R^2 - r^2}} > 0, \qquad w'' + 2 = \frac{2aR^2}{(R^2 - r^2)^{3/2}},$$

(note $w'' - w'/r > 0$) and so

$$\det(w_{ij} + 2\delta_{ij}) = (2a)^n R^2/(R^2 - r^2)^{(n+2)/2}.$$

On the other hand, $(1 + \frac{1}{4}w'^2/w)^{(n+2)/2} = R^{n+2}/(R^2 - r^2)^{(n+2)/2}$. Thus the graph $y = f(x)$ (with $w = x_{n+1}^2 = e^{2f}$) has constant Gauss curvature K, if R, a are related by

$$(K + 1)R^n = (2a)^n, \qquad K \in (-1, 0). \tag{4.7}$$

From (4.6) and (4.7), we see that

$$a = \frac{c + \sqrt{(1 + \lambda)c^2 + \lambda\delta^2}}{\lambda}, \qquad \lambda + 1 = 4(K + 1)^{-2/n} > 0. \tag{4.8}$$

Lemma 4.1. *Let $\overline{B}_{\delta_0}(0) \subset \Omega$. Then $u > w(r, \delta_0)$ in $\overline{B}_{\delta_0}(0)$.*

Proof. For $0 < \delta < \delta_0$ sufficiently small we have $u > w(r, \delta)$ in $\overline{B}_\delta(0)$. Let $\delta^* = \sup\{\delta \in (0, \delta_0) : u > w(r, \delta) \text{ in } \overline{B}_\delta(0)\}$. By continuity, $u \geq w(r, \delta^*)$ in $\overline{B}_{\delta^*}(0)$. Hence by the maximum principle, $u > w$ in $B_{\delta^*}(0)$. But we also have $u > c^2 = w(r, \delta)$ on $\partial B_\delta(0)$ for all $\delta \in (0, \delta_0)$. Thus $\delta^* = \delta_0$ and the strict inequality holds.

Corollary 4.2. *Let $y \in \Omega$ with $\operatorname{dist}(y, \partial\Omega) = \rho$. Then $u \geq c^2 + \alpha(K)\rho^2$, $\alpha(K) > 0$.*

Proof. From (4.7), (4.8) with $\delta = \rho$, we have

$$R - a = (\sqrt{\lambda + 1} - 1)a = \frac{c + \sqrt{(1 + \lambda)c^2 + \lambda\rho^2}}{\sqrt{1 + \lambda} + 1}.$$

This implies for suitable $\alpha = \alpha(K) > 0$

$$R - a \geq c + \frac{\alpha}{2}\frac{\rho^2}{c}. \tag{4.9}$$

Choosing y as the origin of our coordinates, we obtain $u(y) \geq w(0, \rho) = (R - a)^2 \geq c^2 + \alpha\rho^2$. q.e.d.

We turn our attention now to the construction of an upper barrier for u. Assume that Ω satisfies *a uniform exterior ball condition*, that is, there exists $\delta = \delta(\Omega)$ such that for each point $P \in \partial\Omega$, $\overline{\Omega} \cap \overline{\Omega} B_\delta(0) = \{P\}$ for suitable choice of origin.

With δ now fixed, set

$$h(x) = h(r) = (c + A(r^2 - \delta^2))^2, \qquad \delta \leq r \leq \delta + \epsilon.$$

Then

$$h'(r) = 4Arc + 4A^2 r(r^2 - \delta^2), \qquad \tfrac{1}{4}h'^2/h = 4A^2 r^2,$$
$$h''(r) = (4Ac + 8A^2r^2) + 4A^2(r^2 - \delta^2),$$

and so by (4.5),

$$\begin{aligned}\det(h_{ij} + 2\delta_{ij}) &= [2 + 4Ac + 4A^2(r^2 - \delta^2)]^{n-1} \\ &\quad \cdot [(2 + 4Ac + 8A^2r^2) + 4A^2(r^2 - \delta^2)] \\ &\leq 2^n(1 + 2Ac + 6\epsilon\delta A^2)^{n-1} \cdot 2(1 + 4A^2\delta^2),\end{aligned}$$

while

$$2^n(K+1)(1 + \tfrac{1}{4}|\nabla h|^2/h)^{(n=2)/2} \geq 2^n(K+1)(1 + 4A^2\delta^2)^{(n+2)/2}.$$

Thus h is a supersolution of (4.2) if

$$(4.10) \qquad (1 + 2Ac + 6\epsilon\delta A^2)^{n-1} \leq \frac{K+1}{2}(1 + 4A^2\delta^2)^{n/2}.$$

Choosing $\epsilon = \theta A^{-(n-2)/(n-1)}$ for $\theta = \theta(\delta, K)$ small enough insures that (4.10) is satisfied for $A \geq A_0$ large independent of c. Note that on $r = \delta + \epsilon$,

$$h = (c + A\epsilon(2\delta + \epsilon))^2 \geq 4\delta^2\theta^2 A^{2/(n-1)} > \sup_\Omega u$$

for $A \geq A_0$ large enough, independent of c.

Denote $\Omega_A \equiv \Omega \cap \{\delta < r < \delta + \epsilon(A)\}$ and note that $h > u$ on $\partial\Omega_A - \{P\}$, $h(P) = u(P) = c^2$.

Lemma 4.3. *$h \geq u$ on Ω_A for $A \geq A_0$.*

Proof. For $A \gg 1$ we have $h > u$ on $\overline{\Omega}_A - \{P\}$. Decrease A continuously. By construction $h > u$ on $\partial\Omega_A - \{P\}$ $\forall A \geq A_0$ and thus by the maximum principle, $h > u$ on $\Omega_A \forall A \geq A_0$.

Corollary 4.4. *Let $y \in \Omega$ with $d(y, \partial\Omega) = \rho \leq \epsilon(A_0)$. Then*

$$u(y) \leq c^2 + \beta(K, \delta)(c\rho + \rho^2).$$

Proof. Let $P \in \partial\Omega$ be such that $|P - y| = \rho$, and let h, Ω_{A_0} be the supersolution constructed above. Then

$$\begin{aligned}u(y) &\leq h(\delta + \rho) = (c + A_0((\delta + \rho)^2 - \delta^2))^2 \\ &\leq c^2 + \beta(K, \delta)(c\rho + \rho^2).\end{aligned}$$

We can now prove the important

Proposition 4.5. *Let Ω satisfy a uniform exterior ball condition with constant δ. Then $|\nabla u|^2/u \le C$ in Ω, with $C = C(\delta, K)$ independent of c.*

Proof. Let $y \in \Omega$ with $\operatorname{dist}(y, \partial\Omega) = \rho > 0$. It suffices to assume $\rho \le \epsilon$. Set $\tilde{u} = u + |x-y|^2$ and note that $\tilde{u}$ is convex since $\{\tilde{u}_{ij}\} = \{u_{ij} + 2\delta_{ij}\} > 0$. Hence

$$|\nabla \tilde{u}(y)| \le \rho^{-1}\big(\sup_{\partial B_\rho(y)} \tilde{u} - \tilde{u}(y)\big).$$

Thus using Corollaries 4.2 and 4.4 (with ρ replaced by 2ρ) and Corollary 4.2 we have

$$|\nabla u(y)| \le \rho^{-1}(-\alpha\rho^2 + \beta(2c\rho + 4\rho^2)) = (4\beta - \alpha)\rho + 2\beta c$$

so that

$$|\nabla u(y)|^2 \le C_1(\rho^2 + c^2).$$

By Corollary 4.2 we deduce

$$\frac{|\nabla u(y)|^2}{u(y)} \le \frac{C_1(\rho^2 + c^2)}{c^2 + \alpha(K)\rho^2} \le C.$$

Remark 4.6. For an arbitrary domain Ω which need not satisfy the uniform exterior ball condition, from the above argument we obtain the interior estimate

$$\sup_D \frac{|\nabla u|^2}{u} \le C = C(\operatorname{dist}(D, \partial\Omega), K).$$

As a consequence,

$$2^n(K+1) \le \det(u_{ij} + 2\delta_{ij}) \le C \quad \text{on } D, \tag{4.11}$$

where C depend only on $\operatorname{dist}(D, \partial\Omega)$ and K.

4.2. Second derivative estimates on $\partial\Omega$. We show in this section that $|D^2 u| \le C$ on $\partial\Omega$ with C independent of c as $c \to 0$. Let $0 \in \partial\Omega$, and as usual choose coordinates with x_n the interior normal to $\partial\Omega$ at 0 and with $x^1 = (x_1, \cdots, x_{n-1})$ such that $\rho_{\alpha\beta} = \kappa_\alpha \delta_{\alpha\beta}$ at 0 (recall near 0 we represent $\partial\Omega$ as a graph $x_n = \rho(x)$ with principal curvatures $\kappa_1, \cdots, \kappa_{n-1}$). Then $(u_{\alpha\beta}) + 2\delta_{\alpha\beta})(0) = (2 - u_n(0)\kappa_\alpha)\delta_{\alpha\beta}$. Since $0 \le u_n(0) \le Cc$ and $|\kappa_\alpha| \le C$, we have

$$u_{\alpha\alpha} + 2 \ge 1 \quad \text{for } c < \frac{1}{C^2}.$$

We must show

Lemma 4.7. $|u_{\alpha n}(0)| \le C$ *independent of* c.

Proof. Set

$$F(D^2u) = (\det(u_{ij} + 2\delta_y))^{1/n},$$
$$f(u, \nabla u) = 2(K+1)^{1/n}(1 + |\nabla u|^2/4u)^{(n+2)/2n}.$$

Differentiating the equation $F(D^2u) = f$ with respect to x_α gives

$$|\mathscr{L}u_\alpha| \le C/c$$

where $\mathscr{L} = F^{ij}\partial_i\partial_j - f_{p_i}\partial_i$. Here we have used Proposition (4.5) to estimate $|f_u u_\alpha| \le C/c$. Set $T = \partial_\alpha + \rho_\alpha \partial_n$. Then

$$\mathscr{L}Tu = \mathscr{L}u_\alpha + \rho_\alpha \mathscr{L}u_n + u_n \mathscr{L}\rho_\alpha + f\rho_{\alpha n}/n - 2\rho_{\alpha i}F^{in}.$$

Hence,

$$\begin{aligned} |\mathscr{L}Tu| &\le C\left(\frac{1}{c} + \sum F^{ii}\right) \quad \text{in } B_\sigma(0), \\ Tu &= 0 \quad \text{on } \partial\Omega \cap B_\sigma(0), \\ |Tu| &\le C(c+\sigma) \quad \text{on } \Omega \cap \partial B_\sigma(0), \end{aligned} \tag{4.12}$$

with C independent of c, σ.

Set $\eta = c^2 - \epsilon|x|^2/2$, $0 < \epsilon < 2$. By the concavity of F,

$$F(\eta) \le F(u) + L(\eta - u),$$

or

$$\begin{aligned} Lu &\le -\epsilon \sum F^{ii} + f(u, \nabla u) - (2-\epsilon) \\ &= -\epsilon \sum F^{ii} + f(u, \nabla u) - f(u, 0) \\ &\quad + 2(K+1)^{1/n} - (2-\epsilon). \end{aligned}$$

Choosing $\epsilon = 1 - (K+1)^{1/n} > 0$ and using the convexity of $f(\cdot, \nabla u)$ we find

$$Lu \le -\epsilon\left(1 + \sum F^{ii}\right) + f_{p_i}u_i$$

or

$$\mathscr{L}u \le -\epsilon\left(1 + \sum F^{ii}\right). \tag{4.13}$$

Consider in $B_\sigma(0) \cap \Omega$ the barrier

$$\varphi = A(u - c^2) + B|x|^2.$$

Then

$$\mathscr{L}\varphi \le -A\epsilon\left(1 + \sum F^{ii}\right) + B\left(2\sum F^{ii} + C\sigma/c\right),$$

since $f_{p_i} = \mathscr{O}(1/c)$. We choose $B = 2C/\sigma$, $A = \Lambda/C$ with $A \gg C$. Then $\varphi \geq |Tu|$ on $\partial(\Omega \cap B_\sigma(0))$ and

$$\mathscr{L}(\varphi \pm Tu) \leq 0 \quad \text{in } \Omega \cap B_\sigma(0).$$

Hence the maximum principle gives $\varphi \geq |Tu|$ in $B_\sigma(0)$, and since $\varphi(0) = Tu(0) = 0$ we have

$$|\partial_n Tu(0)| \leq \partial_n \varphi(0),$$

or

$$|u_{\alpha n}(0)| \leq A u_n(0) \leq C$$

with C independent of c. q.e.d.

Returning to our equation

$$\det(u_{ij} + 2\delta_{ij})(0) = \mathscr{O}(1)$$

and expanding by cofactors we find

$$A^{nn}(u_{nn} + 2) = \mathscr{O}(1)$$

uniformly as $c \to 0$. As we saw earlier $A^{nn} \geq 1$ for c sufficiently small, $0 < u_{nn} + 2 \leq C$ independent of c. Thus we have proved

Proposition 4.8. *$\sum |u_{ij}| \leq C$ on $\partial\Omega$ independent of c as $c \to 0$.*

4.3. Global second derivative bounds. Unfortunately, we must redo the global maximum principle for $D^2 u$ to make certain that we obtain an estimate independent of c.

We rewrite (4.2) as

$$F(D^2 u) = f(u, \nabla u) \quad \text{in } \Omega(1) \tag{4.14}$$

with

$$\begin{aligned} F(D^2 u) &= \log\det(u_{ij} + 2\delta_{ij}), \\ f(u, \nabla u) &= \log 2^n (K+1) + \left(\frac{n+2}{2}\right) \log\left(1 + \frac{|\nabla u|^2}{4u}\right). \end{aligned}$$

Let

$$M = \max_{\xi \in S^{n-1}, x \in \Omega} e^{\mu |\nabla u|^2/(2u)} (u_{\xi\xi} + 2),$$

where $\mu > 0$ will be chosen later.

If M is achieved on $\partial\Omega$, we are done by Proposition 4.7. Thus we may assume M is achieved at $x_0 \in \Omega$ for a direction $\xi = e_1$, and as before

$(u_{ij}(x_0))$ is diagonal. Thus, $\mu|\nabla u|^2/(2u) + \log(u_{11}+2)$ has a maximum at x_0. Set $\lambda_i = u_{ii} + 2 > 0$; then at x_0 there holds

$$\mu\left(-\frac{|\nabla u|^2}{2u^2}u_i + \frac{u_i u_{ii}}{u}\right) + \frac{u_{11i}}{\lambda_1} = 0 \quad \forall i, \tag{4.15}$$

$$\frac{\mu}{u}\left(-\frac{|\nabla u|^2}{2u}u_{ii} - 2\frac{u_i^2}{u}u_{ii} + \frac{|\nabla u|^2}{u^2}u_i^2 + u_{ii}^2 + \sum_k u_k u_{kii}\right) + \frac{u_{11ii}}{\lambda_1} - \frac{u_{11i}^2}{\lambda_1^2} \le 0. \tag{4.16}$$

Multiplying (4.16) by λ_1/λ_i and summing give

$$\sum\left(\frac{u_{11ii}}{\lambda_i} - \frac{u_{11i}^2}{\lambda_1\lambda_i}\right) + \frac{\mu\lambda_1}{u}\left(\sum\frac{u_{ii}^2}{\lambda_i} + \sum_{k,i} u_k \frac{u_{kii}}{\lambda_i}\right) \le C\mu\lambda_1\frac{|\nabla u|^2}{u^2}. \tag{4.17}$$

We now differentiate (4.14):

$$\sum_i \frac{u_{kii}}{\lambda_i} = f_u u_k + f_{p_k} u_{kk} \quad \forall k, \tag{4.18}$$

$$\sum_i \frac{u_{11ii}}{\lambda_i} - \sum_{i,j}\frac{u_{1ij}^2}{\lambda_i\lambda_j} = f_u u_1^2 + 2f_{up_1}u_1u_{11} + f_u u_{11} + f_{p_1p_1}u_{11}^2 + f_{p_i}u_{i11}. \tag{4.19}$$

Note that

$$\sum_{i,j}\frac{u_{1ij}^2}{\lambda_i\lambda_j} \ge \sum_i \frac{u_{1ii}^2}{\lambda_1\lambda_i} + \sum_{i>1}\frac{u_{1ii}^2}{\lambda_1\lambda_i}. \tag{4.20}$$

From (4.18) it follows that

$$\mu\lambda_1\sum_{k,i}\frac{u_k u_{kii}}{\lambda_i} = \mu\lambda_1\left(f_u|\nabla u|^2 + \sum_k f_{p_k}u_k u_{kk}\right), \tag{4.21}$$

while by (4.15) we obtain

$$\sum f_{p_i}u_{i11} = -\lambda_1\mu\sum f_{p_i}\left(\frac{u_i u_{ii}}{u} - \frac{|\nabla u|^2}{2u^2}u_i\right). \tag{4.22}$$

Combining (4.17), (4.19)–(4.22) and Proposition 4.5 gives the estimate
(4.23)

$$\left(\frac{\mu}{u}+f_{p_1p_1}\right)u_{11}^2\frac{\mu\lambda_1}{u}f_u|\nabla u|^2+f_uu_1^2+2f_{up_1}u_1u_{11}-\mu\lambda_1\frac{|\nabla u|^2}{2u^2}\sum u_if_{p_i} \leq C\lambda_1\mu\frac{|\nabla u|^2}{u^2}.$$

One easily checks, using Proposition 4.5, that $f_uu_1^2=\mathscr{O}(1)$, $u_1f_{up_1}=\mathscr{O}(1/u)$, $f_u=\mathscr{O}(1/u)$, $\sum u_if_{p_i}=\mathscr{O}(1)$, $f_u|\nabla u|^2=\mathscr{O}(1)$, $f_{p_1p_1}\geq -c/u$. Hence from (4.23) we obtain

$$((\mu-C)/u)u_{11}^2\leq C\lambda_1(\mu/u+1).$$

Choosing $\mu=C+1$ yields a bound for λ_1 and thus also a bound for M independent of c. Therefore we have proved

Proposition 4.9. $\sum|u_{ij}(x,c)|\leq C$ *in* $\Omega(1)$ *where* C *is independent of* c.

Hence the proof of Theorem 2 is complete.

5. Proof of Theorem 3

In this section we remove the smoothness hypotheses of Theorem 3 by an approximation process.

Proof of Theorem 3a. Let Ω_k be a monotone increasing sequence of smooth domains converging to $\Omega(1)$ in the sense of Hausdorff distance, where as in the proof of Theorem 2, $\Omega(1)$ is the vertical translation of Ω to $P(1)$. As in the proof of Theorem 2, there is a smooth solution u^k of (see (4.2))

$$\det(u_{ij}+2\delta_{ij})=2^n(K+1)\left(1+\frac{|\nabla u|^2}{4u}\right)^{(n+2)/2}\quad\text{in }\Omega_k, \tag{5.1}$$
$$u=c^2\quad\text{on }\Gamma_k=\partial(\Omega_k).$$

We now recall from Remark 4.6 and in particular estimate (4.11) that for any compact subdomain D of $\Omega(1)$ there holds

$$2^n(K+1)\leq\det(u_{ij}^k+\delta_{ij})\leq C, \tag{5.2}$$

where C depends only on $\operatorname{dist}(D,\partial\Omega$. Recall also from Lemma 2.1 that

$$u^k\leq h^k=|x|^2\quad\text{in }\Omega_k,$$

where h^k is harmonic in Ω_k, $h^k = c^2 + |x|^2$ on $\partial\Omega_k$. Because of the monotonicity of the Ω_k, the h^k are monotone increasing. Since also $u^k \geq c^2$, the u^k are uniformly bounded independent of c and k. Thus in the two-dimensional case $n = 2$, we may appeal to a result of Heinz [7] which implies that

$$\|u^k\|_{C^2(D)} \leq C, \tag{5.3}$$

where C depends only on $\operatorname{dist}(D, \partial\Omega(1))$. Using the interior higher regularity results of Evans and Krylov [9], [3], from (5.3) we obtain the estimate

$$\|u^k\|_{C^{2+\alpha}(D)} \leq C, \tag{5.4}$$

where again C depends only on $\operatorname{dist}(D, \partial\Omega(1))$. Thus a subsequence of the u^k converges to a C^∞ solution $u = u(x, c)$ of (5.1), where the convergence is locally in $C^{2+\alpha}$. Of course, u satisfies (5.3). The point in question is whether $u \in C^0(\overline{\Omega(1)})$ and $u = c^2$ on Γ. To show this, extend h^k to be $c^2 + |x|^2$ outside Ω_k; then h^k is globally subharmonic and uniformly bounded independent of k. Thus h^k converges to a harmonic function h in $\Omega(1)$. To show that $h = \phi \equiv c^2 + |x|^2$ we use a standard barrier argument. Namely, for each $x_0 \in \Gamma$ there is a superharmonic function w with $w(x_0) = 0$ and $w > 0$ in $\overline{\Omega(1)} - \{x_0\}$. Given $\epsilon > 0$, choose a neighborhood N of x_0 so that $\phi(x) - \phi(x_0) \leq \epsilon$ in N. Now choose λ (independent of k) so large that

$$\sup_{\Gamma - \Gamma\cup N} h^k \leq \lambda \inf_{\Gamma - \Gamma\cup N} w.$$

Then by the maximum principle,

$$h^k \leq \phi(x_0) + \epsilon + \lambda w \quad \text{on } \Omega(1), \tag{5.5}$$

and thus (5.1) shows that the h^k converge uniformly to h in $\overline{\Omega(1)}$. It follows that if we extend the u^k to be C^2 outside Ω_k, then the u^k converge uniformly to $u(x, c)$ in $\overline{\Omega(1)}$. Finally, letting c tend to zero, we can abstract a subsequence of the $u(x, c)$ to obtain the required solution u.

Proof of Theorem 3b. We modify the above argument by replacing the two-dimensional Heinz interior second derivative estimate with one valid in all dimensions; then the remainder of the argument is valid in all dimensions.

Let η_k^r, for $r = 1, 2$, be the unique admissible smooth solutions of

$$\begin{aligned} \det(\eta_{ij}^r + 2\delta_{ij}) &= 2^{n-r}(K+1) \quad \text{in } \Omega_k, \\ \eta^r &= c^2 \quad \text{on } \Gamma_k. \end{aligned} \tag{5.6}$$

Then as in §2,

$$c^2 \le \eta^1 \le \eta^2 \le h^k - |x|^2,$$

and η^r, $r = 1, 2$, are uniformly locally Lipschitz (independent of c and k) on compact subdomains of $\Omega(1)$. By assumption, every point of $\partial\Omega$ is a regular point for Laplace's equation, and thus a barrier exists at each point of $\partial\Omega$. Therefore as in the proof of Theorem 3a, the h^k converge uniformly to h in $\overline{\Omega(1)}$. Hence we may conclude that the η_k^r converge uniformly to η^r in $\overline{\Omega(1)}$. Moreover, the η^r are solutions in the viscosity sense [4] of the limiting problems

$$\begin{aligned} \det(\eta_{ij}^r + 2\delta_{ij}) &= 2^{n-r}(K+1) \quad \text{in } \Omega(1), \\ \eta &= c^2 \quad \text{on } \partial\Omega(1), \end{aligned} \tag{5.7}$$

and thus $\eta^2 > \eta^1$ in $\Omega(1)$. In particular, given D a fixed compact subdomain of $\Omega(1)$, we obtain

$$\epsilon_k \equiv \frac{1}{2}\inf_D(\eta_k^2 - \eta_k^1) \to \epsilon \equiv \frac{1}{2}\inf_D(\eta^2 - \eta^1) > 0.$$

We now modify the calculations of §4.3 to show that

$$|D^2 u^k| \le C \quad \text{on } D \tag{5.8}$$

with C independent of k and c. To this end we choose ζ of the form $\zeta = (\eta_k^2 - u^k - \epsilon)_+$, and note that $\zeta > \eta_k^2 - \eta_k^1$, and also that since $\zeta \le (h^k - (|x|^2 + c^2))_+ \to (h - (|x|^2 + c^2))_+$, the support of ζ is contained in a fixed compact subdomain of $\Omega(1)$ independent of k and c.

Using the concavity of $F(D^2u)$ (recall (4.14)) we have

$$F(D^2\eta_k^2) \le F(D^2u^k) + F^{ij}(\eta_k^2 - u^k)_{ij},$$

and so at x_0

$$\sum \frac{\zeta_{ii}}{\lambda_i} \ge \log\frac{1}{4} - \frac{n+2}{2}\log\left(1 + \frac{|\nabla u^k|^2}{4u^k}\right) \ge -C. \tag{5.19}$$

Let

$$M = \max_{\xi\in S^{n-1}, x\in\Omega} \zeta e^{\mu(|\nabla u|^2+4u)/2}(u_{\xi\xi} + 2),$$

where $\mu > 0$ will be chosen later, and ζ is as described.

Clearly M is achieved at $x_0 \in \Omega$ for a direction $\xi = e_1$, and as before $(u_{ij}(x_0))$ is diagonal. Thus,

$$\log \zeta + (\mu/2)(|\nabla u|^2 + 4u) + \log(u_{11} + 2)$$

has a maximum at x_0. Set $\lambda_i = u_{ii} + 2 > 0$; then at x_0 there hold

$$\zeta_i/\zeta + \mu u_i \lambda_i + \frac{u_{11i}}{\lambda_1} = 0 \quad \forall i, \tag{5.10}$$

$$\frac{\zeta_{ii}}{\zeta} - \frac{\zeta_i^2}{\zeta^2} + \mu u_{ii}\lambda_i + \mu \sum_k u_k u_{kii} + \frac{u_{11ii}}{\lambda_1} - \frac{u_{11i}^2}{\lambda_1^2} \le 0. \tag{5.11}$$

Multiplying (4.16) by λ_1/λ_i and summing give

$$\begin{aligned} &\lambda_1 \sum \frac{1}{\lambda_i}\left(\frac{\zeta_{ii}}{\zeta} - \frac{\zeta_i^2}{\zeta^2}\right) + \sum \left(\frac{u_{11ii}}{\lambda_i} - \frac{u_{11i}^2}{\lambda_1 \lambda_i}\right) + \mu\lambda_1 \sum u_{ii} \\ &\quad + \mu\lambda_1 \sum_{k,i} u_k \frac{u_{kii}}{\lambda_i} \le 0. \end{aligned} \tag{5.12}$$

We now differentiate (4.14):

$$\sum_i \frac{u_{kii}}{\lambda_i} = f_u u_k + f_{p_k} u_{kk} \quad \forall k, \tag{5.13}$$

$$\begin{aligned} &\sum_i \frac{u_{11ii}}{\lambda_i} - \sum_{i,j} \frac{u_{1ij}^2}{\lambda_i \lambda_j} \\ &\quad = f_u u_1^2 + 2 f_{up_1} u_1 u_{11} + f_u u_{11} + f_{p_1 p_1} u_{11}^2 + f_{p_i} u_{i11}. \end{aligned} \tag{5.14}$$

Note that

$$\sum_{i,j} \frac{u_{1ij}^2}{\lambda_i \lambda_j} \ge \sum_i \frac{u_{1ii}^2}{\lambda_1 \lambda_i} + \sum_{i>1} \frac{u_{1ii}^2}{\lambda_1 \lambda_i}. \tag{5.15}$$

From (5.13) it follows that

$$\mu\lambda_1 \sum_{k,i} \frac{u_k u_{kii}}{\lambda_i} = \mu\lambda_1 \left(f_u |\nabla u|^2 + \sum_k f_{p_k} u_k u_{kk}\right), \tag{5.16}$$

while by (5.10) we obtain

$$\begin{aligned} \sum f_{p_i} u_{i11} &= -\lambda_1 \sum f_{p_i} \left(\frac{\zeta_i}{\zeta} + \mu u_i (u_{ii} + 2)\right) \\ &= -\lambda_1 \sum f_{p_i} \frac{\zeta_i}{\zeta} - \mu\lambda_1 \sum f_{p_i} u_i u_{11} - 2\mu\lambda_1 \sum u_i f_{p_i}, \end{aligned} \tag{5.17}$$

and

$$
\begin{aligned}
\sum \frac{1}{\lambda_i}\frac{\zeta_i^2}{\zeta^2} &= \frac{1}{\lambda_1}\frac{\zeta_1^2}{\zeta^2} + \sum_{i>1}\frac{1}{\lambda_i}\left(\mu u_i\lambda_i + \frac{u_{11i}}{\lambda_1}\right)^2 \\
&= \frac{1}{\lambda_1}\frac{\zeta_2^2}{\zeta^2} + \sum_{i>1}\left(\frac{1}{\lambda_i}\frac{u_{11i}^2}{\lambda_1^2} + \mu^2 u_i^2\lambda_i + 2\mu u_i \frac{u_{11i}}{\lambda_1}\right) \\
&= \frac{1}{\lambda_1}\frac{\zeta_1^2}{\zeta^2} + \sum_{i>1}\left(\frac{1}{\lambda_i}\frac{u_{11i}^2}{\lambda_1^2} - \mu^2 u_i^2\lambda_i - 2\mu u_i \frac{\zeta_i}{\zeta}\right) \\
&\le \frac{1}{\lambda_1}\frac{\zeta_1^2}{\zeta^2} + \sum_{i>1}\frac{1}{\lambda_i}\frac{u_{11i}^2}{\lambda_1^2} - 2\mu\sum_{i>1} u_i\frac{\zeta_i}{\zeta}.
\end{aligned}
\tag{5.18}
$$

Combining (5.9), (5.12), and (5.14)-(5.18), gives the estimate

$$
\begin{aligned}
&(\mu + f_{p_1p-1})\lambda_1^2 - 2n\mu\lambda_1 + f_u u_1^2 + 2f_{up_1}u_1u_{11} + f_u u_{11} \\
&\quad - 2\mu\lambda_1\sum u_i f_{p_i} + \mu\lambda_1 f_u|\nabla u|^2 - C\frac{\lambda_1}{\zeta} \\
&\quad - \frac{\zeta_1^2}{\zeta^2} + 2\mu\lambda_1\sum_{i>1} u_i\frac{\zeta_i}{\zeta} - \lambda_1\sum f_{p_i}\frac{\zeta_i}{\zeta} \le 0.
\end{aligned}
\tag{5.19}
$$

Since the support of ζ is fixed, the quantities f_{p_1p-1}, f_u, f_{up_1}, f_{p_i}, ζ_i, u_i are uniformly bounded on the support of ζ independent of k and c. Thus multiplying (5.15) by ζ^2 and choosing μ sufficiently large, we find that M is uniformly bounded independent of k and c. Since $\zeta \ge \epsilon/2$ on D for k large, the interior estimate (5.8) is valid. This completes the proof of Theorem 3b.

6. Uniqueness theorems

In this section we shall show that a Jordan curve Γ bounds exactly two K-surfaces, when Γ is on a horosphere or on P_∞, the asymptotic boundary of $\mathbf{H}^3$ (assuming $-1 < K < 0$). Each of the K-surfaces is an embedded disk and is a graph in a horospherical coordinate system.

In general, a Jordan curve Γ in $\mathbf{H}^3$ need not bound any K-surface, since there are topological obstructions [13]. Also Γ can bound immersed (and embedded) K-surfaces of higher genus. For example, let S be a sphere in $\mathbf{H}^3$ of curvature K (S is compact if $K > 0$, and is an equidistant, noncompact, sphere if $-1 < K \le 0$). Let C_1, C_2 be circles on S that meet in two points, and let N_1, N_2 be small tubular neighborhoods of C_1, C_2 on S. Let P be one of the components of $N_1 \cap N_2$. Displace N_2 off N_1 near P, so the new $\widetilde{N}_2 \cup N_1$ is topologically a torus minus a disk, and let Γ be the (smoothed) boundary of $N_1 \cup \widetilde{N}_2$. Before the displacement of N_2, the corresponding Γ (immersed into S) bounds the immersed K-surface M (a torus minus a disk) in S. If $-1 < K < 0$ any small perturbation of the boundary values of a K-surface comes from a perturbation of the surface, so the embedded $\Gamma = \partial(N_1 \cup \widetilde{N}_2)$ bounds an embedded K-surface of genus one. One can also make this work when $K \geq 0$.

Let the upper half-space of $\mathbf{R}^3$, $x_3 \geq 0$, model $\mathbf{H}^3 \cup P_\infty$, with P_∞ the extended plane $x_3 = 0$. For $c > 0$, let $P(c)$ denote the horosphere $x_3 = c$. We shall say a curve Γ in P_∞ is the asymptotic homological boundary of a surface M in $\mathbf{H}^3$, if for $c > 0$ sufficiently small, $M \cap P(c)$ contains a connected component $\Gamma(c)$ such that $\Gamma(c)$ converges to Γ as $c \to 0$, and $\Gamma(c)$ is homologous to zero on M, i.e., there exists a compact submanifold $M(c)$ of M and $\Gamma(c) = \partial M(c)$. We write $\Gamma = \partial_\infty(M)$ for the asymptotic homological boundary Γ of M. When we speak of graphs we mean graphs in this coordinate system: $x_3 = f(x_1, x_2)$.

Theorem 6.1. *Let Γ be a Jordan curve in P_∞, and K a constant between -1 and 0. There are exactly two embedded K-surfaces M in $\mathbf{H}^3$ with $\partial_\infty M = \Gamma$. Each surface is an embedded disk and is a graph over one of the components of $P_\infty - \Gamma$. If M is any immersed K-surface in $\mathbf{H}^3$ with $\partial_\infty M = \Gamma$, then M is embedded, and is hence one of the two graphical disks.*

Proof. The existence of one of the two such K-surfaces follows immediately from Theorem 3a. To obtain the second K-surface with boundary Γ, we choose a horospherical coordinate system so that the other connected component of $P_\infty - \Gamma$ is bounded, and again apply Theorem 3a.

It remains to prove the uniqueness of embedded M, with $\partial_\infty M = \Gamma$ and the embeddedness of an immersed such surface. First we establish some properties of K-surfaces in $\mathbf{H}^3$.

Lemma 6.2. *Let Γ be a smooth Jordan curve embedded in the horosphere $P(c)$, $c > 0$. Let M be a compact K-surface in $\mathbf{H}^3$ with $\partial M = \Gamma$. Then $x_3 \geq c$ on M, and M is transverse to $P(c)$.*

Proof. Assume to the contrary, that M is not above the horosphere $P(c)$. Let p be a lowest point of M, so that $x_3(p) < c$. First observe that the mean curvature vector of M at p (denoted by $H(M, p)$) cannot point up: for the vector $H(P(x_3(p)), p)$ points up and the curvature of $P(x_3(p))$ is zero, and therefore greater than K. So $P(x_3(p))$ should be above M in a neighborhood of p if $H(M, p)$ points up. Hence $H(M, p)$ must point down. Consider the hyperbolic plane L, tangent to M at p and below M. M has more curvature than L, so M should be below L in a neighborhood of p. This contradiction shows M is above $x_3 = c$.

Now suppose M is not transverse to $P(c)$ at some point $p \in \Gamma$. Consider vertical planes passing through p (vertical Euclidean planes are hyperbolic planes too) and their trace curves α on M on β on $P(c)$. The mean curvature vector of M at p is vertically upward, and each curve α is tangent to the corresponding β at p, so the curvature of each α is greater than or equal to the curvature of each β, which is one. The curvatures of the α-curves at p (as the vertical planes rotate about the vertical line through p) are between the principal curvatures κ_1 and κ_2 of M at p, since they are normal sections of M at p. We know $K = \kappa_1\kappa_2 - 1$ and $\kappa_1 > 0$, $\kappa_2 > 0$. Since $K < 0$, at least one of κ_1, κ_2 is less than one. Since each normal curvature is of the form $\kappa_1 \cos^2\theta + \kappa_2 \sin^2\theta$ for some θ, in any θ interval of length π, there is a normal curvature less than one. Hence some α curve has curvature less than one, a contradiction. This proves transversality and Lemma 6.2.

Lemma 6.3. *Let Γ be a smooth Jordan curve in $P(c)$, $c > 0$, and let Ω be the bounded domain in $P(c)$ with boundary Γ. Then there is a unique K-surface M embedded in $\mathbf{H}^3$, with $\partial M = \Gamma$, and whose mean curvature vector points up, and M is a graph over Ω.*

Proof. The existence of M has been proved in Corollary 1; it remains to prove uniqueness.

Let C_0 and C_1 be circles in $P(c)$ that bound an annulus A in $P(c)$, containing Γ in its interior. Let C_t, $0 \le t \le 1$, be a smooth foliation of A by Jordan curves such that, for some τ, $C_\tau = \Gamma$.

Let M_0 and M_1 be the equidistant spherical caps of curvature K such that $\partial M_0 = C_0$, $\partial M_1 = C_1$, and the mean curvature vectors of M_0, M_1 point up. Choose C_0, C_1 so that M_0 is below M and M is below M_1. This is easy to do by Lemma 6.2: once M_0 is chosen below M, M_1 can be chosen to be the image of a spherical cap containing M_0 by a hyperbolic isometry which is a homothety from a point on P_∞.

We now observe that there is a foliation F of the compact region bounded by $A \cup M_0 \cup M_1$, by K-surfaces N_t, $0 \le t \le 1$, with $\partial N_t = C_t$, each N_t is a graph, and $N_0 = M_0$, $N_1 = M_1$. We obtain F as follows. Start at M_0. By Corollary 1 and the Appendix, for t near 0, $t > 0$, C_t bounds a K-surface N_t (a graph), $N_0 = M_0$, and N_t varies continuously with t (for compact K-surfaces, $K \in (-1, 0)$, small variations of the boundary values come from variations of the surface). There are no non-trivial Jacobi fields on any N_t, so they are pairwise disjoint and foliate a neighborhood of M_0. By the compactness Theorem 2.6, the set of t for which N_t exists is closed. Hence the foliation can be extended to $t = 1$. It remains to prove $N_1 = M_1$. There are several ways to see this. N_1 is above $P(c)$, so one can use the Alexandrov reflection technique with vertical planes to prove N_1 has all the symmetries of its boundary, the circle C_1. Hence N_1 is rotational. The details of this approach are in [11]. Another way to prove $N_1 = M_1$ using a foliation, will be clear later.

Now using the foliation F we will prove $M = N_\tau$, hence is unique. M_0 is disjoint from M and below M. As t increases from 0 to τ, $t < \tau$, no M_t can intersect M, otherwise, consider the smallest such t, M_t is on one side of M at an intersection point (necessarily interior to M_t and M), and their mean curvature vectors are both pointing up at this point, so they would be equal by the maximum principle. This is impossible since $\partial M_t \neq \partial M$, for $t < \tau$. Thus M is above N_τ (maybe equal to it). Now do the same argument starting with M_1 and letting t decrease from 1 to τ. As before, we conclude N_τ is above M. Thus $M = N_\tau$, and we have proved Lemma 6.3.

Remarks. 1. Notice that the above argument can be used to give another proof that $N_1 = M_1$: foliate a region containing M_1 by equidistant K-spheres whose boundaries foliate an annulus on $P(c)$ containing C_1. Then the above argument shows N_1 is a leaf of the foliation, hence equal to M_1.

2. The above proof also implies that a compact embedded K-surface M in $\mathbf{H}^3$ whose boundary is a round circle is part of a sphere. After an ambient isometry, one can assume C is contained in a horosphere P. If the mean curvature vector of M points up, then M is part of a sphere as explained in Lemma 6.3. It if points down, then let P_1 be a horosphere in $\mathbf{H}^3$ with $P_1 \cap P = C$. Thus M points up with respect to S_1 in a suitable system of horospherical coordinates.

3. We will see later that one need only assume M immersed in order to conclude that M is spherical, that is, M being an immersed K-surface with ∂M in a horosphere implies that M is embedded.

Now we can prove the uniqueness part of Theorem 6.1. Assume first that $\Gamma \subset P_\infty$ is a smooth Jordan curve, and Ω the bounded component of $\{x_3 = 0\}$ with boundary Γ. Foliate an annulus A in P_∞, by Jordan curves C_t so that C_0 and C_1 are circles, and $C_\tau = \Gamma$ for some τ, $0 < \tau < 1$. Let M be a K-surface embedded in $\mathbf{H}^3$ with $\partial_\infty M = \Gamma$, and the mean curvature vector of M points up. Let M_0 and M_1 be equidistant K-spheres in $\mathbf{H}^3$, with $\partial_\infty M_0 = C_0$, $\partial_\infty M_1 = C_1$, and whose mean curvature vectors point up. Choose C_0, C_1 so that M_0 is below M, and M is below M_1. We foliate the region between M_0 and M_1 by K-surfaces N_t, $N_0 = M_0$, $N_1 = M_1$, each N_t a graph; and $\partial_\infty N_t = C_t$, for $0 \le t \le 1$. Assuming such a foliation F exists, it follows, as in the proof of Lemma 6.3, that $M = N_\tau$, hence is unique. As t goes from 0 to τ, $t < \tau$, N_t cannot touch M and is below M (a first point of contact of N_t and M cannot be at infinity since this would oblige $\partial N_t \cap \partial M \neq \varnothing$). Similarly, decreasing t from 1 to τ, we conclude N_τ is above M. Hence $M = N_\tau$ as desired.

Now we construct the foliation F. For $c > 0$, let $C_t(c)$ be the foliation in $P(c)$ obtained by vertical translation of C_t. As in the proof of Lemma 6.3, there is a foliation $F(c)$, by compact K-surfaces $N_t(c)$, $0 \le t \le 1$, satisfying: $N_t(c)$ is a graph, $\partial N_t(c) = C_t(c)$, and $N_0(c)$, $N_1(c)$ are equidistant spherical caps that converge to M_0 and M_1 respectively, as $c \to 0$. By the compactness results of §4, each $N_t(c)$ converges to a K-surface graph N_t, as $c \to 0$ uniformly on compact sets. Clearly the N_t are pairwise disjoint for $t_1 \neq t_2$ (otherwise $N_{t_1}(c) \cap N_{t_2}(c) \neq \varnothing$ for some $c > 0$), and they vary smoothly with t, hence they form a foliation F as desired.

Now suppose $\Gamma \subset S_\infty$ is a Jordan curve, not necessarily smooth. Let C_t, $0 \le t \le 1$, be a topological foliation of an annulus in P_∞, with C_0, C_1 circles and $C_\tau = \Gamma$ for some τ. This foliation can be obtained using a homeomorphism $\phi\colon P_\infty \to P_\infty$, taking Γ to a circle and with ϕ equal to the identity in two small disks, one in each connected component of $P_\infty - \Gamma$. Then the preimage by ϕ of a foliation by circles in P_∞ will give the C_t. For $c > 0$, let $C_t(c)$ be a smooth foliation by Jordan curves, $0 \le t \le 1$, chosen so that $C_t(c) \to C_t$ as $c \to 0$. The foliation $C_t(c)$ bounds a smooth foliation by K-surfaces $N_t(c)$, $\partial N_t(c) = C_t(c)$, each $N_t(c)$ a graph. This was proved in Lemma 6.3. By the compactness results of §5, $N_t(c)$ converges to a graph N_t, as $c \to 0$, $\partial N_t = C_t$. The foliation $N_t(c)$ converge to the foliation by N_t. As in the smooth case, this implies that any K-surface M with $\partial_\infty M = \Gamma$ and mean curvature vector pointing up, is the leaf N_τ of this foliation.

Remark 6.4. We remark that there may exist an embedded K-surface M with asymptotic boundary a circle Γ (not homologically) and M not a graph. It is not hard to see that a rotational surface of this type does not exist; one obtained by rotating a "drop-like" curve about an axis.

Our argument fails since one cannot find an equidistant sphere below such an M, and one above; the mean curvature vectors point in opposite directions when one uses the foliation. One can still say something about $M : M$ is invariant by symmetry in the hyperbolic plane P with $\partial_\infty P = \Gamma$. Each component of M in $\mathbf{H}^3 - P$ is a graph over a domain in P. We refer the reader to [11] where this is proved for H-surfaces. The proof uses Alexandrov reflection in hyperbolic planes "parallel" to P, and works exactly the same way for K-surfaces; the maximum principle is the basic tool.

In fact, all the theorems of [11] that are proved using Alexandrov reflection apply verbatim for K-surfaces in $\mathbf{H}^3$. For example, if M is an embedded K-surface and $\partial_\infty M$ is one point, then M is a horosphere. If $\partial_\infty M$ consists of two disjoint circles, then M is a rotational surface. Similarly if $\partial_\infty M$ equals two points, M is rotational.

Finally, to complete the proof of Theorem 6.1, we will show that when $\partial_\infty M = \Gamma$ and M is an immersed K-surface, then M is embedded.

Choose $c > 0$ so that $M \cap P(c)$ contains an embedded curve $\Gamma(c)$ and $\Gamma(c) = \partial N$, $N \subset M$, N compact. By Lemma 6.2, we know that N is above $P(c)$ and is transverse to $P(c)$ along $\Gamma(c)$. Let S be a compact sphere, sufficiently close to $P(c)$, so that S is transverse to M, $S \cap M$ is a Jordan curve $\tilde{\Gamma}$, close to $\Gamma(c)$, and $\tilde{\Gamma}$ bounds a compact submanifold $\tilde{N}$ of M, $\tilde{N}$ contained in the ball of $\mathbf{H}^3$ bounded by S. It suffices to prove $\tilde{N}$ is embedded. For notational convenience we will call $\tilde{\Gamma}$, $\tilde{N}$, by Γ, N, for the rest of this proof.

Γ separates S into two connected components A and B. The idea is to show that one can smooth, either $A \cup N$ or $B \cup N$, along Γ, to obtain a smooth immersed compact surface of positive curvature. Then by Hadamard's theorem, the surface is an embedded sphere.

Orient S and N so that their unit normal vectors n_S and n_N, point to the convex side of each surface (so n_S points into the ball bounded by S). Let ν be a unit vector field along Γ, that is tangent to M and points into A, and let $P(x)$ be the plane generated by $n_S(x)$ and $\nu(x)$. Denote $L(x) = T_x(M) \cap P(x)$. $L(x)$ is one-dimensional since the tangent vector $\Gamma'(x)$ to Γ at x, is orthogonal to $P(x)$, and in $T_x(M)$. M is transverse to S along Γ, so $L(x)$ is never orthogonal to $n_S(x)$. Hence $n_N(x)$ is never parallel to $n_S(x)(T_x(N)$ is generated by $\Gamma'(x)$ and $L(x))$. We know $n_N(x)$ is orthogonal to $\Gamma'(x)$, and $L(x)$ hence $n_N(x)$ has a positive projection onto A or B and this is independent of x: $\langle n_N(x), \nu(x)\rangle \neq 0$ for $x \in \Gamma$.

Suppose n_N projects positively onto A. We claim that $N \cup A$ can be smoothed along Γ to have positive curvature. First observe that Γ is a curve on (the convex) surface M, so its curvature vector $\Gamma''(x)$ has a positive scalar product with $n_N(x)$ for each $x \in \Gamma$; i.e., Γ curves towards the convex side of M. Hence, in a neighborhood of x, $N \cup A$ is in the half-space defined by $\Gamma'(x)$, $L(x)$ and $n_M(x)$. So the plane $\Gamma'(x)$, $L(x)$ is a local support plane for $N \cup A$, and $N \cup A$ can be smoothed to be locally convex.

Appendix: The linearized operator and stability

Let $f\colon M \to N$ be an immersion, M and N Riemannian manifolds, M compact, and ∂M nonempty. Let exp denote the usual exponential map of the normal bundle of M in N into N, and let $n(x)$, $x \in M$, denote a unit normal vector field along M in N.

For $u \in C_0^{2+\alpha}(M)$, $-1 \leq t \leq 1$, we define $f(t)\colon M \to N$ to be the maps $x \mapsto \exp_{f(x)}(tu(x)n(x))$. For t near zero, $f(t)$ is an immersion.

Let K be a (curvature) function and define $J = J_f\colon C_0^{2+\alpha}(M) \to C^\alpha(M)$ by

$$J_f(u)(x) = \left.\frac{d}{dt}\right|_{t=0} (K(f_t(x))).$$

J_f is the linearized operator of K at f associated to normal variations given by n. It is also called the Jacobi operator, and elements of its kernel are called the Jacobi fields. M (i.e., $f\colon M \to N$) is said to be stable when the kernel is trivial.

Now suppose $N = N^{m+1}(c)$ is one of the simply connected space forms $\mathbf{R}^{m+1}$, S^{m+1} or $\mathbf{H}^{m+1}$ ($c = 0, +1$, or -1), and $M = M^m$ is of codimension one. Let $0 \leq r < m$, and $K = S_{r+1}$ be the $(r+1)$st symmetric curvature function of M in N. Then we have an explicit formula for the Jacobi operator (cf. [12], [13]):

$$J(u) = L_r(u) + (c(m-r)S_r + S_1 S_{r+1} - (r+2)S_{r+2})u,$$

where $l_r(u) = \operatorname{div}(T_r \nabla u)$, T_r is the rth Newton tensor of the shape operator A of M in N, $T_0 = I$, and $T_r = S_r I - A T_{r-1}$.

When the linear term has a negative coefficient (i.e., when $c(m-r)S_r + S_1 S_{r+1} - (r+2)S_{r+2} < 0$ on M), and when L_r is an elliptic operator, the usual maximum principle implies that the kernel of J is trivial. For example, this is always the case where $c = -1$ and $m = 2$, with $0 < S_2 < 1$ (these are convex surfaces in $\mathbf{H}^3$). The coefficient of u is $2HK = S_1(-1 + S_2) < 0$; the direction of the normal to M is that for which the principal curvatures are positive, so $S_1 > 0$. The same reasoning shows M is stable when $r = 1$, m arbitrary, $0 < S_2 < 1$ and $S_3 \geq 0$. In particular, if any S_k, $k > 3$, is positive, then so is S_3 [13]. In general, however, there may be nontrivial Jacobi fields for S_2 or $S_m(r = m - 1)$.

When M is stable (and the linearized equation is elliptic), then small variations of the boundary values of M come from small variations of M. We now make this precise.

Assume $f\colon M^m \to N^{m+1}$ and let n be a normal vector field along M in N. Consider N as isometrically immersed in some Euclidean space $\mathbf{R}^l$, and let $\pi\colon T \to N$ be the projection of a (small) tubular neighborhood T of N in $\mathbf{R}^l$, for $y \in T$, $\pi(y)$ is the closest point of N to y. Let $\gamma_0\colon \partial M \to N \subset \mathbf{R}^l$ be the restriction of f to ∂M, and for $\gamma \in C^{2+\alpha}(\partial M, N)$, let $h(\gamma) : M \to \mathbf{R}^l$ denote the harmonic extension of $\gamma - \gamma_0$ to M.

For γ in a neighborhood U of γ_0, $U \subset C^{2+\alpha}(\partial M, N)$ and $u \in C_0^{2+\alpha}(M)$ in a neighborhood V of zero, the map $M \to \mathbf{R}^l$, $x \mapsto f(x) + h(\gamma)(x) + u(x)n(x)$ will be an immersion of M into T. We define

$$U \times V \xrightarrow{F} C^\alpha(M), \qquad F(\gamma, u) = K(\pi(f + h(\gamma) + un)).$$

F is C^∞ and

$$D_2F(\gamma_0, 0)(u) = J_f(u).$$

Suppose J_f is elliptic, and M has constant curvature c. Then J_f is a Fredholm operator of index zero, so $D_2F(\gamma_0, 0)$ is an isomorphism. By the implicit function theorem, there is a neighborhood $U_0 \subset U$ of γ_0 a neighborhood $V_0 \subset V$ of 0, and a smooth map $u\colon U_0 \to V_0$ such that $F(\gamma, u(\gamma)) = c$, for $\gamma \in U_0$.

Define $H(\gamma) = \pi(f + h(\gamma) + u(\gamma)n)$ for $\gamma \in U_0$. Then $K(H(\gamma)) = c$ and $H(\gamma_0) = \pi(f + 0) = f$, $H(\gamma)/\partial M = \pi(f + \gamma - \gamma_0) = \pi(\gamma) = \gamma$. Thus the solutions of the equation $K = c$ depend smoothly on the boundary values.

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