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HIGHER ORDER CONSERVATION LAWS. II

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1. Introduction

Suppose that θ is an exact differential form of degree 1 and h is a vector 1-form. Then θ is a conservation law for $\underline{h}$ if $\underline{h}\theta$ is also exact. The Nijenhuis tensor $[\underline{h}, \underline{h}]$ of a vector 1-form $\underline{h}$ with distinct eigenvalues plays an important role in the study of conservation laws and their generalizations on an analytic manifold. For example, the vanishing of this tensor guarantees the existence of a basis of exact 1-forms $(dv^1, \cdots, dv^n)$ which are also eigenforms of $\underline{h}$. An important consequence of this fact is that a differential from θ of degree p is a higher order conservation law if and only if it has a representation

$$\sum_{i_1<\cdots<i_p} \mathscr{A}_{i_1\cdots i_p}(v^{i_1}, \cdots, v^{i_p})dv^{i_1} \wedge \cdots \wedge dv^{i_p} .$$

The preceding result is established in *Higher order conservation laws* [5], hereafter referred to as HOCL.

In the present paper higher order conservation laws for vector 1-forms $\underline{h}$ and $\underline{k}$ which commute are studied. The results which are obtained include as special cases certain theorems which appear in [4]. Some of the notation which was established in HOCL is reviewed briefly in § 2 of this paper.

2. Notation and definitions

The ring of germs of analytic functions at some point of an analytic manifold is denoted by A, and the localization of the A-module of differential forms on this manifold is denoted by $\mathscr{E}$. The exterior algebra $\Lambda^*\mathscr{E}$ generated by $\mathscr{E}$ is a direct sum

$$\Lambda^*\mathscr{E} = \Lambda^0\mathscr{E} \oplus \Lambda^1\mathscr{E} \oplus \cdots \oplus \Lambda^n\mathscr{E} ,$$

where $\Lambda^0\mathscr{E} = A$ and $\Lambda^1\mathscr{E} = \mathscr{E}$. An element $\underline{h} \in \operatorname{Hom}_A(\mathscr{E}, \mathscr{E})$ induces homomorphisms

$$\Lambda^p\mathscr{E} \xleftarrow{h^{(q)}} \Lambda^p\mathscr{E} , \qquad 0 \leq q \leq p ,$$

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of $\Lambda^*\mathscr{E}$ which are defined by setting

$$\text{(2.1)}\qquad \begin{aligned} &h^{(q)}(\theta_1 \wedge \cdots \wedge \theta_p) \\ &= \frac{1}{(p-q)!q!} \sum_\pi |\pi|\cdot(\underline{h}\theta_{\pi(1)} \wedge \cdots \wedge \underline{h}\theta_{\pi(q)}) \wedge \theta_{\pi(q+1)} \wedge \cdots \wedge \theta_{\pi(p)} \,, \end{aligned}$$

where $\theta_i \in \mathscr{E}$, and π runs through all permutations of $(1, \cdots, p)$. The signature of the permutation π is denoted by $|\pi|$, and the transformation $h^{(0)}$ is taken to be the identity on $\Lambda^p\mathscr{E}$.

The following result which is utilized in § 3 asserts that $h^{(q)}$ can be expressed in terms of $h^{(1)}, h^{(2)}, \cdots, h^{(q-1)}$.

Lemma 2.1. Let $\theta \in \Lambda^q\mathscr{E}$. Then

$$\text{(2.2)}\qquad \begin{aligned} &[(h^q)^{(1)} - (h^{q-1})^{(1)}h^{(1)} + \cdots + (-1)^{q-1}h^{(1)}h^{(q-1)}]\theta \\ &= \begin{cases} 0, & q > p \,, \\ (-1)^{q-1}qh^{(q)}\theta, & q \leq p \,. \end{cases} \end{aligned}$$

The proof of Lemma 2.1 appears in HOCL. The result may also be established in the Case $q = n$ by an application of the Cayley-Hamilton Theorem. Let $\underline{h}$ have characteristic equation

$$\underline{h}^n = \mathscr{A}_0 I + \mathscr{A}_1\underline{h} + \mathscr{A}_2\underline{h}^2 + \cdots + \mathscr{A}_{n-1}\underline{h}^{n-1} \,,$$

where $\mathscr{A}_i$ is the sum of the diagonal minors of order i of $(\underline{h})$, the matrix which represents $\underline{h}$ in some given basis of $\mathscr{E}$. Since the trace function is linear, one then obtains from the characteristic equation the result that

$$\operatorname{tr}\underline{h}^n = n\mathscr{A}_0 + \mathscr{A}_1 \operatorname{tr}\underline{h} + \mathscr{A}_2 \operatorname{tr} h^2 + \cdots + \mathscr{A}_{n-1}\operatorname{tr}\underline{h}^{n-1} \,,$$

where $\operatorname{tr}\underline{h}$ denotes the trace of $\underline{h}$. If $p = n$ in equation (2.1), then

$$\begin{aligned} h^{(q)}(\theta_1 \wedge \cdots \wedge \theta_n) &= (\operatorname{tr} h^{(q)})(\theta_1 \wedge \cdots \wedge \theta_n) \\ &= (-1)^{q-1}\mathscr{A}_{n-q}(\theta_1 \wedge \cdots \wedge \theta_n), \quad 1 \leq q \leq n \,, \end{aligned}$$

and hence

$$\begin{aligned} &\operatorname{tr}\underline{h}^n - (\operatorname{tr}\underline{h})(\operatorname{tr}\underline{h}^{n-1}) + (\operatorname{tr} h^{(2)})(\operatorname{tr}\underline{h}^{n-2}) \\ &\qquad - \cdots + (-1)^{n-1}(\operatorname{tr} h^{(n-1)})(\operatorname{tr}\underline{h}) = (-1)^{n-1} n \det \underline{h} \,. \end{aligned}$$

The last equation can then be interpreted as an operator on n-forms, and we obtain

$$\begin{aligned} &(h^n)^{(1)} - (h^{n-1})^{(1)}h^{(1)} + (h^{n-2})^{(1)}h^{(2)} \\ &\qquad - \cdots + (-1)^{n-1}(\operatorname{tr} h^{(n-1)})(\operatorname{tr}\underline{h}) = (-1)^{n-1} n \det h \end{aligned}$$

as a result.

A mapping $D\colon \Lambda^*\mathscr{E} \to \Lambda^*\mathscr{E}$ is said to be a derivation of degree r if $D(\Lambda^p\mathscr{E}) \subset \Lambda^{p+r}\mathscr{E}$, $D(\theta_p \wedge \theta_q) = D\theta_p \wedge \theta_q + (-1)^{pr}\theta_p \wedge D\theta_q$, and $D(\theta + \varphi) = D\theta + D\varphi$. The subscripts denote the degrees of the forms. It is clear that $h^{(1)}$ as defined by (2.1) is a derivation of degree 0. Further examples are obtained by taking special cases of formula (5.9) in [1]. The following definition will be of importance in this paper.

Let $\theta \in \Lambda^p\mathscr{E}$ and $\underline{h}$ and $\underline{k}$ be any elements of $\operatorname{Hom}_A(\mathscr{E}, \mathscr{E})$. A mapping $[\underline{h}, k]\colon \Lambda^p\mathscr{E} \to \Lambda^{p+1}\mathscr{E}$ may be defined by setting

$$(2.3)\qquad \begin{aligned}[\underline{h}, k]\theta = \tfrac{1}{2}\{&-[h^{(1)}k^{(1)} - (hk)^{(1)}]d\theta \\ &+ [h^{(1)}dk^{(1)}\theta + k^{(1)}dh^{(1)}\theta] \\ &+ [d\{(hk)^{(1)} + k^{(1)}h^{(1)}\}\theta]\} .\end{aligned}$$

If one observes that

$$h^{(1)}(\theta_p \wedge \theta_q) = h^{(1)}\theta_p \wedge \theta_q + \theta_p \wedge h^{(1)}\theta_q ,$$

and

$$h^{(2)}(\theta_p \wedge \theta_q) = h^{(2)}\theta_p \wedge \theta_q + \theta_p \wedge h^{(2)}\theta_q + h^{(1)}\theta_p \wedge h^{(1)}\theta_q ,$$

then it is easily verified that

$$(2.4a)\qquad [\underline{h}, \underline{k}](\theta_p \wedge \theta_q) = [\underline{h}, \underline{k}]\theta_p \wedge \theta_q + (-1)^p\theta_p \wedge [\underline{h}, \underline{k}]\theta_q ,$$

$$(2.4b)\qquad [\underline{h}, \underline{k}]\mathscr{A}\theta_p = \mathscr{A}[\underline{h}, \underline{k}]\theta_p , \qquad \mathscr{A} \in A ,$$

and consequently $[\underline{h}, \underline{k}]$ is a derivation of $\Lambda^p\mathscr{E}$ of degree 1, and

$$[\underline{h}, \underline{k}] \in \operatorname{Hom}_A(\Lambda^p\mathscr{E}, \Lambda^{p+1}\mathscr{E}) .$$

It should be observed that if $p > 1$ and $\underline{h} = \underline{k}$, the formula

$$(2.5)\qquad [\underline{h}, \underline{h}]\theta = -h^{(2)}d\theta + h^{(1)}dh^{(1)}\theta - d[(h^2)^{(1)} + h^{(2)}]\theta$$

is the special case of formula (5.9) of [1] obtained by setting $L = M = \underline{h}$. That is, $[\underline{h}, \underline{k}]\theta = \frac{1}{2}i_{[h,k]}\theta$ with $\theta \in \Lambda^p\mathscr{E}$ and $p \geq 2$. The cases $p = 1$ and $\underline{h} = \underline{k}$ yield the usual formulas

$$(2.6a)\qquad \begin{aligned}[\underline{h}, \underline{k}]\theta = \tfrac{1}{2}\{&-[h^{(1)}k^{(1)} - (hk)^{(1)}]d\theta \\ &+ [h^{(1)}d\underline{k}\theta + k^{(1)}d\underline{h}\theta] \\ &- [d\{\underline{h}\underline{k} + \underline{k}\underline{h}\}\theta]\} ,\end{aligned}$$

and

(2.6b) $$[\underline{h}, \underline{h}]\theta = -h^{(2)}d\theta + h^{(1)}d\underline{h}\theta - d\underline{h}^2\theta \ .$$

Note that $[\underline{h}, \underline{k}]$ as defined by (2.3) acts trivially on $\Lambda^0\mathscr{E} = A$, and its action on $\Lambda^1\mathscr{E} = \mathscr{E}$ determines it completely. Thus if $[\underline{h}, \underline{k}] = 0$ on 1-forms, it also vanishes on p-forms when $p \geq 2$. However if $[\underline{h}, \underline{k}]$ vanishes on p-forms when $p \geq 2$, it does not follows that $[\underline{h}, \underline{k}]$ vanishes on 1-forms. For example let $p = 2$ and suppose $\{\partial/\partial u_1, \partial/\partial u_2, \partial/\partial u_3\}$ forms a basis for $E = \operatorname{Hom}_A(\mathscr{E}, A)$. If $\underline{h}$ is described by setting

$$\frac{\partial}{\partial u_1}\underline{h} = \frac{\partial}{\partial u_1} + \frac{\partial}{\partial u_2} - \frac{\partial}{\partial u_3} \ ,$$
$$\frac{\partial}{\partial u_2}\underline{h} = u_2\frac{\partial}{\partial u_2} \ ,$$
$$\frac{\partial}{\partial u_3}\underline{h} = u_3\frac{\partial}{\partial u_3} \ ,$$

then

$$\left(\frac{\partial}{\partial u_1} \wedge \frac{\partial}{\partial u_2} \wedge \frac{\partial}{\partial u_3}\right)[\underline{h}, \underline{h}] \ ,$$
$$= \left(\frac{\partial}{\partial u_1} \wedge \frac{\partial}{\partial u_2}\right)[\underline{h}, \underline{h}] \wedge \frac{\partial}{\partial u_3} - \left(\frac{\partial}{\partial u_1} \wedge \frac{\partial}{\partial u_3}\right)[\underline{h}, \underline{h}] \wedge \frac{\partial}{\partial u_2}$$
$$+ \left(\frac{\partial}{\partial u_2} \wedge \frac{\partial}{\partial u_3}\right)[\underline{h}, \underline{h}] \wedge \frac{\partial}{\partial u_1} = 0 \ ,$$

while

$$\left(\frac{\partial}{\partial u_1} \wedge \frac{\partial}{\partial u_2}\right)[\underline{h}, \underline{h}] = \frac{\partial}{\partial u_2} \neq 0 \ .$$

§ 2 is concluded with a definition of the notion of a higher order conservation law.

An element $\theta \in \Lambda^p\mathscr{E}$ is called a conservation law of order p for $h^{(q)}, 0 \leq q \leq p \leq n$, if and only if θ and $h^{(q)}\theta$ are all (locally) exact forms.

3. The endomorphisms $\underline{h}$ and $\underline{k}$

Let $\underline{h} \in \operatorname{Hom}_A(\mathscr{E}, \mathscr{E})$ have distinct eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$. If $\underline{k} \in \operatorname{Hom}_A(\mathscr{E}, \mathscr{E})$ and $\underline{h}\underline{k} = \underline{k}\underline{h}$, then $\underline{k}$ is of the form

(3.1) $$\underline{k} = f(\underline{h}) = \mathscr{A}_0 I + \mathscr{A}_1\underline{h} + \cdots + \mathscr{A}_{n-1}\underline{h}^{n-1} \ ,$$

where $\mathscr{A}_i \in A$. Any eigenvectors of $\underline{h}$ are also eigenvectors of $\underline{k}$, while

$$\beta_i = f(\lambda_i) = \mathscr{A}_0 + \mathscr{A}_1\lambda_i + \cdots + \mathscr{A}_{n-1}\lambda_i^{n-1} \tag{3.2}$$

are the eigenvalues of $\underline{k}$. The following lemma appears in [4].

Lemma 3.1. *Let $\underline{h} \in \operatorname{Hom}_A(\mathscr{E}, \mathscr{E})$ have distinct eigenvalues $\lambda_1, \cdots, \lambda_n$. If $\underline{h}\underline{k} = \underline{k}\underline{h}$ and $[\underline{h}, \underline{h}] = [\underline{h}, \underline{k}] = 0$, then there exist coordinates $v^1, v^2, \cdots, v^n$ such that $\{dv^1, dv^2, \cdots, dv^n\}$ are eigenforms of $\underline{h}$ and $\underline{k}$, and the corresponding eigenvalues λ_i and β_i of $\underline{h}$ and $\underline{k}$ respectively are functions of v^i alone.*

Lemma 3.2. *Let $\underline{h} \in \operatorname{Hom}_A(\mathscr{E}, \mathscr{E})$ have distinct eigenvalues and satisfy $[\underline{h}, \underline{h}] = 0$. An element $\theta \in \Lambda^p\mathscr{E}$ is a conservation law for $h^{(1)}, \cdots, h^{(p)}$ if and only if*

$$\theta = \sum_{i_1 < \cdots < i_p} \mathscr{A}_{i_1 \cdots i_p}(v^{i_1}, \cdots, v^{i_p}) dv^{i_1} \wedge \cdots \wedge dv^{i_p} ,$$

where $\{dv^i\}$ are the eigenforms of Lemma 3.1.

The proof of Lemma 3.2 appears in HOCL.

Theorem 3.3. *Let $\underline{h} \in \operatorname{Hom}_A(\mathscr{E}, \mathscr{E})$ have distinct eigenvalues. If $\underline{h}\underline{k} = \underline{k}\underline{h}$, $[\underline{h}, \underline{h}] = [\underline{h}, \underline{k}] = 0$, and $\theta \in \Lambda^p\mathscr{E}$ is a conservation law for $h^{(q)}$ with $0 \leq q \leq p \leq n$, then θ is also a conservation law for $(k^j)^{(l)}$ where $0 \leq j \leq p$ and $0 \leq l \leq p$.*

Proof. Let θ have the form

$$\theta = \sum_{i_1 < \cdots < i_p} \mathscr{A}_{i_1 \cdots i_p}\{dv^{i_1} \wedge \cdots \wedge dv^{i_p}\} .$$

It is a consequence of Lemma 3.2 that the functions $\mathscr{A}_{i_1 \cdots i_p}$ depend only on $(v^{i_1}, \cdots, v^{i_p})$ and therefore θ is exact. Hence

$$k^{(1)}\theta = \sum_{i_1 < \cdots < i_p} \mathscr{A}_{i_1 \cdots i_p}(\beta_{i_1} + \cdots \beta_{i_p}) dv^{i_1} \wedge \cdots \wedge dv^{i_p} ,$$

where the β_i are the eigenvalues of $\underline{k}$. As a consequence of Lemma 3.1, the form $k^{(1)}\theta$ is also exact. Similarly the forms $(k^j)^{(1)}\theta$ and hence $(k^j)^{(l)}\theta$ are exact when j and l are nonnegative integers in the set $(0, 1, 2, \cdots \cdot p)$.

It should be noted that whenever $[\underline{k}, \underline{k}] = 0$ it is automatically true that $[\underline{k}^i, \underline{k}^j] = 0$ for any pair of nonnegative integers (i, j) and any vector 1-form $\underline{k}$. The converse of this statement is not true. Moreover, the vanishing of $[\underline{h}, \underline{h}]$ and $[\underline{k}, \underline{k}]$ does not in general guarantee the vanishing of $[\underline{k}, \underline{k}]$. However if certain conditions are imposed on $\underline{h}$ and $\underline{k}$ as in Theorem 3.3, then the following result is obtained.

Theorem 3.4. *Let $h \in \operatorname{Hom}_A(\mathscr{E}, \mathscr{E})$ have distinct eigenvalues. If $\underline{h}\underline{k} = \underline{k}\underline{h}$ and $[\underline{h}, \underline{h}] = [\underline{h}, \underline{k}] = 0$, then $[\underline{k}^i, \underline{k}^j] = [\underline{h}^i, \underline{k}^j] = 0$ for any pair of nonnegative integers i and j.*

Proof. The vanishing of $[\underline{h}, \underline{h}]$ assures the existence of a basis $\{dv^1, \cdots, dv^n\}$ of eigenforms for $\mathscr{E}$. If $\theta = dv^1 + \cdots + dv^n$, then the set of forms $\{\theta, h\theta, \cdots, h^{n-1}\theta\}$ is also a basis of $\mathscr{E}$, and this basis consists of conservation laws for $\underline{h}, \underline{h}^2, \cdots, \underline{h}^{n-2}$. Since θ is then a conservation law for $\underline{k}^i$, it is easily demonstrated that $[\underline{k}^i, \underline{k}^j](\underline{h}^l\theta) = [\underline{h}^i, \underline{k}^j](\underline{h}^l\theta) = \theta$, for $l = 0, 1, 2, \cdots, (n-1)$ and any nonnegative integers i and j.

The fact that $[\underline{h}^i, \underline{k}^j]$ vanishes is a consequence primarily of the algebraic conditions imposed on $\underline{h}$ and $\underline{k}$. That is, it is possible that $[\underline{h}, \underline{h}] = [\underline{k}, \underline{k}] = [\underline{h}, \underline{k}] = 0$ and $[\underline{h}^i, \underline{k}^j] \neq 0$ for some choice of $\underline{h}$ and $\underline{k}$ and nonnegative integers i and j. For example if $\underline{h}$ and $\underline{k}$ are noncommuting vector 1-forms defined in the two dimensional case by

$$\frac{\partial}{\partial x}\underline{h} = x\frac{\partial}{\partial x} + (y-x)\frac{\partial}{\partial y}, \qquad \frac{\partial}{\partial y}\underline{h} = x\frac{\partial}{\partial y},$$

and

$$\frac{\partial}{\partial x}\underline{k} = x\frac{\partial}{\partial x}, \qquad \frac{\partial}{\partial y}\underline{k} = y\frac{\partial}{\partial y},$$

then it is easily established by routine calculations that $[\underline{h}, \underline{h}] = [\underline{k}, \underline{k}] = [\underline{h}, \underline{k}] = 0$, while $[\underline{h}, \underline{k}^2] \neq 0$.

The following corollary yields many additional conservation laws once it is known that $\theta \in \Lambda^p\mathscr{E}$ is a conservation law for $h^{(q)}$.

Corollary 3.5. *If $\theta \in \Lambda^p\mathscr{E}$ is a conservation law for $h^{(q)}$, then θ is also a conservation law for $(hk)^{(j)}$ where j is any nonnegative integer.*

Proof. Since θ is a conservation law for $k^{(j)}$, formula (2.3) implies $d[(hk)^{(1)} + k^{(1)}h^{(1)}]\theta = 0$, and hence θ is a conservation law for $(hk)^{(1)}$. Since $[\underline{h}^i, \underline{k}^j] = 0$, it follows from (2.3) again that $d[(h^ik^j)^{(1)} + (h^i)^{(1)}(k^j)^{(1)}]\theta = 0$ and by repeated use of (2.2) and (2.3) one obtains the result that θ is also a conservation law for $(hk)^{(j)}$ when j is any nonnegative integer.

It should be noted that if $\underline{l} \in \operatorname{Hom}_A(\mathscr{E}, \mathscr{E})$, and a p-form θ is a conservation law $l^{(1)}$, then θ need not be a conservation law for $l^{(j)}$ or $(l^j)^{(1)}$ when $j \geq 2$, and consequently the conclusion of Theorem 3.3 and Corollary 3.5 are nontrivial. An example illustrating this last comment is given in HOCL.

If the hypothesis that the eigenvalues β_i of $\underline{k}$ are also distinct is added, then the following converse of Theorem 3.3 is obtained.

Theorem 3.6. *Let $\underline{h}$ and $\underline{k}$ be elements of* $\operatorname{Hom}_A(\mathscr{E}, \mathscr{E})$ *and $\underline{h}\underline{k} = \underline{k}\underline{h}$. If $\underline{h}$ and $\underline{k}$ have distinct eigenvalues λ_i and β_i respectively, and $[\underline{h}, \underline{h}] = [\underline{h}, \underline{k}] = 0$, then any conservation law for $k^{(j)}$ is also a conservation law for $h^{(j)}$, where $0 \leq j \leq p \leq n$.*

Proof. Let $\theta \in \Lambda^p\mathscr{E}$ be a conservation law for $k^{(j)}, 0 \leq j \leq p \leq n$. Since $\{dv^1, \cdots, dv^n\}$ is an eigenform basis for $\mathscr{E}$, the $\binom{n}{p}$ elements $(dv^{i_1} \wedge \cdots \wedge dv^{i_p})$ with $i_1 < \cdots < i_p$ form a basis for $\Lambda^p\mathscr{E}$. Hence θ has the form

$$\theta = \sum_{i_1 < \cdots < i_p} C_{i_1 \cdots i_p} dv^{i_1} \wedge \cdots \wedge dv^{i_p},$$

where the functions $C_{i_1\cdots i_p}$ will depend in general on $v^1, \cdots, v^n$. In order to show that θ is a conservation law for $h^{(j)}$ it is sufficient to prove that these functions depend only on the variables $v^{i_1}, \cdots, v^{i_p}$. Consequently, one is led to a study of the equations $d(k^j)^{(1)}\theta = 0, 0 \leq j \leq p$. These equations in turn lead to $\binom{n}{p+1}$ systems of homogeneous partial differential equations, and each system contains $(p + 1)$ equations in $(p + 1)$ unknowns. The analysis is identical to that contained in Theorem 3.4 of HOCL, and since the eigenvalues β_i are distinct by our hypothesis, the result that $C_{i_1\cdots i_p} = C_{i_1\cdots i_p}(v^{i_1}, \cdots, v^{i_p})$ is then obtained.

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