J. DIFFERENTIAL GEOMETRY
4 (1970) 477-485

A CHARACTERIZATION OF A STANDARD TORUS IN E^3

KATSUHIRO SHIOHAMA & RYOICHI TAKAGI

0. Introduction

Let M be a two dimensional, connected, complete and orientable Riemannian manifold of class C^∞, and $\iota\colon M \to E^3$ be an isometric immersion of M into a Euclidean three space. The purpose of the present paper is to find some conditions for M to be congruent to a standard torus in E^3; by a standard torus in E^3 we mean a surface of revolution defined by

$$x = (a + b\cos u)\cos v\,, \qquad y = (a + b\cos u)\sin v\,, \qquad z = b\sin u\,,$$
$$a > b > 0, 0 \leq u < 2\pi, 0 \leq v < 2\pi\,,$$

which we shall denote by $T(a, b)$. One of the properties of a standard torus is that one of its principal curvatures is constant everywhere. There are a lot of classes of surfaces with such property, for example, sphere, right circular cylinder, standard torus, etc.. A characterization of a standard torus seems to be more complicated than those of a sphere or right circular cylinder under the condition that one of the principal curvatures is constant everywhere, since a standard torus has non-constant mean curvature and its Gaussian curvature changes sign. The authors were inspired on this subject by one of the problems stated by Willmore in [4], and were informed of this problem by Professor M. Obata.

Problem (*Willmore* [4]). Let $\iota\colon M \to E^3$ be an imbedding of a compact and orientable manifold M of genus 1 into E^3, and H be the mean curvature of $\iota(M)$ with respect to the induced metric from E^3. Then, does the following equality hold?

$$\inf_{\iota} \int_{\iota(M)} H^2 dA = 2\pi^2\,,$$

where dA denotes the area element of M and ι ranges over all imbeddings of M into E^3.

The main theorem of the present paper gives a partial solution to the above problem, and can be stated as follows:

Received July 8, 1969.

Main theorem. *Let M be a two-dimensional, connected, compact and orientable Riemannian manifold of class C^∞ and nonzero genus, and $\iota\colon M \to E^3$ be an isometric immersion. Suppose that one of the principal curvatures of $\iota(M)$ is a constant R everywhere. Then we have*

$$\int_M H^2 \circ \iota dA \geq 2\pi^2 ,$$

where the equality holds if and only if ι is an imbedding, and $\iota(M)$ is congruent to the standard torus $T(\sqrt{2}/|R|, 1/|R|)$.

In § 2 we shall classify the surfaces satisfying that one of the principal curvatures is a constant everywhere, and a proof of the main theorem will be given in § 3.

The authors wish to express their gratitude to Professor H. Nakagawa for his valuable advice about the treatment of differential equations (2.5) in § 2, and to Professor M. Obata for his constant encouragement.

1. Definitions and notation

Throughout this paper let M be a two dimensional, connected, complete and orientable Riemannian manifold of class C^∞, and $\iota\colon M \to E^3$ be an isometric immersion of M into a Euclidean 3-space. When the argument is local in nature, a point $p \in M$ may be identified with $\iota(p)$. Let $\mathscr{F}(M)$ and $\mathscr{F}(E^3)$ be the orthonormal frame bundles on M and E^3 respectively, and B the subset of $\mathscr{F}(E^3)$ defined by $B=\{b=(p,e_1,e_2,e_3) \mid (p,e_1,e_2) \in \mathscr{F}(M),(\iota(p),\iota_*(e_1),\iota_*(e_2),e_3) \in \mathscr{F}(E^3)\}$. Then, $\tilde{\iota}\colon B \to \mathscr{F}(E^3)$ is naturally defined by $\iota(b) = (\iota(p), \iota_*(e_1), \iota_*(e_2), e_3)$ where $b = (p, e_1, e_2, e_3)$. We may identify $\iota_*(e_1)$ and $\iota_*(e_2)$ with e_1 and e_2 respectively. The structure equations of E^3 are given by

$$(1.1) \qquad \begin{aligned} dp &= \sum_{\alpha=1}^{3} \tilde{\omega}_\alpha e_\alpha , & de_\alpha &= \sum_{\beta=1}^{3} \tilde{\omega}_{\alpha\beta} e_\beta , & & \\ d\tilde{\omega}_\alpha &= \sum_{\beta=1}^{3} \tilde{\omega}_{\alpha\beta} \wedge \tilde{\omega}_\beta , & d\tilde{\omega}_{\alpha\beta} &= \sum_{\gamma=1}^{3} \tilde{\omega}_{\alpha\gamma} \wedge \tilde{\omega}_{\gamma\beta} , & \tilde{\omega}_{\alpha\beta} + \tilde{\omega}_{\beta\alpha} &= 0 , \end{aligned}$$

where $\tilde{\omega}_\alpha$ and $\tilde{\omega}_{\alpha\beta}$ are differential 1-forms on $\mathscr{F}(E^3)$. Putting $\tilde{\iota}^*(\tilde{\omega}_\alpha) = \omega_\alpha$ and $\tilde{\iota}^*(\tilde{\omega}_{\alpha\beta}) = \omega_{\alpha\beta}$ where $\tilde{\iota}^*$ is the dual map of $\tilde{\iota}_*$, we get

$$(1.2) \qquad \begin{aligned} &\omega_3 = 0 , \\ &\omega_{i3} = \sum_{j=1}^{2} h_{ij}\omega_j, \qquad h_{ij} = h_{ji} \qquad (i, j = 1, 2) . \end{aligned}$$

The quadratic form $\sum h_{ij}\omega_i\omega_j$ is called the second fundamental form of M. A point $x \in M$ is called a umbilical point if the matrix (h_{ij}) takes the form

$$(h_{ij}) = \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix} \tag{1.3}$$

at the point, where r is a real number. If a point is not umbilical, there exists a neighborhood all of whose points are not umbilical. In such a neighbourhood, we can take an orthonormal frame field with respect to which (h_{ij}) takes the form

$$(h_{ij}) = \begin{pmatrix} r_1 & 0 \\ 0 & r_2 \end{pmatrix}, \qquad r_1 > r_2 . \tag{1.4}$$

In a neighborhood containing no umbilical point we always use only such frame field, which can be considered as a local cross section, to be denoted by σ, of M to the bundle space B. For simplicity we identify $\sigma^*\omega_{ij}$ and $\sigma^*\omega_i$ with ω_{ij} and ω_i respectively, $i, j = 1, 2, 3$. Then, we have

$$\begin{aligned} d\omega_i &= \sum_j \omega_{ij} \wedge \omega_j , \\ d\omega_{ij} &= \sum_k \omega_{ik} \wedge \omega_{kj} \qquad (i, j, k = 1, 2, 3) . \end{aligned} \tag{1.5}$$

Denoting by K and H the Gaussian curvature and the mean curvature of M respectively, we have the following well known formulas

$$K = r_1 \cdot r_2 , \qquad 2H = r_1 + r_2 . \tag{1.6}$$

Since M is orientable, a unit normal vector field e_3 can be globally defined on M. Then we can consider r_1, r_2 as continuous functions on M satisfying $r_1 \geq r_2$, and can reduce the assumption that one of the principal curvatures is everywhere a constant R to one of the following:

$$\text{(i) } r_1 \equiv R \geq r_2 , \qquad \text{(ii) } r_1 \geq r_2 \equiv R . \tag{1.7}$$

Furthermore we may assume that $R \geq 0$ (by replacing the unit normal vector field e_3 by $-e_3$, if necessary). It follows from a theorem of Massey [2] that $\iota(M)$ is a cylinder if $R = 0$. We shall classify the surfaces with the properties (1.7) and $R > 0$ in the next section.

2. Surfaces one of whose principal curvatures is a positive constant

First, let M possess the property (i) of (1.7). Later, it will be seen that the discussion on $r_1 \equiv R$ essentially covers the one on $r_2 \equiv R$. Suppose that there is a non-umbilical point p on M. Then, there exists a neighborhood V of p in which every point is nonumbilical. From (1.2) and (1.4) we observe

$$\omega_{13} = R\omega_1 , \tag{2.1}$$

$$\omega_{23} = r_2\omega_2 , \qquad R > r_2 , \tag{2.2}$$

where r_2 is differentiable on V.

Lemma 1. *There are differentiable functions u and f defined on V satisfying*

$$\omega_{12} = f\omega_2 , \tag{2.3}$$

$$\omega_1 = du . \tag{2.4}$$

Proof. Taking exterior differentiation of (2.1) and making use of the structure equations (1.5), we have

$$Rd\omega_1 = R\omega_{12} \wedge \omega_2 = r_2\omega_{12} \wedge \omega_2 .$$

Since $R - r_2 \neq 0$, we have $\omega_{12} \wedge \omega_2 = 0$ and $d\omega_1 = 0$, from which the Lemma follows.

Taking exterior differentiation of (2.2) and (2.3) and making use of the structure equations (1.5) again, along every integral curve of e_1 we get

$$\begin{aligned} \partial r_2/\partial u &= f(R - r_2) , \\ \partial f/\partial u &= -(Rr_2 + f^2) , \end{aligned} \tag{2.5}$$

which imply

$$(R - r_2)\frac{\partial^2 r_2}{\partial u^2} + 2\left(\frac{\partial r_2}{\partial u}\right)^2 + Rr_2(R - r_2)^2 = 0 ,$$

or

$$\phi\left(\frac{\partial^2\phi}{\partial u^2} + R^2\phi - R\right) = 0 , \tag{2.6}$$

where we have put

$$\phi = 1/(R - r_2) . \tag{2.7}$$

By solving (2.6) and making use of (2.7), (2.5) we thus have

$$\begin{aligned} r_2 &= \frac{R(a\cos Ru + b\sin Ru)}{a\cos Ru + b\sin Ru + 1/R} , \\ f &= \frac{R(-a\sin Ru + b\cos Ru)}{a\cos Ru + b\sin Ru + 1/R} . \end{aligned} \tag{2.8}$$

Lemma 2. *Every integral curve γ of e_1 in V is a geodesic of M, and moreover is a part of a circle of radius $1/R$.*

Proof. From (2.3) we have $(de_1)(e_1) = \omega_{12}(e_1)\cdot e_1 = 0$, which shows that γ is a geodesic of M. Moreover we have along γ,

$$\begin{aligned} dp &= e_1 du\ , \\ de_1 &= \qquad Re_3 du\ , \\ de_3 &= -Re_1 du\ , \\ de_2 &= 0\ . \end{aligned}$$

Therefore γ is a part of a circle of radius $1/R$. q.e.d.

Suppose again that there is a non-umbilical point p on M. From now on V_0 denotes the set of all non-umbilical points on M, and V is the connected component of V_0 containing p.

Lemma 3. *The integral curve γ_p of e_1 through p is a closed geodesic, and is a circle of radius $1/R$. Furthermore there does not exist any umbilical point on γ_p.*

Proof. By Lemma 2 it is sufficient to show that there is not a sequence $\{u_n\}$ $(n = 1, 2, \cdots)$ of parameters of γ_p converging to u_0 such that $\lim_{n\to\infty} \gamma_p(u_n)$ is a umbilical point. Assume that there is such a sequence. From (2.8), we have

$$R - r_2(\gamma_p(u)) = 1/(a_0 \cos Ru + b_0 \sin Ru + 1/R)\ ,$$

where a_0, b_0 are constants along γ_p. Noting that $R - r_2$ is a continuous function on M we see

$$\lim_{u\to u_0} [R - r_2(\gamma_p(u))] \neq 0\ .$$

This fact implies $\lim_{n\to\infty} \gamma_p(u_n) \in V$, which contradicts the assumption.

Proposition 4. *Let M be a surface with $r_1 \equiv R \geq r_2$. Then M is either totally umbilical or else umbilic free.*

Proof. Suppose that there is a non-umbilical point p on M. For any point $q \in V$ let γ_q denote the integral curve of e_1 which is a closed geodesic. γ_q is a circle of radius $1/R$ and contained entirely in V by Lemma 3. Since V is open in M, it suffices to show that V is closed in M. Let $\{p_n\}$ $(n = 1, 2, \cdots)$ be a sequence of points belonging to V such that $\lim_{n\to\infty} p_n = p_0 \in M$, and set $\gamma_n = \gamma_{p_n}$. By completeness of M we can choose a subsequence $\{\bar{\gamma}_n\}$ of $\{\gamma_n\}$ converging to some closed geodesic γ_0 through p_0. It follows from (2.8) that for each n there exists a point q_n on $\bar{\gamma}_n$ for which $r_2(q_n) = 0$ holds. Then we can choose a subsequence $\{\bar{q}_n\}$ of $\{q_n\}$ converging to a point q_0 on γ_0. Thus we have $r_2(q_0) = \lim_{n\to\infty} r_2(\bar{q}_n) = 0$ by continuity of r_2, and hence $q_0 \in V$. This fact together with $\lim_{n\to\infty} \gamma_n = \gamma_0$ implies that the tangent vector of γ_0 at q_0 coincides with $e_1(q_0)$. Thus we have $\gamma_0 = \gamma_{q_0}$, in particular $p_0 \in V$.

Corollary to Proposition 4. *Let M be a surface with $r_1 \equiv R \geq r_2$. If there is a umbilical point on M, then M is totally umbilical and hence M is isometric to the sphere $S^2(R)$ of radius $1/R$.*

Proposition 5. *Let M be a surface with $r_1 \equiv R \geq r_2$. If r_2 does not change sign, then $\iota(M)$ is congruent to either $S^2(R)$ or the right circular cylinder $S^1(R) \times E^1$, where $S^1(R)$ is a circle of radius $1/R$.*

Proof. By virtue of Proposition 4 it suffices that $f \equiv 0$ and $r_1 \equiv 0$ hold if M is umbilic free. Then we may assume that the orthonormal frame field (p, e_1, e_2, e_3) under consideration is globally defined on M. For any point $p \in M$, the integral curve γ_p of e_1 through p is a circle of radius $1/R$. From (2.8) we have, along γ_p,

$$r_2 = \frac{R\sqrt{a^2 + b^2}\sin(Ru + \Phi)}{\sqrt{a^2 + b^2}\sin(Ru + \Phi) + 1/R},$$

where $\cos\Phi = a/\sqrt{a^2 + b^2}, \sin\Phi = b/\sqrt{a^2 + b^2}$. Since u can take all real numbers, r_2 changes sign if $a^2 + b^2 \neq 0$, from which we must have $a = 0, b = 0$. Therefore a and b must vanish identically on M and then we have $f \equiv 0$ and $r_2 \equiv 0$. The remainder of proof follows immediately from the structure equations. q.e.d.

Now let M possess the property (ii) of (1.7), i.e., $r_1 \geq r_2 \equiv R > 0$. Also in this case the previous discussions are valid by exchanging the role of r_1 and the one of r_2 mutually. Assume that there is a non-umbilical point p on such M. Then there is a point p_0 on the integral curve of e_2 such that $r_1(p_0) = 0$, which contradicts $r_1 > 0$. Thus we have proved

Proposition 6. *If M is a surface with $r_1 \geq r_2 \equiv R$, then M is isometric to $S^2(R)$.*

As a contrapositive of Proposition 4, we state that r_2 changes sign if M is neither isometric to a sphere nor to a right circular cylinder. Thus if M is compact, which is not isometric to a sphere, and possesses the property $r_1 \equiv R \geq r_2 (R > 0)$, then r_2 changes sign. This case will be dealt with in the next section.

3. A characterization of a standard torus

Throughout this section let $\iota\colon M \to E^3$ be an isometric immersion of a connected, compact and orientable riemannian manifold M of nonzero genus, and let M possess the property $r_1 \equiv R \geq r_2 (R > 0)$. First we shall prove the following

Theorem 7. *If ι is an imbedding, then M is diffeomorphic to a standard torus.*

Proof. We may assume that the orthonormal frame field (p, e_1, e_2, e_3) is globally defined on M. Fix an arbitrary point p of M, and let λ_p be the integral curve of e_2 with $\lambda_p(0) = p$. Then there is a positive number L such that the restriction $\lambda_p | [0, L)$ traverses every circle $\gamma_q (q \in M)$ just once and $\lambda_p(L) \in \gamma_p$.

Making use of λ_p we can easily construct a simple and closed curve $\tilde{\lambda}_p$: $[0, L] \to M$ which traverses every circle $\gamma_q(q \in M)$ just once. $\tilde{\lambda}_p$ can be considered as a cross section of the base space $\tilde{\lambda}_p([0, L])$ to the circle bundle space M. Hence M is diffeomorphic to a standard torus. q.e.d.

Define an orthonormal frame field $(p, \bar{e}_1, \bar{e}_2)$ on M by $\iota_*(\bar{e}_i) = e_i, i = 1, 2$. Furthermore, define a mapping ϕ: $M \to E^3$ by $q = \phi(p) = \iota(p) + (1/R)e_3(\iota(p))$. Then we have $dq = dp + (1/R)e_3 = (1 - r_2/R)e_2\omega_2$, which shows that the curve $C(\bar{v}) = \phi(p(\bar{u}, \bar{v}))$ is regular because $r_2 \neq R$, where $\bar{u}$ (resp. $\bar{v}$) denotes the parameter of some integral curve of $\bar{e}_1$ (resp. $\bar{e}_2$). We shall call the curve C the central curve of M. Since M is compact, C is closed (not necessarily simple). It can be seen by a straightforward calculation that $C(\bar{v})$ is a circle, i.e., $\iota(M)$ is congruent to a standard torus if and only if both functions a and b in §2 are nonzero constants. Here we shall estimate $\int_M H^2 \circ \iota dA$. It is evident that the following inequality holds:

$$\int_M H^2 \circ \iota dA \geq \int_{\iota(M)} H^2 dA \ , \tag{3.1}$$

where the equality holds if and only if there does not exist any open subset W of $\iota(M)$ whose inverse image $\iota^{-1}(W)$ has at least two components.

In order to compute $\int_{\iota(M)} H^2 dA$, we will use the formulas of Frenet-Serrét (C, ξ_1, ξ_2, ξ_3) for the central curve C. Retake the parameter of C so that it represents arc length from a fixed point on C. Then we obtain an immersion ιp of $S^1(1) \times S^1(2\pi/l)$ onto $\iota(M)$ defined by

$$(\iota p)(u, v) = C(v) + \frac{1}{R}(\xi_2 \cos u + \xi_3 \sin u) \ .$$

Denoting curvature and torsion of C by κ and τ, the formulas of Frenet-Serrét for C are

$$\begin{aligned} dC &= \xi_1 dv \ , \\ d\xi_1 &= \qquad\quad \kappa\xi_2 dv \ , \\ d\xi_2 &= -\kappa\xi_1 dv \qquad\quad + \tau\xi_3 dv \ , \\ d\xi_3 &= \qquad\quad -\tau\xi_2 dv \ . \end{aligned} \tag{3.2}$$

Then the orthonormal frame field and the basic forms on M can be expressed as

$$\begin{aligned} e_1 &= -\xi_2 \sin u + \xi_3 \cos u \ , \\ e_2 &= \xi_1 \ , \\ e_3 &= -\xi_2 \cos u - \xi_3 \sin u \ ; \end{aligned} \tag{3.3}$$

$$
\begin{aligned}
\omega_1 &= \frac{1}{R}(du + \tau dv) , \\
\omega_2 &= \left(1 - \frac{\kappa}{R}\cos u\right) dv .
\end{aligned} \tag{3.4}
$$

Making use of (3.1) and (3.2), the connection form ω_{12} and other forms of $\iota(M)$ can be expressed as

$$
\begin{aligned}
\omega_{12} &= \frac{R\kappa \sin u}{R - \kappa \cos u}\omega_2 , \\
\omega_{13} &= R\omega_1 , \\
\omega_{23} &= \frac{R\kappa \cos u}{R - \kappa \cos u}\omega_2 .
\end{aligned} \tag{3.5}
$$

Form (2.2), (3.4) and (3.5) we have $r_2 = \dfrac{-R\kappa \cos u}{R - \kappa \cos u}$. Thus,

$$
4(H^2 - K) = \frac{R^4}{(R - \kappa \cos u)^2} .
$$

From (3.4) the area element dA of $\iota(M)$ is given by

$$
\omega_1 \wedge \omega_2 = \frac{1}{R^2}(R - \kappa \cos u) du \wedge dv . \tag{3.6}
$$

On the other hand, the Gauss-Bonnet theorem implies

$$
\int_{\iota(M)} K dA = 4\pi(1 - g) = 0 . \tag{3.7}
$$

Taking account of these facts, we have

$$
4\int_{\iota(M)} H^2 dA = \int_{\iota(M)} 4(H^2 - K) dA = \int_0^l \left\{\int_{-\pi}^{\pi} \frac{R^2}{R - \kappa \cos u} du\right\} dv , \tag{3.8}
$$

where l denotes the length of the central curve C.

Since (3.4) implies necessarily $|\kappa| < R$ and κ depends only on v, we find

$$
\int_{\iota(M)} H^2 dA = \frac{\pi R^2}{2}\int_0^l \frac{dv}{\sqrt{R^2 - \kappa^2}} . \tag{3.9}
$$

By virtue of the Schwarz's inequality, we have

$$
\int_0^l \frac{dv}{\sqrt{R^2 - \kappa^2}} \geq l^2 \Big/ \int_0^l \sqrt{R^2 - \kappa^2}\, dv . \tag{3.10}
$$

Recalling the inequality, we have

$$(3.11)\qquad \left(\int_0^l \sqrt{R^2 - \kappa^2}\, dv\right)^2 \leq R^2l^2 - \left(\int_0^l \kappa dv\right)^2 .$$

Here by virtue of the generalized Fenchel's theorem according to Milnor [3], we have

$$(3.12)\qquad \int_0^l \kappa dv \geq 2\pi ,$$

where the equality holds if and only if C is a convex curve in a plane.

Combining the inequalities (3.1), (3.10), (3.11) and (3.12), we obtain

$$\int_M H^2 \circ \iota dA \geq \frac{R^2l^2\pi}{2\sqrt{R^2l^2 - 4\pi^2}} ,$$

where the equality holds if and only if the equalities in (3.1), (3.10), (3.11) and (3.12) hold simultaneously. In other words, C is a circle of radius $l/(2\pi)$ and ι is an imbedding. Summing up the above results, we can state as follows:

Theorem 8. *Let M be a two-dimensional, connected, compact and orientable Riemannian manifold of nonzero genus, and ι: $M \to E^3$ be an isometric immersion. Suppose that one of the principal curvatures of M is a positive constant R everywhere, and let l be the length of the central curve of M. Then we have*

$$\int_M H^2 \circ \iota dA \geq \frac{R^2l^2\pi}{2\sqrt{R^2l^2 - 4\pi^2}} ,$$

where the equality holds if and only if ι is an imbedding, and $\iota(M)$ is congruent to a standard torus $T(l/(2\pi), 1/R)$.

Proof of main theorem. Considering the right hand side of the inequality in Theorem 8 as a function of Rl, we can easily see that it attains the minimum $2\pi^2$ for $Rl = 2\sqrt{2}\pi$, and hence our main theorem follows from Theorem 8.

References

[1] W. Fenchel, *Über Krümmung und Windung geschlossener Raumkurven,* Math. Ann. **101** (1929) 238–252.

[2] W. S. Massey, *Surfaces of Gaussian curvature zero in Euclidean* 3-*space,* Tôhoku Math. J. **14** (1962) 73–79.

[3] J. W. Milnor, *On the total curvature of knots,* Ann. of Math. **52** (1950) 248–257.

[4] T. J. Willmore, *Note on embedded surfaces,* An. Şti. Univ. "Al. I. Cuza" Iaşi Şect. I a Mat. **11** (1965) 493–496.

[5] ——, *Mean curvature of immersed surfaces,* An. Şti. Univ. "Al. I. Cuza" Iaşi Şect. I a Mat. **14** (1968) 99–103.

Tokyo Institute of Technology
Tokyo University of Education