JANUS-LIKE ALGEBRAIC VARIETIES

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1. Introduction—the examples

Let $\mathscr{D}$ be a bounded, symmetric domain (or equivalently, a Hermitian symmetric space of noncompact type), and $\Gamma \subset \mathrm{Aut}(\mathscr{D})$ (holomorphic isometries) be a discrete group acting properly discontinuously on $\mathscr{D}$. Supposing further that Γ acts freely, $V = \Gamma\backslash\mathscr{D}$ is a complex manifold. If Γ is in addition cocompact, then this complex manifold is an algebraic variety (complete, and of general type) as Kodaira proved in 1954 as an application of his embedding theorem. More generally, we may consider the case where $\Gamma\backslash\mathscr{D}$ has finite volume. In 1960 it was shown that if Γ is arithmetic, then $\Gamma\backslash\mathscr{D}$ is always algebraic (quasi-projective); in fact the Baily-Borel embedding of the Satake compactification $(\Gamma\backslash\mathscr{D})^*$ realizes $\Gamma\backslash\mathscr{D}$ as a Zariski open subset of a normal algebraic variety. Somewhat later on (1975) the theory of toroidal embeddings was applied to yield *smooth* compactifications $\overline{(\Gamma\backslash\mathscr{D})}$, which dominate $(\Gamma\backslash\mathscr{D})^*$ and have the property that the complement $\overline{(\Gamma\backslash\mathscr{D})} - (\Gamma\backslash\mathscr{D})$ is a normal crossings divisor.

If we are given that the projective variety V is of the form $\Gamma\backslash\mathscr{D}$ for some bounded symmetric domain $\mathscr{D}$ and some group Γ as above (i.e., *compact*), then we may recover both $\mathscr{D}$ and Γ from V, simply by calculating Chern numbers and applying Hirzebruch proportionality. However, in the case $V = \overline{\Gamma\backslash\mathscr{D}}$, this is not necessarily true. A priori, it is possible that there exists a locally symmetric V with two sets of normal crossings divisors, say Δ_1, Δ_2, such that

$$V - \Delta_1 = \Gamma_1\backslash\mathscr{D}_1; \qquad V - \Delta_2 = \Gamma_2\backslash\mathscr{D}_2,$$

and $\mathscr{D}_1$ and $\mathscr{D}_2$ are two completely unrelated bounded symmetric domains. (This statement can also be formulated as: $\overline{(\Gamma_1\backslash\mathscr{D}_1)} \cong \overline{(\Gamma_2\backslash\mathscr{D}_2)}$.) We call such a variety V *Janus-like.*[1] It is the object of this paper to

[1] The reader will recall that the Roman god Janus was depicted as one having two faces. This name is also the origin of the word January, for the month looking back into the last as well as forward towards the next year.

give examples of Janus-like varieties V. In our main examples, $\mathscr{D}_1 \cong \mathbb{B}_3$, the complex 3-ball, and $\mathscr{D}_2 \cong \mathbb{S}_2$, the Siegel space of degree 2. Recall that $\mathrm{Aut}(\mathscr{D}_1) = \mathrm{PU}(3, 1)$, and $\mathrm{Aut}(\mathscr{D}_2) = \mathrm{PSp}(4, \mathbb{R})$. Let $\mathscr{O}_K$ be the ring of algebraic integers in the field $K = \mathbb{Q}(\sqrt{-3})$, and for an ideal $n\mathscr{O}_K$ let $\Gamma_K(n)$ denote the principal congruence subgroup of level n in the Picard modular group $\mathrm{U}(3, 1; \mathscr{O}_K)$. Let $\Gamma(N)$ be the principal congruence subgroup of level N in the Siegel modular group $\mathrm{Sp}(4, \mathbb{Z})$. In order to state our main result we introduce the following notation: For a lattice Γ in $\mathrm{Sp}(4, \mathbb{Z})$ we let $X(\Gamma) = \Gamma\backslash\mathbb{S}_2$, $X^*(\Gamma) = (\Gamma\backslash\mathbb{S}_2)^*$, and $\overline{X(\Gamma)} = \overline{\Gamma\backslash\mathbb{S}_2}$, and for a lattice Γ_K in $\mathrm{U}(3, 1; \mathscr{O}_K)$ we let $Y(\Gamma_K) = \Gamma_K\backslash\mathbb{B}_3$, $Y^*(\Gamma_K) = (\Gamma_K\backslash\mathbb{B}_3)^*$, $\overline{Y(\Gamma_K)} = \overline{(\Gamma_K\backslash\mathbb{B}_3)}$. We write $X(N)$ for $X(\Gamma(N))$, and similarly for $X^*(N)$, $\overline{X(N)}$, $Y(n)$, $Y^*(n)$, and $\overline{Y(n)}$.

Our main result is the following.

Main Theorem. *There are isomorphisms*

(i) $\overline{X(1)} \cong \overline{Y(1)}$,

(ii) $\overline{X(2)} \cong \overline{Y(\sqrt{-3})}$,

(iii) $\overline{X(3)} \cong \overline{Y(2)}$,

(iv) $\overline{X(6)} \cong \overline{Y(2\sqrt{-3})}$.

Further, under these isomorphisms the boundary components of $\overline{X(N)}$ are mapped isomorphically onto modular subvarieties of $\overline{Y(n)}$, i.e., compactifications of quotients of subdomains of $\mathbb{B}_3$, and the boundary components of $\overline{Y(n)}$ are mapped isomorphically onto modular subvarieties of $\overline{X(N)}$, i.e., compactifications of quotients of subdomains of $\mathbb{S}_2$. Indeed, establishing these two sets of isomorphisms is an essential step in our proof. We note that these isomorphisms themselves produce examples of Janus-like surfaces, the first with $\mathscr{D}_1 = \mathbb{S}_1 \times \mathbb{C}$ and $\mathscr{D}_2 = \mathbb{B}_2$, and the second with $\mathscr{D}_1 = \mathbb{C}^2$ and $\mathscr{D}_2 = \mathbb{S}_1 \times \mathbb{S}_1$. (Of course, $\mathbb{S}_1 \cong \mathbb{B}_1$). We shall not give the relevant lattices here, to avoid excessive notation at this point, but just refer the reader to 3.8 and 3.19 below. In turn, to produce these we first find Janus-like curves, with $\mathscr{D}_1 = \mathbb{S}_1$ and $\mathscr{D}_2 = \mathbb{C}$, in 3.1 and 3.2.

We should note that, strictly speaking, some of these examples leave the discussion at the beginning of the introduction. In cases (i) and (ii), both groups act with fixed points, and in (iii), $\Gamma_K(2)$ acts with fixed points while $\Gamma(3)$ does not. The quotient in case (i) is a " V-manifold", i.e., has finite quotient singularities (and is in fact *globally* the quotient of a smooth variety), but in cases (ii) and (iii) is nonsingular. In case (iv), however, we are in the situation discussed above; both groups act freely.

Case (iv) of the theorem is particularly interesting because the variety there is of general type; also both Zariski open parts are log-general type.

In cases (ii) and (iii) the variety is rational, but has a rich structure, never-theless. Indeed, these two varieties have alternate descriptions. They are isomorphic to varieties which have been known since the 19th century—the desingularizations of the Segre cubic and Burkhardt quartic respectively. The original proof of the isomorphism in case (ii), from which that in case (i) easily follows, was based on the explicit identification of the Satake compactification $Y^*(\sqrt{-3})$ with the Segre cubic and of $X^*(2)$ with the projectively dual variety. Similarly a proof of case (iii) may be based on the identification of the Satake compactification $Y^*(2)$ with the Burkhardt quartic and of $X^*(3)$ with (the normalization of) its projective dual. Our proofs here are much simpler, being based on the moduli interpretations of $\overline{X(N)}$ and $\overline{Y(n)}$. However, since these identifications are themselves interesting, and formed part of our original motivation for studying these varieties, we devote the remainder of this introduction to their description (though we make no use of it in the proof of the main theorem above).

The Segre cubic.   The first variety which came to our attention ([10, §5] contains the first observation to this effect) was the Segre cubic. This is the cubic 3-fold given in $\mathbb{P}^5 = \{[x_0, \cdots, x_5]\}$ by the following two equations

$$\mathscr{S} = \left\{ \sum x_i = 0, \sum x_i^3 = 0 \right\}.$$

Since the first equation has degree 1 we see that the Segre cubic is a hypersurface. One sees easily that $\mathscr{S}$ has 10 ordinary double points $([1, 1, 1, -1, -1, -1]$ and its transforms under the symmetric group Σ_6, which obviously acts on $\mathscr{S})$. It has been known since the last century that there is a unique cubic hypersurface in $\mathbb{P}^4$ with 10 nodes (see [9, Theorem 1.11] for a proof). The following 15 $\mathbb{P}^2$'s lie entirely on $\mathscr{S}$:

$$x_{\sigma(0)} + x_{\sigma(3)} = x_{\sigma(1)} + x_{\sigma(4)} = x_{\sigma(2)} + x_{\sigma(5)}, \qquad \sigma \in \Sigma_6.$$

In fact, each such $\mathbb{P}^2$ contains exactly 4 of the 10 nodes, and the inter-sections with the other $\mathbb{P}^2$'s yield the line arrangement shown in Figure A (next page) in each $\mathbb{P}^2$.

To visualise $\mathscr{S}$ more precisely, one can utilize the fact that $\mathscr{S}$ is rational. The explicit birational map is as follows: let $H_1, \cdots, H_{10}$ be the arrangement of 10 planes in $\mathbb{P}^3$ given by the equations:

$$x_i = 0, \quad i = 0, \cdots, 3; \qquad x_i = x_j, \quad i \neq j.$$

These planes are the faces and symmetry planes of a tetrahedron in $\mathbb{P}^3$. Blow up: (1) the four corners and center, $(1, 0, 0, 0), \cdots, (0, 0, 0, 1)$

☐ = node
─── = intersection
      with other $\mathbb{P}^2$
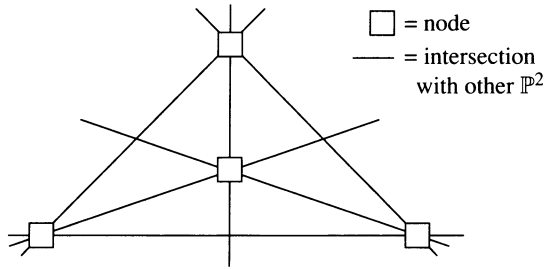
FIGURE A

and $(1, 1, 1, 1)$; (2) the 10 lines joining two of the five points. Note that after (1), the normal bundle of each such line in (2) is $\mathscr{O}(-1) \oplus \mathscr{O}(-1)$; (3) hence each exceptional $\mathbb{P}^1 \times \mathbb{P}^1$ can be blown down to an ordinary double point. The result of (1)–(3) is precisely the Segre cubic.

Theorem 1. $\mathscr{S}$ *is the Satake compactification* $Y^*(\sqrt{-3})$.

The fact that $\mathscr{S}$ is a ball quotient is mentioned in [9]; the determination of the group is a straightforward 3-dimensional generalisation of the work of Holzapfel [6].

Let $\mathscr{Q}$ denote the dual variety to $\mathscr{S}$. This is a hypersurface of degree $3(3-1)^3 - 10 \cdot 2 = 4$ (see [5, 4.4.3]). The 10 double points on $\mathscr{S}$ determine 10 double tangents of $\mathscr{Q}$, and the 15 $\mathbb{P}^2$'s determine 15 singular lines on $\mathscr{Q}$. One sees without too much trouble that $\mathscr{Q}$ is the quartic known as the Igusa quartic, given in $\mathbb{P}^9$ by five linear relations and the single nonlinear relation:

$$\sigma_2^2 - 4\sigma_4 = 0,$$

where the σ_i are the elementary symmetric functions. The following was known essentially in the last century from work on theta functions (see for example [15, p. 505]), and rigorously proved by Igusa [8].

Theorem I$'$. $\mathscr{Q}$ *is the Satake compactification* $X^*(2)$.

Now dual varieties (of degree ≥ 3) are birational, so these two theorems yield a big step toward the proof of the main theorem above in case (ii).

The Burkhardt quartic. The next example that came to our attention was the Burkhardt quartic, which is given in $\mathbb{P}^5$ by the following equations:

$$\mathscr{B} = \left\{ \sum x_i = 0, \ \sum_{i \neq j \neq k \neq l} x_i x_j x_k x_l = 0 \right\}.$$

As above, $\mathscr{B}$ is a hypersurface. It has 45 nodes:

15 nodes, $\quad (ij) := \{x_i = 1, x_j = -1, x_k = 0, k \neq i, k \neq j\}.$

30 nodes, $\quad (ij.kl.mn) := \{x_i = x_j = 1, x_k = x_l = \rho, x_m = x_n = \rho^2,$
$$\rho = e^{2\pi i/3}, i \neq j \neq k \neq l \neq m \neq n\}.$$

It has recently been proven that this property uniquely determines $\mathscr{B}$ [4], i.e., any quartic hypersurface in $\mathbb{P}^4$ with 45 nodes is projectively equivalent to the Burkhardt quartic (and 45 is the maximal number of nodes which a quartic 3-fold can have—in this case the Varchenko bound is sharp). In addition there are 40 planes lying entirely on $\mathscr{B}$, given by the equations

$$x_i = x_j/\rho = x_k/\rho^2, \quad i < j < k \qquad (20),$$
$$x_i = x_j/\rho^2 = x_k/\rho, \quad i < j < k \qquad (20).$$

Each such $\mathbb{P}^2$ contains 9 of the 45 nodes, and these 9 are in fact the 9 base points of the Hesse pencil of elliptic curves. Recall that this pencil has four degenerate cubics which are each a triangle of lines. These 12 lines in each $\mathbb{P}^2$ are the intersections with the other 39 of the 40 planes. From the equation of $\mathscr{B}$ one sees immediately that the symmetric group Σ_6 acts on it. But in fact a much larger group acts: the group of the even permutations of the 27 lines lying on a smooth cubic surface in $\mathbb{P}^3$, the simple group of order 25,920 which we denote $G_{25,920}$. Namely, the 45 nodes of $\mathscr{B}$ correspond to the 45 tritangents of a smooth cubic surface, and the 27 lines on such a cubic correspond to 27 sets of 5 of the 45 nodes not lying in a $\mathbb{P}^3$ (recall each line is contained in five tritangents; here the inclusion relation is reversed, in some sense). These sets of five of the nodes spanning $\mathbb{P}^4$ are called *Jordan pentahedra*.

The 45 nodes are joined by lines containing two, resp. three, of them, which are denoted ε-*lines* (containing two nodes) and κ-*lines* (containing three nodes). (One sees from the coordinates for the nodes that no more than 3 can lie on a line.) The 27 Jordan pentahedra are spanned by sets of five nodes, all of which are joined by ε-lines, such as the set:

$$(ij), (kl), (mn), (ij.kl.mn), (ij.mn.kl).$$

Thus these five nodes are the vertices of a coordinate simplex, and the 10 ε-lines joining them in pairs are the edges (1-dimensional faces) of the simplex. The 2-dimensional faces of the pentahedron, containing three nodes and 3 ε-lines, are called *f-planes* and intersect $\mathscr{B}$ in 3-nodal (rational) quartic curves:
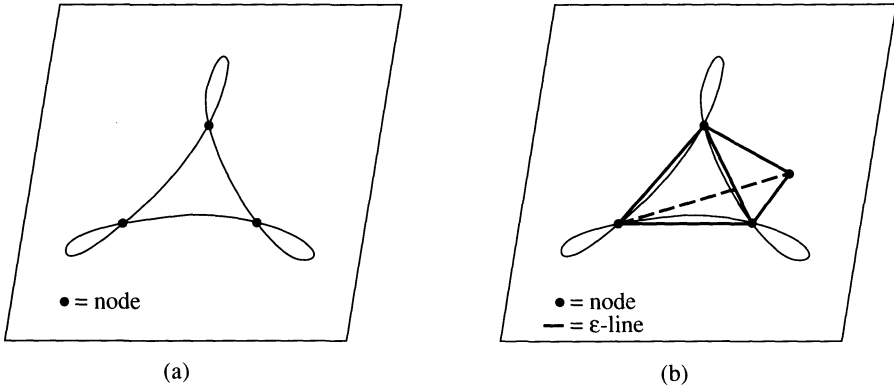
FIGURE 1. (a) f-PLANE; (b) COORDINATE TETRAHE-
DRON IN JORDAN PRIME

The five sides (3-dimensional faces) of the simplex are so-called *Jordan primes*, containing four of the nodes, six of the ε-lines, and four of the f-planes of the pentahedron of reference. For each of the nodes of the simplex there is a Jordan prime which is the opposite side of the simplex. There are given by the equations:

$$\text{node} = (ij), \qquad \text{opposite face}: \quad J_{ij} = \{x_i - x_j = 0\},$$

$$\text{node} = (ij.kl.mn), \qquad \text{opposite face}:$$

$$J_{(ij.kl.mn)} = \{x_i - x_j + \rho^2(x_k - x_l) + \rho(x_m - x_n) = 0\}.$$

The intersection of this Jordan prime with $\mathscr{B}$ contains, in addition to these four, two other sets of four nodes, each itself the set of vertices of a (different) coordinate simplex. The intersection $\mathscr{B} \cap J_{ij}$ is therefore a 12-nodal quartic surface in $\mathbb{P}^3$, and in fact is the famous "desmic surface", so called because of the triad of desmic tetrahedra whose vertices are the nodes.

The relation to the 27 lines and 45 tritangents of the cubic surface can now be precisely formulated: The 45 nodes correspond to the 45 tritangents such that:

two nodes are joined by an ε-line

(i.e., are the vertices of a Jordan pentahedron)

$\Leftrightarrow$

the corresponding tritangents meet in one of the 27 lines.

(45)

Hence the five nodes spanning a Jordan pentahedron correspond to the

five tritangents passing through a fixed line.

It is also known that $G_{25,920}$ is the subgroup of index 2 in the Weyl group of E_6 consisting of even reflections. The following table gives a translation between these different realizations of $G_{25,920}$ and its subgroups:

Order	Normalizer on cubic surface	Normalizer on $\mathscr{B}$	Subgroup of Weyl group $W^+(E_6)$
960	Line	Jordan pentahedron	$W^+(D_5)$
720	double-six	set of 30 nodes	$W^+(A_5)$
648	triad of trihedral pairs	one of the 40 planes	
576	tritangent	node	$W^+(F_4)$
120	skew pair of lines	10 nodes lying in a $\mathbb{P}^3$	$W^+(A_4)$

It is also known that $\mathscr{B}$ is rational; however, as opposed to the Segre cubic where the rationalization is Σ_6-equivariant, this rationalization of $\mathscr{B}$ is *not* $G_{25,920}$-equivariant. More useful for the study of $\mathscr{B}$ is the projection from a node, which displays $\mathscr{B}$ as a double cover of $\mathbb{P}^3$. The branch locus (of degree 6) splits into a quadric surface and a special 12-nodal quartic surface (the desmic surface referred to above, see Figure 1b)). By the way, this property is essentially what is used in [4] to prove uniqueness of $\mathscr{B}$. The following, although known to us since 1987, has not yet appeared in print:

Theorem II. *$\mathscr{B}$ is the Satake compactification $Y^*(2)$.*

Although we will not prove this theorem here let us give a few remarks on how to do it. First, one applies the Yau equality (in its logarithmic form) to show that $\mathscr{B}$-{45 nodes} is a ball quotient, hence $\mathscr{B}$ is the Satake compactification of a ball quotient. Of course, the (logarithmic) Yau equality holds only for *free* actions, so modifications for fixed points are necessary. It remains to identify the group. For this one can use the moduli interpretation via moduli problems as in this paper. An alternative method would be to identify the group "piece by piece", as in §§3–4 of this paper, first checking the parabolics, then modular subgroups. Note, however, that Theorem II follows from II′ below, our Main Theorem, and some elementary arguments on birational modifications. Finally we remark that recently Bert van Geemen has succeeded in proving Theorem II directly, using theta functions. His method works in the present case because of the fact that $Y^*(2)$ is a moduli space of abelian varieties which have *automorphisms*, not just endomorphisms. For details see [22], which treats this side of the Janus-like behavior from the standpoint of theta functions.

We consider also the dual variety to $\mathscr{B}$, call it $\mathscr{R}$. This is a variety of degree $4(4-1)^3 - 2 \cdot 45 = 18$. The 45 nodes of $\mathscr{B}$ determine 45 double tangents, i.e., $\mathbb{P}^1 \times \mathbb{P}^1$'s lying on $\mathscr{R}$. Furthermore, the 40 planes determine 40 singular lines on $\mathscr{R}$. These singularities are, as it turns out, not normal [18]. Let $\widetilde{\mathscr{R}}$ be the normalization of $\mathscr{R}$. Then the following is proved in [18]:

Theorem II$'$. $\widetilde{\mathscr{R}} \cong$ *the Satake compactification* $X^*(3)$.

Once again we remark that these two results imply that $\overline{Y(2)}$ and $\overline{X(3)}$ are birational, and represent the main step in the original proof of the main theorem above in case (iii).

Let us now sketch the strategy of the proof presented in this paper. We could paraphrase this approach as a "consequent application of moduli interpretations." Since the work of Shimura in the early 1960s, it has been known that Picard modular varieties (quotients of $\mathbb{B}_n$ by Picard modular groups) are natural moduli spaces of abelian $(n+1)$-folds with certain complex multiplications. In our case we have abelian 4-folds (principally polarized), with complex multiplication by $K = \mathbb{Q}(\sqrt{-3})$. But the field K is very special among imaginary quadratic fields, being itself cyclotomic, $K = \mathbb{Q}(\rho)$, $\rho = \exp(2\pi i/3)$. This makes it possible to identify the abelian 4-folds as the Jacobians of certain genus 4, trigonal curves. On the other hand, it is well known that $(\Gamma(N)\backslash \mathbb{S}_2)$ is the moduli space of abelian surfaces (principally polarized) with level N structure. By general theory, then, these are the moduli spaces of genus 2 curves with level structures and, as is well-known, all genus 2 curves are hyperelliptic. One checks easily that both a genus 4 ($\mathbb{Z}_3$-Galois-) trigonal curve as well as a genus 2 hyperelliptic curve has six branch points on $\mathbb{P}^1$; thus the main part of our proof of the main theorem lies in identifying a level 2 (respectively level 3, 6) structure on a hyperelliptic curve with a level $\sqrt{-3}$ (respectively level 2, $2\sqrt{-3}$) structure on a trigonal curve. This observation then yields the isomorphisms of the main theorem on Zariski open sets. On the complements of these we determine precisely the structure of the divisors, and show they can be identified with each other also, yielding the isomorphisms of the theorem.

2. The domains and groups

2.1. Domains.

In this section we introduce 2 types of Hermitian symmetric spaces $\mathscr{D} = G_0/K$ of noncompact type, dual to $\check{\mathscr{D}} = G_u/K$ and embedded

$\mathscr{D} \subset \check{\mathscr{D}}$ by means of the Borel embedding. If $\mathfrak{p}^+$ denotes the holomorphic tangent space of $\mathscr{D}$ at the identity, the Harish-Chandra map $\zeta: \mathscr{D} \to \mathfrak{p}^+$ realises $\mathscr{D}$ as a bounded symmetric domain $\mathbf{D}$ in $\mathbb{C}^n \cong \mathfrak{p}^+$. The topological closure $\overline{\mathbf{D}} \supset \mathbf{D}$ in $\mathfrak{p}^+$ (with the Euclidean topology) is called the natural compactification of $\mathbf{D}$ and the maximal irreducible holomorphic arc components of the boundary $\partial \mathbf{D} = \overline{\mathbf{D}} - \mathbf{D}$ are the boundary components; the length of a maximal flag $F_0 \subset F_1 \subset \cdots \subset F_k$ of such boundary components is the $\mathbb{R}$-rank of the group G_0 (or of the domain $\mathscr{D}$). Via Cayley transforms $c_F: \mathbf{D} \to \mathbf{D}_F$ (inside $\mathfrak{p}^+$) the domain has different unbounded realizations which correspond to certain standard boundary components and which fibre onto them $\mathbf{D}_F \to F$. In particular there is a total Cayley transform corresponding to a 0-dimensional boundary component. Group theoretically the boundary components correspond in a 1 to 1 fashion to parabolic subgroups of G_0. There are other subgroups of G_0 of special interest, namely the stabilizers of certain subdomains $\mathbf{D}_i \subset \mathbf{D}$, which are reductive subgroups of G_0. For details on these matters see [2], [21] for the general theory and [25] and [24] for the special cases interesting us.

2.1.1. The ball. We consider first the homogeneous space $\mathscr{D} = U(3, 1)/U(3) \times U(1)$ with compact dual $\check{\mathscr{D}} = \mathbb{P}^3$. $\mathscr{D}$ has the well-known bounded realization as the 3-dimensional ball $\mathbb{B}_3$. We regard $\mathbb{B}_3$ as embedded in $\mathbb{P}^3$ by

$$\mathbb{B}_3 = \left\{ \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ 1 \end{bmatrix} \mid \sum |z_i|^2 < 1 \right\}.$$

The automorphism group of $\mathbb{P}^3$ is $\mathrm{PGL}(4, \mathbb{C})$, and the automorphism group of $\mathbb{B}_3$ is the subgroup of $\mathrm{PGL}(4, \mathbb{C})$ leaving $\mathbb{B}_3$ invariant, which is $\mathrm{PU}(3, 1)$, i.e., the group of matrices preserving the Hermitian form

$$\mathbf{H} = \mathrm{diag}(1, 1, 1, -1),$$

acting by projective linear transformations on $\mathbb{B}_3$. (More precisely, we should write $\mathrm{PU}_{\mathbf{H}}(3, 1)$). Its closure $\overline{\mathbb{B}}_3$ is just the closed ball, and all boundary components are 0-dimensional, as $U(3, 1)$ has $\mathbb{R}$-rank 1.

A typical Cayley transform is

$$c_F: \mathbb{B}_3 \to \mathbb{D}_F = \left\{ (u, v_1, v_2) \mid \mathrm{Im}(u) - \frac{1}{2} \sum |v_i|^2 > 0 \right\}$$

given by

$$\begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ 1 \end{bmatrix} \mapsto \begin{bmatrix} u_1 \\ v_1 \\ v_2 \\ 1 \end{bmatrix} = \begin{bmatrix} \mathbf{T} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ 1 \end{pmatrix} \end{bmatrix},$$

where $\mathbf{T}$ is the matrix

$$\mathbf{T} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & -i\sqrt{2} & 0 & 0 \\ 0 & 0 & -i\sqrt{2} & 0 \\ i & 0 & 0 & -i \end{pmatrix}.$$

Then the automorphism group of $\mathbf{D}_F$ is just $\mathbf{T}^{-1}\,\mathrm{PU}(3, 1)\mathbf{T} = \mathrm{PU}_J(3, 1)$, the group of matrices preserving the Hermitian form $\mathbf{J} = \mathbf{T}\mathbf{H}\mathbf{T}^*$. Concretely,

$$\mathbf{J} = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}.$$

The domain $\mathbb{B}_3$ has as a boundary component $(1, 0, 0) = [1, 0, 0, 1]$, and the corresponding parabolic in $\mathrm{PU}_J(3, 1)$ takes triangular form. $\mathrm{PU}(3, 1)$ operates transitively on the boundary components, and so any two parabolics are conjugate.

As well, the domain $\mathbb{B}_3$ has a subdomain $\mathbb{B}_2$ given by $z_3 = 0$, which under c_F maps to the subdomain $v_2 = 0$ in $\mathbf{D}_F$. The group $\mathrm{PU}(2, 1)$, regarded as a subgroup of $\mathrm{PU}(3, 1)$ (under the obvious inclusion) leaves $\mathbb{B}_2$ invariant and acts transitively on it. The stabilizer of $\mathbb{B}_2$ is $\mathrm{PU}(2, 1) \times \{z \mid |z| = 1\}$, the second factor acting trivially. $\mathrm{PU}(3, 1)$ acts transitively on the subsets of $\mathbb{B}_3$ isomorphic to $\mathbb{B}_2$, and so any two subgroups isomorphic to $\mathrm{PU}(2, 1)$ are conjugate. An analogous statement holds for $\mathbb{B}_1 = \{z_2 = z_3 = 0\}$ and corresponding subgroup $\mathrm{PU}(1, 1)$. If we let $(\mathbf{D}_F)_2 = c_F(\mathbb{B}_2)$ and $\mathrm{PU}_J(2, 1) = \mathbf{T}^{-1}\,\mathrm{PU}(2, 1)\mathbf{T}$, and $(\mathbf{D}_F)_1 = c_F(\mathbb{B}_1)$ and $\mathrm{PU}_J(1, 1) = \mathbf{T}^{-1}\,\mathrm{PU}(1, 1)\mathbf{T}$, similar statements hold as well (with appropriate matrices $\mathbf{T}$).

Note also that the given boundary component is contained in the closure of the given $\mathbb{B}_2$ (or $\mathbb{B}_1$).

2.1.2. Siegel space (of degree 2). Here we are concerned with $\mathscr{D} = \mathrm{Sp}(4, \mathbb{R})/\mathrm{U}(2)$, which has the following bounded realization:

$$\mathbf{D} = \{Z \in M_2(\mathbb{C}) \mid Z = {}^t Z, ZZ^* - 1 < 0\}.$$

Its closure $\overline{\mathbf{D}}$ consists of symmetric Z's such that $ZZ^* - 1$ is negative semidefinite. The holomorphic arc components have representatives

$$(1) \qquad F_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad F_1 = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & \tau \end{pmatrix} \mid |\tau|^2 < 1 \right\} \cong \mathbb{B}_1 \subset \mathbf{D}$$

of dimensions 0 and 1 respectively (note that $\mathrm{Sp}(4, \mathbb{R})$ has both 0- and 1-dimensional boundary components, as this group has $\mathbb{R}$-rank 2), and every boundary component of $\mathbf{D}$ is equivalent to one of these under the action of its automorphism group $\mathrm{Sp}_{\mathbf{H}}(4, \mathbb{R})$, by which is meant the set of all complex symplectic matrices fixing the Hermitian form

$$\mathbf{H} = \begin{pmatrix} i1_2 & 0 \\ 0 & -i1_2 \end{pmatrix}.$$

The total Cayley transform $Z \mapsto i(1_2 + Z)(1_2 - Z)^{-1}$ maps $\mathbf{D}$ onto the well known unbounded realization $\mathbf{D}_F = \mathbb{S}_2 = \{ Z = {}^t Z \mid \mathrm{Im}(Z) > 0 \}$, and the usual symplectic group $\mathrm{Sp}(4, \mathbb{R})$ is the automorphism group corresponding to the usual symplectic form

$$\mathbf{J} = \begin{pmatrix} 0 & -1_2 \\ 1_2 & 0 \end{pmatrix},$$

acting in the well-known manner: $Z \mapsto (AZ + B)(CZ + D)^{-1}$ for an element $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(4, \mathbb{R})$. In this realization representatives of the cusps are given by

$$F_0 = \left\{ \lim_{\mathrm{Im}(\tau_1), \mathrm{Im}(\tau_2) \to \infty} \begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{pmatrix} \right\} = \mathrm{pt.},$$

$$(2) \qquad F_1 = \left\{ \lim_{\mathrm{Im}(\tau_1) \to \infty} \begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{pmatrix} \right\} \cong \mathbb{S}_1 \subset \overline{\mathbb{S}}_2.$$

The corresponding parabolic subgroups are

$$P_0 = \left\{ \left(\begin{array}{c|c} * & * \\ \hline 0 & * \end{array} \right) \right\}, \qquad P_1 = \left\{ \left(\begin{array}{cc|cc} a_{11} & a_{21} & & \\ 0 & a_{22} & & \ast \\ \hline 0 & 0 & a_{33} & 0 \\ 0 & a_{42} & a_{43} & a_{44} \end{array} \right) \right\}.$$

Notice that P_0 is the stabilizer of a (totally) isotropic plane, while P_1 is the stabilizer of an isotropic line in $\mathbb{R}^4$.

Consider the following subdomain

$$\mathbb{S}_1 \times \mathbb{S}_1 = \left\{ \begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{pmatrix} \right\} \subset \mathbb{S}_2.$$

The automorphism group $SL(2, \mathbb{R}) \times SL(2, \mathbb{R}) \subset Sp(4, \mathbb{R})$ acts transitively on $\mathbb{S}_1 \times \mathbb{S}_1$ with embedding given by

$$\left(\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \right) \mapsto \begin{pmatrix} a_1 & 0 & b_1 & 0 \\ 0 & a_2 & 0 & b_2 \\ c_1 & 0 & d_1 & 0 \\ 0 & c_2 & 0 & d_2 \end{pmatrix} \in Sp(4, \mathbb{R}).$$

The stabilizer of $\mathbb{S}_1 \times \mathbb{S}_1$ is an extension of $\mathbb{Z}/2$ by this group, the nontrivial element of $\mathbb{Z}/2$ being represented by the matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix},$$

which sends

$$\begin{pmatrix} z_1 & 0 \\ 0 & z_2 \end{pmatrix} \quad \text{to} \quad \begin{pmatrix} z_2 & 0 \\ 0 & z_1 \end{pmatrix}.$$

Note that the closure of $\mathbb{S}_1 \times \mathbb{S}_1$ in $\overline{\mathbb{S}_2}$ contains the 1-dimensional boundary component F_1 above. Any other subdomain of $\mathbb{S}_2$ isomorphic to $\mathbb{S}_1 \times \mathbb{S}_1$ is the image of this one under the action of an element of $Sp(4, \mathbb{R})$, and so any two subgroups isomorphic to $SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$ are conjugate.

2.2. Lattices.

2.2.1. Picard lattices.
We consider first the matrix group $U_H(3, 1)$ from 2.1.1. Let K be the imaginary quadratic field $K = \mathbb{Q}(\sqrt{-3})$, and let $\mathcal{O}_K$ denote its ring of integers.

Definition 2.1. The *Picard modular group* for the field K is the group

$$\Gamma_K := U_H(3, 1) \cap M_4(\mathcal{O}_K),$$

i.e., the set of all matrices in $U_H(3, 1)$ with coefficients in $\mathcal{O}_K$.

Note that Γ_K is a maximal lattice in the $\mathbb{Q}$-group $U_H(3, 1) \cap M_4(K)$. This group has $\mathbb{Q}$-rank 1 and therefore has only 0-dimensional boundary components. Now consider a subgroup Γ'_K of Γ_K of finite index (possibly $\Gamma'_K = \Gamma_K$). Each Γ'_K-equivalence class of (0-dimensional) boundary components is called a cusp of Γ'_K. The theory of compactifications (see §3) tells us that the cusps of Γ'_K are in $1 - 1$ correspondence with the irreducible components of the divisor $\overline{(\Gamma'_K \backslash \mathscr{D})} - (\Gamma'_K \backslash \mathscr{D})$. The cusps (or equivalently, the maximal $\mathbb{Q}$-parabolics, each such being the stabilizer of a cusp) correspond to the Γ'_K-orbits of 1-dimensional subspaces of K^4 isotropic with respect to the form **H**. We shall call these *isotropic lines*, and denote them by l.

Similarly, each Γ'_K-orbit of 3-dimensional subspaces on which $\mathbf{H}$ has signature $(2,1)$ corresponds to the divisor which is the closure in $\overline{(\Gamma'_K \backslash \mathscr{D})}$ of a modular subvariety $(\Gamma'_K)_2 \backslash \mathbb{B}_2$ of $(\Gamma'_K \backslash \mathscr{D})$. We call these subspaces $(2,1)$ *solids* and denote them by Δ_3. We may also consider $(1,1)$ *planes* Δ_2, Γ'_K-orbits of 2-dimensional subspaces on which $\mathbf{H}$ has signature $(1,1)$; these correspond to the closure in $(\Gamma'_K \backslash \mathscr{D})$ of subvarieties $(\Gamma'_K)_1 \backslash \mathbb{B}_1$ of $(\Gamma'_K \backslash \mathscr{D})$. Note that Δ_3 (respectively Δ_2) determines and is determined by its orthogonal complement, a 1- (respectively 2-) dimensional subspace of K^4 on which $\mathbf{H}$ is positive definite.

The Γ'_K of interest to us will be the groups $\Gamma_K(n)$, defined as follows: consider an ideal of $\mathscr{O}_K$, necessarily of the form $n\mathscr{O}_K$, for some $n \in \mathscr{O}_K$ (as $\mathscr{O}_K$ is a principal ideal domain), which is invariant under conjugation, and let $\mathscr{O}_K / n\mathscr{O}_K$ be its residue ring. Note that $\mathscr{O}_K / n\mathscr{O}_K$ inherits a conjugation from $\mathscr{O}_K$.

Definition 2.2. The *principal congruence subgroup of level* n, $\Gamma_K(n)$, is the kernel of the map

$$\Gamma_K \to U(3,1; \mathscr{O}_K / n\mathscr{O}_K)$$

given by reducing matrix entries modulo $n\mathscr{O}_K$.

It is known that this map is an epimorphism, so there is an exact sequence

$$1 \to \Gamma_K(n) \to \Gamma_K \to U(3,1; \mathscr{O}_K / n\mathscr{O}_K) \to 1.$$

We shall be considering the action of these groups on $\mathbb{B}_3$, and so we must take their images in $PU(3,1)$, giving the exact sequence

$$1 \to P\Gamma_K(n) \to P\Gamma_K \to PU(3,1; \mathscr{O}_K / n\mathscr{O}_K) \to 1.$$

2.2.2. Siegel lattices. We now turn our attention to the group $Sp(4, \mathbb{R})$ of §2.1.2. The Siegel modular group $\Gamma = Sp(4, \mathbb{Z})$ is the subgroup of integral matrices; Γ is contained in the $\mathbb{Q}$-group $Sp(4, \mathbb{Q})$ which has $\mathbb{Q}$-rank 2. Again we consider a subgroup Γ' of Γ of finite index (possibly $\Gamma' = \Gamma$). As we have observed, Γ (or Γ') has both 0- and 1-dimensional boundary components, and we call the Γ'-orbits of these the 0- and 1-dimensional cusps of Γ'. Again the cusps (or the maximal parabolics, each such being the stabilizer of a cusp) correspond to the Γ'-orbits of isotropic subspaces in $\mathbb{Q}^4$ with respect to the usual symplectic form. Such a subspace is either 2-dimensional, in which case we call it an *isotropic plane* and denote it by h, or is 1-dimensional, in which case we call it an *isotropic line* and denote it by l; these correspond to the 0- and 1-dimensional cusps, respectively. Here the 1-dimensional cusps of Γ' are

in $1 - 1$ correspondence with the irreducible components of the divisor $\overline{(\Gamma'\backslash\mathscr{D})} - (\Gamma'\backslash\mathscr{D})$.

Similarly, we may consider proper subspaces of $\mathbb{Q}^4$ on which the symplectic form is nonsingular over $\mathbb{Z}$. Such a subspace δ is necessarily a plane, and has an orthogonal complement $\delta^\perp$ on which the form is also nonsingular. Each Γ'-orbit class of pairs $\{\delta, \delta^\perp\}$ corresponds to the divisor which is the closure in $\overline{(\Gamma'\backslash\mathbb{S}_2)}$ of a modular subvariety $(\Gamma')_2\backslash\mathbb{S}_1 \times \mathbb{S}_1$ in $\Gamma'\backslash\mathbb{S}_2$. We call such pair a *nonsingular pair* and denote it by Δ.

The subgroups Γ' of interest to us will be the groups $\Gamma(N)$ defined as follows:

Definition 2.3. The *principal congruence subgroup of level* N, $\Gamma(N)$, is the kernel of the map

$$\Gamma \to \mathrm{Sp}(4, \mathbb{Z}/N\mathbb{Z})$$

given by reducing matrix entries modulo $N\mathbb{Z}$.

Again this map is an epimorphism, and there is an exact sequence

$$1 \to \Gamma(N) \to \Gamma \to \mathrm{Sp}(4, \mathbb{Z}/N\mathbb{Z}) \to 1,$$

and again, as we are considering the action on $\mathbb{S}_2$, we must take the images of these lattices in $PSp(4, \mathbb{R}) = \mathrm{Sp}(4, \mathbb{R})/\pm 1$, to obtain an exact sequence

$$1 \to P\Gamma(N) \to P\Gamma \to P\mathrm{Sp}(4, \mathbb{Z}/N\mathbb{Z}) \to 1.$$

2.3. Moduli interpretations.

2.3.1. Abelian varieties with complex multiplication. We now wish to consider lattices acting on nonbounded domains $\mathbf{D}_F$ which are Cayley transforms of $\mathbb{B}_3$. Of course one can just set

$$(\Gamma_K)_F = U_\mathbf{J}(3, 1) \cap M_4(\mathscr{O}_K),$$

where $\mathbf{J} = \mathbf{THT}^*$ is a transformed form as in §2.1. This group, however, will not in general be commensurable with the lattice defined in the last section, depending on the field of definition of $\mathbf{T}$; for this reason some care must be taken in defining the domain $\mathbf{D}_F$. Let $\mathbf{J}$ be the following Hermitian form:

$$\mathbf{J} = \begin{pmatrix} \sqrt{3} & 0 & & \\ 0 & \sqrt{3} & & 0 \\ & & 0 & -i \\ & 0 & i & 0 \end{pmatrix}, \quad \mathbf{J} = -i\mathbf{R}, \ \mathbf{R} = \begin{pmatrix} \sqrt{-3} & 0 & & \\ 0 & \sqrt{-3} & & 0 \\ & & 0 & 1 \\ & 0 & -1 & 0 \end{pmatrix}.$$

Furthermore let $\mathbf{D}_F$ be the following domain:

$$\mathbf{D}_F = \{(u, v_1, v_2) \in \mathbf{C}^3 \,|\, \mathrm{Im}(u) - \frac{\sqrt{3}}{2}\sum |v_i|^2 > 0\}.$$

It is easy to see that $U_J(3, 1) \cap M_4(\mathcal{O}_K)$ is isomorphic to the Picard modular group defined above. We have introduced the domain $\mathbf{D}_F$ to be able to draw on results of Shimura giving $(\Gamma_K)_F \backslash \mathbf{D}_F$ a moduli interpretation.

To state Shimura's result we must briefly digress to explain the notion of abelian varieties with (complex) multiplication. Let K^m be the m-dimensional vector space over our field K, Φ a representation of K by $(m \times m)$ matrices with complex coefficients. We say Φ has signature (p, q) if for all $a \in K$, $\Phi(a)$ has eigenvalues

$$(\underbrace{a, \cdots, a}_{p\text{-times}}, \underbrace{\overline{a}, \cdots, \overline{a}}_{q\text{-times}}).$$

Let A be an abelian variety and recall that a polarization $\mathscr{C}$ (in the sense of Weil, i.e., an algebraic equivalence class of a hyperplane section) determines a positive involution of $\operatorname{End}_{\mathbb{Q}}(A)$, $a \mapsto a^\rho$, defined as follows. If $\operatorname{Pic}(A)$ is the Picard variety, then points of $\operatorname{Pic}(A)$ are divisors on A algebraically equivalent to 0; if X is such a divisor on A, $\operatorname{Cl}(X) \in \operatorname{Pic}(A)$ denotes the corresponding point on $\operatorname{Pic}(A)$. If $X \subset A$ is any ample divisor on A, the map

$$\varphi_X \colon A \to \operatorname{Pic}(A)$$
$$u \to \operatorname{Cl}(X_u - X), \qquad X_u = [X + u],$$

defines an isogeny, and taking in particular $X \in \mathscr{C}$ (the given polarization), a^ρ ($a \in \operatorname{End}_{\mathbb{Q}}(A)$) is defined by:

$$a^\rho := \varphi_X^{-1} a \varphi_X.$$

Definition 2.4. Let $A = \mathbb{C}^m / \Lambda$ be a polarized abelian variety (with Riemann form $E(x, y)$). A has *complex multiplication by* $\mathcal{O}_K$ if there is a representation $\Phi \colon K \to \operatorname{GL}(m, \mathbb{C})$ as above, such that $\Phi(\mathcal{O}_K) \subset \operatorname{End}(A)$, and which is compatible with the polarization, i.e.,

$$E(\Phi(a)x, y) = E(x, \Phi(a^\rho)y).$$

We also speak of complex multiplication with signature (p, q) if Φ has signature (p, q) as above.

Remark. This is a special type of complex multiplication; for the more general formulation see Shimura. We should also remark the notion of "complex multiplication in the sense of Shimura-Taniyama" is usually used to refer to the case in which K itself is a totally imaginary extension of a totally real field F, $[F : \mathbb{Q}] = m$. Such abelian varieties do not have moduli, hence this is *not* the case we are studying, although among all A's with complex multiplication as defined above, there *are* some (in fact a

dense set) which do have complex multiplication in the sense of Shimura-Taniyama.

We can now formulate Shimura's result:

Theorem 2.1 [19, Theorems 1, 2]. *Fix an abelian variety A with complex multiplication in the sense of 2.4, i.e., a triple (A, E, Φ), where Φ has signature $(3, 1)$. Then the space $(\Gamma_K)_F \backslash \mathbf{D}_F$ is the moduli space of principally polarized abelian 4-folds which are deformations of the fixed A, with complex multiplication by $\mathscr{O}_K$ (as just defined), i.e., of triples (A', E, Φ), where E and Φ are fixed, and A' has a varying complex structure.*

Remark. The skew-Hermitian form $\mathbf{R}$ above defines the Riemann form E by the formula $\mathbf{R} = (r_{ij})$,

$$E(\Phi(a)x_i, x_j) = \operatorname{tr}(ar_{ij}), \qquad x_i \text{ a } \mathbb{Z}\text{-base of } \Lambda.$$

Note that since $\Gamma_K \backslash \mathbb{B}_3 \cong (\Gamma_K)_F \backslash \mathbf{D}_F$, the same holds for $\Gamma_K \backslash \mathbb{B}_3$. Recall the subdomain $\mathbb{B}_2 \subset \mathbb{B}_3$; an application of the same result of Shimura shows that its quotient $(\Gamma_K)_2 \backslash \mathbb{B}_2$ is the moduli space of polarized abelian 3-folds (Φ has signature $(2, 1)$), hence of abelian 4 folds which *split* $A \cong A^3 \times A^1$. Note that the 1-dimensional factor A^1, an elliptic curve with complex multiplication (here in the sense of Shimura-Taniyama) has no moduli. In fact more can be said: A^1 actually has an automorphism of order 3, hence $A^1 \cong E_\rho = \mathbb{C}/\mathscr{O}_K$.

We digress again briefly to describe level structures. Let A be an abelian variety with complex multiplication by $\mathscr{O}_K$, and $n\mathscr{O}_K$ an ideal. Then the map "multipication by n" is well defined: $\varphi_n: A \xrightarrow{n} A$, and its kernel is called the "set of points of order n".

Definition 2.5. A *level n structure* on A is a choice of isomorphism $(\mathscr{O}_K/n\mathscr{O}_K)^4 \xrightarrow{\sim} \operatorname{Ker}(\varphi_n)$ respecting the natural symplectic and Hermitian forms on both sides.

Let $\Gamma_K(n) \subset \Gamma_K$ be a principal congruence subgroup, and let $Y(n)$ be the corresponding quotient. Then, as one easily sees, the inverse image of the abelian variety corresponding to $\tau \in Y(1)$ under the natural covering $Y(n) \to Y(1)$ consists of points corresponding to the same abelian variety with markings of $\operatorname{Ker}(\varphi_n)$. Hence, as a corollary of Shimura's result we immediately obtain

Corollary 2.2. *The quotient $Y(n)$ is the moduli space of abelian 4-folds with complex multiplication as above and with a level n structure.*

In particular, for $n = 2$, $\operatorname{Ker}(\varphi_2)$ is just the set of points of order 2, since $(\mathscr{O}_K/2\mathscr{O}_K)^4 \cong (\mathbb{F}_4)^4 \cong (\mathbb{Z}_2)^8$, the second isomorphism being of

the underlying additive groups. For $n = \sqrt{-3}$, $\text{Ker}(\varphi_{\sqrt{-3}})$ is a subgroup consisting of 9 of the 81 points of order 3.

2.3.2. Picard curves. The above moduli interpretation of ball quotients derived from the fact that K is an imaginary quadratic field. In this section we describe a moduli interpretation which derives from the fact that K is, in fact, *cyclotomic.* Consider the following family of trigonal curves:

$$(3) \qquad\qquad y^3 = \prod_{i=1}^{6}(x - \xi_i).$$

These curves have genus 4 and, being (Galois-) trigonal, have automorphisms of order 3 given by $(y, x) \mapsto (\rho y, x)$. Note that $\mathbb{Q}(\rho) = \mathbb{Q}(\sqrt{-3})$ $= K$; since this automorphism of the curve passes on to the Jacobian $\mathscr{J}$, $\text{Aut}(\mathscr{J}) \supset \mathbb{Z}_3$. One easily sees that there is $\Phi: K \to \text{GL}(4, \mathbb{C})$ with $\Phi(\mathscr{O}_{\mathbb{Q}(\rho)}) \subset \text{End}(\mathscr{J})$, hence the Jacobian variety of this trigonal curve has complex multiplication by K. Furthermore, $\Phi: K \to \text{GL}(4, \mathbb{C})$ is equivalent to the one in Theorem 2.1, where we take the fixed A to be $(E_\rho)^4$, and from the results of the previous section follows immediately

Corollary 2.3. *The quotient $Y(n)$ is a moduli space of (Galois-) trigonal curves (Picard curves) of genus 4 and with a level n structure. The modular subvarieties $(\Gamma_K(n))_2 \backslash \mathbb{B}_2$ parameterize those genus 4 curves which split into a component of genus 3 (a Picard curve of genus 3, i.e., one whose Jacobian has complex multiplication) and a component of genus 1 (an elliptic curve with complex multiplication) meeting at a point.*

(Note that in general $\Gamma_K(n) \backslash \mathbb{B}_3$ will have a finite number of distinct modular subvarieties.) In fact, the elliptic curve in Corollary 2.3 is actually $\mathbb{C}/\mathscr{O}_K$, the unique elliptic curve with automorphism group $\mathbb{Z}_6$, since this curve definitely occurs, but has no deformations (this is the complex multiplication in the sense of Shimura-Taniyama mentioned earlier, which makes a dense subset in moduli space but consists of isolated points). For later use we make

Observation 2.4. *A level $\sqrt{-3}$ structure corresponds to an ordering of the six points ξ_i over which the trigonal curve in (3) is branched. Hence the moduli space of ordered sets of six distinct points in $\mathbb{P}^1$ is isomorphic to a Zariski open subset of $Y(\sqrt{-3})$.*

To legitimate this claim we make the following remarks. Let p be a branch point; clearly $3p = 0$ (in the Jacobian). Hence $\sqrt{-3}p$ is a point of order $\sqrt{-3}$, i.e., $\sqrt{-3}p \in \text{Ker}(\varphi_{\sqrt{-3}})$. Recall that the fibre of $\tau \in Y(1)$ under $Y(\sqrt{-3}) \to Y(1)$ is the set of level $\sqrt{-3}$-structures; since the symmetric group Σ_6 is the Galois group (see below), it is clear that

an ordering of the six branch points determines an inverse image of τ in $Y(\sqrt{-3})$ and hence a level structure.

2.3.3. Hyperelliptic curves. The moduli interpretation of $\Gamma(N)\backslash \mathbb{S}_2$ is well known.

Theorem 2.5. *The space $X(N)$ is the moduli space of principally polarized abelian surfaces with level N structure.*

Now the generic principally polarized abelian surface is the Jacobian of a nonsingular genus 2 curve. Every such curve is hyperelliptic and so is given by the equation

$$(4) \qquad\qquad y^2 = \prod_{i=1}^{6}(x - \xi_i),$$

and since the branch points are points of order 2, we observe (see [11, §8])

Observation 2.6. *A level 2 structure on the hyperelliptic curve (4) corresponds to an ordering of the six branch points.*

We thus see

Corollary 2.7. *$X(2)$ and $Y(\sqrt{-3})$ have Zariski open sets which are isomorphic.*

Remark. The 15 copies of $\mathbb{S}_1 \times \mathbb{S}_1$-quotients on $X(2)$ correspond to curves (4) for which 3 of the $\{\xi_i\}$ coincide; these are curves which split into two elliptic curves, with corresponding Jacobians which are products $E_1 \times E_2$. On the Zariski open set in $X(2)$ which is the complement of the union of these divisors all $\{\xi_i\}$ are distinct.

2.4. Groups and complexes.

We have so far proceeded in parallel for Γ and Γ_K. In this subsection we present some isomorphisms between the two situations which form part of the reason for the isomorphisms given by our main theorem.

2.4.1. Groups. For $N \in \mathbb{Z}$, we have $P\Gamma/P\Gamma(N) \cong \mathrm{P}\,\mathrm{Sp}(4, \mathbb{Z}/N\mathbb{Z})$, and for $n \in \mathscr{O}_K$, we have

$$P\Gamma_K/P\Gamma_K(n) \cong \mathrm{PU}(3, 1; \mathscr{O}_K/n\mathscr{O}_K).$$

We observe that $\mathscr{O}_K/n\mathscr{O}_K$ is a finite ring of cardinality the norm of n, and if n is a prime it is a field (as K is a principal ideal domain). This is in particular the case if $n = \sqrt{-3}$, of norm 3, and if $n = 2$, of norm 4. In the first case the conjugation on $\mathscr{O}_K$ descends to the trivial automorphism of $\mathbb{F}_3$ (of course), in the second to the unique nontrivial automorphism of $\mathbb{F}_4$ (as may easily be checked).

Proposition 2.8. (i) $\mathrm{PSp}(4, \mathbb{Z}/2\mathbb{Z}) \cong \mathrm{PU}(3, 1; \mathbb{F}_3) \cong \Sigma_6$, *the symmetric group on six elements.*

(ii) $\mathrm{PSp}(4, \mathbb{Z}/3\mathbb{Z}) \cong \mathrm{PU}(3, 1; \mathbb{F}_4) \cong G_{25,920}$, *the simple group of order 25,920.*

(iii) $\mathrm{PSp}(4, \mathbb{Z}/6\mathbb{Z}) \cong \mathrm{PU}(3, 1; \mathbb{F}_3 \times \mathbb{F}_4) \cong \Sigma_6 \times G_{25,920}$.

Proof. For (i) and (ii) see [3, pp. 4, 26]; these immediately imply (iii). q.e.d.

Here we must remark that our notation $\mathrm{PU}(3, 1; \mathscr{O}_K/n\mathscr{O}_K)$ is non-standard. We have choosen this notation as this group is a quotient of $\mathrm{PU}(3, 1; \mathscr{O}_K)$. However, we see that for $n = \sqrt{-3}$, $U(3, 1; \mathscr{O}_K/n\mathscr{O}_K)$ is the group preserving the form $x_1^2 + x_2^2 + x_3^2 - x_4^2$ (conjugation being trivial here), and this group is denoted $GO_4^-(3)$ in [3, p. 4]. Similarly, for $n = 2$, $U(3, 1; \mathscr{O}_K/n\mathscr{O}_K)$ is the group preserving the form $x_1\overline{x}_1 + x_2\overline{x}_2 + x_3\overline{x}_3 + x_4\overline{x}_4$ (as $1 = -1$ here), and this group is denoted $GU_4(2)$ in [3, p. 26].

The groups $P\Gamma/P\Gamma(N)$ (respectively $P\Gamma_K/P\Gamma_K(n)$) are the Galois groups of the ramified covers $X(N) \to X(1)$ (respectively $Y(n) \to Y(1)$), which in fact extend to $\overline{X(N)} \to \overline{X(1)}$ (respectively $\overline{Y(n)} \to \overline{Y(1)}$), and the reader will note that the groups in parts (i), (ii), and (iii) of the above proposition are the Galois groups of the covers in parts (ii), (iii), and (iv) of our main theorem.

2.4.2. Complexes. We have seen that equivalence classes of subspaces of K^4 (respectively $\mathbb{Q}^4$) correspond to cusps of modular subvarieties for the action of Γ_K on $\mathbb{B}_3$ (respectively of Γ on $\mathbb{S}_2$). Let us formalize and generalize this notion.

Let $V = K^4$ (respectively $\mathbb{Q}^4$) and Λ be a subgroup of Γ_K (respectively of Γ) of finite index. We consider certain non-trivial proper subspaces of V and put a partial order on them as follows:

  (i) Siegel case: $\{l, h, \Delta\}$ with $l < h$ if $l \subset h$, and $l < \Delta = \{\delta, \delta^\perp\}$ if $l \subset \delta$ or $l \subset \delta^\perp$.

  (ii) Picard case: $\{l, \Delta_2, \Delta_3\}$ with $\Delta_2 < \Delta_3$ if $\Delta_2 \subset \Delta_3$, and $l < \Delta_3$ if $l \subset \Delta_3$.

Note that the action of Λ on V induces an action of Λ on the above sets, preserving the partial order.

Definition 2.6. The simplicial complex $\mathscr{T}_+(V)$ is the simplicial realization of the partially ordered set above. The simplicial complex $\mathscr{T}_+(\Lambda)$ is the quotient $\Lambda \backslash \mathscr{T}_+(V)$.

Observe that $\mathscr{T}_+(V)$ contains the full subcomplex $\mathscr{T}(V)$ whose vertices are the isotropic subspaces of V, and the full subcomplex $\mathscr{S}(V)$ whose vertices are the nonsingular (with respect to the given form) subspaces of V. The reader familiar with Tits buildings will recognize $\mathscr{T}(V)$ as the Tits building of the $\mathbb{Q}$-group Γ_K; we call $\mathscr{S}(V)$ the *scaffolding* and the complex $\mathscr{T}_+(V)$ the *Tits building with scaffolding* for Γ_K. Simplices of

$\mathscr{T}_+(V)$ not lying in $\mathscr{T}(V) \cup \mathscr{S}(V)$ we call *cross-simplices*. We employ similar terminology for $\mathscr{T}_+(\Lambda)$. We see here that in the Siegel (Picard) cases, $\mathscr{T}(V)$ is 1- (0-) dimensional and $\mathscr{S}(V)$ is 0- (1-)dimensional, while in both cases the cross-simplices are 1-dimensional.

2.4.3. Duality.

Definition 2.7. For two groups Λ_1 and Λ_2, $\mathscr{T}_+(\Lambda_1)$ and $\mathscr{T}_+(\Lambda_2)$ are *dual* if there is a simplicial isomorphism $\alpha\colon \mathscr{T}_+(\Lambda_1) \to \mathscr{T}_+(\Lambda_2)$ with $\alpha(\mathscr{T}(\Lambda_1)) = \mathscr{S}(\Lambda_2)$ and $\alpha(\mathscr{S}(\Lambda_1)) = \mathscr{T}(\Lambda_2)$.

We will see that we have examples of dual Tits buildings with scaffoldings here, and this is part of the reason for our Janus-like behavior. Note from what we have said in §2.2 that the vertices of $\mathscr{T}(V)$ (respectively $\mathscr{S}(V)$) parameterize cusps or (as we shall see in §3.2) boundary components (respectively modular subvarieties) of the compactification $\overline{(\Lambda\backslash\mathscr{D})}$, and the edges parameterize inclusions of components/subvarieties or (for the cross-edges) intersections of modular and boundary divisors. First we count these:

Lemma 2.9. *The following table is correct*:

N	n	$x_b = y_m$	$y_b = x_m$
1	1	1	1
2	$\sqrt{-3}$	15	10
3	2	40	45
6	$2\sqrt{-3}$	600	450

In this table x_b (respectively y_b) denotes the number of vertices in $\mathscr{T}_+(\Gamma(N))$ (respectively $\mathscr{T}_+(\Gamma_K(n)))$ corresponding to isotropic lines, and x_m (respectively y_m) the number of vertices in $\mathscr{T}_+(\Gamma(N))$ (respectively $\mathscr{T}_+(\Gamma_K(n)))$ corresponding to maximal nonsingular pairs (subspaces).

Proof. We show that the values of $x_b(N)$ and $y_b(n)$ (in the obvious notation) are correct; the values for $x_m(N)$ and $y_m(n)$ follow by similar arguments.

$x_b(1) = 1$: Note that every line in $\mathbb{Q}^4$ is isotropic, as we have a symplectic form here. Each such is generated by a primitive vector in $\mathbb{Z}^4$, well-defined up to multiplication by ± 1 (the group of units in $\mathbb{Z}$). Thus this claim is given by the well-known fact that $\mathrm{Sp}(4, \mathbb{Z})$ operates transitively on those vectors.

$x_b(2) = 15$: There are 15 nonzero elements in $(\mathbb{Z}/2\mathbb{Z})^4$, and ± 1 acts trivially.

$x_b(3) = 40$: There are 80 nonzero elements in $(\mathbb{Z}/3\mathbb{Z})^4$, and ± 1 acts effectively.

$y_b(1) = 1$: This is proved in [25, p. 50].

$y_b(\sqrt{-3}) = 10$: We may take $\mathbb{F}_3 = \{0, 1, -1\}$; note that the units of $\mathscr{O}_K$ act transitively on the nonzero elements of $\mathbb{F}_3$, and conjugation is trivial. We are looking for (x_1, x_2, x_3, x_4) with $x_1^2 + x_2^2 + x_3^2 - x_4^2 = 0$. If $x_4 = 0$, x_1, x_2, x_3 may be choosen arbitrarily, not all zero, giving eight possibilities, and if $x_4 = \pm 1$, exactly one of x_1, x_2, and x_3 must be nonzero, giving six choices, and $(8 + 2 \cdot 6)/2 = 10$. Then $y_b(\sqrt{-3}) = 10 \cdot y_b(1) = 10$.

$y_b(2) = 45$: We may take $\mathbb{F}_4 = \{0, 1, \rho, \bar{\rho}\}$; note that the units of $\mathscr{O}_K$ act transitively on the nonzero elements and conjugation is nontrivial. We are looking for $(x_1, \cdots, x_4)$ with $x_1\bar{x}_1 + x_2\bar{x}_2 + x_3\bar{x}_3 - x_4\bar{x}_4 = 0$. If $x_4 = 0$, exactly one of x_1, x_2, x_3 must be zero and the others arbitrary nonzero elements, giving 27 possibilities, and if $x_4 = 1$, ρ, or $\bar{\rho}$, either exactly one of x_1, x_2, and x_3 is nonzero, and may be arbitrary, giving nine choices, or all of them are nonzero and arbitrary, giving 27 choices; $(27 + 3(9 + 27))/3 = 45$, and $y_b(2) = 45 \cdot y_b(1) = 45$.

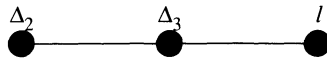Clearly $x_b(6) = x_b(2)x_b(3)$ and $y_b(2\sqrt{-3}) = y_b(2)y_b(\sqrt{-3})$. This completes the proof.

Theorem 2.10.   *The following Tits buildings with scaffoldings are dual:*

   (i) $\mathscr{T}_+(\Gamma(1))$ *and* $\mathscr{T}_+(\Gamma_K(1))$.

   (ii) $\mathscr{T}_+(\Gamma(2))$ *and* $\mathscr{T}_+(\Gamma_K(\sqrt{-3}))$.

   (iii) $\mathscr{T}_+(\Gamma(4))$ *and* $\mathscr{T}_+(\Gamma_K(2))$.

   (iv) $\mathscr{T}_+(\Gamma(6))$ *and* $\mathscr{T}_+(\Gamma_K(2\sqrt{-3}))$.

Proof.   (i) It is easy to check from Lemma 2.9 that for the Siegel case we have the Tits building with scaffolding:
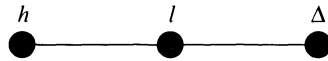
and in the Picard case

where in the Siegel (resp. Picard) case the full subcomplex containing the leftmost two vertices (resp. rightmost vertex) is the building and the full subcomplex containing the rightmost vertex (resp. leftmost two vertices) is the scaffolding. Putting these together we have the diagram:
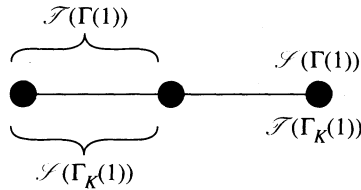
Then the simplicial isomorphism α from $\mathscr{T}_+(\Gamma(1))$ to $\mathscr{T}_+(\Gamma_K(1))$ which is determined by $\alpha(h) = \Delta_2$, $\alpha(l) = \Delta_3$, $\alpha(\Delta) = l$ is obviously a duality, giving (i).

Let a group G act on a set T. Then for $t \in T$ we let $P(t)$ be the stabilizer of t, i.e.,

$$P(t) = \{g \in G | g(t) = t\}.$$

We have just established that $\Gamma/\Gamma(N) = \Gamma_K/\Gamma_K(n)$ for $(N,n) = (2,\sqrt{-3})$, $(3,2)$, $(6,2\sqrt{-3})$. Let G be this common group. We concentrate on the first two cases, those of parts (ii) and (iii) respectively; part (iv) then follows immediately.

Let π be a vertex of $\mathscr{T}_+(\Gamma(N))$ and π_* a vertex of $\mathscr{T}_+(\Gamma_K(n))$. Then as G acts on each of these, we have correspondences $\pi \leftrightarrow P(\pi)$ and $\pi_* \leftrightarrow P(\pi_*)$. We will show that for proper choices of π and π_* the subgroups $P(\pi)$ and $P(\pi_*)$ are equal, thus giving a 1-1 map from the simplices of $\mathscr{T}_+(\Gamma(N))$ to the simplices of $\mathscr{T}_+(\Gamma_K(n))$ by $\pi \to P(\pi) = P(\pi_*) \to \pi_*$, which respects the incidence relations, in other words is a simplicial isomorphism. We have for the vertices in $\mathscr{T}_+(\Gamma(N))$ and $\mathscr{T}_+(\Gamma_K(n))$:

(5) $P(l) = P(\Delta_3)$,

(6) $P(h) = P(\Delta_2)$,

(7) $P(\Delta) = P(l)$.

Assuming this for the moment we then also have for the edges (l,h)

$$l < h \leftrightarrow P(l,h) = P(l) \cap P(h)$$
$$= P(\Delta_3) \cap P(\Delta_2) \leftrightarrow P(\Delta_3, \Delta_2) \leftrightarrow (\Delta_3, \Delta_2)$$

with $\Delta_2 < \Delta_3$, and for the edges (l,Δ)

$$l < \Delta \leftrightarrow P(l,\Delta) = P(l) \cap P(\Delta)$$
$$= P(\Delta_3) \cap P(l) \leftrightarrow P(\Delta_3, l) \leftrightarrow (\Delta_3, l)$$

with $l < \Delta_3$, as required. Note also that this isomorphism maps $\mathscr{T}(\Gamma(N))$ to $\mathscr{S}(\Gamma_K(n))$ and $\mathscr{S}(\Gamma(N))$ to $\mathscr{T}(\Gamma_K(n))$, so is a duality.

It remains to establish the equalities in (5)–(7) above. For this we may refer to the descriptions of these subgroups given in [3]. We do $N = 3$, $n = 2$, the complicated case; the case $N = 2$, $n = \sqrt{-3}$ is similar. First we note that [3] works projectively, and we do not, so that what is here called an isotropic plane (respectively line) is there called an isotropic line (respectively point). We refer henceforth to [3, p. 26].

First consider (5). Then, using our language, an isotropic line l (or simply a line, all lines being isotropic with respect to a symplectic form), corresponds to a nonisotropic line. But as we have observed, a nonisotropic line determines, and is determined by, its orthogonal complement Δ_3.

Next consider (6). Then an isotropic plane h corresponds to a base. But this is just a 2-dimensional subspace on which the form is $x_1\bar{x}_1 + x_2\bar{x}_2$, so its orthogonal complement is Δ_2.

Finally consider (7). Then we see "$S_2(3)wr2$" corresponds to an isotropic line l. Thus to complete the proof we must show that this subgroup is the stabilizer of a nonsingular pair Δ. But $S_2(3)wr2$ is the subgroup of $\mathrm{Sp}_4(3)$ ($\mathrm{Sp}(4, \mathbb{F}_3)$ in our notation) given by

$$\left\{ \begin{pmatrix} a_1 & & b_1 & \\ & a_2 & & b_2 \\ c_1 & & d_1 & \\ & c_2 & & d_2 \end{pmatrix} \middle| \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix} \in \mathrm{Sp}(2, \mathbb{F}_3), i = 1, 2 \right\} \rtimes \Sigma_2$$

where $\rtimes$ denotes semidirect product, and $\Sigma_2 = \mathbb{Z}_2$ is generated by the matrix

$$\begin{pmatrix} 0 & 1 & & \\ -1 & 0 & & \\ & & 0 & 1 \\ & & -1 & 0 \end{pmatrix} ;$$

it is easy to check that this subgroup is the stabilizer of the nonsingular pair Δ given by

$$\{(1, 0, 0, 0) \wedge (0, 0, 1, 0), (0, 1, 0, 0) \wedge (0, 0, 0, 1)\}.$$

(The above group is a subgroup of $\mathrm{Sp}(4, \mathbb{F}_3)$; we take its image in $\mathrm{PSp}(4, \mathbb{F}_3)$.)

Theorem 2.11. *The following table is correct:*

N	n	$v(t) = v_K(s)$	$e(t) = e_K(s)$	$v(s) = v_K(t)$	$e(s) = e_K(t)$	$e(c) = e_K(c)$
1	1	2	1	1	0	1
2	$\sqrt{-3}$	30	45	10	0	60
3	2	80	160	45	0	360
6	$2\sqrt{-3}$	1200	7200	450	0	21,600

Here $v(t)$, $e(t)$, $v(s)$, $e(s)$, *and* $e(c)$ *denote the number of vertices and edges in* $\mathscr{T}(\Gamma(N))$, *the number of vertices and edges in* $\mathscr{S}(\Gamma(N))$ *and the number of cross-edges in* $\mathscr{T}_+(\Gamma(N))$, *and similarly for* $\mathscr{T}_+(\Gamma_K(n))$.

Proof. Again we concentrate on the second and third rows of the table. It is easy to calculate that each line l is contained in $\mu = \mu(N)$ isotropic planes h and each plane (isotropic or nonsingular) contains the same number $\mu(N)$ of lines, where $\mu(2) = 2^2 - 1 = 3$ and $\mu(3) = (3^2 - 1)/2 = 4$. (Since $\mathrm{Sp}(4, \mathbb{Z})$ operates transitively on $\{l\}$, it suffices to check this for any given l; taking $l = \pm(1, 0, 0, 0)$ makes the calculation simple.) Hence there are the same number of l-vertices and h-vertices, this number being x_b, so $v(t) = 2x_b$, and $e(t) = \mu x_b$. Also, $v(s) = x_m$ and $e(c) = 2\mu x_m$ (the factor of 2 arising as $\Delta = \{\delta, \delta^\perp\}$ contains two planes). The values for the bottom row are then consequences of Lemma 2.9.

3. Compactifications: Minimal and smooth compactifications

For a Hermitian symmetric space $\mathscr{D}$ of noncompact type and an arithmetic group Γ acting freely on it there is a natural (singular) compactification $\Gamma\backslash\mathscr{D}^*$, the Satake compactification. A fundamental result is the theorem of Baily-Borel, that $\Gamma\backslash\mathscr{D}^*$ can be realized as a normal algebraic variety by means of modular forms. $\Gamma\backslash\mathscr{D}^*$ can be desingularized by means of toroidal embeddings, or, equivalently, one can find a smooth compactification $\overline{\Gamma\backslash\mathscr{D}}$ which dominates $\Gamma\backslash\mathscr{D}^*$. These compactifications depend on a choice of combinatorial nature (polyhedral cone decompositions), and it is a fundamental result of [1] that one can find such cone decompositions such that (a) $\overline{\Gamma\backslash\mathscr{D}}$ is smooth and projective, and (b) $(\overline{\Gamma\backslash\mathscr{D}}) - (\Gamma\backslash\mathscr{D})$ is a divisor with normal crossings. If the action of Γ is not free, the same methods and results apply, with the exception that in this case $\Gamma\backslash\mathscr{D}$ and $\overline{\Gamma\backslash\mathscr{D}}$ may only be almost smooth, i.e., may have finite quotient singularities. In this section we describe the compactifications $\Gamma\backslash\mathscr{D}^*$ and $\overline{\Gamma\backslash\mathscr{D}}$ in our cases. We will assume the reader is familar with the general theory.

3.1. Curves.

3.1.1. Siegel case.

First we consider curves in the Siegel setting. Let E_ρ be the elliptic curve which is the quotient of $\mathbb{C}$ by the lattice generated by 1 and ρ. Note that this lattice is precisely $\mathcal{O}_K$, so we have $E_\rho = \mathbb{C}/\mathcal{O}_K$.

Lemma 3.1. *Let* $\Gamma(6)$ *be the principal congruence subgroup of level* 6 *in* $\mathrm{SL}(2,\mathbb{Z})$. *Then* $(\overline{\Gamma(6)\backslash\mathbb{S}_1})$ *is isomorphic to* E_ρ.

Proof. Observe E_ρ has an automorphism of order 6 given by multiplication by $-\rho$. It is easy to check that $(\overline{\Gamma(6)\backslash\mathbb{S}_1})$ is a curve of genus 1 (with $\nu(6) = 12$ cusps) and has an automorphism of order 6 given by the action of the element $\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$ of $\mathrm{SL}(2,\mathbb{Z})$. Since, as is well known, there is a *unique* elliptic curve with such an automorphism, these two must be identical.

Remark. Since the elliptic curve E_ρ is known to be the plane cubic $y^2 = x^3 - 1$, we have shown that this cubic is Janus-like.

Corollary 3.2. *The curves* $E_\rho/\mathbb{Z}_3$ *and* $(\overline{\Gamma(2)\backslash\mathbb{S}_1})$ *(respectively* $E_\rho/\mathbb{Z}_2$ *and* $(\overline{\Gamma(3)\backslash\mathbb{S}_1})$, *and* $E_\rho/\mathbb{Z}_6$ *and* $(\overline{\Gamma(1)\backslash\mathbb{S}_1})$) *are isomorphic. This curve is* $\mathbb{P}^1$ *with* $\nu(N)$ *cusps, where* $\nu(1) = 1$, $\nu(2) = 3$, $\nu(3) = 4$.

Proof. The curve $(\overline{\Gamma(6)\backslash\mathbb{S}_1})$ has the following quotients (see Figure 2). The Galois groups of the coverings $p(N)$, $N = 1, 2, 3$, are $P\Gamma/P\Gamma(6)$, $P\Gamma(2)/P\Gamma(6)$, and $P\Gamma(3)/P\Gamma(6)$, respectively (recall that the action of $\mathrm{SL}(2,\mathbb{Z})$ factors through $\mathrm{PSL}(2,\mathbb{Z})$), which are groups of order 72, 12, and 6, respectively. We now describe the actions explicitly.

$p(3)$. $P\Gamma(3)/P\Gamma(6)$ acts by first translating by three of the nine 3-division points (of course those "contained in" $\mathcal{O}_K$, i.e., yielding the automorphism of order 3 on E_ρ). This translation yields an isogeny of degree 3: $i_3 \colon E_\rho \to E_\rho/(\mathbb{Z}_3)$; the quotient $E_\rho/(\mathbb{Z}_3)$ is again isomorphic to E_ρ, and so composing with this isomorphism yields an isogeny $i_3 \colon E_\rho \to E_\rho$. Secondly, one divides by the involution (or order 2) of E_ρ, yielding a 2:1 map $\pi_2 \colon E_\rho \to \mathbb{P}^1$, branched at the 2-division points of E_ρ. One then sees easily that these four branch points on $\mathbb{P}^1$ are acted on by the tetrahedral group, i.e., without loss of generality are $(0, 1, \rho, \overline{\rho}) \in \mathbb{P}^1$. Hence $p(3) = \pi_2 \circ i_3$ (see Figure 2, next page).

$p(2)$. $P\Gamma(2)/P\Gamma(6)$ acts by translating by the subgroup consisting of the 2-division points (all of them). This translation yields an isogeny of degree 4:

$$i_4 \colon E_\rho \to E_\rho/(\mathbb{Z}_2 \oplus \mathbb{Z}_2)),$$

and again the quotient is isomorphic to E_ρ, so composing with this
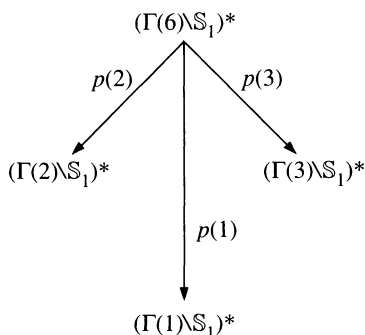
FIGURE 2. QUOTIENTS OF E_ρ

isomorphism yields an isogeny: $i_4\colon E_\rho \to E_\rho$. Secondly, one divides by the action of the automorphism of order 3, $\pi_3\colon E_\rho \to \mathbb{P}^1$, branched now at the three 3-division points used to translate in the step above. Since any three points on $\mathbb{P}^1$ are equivalent, these branch points on $\mathbb{P}^1$ can be taken to be $(0, 1, \infty)$ (note that E_ρ is the *unique* elliptic curve which is a 3:1 cover of $\mathbb{P}^1$ with Galois group $\mathbb{Z}_3$, hence no choices are involved). Then $p(2) = \pi_3 \circ i_4$.

$p(1)$: Here we combine the two situations above and have an isogeny of degree 12: $i_{12} = i_3 \cdot i_4\colon E_\rho \to E_\rho$, then divide by the automorphism group $\mathbb{Z}_6$, $\pi_6\colon E_\rho \to \mathbb{P}^1$, and $p(1) = \pi_6 \circ i_{12}$. For this cover we have the following branch behavior (see Figure 3) where 0 is the image of 0 on the elliptic curve, **a** is the image of the three 3-division points and **b** is the image of the 2-division points. The corollary follows from these descriptions.
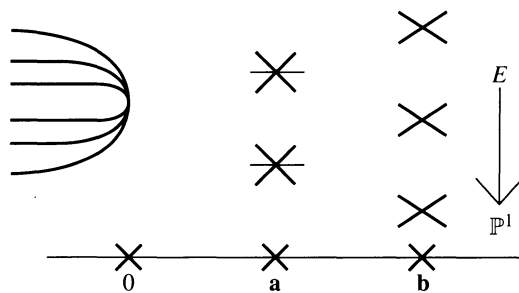

FIGURE 3. THE BRANCH LOCUS

3.1.2. Picard case. Next, we want to consider curves which are quotients of $\mathbb{B}_1$ by lattices in $PU(1,1)$. Let $\Gamma_K = PU(1,1;\mathscr{O}_K)$, $\Gamma_K(n) =$ the principal congruence subgroup of level n in Γ_K. Since $PU(1,1) \cong PSL(2,\mathbb{R})$, Γ_K and $\Gamma_K(n)$ may be identified with lattices in $PSL(2,\mathbb{R})$. This works as follows (see [24, II. 4.13]). An explicit isomorphism is given by:

$$\Psi \colon \mathbb{B}_1 \to \mathbb{S}_1$$

$$\begin{bmatrix} z \\ 1 \end{bmatrix} \mapsto \begin{pmatrix} -1 & 1 \\ \sqrt{3} & \sqrt{3} \end{pmatrix} \begin{bmatrix} z \\ 1 \end{bmatrix} = \left[\frac{-z+1}{\sqrt{3}(z+1)}, 1 \right].$$

The cusps of $\mathbb{B}_1$ are points of the form $[(a+b\sqrt{-3})/(a-b\sqrt{-3}), 1]$ and are mapped via Ψ onto the usual cusps $b/a \in \mathbb{Q} \cup \{\infty\}$ of $\mathbb{S}_1$. Ψ induces a group isomorphism

$$\Psi^* \colon SU(1,1) \to SL(2,\mathbb{R})$$

$$g \mapsto AgA^{-1}, \qquad A = \begin{pmatrix} -1 & 1 \\ \sqrt{3} & \sqrt{3} \end{pmatrix}$$

as above. Then the image of Γ_K is:

$$\widetilde{\Gamma}_3 := \left\{ \begin{pmatrix} \alpha & \beta \\ 3\gamma & \delta \end{pmatrix} \in SL(2,\mathbb{Z}) \,|\, \alpha \equiv \delta(2), \, \beta \equiv \gamma(2) \right\},$$

which is a subgroup of the congruence group

$$\Gamma_0(3) = \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL(2,\mathbb{Z}) \,|\, \gamma \equiv 0(3) \right\},$$

of index 3. Zeltinger shows that $\widetilde{\Gamma}_3$, and so Γ_K, has only one cusp (as does $\Gamma_0(3)$). Let us now consider $\Gamma_K(n)$, $n = \sqrt{-3}$, 2, and $2\sqrt{-3}$. To find the number of $\Gamma_K(n)$-cusps one has to calculate the number of isotropic vectors in $(\mathbb{F}_3)^2$, $(\mathbb{F}_4)^2$, and $(\mathbb{F}_3 \times \mathbb{F}_4)^2$ respectively. The forms will be $x_1^2 - x_2^2$ for $\mathbb{F}_3$ (since conjugation on $\mathbb{F}_3$ is trivial), $x_1\bar{x}_1 + x_2\bar{x}_2$ over $\mathbb{F}_4$ (since $1 = -1$ over $\mathbb{F}_4$), and $x_1\bar{x}_1 - x_2\bar{x}_2$ over $(\mathbb{F}_3 \times \mathbb{F}_4)^2$. Obviously the number of cusps of $\Gamma_K(2\sqrt{-3})$ will be the product of those for $\Gamma_K(\sqrt{-3})$ and $\Gamma_K(2)$, so we only need to calculate those. One sees easily:

isotropic lines $\subset (\mathbb{F}_3)^2$: $\{(1,1), (-1,1)\}$ $(\mathbb{F}_3 = (0,1,-1))$,

isotropic lines $\subset (\mathbb{F}_4)^2$: $\{(1,x), (1,1), (1,\bar{x})\}$ $(\mathbb{F}_4 = (0,1,x,\bar{x}))$,

hence we have two cusps for $\Gamma_K(\sqrt{-3})$, three cusps for $\Gamma_K(2)$ and six cusps for $\Gamma_K(2\sqrt{-3})$. Furthermore, calculations with the Euler numbers together with the fact that $(\Gamma_K \backslash \mathbb{B}_1)^* \cong \mathbb{P}^1$ shows that all curves involved are rational.

Lemma 3.3. (i) *The six cusps of* $(\Gamma_K(2\sqrt{-3})\backslash\mathbb{B}_1)^* \cong \mathbb{P}^1$ *are the sixth roots of unity on* $\mathbb{P}^1$.

(ii) *The three cusps of* $(\Gamma_K(2)\backslash\mathbb{B}_1)^* \cong \mathbb{P}^1$ *are the third roots of unity on* $\mathbb{P}^1$.

(iii) *The two cusps of* $(\Gamma_K(\sqrt{-3})\backslash\mathbb{B}_1)^* \cong \mathbb{P}^1$ *are* ± 1 *on* $\mathbb{P}^1$.

Proof. Since $(\Gamma_K\backslash\mathbb{B}_1)^*$ has only one cusp, we may without restricting generality, take it to be $1 \in \mathbb{P}^1$. Now note that the covers are given as:

$$(\Gamma_K(\sqrt{-3})\backslash\mathbb{B}_1)^* \to (\Gamma_K\backslash\mathbb{B}_1)^*,$$
$$z \mapsto z^2,$$
$$(\Gamma_K(2)\backslash\mathbb{B}_1)^* \to (\Gamma_K\backslash\mathbb{B}_1)^*,$$
$$z \mapsto z^3,$$
$$(\Gamma_K(2\sqrt{-3})\backslash\mathbb{B}_1)^* \to (\Gamma_K(\sqrt{-3})\backslash\mathbb{B}_1)^*,$$
$$z \mapsto z^3,$$
$$(\Gamma_K(2\sqrt{-3})\backslash\mathbb{B}_1)^* \to (\Gamma_K\backslash\mathbb{B}_1)^*,$$
$$z \mapsto z^6,$$

which are all branched at 0 and infinity. Since cusps map to cusps, the lemma follows from this.

3.2. Surfaces.

3.2.1. Siegel case.

Lemma 3.4. (i) $\overline{X}_m(6) := \overline{(\Gamma(6) \times \Gamma(6)\backslash(\mathbb{S}_1 \times \mathbb{S}_1))} = E_\rho \times E_\rho$.

(ii) $\overline{X}_m(3) := \overline{(\Gamma(3) \times \Gamma(3)\backslash(\mathbb{S}_1 \times \mathbb{S}_1))} = \mathbb{P}^1 \times \mathbb{P}^1$.

(iii) $\overline{X}_m(2) := \overline{(\Gamma(2) \times \Gamma(2)\backslash(\mathbb{S}_1 \times \mathbb{S}_1))} = \mathbb{P}^1 \times \mathbb{P}^1$.

Each of these (product) surfaces has a configuration of curves in which each curve is either $\{cusp\} \times C$ *or* $C \times \{cusp\}$, $C = E_\rho$ *in the first case,* $\mathbb{P}^1$ *in the other two, and in which there are a total of* 24 *(respectively* 8, 6*) curves.*

Proof. Obvious.

Now we consider the domain $\mathbb{C} \times \mathbb{S}_1$. This has as automorphism group $\mathrm{SL}(2, \mathbb{R}) \ltimes \mathbb{R}^2$. Concretely, this is the matrix group

$$\left\{ \begin{pmatrix} 1 & m & n \\ 0 & a & b \\ 0 & c & d \end{pmatrix} \middle| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{R}), (m, n) \in \mathbb{R}^2 \right\},$$

and an element as above acts on a point (w, z) of $\mathbb{C} \times \mathbb{S}_1$ by

$$(w, z) \mapsto \left(\frac{w + mz + n}{cz + d}, \frac{az + b}{cz + d} \right).$$
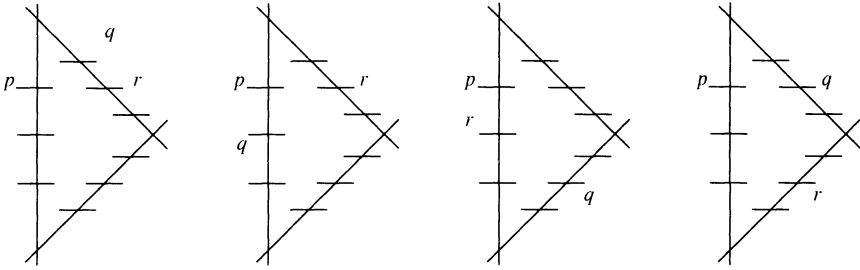
FIGURE 4. THE ELLIPTIC MODULAR SURFACE OF LEVEL 3.

Lemma 3.5. *For* $N = 6$, 3 *(respectively* 2*),* $\overline{X}_b(N) := \overline{(\Gamma(N)\backslash(\mathbb{C} \times \mathbb{S}_1))}$ *is the elliptic modular surface (respectively Kummer modular surface) of level* N*, a holomorphic fibre space over* $\overline{(\Gamma(N)\backslash\mathbb{S}_1)}$ *with general fibre an elliptic curve (respectively* $\mathbb{P}^1$*), and with exceptional fibres of type* I_N *(respectively the union of* 2 *copies of* $\mathbb{P}^1$ *meeting at a point) over the cusps.*

Proof. This is well known; see for example Shioda [21] or Livne [14].

Lemma 3.6. *These surfaces* $\overline{X}_b(N)$ *have the following configurations of curves:*

(i) *A configuration of* N $\mathbb{P}^1$ *'s forming for* $N > 2$ *(resp.* $N = 2$*) an* N-gon *(resp. two* $\mathbb{P}^1$ *'s meeting at a point) lying over each of the cusps and*

(ii) N^2 *sections of the projection*

$$\Gamma(N)\backslash(\mathbb{C} \times \mathbb{S}_1) \to \Gamma(N)\backslash\mathbb{S}_1$$

which extend over the cusps,

each of the $\mathbb{P}^1$ *'s in each cusp being intersected by* N *of the sections.*

In case $N = 3$ the labels p, q, and r each denote a section in the schematic of the configuration depicted in Figure 4, there being nine sections and four cusps in this case.

In case $N = 2$, the entire configuration is as in Figure 5 (next page) (where of course the sections do not intersect). For $N = 3$ (respectively 2) the curves in the exceptional fibres (which are $\mathbb{P}^1$'s) have self-intersection -2 (respectively -1), while in both cases the sections (which are $\mathbb{P}^1$'s) have self-intersection -1.

Proof. This is well known; see, for example, [21], and [10] for $N = 2$.

Proposition 3.7. (i) *The surface* $\overline{X}_b(2)$ *is isomorphic to* $\mathbb{P}^2$ *blown up at four points.*

(ii) *The surface* $\overline{X}_b(3)$ *is isomorphic to* $\mathbb{P}^2$ *blown up at the nine inflection points of a smooth cubic.*
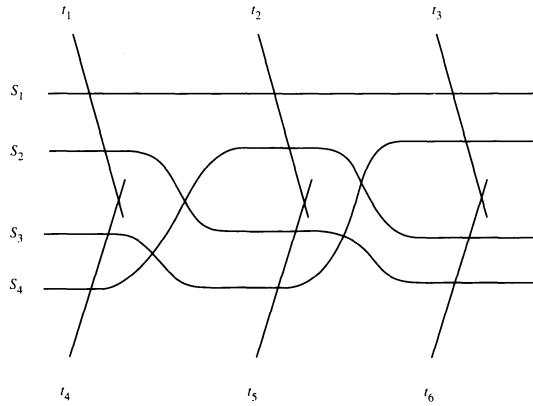
FIGURE 5. THE KUMMER MODULAR SURFACE

Proof. (i) Blowing down the four sections (which are disjoint (-1)-curves) we see that the proper transform of each fibre has self-intersection 4 (as a quadric in $\mathbb{P}^2$), and the proper transforms of the six components of the singular fibres have self-intersection 1 (as a line in $\mathbb{P}^2$). Conversely, start with $\mathbb{P}^4$, four points in it, take the pencil of all quadrics through the four points; blowing up the base locus the result is $\overline{X}_b(2)$, with fibre structure onto $\mathbb{P}^1$ given by the pencil. The proper transforms of the six lines in $\mathbb{P}_2$ joining pairs of the four points are the components of the singular fibres.

(ii) It is well known how to get $\overline{X}_b(3)$ geometrically. Consider the Hesse pencil of cubics in $\mathbb{P}^2$; this pencil has nine base points and four singular cubics which degenerate into the union of three lines. Blowing up the base points fibres the family of elliptic curves over $\mathbb{P}^1$ with four singular fibres of type I_3. The nine exceptional $\mathbb{P}^1$'s are the nine sections of $\overline{X}_b(3)$. Note there is a natural action of the group $SL(2, \mathbb{Z}/3) \times (\mathbb{Z}/3)^2$ on $\overline{X}_b(3)$; this group has order $24 \cdot 9 = 216$. This action is actually effective (although $-1 \in SL(2, \mathbb{Z}/3)$ acts trivially on the base curve, this element induces a *reflection* on the sections, as opposed to the translations).   q.e.d.

It is known that the arrangement of the 12 lines (four triangles) of the Hesse pencil is defined by a unitary reflection group of order 216; this is of course the same group.

3.2.2. Picard case. Now we consider the surfaces that arise in the Picard case. Let $\overline{Y}_b(2\sqrt{-3})$ be $(\mathbb{C}/\mathscr{O}_K) \times (\mathbb{C}/\mathscr{O}_K)$. Let $\overline{Y}_b(2)$ be the quotient of $\overline{Y}_b(2\sqrt{-3})$ by $\mathbb{Z}_2 \times \mathbb{Z}_2$, where the generator of each copy of $\mathbb{Z}_2$ acts by multiplication by -1 on the corresponding copy of $\mathbb{C}$. Let

$\overline{Y}_b(\sqrt{-3})$ be the quotient of $\overline{Y}_b(2\sqrt{-3})$ by $\mathbb{Z}_3 \times \mathbb{Z}_3$, where the generator of each copy of $\mathbb{Z}_3$ acts by multiplication by ρ on the corresponding copy of $\mathbb{C}$. Our notation is justified by Lemma 3.20 below.

We have the following examples of Janus-like surfaces:

Theorem 3.8.   (i) $\overline{Y}_b(2\sqrt{-3}) = \overline{X}_m(6)$.

(ii) $\overline{Y}_b(2) = \overline{X}_m(3)$.

(iii) $\overline{Y}_b(\sqrt{-3}) = \overline{X}_m(2)$.

(iv) $\overline{Y}_b(1) = \overline{X}_m(1)$.

Proof. $\mathscr{O}_K$ is generated over $\mathbb{Z}$ by 1 and ρ, so $\overline{Y}_b(2\sqrt{-3}) = E_\rho \times E_\rho$. The theorem now follows immediately from Corollary 3.2   q.e.d.

Of course, these examples are not too interesting, merely being products of Janus-like curves. Our next examples, however, are much less trivial. Let $\mathrm{PU}(2, 1; \mathscr{O}_K)$ act on $\mathbb{B}_2$, and let $\Gamma_K(n)$ be the principal congruence subgroup of level n in this group. Let $Y_m(n) = \Gamma_K(n)\backslash\mathbb{B}_2$. We have the following result of Holzapfel.

Proposition 3.9 [6, 0.3.3].   (i) $Y_m(\sqrt{-3}) \cong \mathbb{P}^2 - \{p_1, \cdots, p_4\}$, where p_1, p_2, p_3, p_4 are the points in $\mathbb{P}^2$ indicated in the following diagram:
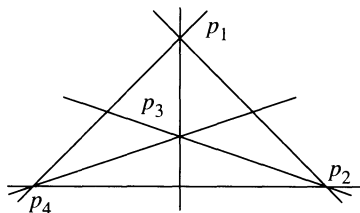
(ii) $Y_m^*(\sqrt{-3}) \cong \mathbb{P}^2$, i.e., the Satake compactification of this $\mathbb{B}_2$-quotient is actually smooth.

(iii) $\overline{Y}_m(\sqrt{-3}) = \widehat{\mathbb{P}^2} = \mathbb{P}^2$ blown up in p_1, p_2, p_3, p_4. The three marked points on each compactification divisor $\mathbb{P}^1$ are the tangents of the three lines in the diagram in (i) passing through each p_i. (This diagram is, of course, the same as that appearing in the introduction.)

Corollary 3.10. $\overline{Y}_m(\sqrt{-3}) = \overline{X}_b(2)$.

Proof.   Immediate from Propositions 3.7 and 3.9.

Corollary 3.11. $\overline{Y}_m(1) = \overline{X}_b(1)$.

Proof. $\overline{Y}_m(1)$ is the quotient of $\overline{Y}_m(\sqrt{-3})$ by $\mathrm{PU}(2, 1; \mathbb{F}_3)$, and $\overline{X}_b(1)$ is the quotient of $\overline{X}_b(2)$ by $\mathrm{SL}(2, \mathbb{Z}/2) \times (\mathbb{Z}/2)^2$; these groups are both isomorphic to the symmetric group Σ_4. Thus it remains to show that the two group actions agree. We use the following principle: two automorphisms of finite order on a smooth variety (or manifold) agree if they agree on some open set (in the complex topology), and furthermore

they agree on an open set containing a fixed point if their derivatives agree on the tangent space to that point. Let us consider the point $t_1 \cap t_4$ in the notation of Figure 5. We claim first that the isomorphisms of these two groups to Σ_4 is in fact given by their actions on the set $\{S_1, \cdots S_4\}$. They certainly both leave $\mathscr{S} := S_1 \cup \cdots \cup S_4$, as well as $\mathscr{T} := t_1 \cup \cdots \cup t_6$ invariant, so we get a map from each group to the group of permutations of the indices $\{1, \cdots, 4\}$, which we claim is an injection, and hence an isomorphism. For suppose we have an automorphism which leaves the indices fixed. Then it leaves t_1 invariant (as this is the only t_i intersected by both S_1 and S_2), and similarly for t_4, so it leaves the points $t_1 \cap S_1$, $t_1 \cap S_2$, $t_4 \cap S_3$, $t_4 \cap S_4$, and $t_1 \cap t_4$ fixed. Hence, it is an automorphism of t_1, and of t_4, fixing three points on each, so must be the identity. But then on the tangent space to $t_1 \cap t_4$ its derivative has two eigenvalues of $+1$, so it is the identity map.

A similar argument shows that Σ_4 can only have a single action fixing the configurations $\mathscr{S}$ and $\mathscr{T}$. Consider, for example, (12) (Σ_4 being generated by transpositions). The same logic shows it leaves t_1 invariant and t_4 invariant, fixing $t_1 \cap t_4$, $t_4 \cap S_3$, $t_4 \cap S_4$, and interchanging $t_1 \cap S_1$ and $t_1 \cap S_2$, so that on the tangent space to $t_1 \cap t_4$ it has an eigenvalue of -1 in the direction tangent to t_1 and $+1$ in the direction tangent to t_4, thereby determining it completely.   q.e.d.

Theorem 3.12. *Let $X_b^0(6)$ consist of $\overline{X}_b(6)$ with the 36 sections removed. Then*

$$Y_m(2\sqrt{-3}) = X_b^0(6), \qquad \overline{Y}_m(2\sqrt{-3}) = \overline{X}_b(6).$$

Proof. First we prove that $X_b^0(6)$ is a ball quotient; for this it suffices to show the proportionality

$$\overline{c}_1^2(\overline{X}_b(6), \Delta) = 3\overline{c}_2(\overline{X}_b(6), \Delta)$$

of the logarithmic Chern numbers, where Δ is the union of the 36 sections. But

$$\overline{c}_1(\overline{X}_b(6), \Delta) = c_1(\overline{X}_b(6)) - \Delta,$$

so

$$\begin{aligned}
\overline{c}_1^2(\overline{X}_b(6), \Delta) &= c_1^2(\overline{X}_b(6)) - 2c_1(\overline{X}_b(6)) \cdot \Delta + \Delta^2 \\
&= 0 + 2 \cdot 6 \cdot 36 + 36(-6) = 3 \cdot 72,
\end{aligned}$$

and

(8) $$\bar{c}_2(\overline{X}_b(6), \Delta) = c_2(\overline{X}_b(6)) = 72,$$

as claimed.

Now, since we have a covering map $X_b^0(6) \to X_b^0(2)$ sending cusps to cusps (i.e., sections to sections), the fundamental group $\pi_1(X_b^0(6))$, which by the above is a lattice in $PU(2, 1)$, actually is a (normal) sublattice of Γ_K with quotient $\cong G_{216} = PU(2, 1; \mathbb{F}_4)$. Let $\Gamma_? = \pi_1(X_b^0(6))$. We have exact sequences

$$1 \to \Gamma_? \overset{\varphi?}{\to} \Gamma_K \longrightarrow G_{216} \to 1$$

$$\| \qquad\qquad \|$$

$$1 \to \Gamma_K(2\sqrt{-3}) \overset{\varphi}{\to} \Gamma_K \longrightarrow G_{216} \to 1$$

and the reader will agree that $\varphi_?(\Gamma_?)$ *must* be equal to $\varphi(\Gamma_K(2\sqrt{-3}))$. However, we know of no general result which would guarantee this, so we will have to work hard to identify the two.

Lemma 3.13. *Let $P_? \subset \Gamma_?$ be a parabolic (stabilizer of a cusp), $P \subset \Gamma_K(2\sqrt{-3})$ a parabolic. Then $\varphi_?(P_?)$ and $\varphi(P)$ are conjugate.*

Proof. We know that $X_b^0(6)$ is an open ball quotient which is compactified by adding the sections; these are all copies of E_ρ. We want to conclude something on the structure of $P_?$ from this. Let $\mathbf{D}_F$ be the (2– dimensional) domain considered in §2.3, a Cayley transform of $\mathbb{B}_2$:

$$\mathbf{D}_F = \{(u, v) \mid \operatorname{Im}(u) - (\sqrt{3}/2)|v|^2 > 0\}.$$

Then the parabolic in $U_J(2, 1)$ of the cusp $\infty = (1, 0, 0)$ has the form

$$P_\infty = \left\{ \begin{pmatrix} 1 & 0 & x \\ \sqrt{-3}\overline{x} & 1 & y + (\sqrt{-3}/2)|x|^2 \\ 0 & 0 & 1 \end{pmatrix} \mid x \in \mathbb{C}, y \in \mathbb{R} \right\}.$$

Let $\Gamma_\infty \subset P_\infty$ be any lattice in P_∞. Γ_∞ acts on $\mathbb{C}^* \times \mathbb{C}$, as follows. The elements x are contained in a lattice Λ_∞, and $\mathbb{C}/\Lambda_\infty$ is the compactifying elliptic curve. The elements y are contained in a lattice of $\mathbb{R}$, generated by some number, q say, $q \in \mathbb{R}$. Then $\mathbb{C}^* \times \mathbb{C}/\Gamma_\infty$ is just the normal bundle of $\mathbb{C}/\Lambda_\infty$ in the ball quotient; the self-intersection number of the elliptic curve $T = \mathbb{C}/\Lambda_\infty$ in the ball quotient is given by the formula (see [6, I.1.2.2]):

$$(T)^2 = -2|\Lambda_\infty|/q,$$

where $|\Lambda_\infty|$ denotes the volume of a fundamental domain of Λ_∞. Hence we see: two parabolics (lattices) in $U_J(2, 1)$ are isomorphic, if

(a) the elliptic curves are the same (the Λ_∞ 's coincide), and

(b) the self-intersection numbers are the same (the q 's coincide also).
Applying this in our situation, (a) is clear (since $E_\rho = \mathbb{C}/\mathscr{O}_K$, which is the compactifying curve for $\Gamma_K(2\sqrt{-3})$), and for (b), both on $X_b(6)$ as well as on $Y_m(2\sqrt{-3})$ the self-intersection number is -6. This proves that $P_?$ and P are isomorphic as abstract groups, hence conjugate in Γ_K, i.e., $\varphi_?(P_?)) = \gamma_0 \varphi(P) \varphi_0^{-1}$ for some $\gamma_0 \in \Gamma_K$. q.e.d.

We shall simply assume γ_0 is the identity. Otherwise our final conclusion would be that

$$\varphi_?(\Gamma_?) = \gamma_0 \varphi(\Gamma_K(2\sqrt{-3}))\gamma_0^{-1} ;$$

but since these are normal subgroups, we have that they are equal if and only if they are conjugate.

Before we can proceed we need the following auxiliary result.

Lemma 3.14. (i) *Two cusps c_1 and c_2 of $\mathbb{B}_2$ are equivalent under $\Gamma_?$ if and only if they are equivalent under the normalizer of B in $\Gamma_?$, B the unique $\mathbb{B}_1$ with c_1 and c_2 in its closure.*

(ii) *Two subballs B_1 and B_2 of $\mathbb{B}_2$ with the same cusp c in their closures are equivalent in $\Gamma_?$ if and only if they are equivalent under $P_?(c)$ =normalizer of c in $\Gamma_?$.*

Proof. (i) First fix for the rest of this proof representatives for the $\Gamma_?$-cusps and modular curves. Suppose that c_1 and c_2 are both contained in one of the representative modular curves. Then (i) follows "by inspection", since we know that the 12 classes of cusps in this $\mathbb{B}_1$ are 12 of the 36 $\Gamma_?$ cusps. Now (i) follows in general, as the property "equivalence of cusps in a $\mathbb{B}_1$" is preserved under equivalence of $\mathbb{B}_1$'s under $\Gamma_?$.

(ii) Suppose B_1 and B_2 are equivalent, i.e., $\exists \gamma \in \Gamma_?$ with $\gamma B_1 = B_2$. Let c be the common cusp of B_1 and B_2. Then $\gamma(c) \in B_2$ and is equivalent to c, hence by (i) $\exists \delta \in N(B_2)$ with $\delta\gamma(c) = c$. It follows that $\delta\gamma(c) = c$, and $\delta_\gamma(B_1) = B_2$, in other words B_1 is equivalent to B_2 by $\delta\gamma \in P_?(c)$. q.e.d.

Geometrically we have the cusp ∞ in $\mathbf{D}_F$ and a finite number of modular subvarieties (curves of the form $\Gamma\backslash\mathbb{B}_1$) which *meet* at this cusp. From Lemma 2.3 we see that the components of the singular fibres of $X_b(6)$ are curves which are quotients $(\Gamma_K(2\sqrt{-3})\backslash\mathbb{B}_1)^*$, whose cusps are the intersections with the sections. Since $X_b(6)$ has 12 singular fibres, there are 12 of these curves meeting at a given cusp. Let $N_1^?\cdots, N_{12}^?$ denote their stabilizers in $\Gamma_?$. In $\Gamma_K(2\sqrt{-3})$ we also have 12 such modular curves and stabilizers $N_1, \cdots, N_{12} \subset \Gamma_K(2\sqrt{-3})$. (This can be seen as follows: on $Y_m(1)$ there is a unique modular curve; on $Y_m(\sqrt{-3})$ three such meet

at a cusp. Consider $Y_m(2\sqrt{-3}) \to Y_m(\sqrt{-3})$; then the 12 modular curves are the inverse images of those 3). Furthermore, the part of $N_i^?$ and N_i acting effectively is just the lattice $\mathrm{PU}(1, 1; 2\sqrt{-3}\mathscr{O}_K)$, hence $N_i^? \cong N_i$ as groups.

Lemma 3.15. $\varphi_?(N_i^?) = \varphi(N_i)$, $i = 1, \cdots, 12$.

Proof. Since the groups are pairwise isomorphic, the question is whether we can *simultaneously* identify their images under $\varphi_?$ and φ respectively. But note that this will follow if we can identify the 12 copies of $c_F(\mathbb{B}_1)$ meeting at the cusp ∞ in $\mathbf{D}_F$, each being (a subgroup of finite index of) the stabilizer of the same object in the same group Γ_K. Each one of these $c_F(\mathbb{B}_1)$'s is a *linear* section of $\mathbf{D}_F$, hence is determined by its *tangent* direction at the cusp. This tangent direction at the cusp is determined by the intersection of the curve with the compactifying curve, which is E_p. Hence the lemma follows from the fact that the 12 points of intersection on E_p (on $Y_m(2\sqrt{-3})$) are just the 2- and 3-division points, which are (on $X_b(6)$) the base points for the singular fibres.

Lemma 3.16. $\Gamma_?$ *is generated by* $P_?$ *and* $N_i^?$, $i = 1, \cdots, 12$.

Proof. Let $\gamma \in \Gamma_?$. Then $\gamma(\infty) = c$, some (other) cusp. If $c = \infty$, we are done, as $\gamma \in P_?$. Otherwise let B be the unique subball containing ∞ and c in its closure. Then by (i) of Lemma 3.14, there is a $\gamma' \in N(B)$ with $\gamma'(\infty) = c$. Thus $\gamma^{-1}\gamma' \in P_?$. However, B is equivalent to some $(\mathbb{B}_1)_i$ (as these are representatives under $\Gamma_?$ of subballs), and both B and $(\mathbb{B}_1)_i$ have ∞ in their closures. Therefore

$$N(B) = \gamma'' N((\mathbb{B}_1)_i)(\gamma'')^{-1} = \gamma''(N_i^?)(\gamma'')^{-1},$$

But by (ii) of 3.14 we may choose $\gamma'' \in P_?$, so $N(B) \subset$ subgroup generated by $P_?$ and $N_i^?$, hence $\gamma' \in$ this subgroup, and since $\gamma^{-1}\gamma' \in P_?$, $\gamma \in$ this subgroup as well.

Corollary 3.17. $\varphi_?(\Gamma_?) = \varphi(\Gamma_K(2\sqrt{-3}))$.

Proof. This follows from the last three lemmas. q.e.d.

This corollary also completes the proof of the theorem. q.e.d.

Corollary 3.18 (of the Theorem). $\overline{Y}_m(2) = \overline{X}_b(3)$.

Proof. As in the last section, $\overline{Y}_m(2) = \Sigma_4 \backslash \overline{Y}_m(2\sqrt{-3})$, $\overline{X}_b(3) = \Sigma_4 \backslash \overline{X}_b(6)$, and we need only check that the actions agree. The argument here is similar to (though more elaborate than) the argument in Corollary 3.11, and so we omit it. q.e.d.

We gather here, for convenience, this last family of Janus-like surfaces. Each are at the same time compactified quotients of $\mathbb{B}_2$ and $\mathbb{C} \times \mathbb{S}_1$. Note that the first of these is properly elliptic, while the others are rational, and

the first three are nonsingular, while the last has singularities described in [6, I.3.6].

Theorem 3.19.   (i) $\overline{Y}_m(2\sqrt{-3}) = \overline{X}_b(6)$.

(ii) $\overline{Y}_m(2) = \overline{X}_b(3)$.

(iii) $\overline{Y}_m(\sqrt{-3}) = \overline{X}_b(2)$.

(iv) $\overline{Y}_m(1) = \overline{X}_b(1)$.

3.3. Threefolds.

3.3.1. Picard case. We consider first the minimal compactification of $\Gamma_K(n)\backslash\mathbb{B}_3$. Since all boundary components are 0-dimensional, it follows that $\Gamma_K(n)\backslash\mathbb{B}_3$ is compactified by adding a finite number of points (cusps). If $\Gamma_K(n)$ acts freely on $\mathbb{B}_3$, then these cusps are the only singularities of $(\Gamma_K(n)\backslash\mathbb{B}_3)^*$. To affect a desingularization we apply the theory of toroidal embeddings. The decomposition of a parabolic as elaborated in [16]:

$$N(F) = (G_r \cdot G_l \cdot M) \cdot V \cdot U$$

has in this case the form (see [17, III, §4])

$$G_r = \{0\}, \quad M = \mathrm{SU}(2) \times U(1), \quad G_l = \mathbb{R}^*, \quad U = \mathbb{R}, \quad V = \mathbb{C}^2.$$

Hence there is no choice of polyhedral decomposition, and the compactification divisor is just $V/\Gamma_V = \mathbb{C}^2/\Lambda$, where Γ_V is a lattice induced by $\Gamma_K(n) \cap N(F)$. To see what this lattice is, one writes down the parabolic as described (in the surface case) in the last section; here it looks as follows (see [20, 1.21]):

$$P_{\mathbb{R}} = \left\{ \begin{pmatrix} * & 0 & 0 & x_1 \\ 0 & * & 0 & x_2 \\ \sqrt{-3}\overline{x}_1 & \sqrt{-3}\overline{x}_2 & * & y + (\sqrt{-3}/2)|x|^2 \\ 0 & 0 & 0 & * \end{pmatrix} \middle| \begin{array}{l} * \in U(1), \\ \\ x \in \mathbb{C}^2, y \in \mathbb{R} \end{array} \right\},$$

$$P_K = \{\gamma \in P_{\mathbb{R}} | * \in U(1) \cap K, x \in K^2, y \in \mathbb{Q}\},$$

$$(\Gamma_K)_F \cap P_{\mathbb{R}} = P_{\mathscr{O}_K} = \{\gamma \in P_K | * \in U(1) \cap K, x \in \mathscr{O}_K^2, y \in \mathbb{Z}\}.$$

From this one sees easily that the compactification divisors are just *products* of those obtained in the case of surfaces, with normal bundle being $p_1^* N \otimes p_2^* N$, where p_i is the projection onto a factor, and N is the normal bundle in the surface case. Explicitly we have

Lemma 3.20. *The compactification divisors for* $\Gamma_K(n)$ *are*
  (i) $n = 2\sqrt{-3}$: $E_\rho \times E_\rho$.
  (ii) $n = 2$: $\mathbb{P}^1 \times \mathbb{P}^1$ *with four points marked on each copy.*
  (iii) $n = \sqrt{-3}$: $\mathbb{P}^1 \times \mathbb{P}^1$ *with three points marked on each copy.*
  (iv) $n = 1$: $\mathbb{P}^1 \times \mathbb{P}^1$ *with one point marked on each copy.*

This also explains the notation $\overline{Y}_b(n)$ of §3.2.2 for these surfaces—the boundary components of $\overline{Y(n)}$.

3.3.2. Siegel case. We now turn to the Satake compactifications of $\Gamma(N)\backslash\mathbb{S}_2$. As these are well known, we will be brief (see [24], [12] for details). The 1-dimensional boundary components are quotients $\Gamma(N)\backslash\mathbb{S}_1$, which are compactified by adding the cusps, which in our situation are the 0-dimensional boundary components of $\Gamma(N)\backslash\mathbb{S}_2$. The number of 1-dimensional (=number of 0-dimensional) boundary components is listed in Lemma 2.9. Obviously several 1-dimensional components must meet at each 0-dimensional one; since the subgroups $\Gamma(N)$ are normal, the situation at each 0-dimensional cusp is the same (at each level). Let $\nu(N)$ denote the number of cusps of $\Gamma(N)\backslash\mathbb{S}_1$, $\mu_i(N)$ the number of i-dimensional cusps of $\Gamma(N)\backslash\mathbb{S}_2$, and $t(N)$ the number of 1-dimensional components meeting at each 0-dimensional cusp. Then

$$\frac{\nu(N) \cdot \mu_1(N)}{t(N)} = \mu_0(N), \qquad \mu_0(N) = \mu_1(N),$$

hence $\nu(N) = t(N)$. We have $\nu(2) = 3$, $\nu(3) = 4$, and $\nu(6) = 12$, and of course $\mu(N)$ is just the number $x_b(N)$ of Lemma 2.9.

We now turn to the smooth compactifications of $\Gamma(N)\backslash\mathbb{S}_2$. We will be brief, and for details strongly recommend [7]. Let us start with the parabolic of a 1-dimensional boundary component. In this case the decomposition of $N(F)$ is as follows:

$$N(F) = (G_r \cdot G_l \cdot M) \cdot U \cdot V,$$

$$G_r = \mathrm{Sp}(2, \mathbb{R}), \quad G_l = \mathrm{GL}(1, \mathbb{R}), \quad M = \{1\}, \quad V = \mathbb{C}, \quad U = \mathbb{R}.$$

An element A of our parabolic P_1 factors as

$$P_1 \ni A = \begin{pmatrix} a & 0 & b & 0 \\ 0 & 1 & 0 & 0 \\ c & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & u & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1/u \end{pmatrix}$$

$$\cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & r \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & n \\ m & 1 & n & 0 \\ 0 & 0 & 1 & -m \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

and G_l acting on U defines the cone $\mathbb{R}^+$. The intersection $\Gamma(N) \cap P_1$ is the corresponding integral group $(\equiv 1(N))$, and according to the general theory G_r acts on the boundary component, while the "rest" acts on $U_{\mathbb{C}} \times V \cong \mathbb{C} \times \mathbb{C}$. First $U_{\mathbb{Z}}$ acts on $U_{\mathbb{C}}$, yielding $\mathbb{C}^* \times \mathbb{C}$; then the lattice part of $V_{\mathbb{Z}}$ acts on $\mathbb{C}$ as $z \mapsto mz + n$. This combines with the action of $\Gamma(N) \cap G_r \subset \mathrm{Sp}(2, \mathbb{Z})$ on $\mathbb{C}^* \cong \mathbb{S}_1$ to yield a quotient of $\mathbb{S}_1 \times \mathbb{C}$, which of course is just an elliptic (respectively Kummer) modular surface described in Lemmas 3.5–3.6. This also explains the notation $X_b(N)$ used there for this surface—the boundary components of $\overline{X(N)}$. This surface is then glued into the open quotient as the divisor $z = 0$, $z \in \mathbb{C}$, $\mathbb{C} = \overline{(\mathbb{C}^*)}$, $\mathbb{C}^* = U_{\mathbb{C}}/U_{\mathbb{Z}}$.

To complete the compactification one must consider the 0-dimensional boundary components and the corresponding parabolics, whose decomposition simplifies to:

$$G_l = \mathrm{GL}(2, \mathbb{R}), \quad U = \mathbb{R}^3,$$

$$P_0 = \left\{ \begin{pmatrix} g & u \\ 0 & {}^t g^{-1} \end{pmatrix} \middle| g \in \mathrm{GL}(2, \mathbb{R}), u \text{ symmetric} \right\}.$$

In this case $U_{\mathbb{C}} = \mathbb{C}^3$, $U_{\mathbb{C}}/U_{\mathbb{Z}} = (\mathbb{C}^*)^3$ and the compactification is effected by locally adding the coordinate axes of $(\mathbb{C}^*)^3$ and dividing out by $\Gamma(N) \cap G_l$. The result is a union of curves (in fact $\mathbb{P}^1$'s) meeting three at a time at a number of points. The intersection diagram is the incidence diagram of the polyhedral cone decomposition in the cone of symmetric, positive definite matrices $(= U \subset \mathbb{R}^3)$. Altogether one has $\overline{(\Gamma(N) \backslash \mathbb{S}_2)} \to (\Gamma(N) \backslash \mathbb{S}_2)^*$ with the following:

(a) The compactification locus on each of these varieties has $\mu_1(N)$ irreducible components, on each of which the map is the projection of an elliptic, for $N \geq 3$ (respectively, Kummer, for $N = 2$), modular surface onto its base.

(b) Each component of $\overline{\Gamma(N) \backslash \mathbb{S}_2}$ has $\nu(N)$ singular fibres of type I_N lying over the cusps of $(\Gamma(N) \backslash \mathbb{S}_1)^*$ (respectively three singular fibres for $N = 2$), yielding the configuration of curves described in Lemma 3.6.

(c) Two such divisors meet along irreducible components of the I_N fibres; three meet at a double point of any one of the I_N-fibres (and analogous statements hold for $N = 2$).

3.3.3. Normal crossings divisors on threefolds. Recall now that we also have in addition to the boundary components the 10 copies of the modular subvariety $\Gamma(2) \backslash \mathbb{S}_1 \times \Gamma(2) \backslash \mathbb{S}_1$ on $X(2)$, and hence 10 copies of $(\Gamma(2) \backslash \mathbb{S}_1)^* \times (\Gamma(2) \backslash \mathbb{S}_1)^* \cong \mathbb{P}^1 \times \mathbb{P}^1$ on $\overline{X(2)}$. We now describe the mutual inter-

sections. Each cusp on each copy of $\Gamma(2)\backslash\mathbb{S}_1$ determines one of the cusps of $\Gamma(2)$ on $\mathbb{S}_2$: (cusp on $\mathbb{S}_2$) = { cusps of $\Gamma(2)$ on $\mathbb{S}_1$ } $\times \mathbb{S}_1$, and since $\Gamma(2)$ has three cusps, it follows that on each ruling of $\mathbb{P}^1 \times \mathbb{P}^1$ we have three fibres which are intersections with the compactification divisors $\overline{X}_b(2)$. These intersections on the divisor $\overline{X}_b(2)$ are sections of the Kummer fibering over $\mathbb{P}^1$. Hence each copy of $\overline{X}_b(2)$ meets six other $\overline{X}_b(2)$'s (along the three singular fibres) and four copies of the $\overline{X}_m(2)$. Let $\mathscr{D}(2) = \bigcup_{i=1}^{15}(\overline{X}_b(2))_i \cup \bigcup_{j=1}^{10}(\overline{X}_m(2))_j$ be the normal crossings divisor consisting of the union of the compactification divisors and the modular subvarieties, and let $\Gamma_{\mathscr{D}(2)}$ be the dual graph. Similarly, let $\mathscr{E}(\sqrt{-3}) = \bigcup_{i=1}^{15}(\overline{Y}_m(\sqrt{-3}))_i \cup \bigcup_{j=1}^{10}(\overline{Y}_b(\sqrt{-3})_j$ be the corresponding divisor on $\overline{Y(\sqrt{-3})}$ and $\Gamma_{\mathscr{E}(\sqrt{-3})}$ the corresponding graph. Then inspections of the intersections as above yields

Lemma 3.21. *The graphs* $\Gamma_{\mathscr{D}(2)}$ *and* $\Gamma_{\mathscr{E}(\sqrt{-3})}$ *are isomorphic.*

Corollary 3.22. *The normal crossings divisors* $\mathscr{D}(2)$ *and* $\mathscr{E}(\sqrt{-3})$ *are isomorphic.*

Proof. From Corollary 3.4 and the discussion of §3.1 we know that the components of $\mathscr{D}(2)$ and $\mathscr{E}(\sqrt{-3})$ are isomorphic; from the lemma we know that the combinatorics of the intersections are the same. It remains to show that the intersection curves are really the same on each copy. But this is clear from our discussions of the surfaces in §3.2: on the $\overline{X}_b(2)$ the curves are either singular fibres or sections; on the $\overline{X}_m(2)$ we have the curves of the form {cusp} $\times \mathbb{P}^1$ or $\mathbb{P}^1 \times$ {cusp}, where "cusp" refers to one of the three $\Gamma(2)$ cusps on $\mathbb{S}_1$. q.e.d.

We define also the following divisors:
$$\mathscr{D}(3) = \bigcup_{i=1}^{40}(\overline{X}_b(3))_i \cup \bigcup_{j=1}^{45}(\overline{X}_m(3))_j,$$
$$\mathscr{D}(6) = \bigcup_{i=1}^{600}(\overline{X}_b(6))_i \cup \bigcup_{j=1}^{900}(\overline{X}_m(6))_j,$$
$$\mathscr{E}(2) = \bigcup_{i=1}^{40}(\overline{Y}_m(2))_i \cup \bigcup_{j=1}^{45}(\overline{Y}_b(2))_j,$$
$$\mathscr{E}(2\sqrt{-3}) = \bigcup_{i=1}^{600}(\overline{Y}_m(2\sqrt{-3}))_i \cup \bigcup_{j=1}^{900}(\overline{Y}_b(2\sqrt{-3}))_j.$$
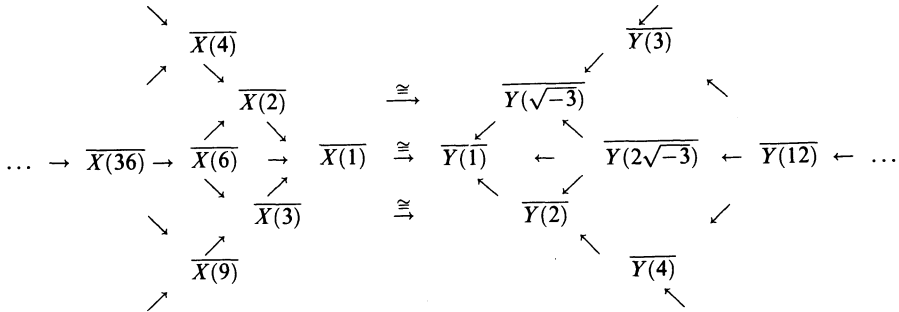We then have

Corollary 3.23. (1) $\mathscr{D}(3)$ *and* $\mathscr{E}(2)$ *are isomorphic.*
(2) $\mathscr{D}(6)$ *and* $\mathscr{E}(2\sqrt{-3})$ *are isomorphic.*

Proof. This follows immediately from the above.

4. The theorem

Theorem 4.1. *We have the following diagram of modular varieties* $\overline{X(N)}$, $\overline{Y(n)}$:

$$\overline{X(4)}\qquad\qquad\qquad\overline{Y(3)}$$

$$\overline{X(2)}\quad\xrightarrow{\ \cong\ }\quad\overline{Y(\sqrt{-3})}$$

$$\cdots\to\overline{X(36)}\to\overline{X(6)}\ \to\ \overline{X(1)}\xrightarrow{\cong}\overline{Y(1)}\ \leftarrow\ \overline{Y(2\sqrt{-3})}\ \leftarrow\ \overline{Y(12)}\ \leftarrow\ \cdots$$

$$\overline{X(3)}\quad\xrightarrow{\cong}\quad\overline{Y(2)}$$

$$\overline{X(9)}\qquad\qquad\qquad\overline{Y(4)}$$

with isomorphisms as indicated, plus the fibre product isomorphism $\overline{X(6)}$ $\xrightarrow{\cong}\overline{Y(2\sqrt{-3})}$.

Proof. $\overline{X(2)}\to\overline{Y(\sqrt{-3})}$:

From Corollary 2.7 we know that $X(2)$ and $Y(\sqrt{-3})$ have isomorphic open subsets. This Zariski open set corresponds to hyperelliptic (respectively Picard) curves

$$y^2=\prod(x-\xi_i)\qquad(y^3=\prod(x-\xi_i))$$

such that *all* ξ_i *are distinct.* From the remark following 2.7 in the hyperelliptic case this is the complement of the union of modular subvarieties. Letting $\mathscr{V}\subset\overline{X(2)}$ denote this Zariski open subset, we thus have $\overline{X(2)}-\mathscr{D}(2)=\mathscr{V}$. From Corollary 2.3 we see that the Zariski open subset on $\overline{Y(\sqrt{-3})}$ is the complement of the modular subvarieties; hence, denoting this set again by $\mathscr{V}$, we have $\overline{Y(\sqrt{-3})}-\mathscr{E}(\sqrt{-3})=\mathscr{V}$. On the other hand, by 3.2.2, $\mathscr{D}(2)=\mathscr{E}(\sqrt{-3})$. Therefore, to show that the isomorphism of $\mathscr{V}$ (onto itself) extends to each compactification, it is enough to show it extends to a homeomorphism of the compact varieties. Since the isomorphisms are given explicitly by the "coordinates" ξ_i, the question is how this correspondence looks at $\mathscr{D}(2)$ and $\mathscr{E}(\sqrt{-3})$. These maps are explicitly described in §5.3 below, from which continuity follows.

$\overline{X(1)}\ \to\ \overline{Y(1)}$: These are the quotients of $\overline{X(2)}$ and $\overline{Y(\sqrt{-3})}$ by $\mathrm{PSp}(4,\mathbb{Z}/2)$ and $\mathrm{PU}(3,1;\mathbb{F}_3)$ respectively, and both of these groups are, by Proposition 2.8, isomorphic to Σ_6. These groups preserve the configurations $\mathscr{D}(2)$ and $\mathscr{E}(\sqrt{-3})$ used above. There is an indexing under which the 15 boundary components $\overline{X}_b(2)$ (respectively the 15 modular components $\overline{Y}_m(2)$) correspond to unordered pairs of elements of
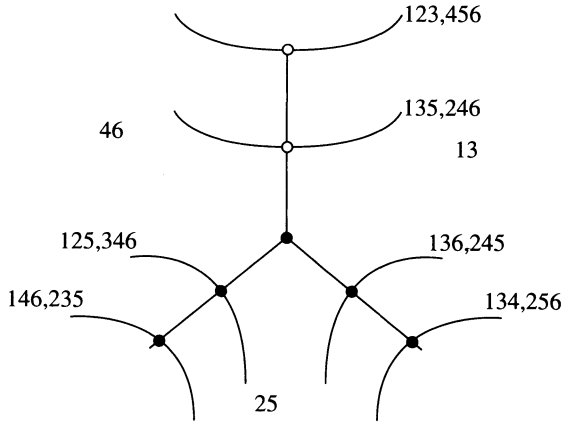
FIGURE 6. CONFIGURATION IN $\overline{X(2)}$

$\{1, \cdots, 6\}$, and under which the 10 modular divisors $\overline{X}_m(2)$ (respectively the 10 boundary components $\overline{X}_b(2)$) correspond to unordered pairs of unordered triples of elements of $\{1, \cdots, 6\}$, and Σ_6 operates as the permutations of $\{1, \cdots, 6\}$ [10]. Again, we need to show that Σ_6 has only possible action preserving the configurations $\mathscr{D}(2)$ and $\mathscr{E}(\sqrt{-3})$, and again it suffices to check the action of a generator on the tangent space to a fixed point (cf. proof of Corollary 3.11).

One has the following configuration in $\overline{X(2)}$ ([10], Fig. 2, reindexed to accord with [13]) as shown in Figure 6. In this configuration each straight line represents a $\mathbb{P}^1$ which is the intersection of two boundary components, and each half of each curved line is a $\mathbb{P}^1$ which is part of the intersection of a modular divisor with the boundary. The dots (solid or open) denote triple points.

The point p in the center of the diagram is indexed by the unordered triple of unordered pairs $\{\{1, 3\}, \{2, 5\}, \{4, 6\}\}$, and there are 15 such points in $\overline{X(2)} = \overline{Y(\sqrt{-3})}$. The action of Σ_6 is given by its natural permutation representation on $1, \cdots, 6$. Thus the image of p under an element of Σ_6 is determined. It remains to check the action of the subgroup of Σ_6 which stabilizes p on the tangent space at p. Now consider the action of the element (25). It leaves each solid point fixed and interchanges the two open points, thereby determining its action at the tangent space to the point in the center of the diagram (eigenvalue $+1$ with eigenspace spanned by the tangents along the two diagonal $\mathbb{P}^1$'s, eigenvalue -1 with eigenspace spanned by the tangent along the vertical $\mathbb{P}^1$). The action of

the other elements of the stabilizer may be checked similarly.

$\overline{X(6)} \to \overline{Y(2(\sqrt{-3})}$: We prove this using the same method as in Theorem 3.12, where we proved $\overline{X}_b(6) \cong \overline{Y}_m(2\sqrt{-3})$. First let $X^0(6)$ be $X(6)$ with the surfaces $\overline{X}_m(6)$ removed (the translates of the diagonal).

Lemma 4.2. $X^0(6)$ *is a ball quotient.*

Proof.   Here we must prove the equality

$$3\overline{c}_1^3(\overline{X(6)}, \Delta) = 8\overline{c}_1\overline{c}_2(\overline{X(6)}, \Delta)$$

for the logarithmic Chern numbers, where $\Delta = \bigcup \overline{X}_m(6)$. Since the abelian surfaces $\overline{X}_m(6)$ are disjoint, one sees easily (using adjunction):

$$\overline{c}_1(\overline{X(6)}, \Delta) = c_1(\overline{X(6)}) - \Delta, \qquad \overline{c}_2(\overline{X(6)}, \Delta) = c_2(\overline{X(6)}).$$

Furthermore, applying adjunction to each component $\overline{X}_m(6)$ of Δ yields

$$c_1(\overline{X(6)}) \cdot \overline{X}_m(6) = c_1(\overline{X}_m(6)) + (\overline{X}_m(6))^2 = (\overline{X}_m(6))^2,$$

so $c_1(\overline{X(6)}) \cdot \Delta = \Delta^2$. Hence for the numbers we have

$$\overline{c}_1^3 = c_1^3(\overline{X(6)}) - 3c_1^2(\overline{X(6)}) \cdot \Delta + 3c_1(\overline{X(6)}) \cdot \Delta^2 - \Delta^3$$
$$= c_1^3(\overline{X(6)}) - \Delta^3,$$
$$\overline{c}_1\overline{c}_2(\overline{X(6)}, \Delta) = c_1c_2(\overline{X(6)}),$$

so the equality to be shown is

$$3c_1^3(\overline{X(6)}) - 8c_1c_2(\overline{X(6)}) = 3(\Delta)^3.$$

All intersection numbers involved can be taken from Yamazaki's paper. Let $\overline{X(N)}$ be the Igusa desingularisation of the Siegel 3-fold of level N. Then

$$c_1^3(\overline{X(N)}) = \frac{N^7 \cdot \prod_{p|N}(1-p^{-2})(1-p^{-4})}{2^6 \cdot 3 \cdot 5}[-9N^3 + 360N - 880],$$

$$c_1c_2(\overline{X(N)}) = \frac{N^7 \cdot \prod_{p|N}(1-p^{-2})(1-p^{-4})}{2^4 \cdot 3 \cdot 5}[-N^3 + 30N - 60],$$

$$(\Delta)^3 = \frac{N^{10} \cdot \prod_{p|N}(1-p^{-2})(1-p^{-4})}{2^6 \cdot 3^2}.$$

Substituting $N = 6$ into the above we get

$$3(-119520) - 8(-69120) = 3 \cdot 64800,$$

which checks. This proves the lemma.   q.e.d.

Now since we know that $X^0(6)$ is a ball quotient (compactified by adding the abelian surfaces $\overline{X}_m(6)$), let $\Gamma_?$ be the lattice in $PU(3,1)$ such that $X^0(6) = \Gamma_?\backslash\mathbb{B}_3$; from the cover $X^0(6) \to X^0(2)$ we see that $\Gamma_? \subset PU(3,1;\sqrt{-3}\mathscr{O}_K)$. It is a normal subgroup with quotient $G_{25,920}$, and again we strongly suspect $\Gamma_? = PU(3,1;2\sqrt{-3}\mathscr{O}_K)$. To prove this, we start as in §3.2.2 with a parabolic. Let $P_?$ denote the parabolic in $\Gamma_?$ for a cusp, and $P \subset PU(3,1;2\sqrt{-3}\mathscr{O}_K)$ the parabolic of a cusp (all parabolics are isomorphic since $\Gamma_?$, $\Gamma_K(2\sqrt{-3})$ are normal subgroups). We have the following exact sequences:

$$
\begin{array}{ccccccc}
1 & \longrightarrow & \Gamma_? & \xrightarrow{\varphi_?} & \Gamma_K & \to & G_{25,920} & \longrightarrow & 1 \\
& & & & \| & & \| & & \\
1 & \longrightarrow & \Gamma_K(2\sqrt{-3}) & \xrightarrow{\varphi} & \Gamma_K & \longrightarrow & G_{25,920} & \longrightarrow & 1
\end{array}
$$

Lemma 4.3. $P_? \cong P$, $\varphi_?(P_?) = \varphi(P)$.

Proof. A parabolic in $PU_J(3,1)$ looks as follows:

$$
P_\infty = \left\{ \begin{pmatrix} 1 & 0 & 0 & x_1 \\ 0 & 1 & 0 & x_2 \\ \sqrt{-3}\overline{x}_1 & \sqrt{-3}\overline{x}_2 & 1 & y+(\sqrt{-3}/2)\Sigma|x_i|^2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \middle| x_i \in \mathbb{C}, y \in \mathbb{R} \right\}.
$$

As in the surface case, $P_\infty \cap \Gamma$ acts on $\mathbb{C}^* \times \mathbb{C}^2$ by dividing $\mathbb{C}^2$ by the lattice generated by x_1, x_2 and acts on $\mathbb{C}^*$ (normal direction) by a lattice with generator some $q \in \mathbb{R}$. One sees clearly that the quotient of $\mathbb{C}^2$ is a product, $\mathbb{C}^2/\Lambda = \mathbb{C}/\mathscr{O}_K \times \mathbb{C}/\mathscr{O}_K$, and that its normal bundle $N \to \mathbb{C}/\mathscr{O}_K \times \mathbb{C}/\mathscr{O}_K$ is $N \cong p_1^*N_1 \otimes p_2^*N_2$, where p_i are the projections and N_i are the bundles over $\mathbb{C}/\mathscr{O}_K$ which are the normal bundles in the surface case. Furthermore, these two data suffice to determine the parabolic, and since both are checked in §3.2.2 (here being just the product situation), the lemma follows.   q.e.d.

Next, consider a fixed cusp ∞ with compactifying surface

$$
S_\infty \cong \overline{Y}_b(2\sqrt{-3}) \cong E_\rho \times E_\rho.
$$

There are 24 subvarieties $B_i \cong Y_m(2\sqrt{-3})$ of $Y(2\sqrt{-3})$ whose closures intersect S_∞, each along a curve of the form $\{p_i\} \times E_\rho$ or $E_\rho \times \{q_i\}$, $i = 1, \cdots, 12$, where the p_i and q_i are the 12 points of order 2 and 3 used in §3.2.2, on the first and second copies of E_ρ in S_∞. Each B_i is uniformized by a subball $\mathbb{B}_2 \subset \mathbb{B}_3$ which passes through ∞. Such a $\mathbb{B}_2$ is a linear section of $\mathbb{B}_3$, hence is determined by its tangent directions

at ∞, or by its intersection with S_∞. Let $N(B_i)$, $i = 1, \cdots, 24$, be the stabilizers of the B_i, $N(B_i) \subset \mathrm{PU}(3, 1; 2\sqrt{-3}\mathscr{O}_K)$, $(N_?)_i \subset \Gamma_?$ the corresponding stabilizers in $\Gamma_?$.

Lemma 4.4. $\varphi(N(B_i)) \cong \varphi_?((N_?)_i)$, $i = 1, \cdots, 24$.

Proof. As in the surface case this will follow if we can verify that the 24 subballs $\mathbb{B}_2 \subset \mathbb{B}_3$ are the same: the argument of Lemma 3.15 carries over here with no change (note that Lemma 3.14 is valid here also (with obvious modifications for the change in dimension), with identical proof). To show that the 24 subballs are the same, it is sufficient to note that the divisors on S_∞ are the same in both cases.   q.e.d.

Finally, we have here as in the surface case

Lemma 4.5. $\varphi_?(\Gamma_?)$ *is generated by* $\varphi_?(P_?)$ *and* $\varphi_?((N_?)_i)$.

The argument is completely analogous to 3.16 and we therefore omit it.   q.e.d.

From the last three lemmas we get $\varphi_?(\Gamma_?) = \varphi(\Gamma_K(2\sqrt{-3}))$ from which the claim $\overline{X(6)} \to \overline{Y(2\sqrt{-3})}$ follows.

$\overline{X(3)} \to \overline{Y(2)}$: This follows from $\overline{X(6)} \to \overline{Y(2\sqrt{-3})}$, Σ_6 acting on both spaces in a natural way, as soon as we have identified both group actions. This is done as above by computing eigenvalues at fixed points, hence we omit the explicit verification.

Remark.   Bert v. Geemen has found a direct proof of this last case, on Zariski open subsets, by showing that the data used to fix a level 3 structure on a genus 2 curve corresponds bijectively to the data used to fix a set of level 2 structures on a genus 4 Picard curve. This rather vague statement is difficult to make precise without going into details, so we just refer to the paper by van Geemen [22] mentioned in the introduction.

Corollary 4.6.   *Consider the isomorphisms* $\overline{X(N)} \overset{\cong}{\to} \overline{Y(n)}$. *Under each such the boundary components of* $\overline{X(N)}$ *are mapped to the modular subvarieties of* $\overline{Y(n)}$, *and the modular subvarieties of* $\overline{X(N)}$ *are mapped to the boundary components of* $\overline{Y(n)}$.

Remark.   The really interesting isomorphism of the theorem is $\overline{X(6)} \to \overline{Y(2\sqrt{-3})}$. This is because in both cases the lattices act freely, hence the open quotients $X(6)$ and $Y(2\sqrt{-3})$ fulfill logarithmic proportionality with $\bar{c}_1^3/\overline{c_1 c_2} = 9/4$ for $X(6)$ and $= 8/3$ for $Y(2\sqrt{-3})$.

5. A remarkable duality

Corollary 4.6 shows that the isomorphisms of Theorem 4.1 actually have more structure than just isomorphisms: there is obviously a duality

of some sort involved, between boundary components and modular subvarieties, or as mentioned in the introduction, projective duality. Probably the most prominent expression of this duality is the duality between Tits buildings with scaffoldings, discussed in §2. Here we sketch three other aspects.

5.1. Modular forms.

For a lattice Γ and domain $\mathscr{D}$ as in §2, let $R_k(\Gamma)$ denote the ring of Γ-modular forms of weight k (belonging to the representation $(\mathrm{ad}_\mathfrak{k})^{\otimes k}$) [2]. It is well known that the Baily-Borel embedding of $\Gamma\backslash\mathscr{D}^*$ is given by $\mathrm{Proj}(\bigoplus_k R_k(\Gamma))$, hence the coordinate ring of $\Gamma\backslash\mathscr{D}^*$ in the sense of algebraic geometry is $\bigoplus_k R_k(\Gamma)$. Consider the dual variety of $\Gamma\backslash\mathscr{D}^*$, and let $\bigoplus_k D_k$ be its coordinate ring. Then we say $\bigoplus_k R_k(\Gamma)$ and $\bigoplus_k D_k$ are *projectively dual* rings. Thus 4.1 and the discussion of the varieties $\mathscr{S}$ and $\mathscr{B}$ in the introduction imply

Proposition 5.1. *For the following levels, the rings of modular forms for* $\Gamma(N)$ *are projectively dual to the ring of modular forms for* $\Gamma_K(n)$: $n = \sqrt{-3}$ *and* $N = 2$; $n = 2$ *and* $N = 3$.

Question 5.2. Is the same true for $n = 2\sqrt{-3}$ and $N = 6$?

From a somewhat different point of view, a modular form on $\overline{\Gamma\backslash\mathscr{D}}$ is a section of the line bundle $\Omega_{\overline{\Gamma\backslash\mathscr{D}}}(\log)^{\otimes k}$, where the logarithmic poles are along the divisor $\overline{\Gamma\backslash\mathscr{D}} - (\Gamma\backslash\mathscr{D})$. So from the above proposition we have:

Proposition 5.3. *The following rings are projectively dual*:

$$\bigoplus_k H^0(\overline{X(N)}, \Omega_{\overline{X(N)}}(\log)^{\otimes k}), \quad \bigoplus_k H^0(\overline{Y(n)}, \Omega_{\overline{Y(n)}}(\log)^{\otimes k}),$$

with N, n *as in* 5.1.

Let us consider now cusp forms, i.e., modular forms of weight k vanishing along the compactification divisors. Hence these are sections of the line bundle $\Omega_{\overline{\Gamma\backslash\mathscr{D}}}(\log)^{\otimes k-1} \otimes \Omega_{\overline{\Gamma\backslash\mathscr{D}}}$. In particular, cusp forms of weight 1 are just sections of the canonical bundle. For $N = 2$, 3; $n = \sqrt{-3}$, 2 there are none such, but for $N = 6$ the number has been calculated by Weissauer, the result being 2906. Note that in this case there is no distinction between the Picard and Siegel modular cases. Therefore the calculation being done for $\Gamma(6)$, we get for free the number of cusp forms of weight 1 for $\Gamma_K(2\sqrt{-3})$: it is 2906. In fact, as a consequence of 4.1 we can state

---

[2] This is perhaps not the usual notion of weight, namely the power of $\det$ which occurs in the representation $(\mathrm{ad}_\mathfrak{k})^{\otimes k}$.

Proposition 5.4. *The following* $\mathbb{C}$-*vector spaces are isomorphic*:

$$\left\{ \begin{array}{c} \textit{cusp forms of } \Gamma(6) \\ \textit{of weight } 1 \end{array} \right\} \cong \left\{ \begin{array}{c} \textit{cusp forms of } \Gamma_K(2\sqrt{-3})) \\ \textit{of weight } 1 \end{array} \right\}.$$

5.2. Metrics.

Consider a bounded symmetric domain $\mathscr{D}$ with its Bergmann metric. For a lattice Γ acting freely on $\mathscr{D}$, this metric, being $G_{\mathbb{R}}$-invariant and in particular Γ-invariant, descends to the quotient $\Gamma\backslash\mathscr{D}$ to give a metric there. By Mumford's extension of the proportionality principle to the non-compact case, this metric extends to a metric on $\overline{\Gamma\backslash\mathscr{D}}$ with logarithmic singularities along the compactification divisor. To apply this we stick with $\overline{X(6)} \cong \overline{Y(2\sqrt{-3})}$, since as mentioned above, both groups act freely in this case. Let us call this variety $\overline{X}$; then we have two natural Bergmann metrics, each with logarithmic singularities along one of the compactification divisors. Now note also that the modular subvarieties have a geometric interpretation in terms of the Bergmann metrics. In fact, the subdomains $\mathbb{S}_1 \times \mathbb{S}_1 \subset \mathbb{S}_2$ and $\mathbb{B}_2 \subset \mathbb{B}_3$ are *totally geodesic* submanifolds with respect to the Bergmann metrics on $\mathbb{S}_2$ and $\mathbb{B}_3$. Furthermore, although there are positive dimensional families of such totally geodesic submanifolds on the universal covers $\mathbb{S}_2$ and $\mathbb{B}_3$, there are *finitely* many such which have the following property: they are invariant under the lattice, so yield totally geodesic submanifolds on the quotients, and are such that they *contain* (a maximal number of) boundary components. We call such subvarieties *parabolic totally geodesic*. Since the intersection of two totally geodesic submanifolds is again totally geodesic, we see that the union of these defines a finite complex, which is easily seen to be the scaffolding as defined in §2.4. Also the locus of logarithmic singularities is a normal crossings divisor; hence we have on the quotients two normal crossings divisors. The conclusion is

Proposition 5.5. *Let* g_1 *and* g_2 *be the Bergmann metrics, extended to* $\overline{g_1}$, $\overline{g_2}$ *on* $\overline{X}$. *Then these two metrics are dual, that is*

$$\left\{ \begin{array}{c} \textit{locus of logarithmic} \\ \textit{singularities of } \overline{g_1} \end{array} \right\} \cong \left\{ \begin{array}{c} \textit{locus of parabolic totally geodesic} \\ \textit{submanifolds of } \overline{g_2} \end{array} \right\},$$

where the isomorphism is of divisors. The same is true for $\overline{g_2}$, $\overline{g_1}$.

5.3. Moduli.

We give here a table of all degenerations occuring in our families of curves of genus 4 and genus 2 respectively; here the degeneration does not see the level structure—the degenerations are independent of the level. Hence we assume $N = 2$, $n = \sqrt{-3}$. We employ the following notation,

the equations (3) and (4) from §2.3 defining the curves:

$$D_i = \text{divisors } \overline{X}_b(2)$$
$$\cong \overline{Y}_m(\sqrt{-3}): \qquad \text{two of the } \xi_k \text{ coincide, } i = 1, \cdots, 15$$
$$D_{ij} = D_i \cap D_j: \qquad \text{two pairs of the } \xi_k \text{ coincide}$$
$$D_{ijk} = D_i \cap D_j \cap D_k: \qquad \text{three pairs of the } \xi_k \text{ coincide}$$
$$E_t = \text{divisors } \overline{X}_m(2)$$
$$\cong \overline{Y}_b(\sqrt{-3}): \qquad \text{three of the } \xi_k \text{ coincide, } t = 1, \cdots, 10$$
$$E_{ti} = D_i \cap E_t: \qquad \text{one pair and one triple of the } \xi_k$$
$$\text{coincide}$$
$$E_{tij} = D_{ij} \cap E_t: \qquad \text{two triples of the } \xi_k \text{ coincide}$$

The reader should have no difficulty verifying these degenerations. To complete the proof of our main theorem, we use the fact that the dependency on the parameters ξ_k is continuous. As an example we describe this on the generic singular locus, i.e., along D_i and E_t.

Picard case.

D_i: If one pair of the ξ_k coincide, we have a 3-fold cover, branched at five points, one of them double. Without restricting generality, we can assume one of the points (say the double zero) is at ∞. The resulting cover splits into two components, one of them a smooth genus 3 curve ($\mathbb{Z}_3$-trigonal) and the other an elliptic curve. Note that the Jacobian of this curve still has complex multiplication, and, in addition, both the genus 3 and the elliptic curve have an extra automorphism. This means, in particular, that the elliptic component is the curve E_ρ, with no variation in the family. More generally, one easily sees that *all* the elliptic curves in the degenerations corresponding to the divisors D_i (i.e., occurring in the second to fifth rows in Table 1) are this particular elliptic curve.

E_t: Here the cover splits into two curves of genus 1, but the automorphism is lost, i.e., one just has an endomorphism. The parameter (moduli) of the elliptic curve is given by the double ratio of the four branch points.

Siegel case.

D_i: If two of the ξ_k coincide, then we have a double cover with five branch points; this is a genus 1 curve with one double point. Since D_i is the Kummer modular surface, its points can be given coordinates (z, w) with $z \in \Gamma(2)\backslash \mathbb{S}_1$, $w \in E_\rho/\mathbb{Z}_3 \cong \mathbb{P}^1$. Then z is the moduli point of the elliptic curve, and w gives the double point.

TABLE 1

Locus	branch points on P^1	genus 4 curve	genus 2 curve
generic	✕✕✕✕✕✕ (1 1 1 1 1 1)	$g = 4$	$g = 2$
D_i	✕✕✕✕ ▫ (1 1 1 1 2)	1 ⟍⟋ 3	1
D_{ij}	✕✕ ▫ ▫ (1 1 2 2)	1 ⟍ 2 1	0
D_{ijk}	▫ ▫ ▫ (2 2 2)	1 ⟍ 1 ⟋ 1 / 1	0 / 0
E_t	△ ✕✕✕ (3 1 1 1)	1 / 1	1 ⟍⟋ 1
E_{ti}	△ ✕ ▫ (3 1 2)	1 / 0 / 1 / 1	0 / 1
E_{tij}	△ △ (3 3)	0 / 0 / 0	0 / 0

E_t: When three of the ξ_k coincide, the curve is $y^2 = \prod_1^3 (x - \xi_k)$, an elliptic curve. More precisely, let

$$y^2 = \prod_{k=1}^{6} (x - \xi_k)$$

be the original equation, and write it as

$$y^2 = (x - \xi_1)(x - \xi_2)(x - \xi_3)(x - \lambda\xi_4)(x - \lambda\xi_5)(x - \lambda\xi_6),$$

and the degeneration is then given by letting $\lambda \to \infty$. The limit curve is thus $y^2 = \prod_1^3 (x - \xi_k)$, with a fourth branch point at infinity, and changing variables to $\tilde{x} := \lambda x$ we get

$$y^2 = (\tilde{x} - \lambda\xi_1)(\tilde{x} - \lambda\xi_2)(\tilde{x} - \lambda\xi_3)(\tilde{x} - \xi_4)(\tilde{x} - \xi_5)(\tilde{x} - \xi_6),$$

which for $\lambda \to \infty$ becomes $y^2 = \prod_4^6 (\tilde{x} - \xi_k)$, with fourth branch point at infinity. Therefore, over $\xi \in E_t$ the corresponding degeneration consists of the two elliptic curves above, meeting at their common branch point.

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