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# THE RIEMANNIAN STRUCTURE OF ALEXANDROV SPACES

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**Abstract**

Let $X$ be an $n$-dimensional Alexandrov space of curvature bounded from below. We define the notion of singular point in $X$, and prove that the set $S_X$ of singular points in $X$ is of Hausdorff dimension $\leq n-1$ and that $X - S_X$ has a natural $C^0$-Riemannian structure.

## 0. Introduction

Let $\mathcal{M}(m, \kappa, D)$ denote the class of $m$-dimensional compact Riemannian manifolds with sectional curvature $\geq -\kappa^2$ and diameter $\leq D$. Any sequence $\{M_i\}_{i=1,2,\dots}$ of $\mathcal{M}(m, \kappa, D)$ contains a subsequence $\{M_{j(i)}\}_i$ converging to a compact metric space $M_\infty$ with respect to the Hausdorff distance $d_H$ (see [12]). Although we could not expect the limit space $M_\infty$ to be a manifold, it inherits several properties of the manifolds in $\mathcal{M}(m, \kappa, D)$, i.e., $M_\infty$ is an Alexandrov space of curvature $\geq -\kappa^2$, diameter $\leq D$, and of Hausdorff dimension $\leq m$. We say a metric space $X$ is an Alexandrov space (of curvature bounded from below) if $X$ is a connected, complete, and locally compact length space of curvature bounded from below and of finite Hausdorff dimension. (In [4] any such space $X$ is called a FSCBB. The precise definition of Alexandrov space will be given in §1.) Therefore the study of Alexandrov spaces makes clear the structure of the $d_H$-closure of $\mathcal{M}(m, \kappa, D)$, and then it is very useful for the study of manifolds in $\mathcal{M}(m, \kappa, D)$.

Assume that $X$ is an Alexandrov space of curvature $\geq k$. For any triple of points $p, q, r \in X$ we denote by $\tilde{\angle} pqr$ the angle at $\tilde{q}$ of a triangle $\triangle \tilde{p}\tilde{q}\tilde{r}$ in the simply connected space form of constant curvature $k$ such that $|\tilde{p}\tilde{q}| = |pq|$, $|\tilde{q}\tilde{r}| = |qr|$, and $|\tilde{r}\tilde{p}| = |rp|$, where $|xy|$ denotes the distance between $x$ and $y$. A point $p \in X$ is called an $(n, \delta)$-*strained point* if there exist points $p_i \in X$, $i = 1, \cdots, 2n$ such that

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$\tilde{\angle} p_i p p_{i+n} > \pi - \delta$ for any $i = 1, \cdots, n$ and $\tilde{\angle} p_i p p_j > \pi/2 - \delta$ for any $i, j = 1, \cdots, 2n$ with $i \not\equiv j \mod n$. We call any such $\{p_i\}$ an $(n, \delta)$-*strainer at* $p$. Let $X_{n,\delta}$ be the set of $(n, \delta)$-strained points in $X$. Burago, Gromov, and Perelman [4, §6] proved that the Hausdorff dimension of $X$ is equal to the maximal number of $n$ such that $X_{n,\delta} \neq \emptyset$ for all sufficiently small $\delta > 0$, and in particular the Hausdorff dimension is an integer and is called the *dimension of* $X$ *denoted by* $\dim X$. They also proved that $X_{n,\delta}$ is open dense in $X$ and is an $n$-dimensional topological manifold, where $n := \dim X$ and $\delta > 0$ is small enough. In [11, §3 $\frac{2}{3}$], Gromov conjectured that $X - X_{n,\delta}$ is of $n$-dimensional Hausdorff measure zero. We give an affirmative answer to that conjecture. A point $p \in X$ is called a *nonsingular point* if it is an $(n, \delta)$-strained point for any $\delta > 0$, and a *singular point* if it is not a nonsingular point. Clearly, $X - X_{n,\delta}$ is contained in the set $S_X$ of singular points in $X$. One of our main theorems is

**Theorem A.** *Let* $X$ *be an* $n$*-dimensional Alexandrov space. The set* $S_X$ *of singular points in* $X$ *is of Hausdorff dimension* $\leq n - 1$.

Recall that the Gromov convergence theorem [12], etc., states that for any sequence $\{M_i\}$ in $\mathscr{M}(m, \kappa, D)$ such that the sectional curvature $K_{M_i}$ of $M_i$ satisfies $|K_{M_i}| \leq \kappa^2$ and the volume of $M_i$ is greater than a positive constant, then the limit $M_\infty$ of some subsequence of $\{M_i\}$ is a $C^{1+\alpha}$-Riemannian manifold for any $0 < \alpha < 1$ (i.e., the metric tensor is $C^1$ and its differential is $C^\alpha$-continuous in the sense of Hölder's condition), and $M_i$ is diffeomorphic to $M_\infty$ for all sufficiently large $i$. It is, therefore, natural to ask whether $X - S_X$ has some differentiable structure and Riemannian structure or not. In this direction we have the following result. We refer to §1 the precise definition of differentiable and Riemannian structures on a space which is not necessarily locally Euclidean.

**Theorem B.** *Let* $X$ *be an* $n$*-dimensional Alexandrov space. Then there exists a* $C^0$*-Riemannian structure on* $X - S_X \subset X$ *satisfying the following*:

(1) *There exists an* $X_0 \subset X - S_X$ *such that* $X - X_0$ *is of* $n$*-dimensional Hausdorff measure zero and that the Riemannian structure is* $C^{1/2}$*-continuous on* $X_0 \subset X$.

(2) *The metric structure on* $X - S_X$ *induced from the Riemannian structure coincides with the original metric of* $X$.

**Remarks.** (1) It was proved in [4] that $X - X_{k,\delta}$ is of topological dimension $\leq k - 1$ for any $0 \leq k \leq n$ and that any nonboundary $(n-1, \delta)$-strained point is $(n, \delta')$-strained, where $\delta' \to 0$ as $\delta \to 0$. By these two statements, the set of singular and nonboundary points is of topological dimension $\leq n-2$. Recently, in [5], independently of Theorem A, this has been proved for Hausdorff dimension instead of topological dimension.

(2) In general, $X_{n,\delta}$ does not have any $C^0$-Riemannian structure satisfying (2) of Theorem B because $X_{n,\delta}$ may contain singular points (see Example (2) below).

(3) The set $X - S_X$ of nonsingular points in $X$ is not necessarily a Riemannian manifold (see Example (2) below). Nevertheless, $X$ is a $C^0$-Riemannian manifold in the ordinary sense whenever $X$ contains no singular points (see the definition of Riemannian structure in §1).

(4) The boundary of any convex body in a Euclidean $(n+1)$-space $\mathbf{R}^{n+1}$ is an Alexandrov space of curvature $\geq 0$, and its structure was investigated very well for many decades (see for example [7]). A point on the boundary of a convex body is called a *singular point* if its support hyperplane is not unique. Note that the notion of singular point described here is a little wider than that for Alexandrov space. A point is called an *r-singular point* if its support hyperplanes have "$n+1-r$ degrees of freedom". Then the sets $S_r$ of $r$-singular points for $r = 0, \cdots, n-1$ satisfy $S_0 \subset S_1 \subset \cdots \subset S_{n-1}$, the Hausdorff dimension of $S_r \leq r$, and the complement of $S_{n-1}$ has $C^1$ differentiable structure (see [2]). Our result is a generalization of this.

(5) A. D. Alexandrov also proved that the boundary of a convex body in $\mathbf{R}^{n+1}$ has an almost everywhere second differentiable structure in the sense of Stolz. In [11], Gromov suggested that any Alexandrov space will have some second differentiable structure. Our result is a partial answer to that. An affirmative answer to the conjecture will be given in the subsequent paper [15] by developing the approach of this paper.

(6) Let $X$ be the limit space of a Cauchy sequence in $\{M \in \mathscr{M}(m, \kappa, D) \mid |K_M| \leq \kappa^2\}$ and let $n := \dim X$. Then, $X$ and $S_X$ have more rigid structures (see [10]), i.e., $X$ has a stratification $S_n := X \supset S_{n-1} := S_X \supset S_{n-2} \supset \cdots \supset S_0 \supset S_{-1} = \emptyset$ such that $S_l - S_{l-1}$ for any $l = 0, \cdots, n$ has a structure of $C^\infty$-Riemannian manifold whose induced metric is close to the original metric (in the sense of the Lipschitz distance).

(7) In [3], Berestovskii proved that any G-space satisfying a certain axiom is a $C^0$-Riemannian manifold. Later, Plaut [17] extended this to the case of geodesically complete Alexandrov spaces having positive injectivity radius. Their proofs are simpler than ours because of the fact that any geodesic is extendable in their cases.

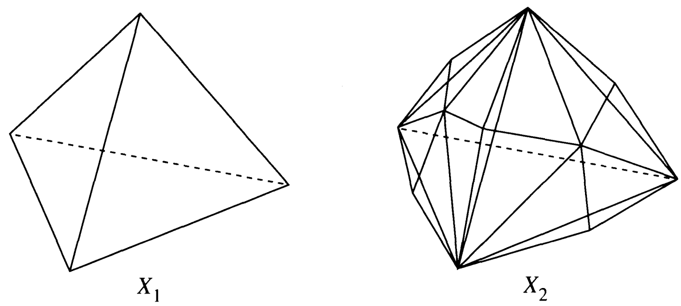

FIGURE 1

**Examples.** (1) Let $X$ be a complete two-dimensional PL-manifold without boundary. For any vertex $p \in X$ denote by $\angle(X, p)$ the sum of all the inner angles at $p$ of faces $F$'s of $X$ such that $p$ is a vertex of $F$. Then, $X$ is of curvature bounded from below (or is an Alexandrov space) if and only if $\angle(X, p) \leq 2\pi$ holds for any vertex $p$ of $X$ (see [7, §17]). In this case, $X$ becomes nonnegatively curved, and a vertex $p$ of $X$ is a singular point if and only if $\angle(X, p) < 2\pi$. Note now that the space $\Sigma_p$ of direction at a vertex $p$ of $X$ is a circle (for the definition of the space of direction see §1.3). It follows that the length of $\Sigma_p$ is equal to $\angle(X, p)$.

(2) Let us construct an example of a two-dimensional Alexandrov space $X$ with the property that the set $S_X$ of singular points of $X$ is dense in $X$. We first define a sequence $\{X_k\}$ of convex polyhedra in $\mathbf{R}^3$ inductively. (The desired $X$ is realized as the Hausdorff limit of $\{X_k\}$.) Let $X_1$ be a regular tetrahedron in $\mathbf{R}^3$, the barycenter of which is the origin $o$. Assume that $X_k$ has been defined. Let us define $X_{k+1}$. Take a monotone decreasing sequence $\{\epsilon_i\}$ of numbers tending to zero in such a way that $0 < \epsilon_i < 1$ for each $i$ and $\epsilon := \prod_{j=1}^{\infty}(1 - \epsilon_j) > 0$. We take the barycentric subdivision of $X_k$ and move all the new vertices outward slightly along rays emanating from $o$ (keeping the original vertices of $X_k$) to obtain the convex tetrahedron $X_{k+1}$ (see Figure 1). We may assume that

$$2\pi - \angle(X_{k+1}, p) \geq (1 - \epsilon_k)(2\pi - \angle(X_k, p))$$

for any vertex $p$ of $X_k$.

Define $X \subset \mathbf{R}^3$ to be the Hausdorff limit of $\{X_k\}$. Then, $X$ is nonnegatively curved. For any $k$ and any vertex $p$ of $X_k$, we obtain

$$\begin{aligned}\lim_{i\to\infty}(2\pi - \angle(X_i, p)) &\geq \prod_{i=0}^{\infty}(1-\epsilon_{k+i})(2\pi - \angle(X_k, p)) \\ &\geq \epsilon\,(2\pi - \angle(X_k, p)) > 0.\end{aligned}$$

The length of the space of direction at $p$ of $X$ is $\lim_{i\to\infty}\angle(X_i, p) < 2\pi$. Thus any vertex of $X_k$ for any $k$ is a singular point of $X$. Since the maximal length of all the edges of $X_k$ tends to zero as $k \to \infty$, the set $S_X$ of singular points is dense in $X$.

(3) Let $X$ be the double of the $n$-dimensional Euclidean unit ball, i.e., the union of two copies $B_1$ and $B_2$ of the $n$-dimensional Euclidean closed unit balls such that $\partial B_1$ and $\partial B_2$ are identified by an isometry, so that $X$ is homeomorphic to the standard sphere $S^n$. Since $X$ contains no singular points, applying Theorem B and recalling Remark (3) we have that $X$ is a $C^0$-Riemannian manifold (and then a $C^1$-differentiable manifold).

The present paper is organized as follows. In §1, we introduce the notion and convention used in this paper and summarize the facts known for Alexandrov space (see [4]). For example we define the angle of two minimal segments emanating from a point in an Alexandrov space $X$, the space of direction $\Sigma_p$ at $p \in X$, the tangent cone $K_p$ at $p$, (weak) $C^r$-differentiable structure and $C^{r-1}$-Riemannian structure on $Y \subset X$ for $r \geq 0$, etc.

In §2, we prove Theorem A. For $\delta > 0$ we define

$$\tilde{S}_{p,\delta} := \{x \in X \mid \tilde{\angle} pxy < \pi - \delta \text{ for any } y \in X - \{x\}\}.$$

Let $B(p, r)$ denote the metric $r$-ball centered at $p \in X$. From Toponogov's convexity, the map $\tilde{S}_{p,\delta} \cap B(p, r) \to \Sigma_p$ which assigns to $x \in \tilde{S}_{p,\delta} \cap B(p, r)$ the element $v_{px} \in \Sigma_p$ corresponding to a minimal segment $px$ is $L$-expanding, i.e.,

$$|v_{px}v_{py}| \geq L|xy| \quad \text{for any } x, y \in \tilde{S}_{p,\delta} \cap B(p, r),$$

where $L$ is a positive constant depending only on $n := \dim X$, $\kappa$, $\delta$, and $r$. Since the Hausdorff dimension of $\Sigma_p$ is equal to $n-1$, the Hausdorff dimension of $\tilde{S}_{p,\delta} \cap B(p, r)$ is $\leq n-1$. Since the Hausdorff measure is completely additive, the Hausdorff dimension of $\tilde{S}_p := \bigcup_{\delta>0}\tilde{S}_{p,\delta}$ is $\leq n-1$. From the splitting theorem for $K_p$ (see [13], [18]), there exists a discrete subset $\{p_i\}_{i=1,2,\cdots} \subset X$ such that for any $x \in S_X$ there is an $i$ with $x \in \tilde{S}_{p_i}$. Hence the Hausdorff dimension of $S_X$ is $\leq n-1$.

In the later sections we prove Theorem B. In §3 we define natural local chart through the distance functions on $X$. For $p \in X$, let

$V_p := \{x \in X \mid px \text{ is unique}\}$, where $px$ denotes a minimal segment joining $p$ and $x$. Let $x \in X - S_X$. Since $K_x$ is linearly isometric to $\mathbf{R}^n$, we identify $K_x$ with $\mathbf{R}^n$. We have that $X - V_x$ is of $n$-dimensional Hausdorff measure zero because of $X - V_x \subset \tilde{S}_x$. Thus, for almost all choices of $p_1, \cdots, p_n \in X$, $x \in V_{p_i}$ for any $i = 1, \cdots, n$ and $\{v_{xp_i}\}_{i=1,\cdots,n}$ is linearly independent in $K_x$. Let us define a map $\varphi : X \to \mathbf{R}^n$ by

$$\varphi(x) := (|p_1 x|, \cdots, |p_n x|)$$

for $x \in X$. We formulate the first variation formula for any triple of points (Theorem 3.5), which enables us to show that $\varphi$ is a bi-Lipschitz homeomorphism on an open neighborhood $U_\varphi$ at $p$. Let $V_\varphi := \bigcap_{i=1}^n V_{p_i} \cap (U_\varphi - S_X)$, and define a map $g_\varphi : V_\varphi \to \mathrm{Mat}(n)$ by

$$g_\varphi(x) := (\cos |v_{xp_i} v_{xp_j}|).$$

Then, $g_\varphi$ is continuous. We introduce the notion of cut locus for Alexandrov space. The cut locus $C_p$ of $p \in X$ is the complement of the set of points $x \in X$ such that $px$ is unique and extends to a minimal segment $py$, $y \neq x$. We prove that the $n$-dimensional Hausdorff measure of $C_p$ is zero and that $g_\varphi$ is $C^{1/2}$-continuous on $V_\varphi - \bigcap_{i=1}^n C_{p_i}$. We may assume that $g_\varphi$ is positive definite. We call any such $\varphi : U_\varphi \to \mathbf{R}^n$ a natural local chart.

In §4, we prove that coordinate transformations of natural local charts are almost everywhere differentiable, and $g_\varphi$ is the Riemannian metric compatible with them, which determines a Riemannian structure on $X - S_X \subset X$ in a weak sense. In §5 we construct new charts which are $C^1$ everywhere by approximating natural local charts. In §6, we prove that $X - S_X$ is locally path connected, and the induced metric on $X - S_X$ from the Riemannian structure coincides with the original metric on $X$. The proof of Theorem B will be completed there.

Finally, in §7, we give an addendum. Concerning [14], we see that if a convergent sequence $\{M_i\}$ of Riemannian manifolds in $\mathscr{M}(m, \kappa, D)$ satisfies that the excess of $M_i$ tends to zero, then the limit space is a $C^{1/2}$-Riemannian manifold.

## 1. Preliminaries

In this section, following [4] we will review known facts, and introduce the notion of differentiable structure and Riemannian structure on topological space which is not necessarily locally Euclidean.

**1.1 Lower curvature bound for length space.** Let $X$ be a complete locally compact length space, i.e., a complete locally compact metric space such that any two points $p, q \in X$ is joined by a minimal segment whose length is equal to the distance $|pq|$ (or $|p, q|$) between $p$ and $q$, where the length $|c|$ of a continuous curve $c : [a, b] \to X$ is defined to be

$$\sup_{a=t_0<\cdots<t_m=b} \sum_{i=0}^{m-1} |c(t_i)\, c(t_{i+1})|.$$

For $p, q \in X$ we denote by $pq$ a minimal segment joining $p$ and $q$. We now fix a number $k \in \mathbf{R}$. For simplicity, we call a complete simply connected surface of constant curvature $k$ a *k-plane.* For any triangle $\triangle pqr$ in $X$, i.e., any triple of points $p, q, r \in X$, we denote by $\tilde{\triangle} pqr$ a triangle $\triangle \tilde{p}\tilde{q}\tilde{r}$ in a $k$-plane such that $|\tilde{p}\tilde{q}| = |pq|$, $|\tilde{q}\tilde{r}| = |qr|$, and $|\tilde{r}\tilde{p}| = |rp|$. Denote by $\tilde{\angle} pqr$ the angle of the triangle $\tilde{\triangle} pqr$ at the vertex corresponding to $q$. Note that $\tilde{\triangle} pqr$ does not necessarily exist in the case of $k > 0$. By definition, $X$ is of *curvature* $\geq k$ if the following axiom holds.

*The Alexandrov Convexity.* For any minimal segments $px$ and $py$ emanating from a common point $p$, the angle $\tilde{\angle} x(s)\, p\, y(t)$ is monotone nonincreasing in $s, t > 0$, where $x(s)$ (resp. $y(t)$) denotes the point on $px$ (resp. $py$) whose distance from $p$ is equal to $s$ (resp. $t$).

This statement is equivalent to the following. Take any triangle $\triangle pqr$ and any points $x \in pq$ and $y \in pr$, where $pq$ and $pr$ are arbitrarily fixed. Then, there exists a triangle $\tilde{\triangle} pqr =: \triangle \tilde{p}\tilde{q}\tilde{r}$ such that if we take the two points $\tilde{x} \in \tilde{p}\tilde{q}$ and $\tilde{y} \in \tilde{p}\tilde{r}$ with $|px| = |\tilde{p}\tilde{x}|$ and $|py| = |\tilde{p}\tilde{y}|$, then we have $|xy| \geq |\tilde{x}\tilde{y}|$.

Let $X$ be of curvature $\geq k$ and fix two minimal segments $px$ and $py$. The Alexandrov convexity implies the existence of the limit of $\tilde{\angle} x(s)\, p\, y(t)$ as $s, t \to 0$, which is called the *angle* $\angle xpy$. We have $\angle xpy \geq \tilde{\angle} xpy$, which is an analogue of Toponogov's comparison theorem for Riemannian manifold and which we call Toponogov's convexity. It is easily verified that any minimal segment in $X$ does not branch.

**1.2 Hausdorff measure and rough volume.** Assume now that $X$ is a metric space and $A$ a subset of $X$. For $\delta > 0$, let $G_\delta$ be any family of Borel subsets of $X$ with diameter $\leq \delta$ and covering $A$. For $a > 0$, the *a-dimensional Hausdorff measure* $V_H{}^a(A)$ *of* $A$ is defined by

$$V_H{}^a(A) := \sup_{\delta>0} \inf_{G_\delta} \sum_{S\in G_\delta} \alpha(a) \left(\frac{\operatorname{diam}(S)}{2}\right)^a ,$$

where $\alpha(a) := \Gamma(1/2)^a/\Gamma(a/2+1)$, and $\Gamma$ is the Euler's function. Note that $a$ is not necessarily an integer. The Hausdorff dimension $\dim_H A$ of $A$ is defined to be $\sup\{a \geq 0 \mid {V_H}^a(A) = +\infty\} = \inf\{a \geq 0 \mid {V_H}^a(A) = 0\}$.

Assume $A \subset X$ to be precompact. A subset $N$ of $A$ is called an *$\epsilon$-discrete net,* $\epsilon > 0$, if the distance between any two different points in $N$ is greater than or equal to $\epsilon$. Then, any $\epsilon$-discrete net of $A$ is of finite number. Denote by $\beta_A(\epsilon)$ the maximal number of $\epsilon$-discrete net of $A$ and define the *$a$-dimensional rough volume $Vr^a(A)$ of $A$* by

$$Vr^a(A) := \limsup_{\epsilon \to 0} \epsilon^a \beta_A(\epsilon).$$

The *rough dimension* $\dim_r A$ *of $A$* is defined in the same manner as the Hausdorff dimension. Obviously we have ${V_H}^a(A) \leq \alpha(a)\, Vr^a(A)$ and $\dim_H A \leq \dim_r A$. The Hausdorff measure is a Borel regular measure. On the other hand, the rough volume is not completely additive and does not measure $X$.

Let $X$ and $Y$ be two metric spaces and $L > 0$ a number. A map $f : X \to Y$ is said to be *$L$-expanding* if $|f(p)f(q)| \geq L|pq|$ holds for any $p, q \in X$. A map $f : X \to Y$ is said to be *$L$-contracting* if $|f(p)f(q)| \leq L|pq|$ holds for any $p, q \in X$ (i.e., $f$ is Lipschitz continuous with Lipschitz constant $L$). If $f : X \to Y$ is $L$-expanding, then ${V_H}^a(f(X)) \geq L^a {V_H}^a(X)$ and $\dim_H f(X) \geq \dim_H X$. If $f : X \to Y$ is $L$-contracting, then ${V_H}^a(f(X)) \leq L^a {V_H}^a(X)$ and $\dim_H f(X) \leq \dim_H X$. The same inequalities hold for the rough volume and the rough dimension.

**1.3 Alexandrov space.** Let $X$ be a complete locally compact length space of curvature $\geq -\kappa^2 > -\infty$ and of Hausdorff dimension $= n < +\infty$. We call any such space $X$ an *Alexandrov space.* Recall (see [4, §6]) that $n$ becomes an integer and that both the Hausdorff dimension and the rough dimension of any metric ball are equal to $n$, and besides the topological dimension coincides with $n$. Thus we call $n$ the *dimension* $\dim X$ *of* $X$. For a point $p \in X$, we denote by $\Sigma'_p$ the set of equivalence classes of minimal segments emanating from $p$, where $px$ is equivalent to $py$ if $\angle xpy = 0$, i.e., one of $px$ and $py$ is contained in the other. The space $\Sigma'_p$ has the distance function naturally induced from the angle between minimal segments from $p$. We call the completion of $\Sigma'_p$ the *space of direction* $\Sigma_p$, and each element of $\Sigma_p$ a *direction at* $p$. It is known that the space of direction $\Sigma_p$ is compact. For any minimal segment $px$ in $X$, the symbol $v_{px}$ denotes the direction at $p$ corresponding to $px$.

With this notation we have $|v_{px}v_{py}| = \angle xpy$ for any $px$ and $py$. The *cone* $K(Y)$ *over a metric space* $Y$ is defined to be the topological cone $Y \times [0, +\infty)/Y \times 0$ equipped with the metric defined by

$$|(x, s), (y, t)| := \sqrt{s^2 + t^2 - 2st\cos\min\{|xy|, \pi\}}$$

for any $(x, s), (y, t) \in K(Y)$. Denote the vertex of $K(Y)$ by $o$. The *tangent cone* $K_p$ *at* $p \in X$ is defined to be the cone $K(\Sigma_p)$ over the space of direction $\Sigma_p$. It is known (see [4, §7]) that the space of direction (resp. the tangent cone) at any point is an Alexandrov space of dimension $n-1$ (resp. $n$) and of curvature $\geq 1$ (resp. $\geq 0$), and that for any fixed $p \in X$, the pointed space $(rX, p)$ converges to $(K_p, o)$ as $r \to +\infty$ in the sense of the pointed Hausdorff distance, where $rX$ denotes $X$ with metric multiplied $r$ times. A point $p \in X$ is called a *singular point* if $K_p$ is not isometric to $\mathbf{R}^n$, or equivalently $\Sigma_p$ is not isometric to the standard $(n-1)$-sphere. Note that the present definition of a singular point is equivalent to the definition given in §0 (to prove this we may use the splitting theorem (see [13], [18])). If a point $p \in X$ is nonsingular, then $K_p$ is identified with $\mathbf{R}^n$ and the scalar multiplication, the inner product $\langle \cdot, \cdot \rangle$, etc. in $K_p$ are assumed to be defined.

**1.4 Radius.** The *radius* $\mathrm{rad}(X)$ *of a length space* $X$ is defined by

$$\mathrm{rad}(X, p) := \sup_{q \in X} |pq|, \qquad \mathrm{rad}(X) := \inf_{p \in X} \mathrm{rad}(X, p).$$

It follows that $\mathrm{diam}(X)/2 \leq \mathrm{rad}(X) \leq \mathrm{diam}(X)$.

Assume now that $X$ is an $n$-dimensional Alexandrov space of curvature $\geq 1$. If a point $p \in X$ satisfies $\mathrm{rad}(X, p) = \pi$, there exists a unique point $q$ such that $|pq| = \pi$. The point $q$ is called the *antipodal point of* $p$. We can easily prove that $\mathrm{rad}(X) = \pi$ if and only if $X$ is isometric to the $n$-dimensional standard sphere. Therefore, a point $p$ in an Alexandrov space is a nonsingular point if and only if $\mathrm{rad}(\Sigma_p) = \pi$.

**1.5 Generalized differentiable structure and Riemannian structure.** Let $X$ be a topological space, and let $Y \subset X$, $n \in \mathbf{N}$, $0 \leq r < 2$. A family $\mathfrak{D} = \{(U_\varphi, V_\varphi, \varphi)\}_{\varphi \in \Phi}$ is called a *weak* $C^r$*-atlas on* $Y \subset X$ if the following hold:

(1) For each $\varphi \in \Phi$, $U_\varphi$ is an open subset of $X$, and $V_\varphi \subset U_\varphi$.

(2) Each $\varphi \in \Phi$ is a homeomorphism from $U_\varphi$ into an open subset of $\mathbf{R}^n$.

(3) $\{V_\varphi\}_{\varphi \in \Phi}$ is a covering of $Y$.

(4) If two maps $\varphi, \psi \in \Phi$ satisfy $V_\varphi \cap V_\psi \neq \emptyset$, then

$$\psi \circ \varphi^{-1} : \varphi(U_\varphi \cap U_\psi) \to \psi(U_\varphi \cap U_\psi)$$

is $C^r$ on $\varphi(V_\varphi \cap V_\psi \cap Y)$.

Note here that, for $0 \leq \alpha < 1$, a map $f$ from an open subset $U$ of $\mathbf{R}^n$ into $\mathbf{R}^n$ is said to be $C^{1+\alpha}$ on a (not necessarily open) subset $V$ of $U$ if the differential of $f$ exists at every point in $V$ and is $C^\alpha$-continuous in the sense of Hölder's condition.

Each $(U_\varphi, V_\varphi, \varphi)$ is called a *local chart.* When $U_\varphi = V_\varphi$ holds for every $\varphi \in \Phi$, we call the weak $C^r$-atlas a *$C^r$-atlas on* $Y \subset X$, and express that $\{(U_\varphi, \varphi)\}_{\varphi\in\Phi}$. When $X = Y$, the existence of a $C^r$-atlas on $Y = X$ implies that $X$ is a $C^r$-manifold in the ordinary sense. Note that our aim is to construct a $C^1$-atlas on $X - S_X \subset X$ and that the concept of weak atlas is needed only for the way of our proof. Two weak $C^r$-atlases on $Y \subset X$ are said to be *equivalent* if the union of these is also a weak $C^r$-atlas on $Y \subset X$. We call each equivalence class of weak $C^r$-atlases a *weak $C^r$-differentiable structure.* A weak $C^r$-differentiable structure is called a *$C^r$-differentiable structure* if it contains at least one $C^r$-atlas. A continuous function $f : X \to \mathbf{R}$ is said to be $C^r$ on $Y$ with respect to a fixed weak $C^r$-differentiable structure on $Y \subset X$ if $f \circ \varphi^{-1}$ is $C^r$ on $\varphi(V_\varphi \cap Y)$ for any local chart $(U_\varphi, V_\varphi, \varphi)$ in the maximal weak $C^r$-atlas. In the same manner, we can define the concepts of $C^r$-map, $C^r$-curve, etc. We can also define the tangent space $T_pX$ for each $p \in Y$ and the concepts of vector field, form, and any other local objects in the standard way.

A family $\mathfrak{g} = \{g_\varphi\}_{\varphi\in\Phi}$ is called a *$C^{r-1}$-Riemannian metric* associated with a weak $C^r$-atlas $\{(U_\varphi, V_\varphi, \varphi)\}_{\varphi\in\Phi}$ on $Y \subset X$ if the following hold:

(1) For each $\varphi \in \Phi$, $g_\varphi$ is a map from $V_\varphi$ to the set of positive symmetric matrices.

(2) For each $\varphi \in \Phi$, $g_\varphi \circ \varphi^{-1}$ is $C^{r-1}$ on $\varphi(V_\varphi \cap Y)$.

(3) For any $x \in V_\varphi \cap V_\psi$, $\varphi, \psi \in \Phi$, we have

$$g_\psi(x) = {}^t d(\varphi \circ \psi^{-1})(\psi(x))\, g_\varphi(x)\, d(\varphi \circ \psi^{-1})(\psi(x)).$$

Let $\mathfrak{D}$ and $\mathfrak{D}'$ be two equivalent $C^r$-atlases on $Y \subset X$. Two $C^{r-1}$-Riemannian metrics $\mathfrak{g}$ and $\mathfrak{g}'$ respectively associated with $\mathfrak{D}$ and $\mathfrak{D}'$ are said to be equivalent if $\mathfrak{g} \cup \mathfrak{g}'$ is a $C^{r-1}$-Riemannian metric associated with $\mathfrak{D} \cup \mathfrak{D}'$. Note that if $\mathfrak{D}$ has a $C^{r-1}$-metric $\mathfrak{g}$, then a unique $C^{r-1}$-metric

$\mathfrak{g}'$ associated with $\mathfrak{D}'$ is naturally induced from $\mathfrak{g}$ such that $\mathfrak{g}$ and $\mathfrak{g}'$ are equivalent. We call the pair of any equivalent class of $C^{r-1}$-Riemannian metric and its associated (resp. weak) $C^r$-differentiable structure on $Y \subset X$ a (resp. *weak*) *$C^{r-1}$-Riemannian structure on* $Y \subset X$. In the case where $X = Y$, the existence of a $C^{r-1}$-Riemannian structure on $Y = X$ implies that $X$ is a $C^{r-1}$-Riemannian manifold.

Now, we fix a (weak) $C^{r-1}$-Riemannian structure on $Y \subset X$. Take any $p \in Y$ and a local chart $(U_\varphi, \varphi) = (U_\varphi, x^1, \cdots, x^n)$ around $p$. (When the structure is weak, we may take $\varphi$ such that $p \in V_\varphi$.) Let $u, v \in T_pX$ be any vectors, and let

$$u = \sum_i u_i \frac{\partial}{\partial x^i} \quad \text{and} \quad v = \sum_i v_i \frac{\partial}{\partial x^i},$$

where $u_i, v_i \in \mathbf{R}$, $i = 1, \cdots, n$. The inner product in the tangent space is defined by

$$\langle u, v \rangle_p := \sum_{i,j} u_i v_j g_{ij}(p),$$

where $(g_{ij}(p)) := g_\varphi(p)$.

**1.6 Landau's symbols.** We sometimes use Landau's symbols $o(\cdot)$ and $O(\cdot)$ (i.e., when $x$ tends to $0$, we have that $o(x)/x$ tends to zero, and $O(x)/x$ is bounded). Define the symbol $\theta(x)$ to be a number which tends to zero as $x \to 0$, (i.e., $\theta(x) = o(x)/x$). Note that $\theta(x) + \theta(x) = \theta(x)$, $c\,\theta(x) = \theta(x)$, etc., hold. The symbols $o_{a,b,\dots}(x)$, $O_{a,b,\dots}(x)$, $\theta_{a,b,\dots}(x)$ mean $o(x)$, $O(x)$, $\theta(x)$ depending on $a, b, \cdots$.

## 2. The mass of singular points

The purpose of this section is to prove Theorem A. Let $X$ be an $n$-dimensional Alexandrov space of curvature $\geq -\kappa^2$, $\kappa > 0$. For $p \in X$ and $\delta > 0$ let

$$\tilde{S}_{p,\delta} := \{\, x \in X \mid \tilde{\angle} pxy < \pi - \delta \text{ for any } y \in X - \{x\} \,\}.$$

It follows that $\tilde{S}_{p,\delta} \subset \tilde{S}_{p,\delta'}$ for any $\delta > \delta' > 0$. Let $\tilde{S}_p := \bigcup_{\delta>0} \tilde{S}_{p,\delta}$. Then we have

**Lemma 2.1.** *There exists a constant $L = L(\kappa, r, \delta) > 0$ depending only on $\kappa, r, \delta > 0$ such that $\tilde{\angle} xpy \geq L|xy|$ for any $x, y \in \tilde{S}_{p,\delta} \cap B(p, r)$.*

*Proof.* Take any $x, y \in \tilde{S}_{p,\delta} \cap B(p, r)$. It follows that

$$\begin{aligned}\frac{|px|+|xy|-|py|}{\kappa} \sinh \frac{|px|+|xy|}{\kappa} &\geq \cosh \frac{|px|+|xy|}{\kappa} - \cosh \frac{|py|}{\kappa} \\ &= (1+\cos \tilde{\angle} pxy) \sinh \frac{|px|}{\kappa} \sinh \frac{|xy|}{\kappa}.\end{aligned}$$

Hence, by $|px|+|xy| < 3r$ and $\tilde{\angle} pxy < \pi - \delta$, we have

$$\frac{|px|+|xy|-|py|}{\kappa} \sinh \frac{3r}{\kappa} \geq (1-\cos\delta) \sinh \frac{|px|}{\kappa} \sinh \frac{|xy|}{\kappa}.$$

On the other hand,

$$\begin{aligned}\frac{(\tilde{\angle} xpy)^2}{2} \sinh \frac{|px|}{\kappa} \sinh \frac{r}{\kappa} &\geq (1-\cos \tilde{\angle} xpy) \sinh \frac{|px|}{\kappa} \sinh \frac{|py|}{\kappa} \\ &= \cosh \frac{|xy|}{\kappa} - \cosh \frac{|py|-|px|}{\kappa} \\ &\geq \frac{1}{2\kappa^2}(|xy|-|py|+|px|)\,|xy|.\end{aligned}$$

Therefore

$$|xy| \cdot \frac{1}{\kappa}\sqrt{\frac{1-\cos\delta}{\sinh(3r/\kappa)\sinh(r/\kappa)}} \leq \tilde{\angle} xpy.$$

**Lemma 2.2.** *The set $\tilde{S}_p$ for any $p \in X$ is of Hausdorff dimension $\leq n-1$.*

*Proof.* Since $\tilde{S}_p = \bigcup_i \tilde{S}_{p,\delta_i} \cap B(p, r_i)$, where $\delta_i \to 0$ and $r_i \to +\infty$, it suffices to prove that $\tilde{S}_{p,\delta} \cap B(p, r)$ for any $\delta, r > 0$ is of Hausdorff dimension $\leq n-1$. In fact, Lemma 2.1 and Toponogov's convexity imply that the map $\tilde{S}_{p,\delta} \cap B(p, r) \ni x \longmapsto v_{px} \in \Sigma_p$ is $L$-expanding, where $L > 0$ is the constant in Lemma 2.1. Since $\Sigma_p$ is $(n-1)$-dimensional, this completes the proof. q.e.d.

For any given $D > 0$ we assign a positive number

$$\begin{aligned}\psi(D) := \max\{\, |\tilde{p}\tilde{r}|/\angle\tilde{p}\tilde{q}\tilde{r} \mid\ & \triangle\tilde{p}\tilde{q}\tilde{r} \text{ is a triangle in the } k\text{-plane} \\ & \text{such that } |\tilde{p}\tilde{q}|, |\tilde{p}\tilde{r}| \leq D \text{ and } |\tilde{p}\tilde{r}| \geq 2\,|\,|\tilde{p}\tilde{q}| - |\tilde{q}\tilde{r}|\,|\,\}.\end{aligned}$$

For the proof of the next lemma we need the following fact, which is an almost immediate consequence of Toponogov's convexity.

**Fact (a)** [4, 9.2]. *Let $C$ be a compact subset of $X$ and $x \in X$ a point. Set $D := \mathrm{diam}(C \cup \{x\})$, $D_1 := \max_{a\in C}|ax| - \min_{a\in C}|ax|$. Then for any $\epsilon > 0$ and any maximal $\epsilon$-discrete net $N$ in $C$, there exists a subset $N'$ of $N$ containing at least $\beta_C(\epsilon)/(2D_1/\epsilon+1)$ points such that $\{v_{xy} \mid y \in N'\}$ is an $\epsilon/\psi(D)$-discrete net of $\Sigma_x$.*

A point $x \in X$ is called a *singular point viewed from a point* $p \in X$ if there exists a $px$ such that $\mathrm{rad}(\Sigma_x, v_{xp}) < \pi$. Denote by $S_p$ the set of singular points viewed from $p$.

**Lemma 2.3.** *Let $C$ be a compact subset of $X$ such that $Vr^n(C) > 0$. Then there exists an $\epsilon = \epsilon(C) > 0$ such that if $\{p_i\}$ is a maximal $\epsilon$-discrete net in $C$, then any singular point in $X$ is a singular point viewed from some $p_i$.*

*Proof.* Let $\epsilon > 0$ and let $\{p_i\}$ be a maximal $\epsilon$-discrete net in $C$. Suppose now that there exists a singular point $x \in X$ which is not a singular point viewed from any $p_i$. Since $\mathrm{rad}(\Sigma_x, v_{xp_i}) = \pi$ for any $i$, there exists the antipodal point $-v_{xp_i}$ of $v_{xp_i}$ (i.e., $|v_{xp_i}, (-v_{xp_i})| = \pi$), so that the bi-ray consisting of the two rays tangent to $v_{xp_i}$ and $-v_{xp_i}$ is a straight line in the tangent cone $K_x$. By applying the splitting theorem (see [13], [18]), the tangent cone $K_x$ is isometric to the product space $K'_x \times \mathbf{R}^k$, where $K'_x$ is an Alexandrov space containing no straight lines. Then, $\Sigma_x$ contains a subspace $S^{k-1}$ isometric to the $(k-1)$-dimensional standard sphere, and every $v_{xp_i}$ is contained in $S^{k-1}$. Note that, since $x$ is a singular point, we have $k < n$. Fact (a) shows that

$$\beta_{S^{k-1}}(\epsilon/\psi(D)) \geq \beta_C(\epsilon)/(2D_1/\epsilon + 1).$$

When $\epsilon$ tends to zero, the left-hand side times $\epsilon^{k-1}$ tends to $\varphi(D)^{k-1} \cdot Vr^{k-1}(S^{k-1})$, and the right-hand side times $\epsilon^{n-1}$ to $Vr^n(C)/2D_1$. Therefore $\epsilon$ must be greater than a positive constant depending only on $C$.

*Proof of Theorem* A. By Toponogov's convexity we have $S_p \subset \tilde{S}_p$ for any $p \in X$. This and Lemmas 2.2 and 2.3 complete the proof.

## 3. Natural local chart

In this section we construct local charts by using the distance functions on $X$. Let $p$ be a point in $X$. A point $x \in X$ is called a *cut point of* $p$ if any minimal segment $py$ emanating from $p$ does not contain $x$ in its interior. We denote by $C_p$ the set of cut points of $p$ and call it the *cut locus* of $p$. Let $W_p := X - C_p$. We have

**Proposition 3.1.** *The cut locus of any point in $X$ is of $n$-dimensional Hausdorff measure zero.*

*Proof.* Fix any $p \in X$. For $\delta > 0$ let

$$W_{p,\delta} := \{x \in X \mid \text{there exists a minimal segment } py \text{ containing } x \text{ such that } |px| \leq (1-\delta)|py|\}.$$

It follows that $W_p = \bigcup_{\delta>0} W_{p,\delta}$. Note that, since a limit of minimal segments is also minimal, $W_{p,\delta}$ is a closed set and $W_p$ is a (topological) Borel set. Define a map $f_{\delta,r} : W_{p,\delta} \cap B(p, (1-\delta)r) \to B(p, r)$ for $\delta, r > 0$ by the following: for any $x \in W_{p,\delta} \cap B(p, (1-\delta)r)$, we take a minimal segment $py$ containing $x$ such that $|px| = (1-\delta)|py|$, so that $f_{\delta,r}(x) := y$. Obviously this is a surjective map. The Alexandrov convexity implies that $f_{\delta,r}$ is $(1+\theta_r(\delta))$-contracting, so that

$$(1+\theta_r(\delta))^n V_H{}^n(W_{p,\delta} \cap B(p, (1-\delta)r)) \geq V_H{}^n(B(p, r)).$$

Letting $\delta \to 0$ we have

$$V_H{}^n(W_p \cap B(p, r)) = V_H{}^n(B(p, r)),$$

which shows that $C_p \cap B(p, r)$ for any $r > 0$ is of $n$-dimensional measure zero. This completes the proof.

**Remark.** Since any singular point viewed from a point $p \in X$ is a cut point of $p$, Proposition 3.1 and Lemma 2.3 show $S_X$ to be of $n$-dimensional Hausdorff measure zero.

For $p, q, x \in X$, define the excess function

$$e_{pq}(x) := |px| + |qx| - |pq|.$$

The following facts are needed for the proof of the next lemma.

**Fact (b)** [14, 4.7]. *Let $p, q, x, y \in X$ be such that $e_{pq}(x) \leq r\delta$, where $r := \min\{|px|, |qx|\}$ and $\delta > 0$ is a small number. Assume that $xp$ and $xy$ are fixed and that $y(s)$ denotes the point on $xy$ whose distance from $x$ is equal to $s$. Then we have*

$$|py(s)| = |px| - s\cos\angle pxy + O(s^2/r + \delta^{1/2}s + r\delta).$$

**Fact (c)** [4, 2.8]. *Let $p_i, q_i, x_i \in X$ tend to $p, q, x \in X$ respectively. Then the following holds:*

(c-1) *If $p_ix_i$ and $q_ix_i$ tend to $px$ and $qx$ respectively, then*

$$\liminf_{i\to+\infty} \angle p_ix_iq_i \geq \angle pxq.$$

(c-2) *Fix a minimal segment $px$. If $x_i \in px$ for any $i$, then*

$$\lim_{i\to+\infty} \tilde{\angle} qxx_i = \min_{qx} \angle pxq.$$

We now prove

**Lemma 3.2.**

(1) *Let $p, q, x, x_i \in X$. If $px_i$ and $qx_i$ tend to $px$ and $qx$ respectively and* $\operatorname{rad}(\Sigma_x, v_{xp}) = \pi$, *then $\angle px_iq$ tends to $\angle pxq$. In particular,*

*if $x$ is a nonsingular point, and $px$ and $qx$ are unique, then the function $y \mapsto \angle pyq$ is continuous at $x$, i.e.,*

$$|\angle pxq - \angle pyq| < \theta_{p,q,x}(|xy|).$$

(2) *Fix any $p, q \in X$ and $x \in W_p \cap W_q$. Then*

$$|\cos\angle pxq - \cos\angle pyq| < O_{p,q,x}(|xy|^{1/2})$$

*for any $y \in X$.*

*Proof.* (1) The first assertion implies the second. It follows from Fact (c-1) that

$$\liminf_{i\to+\infty} \angle px_iq \geq \angle pxq.$$

Take any fixed $\epsilon > 0$. By $\mathrm{rad}(\Sigma_x, v_{xp}) = \pi$, there exists a point $a \in W_x$ such that $\angle axp > \pi - \epsilon$. Obviously, $ax$ is unique. By Fact (c-1),

$$\liminf_{i\to+\infty} \angle ax_ip \geq \angle axp > \pi - \epsilon,$$

and hence

$$\angle ax_ip > \pi - \epsilon$$

for all sufficiently large $i$. Therefore by remarking that the curvatures of $\Sigma_x$ and $\Sigma_{x_i}$ are greater than or equal to $1$, we have

$$|\angle axq + \angle pxq - \pi| < \epsilon \quad \text{and} \quad |\angle ax_iq + \angle px_iq - \pi| < \epsilon$$

for all sufficiently large $i$. Since $\liminf_{i\to+\infty} \angle ax_iq \geq \angle axq$,

$$\limsup_{i\to+\infty} \angle px_iq < \angle pxq + 2\epsilon,$$

which completes the proof of (1).

(2) By $x \in W_p \cap W_q$ we can extend $px$ and $qx$ to $pp'$ and $qq'$ such that $0 < c := |p'x| = |q'x| \leq \min\{|px|, |qx|\}$. Assume that $t := |xy|$ is small enough and let $x' \in X$ be the point on $px$ such that $|xx'| = t^{1/2}$. Applying Fact (b) yields

$$|q'x'| = c + t^{1/2}\cos\angle pxq + O(t).$$

Let $y' \in py$ be such that $|yy'| = t^{1/2}$ (see Figure 2, next page). Since $e_{qq'}(y) \leq 2t$, by Fact (b) we have

$$|q'y'| = c + t^{1/2}\cos\angle pyq + O(t).$$

The above two formulas imply

$$(*) \qquad \cos\angle pxq - \cos\angle pyq = t^{-1/2}(|q'x'| - |q'y'|) + O(t^{1/2}).$$

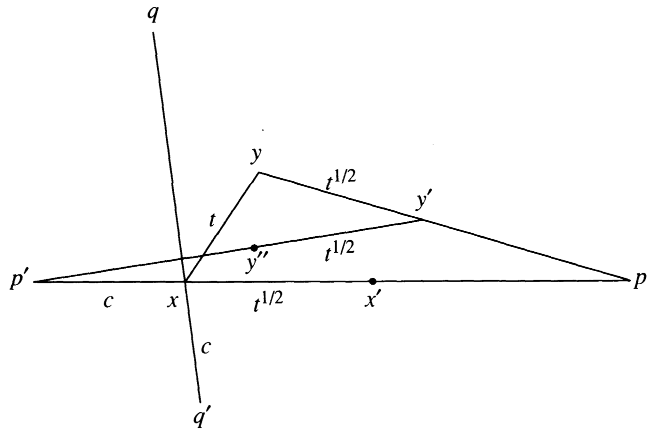


FIGURE 2

Letting $\delta := \pi - \tilde{\angle} pyp'$ and $\delta' := \pi - \tilde{\angle} py'p'$, later we shall prove that $\delta' \le (1+\theta(t))\delta$. Denoting by $y''$ the point on $p'y'$ such that $|y'y''| = t^{1/2}$, we obtain

$$(1+\theta(t))\frac{|yy''|}{t^{1/2}} = \tilde{\angle} yy'y'' \le \angle yy'y'' = \pi - \angle py'p' \le \delta' \le (1+\theta(t))\delta.$$

Let $\triangle \tilde{p}\tilde{y}\tilde{p}' := \tilde{\triangle} pyp'$ and let $\tilde{x}$ and $\tilde{p}''$ be the points on $\tilde{p}\tilde{p}'$ such that $|\tilde{p}'\tilde{x}| = c$ and $|\tilde{p}'\tilde{p}''| = 2c$. Then the Alexandrov convexity shows that $|\tilde{x}\tilde{y}| \le t$, so that $\delta \le \pi - \angle \tilde{p}'\tilde{y}\tilde{p}'' \le O(t)$ and

$$|yy''| \le O(t^{3/2}).$$

Since $(1-\theta(|x'y'|))|x'y'| \le |xy''| \le t + |yy''| \le O(t)$, we have

$$|x'y'| \le O(t),$$

which together with $(\,*\,)$ implies the claim.

Let us last prove $\delta' \le (1+\theta(t))\delta$. Then

$$\begin{aligned}\cosh\frac{|pp'|}{\kappa} &= \cosh\frac{|py|}{\kappa}\cosh\frac{|p'y|}{\kappa} + \sinh\frac{|py|}{\kappa}\sinh\frac{|p'y|}{\kappa}\cos\delta \\ &= \cosh\frac{|py|+|p'y|}{\kappa} - \delta^2\left(\frac{1}{2}+O(\delta^2)\right)\sinh\frac{|py|}{\kappa}\sinh\frac{|p'y|}{\kappa},\end{aligned}$$

and the same formula holds for $y'$ and $\delta'$ instead of $y$ and $\delta$. Since by the triangle inequality $|py| + |p'y| \ge |py'| + |p'y'|$,

$$\begin{aligned}&\delta^2\left(\frac{1}{2}+O(\delta^2)\right)\sinh\frac{|py|}{\kappa}\sinh\frac{|p'y|}{\kappa} \\ &\qquad \ge \delta'^2\left(\frac{1}{2}+O(\delta'^2)\right)\sinh\frac{|py'|}{\kappa}\sinh\frac{|p'y'|}{\kappa},\end{aligned}$$

which completes the proof. q.e.d.

For $p \in X$ let $V_p := \{ q \in X \mid pq \text{ is unique} \}$. Obviously $W_p \subset V_p$, so $X - V_p$ is of $n$-dimensional Hausdorff measure zero in particular. The following proposition and Lemma 2.2 imply that $X - V_p$ is of Hausdorff dimension $\leq n-1$.

**Proposition 3.3.** *For any* $p \in X$ *we have*

$$\tilde{S}_p = S_p \cup (X - V_p).$$

*Proof.* Let us first prove $\tilde{S}_p \subset S_p \cup (X - V_p)$. We may show that $(X - S_p) \cap V_p \subset X - \tilde{S}_p$. Take any $x \in (X - S_p) \cap V_p$. Then, $px$ is unique. For any $\delta > 0$ there exist $y \in X - \{x\}$ and a minimal segment $xy$ such that $\angle pxy \geq \pi - \delta/2$. From Fact (c-2) it follows that there exists $y' \in xy$ close enough to $x$ such that $\tilde{\angle} pxy' \geq \pi - \delta$, that is, $x \in X - \tilde{S}_{p,\delta}$. By the arbitrariness of $\delta$, we have $x \in X - \tilde{S}_p$.

Next we shall prove $\tilde{S}_p \supset S_p \cup (X - V_p)$. Since $S_p \subset \tilde{S}_p$, it suffices to show that $X - V_p \subset \tilde{S}_p$. Suppose that $x \in (X - V_p) - \tilde{S}_p$ exists. Then, at least two different minimal segments $\sigma$ and $\tau$ connect $x$ to $p$. Since $x \notin \tilde{S}_p$, for any $\delta > 0$ there exists $y \in X - \{x\}$ such that $\tilde{\angle} pxy \geq \pi - \delta$. By Toponogov's convexity we have that $\min\{ |v_\sigma v_{xy}|, |v_\tau v_{xy}| \} \geq \pi - \delta$. The existence of the triangle $\tilde{\Delta} v_\sigma v_\tau v_{xy}$ in $S^2$ shows that $\delta \geq |v_\sigma v_\tau|/2$, which contradicts the arbitrariness of $\delta$. q.e.d.

For $p_1, \cdots, p_n \in X$ let $\varphi : X \supset U_\varphi \to \mathbf{R}^n$ be the map defined by $\varphi(x) := {}^t(|p_1 x|, \cdots, |p_n x|)$ for any $x \in U_\varphi$ (we consider $\mathbf{R}^n$ to be the set of column vectors unless stated otherwise). Assume that $U_\varphi$ is open. We call $\varphi$ a *natural local chart* if for any $x \in U_\varphi - S_X$ such that $xp_i$ is unique for every $i = 1, \cdots, n$, the symmetric matrix $g_\varphi(x) := (\langle v_{xp_i}, v_{xp_j} \rangle)_{ij}$ is positive definite. The points $p_1, \cdots, p_n$ are called the base points of $\varphi$.

Let $\varphi : U_\varphi \to \mathbf{R}^n$ be a natural local chart with base points $p_1, \cdots, p_n$. Put $V_\varphi := U_\varphi \cap \bigcap_{i=1}^n V_{p_i} \cap (X - S_X)$ and $W_\varphi := U_\varphi \cap \bigcap_{i=1}^n W_{p_i}$. It follows that $V_\varphi \supset W_\varphi$, so that Lemma 3.2 yields

**Lemma 3.4.** *The map* $g_\varphi : V_\varphi \to \mathrm{Mat}(n)$ *is continuous, and is* $C^{1/2}$-*continuous on* $W_\varphi$.

To investigate natural local chart, we need

**Theorem 3.5** (The first variation formula). *Fix any* $p, x \in X$ *and* $xy$ *for each* $y \in X$. *Then*

$$|py| - |px| = -|xy| \cos \min_{px} \angle pxy + o(|xy|)$$

*for any* $y \in X$.

*Proof.* Take any fixed $\epsilon > 0$. Let $xa_i$ be minimal segments such that $\{v_{xa_i}\}$ is an $\epsilon$-dense net of $\Sigma_x$, i.e., for any $u \in \Sigma_x$ there exists an $i$ such that $|uv_{xa_i}| < \epsilon$. The compactness of $\Sigma_x$ implies that $xa_i$ can be assumed to be of finite number. By Fact (c-2), $\tilde{\angle} pxa_i(t)$ tends to $\min_{px} \angle pxa_i$ as $t \to 0$ for any $i$, where $a_i(t)$ denotes the point on $xa_i$ such that $|xa_i(t)| = t$. Hence, there exists $t_\epsilon > 0$ such that for any $i$ and $t \le t_\epsilon$,

$$(*) \qquad |\tilde{\angle} pxa_i(t) - \min_{px} \angle pxa_i| < \epsilon.$$

Take any $y \in X$ with $|xy| \le t_\epsilon$, and note that $\epsilon = \theta(|xy|)$. Then there exists $i(y)$ such that $\tilde{\angle} yxy' \le \angle a_{i(y)}xy < \epsilon$, where $y' := a_{i(y)}(|xy|)$, so that $|yy'| < \theta(\epsilon)\,|xy|$. Hence we have

$$|\tilde{\angle} pxy - \tilde{\angle} pxy'| < \theta(\epsilon)$$

and also

$$|\min_{px} \angle pxa_{i(y)} - \min_{px} \angle pxy| < \epsilon.$$

Moreover, from $(*)$ it follows that

$$|\tilde{\angle} pxy' - \min_{px} \angle pxa_{i(y)}| < \epsilon.$$

The above three inequalities imply

$$|\min_{px} \angle pxy - \tilde{\angle} pxy| < \theta(\epsilon).$$

Thus applying Fact (b) to the $-\kappa^2$-plane yields

$$\begin{aligned} |py| - |px| &= -|xy| \cos \tilde{\angle} pxy + O(|xy|^2) \\ &= -|xy| \cos \min_{px} \angle pxy + |xy|\,\theta(\epsilon) + O(|xy|^2), \end{aligned}$$

which completes the proof, since $\epsilon = \theta(|xy|)$.

**Remarks.** (1) Theorem 3.5 is similar to Fact (b). However, Fact (b) does not imply Theorem 3.5 in the case where $e_{pq}(x) > 0$.

(2) Theorem 3.5 is closely related to the Rademacher theorem, which states that any Lipschitz function on the Euclidean space is almost everywhere differentiable. Compare the above proof with [8, 3.1.2] and [9, 3.1.6].

For $x \in V_\varphi$ we define an inner product on $\mathbf{R}^n$ by

$$\langle u, v \rangle_{\varphi(x)} := {}^t u\, g_\varphi(x)^{-1} v$$

for any $u, v \in \mathbf{R}^n$, and define the map $I_{\varphi(x)} : K_x \ni u \longmapsto (\langle u, v_{xp_i} \rangle)_i \in \mathbf{R}^n$. We have

**Lemma 3.6.** (1) $I_{\varphi(x)} : K_x \to (\mathbf{R}^n, \langle\cdot, \cdot\rangle_{\varphi(x)})$ *is an isometry for any* $x \in V_\varphi$.

(2) *For any* $x \in V_\varphi$ *and* $y \in U_\varphi$, *letting* $h := (\langle v_{xy}, v_{xp_i}\rangle)_i$, *we have*

$$\varphi(y) - \varphi(x) = -|xy|\, h + o_x(|xy|).$$

(3) *For any* $x, y \in U_\varphi$ *and* $z \in V_\varphi$,

$$|\,\varphi(x) - \varphi(y)\,|_{\varphi(z)} = (1 + \theta_z(\max\{|xz|, |yz|\}))\, |xy|.$$

(4) *For any* $x \in X - S_X$ *there exists a bi-Lipschitz natural local chart* $\varphi : U_\varphi \to \mathbf{R}^n$ *such that* $x \in V_\varphi$.

*Proof.* (1) Any $u \in K_x$ is written as $u = \sum_{i=1}^n \xi_i v_{xp_i}$, $\xi_i \in \mathbf{R}$. Since $\langle u, v_{xp_i}\rangle = \sum_j \xi_j \langle v_{xp_i}, v_{xp_j}\rangle$, letting $\xi := (\xi_i)$, we have

$$I_{\varphi(x)}(u) = g_\varphi(x)\,\xi,$$

and

$$|u|^2 = {}^t\xi\, g_\varphi(x)\,\xi = {}^tI_{\varphi(x)}(u)\, g_\varphi(x)^{-1} I_{\varphi(x)}(u) = |\,I_{\varphi(x)}(u)\,|^2_{\varphi(x)}.$$

(2) The first variation formula implies that

$$|p_iy| - |p_ix| = -|xy|\, h_i + o_x(|xy|),$$

where $(h_i) := h$. This proves (2).

(3) Fix any $z \in V_\varphi$ and let $i = 1, \cdots, n$ be any number. Since $z \in V_{p_i} \cap (X - S_{p_i})$, it follows from Proposition 3.3 that $z \notin \tilde{S}_{p_i}$. Hence, for any $\epsilon > 0$ there exists a point $p_i' \in X$ such that

$$(*) \qquad \tilde{\angle} p_i z p_i' > \pi - \epsilon.$$

We take any $x, y \in U_\varphi$ such that $\max\{|xz|, |yz|\}/|p_i'z| < \epsilon$ for any $i$. Note here that we can assert $\epsilon = \theta_z(\max\{|xz|, |yz|\})$. Take a minimal segment $xy$ and denote the point on $xy$ by $y(t)$ whose distance to $x$ is equal to $t > 0$. For any $i$ and $0 < t < |xy|$, let $\alpha_i := \angle p_i xy$, $\alpha_i(t) := \angle p_i y(t) y$, $\alpha_i' := \angle p_i' xy$, and $\alpha_i'(t) := \angle p_i' y(t) y$. Note that these angles are not uniquely determined because $p_ix$, $p_iy(t)$, etc., are not necessarily unique. Toponogov's convexity shows that $\alpha_i(t) \le \alpha_i + \theta(\epsilon)$ and $\alpha_i'(t) \le \alpha_i' + \theta(\epsilon)$ for any $0 < t \le |xy|$. Moreover, it follows from $(*)$ that $|\,\alpha_i + \alpha_i' - \pi\,| < \theta(\epsilon)$ and $|\,\alpha_i(t) + \alpha_i'(t) - \pi\,| < \theta(\epsilon)$. Therefore

$$|\,\alpha_i(t) - \alpha_i\,| < \theta(\epsilon)$$

for any $0 < t < |xy|$ and $i$. By remarking that this holds for any choice of $\alpha_i$ and $\alpha_i(t)$ (or $p_ix$ and $p_iy(t)$) and by the first variation formula, we obtain

$$\left|\frac{1}{\delta}(|p_iy(t+\delta)| - |p_i(t)|) + \cos\alpha_i\right| < \theta(\epsilon) + \theta_i(\delta)$$

for any $0 \le t \le |xy|$ and any $\delta \in \mathbf{R} - \{0\}$ with $t + \delta \in [0, |xy|]$. Using the compactness of the interval $[0, |xy|]$, we can show that there exists a division $0 = t_0 < \cdots < t_m = |xy|$ such that

$$\left|\frac{|p_iy(t_{j+1})| - |p_iy(t_j)|}{t_{j+1} - t_j} + \cos\alpha_i\right| < \theta(\epsilon)$$

for any $i$ and $j$. By adding up over all $j$,

$$|p_iy| - |p_ix| = -|xy|(\cos\alpha_i + \theta(\epsilon)).$$

Thus

$$\varphi(y) - \varphi(x) = -|xy|(h_{xy} + \theta(\epsilon)),$$

where $h_{xy} := (\cos\alpha_i)_i$. Since $|h_{xy}|_{\varphi(z)} = 1$ holds in the case where $x = z$, Lemma 3.2(1) shows that $|h_{xy}|_{\varphi(z)} = 1 + \theta(|xz|)$, so that

$$|\varphi(y) - \varphi(x)|_{\varphi(z)} = (1 + \theta(\epsilon) + \theta(|xz|))|xy|.$$

Since as stated above $\epsilon = \theta(\max\{|xz|, |yz|\})$, the proof of (3) is completed.

(4) Let $x \in X - S_X$ be any fixed point. Since $(rX, x)$ tends to $(\mathbf{R}^n, o)$ as $r \to +\infty$ in the sense of the pointed Hausdorff distance, and since $X - V_x$ is of measure zero, there exist points $p_1, \cdots, p_n \in V_x$ such that $(\langle v_{p_ix}, v_{p_jx}\rangle)_{ij}$ is positive definite (for example, an $(n, \delta)$-strainer at $x$ satisfies such the condition). Lemma 3.2(1) implies that $(\langle v_{p_iy}, v_{p_jy}\rangle)_{ij}$ is also positive definite for all $y \in V_\varphi$ close enough to $x$. Hence, taking a sufficiently small neighborhood $U_\varphi$ at $x$ we obtain the natural local chart $\varphi : U_\varphi \to \mathbf{R}^n$ with the base points $p_1, \cdots, p_n$. By (3), replacing $U_\varphi$ by a smaller one we conclude (4).

## 4. Natural atlas

Let $\varphi : U_\varphi \to \mathbf{R}^n$ be a bi-Lipschitz natural local chart with base points $p_1, \cdots, p_n$. For $q \in X$, define the function $d_q : X \to \mathbf{R}$ by $d_q(x) := |qx|$ for any $x \in X$. Let $D_\varphi(d_q) : V_\varphi \cap V_q \to \mathbf{R}^n$ (where we consider $\mathbf{R}^n$ to be the set of row vectors) be the map defined by

$$D_\varphi(d_q)(x) := {}^t\xi\, g_\varphi(x)^{-1}$$

for any $x \in V_\varphi \cap V_q$, where $\xi_i := \langle v_{xq}, v_{xp_i} \rangle$ and $\xi := (\xi_i)$. Let $\psi : U_\psi \to \mathbf{R}^n$ be another bi-Lipschitz natural local chart with base points $q_1, \dots, q_n$ such that $V_\varphi \cap V_\psi \neq \emptyset$. Define a map $D_\varphi(\psi) : V_\varphi \cap V_\psi \to \mathrm{Mat}(n)$ by

$$D_\varphi(\psi)(x) := A\, g_\varphi(x)^{-1} = \begin{pmatrix} D_\varphi(d_{q_1})(x) \\ \vdots \\ D_\varphi(d_{q_n})(x) \end{pmatrix}$$

for any $x \in V_\varphi \cap V_\psi$, where $a_{ij} := \langle v_{xq_i}, v_{xp_j} \rangle$ and $A := (a_{ij})$.

**Lemma 4.1.** (1) *The function $d_q \circ \varphi^{-1} : \varphi(U_\varphi) \to \mathbf{R}$ is differentiable on $\varphi(V_\varphi \cap V_q)$, and its differential is equal to $D_\varphi(d_q) \circ \varphi^{-1}$. The map $\psi \circ \varphi^{-1} : \varphi(U_\varphi \cap U_\psi) \to \psi(U_\varphi \cap U_\psi)$ is differentiable on $\varphi(V_\varphi \cap V_\psi)$, and its differential is equal to $D_\varphi(\psi) \circ \varphi^{-1}$.*

(2) *The map $D_\varphi(d_q) \circ \varphi^{-1}$ is continuous on $\varphi(V_\varphi \cap V_q)$ and is $C^{1/2}$-continuous on $\varphi(W_\varphi \cap W_q)$. The map $D_\varphi(\psi) \circ \varphi^{-1}$ is continuous on $\varphi(V_\varphi \cap V_\psi)$ and is $C^{1/2}$-continuous on $\varphi(W_\varphi \cap W_\psi)$.*

(3) *For any $x \in V_\varphi \cap V_\psi$, we have*

$$g_\psi(x) = {}^t D_\psi(\varphi)(x)\, g_\varphi(x)\, D_\psi(\varphi)(x),$$
$$D_\varphi(\psi)(x)^{-1} = D_\psi(\varphi)(x).$$

**Remark.** Since $d_q \circ \varphi^{-1} : \varphi(U_\varphi) \to \mathbf{R}$ and $\psi \circ \varphi^{-1} : \varphi(U_\varphi \cap U_\psi) \to \psi(U_\varphi \cap U_\psi)$ are Lipschitz continuous, Rademacher's theorem implies that these maps are almost everywhere differentiable. However Rademacher's theorem does not tell us where these maps are differentiable. Lemma 4.1 gives detailed information for it.

*Proof.* (1) Take any fixed $\bar{x} \in \varphi(V_\varphi \cap V_q)$ and set $x := \varphi^{-1}(\bar{x})$. Let $\bar{h} \in \mathbf{R}^n$ be any vector such that $|\bar{h}|$ is small enough, where $|\cdot|$ denotes the canonical norm on $\mathbf{R}^n$. We put $y := \varphi^{-1}(\bar{x} + \bar{h})$ and $h := (\langle v_{xy}, v_{xp_i} \rangle)_i$. Then Lemma 3.6(2) implies

$$\bar{h} = -|xy|\, h + o_x(|xy|).$$

By $I_{\varphi(x)}(v_{xq}) = \xi$ and $I_{\varphi(x)}(v_{xy}) = h$, we have

$$\langle v_{xq}, v_{xy} \rangle = {}^t\xi\, g_\varphi(x)^{-1} h = D_\varphi(d_q)(x)\, h.$$

Hence use of the first variation formula leads to

$$\begin{aligned} d_q(y) - d_q(x) &= -|xy| \langle v_{xq}, v_{xy} \rangle + o_x(|xy|) \\ &= -|xy| D_\varphi(d_q)(x)\, h + o_x(|xy|) \\ &= D_\varphi(d_q)(x)\, \bar{h} + o_x(|xy|). \end{aligned}$$

Since $o_x(|h|) = o_x(|h|_{\varphi(x)})$, by Lemma 3.6(3) we obtain $o_x(|xy|) = o_x(|\bar{h}|)$ and therefore

$$d_q \circ \varphi^{-1}(\bar{x} + \bar{h}) - d_q \circ \varphi^{-1}(\bar{x}) = D_\varphi(d_q)(x)\, \bar{h} + o_x(|\bar{h}|),$$

which means the first assertion.

Applying the above formula for $q = q_1, \cdots, q_n$ yields

$$\psi \circ \varphi^{-1}(\bar{x} + \bar{h}) - \psi \circ \varphi^{-1}(\bar{x}) = D_\varphi(\psi)(x)\, \bar{h} + o_x(|\bar{h}|)$$

for any fixed $\bar{x} \in \varphi(V_\varphi \cap V_\psi)$.

(2) is a direct consequence of Lemma 3.2. (Recall that $\varphi$ is a bi-Lipschitz homeomorphism.)

(3) follows from an easy calculation. q.e.d.

Denote by $\Phi_X$ the set of all bi-Lipschitz natural local charts on $X$ and let $\mathfrak{D}_X := \{(U_\varphi, V_\varphi, \varphi)\}_{\varphi \in \Phi_X}$ and $\mathfrak{g}_X := \{g_\varphi\}_{\varphi \in \Phi_X}$. We have

**Theorem 4.2.** (1) *$\mathfrak{D}_X$ is a weak $C^1$-atlas on $X - S_X \subset X$ with the $C^0$-metric $\mathfrak{g}_X$.*

(2) *There exist $X_0 \subset X - S_X$ and $\Phi_{X_0} \subset \Phi_X$ such that $X - X_0$ is of $n$-dimensional Hausdorff measure zero, and that $\{(U_\varphi, \varphi)\}_{\varphi \in \Phi_{X_0}}$ is a $C^{1+1/2}$-atlas on $X_0 \subset X$ with the $C^{1/2}$-metric $\{g_\varphi\}_{\varphi \in \Phi_{X_0}}$.*

*Proof.* (1) follows from Lemmas 3.6(4) and 4.1.

(2) Since $X$ is finite dimensional, any metric ball of $X$ is precompact and hence $X$ is separable. We take a countable dense subset $\{x_i\} \subset X$ and a $\varphi_i \in \Phi_X$ with $x_i \in V_{\varphi_i}$ for each $i$. Let $Z := \bigcup_i (U_{\varphi_i} - W_{\varphi_i})$. Since each $U_{\varphi_i} - W_{\varphi_i}$ is of $n$-dimensional Hausdorff measure zero, so is $Z$. By letting $X_0 := X - S_X - Z$ and $\Phi_{X_0} := \{\varphi_i\}$, the proof is completed.

## 5. Construction of a $C^1$-atlas

The purpose of this section is to construct a $C^1$-atlas equivalent to the weak $C^1$-atlas $\mathfrak{D}_X$.

Let $q \in X - S_X$ and $\delta > 0$. Define a function $d_q^\delta : X \to \mathbf{R}$ by

$$d_q^\delta(x) := \frac{1}{{V_H}^n(B(q, \delta))} \int_{B(q,\delta) \ni y} |xy|$$

for any $x \in X$, where the integral is relative to the Hausdorff measure. For $\varphi \in \Phi_X$, define a function $D^\delta_\varphi(d_q) : V_\varphi \to \mathbf{R}^n$ by

$$D^\delta_\varphi(d_q)(x) := \frac{1}{{V_H}^n(B(q,\delta))} \int_{B(q,\delta)\ni y} D_\varphi(d_y)(x)$$

for any $x \in V_\varphi$. Note that since the map $y \mapsto D_\varphi(d_y)(x)$ is continuous on $V_x$ and since $B(q,\delta) \cap V_x$ has the full measure in $B(q,\delta)$, the above integral has a meaning. For any $\varphi, \psi \in \Phi_X$, let $\psi^\delta := {}^t(d^\delta_{q_1}, \cdots, d^\delta_{q_n}) : U_\psi \to \mathbf{R}^n$ and $D^\delta_\varphi(\psi) := {}^t(D^\delta_\varphi(d_{q_1}), \cdots, D^\delta_\varphi(d_{q_n})) : V_\varphi \to \mathrm{Mat}(n)$. We have

**Lemma 5.1.** (1) *The function $d^\delta_q \circ \varphi^{-1} : \varphi(U_\varphi) \to \mathbf{R}$ is $C^1$ on $\varphi(V_\varphi)$ and its differential is equal to $D^\delta_\varphi(d_q) \circ \varphi^{-1}$. The map $\psi^\delta \circ \varphi^{-1} : \varphi(U_\varphi \cap U_\psi) \to \psi^\delta(U_\varphi \cap U_\psi)$ is $C^1$ on $\varphi(V_\varphi \cap U_\psi)$ and its differential is equal to $D^\delta_\varphi(\psi) \circ \varphi^{-1}$.*

(2) *For any fixed $\varphi, \psi \in \Phi_X$ and $x \in V_\varphi \cap V_\psi$, we have*

$$|D^\delta_\varphi(\psi)(x) - D_\varphi(\psi)(x)| < \theta(\delta).$$

(3) *For any $\varphi \in \Phi_X$ and $x \in V_\varphi$, there exist $\delta(\varphi, x) > 0$ and a neighborhood $U_{\tilde\varphi_x} \subset U_\varphi$ at $x$ such that $\tilde\varphi_x := \varphi^{\delta(\varphi,x)} : U_{\tilde\varphi_x} \to \mathbf{R}^n$ is a bi-Lipschitz into homeomorphism.*

(4) *For any $\varphi, \psi \in \Phi_X$, $x \in V_\varphi$, and $y \in V_\psi$, the maps $\tilde\psi_y \circ \varphi^{-1}$, $\varphi \circ \tilde\psi_y^{-1}$, and $\tilde\psi_y \circ \tilde\varphi_x^{-1}$ are $C^1$ respectively on $\varphi(V_\varphi \cap U_{\tilde\psi_y})$, $\tilde\psi_y(V_\varphi \cap U_{\tilde\psi_y})$, and $\tilde\varphi_x(U_{\tilde\varphi_x} \cap U_{\tilde\psi_y} \cap (X - S_X))$.*

*Proof.* (1) Fix any $x \in V_\varphi$ and let $\bar x := \varphi(x)$. For any $y \in W_x$, since $x \in V_y$, Lemma 4.1(1) implies that

$$(*) \qquad d_y \circ \varphi^{-1}(\bar x + \bar h) - d_y \circ \varphi^{-1}(\bar x) = D_\varphi(d_y)(x)\,\bar h + o(|\bar h|).$$

for any $\bar h \in \mathbf{R}^n$. Since $d_y \circ \varphi^{-1}(\bar x + \bar h)$, $d_y \circ \varphi^{-1}(\bar x)$, and $D_\varphi(d_y)(x)\,\bar h$ are integrable in $y \in B(q,\delta)$, so is the above $o(|\bar h|)$. Thus

$$\lim_{\bar h \to 0} \frac{1}{|\bar h|} \int_{B(q,\delta)\ni y} o(|\bar h|) = \int_{B(q,\delta)\ni y} \lim_{\bar h \to 0} \frac{o(|\bar h|)}{|\bar h|} = 0.$$

Integrating $(*)$ over all $y \in B(q,\delta)$ yields

$$d^\delta_q \circ \varphi^{-1}(\bar x + \bar h) - d^\delta_q \circ \varphi^{-1}(\bar x) = D^\delta_\varphi(d_q)(x)\,\bar h + o(|\bar h|),$$

which means that $d^\delta_q \circ \varphi^{-1}$ is differentiable at $\bar x$. The continuity of the differential $D^\delta_\varphi(d_q) \circ \varphi^{-1}$ is implied by that of $D_\varphi(d_y)$ (see Lemma 4.1(2)).

(2) follows from the continuity of the map $y \mapsto D_\varphi(d_y)(x)$ at $q_i$, where $x \in V_\varphi \cap V_\psi$.

(3) Fix any $\varphi \in \Phi_X$ and $x \in V_\varphi$, and let $\bar{x} := \varphi(x)$. It is easily verified that $\varphi^\delta$ is Lipschitz continuous with Lipschitz constant $\sqrt{n}$. Since $\varphi$ is bi-Lipschitz, it suffices to prove that

$$(*) \qquad |\varphi^\delta \circ \varphi^{-1}(\bar{y}) - \varphi^\delta \circ \varphi^{-1}(\bar{z})| \geq L|\bar{y} - \bar{z}|$$

for any $\bar{y}, \bar{z} \in B(\bar{x}, r)$, where $r, \delta > 0$ are sufficiently small constants, and $L > 0$ is a constant. It follows from (2) and Lemma 4.1(1) that the differential, $D^\delta_\varphi(\varphi)(x)$, of $\varphi^\delta \circ \varphi^{-1}$ at $\bar{x}$ is a regular matrix, which is in fact close to the identity matrix. Thus, the rest of the proof is to modify a standard proof of the inverse function theorem. Define a map $f : \varphi(U_\varphi) \to \mathbf{R}^n$ by

$$f(\bar{y}) := \varphi^\delta \circ \varphi^{-1}(\bar{y}) - D^\delta_\varphi(\varphi)(x)\,\bar{y}$$

for any $\bar{y} \in \varphi(U_\varphi)$. Then, $f$ is Lipschitz continuous on $\varphi(U_\varphi)$, and $C^1$ on $\varphi(V_\varphi)$. We now fix any $r > 0$ and $\bar{y}, \bar{z} \in B(\bar{x}, r)$. Let $P : \mathbf{R}^n \to \mathbf{R}^{n-1}$ be the orthogonal projection from $\mathbf{R}^n$ to the hyperplane (which is identified with $\mathbf{R}^{n-1}$) containing the origin $o$ and normal to the vector $\bar{y} - \bar{z}$. By the coarea formula (cf. [9, 2.10.25]), we have

$$\int_{\mathbf{R}^{n-1} \ni \xi} {V_H}^1(\varphi(U_\varphi - V_\varphi) \cap P^{-1}(\xi)) \leq c\, {V_H}^n(\varphi(U_\varphi - V_\varphi)) = 0,$$

where $c > 0$ is a constant. This implies that ${V_H}^1(\varphi(U_\varphi - V_\varphi) \cap P^{-1}(\xi)) = 0$ for almost all $\xi \in \mathbf{R}^{n-1}$, so that there exist two sequences $\{\bar{y}_i\}$ and $\{\bar{z}_i\}$ of points in $B(\bar{x}, r)$ such that $P(\bar{y}_i) = P(\bar{z}_i) =: \xi_i$ and ${V_H}^1(\varphi(U_\varphi - V_\varphi) \cap P^{-1}(\xi_i)) = 0$ for every $i$ and that $\lim_{i\to\infty} \bar{y}_i = \bar{y}$ and $\lim_{i\to\infty} \bar{z}_i = \bar{z}$. Thus, $f$ is $C^1$ at almost all points on the line $P^{-1}(\xi_i)$. Since $df(\bar{x}) = 0$, $\sup_{\bar{w} \in B(\bar{x}, r)} |df(\bar{w})| < \theta(r)$. By remarking that the line segment $\bar{y}_i\bar{z}_i =: I_i$ is contained in $B(\bar{x}, r) \cap P^{-1}(\xi_i)$, we have

$$|f(\bar{y}_i) - f(\bar{z}_i)| \leq \int_{I_i \ni \bar{w}} |df(\bar{w})| \leq \theta(r)|\bar{y}_i - \bar{z}_i|$$

for any $i$ and therefore, by letting $i \to \infty$,

$$|f(\bar{y}) - f(\bar{z})| \leq \theta(r)|\bar{y} - \bar{z}|.$$

Since

$$|\varphi^\delta \circ \varphi^{-1}(\bar{y}) - \varphi^\delta \circ \varphi^{-1}(\bar{z})| \geq |D^\delta_\varphi(\varphi)(x)(\bar{y} - \bar{z})| - |f(\bar{y}) - f(\bar{z})|,$$

we obtain $(*)$ provided $r, \delta > 0$ are small enough.

(4) The differentiability of $\tilde{\psi}_y \circ \varphi^{-1}$ follows from (1). We shall prove the differentiability of $\varphi \circ \tilde{\psi}_y^{-1}$. Take any fixed $\bar{x} \in \tilde{\psi}_y(V_\varphi \cap U_{\tilde{\psi}_y})$ and any $\bar{h} \in \mathbf{R}^n$ such that $|\bar{h}|$ is small enough. Let

$$\bar{\bar{h}} := \varphi \circ \tilde{\psi}_y^{-1}(\bar{x} + \bar{h}) - \varphi \circ \tilde{\psi}_y^{-1}(\bar{x}) \quad \text{and} \quad \bar{\bar{x}} := \varphi \circ \tilde{\psi}_y^{-1}(\bar{x}).$$

The differentiability of $\tilde{\psi}_y \circ \varphi^{-1}$ at $\bar{\bar{x}}$ implies that

$$\bar{h} = \tilde{\psi}_y \circ \varphi^{-1}(\bar{\bar{x}} + \bar{\bar{h}}) - \tilde{\psi}_y \circ \varphi^{-1}(\bar{\bar{x}}) = D_\varphi^{\delta(\psi, y)}(\psi) \circ \varphi^{-1}(\bar{\bar{x}})\, \bar{\bar{h}} + o(|\bar{\bar{h}}|),$$

so that $o(|\bar{\bar{h}}|) = o(|\bar{h}|)$ and

$$\bar{\bar{h}} = (D_\varphi^{\delta(\psi, y)}(\psi) \circ \tilde{\psi}_y^{-1}(\bar{x}))^{-1}\, \bar{h} + o(|\bar{h}|),$$

which means the differentiability of $\varphi \circ \tilde{\psi}_y^{-1}$ at $\bar{x}$.

For any $z \in U_{\tilde{\varphi}_x} \cap U_{\tilde{\psi}_y} \cap (X - S_X)$ there exists $\rho \in \Phi_X$ such that $z \in V_\rho$. Then, $\tilde{\psi}_y \circ \tilde{\varphi}_x^{-1} = (\tilde{\psi}_y \circ \rho^{-1}) \circ (\tilde{\varphi}_x \circ \rho^{-1})^{-1}$ is $C^1$ at $\tilde{\varphi}_x(z)$. The arbitrariness of $z$ implies that $\tilde{\psi}_y \circ \tilde{\varphi}_x^{-1}$ is $C^1$ on $\varphi(U_{\tilde{\varphi}_x} \cap U_{\tilde{\psi}_y} \cap (X - S_X))$. q.e.d.

From Lemma 5.1 it follows that $\tilde{\mathfrak{D}}_X := \{ (U_{\tilde{\varphi}_x}, \tilde{\varphi}_x) \mid \varphi \in \Phi_X,\ x \in V_\varphi \}$ is a $C^1$-atlas on $X - S_X \subset X$ equivalent to the weak $C^1$-atlas $\mathfrak{D}_X$. For any $p \in X - S_X$, the tangent space $T_pX$ and the inner product $\langle \cdot, \cdot \rangle_p$ on $T_pX$ are induced from the weak $C^0$-Riemannian structure connected with $\mathfrak{D}_X$ and $\mathfrak{g}_X$. For any $\varphi \in \Phi_X$, $x \in V_\varphi$, and $p \in U_{\tilde{\varphi}_x} \cap (X - S_X)$, let us define

$$g_{\tilde{\varphi}_x}(p) := \left( \left\langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right\rangle_p \right)_{ij},$$

where $(x^1, \cdots, x^n) := \tilde{\varphi}_x$. Letting $\tilde{\mathfrak{g}}_X := \{ g_{\tilde{\varphi}_x} \mid \varphi \in \Phi_X,\ x \in V_\varphi \}$, we directly have

**Theorem 5.2.** *$\tilde{\mathfrak{D}}_X$ with $\tilde{\mathfrak{g}}_X$ is a $C^1$-atlas with a $C^0$-metric on $X - S_X \subset X$ equivalent to the weak $C^1$-atlas $\mathfrak{D}_X$ with the metric $\mathfrak{g}_X$.*

Theorems 4.2 and 5.2 imply Theorem B except (2).

**Remark.** The tangent space $T_pX$ at any $p \in X - S_X$ is naturally identified with $K_p$. In fact, the inner product $\langle \cdot, \cdot \rangle_p$ in $T_pX$ is induced from $\langle \cdot, \cdot \rangle_{\varphi(p)}$ introduced in §3, and $K_p$ is identified with $(\mathbf{R}^n, \langle \cdot, \cdot \rangle_{\varphi(p)})$ by the isometry $I_{\varphi(p)}$ (see Lemma 3.6(1)).

## 6. Compatibility between the length metric and the Riemannian structure

The main purpose of this section is to prove Theorem B(2). From now on, assume $X - S_X \subset X$ to be equipped with the Riemannian structure constructed above.

**Proposition 6.1.** *Let $\gamma : [a, b] \to X - S_X$ be any minimal segment. Then, $\gamma|_{(a,b)}$ is a $C^1$-curve.*

*Proof.* Fix any $t_0 \in (a, b)$ and take a $\varphi \in \Phi_X$ such that $\gamma(t_0) \in V_\varphi$. Denote by $p_1, \cdots, p_n$ the base points associated with $\varphi$. Set $\dot\gamma(t) := v_{\gamma(t)\gamma(t+h)}$ for any $t \in (a, b)$ and an $h > 0$. Note that $\dot\gamma(t)$ is independent of $h$. We now fix $\gamma(t)p_i$ such that $|\dot\gamma(t)\, v_{\gamma(t)p_i}|$ is minimal. Then the first variation formula implies that

$$|\gamma(t+h)\, p_i| - |\gamma(t)\, p_i| = -h\langle\dot\gamma(t), v_{\gamma(t)p_i}\rangle + o_{\gamma(t)}(h)$$

for any $t \in (a, b)$ and any $h \in \mathbf{R}$ with $t + h \in [a, b]$, so that

$$\varphi \circ \gamma(t+h) - \varphi \circ \gamma(t) = -h\,(\langle\dot\gamma(t), v_{\gamma(t)p_i}\rangle)_i + o_{\gamma(t)}(h),$$

which means that $\varphi \circ \gamma$ is differentiable and not necessarily $C^1$. Moreover, by $\gamma(t_0) \in V_\varphi$ and Lemma 3.2(1), the function $t \mapsto \langle\dot\gamma(t)\langle v_{\gamma(t)p_i}\rangle$ is continuous at $t = t_0$, and hence $\gamma$ is $C^1$ at $t_0$. q.e.d.

Denote by $L(\cdot)$ the length of a $C^1$-curve induced from the Riemannian structure. We have

**Proposition 6.2.** *Any $C^1$-curve $c : [a, b] \to X - S_X$ satisfies $L(c) = |c|$.*

*Proof.* For any $t \in [a, b]$ we can find $\varphi_t \in \Phi_X$ such that $c(t) \in V_{\varphi_t}$. By remarking that $\varphi_t \circ c$ is differentiable at $t$ and using Lemma 3.6(3), we can prove the claim in the standard way. q.e.d.

In order to prove the next theorem we need

**Lemma 6.3.** *Let $\delta > 0$, $p \in X$, and $F \subset X$. Assume that the metric of $F$ is the restriction of the metric of $X$, and that $||px| - |py|| < \delta\,|xy|$ for any $x, y \in F$. Set $A := B(o, |pF|; K(F/|pF|))$, and the $|pF|$-ball centered at $o$ in the cone over the space $F/|pF|$. We fix a minimal segment $px$ joining $p$ and each $x \in F$, and define a map $f : A \to X$ by the following: for any $(t, x) \in A$ with $t \geq 0$ and $x \in F$ we assign to $f(t, x)$ the point $y \in px$ with $|py| = t$. Then, $f$ is an $L$-expanding map, where $L > 0$ depends only on $k$, $|pF|$, and $\delta$.*

*Proof.* The lemma is a straightforward consequence of the Alexandrov convexity.

**Theorem 6.4.** *For any $p, q \in X$ and $\epsilon > 0$, there exist a point $x \in X - S_X$ and minimal segments $px$, $qx$ entirely contained in $X - S_X$ such* *that* $|px|+|qx| < |pq|+\epsilon$. *In particular,* $X - S_X$ *is locally path-connected.*

*Proof.* Take $p, q \in X$ and fix a minimal segment $pq$ joining them. We take a nonsingular point $z$ close enough to $pq$. By Lemma 3.6(4), there exists a bi-Lipschitz natural chart $\varphi : U \to \mathbf{R}^n$ such that $U$ is a sufficiently small neighborhood of $z$. We may assume that one of the base points of $\varphi$ is taken to be $p$ (cf. the proof of Lemma 3.6(4)). Let $F_t := \{x \in U \mid |px| = |pz| + t\}$ for any $t \in \mathbf{R}$. Since $F_t$ is the inverse image of a piece of a hyperplane of $\mathbf{R}^n$ through $\varphi$, for a small $\delta > 0$ and any $t \in \mathbf{R}$ with $|t| < \delta$, $F_t$ has finite and positive $(n-1)$-dimensional Hausdorff measure. By the coarea formula and Theorem A, we obtain

$$\int_{-\delta}^{\delta} V_H{}^{n-1}(F_t \cap S_X)\, dt \le c\, V_H{}^n(\{x \in U \,\big|\, ||px| - |pz|| < \delta\} \cap S_X) = 0,$$

where $c$ is a constant depending only on $n$. Hence, one can choose a $t \in \mathbf{R}$ with $|t| < \delta$ such that

$$(*) \qquad V_H{}^{n-1}(F \cap S_X) = 0,$$

where $F := F_t$. We fix a minimal segment $px$ joining $p$ and each $x \in F$, and then define the map $f : A \to X$ as in Lemma 6.3, so that $f$ is $L$-expanding. Let $\gamma_x$ for $x \in F$ be the ray in $K(F/|pF|)$ from $o$ of direction $x$. Then applying the coarea formula and Theorem A yields

$$\begin{aligned}\int_{F \ni x} V_H{}^1(\gamma_x \cap f^{-1}(S_X) - B(o, r)) &\le c\, V_H{}^n(f^{-1}(S_X) - B(o, r)) \\ &\le \frac{c}{L^n} V_H{}^n(f(A) \cap S_X - B(p, r)) = 0\end{aligned}$$

for any $r > 0$, where $c$ is a positive constant depending only on $n$ and the Lipschitz constant of the central projection from $A - B(o, r)$ to $F$. Therefore, for any $r > 0$ and almost all $x \in F$,

$$V_H{}^1(px \cap S_X - B(p, r)) = 0.$$

Since $px \cap S_X = \bigcup_i (px \cap S_X - B(p, r_i))$, where $r_i \to 0$, we have

$$V_H{}^1(px \cap S_X) = 0$$

for almost all $x \in F$. Recall (see [4, 7.16]) that the space of direction is continuous along the interior of any minimal segment with respect to the Gromov-Hausdorff distance. Thus, $px - \{x\}$ contains no singular points for almost all $x \in F$.

It follows that $||qx| - |qy|| < \delta' |xy|$ for any $x, y \in F$, where $\delta' > 0$ is a constant tending to zero as $\delta$ and $\operatorname{diam}(U)$ both tend to zero. Hence we can prove that $qx - \{x\}$ contains no singular points for almost all $x \in F$ in the same way as above. Moreover, $(*)$ states that almost all $x \in F$ are nonsingular. Therefore, $px \cup qx$ contains no singular points for almost all $x \in F$. Hence the proof of the first assertion is completed.

Now let us prove the second assertion. Take any fixed $p \in X$ and $r, \epsilon > 0$. Any point $q \in B(p, r) - S_X$ and $p$ are joined by a curve $c_q \subset X - S_X$ with length less than $r + \epsilon$. The subset $U := \bigcup_{q \in B(p,r)} c_q$ of $X - S_X$ is path-connected (in fact contractible) and satisfies

$$B(p, r) - S_X \subset U \subset B(p, r + \epsilon/2) - S_X,$$

which completes the proof.

*Proof of Theorem* B (2). Denote by $d$ the distance function on $X - S_X$ induced from the Riemannian structure. Let $p, q \in X - S_X$ be any points. By Propositions 6.1 and 6.2 we have

$$\begin{aligned} d(p, q) &= \inf\{ L(c) \mid c \text{ is a } C^1\text{-curve joining } p \text{ and } q \} \\ &\geq \inf\{ |c| \mid c \text{ is a continuous curve joining } p \text{ and } q \} \\ &= |pq|. \end{aligned}$$

On the other hand, for any $\epsilon > 0$ we take minimal segments $px$ and $qx$ as in Theorem 6.4 and have

$$\begin{aligned} d(p, q) \leq d(p, x) + d(q, x) &\leq L(px) + L(qx) \\ &= |px| + |qx| < |pq| + \epsilon, \end{aligned}$$

which completes the proof.

## 7. Addendum

**7.1 Volume and Hausdorff measure.** Let $X$ be an $n$-dimensional Alexandrov space. The volume $\mathrm{vol}_\varphi(\cdot)$ of any (topological) Borel subset of $X - S_X$ entirely contained in a local chart $(U_\varphi, \varphi)$ is defined in the standard way. The volume $\mathrm{vol}(A)$ of any Borel subset $A \subset X - S_X$ is defined by

$$\mathrm{vol}(A) := \sum_i \mathrm{vol}_{\varphi_i}\left(A \cap \left(U_{\varphi_i} - \bigcup_{j<i} U_{\varphi_j}\right)\right),$$

where $\{(U_{\varphi_i}, \varphi_i)\}_{i=1,2,\cdots}$ is a $C^1$-atlas on $X - S_X \subset X$. Note that the existence of the countable family $\{(U_{\varphi_i}, \varphi_i)\}_{i=1,2,\cdots}$ is guaranteed by the separability of $X$ (see the proof of Theorem 4.2(2)), and that $\mathrm{vol}(A)$ is independent of the family $\{(U_{\varphi_i}, \varphi_i)\}_{i=1,2,\cdots}$.

**Proposition.** *For any Borel subset* $A \subset X - S_X$ *we have*

$$\mathrm{vol}(A) = V_H{}^n(A).$$

*Proof.* The proposition is proved in the standard way by using Lemma 3.6(3), so its proof is left to the reader.

**7.2 The limit of manifolds of small excess.** For a length space $X$ and $0 < d \le \mathrm{rad}(X)$, we define the *d-excess* $e^d(X)$ *of* $X$ (see [14]) by

$$e^d(X) := \sup_{p \in X} \sup_{x \in B(p,d)} \inf_{q \in \partial B(p,d)} e_{pq}(x).$$

The *injectivity radius* $\mathrm{inj}(X)$ of an Alexandrov space $X$ is defined to be the supremum of all $r \ge 0$ such that for any $p \in X$ and $v \in \Sigma_p$ there exists a unique minimal segment $pq$ such that $v = v_{pq}$ and $|pq| \ge r$. In the case where $X$ is a Riemannian manifold, our definition of the injectivity radius is equivalent to the usual definition. We can easily prove that $\mathrm{inj}(X) \ge d$ if and only if $\mathrm{rad}(X) \ge d$ and $e^d(X) = 0$ for any Alexandrov space $X$ and $d > 0$. Concerning [14], we have the following corollary.

**Corollary to Theorem B.** *Given* $m \in \mathbf{N}$, $\kappa \ge 0$, *and* $d > 0$, *let* $\{M_i\}$ *be a sequence of m-dimensional* $C^\infty$*-Riemannian manifolds such that* $K_{M_i} \ge -\kappa^2$ *and* $\mathrm{rad}(M_i) \ge d$ *for any* $i$, *and that* $e^d(M_i)$ *tends to zero as* $i \to \infty$. *Then, there exists a convergent subsequence* $\{M_{j(i)}\}$ *of* $\{M_i\}$ *such that the limit space* $X$ *of* $\{M_{j(i)}\}$ *is a* $C^{1/2}$*-Riemannian manifold of dimension* $1 \le \dim X \le m$, *curvature* $\ge -\kappa^2$ *(in the sense of the Alexandrov convexity), and of injectivity radius* $\mathrm{inj}(X) \ge d$. *Furthermore* $M_{j(i)}$ *for every large enough* $i$ *is a fibre bundle over* $X$ *with the fibre whose fundamental group is almost nilpotent.*

*Proof.* Since the limit space $X$ is an Alexandrov space of dimension $1 \le n := \dim X \le m$, curvature $\ge -\kappa^2$, and of $d$-excess $e^d(X) = 0$, and $\mathrm{inj}(X) \ge d$ and $S_X = \varnothing$, for any $x \in X$ we can find a natural local chart $(U_\varphi, \varphi)$ in such a way that $x \in U_\varphi \subset \bigcap_{i=1}^n B(p_i, d)$, where $p_1, \cdots, p_n \in X$ are the base points of $\varphi$. Thus we have $U_\varphi = W_\varphi$, and can construct a $C^{1+1/2}$-atlas of $X$ with $C^{1/2}$-metric (see §4). An almost Riemannian submersion $f_{j(i)} : M_{j(i)} \to X$ is constructed by the method in [14].

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