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A CLASS OF SCHUBERT VARIETIES

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1. Introduction

Recently, the author proved that in a real, complex or quaternionic Grassmann manifold provided with an invariant metric, the minimum locus is a Schubert variety and the conjugate locus is the union of two Schubert varieties (cf. [7], [8]). The purpose of this paper is to study these Schubert varieties in detail.

Let F be the field R of real numbers, the field C of complex numbers, or the field H of real quaternions, F^{n+m} $(n \geq 1, m \geq 1)$ an $(n+m)$-dimensional left vector-space over F provided with a positive definite hermitian inner product, and $G_n(F^{n+m})$ the Grassmann manifold of n-planes in F^{n+m}. The Schubert varieties which we shall study are defined by

$$V_l = \{\boldsymbol{Z} \in G_n(F^{n+m})\colon \dim(\boldsymbol{Z} \cap \boldsymbol{P}) \geq l\}\,,$$

where $\boldsymbol{P}$ is a fixed p-plane in $F^{n+m}, 0 < p < n+m$, and l is a nonnegative integer. It is easy to see that $V_l = G_n(F^{n+m})$ if $l = \max(0, p-m)$, and V_l is empty if $l > \min(n, p)$.

Let $W_l = V_l \backslash V_{l+1}$ and let k be an integer such that $\max(1, p-m+1) \leq k \leq \min(n, p)$. Then

$$V_k = W_k \cup W_{k+1} \cup \cdots \cup W_{\min(n,p)}\,.$$

Roughly speaking, our main result is:

V_k is the disjoint union of a Grassmann manifold $W_{\min(n,p)}$ (which reduces to a point if $p = n$) and $\min(n,p) - k$ "tensor" bundles $W_l(k \leq l \leq \min(n,p) - 1)$ whose base space is $G_l(F^p) \times G_{n-l}(F^{n+m-p})$, whose standard fiber is the tensor product $(F^{n-l})^* \otimes F^{p-l}$ of an $(n-l)$-dimensional right vector space and a $(p-l)$-dimensional left vector space, and whose group is the tensor product $GL(n-l, F) \otimes GL(p-l, F)$.

In § 2, we describe a covering of $G_n(F^{n+m})$ by coordinate neighbourhoods. In § 3, we prove that each V_l is a Schubert variety and obtain the local equations of V_l in a coordinate neighbourhood in $G_n(F^{n+m})$, which show that V_{l+1} is the singular locus of V_l. In § 4, we obtain a covering of the manifold W_l

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by coordinate neighborhoods. In § 5, we complete our study of V_k by analysing the bundle structure of W_l.

2. Local coordinate systems in a Grassmann manifold

For the moment, we use the symbol $G_n(F^{n+m})$ $(n \geq 1, m \geq 1)$ to denote the set of all n-planes in F^{n+m}. We shall define on it an atlas which will turn it into an analytic manifold. In F^{n+m}, let $\{x_1, \cdots, x_{n+m}\}$ be a fixed system of rectangular coordinates defined by an orthonormal basis $\{\boldsymbol{e}_1, \cdots, \boldsymbol{e}_{n+m}\}$. Denote by $U_{i_1 \cdots i_n}$ the subset of $G_n(F^{n+m})$ consisting of all those n-planes with equations of the form

$$x_{\alpha_\gamma} = \sum_k x_{i_k} z_{i_k \alpha_\gamma}, \tag{2.1}$$

where $z_{i_k \alpha_\gamma}$ are scalar constants, $1 \leq k \leq n, 1 \leq \gamma \leq m$, and $(i_1, \cdots, i_n, \alpha_1, \cdots, \alpha_m)$ is a certain derangement of $(1, \cdots, n + m)$ such that $i_1 < \cdots < i_n$, $\alpha_i < \cdots < \alpha_m$. This determines a local chart $(U_{i_1 \cdots i_n}, Z_{i_1 \cdots i_n})$ in $G_n(F^{n+m})$, whose coordinate neighbourhood is $U_{i_1 \cdots i_n}$ and whose local coordinates are the nm elements of the $n \times m$ matrix $Z_{i_1 \cdots i_n} = [z_{i_k \alpha_\gamma}]$. By means of the coordinates $z_{i_k \alpha_\gamma}$, we identify $U_{i_1 \cdots i_n}$ with a Euclidean nm-space.

The following lemma will be proved:

Lemma 2.1. (a) *The coordinate neighbourhoods $U_{i_1 \cdots i_n}$, for all possible choice of $(i_1, \cdots, i_n)$ from $(1, \cdots, n + m)$ such that $i_1 < \cdots < i_n$, form a covering of $G_n(F^{n+m})$.*

(b) *The two sets of local coordinates for an n-plane belonging to $U_{i_1 \cdots i_n} \cap U_{i'_1 \cdots i'_n}$ are rationally and analytically related.*

Thus, provided with the atlas determined by the local charts $(U_{i_1 \cdots i_n}, Z_{i_1 \cdots i_n})$ whose indices $i_1 < \cdots < i_n$ take on all their possible values, the set $G_n(F^{n+m})$ becomes an analytic manifold of F-dimension nm, the *Grassmann manifold* $G_n(F^{n+m})$. An important special case is the projective space $FP^m = G_1(F^{m+1})$ of F-dimension m. In particular, FP^1 is the "circle".

For the proof of Lemma 2.1(a) and for later use, we first give a definition and prove Lemma 2.2 below.

Let $\boldsymbol{B}$ be an n-plane in F^{n+m}, and $\boldsymbol{B}^\perp$ its orthogonal complement, so that $F^{n+m} = \boldsymbol{B} \oplus \boldsymbol{B}^\perp$ (direct sum). Then the projection $\pi_{\boldsymbol{B}}$: $F^{n+m} \to \boldsymbol{B}$ is the map which sends each element of F^{n+m} into its component in $\boldsymbol{B}$. For another n-plane $\boldsymbol{Z}$ in F^{n+m}, we say that $\boldsymbol{Z}$ *projects onto* $\boldsymbol{B}$ if the restriction $\pi_{\boldsymbol{B}} | \boldsymbol{Z}$: $\boldsymbol{Z} \to \boldsymbol{B}$ is an onto map.

Lemma 2.2. *An n-plane $\boldsymbol{Z}$ in F^{n+m} belongs to $U_{i_1 \cdots i_n}$ iff $\boldsymbol{Z}$ projects onto the n-plane spanned by the vectors $\boldsymbol{e}_{i_1}, \cdots, \boldsymbol{e}_{i_n}$.*

Proof. By definition, $\boldsymbol{Z} \in U_{i_1 \cdots i_n}$ iff the equations of $\boldsymbol{Z}$ can be reduced to the form (2.1). Thus, an n-plane $\boldsymbol{Z} \in U_{i_1 \cdots i_n}$ is the set of vectors

$$\sum_k x_{i_k} \boldsymbol{e}_{i_k} + \sum_\gamma \left(\sum_k x_{i_k} z_{i_k \alpha_\gamma} \right) \boldsymbol{e}_{\alpha_\gamma}, \tag{2.2}$$

where x_{i_k} are scalar parameters. Since the projection of this set of vectors in the n-plane $\boldsymbol{B}$ spanned by $\boldsymbol{e}_{i_1}, \cdots, \boldsymbol{e}_{i_n}$ is the set of vectors $\sum_k x_{i_k}\boldsymbol{e}_{i_k}$, $\boldsymbol{Z}$ projects onto $\boldsymbol{B}$.

To prove the converse, we assume that $\boldsymbol{Z}$ projects onto $\boldsymbol{B}$, and let $\boldsymbol{f}_{i_k}$ be the vectors of $\boldsymbol{Z}$ which project onto the vectors $\boldsymbol{e}_{i_k}$ of $\boldsymbol{B}$. Then we have

$$\boldsymbol{f}_{i_k} - \boldsymbol{e}_{i_k} = \sum_k z_{i_k \alpha_r} \boldsymbol{e}_{\alpha_r} ,$$

where $z_{i_k\alpha_r}$ are nm scalars. Obviously, the n vectors $\boldsymbol{f}_{i_k}$ are linearly independent, and therefore span the n-plane $\boldsymbol{Z}$. Hence $\boldsymbol{Z}$ is the set of vectors (2.2), and consequently, has equations (2.1).

We now prove Lemma 2.1(a). By Lemma 2.2, it suffices to show that, for any given n-plane $\boldsymbol{Z}$ in F^{n+m}, there exist, among the vectors $\boldsymbol{e}_1, \cdots, \boldsymbol{e}_{n+m}$, n vectors $\boldsymbol{e}_{i_1}, \cdots, \boldsymbol{e}_{i_n}$ such that $\boldsymbol{Z}$ projects onto the n-plane spanned by them. Let

$$\tilde{\boldsymbol{f}}_j = \sum_\lambda f_{j\lambda}\boldsymbol{e}_\lambda \qquad (1 \le j \le n, 1 \le \lambda \le n+m)$$

be a set of n linearly independent vectors which span $\boldsymbol{Z}$. Then there exist suitable linear combinations $\boldsymbol{f}_{i_1}, \cdots, \boldsymbol{f}_{i_n}$ of $\tilde{\boldsymbol{f}}_1, \cdots, \tilde{\boldsymbol{f}}_n$ such that

$$\boldsymbol{f}_{i_k} = \boldsymbol{e}_{i_k} + \mathscr{L}_k(\boldsymbol{e}_{\alpha_1}, \cdots, \boldsymbol{e}_{\alpha_m}) \qquad (1 \le k \le n) , \tag{2.3}$$

where each of the $\mathscr{L}_k$ means "some linear combination of", and $(i_1, \cdots, i_n, \alpha_1, \cdots, \alpha_m)$ is a derangement of $(1, \cdots, n+m)$. Now we can see at once from (2.3) that $\boldsymbol{Z}$ projects onto the n-plane spanned by the vectors $\boldsymbol{e}_{i_1}, \cdots \boldsymbol{e}_{i_n}$, as was to be proved.

To prove (b) of Lemma 2.1, let $\boldsymbol{Z}$ be an n-plane belonging to $U_{i_1\cdots i_n} \cap U_{i'_1\cdots i'_n}$. Then $\boldsymbol{Z}$ can be represented by either of the following two sets of equations

$$x_{\alpha_r} = \sum_k x_{i_k} z_{i_k\alpha_r} , \tag{2.4}$$

$$x_{\alpha'_r} = \sum_k x_{i'_k} \tilde{z}_{i'_k\alpha'_r} . \tag{2.5}$$

Let us eliminate the m variables x_{α_r} from these $2m$ equations by substituting (2.4) in (2.5). Then the result is a set of m homogeneous and linear equations in the n independent variables x_{i_k}. Equating the coefficients of x_{i_k} to zero, we obtain a set (*) of nm equations in the local coordinates $z_{i_k\alpha_r}$ and $\tilde{z}_{i'_k\alpha'_r}$, each of the terms in these equations being of the form

$$1, z_{i_k\alpha_r}, \tilde{z}_{i'_k\alpha'_r} , \quad \text{or} \quad z_{i_k\alpha_r}\tilde{z}_{i'_k\alpha'_r} . \tag{2.6}$$

Since (2.4) and (2.5) are two sets of equations representing the same n-plane belonging to $U_{i_1\cdots i_n} \cap U_{i'_1\cdots i'_n}$, (2.5) is uniquely determined when (2.4) is given and vice versa. Therefore, equation $(*)$ must admit, in $U_{i_1\cdots i_n} \cap U_{i'_1\cdots i'_n}$, a unique solution for $\tilde{z}_{i'_k\alpha'_r}$ in terms of $z_{i_k\alpha_r}$ and a unique solution for $z_{i_k\alpha_r}$ in terms of $\tilde{z}_{i'_k\alpha'_r}$; moreover, because of the special forms (2.6) of the terms in $(*)$, the $\tilde{z}_{i'_k\alpha'_r}$ and the $z_{i_k\alpha_r}$ are rational functions of each other (see Van der Waerden [6, § 37]). This completes the proof of (b).

The above proof of Lemma 2.1 may seem trivial at first sight, but we have made sure that it is valid not only for the cases $F = R$ and $F = C$ but also for (the non-commutative) case of $F = H$.

3. The Schubert variety V_l and its local equations

We first explain what the Schubert varieties are (cf. [1, Chap. 4], [3, Vol. II, Chap. 14]). Let

$$0 \leq a_1 \leq \cdots \leq a_n \leq m \tag{3.1}$$

be a non-decreasing sequence of integers, and

$$L_{a_1+1} \subset \cdots \subset L_{a_n+n} \subset F^{n+m} \tag{3.2}$$

be a nested sequence of vector subspaces of F^{n+m}, whose dimensions are indicated by their subscripts, and suppose that

$$(a_1, \cdots, a_n) = \{Z \in G_n(F^{n+m})\colon \dim (Z \cap L_{a_j+j}) \geq j \qquad (1 \leq j \leq n)\} .$$

Then $(a_1, \cdots, a_n)$ is a closed sub-variety of $G_n(F^{n+m})$, whose F-dimension is equal to the sum $a_1 + \cdots + a_n$. We call $(a_1, \cdots, a_n)$ a *Schubert variety*, and

$$\dim (Z \cap L_{a_j+j}) \geq j \qquad (1 \leq j \leq n) , \tag{3.3}$$

the *Schubert conditions*.

The Schubert variety $(a_1, \cdots, a_n)$ depends not only on the sequence of integers $a_1, \cdots, a_n$, but also on the choice of the sequence of vector subspaces (3.2). However, with a fixed sequence of integers $a_1, \cdots, a_n$ satisfying (3.1), the Schubert varieties defined by different sequences (3.2) are congruent to one another in F^{n+m}, so that they are also congruent under the induced group of transformations in $G_n(F^{n+m})$.

We now prove

Theorem 3.1. *Let $\{x_1, \cdots, x_{n+m}\}$ be a fixed rectangular coordinate system in F^{n+m} determined by the orthonormal basis $\{e_1, \cdots, e_{n+m})$, p an integer such that $0 < p < n + m$, and P the p-plane spanned by the vectors $e_1, \cdots, e_p$. Let l be any integer such that* $\max(1, p - m + 1) \leq l \leq \min(n, p)$. *Then*

$$V_l = \{Z \in G_n(F^{n+m})\colon \dim (Z \cap P) \geq l\}$$

is the Schubert variety

$$(a_1, \cdots, a_l, a_{l+1}, \cdots, a_n) = (p - l, \cdots, p - l, m, \cdots, m) ,$$

whose F-dimension is equal to $nm - l(m - p + l)$.

Proof. Let us construct the sequence (3.1) by using the following integers:

$$a_1 = \cdots = a_l = p - l , \qquad a_{l+1} = \cdots = a_n = m ,$$

and the sequence (3.2) by using the following vector subspaces of F^{n+m}:

$$\begin{aligned}
\boldsymbol{L}_{a_1+1} &= \mathscr{L}(\boldsymbol{e}_1, \cdots, \boldsymbol{e}_{p-l+1}) , \\
\boldsymbol{L}_{a_2+2} &= \mathscr{L}(\boldsymbol{e}_1, \cdots, \boldsymbol{e}_{p-l+2}) , \\
&\cdots\cdots\cdots\cdots \\
\boldsymbol{L}_{a_l+l} &= \mathscr{L}(\boldsymbol{e}_1, \cdots, \boldsymbol{e}_p) = \boldsymbol{P} , \\
\boldsymbol{L}_{a_{l+1}+l+1} &= \mathscr{L}(\boldsymbol{e}_1, \cdots, \boldsymbol{e}_p, \boldsymbol{e}_{p+1}, \cdots, \boldsymbol{e}_{m+l+1}) , \\
&\cdots\cdots\cdots\cdots \\
\boldsymbol{L}_{a_n+n} &= \mathscr{L}(\boldsymbol{e}_1, \cdots, \boldsymbol{e}_{p+q}) = \boldsymbol{F}^{n+m} ,
\end{aligned}$$

where $\mathscr{L}$ means "the vector subspace spanned by". Now it can easily be verified that in this case the set (3.3) of Schubert conditions is equivalent to the single condition $\dim(\boldsymbol{Z} \cap \boldsymbol{P}) \geq l$. Hence our theorem is proved.

We have seen in § 2 that when a rectangular coordinate system is fixed in F^{n+m}, the Grassman manifold $G_n(F^{n+m})$ is covered by the local coordinate systems $(U_{i_1\cdots i_n}, Z_{i_1\cdots i_n})$, where $i_1 < \cdots < i_n$ run through all the integers $1, \cdots, n + m$. In the following, we shall use the coordinate neighbourhoods $U_{i_1\cdots i_h\alpha_1\cdots\alpha_{n-h}}$, where h is some integer such that $1 \leq h \leq \min(n, p)$, $i_1 < \cdots < i_h$ run through the integers $1, \cdots, p$, and $\alpha_1 < \cdots < \alpha_{n-h}$ run through the integers $p + 1, \cdots, n + m$.

We now prove

Theorem 3.2. *Let l and h be two integers such that*

$$\max(1, p - m + 1) \leq l, h \leq \min(n, p) .$$

Then the Schubert variety V_l defined in Theorem 3.1 *has the following properties*:

(a) *If $h < l$, then $V_l \cap U_{i_1\cdots i_h\alpha_1\cdots\alpha_{n-h}}$ is empty.*

(b) *If $h \geq l$, then the equations of $V_l \cap U_{i_1\cdots i_h\alpha_1\cdots\alpha_{n-h}}$ express the condition for the $h \times (m - p + h)$ matrix*

$$\text{(3.4)} \qquad \begin{bmatrix} z_{i_1\alpha'_1} & \cdots & z_{i_1\alpha'_{m-p+h}} \\ & \cdots\cdots & \\ z_{i_h\alpha'_1} & \cdots & z_{i_h\alpha'_{m-p+h}} \end{bmatrix}$$

to be of rank $\leq h - l$.

(For definition and main properties of the rank of a matrix with elements in a field, not necessarily commutative, see [3, Vol. I, pp. 66–70].)

Proof. Let us use the following index systems:

$$1 \leq a \leq h, \quad 1 \leq a' \leq p - h; \quad 1 \leq b \leq n - h, \quad 1 \leq b' \leq m - p + h;$$
$$i_1 < \cdots < i_h, \qquad i'_1 < \cdots < i'_{p-h} \text{ are complementary in } (1, \cdots, p);$$
$$\alpha_1 < \cdots < \alpha_{n-h}, \qquad \alpha'_1 < \cdots < \alpha'_{m-p+h}$$
$$\text{are complementary in } (p + 1, \cdots, n + m).$$

The equations of an n-plane $Z \in U_{i_1 \cdots i_h \alpha_1 \cdots \alpha_{n-h}}$ are

$$x_{i'_{a'}} = \sum_a x_{i_a} z_{i_a i'_{a'}} + \sum_b x_{\alpha_b} z_{\alpha_b i'_{a'}}, \tag{3.5}$$
$$x_{\alpha'_{b'}} = \sum_a x_{i_a} z_{i_a \alpha'_{b'}} + \sum_b x_{\alpha_b} z_{\alpha_b \alpha'_{b'}}.$$

Therefore, the equations of $Z \cap P$ are these and the following together:

$$x_{\alpha_b} = 0, \qquad x_{\alpha'_{b'}} = 0. \tag{3.6}$$

But on account of (3.6), equations (3.5) split up into the following two sets of equations

$$x_{i'_{a'}} = \sum_a x_{i_a} z_{i_a i'_{a'}}, \tag{3.7}$$

$$0 = \sum_a x_{i_a} z_{i_a \alpha'_{b'}}. \tag{3.8}$$

Since the $m + n - h$ equations (3.6) and (3.7) are independent, $\dim (Z \cap P) \leq h$. On the other hand, it is seen from (3.8) that $\dim (Z \cap P)$ is $h, h - 1, \cdots$ according as the $h \times (m - p + h)$ matrix $[z_{i_a \alpha'_{b'}}]$ is of rank $0, 1, \cdots$. From this it follows that if $h < l$, then $\dim (Z \cap P)$ cannot be $\geq l$, i.e., if $h < l$, then $V_l \cap U_{i_1 \cdots i_h \alpha_1 \cdots \alpha_{n-h}}$ is empty. For $h \geq l$, $\dim (Z \cap P) \geq l$ iff the above matrix is of rank $\leq h - l$. Hence our theorem is proved.

It follows from the definition of V_l that V_{l+1} is a subset of V_l for which $\dim (Z \cap P) \geq l + 1$, and its complement $W_l = V_l \backslash V_{l+1}$ is the set for which $\dim (Z \cap P) = l$. By Theorem 3.2, if $h \geq l + 1$, $V_{l+1} \cap U_{i_1 \cdots i_h \alpha_1 \cdots \alpha_{n-h}}$ is the subset of $V_l \cap U_{i_1 \cdots i_h \alpha_1 \cdots \alpha_{n-h}}$ whose points are such that the matrix (3.4) is of rank $\leq h - l - 1$, and its complement $W_l \cap U_{i_1 \cdots i_h \alpha_1 \cdots \alpha_{n-h}}$ is the set of points for which the matrix (3.4) is of rank $h - l$. For these reasons, we may call the points in W_l the *simple points* of V_l, and those in V_{l+1} the *singular points* of V_l. Of course, this definition is legitimate; moreover, it can easily be verified that, in the case where $F = R$ or $F = C$, V_{l+1} is indeed the set of singular points of V_l as defined in algebraic geometry (cf. [3, Vol. II, Chap. 10, § 14]).

4. The set $W_l = V_l \backslash V_{l+1}$ as a manifold

Let V_l be as defined in Theorem 3.1, $W_l = V_l \backslash V_{l+1}$, and k any integer such that $\max(1, p - m + 1) \leq k \leq \min(n, p)$. Then

$$
\begin{aligned}
W_l &= \{Z \in G_n(F^{n+m}) \colon \dim(Z \cap P) = l\} , \\
V_k &= W_k \cup W_{k+1} \cup \cdots \cup W_{\min(n,p)} ,
\end{aligned}
\tag{4.1}
$$

where the W's are all disjoint, and

$$
W_{\min(n,p)} \begin{cases} = G_n(F^p) & \text{if } p > n , \\ = \{P\} & \text{if } p = n , \\ \approx G_m(F^{n+m-p}) & \text{if } p < n . \end{cases}
$$

In fact, if $p > n$, then $W_n = \{Z\colon Z \subset P\}$. If $p = n$, then $W_n = \{Z\colon Z = P\}$. If $p < n$, then $W_p = \{Z\colon Z \supset P\} = \{Z\colon Z^{\perp} \subset P^{\perp}\}$ which is homeomorphic to $G_m(F^{n+m-p})$. Here $Z^{\perp}, P^{\perp}$ are respectively the orthogonal complements of Z, P in F^{n+m}.

We now prove

Theorem 4.1. (a) *The subset W_l of $G_n(F^{n+m})$ defined by* (4.1) *can be covered by the coordinate neighbourhoods*

$$
U_{i_1 \cdots i_l \alpha_1 \cdots \alpha_{n-l}} ,
$$

where the indices have the ranges $1 \leq i_1 < \cdots < i_l \leq p$, $p + 1 \leq \alpha_1 < \cdots < \alpha_{n-l} \leq n + m$.

(b) *The equations of $W_l \cap V_{i_1 \cdots i_l \alpha_1 \cdots \alpha_{n-l}}$ in the local coordinates*

$$
\begin{bmatrix} [z_{i_a i'_{a'}}] & [z_{i_a \alpha'_{b'}}] \\ [z_{\alpha_b i'_{a'}}] & [z_{\alpha_b \alpha'_{b'}}] \end{bmatrix}
$$

in $U_{i_1 \cdots i_l \alpha_1 \cdots \alpha_{n-l}}$ are

$$
[z_{i_a \alpha'_{b'}}] = 0 ,
$$

where the indices are as in the proof of Theorem 3.2 *only with h replaced by l.*

(c) *W_l is an analytic submanifold of $G_n(F^{n+m})$ of F-dimension*

$$
nm - l(m - p + l) .
$$

Proof. (b) is a special case of Theorem 3.2, and (c) follows immediately from (a) and (b). Therefore, we need only prove (a).

For convenience, we put $q = n + m - p$, and denote by Q the orthogonal complement of P in F^{n+m} and by π_Q the orthogonal projection $F^{n+m} \to Q$. We recall that P is the p-plane spanned by the vectors $e_1, \cdots, e_p$, so that Q is the q-plane spanned by the vectors $e_{p+1}, \cdots, e_{p+q} (= e_{n+m})$.

By Lemma 2.2, to prove (a), it suffices to prove that for any n-plane $Z \in W_l$, there exist, among the vectors $e_1, \cdots, e_{n+m}$, n vectors $e_{i_1}, \cdots, e_{i_l}, e_{\alpha_1}, \cdots, e_{\alpha_{n-l}}$ such that Z projects onto the n-plane spanned by them. Let $Z \in W_l$ and $X = Z \cap P$, and let Y be the orthogonal complement of X in Z. Since X is an l-plane in P, it follows from Lemmas 2.1(a) and 2.2 that there exist among the vectors $e_1, \cdots, e_p$, l vectors $e_{i_1}, \cdots, e_{i_l}$ such that X projects onto the l-plane spanned by them. Let $f_{i_1}, \cdots, f_{i_l}$ be the vectors in X, which project on $e_{i_1}, \cdots, e_{i_l}$, respectively. Then

$$(4.2)\qquad \begin{aligned} f_{i_1} &= e_{i_1} + \mathscr{L}_1(e_{i'_1}, \cdots, e_{i'_{p-l}}) , \\ &\cdots\cdots\cdots \\ f_{i_l} &= e_{i_l} + \mathscr{L}_l(e_{i'_1}, \cdots, e_{i'_{p-l}}) . \end{aligned}$$

Here and in what follows, $\mathscr{L}, \mathscr{L}', \mathscr{L}''$ each mean "a linear combination of".

Let

$$(4.3)\qquad \begin{aligned} \tilde{f}_1 &= \mathscr{L}'_1(e_{p+1}, \cdots, e_{p+q}) + \mathscr{L}''_1(e_1, \cdots, e_p) , \\ &\cdots\cdots\cdots\cdots \\ \tilde{f}_{n-l} &= \mathscr{L}'_{n-l}(e_{p+1}, \cdots, e_{p+q}) + \mathscr{L}''_{n-l}(e_1, \cdots, e_p) \end{aligned}$$

be any set of $n - l$ vectors which span the $(n - l)$-plane Y. Since $Z \cap P$ is the kernel of the projection $\pi_Q | Z$ and $\dim (Z \cap P) = l$, $\pi_Q Z$ is an $(n - l)$-plane in Q. But it is seen from (4.2) and (4.3) that this $(n - l)$ plane is spanned by the $n - l$ vectors

$$\mathscr{L}'_1(e_{p+1}, \cdots, e_{p+q}), \cdots, \mathscr{L}'_{n-l}(e_{p+1}, \cdots, e_{p+q}) ,$$

which must therefore be independent. Hence, it follows from (4.3) that there exist suitable combinations $f_{\alpha_1}, \cdots, f_{\alpha_{n-l}}$ of $\tilde{f}_1, \cdots, \tilde{f}_{n-l}$ such that

$$(4.4)\qquad \begin{aligned} f_{\alpha_1} &= e_{\alpha_1} + \mathscr{L}_{l+1}(e_1, \cdots, e_p, e_{\alpha'_1}, \cdots, e_{\alpha'_{q-n+l}}) , \\ &\cdots\cdots\cdots\cdots \\ f_{\alpha_{n-l}} &= e_{\alpha_{n-l}} + \mathscr{L}_n(e_1, \cdots, e_p, e_{\alpha'_1}, \cdots, e_{\alpha'_{q-n+l}}) . \end{aligned}$$

We have thus constructed, in (4.2) and (4.4), a set of n vectors $f_{i_1}, \cdots, f_{i_l}, f_{\alpha_1}, \cdots, f_{\alpha_{n-l}}$, which span the n-plane Z. It is easy to see from the expressions of these vectors that Z projects onto the n-plane spanned by $e_{i_1}, \cdots, e_{i_l}, e_{\alpha_1}, \cdots, e_{\alpha_{n-l}}$. Therefore, by Lemma 1.2 we see that $Z \in U_{i_1 \cdots i_l \alpha_1 \cdots \alpha_{n-l}}$, and part (a) of our theorem is proved.

5. The manifold $W_l = V_l \backslash V_{l+1}$ as a fiber bundle

We now prove the following main

Theorem 5.1. *Let P be any fixed p-plane in F^{n+m} $(0 < p < n + m)$, and*

l any integer such that $\max(1, p - m + 1) \leq l \leq \min(n, p) - 1$. *Then the* $[nm - l(m - p + l)]$*-dimensional submanifold*

$$W_l = \{Z \in G_n(F^{n+m}) : \dim(Z \cap P) = l\}$$

of $G_n(F^{n+m})$ *is a "tensor" bundle whose base space is the product manifold* $G_l(F^p) \times G_{n-l}(F^{n+m-p})$, *whose standard fiber is the tensor product* $(F^{n-l})^* \otimes F^{p-l}$ *of an* $(n-l)$*-dimensional right vector space and a* $(p-l)$*-dimensional left vector space, and whose group is the "tensor" product* $GL(n-l, F) \otimes GL(p-l, F)$ *with the two subgroups acting on* $(F^{n-l})^*$ *and* F^{p-l}, *respectively.*

Let (x, y) *be any point in* $G_l(F^p) \times G_{n-l}(F^{n+m-p})$. *Then the fiber of* W_l *over* (x, y) *is the set of n-planes* Z *in* F^{n+m} *such that* $Z \cap P$ *is the fixed l-plane* x *in* P, *and* $\pi_Q Z$, i.e., *the projection of* Z *in* Q, *is the fixed* $(n-l)$*-plane* y *in* Q.

Let $U(n+m, F)$ *be the group of motions in* F^{n+m} *regarded as a group of transformations in* $G_n(F^{n+m})$. *Then the subgroup* $U(p, F) \times U(q, F)$ *of* $U(n+m, F)$, *which leaves* P *invariant, leaves* W_l *invariant. It does not act on* W_l *transitively, but induces a transitive group of transformations in the set of fibers of* W_l.

Proof. We first prove some preliminary results. For brevity, we shall denote $U_{i_1 \cdots i_l \alpha_1 \cdots \alpha_{n-l}}$ by $U_{(i\alpha)}$. Let us first consider the transformations of local coordinates in $G_n(F^{n+m})$. Any n-plane Z belonging to

$$U_{(i\alpha)} \cap U_{(j\beta)} \subset G_n(F^{n+m})$$

has the following two equivalent sets of equations:

$$\begin{aligned} x_{i'_{a'}} &= \sum_a x_{i_a} z_{i_a i'_{a'}} + \sum_b x_{\alpha_b} z_{\alpha_b i'_{a'}} , \\ x_{\alpha'_{b'}} &= \sum_a x_{i_a} z_{i_a \alpha'_{b'}} + \sum_b x_{\alpha_b} z_{\alpha_b \alpha'_{b'}} ; \end{aligned} \tag{5.1}$$

$$\begin{aligned} x_{j'_{a'}} &= \sum_a x_{j_a} \tilde{z}_{j_a j'_{a'}} + \sum_b x_{\beta_b} \tilde{z}_{\beta_b j'_{a'}} , \\ x_{\beta'_{b'}} &= \sum_a x_{j_a} \tilde{z}_{j_a \beta'_{b'}} + \sum_b x_{\beta_b} \tilde{z}_{\beta_b \beta'_{b'}} , \end{aligned} \tag{5.2}$$

where the z's and $\tilde{z}$'s are the two sets of coordinates of Z and

$$1 \leq a \leq l , \qquad 1 \leq a' \leq p - l , \qquad 1 \leq b \leq n - l ,$$
$$1 \leq b' \leq p - n + l (= m - p + l) ;$$

$i_1 < \cdots < i_l$, $\quad i'_1 < \cdots < i'_{p-l}$ are complementary in $(1, \cdots, p)$;

$\alpha_1 < \cdots < \alpha_{n-l}$, $\quad \alpha'_1 < \cdots < \alpha'_{q-n+l}$

are complementary in $(p + 1, \cdots, p + q (= n + m))$.

The transformation $(\tilde{z}) = (z)g_{(j\beta)(i\alpha)}$ between the coordinates z's and $\tilde{z}$'s are obtained by eliminating $x_{i'_a}$ and $x_{\alpha'_b}$ from the $2m$ equations (5.1) and (5.2) and then equating to zero the coefficients of x_{i_a} and x_{α_b} in the resulting equations (cf. proof of Lemma 2.1(b)). As is well known, in

$$U_{(i\alpha)} \cap U_{(j\beta)} \cap U_{(k\gamma)} \subset G_n(F^{n+m}) ,$$

the transformations between the three sets of coordinates satisfy the following compatibility condition:

$$g_{(j\beta)(i\alpha)} \circ g_{(k\gamma)(j\beta)} = g_{(k\gamma)(i\alpha)} . \tag{5.3}$$

Let us now consider W_l which is covered by the coordinate neighbourhoods $U_{(i\alpha)} \equiv U_{i_1\cdots i_l\alpha_1\cdots\alpha_{n-l}}$ (cf. Theorem 4.1). Since the equations of $W_l \cap U_{(i\alpha)}$ in $U_{(i\alpha)}$ are $z_{i_a\alpha'_b} = 0$, and those of $W_l \cap U_{(j\beta)}$ in $U_{(j\beta)}$ are $\tilde{z}_{j_a\beta'_b} = 0$, the relations between the two sets of coordinates for the same n-plane in $(W_l \cap U_{(i\alpha)}) \cap (W_l \cap U_{(j\beta)})$ is obtained in a similar way from the following two sets of equations (cf. (5.1) and (5.2)):

$$\begin{aligned} x_{i'_a} &= \sum_a x_{i_a} z_{i_a i'_a} + \sum_b x_{\alpha_b} z_{\alpha_b i'_a} , \\ x_{\alpha'_b} &= \phantom{\sum_a x_{i_a} z_{i_a i'_a} +{}} \sum_b x_{\alpha_b} z_{\alpha_b \alpha'_b} ; \end{aligned} \tag{5.4}$$

$$\begin{aligned} x_{j'_a} &= \sum_a x_{j_a} \tilde{z}_{j_a j'_a} + \sum_b x_{\beta_b} \tilde{z}_{\beta_b j'_a} , \\ x_{\beta'_b} &= \phantom{\sum_a x_{j_a} \tilde{z}_{j_a j'_a} +{}} \sum_b x_{\beta_b} \tilde{z}_{\beta_b \beta'_b} , \end{aligned} \tag{5.5}$$

where the z's and $\tilde{z}$'s are the two sets of coordinates of the n-plane $\boldsymbol{Z}$. In this case, however, there are the following special properties:

(a) The two sets of coordinates $z_{\alpha_b\alpha'_b}$ and $\tilde{z}_{\beta_b\beta'_b}$ transform into each other rationally and analytically in the same way as the two sets of coordinates for an $(n-l)$-plane in

$$U_{\alpha_1\cdots\alpha_{n-l}} \cap U_{\beta_1\cdots\beta_{n-l}} \subset G_{n-l}(F^{n+m-p}) .$$

(b) The two sets of coordinates $z_{i_a i'_a}$ and $\tilde{z}_{j_a j'_a}$ transform into each other rationally and analytically in the same way as the two sets of coordinates for an l-plane in

$$U_{i_1\cdots i_l} \cap U_{j_1\cdots j_l} \subset G_l(F^p) .$$

(c) The remaining relations between the two sets of coordinates for the n-plane $\boldsymbol{Z}$ are reducible to

$$\tilde{z}_{\beta_b j'_a} = \text{sum of terms of the form } f(z_{\alpha_b\alpha'_b}) z_{\alpha_b i'_a} g(z_{i_a i'_a}) , \tag{5.6}$$

where f (resp. g) is some rational and analytic function of the coordinates $z_{\alpha_b\alpha'_b}$

(resp. $z_{i_a i'_{a'}}$). (Consequently, when $z_{i_a i'_{a'}}$ and $z_{\alpha_b \alpha'_{b'}}$ are held fixed, the two sets of coordinates $z_{\alpha_b i'_{a'}}$ and $\tilde{z}_{\beta_b j'_{a'}}$ transform into each other by a homogeneous and linear transformation with two-sided coefficients.)

Thus, the transformation between the two sets of coordinates z's and $\tilde{z}$'s in $(W_l \cap U_{(i\alpha)}) \cap (W_l \cap U_{(j\beta)})$ is split up into three parts. Moreover, on account of the compatibility condition (5.3) for the three sets of coordinates in $U_{(i\alpha)} \cap U_{(j\beta)} \cap U_{(k\gamma)} \subset G_n(F^{n+m})$, each of the transformations of coordinates described in (a), (b) and (c) above is compatible when three sets of coordinates are involved. From these we can already see that W_l has the structure of a vector bundle whose base manifold is $G_l(F^p) \times G_{n-l}(F^{n+m-p})$ and whose fiber is the space of ordered $(p-l)(n-l)$-tuples of F-numbers. Let us now study this structure of W_l more carefully.

We take first the fibers in W_l. For brevity, let us denote $U_{i_1 \cdots i_l}$ and $U_{\alpha_1 \cdots \alpha_{n-l}}$ by $U_{(i)}$ and $U_{(\alpha)}$, respectively. Then the fibre over a point $(x, y) \in U_{(i)} \times U_{(\alpha)}$ of the base manifold $G_l(F^p) \times G_{n-l}(F^{n+m-p})$ consists of those n-planes Z in $W_l \cap U_{(i\alpha)}$ whose coordinates $z_{i_a i'_{a'}}$, $z_{\alpha_b \alpha'_{b'}}$ and $z_{\alpha_b i'_{a'}}$ are such that $z_{i_a i'_{a'}}$ and $z_{\alpha_b \alpha'_{b'}}$ are respectively the coordinates of x in $U_{(i)}$ and y in $U_{(\alpha)}$, whereas $z_{\alpha_b i'_{a'}}$ are arbitrary. To find out what this fiber actually is, consider the n-plane $Z \in W_l \cap U_{(i\alpha)}$ whose equations in F^{n+m} are (cf. (5.4)):

$$(5.7)\qquad \begin{aligned} x_{i'_{a'}} &= \sum_a x_{i_a} z_{i_a i'_{a'}} + \sum_b x_{\alpha_b} z_{\alpha_b i'_{a'}} , \\ x_{\alpha'_{b'}} &= \qquad\qquad\qquad \sum_b x_{\alpha_b} z_{\alpha_b \alpha'_{b'}} . \end{aligned}$$

Since the equations of P are $x_{\alpha_b} = 0, x_{\alpha'_{b'}} = 0$, the equations of the l-plane $Z \cap P$ in P are

$$x_{i_{a'}} = \sum_a x_{i_a} z_{i_a i'_{a'}} .$$

Therefore $Z \cap P$ is the point x of $G_l(F^p)$ in $U_{(i)}$ with coordinates $z_{i_a i'_{a'}}$. On the other hand, since the equations of $Q = P^\perp$ are $x_{i_a} = 0$ and $x_{i_{a'}} = 0$, the equations of the $(n-l)$-plane $\pi_Q Z$ in Q can easily be seen to be

$$x_{\alpha'_{b'}} = \sum_b x_{\alpha_b} z_{\alpha_b \alpha'_{b'}} .$$

Hence $\pi_Q Z$ is the point y of $G_{n-l}(F^{n+m-p})$ in $U_{(\alpha)}$ with coordinates $z_{\alpha_b \alpha'_{b'}}$.

Thus we have found that the fiber of the bundle W_l over the point $(x, y) \in G_l(F^p) \times G_{n-l}(F^{n+m-p})$ consists of those n-planes Z in F^{n+m} such that $Z \cap P$ is the fixed l-plane x and $\pi_Q Z$ is the fixed $(n-l)$-plane y.

Let us now find the standard fiber F_0 and the group G of the fiber bundle W_l. Guided by equations (5.6) and what we have just found out about the fibers in W_l, we take as F_0 the tensor product $(F^{n-l})^* \otimes F^{p-l}$ of an $(n-l)$-dimensional right vector space $(F^{n-l})^*$ over F, and a $(p-l)$-dimensional left vector space F^{p-l} over F. As the group G of the fibre bundle, we take the tensor product $GL(n-l, F) \otimes GL(p-l, F)$ and define its action on F_0 by (cf. [4, Prop. 2.4.1]):

$$z_1 \otimes z_2 \to (g_1 \otimes g_2)(z_1 \otimes z_2) = g_1(z_1) \otimes g_2(z_2) ,$$

where $z_1 \in (F^{n-l})^*, z_2 \in F^{p-l}, g_1 \in GL(n-l, F)$, and $g_2 \in GL(p-l, F)$. More precisely, this means the following. Let $f^*_{\alpha_b}$ be a basis of $(F^{n-l})^*$ and $f_{i'_{a'}}$ a basis of F^{p-l}. Then $f^*_{\alpha_b} \otimes f_{i'_{a'}}$ form a "basis" of $(F^{n-l})^* \otimes F^{p-l}$ such that every element z of F_0 can be expressed uniquely in the form

$$z = \sum_{b,a'} (f^*_{\alpha_b} z_{\alpha_b i'_{a'}} \otimes f_{i'_{a'}}) = \sum_{b,a'} (f^*_{\alpha_b} \otimes z_{\alpha_b i'_{a'}} f_{i'_{a'}}) ,$$

where $z_{\alpha_b i'_{a'}} \in F$ are the components of z. If

$$g_1 \colon f^*_{\alpha_b} \to g_1(f^*_{\alpha_b}) = \sum_{\tilde{b}} f^*_{\alpha_{\tilde{b}}} g^{(1)}_{\alpha_{\tilde{b}} \alpha_b} ,$$

$$g_2 \colon f_{i'_{a'}} \to g_2(f_{i'_{a'}}) = \sum_{\tilde{a}'} g^{(2)}_{i'_{a'}, i'_{\tilde{a}'}} f_{i'_{\tilde{a}'}} ,$$

then we set

$$\begin{aligned} z \to \tilde{z} = (g_1 \otimes g_2)(z) &= \sum_{b,a'} \left(\sum_{\tilde{b}} f^*_{\alpha_{\tilde{b}}} g^{(1)}_{\alpha_{\tilde{b}} \alpha_b} z_{\alpha_b i'_{a'}} \otimes \sum_{\tilde{a}'} g^{(2)}_{i'_{a'}, i'_{\tilde{a}'}} f_{i'_{\tilde{a}'}} \right) \\ &= \sum_{\tilde{b}, \tilde{a}'} \left[f^*_{\alpha_{\tilde{b}}} \sum_{b,a'} (g^{(1)}_{\alpha_{\tilde{b}} \alpha_b} z_{\alpha_b i'_{a'}} g^{(2)}_{i'_{a'}, i'_{\tilde{a}'}}) \otimes f_{i'_{\tilde{a}'}} \right] . \end{aligned}$$

We note that the components of $\tilde{z} = (g_1 \otimes g_2)(z)$ are

$$\tilde{z}_{\alpha_{\tilde{b}} i'_{\tilde{a}'}} = \sum_{b,a'} g^{(1)}_{\alpha_{\tilde{b}} \alpha_b} z_{\alpha_b i'_{a'}} g^{(2)}_{i'_{a'}, i'_{\tilde{a}'}} ,$$

and equations (5.6) are of this from.

We can now describe the structure of W_l as a "tensor" bundle by exhibiting its main ingredients [5, § 2.3]:

(1) The protection $\pi \colon W_l \to G_l(F^p) \times G_{n-l}(F^{n+m-p})$ from W_l to the base space is defined by

$$Z \to (Z \cap P, \pi_Q Z) .$$

(2) The standard fiber F_0 is the tensor product $(F^{n-l})^* \otimes F^{p-l}$ of an $(n-l)$-dimensional right vector space and a $(p-l)$-dimensional left vector space. We assume that a "basis" $f^*_{\alpha_b} \otimes f_{i'_{a'}}$ has been fixed in F_0.

(3) The group G of the bundle is the tensor product

$$GL(n-l, F) \otimes GL(p-l, F)$$

which acts on $F_0 = (F^{n-l})^* \otimes F^{p-l}$ by $(g_1 \otimes g_2)(z_1 \otimes z_2) = g_1(z_1) \otimes g_2(z_2)$.

(4) The base manifold $G_l(F^p) \times G_{n-l}(F^{n+m-p})$ is covered by the family of coordinate neighbourhoods $U_{(i)} \times U_{(\alpha)}$ such that, for each $(i\alpha)$, there is a homeomorphism

$$\phi_{(i\alpha)}: (U_{(i)} \times U_{(\alpha)}) \times F_0 \to \pi^{-1}(U_{(i)} \times U_{(\alpha)}) = W_l \cap U_{(i\alpha)} ,$$

defined as follows. Let $((x, y), z)$ be any element of $(U_{(i)} \times U_{(\alpha)}) \times F_0$, and $z_{i_a i'_{a'}}$, $z_{\alpha_b \alpha'_{b'}}$, and $z_{\alpha_b i'_{a'}}$ the respective coordinates of x, y, z in $U_{(i)}, U_{(\alpha)}$, and F_0. Then $\phi_{(i\alpha)}((x, y), z)$ is the n-plane Z whose equations in F^{n+m} are (5.7).

(5) The homeomorphisms $\phi_{(i\alpha)}$ defined above have the following properties:

(i) $\pi \circ \phi_{(i\alpha)}((x, y), z) = (x, y)$.

(ii) Let

$$\phi_{(i\alpha)(x,y)}: F_0 \to F_0$$

be defined by setting

$$\phi_{(i\alpha)(x,y)}(z) = \phi_{(i\alpha)}((x, y), z) .$$

Then, for any two $(i\alpha)$ and $(j\beta)$ and for each $(x, y) \in (U_{(i)} \times U_{(\alpha)}) \cap (U_{(j)} \times U_{(\beta)})$, the homeomorphism

$$\phi^{-1}_{(j\beta)(x,y)} \circ \phi_{(i\alpha)(x,y)}: F_0 \to F_0$$

is given by (5.6); therefore, it coincides with an element of G, defined above in (3).

(6) Finally, for any $(i\alpha)$ and $(j\beta)$, the map

$$g_{(j\beta)(i\alpha)}: (U_{(i)} \times U_{(\alpha)}) \cap (U_{(j)} \times U_{(\beta)}) \to G$$

defined by

$$g_{(j\beta)(i\alpha)}(x, y) = \phi^{-1}_{(j\beta)(x,y)} \circ \phi_{(i\alpha)(x,y)}$$

is analytic because the linear transformation (5.6) depends on $z_{i_a i'_{a'}}$ and $z_{\alpha_b \alpha'_{b'}}$ rationally and analytically.

This completes our proof that W_l has a fiber bundle structure.

Finally, let $U(n + m, F)$ be the unitary group of transformations leaving invariant the hermitian inner product of F^{n+m}, i.e., the group of motions in F^{n+m}. Then the subgroup $U(p, F) \times U(q, F)$ of $U(n + m, F)$, which leaves invariant the vector subspaces P and Q, leaves invariant the manifold W_l, but it does not act on it transitively; this is because for any two n-planes $Z_1, Z_2 \in W_l$, we have in general $\dim (Z_1 \cap Q) \neq \dim (Z_2 \cap Q)$. However, the group $U(p, F) \times U(q, F)$ carries fibers into fibers and acts transitively on the set of fibers. In fact, if x, x' are any two l-planes in P, and y, y' are any two $(n - l)$-planes in Q, then there always exist some element $h_1 \in U(p, F)$, which carries x onto

x', and some element $h_2 \in U(q, F)$ which carries y onto y'. Thus the element $(h_1, h_2) \in U(p, F) \times U(q, F)$ carries the fiber over (x, y) onto the fiber over (x', y'). This completes the proof of Theorem 5.1.

Up to now, we have excluded the case $l = \max(0, p - m) = l_0$ because $V_{l_0} = \{Z \in G_n(F^{n+m})\colon \dim(Z \cap P) \geq l_0\}$ is $G_n(F^{n+m})$ itself and our results obtained so far do not hold in this case. Now $G_n(F^{n+m})$ is of F-dimension nm, and

$$V_{l_0+1} = \{Z \in G_n(F^{n+m})\colon \dim(Z \cap P) \geq l_0 + 1\}$$

is a Schubert variety of F-dimension $nm - (l_0 + 1)(m - p + l_0 + 1) < nm$. Therefore, $W_{l_0} = G_n(F^{n+m}) \backslash V_{l_0+1}$ is an open submanifold of $G_n(F^{n+m})$, and

$$G_n(F^{n+m}) = W_{l_0} \cup W_{l_0+1} \cup \cdots \cup W_{\min(n,p)} ,$$

where $W_{l_0+1}, \cdots, W_{\min(n,p)-1}$ are "tensor" bundles as described in Theorem 5.1, and $W_{\min(n,p)}$ is a Grassmann manifold or a point as shown at the beginning of § 4. Hence we have

Theorem 5.2. *Corresponding to each integer p such that* $0 < p < n + m$, *there is a decomposition of* $G_n(F^{n+m})$ *into a disjoint union of a sequence of* $\min(n, p) - \max(0, p - m) + 1$ *submanifolds of decreasing dimensions, consisting of an open submanifold, a number of "tensor" bundles, and a Grassmann manifold (which reduces to a point if* $p = n$*).*

There are three cases of special interest.

Case 1. $p = m$ (so that $q = n$). Let us take P to be the m-plane $\mathbf{0}^\perp$ spanned by the vectors $e_{n+1}, \cdots, e_{n+m}$, and let

$$V_l = \{Z \in G_n(F^{n+m})\colon \dim(Z \cap \mathbf{0}^\perp) \geq l\} ,$$
$$W_l = \{Z \in G_n(F^{n+m})\colon \dim(Z \cap \mathbf{0}^\perp) = l\} .$$

Then V_0 is $G_n(F^{n+m})$ itself, and W_0 coincides with the coordinate neighbourhood $U_{1\cdots n}$; both of these are of dimension nm. Moreover, $V_1 = V_0 \backslash W_0$, which is of dimension $nm - 1$, is the boundary of $W_0 = U_{1\cdots n}$. It turns out that V_1 is the cut locus of the point $\mathbf{0} \in G_n(F^{n+m})$ (see [7, Theorem 9(b)]).

Case 2. $p = n$ (so that $q = m$). Let us take P to be the n-plane $\mathbf{0}$ spanned by the vectors $e_1, \cdots, e_n$, and let

$$\tilde{V}_l = \{Z \in G_n(F^{n+m})\colon \dim(Z \cap \mathbf{0}) \geq l\} .$$

It turns out that the conjugate locus of the point $\mathbf{0}$ in a $G_n(R^{n+m})$ is $V_2 \cup \tilde{V}_1$ if $n < m$, is $V_2 \cup \tilde{V}_2$ if $n = m$, and is $V_2 \cup \tilde{V}_{n-m+1}$ if $n > m$, whereas the conjugate locus of the point $\mathbf{0}$ in a $G_n(C^{n+m})$ or a $G_n(H^{n+m})$ is $V_1 \cup \tilde{V}_1$ if $n \leq m$, and is $V_1 \cup \tilde{V}_{n-m+1}$ if $n > m$ (see [8]).

Case 3. $p = n = l + 1$ (then $q = m$). This is the only case in which $\tilde{W}_l$ can be a line bundle. In this case, $\tilde{W}_{n-1}$ is a line-bundle whose base manifold is the product of the two projective spaces FP^n and FP^m. Let us choose $\boldsymbol{P}$ as the n-plane $\mathbf{0}$ as in Case 2. Then $\tilde{V}_n$ consists of the single point $\mathbf{0}$. Hence, $\tilde{V}_{n-1}$ is the union of the line bundle $\tilde{W}_{n-1}$ and a point. In particular, for the $G_2(R^4)$, $\tilde{W}_1$ is a line-bundle over a 2-dimensional torus which can be made compact by adding a point.

6. A remark

In their theory of harmonic functions on classical domains (i.e., the four non-special types of irreducible bounded symmetric domains considered in the theory of several complex variables), Hua and Look [2] proved that the boundary of each of these domains is the disjoint union of a finite number of product spaces (i.e., trivial fibre bundles), so that the closure of each of the classical domains is a "chain of slit spaces" with the closures of the product spaces as slits. Our results in this paper would seem to suggest that the Grassman manifolds and certain Schubert varieties are "chains of slit spaces" of a more general type on which a similar theory of harmonic functions might be constructed. However, the referee kindly points out that this is not the case because the more recent works of I.I. Pjatetski-Shapiro, A. Korányi and J.A. Wolf have shown that the more general spaces do not carry nonconstant holomorphic functions, nor do they carry much of a space of harmonic functions.

References

[1] S. S. Chern, *Topics in differential geometry*, Mimeographed notes, Institute for Advanced Study, Princeton, 1951.
[2] L. K. Hua & K. H. Look, *Theory of harmonic functions in classical domains*, Sci. Sinica **8** (1959) 1031–1091.
[3] W. V. D. Hodge & D. Pedoe, *Methods of algebraic geometry*, Vols. I, II, Cambridge University Press, Cambridge, 1952.
[4] S. Kobayashi & K. Nomizu, *Foundations of differential geometry*, Vol. I, Interscience, New York, 1963.
[5] N. E. Steenrod, *The topology of fibre bundles*, Princeton University Press, Princeton, 1951.
[6] Van der Waerden, *Algebra* I, 5th ed., Springer, Berlin, 1960.
[7] Y. C. Wong, *Differential geometry of Grassmann manifolds*, Proc. Nat. Acad. Sci. U.S.A. **57** (1967) 589–594.
[8] ——, *Conjugate loci in Grassmann manifolds*, Bull. Amer. Math. Soc. **74** (1968) 240–245.

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