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HOMOGENEOUS SUBMANIFOLDS OF HIGHER RANK AND PARALLEL MEAN CURVATURE

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Abstract

Let M^n, $n \geq 2$, be an orbit of a representation of a compact Lie group which is irreducible and full as a submanifold of the ambient space. We prove that if M admits a nontrivial (i.e., not a multiple of the position vector) locally defined parallel normal vector field, then M is (also) an orbit of the isotropy representation of a simple symmetric space. So, in particular, compact homogeneous irreducible submanifolds of the Eucildean space with parallel mean curvature (not minimal in a sphere) are characterized (and classified). The proof is geometric and related to the normal holonomy groups and the theorem of Thorbergsson.

0. Introduction

Riemannian manifolds of nonpositive curvature and submanifolds of the Euclidean space seem to be related. There are several theorems for the fist class of spaces that have a (formal) analogous result in the context of submanifolds. Their proofs seem also to have some similarities, though the concepts involved are of a quite different nature (e.g., holonomy groups of the tangent or normal connection). In the first case a very important role is played by the symmetric spaces. In the case of submanifolds this role is played by all the orbits of the isotropy representation of semisimple symmetric spaces (s-representations) (see [14]). For manifolds of nonpositive curvature with finite volume and higher rank one has the theorem of Ballmann/Burns-Spatzier [1], [2], which asserts that they are locally symmetric. On the other hand, for compact isoparametric submanifolds of higher rank one has the theorem of Thorbergsson [17] which assets that they are orbits of s-representations. The proofs of Burns-Spatzier and Thorbergsson rely on the topological Tits buildings. There is also another proof of the result of Thorbergsson in [12] which does not use Tits buildings and is related to the normal holonomy groups. (For any

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submanifold of the Eucildean space the normal holonomy representation is an s-representation by [11]. This is also the case for the tangent holonomy of spaces of nonpositive curvature.)

Recently, J. Heber [5] proved that an irreducible homogeneous manifold of nonpositive curvature and higher rank is locally symmetric. One could wonder if there is some theorem of this kind in the context of compact homogeneous submanifolds. An answer to this question is given by

Theorem A. *Let M^n, $n \geq 2$, be a compact homogeneous irreducible full submanifold of the Euclidean space with* $\operatorname{rank}(M^n) \geq 2$. *Then M^n is an orbit of the isotropy representation of a simple symmetric space.*

Let us say that the rank of a submanifold is defined to be the maximal number of linearly independent (locally defined) parallel normal vector fields (see §1). Observe that for $n = 1$ Theorem A does not hold since any (homogeneous) curve has flat normal bundle.

This theorem has an immediate corollary, which provides an answer to a classical problem, namely,

Theorem B. *Let M^n be a compact homogeneous irreducible submanifold of the Eucildean space with parallel mean curvature vector which is not minimal in a sphere. Then M^n is an orbit of the isotropy representation of a simple symmetric space.*

Since any representation of a compact Lie group has a minimal orbit in the sphere, Theorem B cannot be improved (see [8]).

We shall give now some ideas of the proof of Theorem A; a fundamental tool is

Theorem C. *Let $M^n = K.v$, $n \geq 2$, be a compact homogeneous irreducible full submanifold of $\mathbb{R}^N$ and let $k \in K$, $p \in M$. Then there exists $c\colon [0, 1] \to M$ piecewise differentiable with $c(0) = p$, $c(1) = k.p$ such that*

$$dk|_{\nu(M)_p} = \tau_c^{\perp},$$

where $\tau^{\perp}$ denotes the $\nabla^{\perp}$-parallel transport.

By means of this theorem we are able to produce submersions $\pi_i\colon M \to M_{\xi_i}$ $(i = 1, \cdots, g)$ onto focal parallel orbits with the property that $TM = \bigoplus_{i=1}^{g} \ker(d\pi_i)$, and $\{\pi_i^{-1}(\pi_i(q))\}_q$ is homogeneous under the normal holonomy group of M_{ξ_i} at $\pi(q)$ for all $q \in M$. It is not hard to prove now that M is a submanifold with constant principal curvatures (see [7]). But there exists orbits which are submanifolds with constant principal curvatures and such that the corresponding isoparametric submanifold is inhomogeneous (see [4]). So, in the last step we have to use the theorem of Thorbergsson [17] in order to conclude our result.

It is easy to see that Theorem A (and hence B) also holds if M^n is not compact but contained in a sphere. Moreover, it is proved in [13] that a homogeneous irreducible full submanifold M^n, $n \geq 2$, of a Euclidean space with $\operatorname{rank}(M) \geq 1$ is always contained in a sphere. This solves completely the problem of classifying, up to minimal immersions, homogeneous submanifolds with parallel mean curvature.

1. Preliminaries and notation

Let M^n be a Riemannian manifold and let $i: M \to \mathbb{R}^N$ be an isometric injective immersion. We say that M is a reducible submanifold of $\mathbb{R}^N$ if $M = M_1 \times M_2$ (Riemannian product) where M_1, M_2 are nontrivial factors and $i = i_1 \times i_2$ where $i_1: M_1 \to \mathbb{R}^{N_1}$, $i_2: M_2 \to \mathbb{R}^{N_2}$ are isometric immersions and $N = N_1 + N_2$. If M is not reducible as a submanifold of $\mathbb{R}^N$, then it is said to be irreducible. Assume now that M is a compact homogeneous submanifold of $\mathbb{R}^N$, i.e., $M = K.v$, where $v \in \mathbb{R}^N$ and K is a compact connected Lie subgroup of $I(\mathbb{R}^N)$. Without loss of generality we may assume that $K \subset SO(N)$. If M is a reducible submanifold, then each factor is also a homogeneous submanifold of the corresponding Euclidean space. So, any compact homogeneous submanifold can be written (uniquely, up to a permutation of the factors) as a product of compact irreducible homogeneous submanifolds.

From [10, Lemma] and the fact that homogeneous Riemannian submanifolds are analytic submanifolds, it is not hard to derive the following.

Lemma 1.1. *Let M^n be a compact homogeneous submanifold of $\mathbb{R}^N$. Assume that there exist an open subset U of M and a nontrivial parallel distribution $\mathfrak{H}$ on U such that $\alpha(X, Y) = 0$ if X lies in $\mathfrak{H}$ and Y lies in $\mathfrak{H}^\perp$, where α is the second fundamental form of M. Then M is a reducible submanifold of $\mathbb{R}^N$.*

Let now $i: M \to \mathbb{R}^N$ be an immersed full Riemannian submanifold (i.e., not contained in an affine hyperplane) and let $\nu(M) = \{(p, w): p \in M,\ w \in (T_pM)^\perp\}$ be its normal bundle. Decompose $\nu(M) = \nu_0(M) \oplus \nu_s(M)$, where $\nu_0(M)$ is the maximal $\nabla^\perp$-parallel subbundle of $\nu(M)$ which is flat and $\nu_s(M) = (\nu_0(M))^\perp$, namely, $\nu_0(M)_p = \{\xi \in \nu(M)_p: \Phi_p^* \cdot \xi = \xi\}$ for all $p \in M$, where Φ^* denotes the restricted normal holonomy group.

Definition 1.2. The dimension (over M) of $\nu_0(M)$ is called the *rank* of (M, i) and is denoted by $\operatorname{rank}(M, i)$.

We shall often write, when there is no possible confusion, $\operatorname{rank}(M)$ instead of $\operatorname{rank}(M, i)$.

Observe that if M^n is a compact homogeneous submanifold of $\mathbb{R}^N$, then M is contained in a sphere and hence $\operatorname{rank}(M) \geq 1$.

Remark 1.3. If M is an isoparametric full submanifold, then $\operatorname{rank}(M)$ coincides with the usual notion of rank, i.e., the codimension. If, in addition, M is homogeneous, then M is the orbit of an s-representation, and $\operatorname{rank}(M)$ is equal to the rank of the corresponding symmetric space.

From here up to the end of this paper, unless otherwise stated, $M^n = K \cdot v$ will denote a compact homogeneous submanifold of $\mathbb{R}^N$ with $\operatorname{rank}(M) \geq 2$, where K is a connected compact Lie subgroup of $SO(N)$ and $v \in \mathbb{R}^N$. By U we will denote an arbitrary open subset of M, which is contractible (in order to make any vector bundle over U trivializable). Since M is an analytic submanifold of $\mathbb{R}^N$, we get that $\operatorname{rank}(M) = \operatorname{rank}(U)$. So, we can find $\xi_1, \cdots, \xi_r \in C^\infty(U, \nu_0(U))$ linearly independent such that $\langle\{\xi_1, \cdots, \xi_r\}\rangle = \nu_0(U)$ and $\nabla^\perp \xi_1 = \cdots = \nabla^\perp \xi_r = 0$, where $r = \operatorname{rank}(M)$. For $q \in U$, the shape operators $\{A_\xi : \xi \in \nu_0(M)_q\}$ are simultaneously diagonalizable and determine, as in the isoparametric case, eigendistributions $E_1, \cdots, E_g$ in U and different $n_1, \cdots, n_g \in C^\infty(U, \nu_0(U))$ (which are not in general $\nabla^\perp$-parallel) such that $TU = E_1 \oplus \cdots \oplus E_g$ and $A_\xi(X_i) = \lambda_i(\xi) X_i = \langle n_i, \xi\rangle X_i$, for all $\xi \in C^\infty(U, \nu_0(U))$, $X_i \in C^\infty(U, E_i)$. We denote the set $\{1, \cdots, g\}$ by I, and observe that, due to the homogeneity, $\langle n_i, n_j\rangle$ is constant on U if $i, j \in I$.

2. On the autoparallelity of the eigendistributions

We keep the assumptions and notation of §1. The aim of §§2–5 is to prove Theorem C (or equivalently Theorem 5.1).

Lemma 2.1. *Let $q \in U$, $i_0 \in I$ and let $\xi \in C^\infty(U, \nu_0(U))$ be parallel. Assume that the function $\lambda_{i_0}(\xi) = \langle n_{i_0}, \xi\rangle$ has a local maximum at q and that $\lambda_{i_0}(\xi)(q) \neq \lambda_i(\xi)(q)$ for all $i \in I \setminus \{i_0\}$. Then E_{i_0} is an autoparallel distribution.*

Proof. Let T be the tensor on U, defined by $T = A_\xi - \lambda_{i_0}(\xi)\mathrm{Id}$. Since $\lambda_{i_0}(\xi)$ achieves a local maximum at q, we have that $w_q(\lambda_{i_0}(\xi)) = 0$ $\forall w_q \in T_q U$. From the fact that A_ξ satisfies the Codazzi identity, it follows that T also satisfies the identity at q, i.e., $\langle(\nabla_{X_1} T)_q X_2, X_3\rangle$ is symmetric in all three variables $\forall\, X_1, X_2, X_3 \in T_q U$. Moreover, the hypothesis implies easily that $\ker(T) = E_{i_0}$ near q. Let $X, Y \in C^\infty(U, E_{i_0})$, $Z \in \mathscr{X}(U)$. Then it is not hard to check that $\langle(\nabla_Z T)X, Y\rangle_q = 0$. Then, by the Codazzi identity, $0 = \langle (\nabla_Y T)(X), Z\rangle_q = -\langle T_q((\nabla_Y X)_q), Z_q\rangle$. Since Z is arbitrary, $(\nabla_Y X)_q \in \ker(T)_q$. Thus E_{i_0} is autoparallel at q, and therefore is autoparallel by homogeneity.

Lemma 2.2. *Let $i_0 \in I$ such that E_{i_0} is autoparallel. If $(E_{i_0})^\perp$ is integrable, and $\langle n_{i_0}, n_i\rangle \neq \langle n_i, n_i\rangle$ for all $i \in I\backslash\{i_0\}$, then $(E_{i_0})^\perp$ is also autoparallel and $M = K.v$ is reducible.*

Proof. We have to show that $\nabla_X Y$ lies in $(E_{i_0})^\perp$ if $X, Y \in C^\infty(U, (E_{i_0})^\perp)$.

Case a. $X, Y \in C^\infty(U, E_i)$ for some $i \in I\backslash\{i_0\}$. Let $q \in U$ and let $\xi_i \in C^\infty(U, \nu_0(U))$ be parallel with $\xi_i(q) = n_i(q)$. Then $\lambda_i(\xi_i) = \langle n_i, \xi_i\rangle \leq \|n_i\|\,\|\xi_i\| = \|n_i\|^2 = \lambda_i(\xi_i)(q)$, and $\lambda_i(\xi_i)$ has a maximum at q. Thus, as in the proof of Lemma 2.1, $T^i = A_\xi - \lambda_i(\xi_i)\mathrm{Id}$ satisfies the Codazzi identity at q. (Observe that, in general, $\ker(T^i)$ does not define a distribution near q if $\lambda_i(\xi_i)(q) = \lambda_j(\xi_i)(q)$ for some $j \neq i$.) Let $Z \in \mathscr{X}(U)$. Then $\langle T^i(\nabla_Y X), Z\rangle_q = 0$ (see the proof of Lemma 2.1). Since Z is arbitrary and $\ker(T^i_q) \subset (E_{i_0})^\perp_q$ by the assumptions we conclude that $(\nabla_Y X)_q \in (E_{i_0})^\perp_q$.

Case b. $X \in C^\infty(U, E_i)$, $Y \in C^\infty(U, E_j)$, $i \neq j$. Let $q \in U$ and let $\xi \in C^\infty(U, \nu_0(U))$ be parallel with $\lambda_i(\xi)(q) \neq \lambda_j(\xi)(q)$. A direct computation, using the Codazzi identity $(\nabla_X A)_\xi(Y) = (\nabla_Y A)_\xi(X)$, shows that

$$\begin{aligned} X(\lambda_j(\xi))Y + \lambda_j(\xi)\nabla_X Y - A_\xi(\nabla_X Y) \\ = Y(\lambda_i(\xi))X + \lambda_i(\xi)\nabla_Y X - A_\xi(\nabla_Y X), \end{aligned}$$

and therefore

$$\begin{aligned} (\lambda_j(\xi) - \lambda_i(\xi))\nabla_X Y \\ = -X(\lambda_j(\xi))Y + Y(\lambda_i(\xi))X - \lambda_i(\xi)[X, Y] + A_\xi([X, Y]). \end{aligned}$$

Since $(E_{i_0})^\perp$ is integrable and invariant under A_ξ, $(\nabla_X Y)_q \in (E_{i_0})^\perp$. Thus we have shown that $(E_{i_0})^\perp$ is autoparallel. It is an easy fact that two orthogonally complementary autoparallel distributions are actually parallel. Hence, by Lemma 1.1, $M = K.v$ is reducible.

Lemma 2.3. *Let $i \in I$. Then the following hold:*

(i) *E_i is autoparallel if and only if $\nabla^\perp_Z n_i = 0$ $\forall Z \in C^\infty(U, (E_i)^\perp)$.*

(ii) *For $\dim(E_i) \geq 2$, E_i is autoparallel if and only if $\nabla^\perp n_i = 0$.*

Proof. Let $q \in U$, $X, Y \in C^\infty(U, E_i)$, $Z \in C^\infty(U, (E_i)^\perp)$, and let $\xi \in C^\infty(U, \nu_0(U))$ be parallel such that $\lambda_i(\xi)(q) \neq \lambda_j(\xi)(q)$ $\forall j \in I \backslash \{i\}$. Then we have

$$\begin{aligned}\langle(\nabla_X A)_\xi Y, Z\rangle &= X(\lambda_i(\xi))\langle Y, Z\rangle + \lambda_i(\xi)\langle\nabla_X Y, Z\rangle \\ &\quad - \langle A_\xi(\nabla_X Y), Z\rangle \\ &= \sum_{j\neq i}(\lambda_i(\xi) - \lambda_j(\xi))\langle[\nabla_X Y]^{E_j}, Z\rangle,\end{aligned}$$

which implies that $\langle(\nabla_X A)_\xi(Y), Z\rangle_q = 0$ $\forall\, Z \in C^\infty(U, (E_i)^\perp)$ if and only if $(\nabla_X Y)_q \in E_i(q)$. Thus, by the Codazzi identity, E_i is autoparallel at q if and only if

$$\begin{aligned}0 = \langle(\nabla_Z A)_\xi(X), Y\rangle_q &= Z(\lambda_i(\xi))(q)\langle X, Y\rangle_q + \lambda_i(\xi)(q)\langle\nabla_Z X, Y\rangle_q \\ &\quad - \langle A_\xi(\nabla_Z X), Y\rangle_q \\ &= Z(\lambda_i(\xi))(q)\langle X, Y\rangle_q,\end{aligned}$$

i.e., if and only if $0 = Z(\lambda_i(\xi))(q) = \langle\nabla_Z^\perp n_i, \xi\rangle_q$. Since $\{\eta_q \in \nu_0(U)_q : \lambda_i(\eta_q) \neq \lambda_j(\eta_q)\ \forall j \in I,\ j \neq i\}$ is open and dense in $\nu_0(U)_q$, we conclude part (i).

Now let $X, Y, W \in C^\infty(U, E_i)$. It is easy to check that

$$\langle(\nabla_X A)_\xi(Y), W\rangle = X(\lambda_i(\xi))\langle Y, W\rangle.$$

If $\dim(E_i) \geq 2$, then we can choose X, Y, W such that $Y = W$, $\|X\|_q = \|Y\|_q = 1$, and $X_q \perp Y_q$. Using the Codazzi identity again we get that $X(\lambda_i(\xi))(q) = 0$. Part (ii) thus follows easily from part (i).

Lemma 2.4. *Let $J = \{i \in I : E_i$ is autoparallel$\}$. Then $J \neq \varnothing$. Moreover, if $J \neq I$, then there exists $j_0 \in J$, so that for simplicity we may assume $j_0 = 1$, such that*

(i) $\dim(E_1) = 1$,

(ii) $\nabla_W^\perp n_1 = 0$ *if and only if* $W \in C^\infty(U, (E_1)^\perp)$ (*or, equivalently, n_1 is not parallel*),

(iii) $(E_1)^\perp$ *is integrable.*

Proof. Choose $q \in U$ and $i \in I$ such that $\|n_i\| = \max\{\|n_k\| : k \in I\}$. Then, by the Cauchy-Schwarz inequality, $\langle n_i, n_i\rangle \neq \langle n_i, n_k\rangle$ for all $k \in I\backslash\{i\}$. Let $\xi \in C^\infty(U, \nu_0(U))$ be parallel with $\xi(q) = n_i(q)$. Then $\lambda_i(\xi)$ achieves a maximum at q. Applying Lemma 2.1 we get that $i \in J$ and hence $J \neq \varnothing$. Assume now that $J \neq I$ and that $\nabla^\perp n_j = 0$ for all $j \in J$. We shall derive a contradiction. Let $\nu_0^J(U)$ be the (parallel) subbundle of $\nu_0(U)$ generated by $\{n_j : j \in J\}$ and consider the complementary subbundle $(\nu_0^J(U))^\perp$ of $\nu_0(U)$. It is clear that $(\nu_0^J(U))^\perp$ is (locally) invariant under the action of K. For each $l \in L = I \backslash J$ let $\tilde{n}_l = \mathrm{pr}(n_l)$, where pr denotes the orthogonal projection to $(\nu_0^J(U))^\perp$. Because of the invariance of $(\nu_0^J(U))^\perp$ under the action of the Lie group K we get that $\langle \tilde{n}_l , \tilde{n}_{l'} \rangle$ is constant for $l , l' \in L$. If $l \in L$ then $\tilde{n}_l \neq 0$; otherwise n_l would be parallel (since $\langle n_j , n_l \rangle$ is constant for all $j \in J$) and hence, by Lemma 2.3, E_l would be autoparallel. Let $l_0 \in L$ be such that $\|\tilde{n}_{l_0}\| = \max\{\|\tilde{n}_l\| : l \in L\}$ and $l_0 , \cdots , l_s$ be the different elements of L such that $\tilde{n}_{l_0} = \cdots = \tilde{n}_{l_s}$ (observe that $i \mapsto n_i$, $i \in I$, is injective but $l \mapsto \tilde{n}_l$, $l \in L$, is not necessarily injective). If $l \in L\backslash\{l_0 , \cdots , l_s\}$ then, due to the Cauchy-Schwarz inequality, $\langle \tilde{n}_{l_0} , \tilde{n}_{l_0} \rangle \neq \langle \tilde{n}_{l_0} , \tilde{n}_l \rangle$. Let now $\tilde{\xi} \in C^\infty(U , \nu_0^J(U))$ be parallel and such that $\langle \tilde{\xi} , n_{l_0} \rangle , \cdots , \langle \tilde{\xi} , n_{l_s} \rangle$ are all different (observe that $n_i - pr(n_i)$ is parallel for all $i \in I$). We can find such a $\tilde{\xi}$ because $n_{l_0} , \cdots , n_{l_s}$ are all different. Let $q \in U$ and let $\eta \in C^\infty(U , (\nu_0^J(U))^\perp)$ be parallel with $\eta(q) = \tilde{n}_{l_0}(q)$. Let $\xi = \tilde{\xi} + \eta$. If $\|\tilde{\xi}\|$ is small, then $\langle \xi , n_{l_0} \rangle_q \neq \langle \xi , n_i \rangle_q$ for all $i \in I\backslash\{l_0\}$. Moreover, $\lambda_{l_0}(\xi) = \langle \xi , n_{l_0} \rangle$ achieves its maximum at q. In fact, $\lambda_{l_0}(\xi) = c + \langle \eta , \tilde{n}_{l_0} \rangle$ where $c = \langle \tilde{\xi} , n_{l_0} - pr(n_{l_0}) \rangle$ is constant. We can now apply Lemma 2.1 to conclude that E_{l_0} is autoparallel. This contradicts the fact that $l_0 \notin J$. Thus there exists $j_0 \in J$ such that $\nabla^\perp n_{j_0} \neq 0$. By Lemma 2.3(ii) we hence obtain parts (i) and (ii) of this lemma. Let now $Z_1 , Z_2 \in C^\infty(U , (E_{j_0})^\perp)$. Then

$$\nabla^\perp_{Z_1} \nabla^\perp_{Z_2} n_{j_0} - \nabla^\perp_{Z_2} \nabla^\perp_{Z_1} n_{j_0} - \nabla^\perp_{[Z_1 , Z_2]} n_{j_0} = R^\perp(Z_1 , Z_2) n_{j_0} = 0 ,$$

because $\nu_0(U)$ is flat. But, by part (ii), $\nabla^\perp_{Z_1} n_{j_0} = 0 = \nabla^\perp_{Z_2} n_{j_0}$, which implies that $\nabla^\perp_{[Z_1 , Z_2]} n_{j_0} = 0$. Hence, again by part (ii), $[Z_1 , Z_2] \in C^\infty(U , (E_{j_0})^\perp)$, which proves part (ii).

3. Constructing a family of orbits of higher rank

We keep the notation and assumptions of §2. Assume that $I \neq J = \{i \in I : E_i$ is autoparallel$\}$. Then, by Lemma 2.4, we may assume that $1 \in J$ and the following:

(i) $\dim(E_1) = 1$,

(ii) $\nabla^\perp n_1 \neq 0$,

(iii) $(E_1)^\perp$ is integrable.

Let now $q \in U$ and let $\xi \in C^{\infty}(U, \nu_0(U))$ be parallel with $\xi(q) = n_1(q)$. Then, due to the Cauchy-Schwarz inequality, $\lambda_1(\xi) = \langle \xi, n_1 \rangle$ achieves its maximum at q. Using Lemma 2.2 we conclude that $M = K.v$ is reducible, provided the following condition holds:

$$(1) \qquad \langle n_i, n_i \rangle \neq \langle n_1, n_i \rangle \qquad \forall i \in I, \ i \geq 2.$$

Unfortunately, the generic condition (1) cannot be "a priori" guaranteed and we need to consider other orbits which are close to $M = K.v$, namely, $M_a = K.(v + a\alpha(X_1, X_1)_v)$, where $a \in \mathbb{R}$, $X_1 \in C^{\infty}(U, E_1)$ with $\|X_1\| = 1$, and α denotes the second fundamental form of M. But now the situation is much more involved than in the case where K acts polar. One must show that the family $\{M_a\}$ has also higher rank, namely, $\mathrm{rank}(M_a) = \mathrm{rank}(M)$ if $a \in \mathbb{R}$ is small. This fact depends strongly on $\dim(E_1) = 1$ and on the nontrivial

Lemma 3.1. *Using the above notation and assumptions we have*:

(i) $\alpha(X_1, X_1) \in C^{\infty}(U, \nu_0(U))$, *where* $X_1 \in C^{\infty}(U, E_1)$ *and* $\|X_1\| = 1$.

(ii) $n_1 = \alpha(X_1, X_1)$.

Proof. If $q \in U$, $\xi_q \in \nu(M)_q$, and $Z \in C^{\infty}(U, (E_1)^{\perp})$, using the Ricci identity and the flatness of $\nu_0(U)$, we get that $A_{\xi_q}(E_1(q)) \subset E_1(q)$. Moreover, it is easy to check that $A_{\xi_q}(X_1(q)) = \langle \alpha(X_1, X_1)_q, \xi_q > X_1(q)$. Choose $\tilde{\xi} \in C^{\infty}(U, \nu(U))$ with $\tilde{\xi}(q) = \xi_q$ such that $(\nabla^{\perp}\tilde{\xi})_q = 0$. Since E_1 is autoparallel, $\langle (\nabla_{X_1} A)_{\tilde{\xi}} X_1, Z \rangle_q = 0$. Then, by the Codazzi identity, $\langle (\nabla_Z A)_{\xi} X_1, X_1 \rangle_q = 0$. It is now easy to see that $\langle (\nabla_Z^{\perp} \alpha(X_1, X_1))_q, \xi_q \rangle = 0$. Since q and ξ_q are arbitrary, we conclude that $\nabla_Z^{\perp} \alpha(X_1, X_1) = 0$ $\forall\, Z \in C^{\infty}(U, (E_1)^{\perp})$. We have seen that E_1 (and hence $(E_1)^{\perp}$) is preserved by all the shape operators of U. Thus, by the Ricci identity, $R^{\perp}(X_1, Z) = 0$ $\forall Z \in C^{\infty}(U, (E_1)^{\perp})$. It is not hard to see now that given $q \in U$, $c\colon [0, 1] \to U$ piecewise differentiable with $c(0) = c(1) = q$, there exist $c_1, c_2\colon [0, 1] \to U$ piecewise differentiable such that (a) $c_1(0) = c_1(1) = q = c_2(0) = c_2(1)$, (b) $c_1'(t) \in (E_1)_{c_1(t)}$, $c_2'(t) \in (E_1)^{\perp}_{c_2(t)}$ $\forall t \in [0, 1]$, (c) $\tau_c^{\perp} = \tau_{c_1}^{\perp} \circ \tau_{c_2}^{\perp}$, where $\tau^{\perp}$ denotes the $\nabla^{\perp}$-parallel transport. (For a proof of this fact see [12, appendix].) Since $\dim(E_1) = 1$, we have $\tau_{c_1}^{\perp} = \mathrm{id}_{\nu_0(U)_q}$ which, together with $\nabla_Z^{\perp} \alpha(X_1, X_1) = 0$ $\forall Z \in C^{\infty}(U, (E_1)^{\perp})$, implies that $\tau_c^{\perp}(\alpha(X_1, X_1)_q) = \alpha(X_1, X_1)_q$. Therefore $\Phi_q^U \alpha(X_1, X_1)_q = \alpha(X_1, X_1)_q$, where Φ_q^U denotes the normal holonomy group of U at q. Hence part (i) is proved. Part (ii) is an easy consequence of (i). q.e.d.

Using the assumptions of this section, choose $X_1 \in C^\infty(U, E_1)$ with $\|X_1\| = 1$ (observe that $\langle\{X_1\}\rangle = E_1$). For $k \in \mathbb{N} \cup \{0\}$ let $n_1^{(k)}$ be defined by

(i) $n_1^{(0)} = n_1$,

(ii) $n_1^{(k+1)} = \nabla^\perp_{X_1} n_1^{(k)}$.

Then, by homogeneity, we get that $\langle n_1^{(k)}, n_i\rangle$ and $\langle n_1^{(k)}, n_1^{(j)}\rangle$ are constant ($i \in I$, $j, k \in \mathbb{N} \cup \{0\}$). Let $\nu_0^1(U)$ be the subbundle of $\nu_0(U)$ defined by

$$\nu_0^1(U)_q = \langle\{n_1^{(k)}(q) \colon k \geq 0\}\rangle, \qquad q \in U.$$

Then we have the following lemma.

Lemma 3.2. *By the same notation and assumptions of this section,*

(i) $\nabla^\perp_Z n_1^{(k)} = 0 \ \forall Z \in C^\infty(U, (E_1)^\perp)$, $k \geq 0$,

(ii) $\nu_0^1(U)$ *is a parallel subbundle of* $\nu_0(U)$.

Proof. (i) By induction on k. If $k = 0$ it is true by Lemma 2.3(i). Before continuing with the induction let us observe that $[X_1, Z] \in C^\infty(U, (E_1)^\perp)$ if $Z \in C^\infty(U, (E_1)^\perp)$. In fact, if ∇ is the Levi-Civita connection in U, then $\langle\nabla_Z X_1, X_1\rangle = 0$ since $\|X_1\| = 1$, and $\langle\nabla_{X_1} Z, X_1\rangle = -\langle Z, \nabla_{X_1} X_1\rangle = 0$ since E_1 is autoparallel. Thus $[X_1, Z] = \nabla_{X_1} Z - \nabla_Z X_1 \in C^\infty(U, (E_1)^\perp)$. Now assume that $\nabla^\perp_Z n_1^{(k)} = 0$. Then

$$\nabla^\perp_Z n_1^{(k+1)} = \nabla^\perp_Z \nabla^\perp_{X_1} n_1^{(k)} = \nabla^\perp_{X_1} \nabla^\perp_Z n_1^{(k)} - \nabla^\perp_{[X_1, Z]} n_1^{(k)} = 0,$$

by the inductive hypothesis and $R^\perp = 0$ on $\nu_0(U)$. Hence, part (i) is proved. Part (ii) is an immediate consequence of (i). q.e.d.

Using Lemma 3.2(i) for $k = 0$, it is not hard to prove the following.

Lemma 3.3. *Let $a \in \mathbb{R}$ and let $f_a \colon U \to \mathbb{R}^N$ be defined by $f_a(q) = q + a n_1(q)$. Then:*

(i) $df_a(Z_i) = (1 - a\lambda_i(n_1))Z_i \ \forall\, Z_i \in C^\infty(U, E_i)$, $i \in I$, $i \geq 2$,

(ii) $df_a(X_1) = (1 - a\lambda_1(n_1))X_1 + a n_1^{(1)}$,

(iii) $\exists\, \varepsilon > 0$ *such that* $|a| < \varepsilon \Rightarrow f_a$ *is an embedding and* $f_a(U)$ *is an open subset of* $M_a = K.(v + a n_1(v))$.

Notation. $E_i^a = df_a(E_i)$, $U_a = f_a(U)$.

Lemma 3.4. *In the notation of Lemma 3.3, if $a \in \mathbb{R}$ is small, then E_1^a defines an autoparallel (one-dimensional) foliation in U_a which is invariant under all the shape operators of U_a.*

Proof. For $q \in U$ let $c_q\colon (-\delta_q, \delta_q) \mapsto U$ be the integral curve of X_1 with $c(0) = q$. Since E_1 is autoparallel, c_q is a geodesic in U. Let us show that the curve $f_a \circ c_q$, which defines an integral manifold of the foliation E_1^a of U_a, is also a geodesic in U_a. At first, we have

$$\begin{aligned}\frac{d}{dt}(f_a \circ c_q(t)) &= \frac{d}{dt}(c_q(t) + an_1 \circ c_q(t))\\ &= (\mathrm{Id} - aA_{n_1 \circ c_q(t)})X_1 \circ c_q(t) + an_1^{(1)} \circ c_q(t)\\ &= (1 - a\langle n_1, n_1\rangle)X_1 \circ c_q(t) + an_1^{(1)} \circ c_q(t).\end{aligned}$$

Due to homogeneity, $\|n_1^{(1)}\|$ is constant, and therefore $\|f_a \circ c_q\|$ is constant. Since, by Lemma 3.1, $d^2 c_q(t)/dt^2 = n_1 \circ c_q(t)$, using Lemma 3.3 it is straightforwarded to show that

$$\frac{d^2}{dt^2} f_a \circ c_q(t) \perp (E_1^a)^{\perp}_{f_a \circ c_q(t)} = df_a(E_1)^{\perp}_{c_q(t)},$$

which implies that $f_a \circ c_q$ is a geodesic in U_a, so that E_1^a is an autoparallel distribution. Regard now $c_q(t)$ as a submanifold of U, and $(E_1)^{\perp}_{c_q(t)}$ as its normal bundle. Let $D^{\perp}/dt$ be the covariant derivative operator on $(E_1)^{\perp}_{c_q(t)}$ induced by the Levi-Civita connection of U. Let $v \in (E_1)^{\perp}_{c_q(0)}$ and let $\tilde{v}$ be its parallel transport along c_q, i.e., $\tilde{v}(t) \in (E_1)^{\perp}_{c_q(t)}$, $D^{\perp}\tilde{v}(t)/dt = 0$, $\tilde{v}(0) = v$, $\forall t \in (-\delta_q, \delta_q)$. Since c_q is a totally geodesic submanifold of U, it is easy to see that $d\tilde{v}(t)/dt = \alpha(c_q'(t), \tilde{v}(t)) = 0$, because E_1 (and hence $(E_q)^{\perp}$) is preserved by all the shape operators of U (see the proof of Lemma 3.1(i)). Thus $\tilde{v}(t)$ is constant. By Lemma 3.3, we get that $\tilde{v}(t)$ can also be regarded as a normal vector field, in U_a, to the geodesic $f_a \circ c_q(t)$. Since $\tilde{v}(t)$ is constant, it follows easily that $\alpha^a((f_a \circ c_q)'(0), v) = 0$, where α^a is the second fundamental form of U_a. Hence $(E_1^a)_{f_a(q)}$ is preserved by all the shape operators of U_a at $f_a(q)$.

Lemma 3.5. *Let $i, j \in I\backslash\{1\}$ and let $X_i \in C^{\infty}(U, E_i)$, $Y_j \in C^{\infty}(U, E_j)$. Then the following hold:*

(i) $\langle \nabla_{X_i} X_1, X_j\rangle = 0$ *if* $i \neq j$.

(ii) *Assume that* $i = j$. *Then* $\langle \nabla_{X_i} X_1, Y_i\rangle = 0$ *if* $\langle X_i, Y_i\rangle = 0$.

(iii) $(\nabla X_1)_{|E_i(q)}\colon E_i(q) \to E_i(q)$ *is proportional to the identity map.*

Proof. Assume that $i \neq j$. Then $\langle \nabla_{X_i} X_1, X_j\rangle = -\langle X_1, \nabla_{X_i} X_j\rangle$. Since $(E_1)^{\perp}$ is integrable, using the Codazzi identity (as in the proof of Lemma

2.2) yields that $\nabla_{X_i} X_j \in C^\infty(U, (E_1)^\perp)$ and hence part (i). Assume now that $i = j$ and that $\langle X_i, Y_i \rangle = 0$. Let $\xi \in C^\infty(U, \nu_0(U))$ be parallel. It is easy to check that $\langle (\nabla_{X_1} A)_\xi X_i, Y_i \rangle = 0$. By the Codazzi identity we obtain that

$$0 = \langle (\nabla_{X_i} A)_\xi Y_i, X_1 \rangle = (\lambda_i(\xi) - \lambda_1(\xi)) \langle \nabla_{X_i} Y_i, X_1 \rangle.$$

Part (ii) now follows easily. Part (iii) is an easy consequence of parts (i) and (ii).

4. The rank of the family U_a and explicit computations

Use the assumptions and notation of §3, let $\{n_1^{(1)}\}^\perp$ be the subbundle of $\nu(U)$ which is orthogonal to $n_1^{(1)}$, and let $(f_a^{-1})^*(\{n_1^{(1)}\}^\perp)$ be its pullback over f_a^{-1}, where $a \in \mathbb{R}$ is small and fixed. If $(q, \xi) \in \{n_1^{(1)}\}^\perp$ then, due to Lemma 3.3, $(f_a(q), \xi) \in \nu(U_a)_{f_a(q)}$. So, we shall always regard $(f_a^{-1})^*(\{n_1^{(1)}\}^\perp)$ as a codimension-1 subbundle of $\nu(U_a)$, and write

$$\nu(U_a) = \mathbf{W}^a \oplus (f_a^{-1})^*(\{n_1^{(1)}\}^\perp),$$

where $\mathbf{W}^a = \langle \{w^a\} \rangle$, and $w^a \in C^\infty(U_a, \nu(U_a))$ is defined by

$$w^a(f_a(q)) = n_1^{(1)}(q) - pr_1^a(n_1^{(1)}(q)),$$

pr_1^a denoting the orthogonal projection to the subspace

$$(E_1^a)_{f_a(q)} = \langle \{(1 - a\langle n_1, n_1 \rangle) X_1(q) + a n_1^{(1)}(q)\} \rangle,$$

(recall that $\langle \{X_1\} \rangle = E_1$, $\|X_1\| = 1$).

Remark 4.1. By homogeneity there exist $\beta_1(a), \beta_2(a) \in \mathbb{R}$ such that $w^a(f_a(q)) = \beta_1(a) X_1(q) + \beta_2(a) n_1^{(1)}(q)$.

Consider now the subbundle

$$\mathscr{H} = \{n_1^{(1)}\}^\perp \cap \nu_0^1(U)$$

of $\nu_0(U)$. Then there exist linearly independent $h_1, \cdots, h_s \in C^\infty(U, \mathscr{H})$ such that $\langle \{h_1, \cdots, h_s\} \rangle = \mathscr{H}$ and

$$(1) \qquad \nabla_Z^\perp h_i = 0 \qquad \forall Z \in C^\infty(U, (E_1^\perp)), \ i = 1, \cdots, s,$$

(see Lemma 3.2 and its preceding paragraph). Thus $(f_a^{-1})^*(\mathscr{H}) \subset \nu(U_a)$ is a trivializable subbundle, namely, $\{h_i \circ f_a^{-1} : 1 \le i \le s\}$ provides a trivialization of this subbundle. Hence $\{w^a, h_i \circ f_a^{-1} : 1 \le i \le s\}$ gives a trivialization of $\mathbf{W}^a \oplus (f_a^{-1})^*(\mathscr{H})$. (Observe that $\mathbf{W}^0 \in (f_0^{-1})^*(\mathscr{H}) = \nu_0^1(U)$.)

Lemma 4.2. $\mathbf{W}^a \oplus (f_a^{-1})^*(\{n_1^{(1)}\}^\perp \cap \nu_0^1(U))$ *is a subbundle of* $\nu_0(U_a)$.

Proof. We shall show that w^a, $h_i \circ f_a^{-1} \in C^\infty(U_a, \nu_0(U_a))$ $(i = 1, \cdots, s)$. The proof is similar to that of Lemma 3.1 after having shown the following:

(a) $(R^a)^\perp(X, Y) = 0$ if $X \in C^\infty(U_a, E_1^a)$, $Y \in C^\infty(U_a, (E_1^a)^\perp)$, where $(R^a)^\perp$ is the normal curvature tensor of U_a.

(b) $\nabla_Z^\perp w^a = 0 = \nabla_Z^\perp h_i \circ f_a^{-1} = 0$ $\forall Z \in C^\infty(U_a, (E_1^a)$, $i = 1, \cdots, s$.

(a) follows from Lemma 3.4 and the Ricci identity. By Lemma 3.3, as subspaces of $\mathbb{R}^N$ we have

$$(E_1)_q^\perp = (E_1^a)_{f_a(q)}^\perp = df_a((E_1)_q^\perp) \qquad \forall q \in U.$$

Thus (1) yields the second equality of part (b). Let $q \in U$, $v \in (E_1)_q^\perp$, let $c\colon [0, 1] \to U$ be C^∞ with $c'(0) = v$, and let $c^a = f_a \circ c$. Then $w^a(c^a(q)) = \beta_1(a)X_1(c(t)) + \beta_2(a)n_1^{(1)}(c(t))$, where $\beta_1(a), \beta_2(a) \in \mathbb{R}$ (see Remark 4.1). Since $\|X_1\|$ is constant and $\alpha(X_1(q), v) = 0$, we have that $d/dt|_0 X_1(c(t)) \in (E_1)^\perp(q)$, and also that $d/dt|_0 n_1^{(1)}(c(t)) = -A_{n_1^{(1)}(q)}v \in (E_1)_q^\perp = (E_1^a)_{f_a(q)}^\perp$. Thus $d/dt|_0 w^a(c^a(t)) \in (E_1^a)_{f_a(q)}^\perp$. Since q is arbitrary and $df_a((E_1)^\perp) = (E_1^a)^\perp$, part (b) is proved.

Lemma 4.3. $\nu_0(U_a) = \mathbf{W}^a \oplus (f_a^{-1})^*(\{n_1^{(1)}\}^\perp \cap \nu_0(U))$.

Proof. We shall regard $(f_a^{-1})^*((\nu_0^1(U))^\perp)$ as a subbundle of $\nu(U_a)$. Let $p, q \in U$, and let $c\colon [0, 1] \to U$ be C^∞ such that $c(0) = p$ and $c(1) = q$. Let $\xi_q \in (\nu_0^1(U))_q^\perp$ and let $\xi \in C^\infty([0, 1], c^*(\nu_0(U)))$ be parallel to $\xi(0) = \xi_q$. (Observe that since $\nu_0^1(U)$ is parallel, $\xi(t) \in (\nu_0^1(U))_{c(t)}^\perp$ $\forall t \in [0, 1]$.) Then

$$\frac{d}{dt}\xi(t) = -A_{\xi(t)}c'(t) = -\langle \xi(t), n_1 \circ c(t)\rangle [c'(t)]^1 - A_{\xi(t)}[c'(t)]^2,$$

where $c'(t) = [c'(t)]^1 + [c'(t)]^2$, $[c'(t)]^1 \in (E_1)_{c(t)}$, $[c_2'(t)]^2 \in (E_1)_{c(t)}^\perp$. From $\langle \xi(t), n_1 \circ c(t)\rangle \equiv 0$ it follows that $\frac{d}{dt}\xi(t) \in (E_1)_{c(t)}^\perp = (E_1^a)_{f_a(c(t))}^\perp$. Thus $\xi(t)$ can also be regarded as a parallel section in $(f_a^{-1})^*((\nu_0^1(U))^\perp)$ along the curve $f_a \circ c(t)$. This shows that, for all $q \in U$,

$$(\Phi^a)^*_{f_a(q)|(\nu_0^1(U))_q^\perp} = \Phi^*_{q|(\nu_0^1(U))_q^\perp},$$

where $(\Phi^a)^*$ denotes the restricted normal holonomy group of U_a, and

$(\nu_0^1(U))_q^\perp = (f_a^{-1})^*((\nu_0(U))^\perp)_{f_a(q)}$ as subspaces of $\mathbb{R}^N$. The proof of this lemma follows now easily from Lemma 4.2.

Lemma 4.4. *Under the assumptions of this section if a is small, then $E_1^a = df_a(E_1), \cdots, E_g^a = df_a(E_g)$ are the different eigendistributions of $\{A_\xi^a : \xi \in C^\infty(U_a, \nu_0(U_a))\}$ where A^a denotes the shape operator of U_a.*

Proof. We shall prove first that each E_i^a $(i \in I)$ is contained in an eigendistribution of the shape operators $\{A_\xi^a : \xi \in C^\infty(\nu_0(U_a))\}$. If $i = 1$, this is true by Lemma 3.4. Let $i \in I$, $i \geq 2$, $q \in U$, and let $v_i \in E_i(q)$. Let $c_i : [0, 1] \to U$ be C^∞ such that $c_i'(t) \in (E_i)_{c_i(t)}$ for all $t \in [0, 1]$ and $c_i'(0) = v_i$. Define $\tilde{c}_i = f_a \circ c_i$ and $\tilde{v}_i = (df_a)_q(v_i) = (1 - a\langle n_1, n_i\rangle)v_i$.

Case a. Let $\xi \in C^\infty(U, \{n_1^{(1)}\}^\perp \cap \nu_0(U))$ and let $\tilde{\xi} = \xi \circ f_a^{-1} \in C^\infty(U_a, (f_a^{-1})^*(\{n_1^{(1)}\}^\perp \cap \nu_0(U))$ (see Lemma 4.3). Then

$$A^a_{\tilde{\xi}(q)}\tilde{v}_i = \left[-\left(\frac{d}{dt}\Big|_0 \tilde{\xi} \circ \tilde{c}_i(t)\right)\right]_{T_{f_a(q)}U_a} = \left[-\left(\frac{d}{dt}\Big|_0 \tilde{\xi} \circ \tilde{c}_i(t)\right)\right]_{(E_1^a)^\perp_{f_a(q)}}$$

(by Lemma 3.4)

$$= \left[\frac{d}{dt}\Big|_0 \xi \circ c_i(t)\right]_{(E_1)_q^\perp} = \left[-\frac{d}{dt}\Big|_0 \xi \circ c_i(t)\right]_{E_i(q)}$$

$$= A_{\xi(q)}v = \langle n_i(q), \xi(q)\rangle v_i = \langle n_i(q), \xi(q)\rangle (1 - a\langle n_1, n_i\rangle)^{-1}\tilde{v}_i.$$

Case b. Let $w^a \in C^\infty(U_a, \nu_0(U_a))$ be defined as at the beginning of this section (see also Lemma 4.3). Using Remark 4.1 we have

$$A^a_{w^a(f_a(q))}\tilde{v}_i = \left[-\frac{d}{dt}\Big|_0 w^a(\tilde{c}_i(t))\right]_{(E_1^a)^\perp_{f_a(q)}}$$

$$= \left[-\frac{d}{dt}\Big|_0 w^a(\tilde{c}_i(t))\right]_{(E_1)_q^\perp} = -\beta_1(a)\nabla_{v_i}X_1 + \beta_2(a)A_{n_1^{(1)}(q)}v_i$$

(by Lemma 3.2, $\nabla^\perp_{v_i}n_1^{(1)} = 0$)

$$= -\beta_1(a)\nabla_{v_i}X_1 + \beta_2(a)\langle n_i(q), n_1^{(1)}(q)\rangle v_i = \lambda(a)\tilde{v}_i$$

(see Lemma 3.5).

Thus we have shown that each E_1^a is contained in some eigendistribution, so that the number of different eigendistributions of $\{A_\xi^a : \xi \in C^\infty(U_a, \nu_0(U_a))\}$ is less than or equal to $g = \#(I)$. Since a continuity argument shows that the number of different eigendistributions, for a small, cannot be less than g, we get the lemma.

Lemma 4.5. *In the assumptions of this section. Let $a \in \mathbb{R}$ be small enough that $E_1^a, \cdots, E_g^a$ be the different eigendistributions of $\{A_\xi : \xi \in C^\infty(U_a, \nu_0(U_a))\}$. Let $\{n_i^a : i \in I\}$ be the corresponding curvature normals (i.e., $A_\xi^a(Y_i) = \langle n_i^a, \xi\rangle Y_i$ if $Y_i \in C^\infty(U_a, E_i^a)$). If $\mu(a) = (1 - a\|n_1\|^2)^2 + a^2\|n_1^{(1)}\|^2$, the following hold:*

(i) $n_1^a \circ f_a = (\mu(a))^{-1}[(1 - a\|n_1\|^2)n_1 + an_1^{(2)}]$.

(ii) $\langle n_1^a, n_i^a\rangle = \frac{[(1-a\|n_1\|^2)\langle n_1, n_i\rangle + a\langle n_1^{(2)}, n_i\rangle]}{\mu(a)(1-a\langle n_1, n_i\rangle)}$, *for* $i \geq 2$.

(iii) $\langle n_i^a, n_i^a\rangle = P_i(a) + c_i(1 - a\langle n_1, n_i\rangle)^{-2}$ *for* $i \geq 2$, *where* $c_i \in \mathbb{R} \geq 0$, *and* $P_i(a)$ *is a rational function on* a *with* $P_i(a) \geq 0\ \forall \in \mathbb{R}$. *Moreover, if* $c_i = 0$, *then* $\langle n_1, n_i\rangle = 0$.

Proof. (i) Let $q \in U$ and let $\gamma: (-\varepsilon, \varepsilon) \to U$ be the integral curve of X_1 with $\gamma(0) = q$. Then $\tilde{\gamma}(t) = \gamma(t) + an_1(\gamma(t))$ is the integral curve of E_1^a through $f_a(q)$. Moreover, since by Lemma 3.4 E_1^a is autoparallel, $\tilde{\gamma}$ is a geodesic in U_a. Then, using Lemma 3.1, we get that

$$n_1^a(f_a(q)) = \|\tilde{\gamma}'(0)\|^{-2}\frac{d^2}{dt^2}|_0\tilde{\gamma}(t).$$

We also have that

$$\begin{aligned}\frac{d^2}{dt^2}\bigg|_0 \tilde{\gamma}(t) &= \frac{d}{dt}\bigg|_0 (\gamma'(t) + an_1^{(1)}(\gamma(t)) - aA_{n_1(\gamma(t))}\gamma'(t)) \\ &= (1 - a\|n_1\|^2)\gamma''(0) + an_1^{(2)}(q) - aA_{n_1^{(1)}(q)}\gamma'(0) \\ &= (1 - a\|n_1\|^2)n_1(q) + an_1^{(2)}(q),\end{aligned}$$

(observe that $A_{n-1^{(i)}(q)}\gamma'(0) = 0$ because of $\langle n_1^{(1)}, n_1\rangle = 0$).

(ii) Let $\pi^a: \nu_0(U_a) \to (f_a^{-1})^*(\{n - 1^{(i)}\}^\perp \cap \nu_0(U_a))$ be the orthogonal projection (see Lemma 4.2). Let $i \in I$, $i \geq 2$, and let $c_i: [0, 1] \to U$ be C^∞ with $c_i'(0) = v_i \in (E_i)_{c_i(0)}$. Let $\xi \in C^\infty(U, \{n_1^{(1)}\}^\perp \cap \nu_0(U))$ and let $\tilde{c}_i = f_a \circ c_i$, $\tilde{\xi} = \xi \circ f_a^{-1} \in (f_a^{-1})^*(\{n_1^{(1)}\}^\perp \cap \nu_0(U))$. Then

$$\begin{aligned}A^a_{\tilde{\xi}(f_a(q))}df_a(v_i) &= \left[-\frac{d}{dt}\bigg|_0 \tilde{\xi}(\tilde{c}_i(t))\right]_{E_i^a(f_a(q))} \\ &= \left[-\frac{d}{dt}\bigg|_0 \xi(c(t))\right]_{E_i(q)} = A_{\xi(q)}v_i = \langle n_i(q), \xi(q)\rangle v_i.\end{aligned}$$

Since $df_a(v_i) = (1 - a\langle n_1, n_i\rangle)v_i$, we get $\pi((1 - a\langle n_1, n_i\rangle)^{-1}n_i(q)) = \pi^a(n_i^a(q))$, $\pi = \pi^0$. Since $\langle n_1, n_1^{(1)}\rangle = 0 = \langle n_1^{(1)}, n_1^{(2)}\rangle$ part (i) yields $\pi^a(n_1^a) = n_1^a$. Part (ii) thus follows easily from (i).

(iii) Let $i \geq 2$ and let $P_i(a) = \|n_i^a - \pi^a(n_i^a)\|^2$, $Q_i(a) = \|\pi^a(n_i^a)\|^2$. Then $\langle n_i^a, n_i^a\rangle = P_i(a)+Q_i(a)$, and it is clear that P_i, Q_i are both rational functions and $P_i, Q_i \colon \mathbb{R} \to \mathbb{R}_{\geq 0} \cup \{+\infty\}$ when extended to $\mathbb{R}$. Let us compute Q_i explicitly:

$$\begin{aligned} Q_i(a) &= \langle \pi^a(n_i^a), \pi^a(n_i^a)\rangle \\ &= \frac{\langle \pi(n_i), \pi(n_i)\rangle}{1 - a\langle n_1, n_i\rangle)^{-2}} \\ &= c_i(1 - a\langle n_1, n_i\rangle)^{-2}, \end{aligned}$$

where $c_i = \|\pi(n_i)\|^2$ is a constant because of the homogeneity. If $c_i = 0$, then $\pi(n_i) = 0$, and hence n_i is proportional to $n_1^{(1)}$, which is perpendicular to n_1.

Corollary 4.6. *$(E_1^0)^\perp = E_1^\perp$ is autoparallel distribution, and hence $M = K.v$ is reducible.*

Proof. Let $i \in I$, $i \geq 2$. Extend $R_i(a) = \langle n_1^a, n_i^a\rangle$ and $S_i(a) = \langle n_i^a, n_i^a\rangle = P_i(a) + Q_i(a)$ to rational functions defined for all $a \in \mathbb{R}$ (see Lemma 4.5). If $\langle n_1, n_i\rangle = 0$, then $R_i(0) \neq S_i(0)$ and hence $R_i \neq S_i$. If $\langle n_1, n_i\rangle \neq 0$, then $c_i \neq 0$. So, $\mu(a)(1 - a\langle n_1, n_i\rangle)S_i(a) \to +\infty$ if $a \to (\langle n_i, n_1\rangle^{-1})^-$. But $\mu(a)(1-a\langle n_1, n_i\rangle)R_i(a)$ for $i \in I$ and $i \geq 2$ is linear on a. If a is small and $a \neq 0$, $S_i(a) \neq R_i(a)$ for all $i \in I$, $i \geq 2$. Since $(E_1^a)^\perp$ is integrable, by Lemma 2.2 we conclude that $M_a = K.(v + an_1)$ is reducible for a small, and $a \neq 0$. It is now clear that $(E_1^0)^\perp = E_1^\perp$ is autoparallel, and hence $M = K.v$ is reducible. q.e.d.

Combining the results of §§2, 3 and 4 gives

Proposition 4.7. *Let M^n, $n \geq 2$, be a compact homogeneous irreducible full submanifold of $\mathbb{R}^N$. Let $U \subset M$ be a simply connected open subset, and let $n_1, \cdots, n_g \in C^\infty(U, \nu_0(U))$ be the curvature normals associated to the different eigenvalues of the shape operator of M restricted to $\nu_0(U)$. Then:*

(i) $\nabla^\perp n_i = 0$ *for* $i = 1, \cdots, g$,

(ii) $\langle\{n_1, \cdots, n_k\}\rangle = \nu_0(U)$.

Observe that (ii) is a consequence of (i) and [3].

Definition 4.8. If N is a submanifold of $\mathbb{R}^m$ we say that N is $\nu_0(N)$-*isoparametric* if the shape operator A_ξ has constant eigenvalues for any parallel local section ξ of $\nu_0(N)$.

Corollary 4.9. *Let M^n, $n \geq 2$, be a compact homogeneous irreducible submanifold of $\mathbb{R}^N$. Then M is $\nu_0(M)$-isoparametric.*

5. Transvections of the normal connection. Proof of Theorem C

In the same way as for a connection on the tangent bundle it is useful to study the transvections of the normal connection (cf. [9]). Let N be a submanifold of $\mathbb{R}^m$, and let $g \in I(\mathbb{R}^m)$ be such that $g(N) = N$. We say that g is a *transvection* of N with respect to the normal connection $\nabla^\perp$ if for any $p \in N$ there exists $c\colon [0, 1] \to N$ piecewise differentiable with $c(0) = p$, $c(1) = g(p)$ such that

$$\text{(A)} \qquad dg|_{\nu(N)_p} = \tau_c^\perp ,$$

where $\tau^\perp$ denotes the $\nabla^\perp$-parallel transport. The set of all transvections of N (with respect to $\nabla^\perp$) will be denoted by $\mathrm{Tr}(N, \nabla^\perp)$. In a similar way we define $\mathrm{Tr}_0(N, \nabla^\perp)$ (resp. $\mathrm{Tr}_s(N, \nabla^\perp)$) by replacing condition (A) by

$$(\mathrm{A}_0) \qquad dg|_{\nu_0(N)_p} = \tau^\perp_{c|_{\nu_0(N)_p}}$$

(resp. by (A_s): $dg|_{\nu_s(N)_p} = \tau^\perp_{c|_{\nu_s(N)_p}}$, where $\nu_s(N) = (\nu_0(N))^\perp$). Clearly, $\mathrm{Tr}(N, \nabla^\perp) \subset \mathrm{Tr}_0(N, \nabla^\perp) \cap \mathrm{Tr}_s(N, \nabla^\perp)$. With the above notation Theorem C can be reformulated as follows.

Theorem 5.1. *Let $M^n = K.v$ be a compact homogeneous submanifold of $\mathbb{R}^N$, where $K \subset I(\mathbb{R}^N)$ is connected. Then*:

(i) $K \subset \mathrm{Tr}_s(M, \nabla^\perp)$,

(ii) $K \subset \mathrm{Tr}(M, \nabla^\perp)$ *if M^n is an irreducible full submanifold of $\mathbb{R}^N$ with $n \geq 2$.*

For the proof of Theorem 5.1 we need the following well-known lemma.

Lemma 5.2. *Let H be a connected Lie subgroup of $SO(N)$, which acts on $\mathbb{R}^N$ as an s-representation, and let $N(H)_0$ be the connected component of the normalizator of H in $SO(N)$. Then $H = N(H)_0$.*

Proof of Theorem 5.1. Let $k \in K$, and let $\tilde{k}_t$ be a differentiable curve in K ($t \in [0, 1]$) with $\tilde{k}_0 = \mathrm{id}$, $\tilde{k}_1 = k$. Let $p \in M$, and let $\gamma(t) = k_t \cdot p$. Clearly, $\Phi^*_{\gamma(t)} = (d\tilde{k}_t)\Phi^*_p(d\tilde{k}_t)^{-1}$, where Φ^* denotes the restricted normal holonomy group. Then

$$\Phi^*_p = (\tau^\perp_{\gamma_t})^{-1}\Phi^*_{\gamma(t)}\tau^\perp_{\gamma_t} = h_t\Phi^*_p h_t^{-1},$$

where $\tau^\perp_{\gamma_t}$ is the $\nabla^\perp$-parallel transport along $\gamma_t = \gamma|_{[0, t]}$, and $h_t = (\tau^\perp_{\gamma_t})^{-1} \circ d\tilde{k}_t$. Hence, $h_1 = (\tau^\perp_\gamma)^{-1} \circ dk \in N_0(\Phi^*_p)$. But, by [11], Φ^*_p acts on $\nu_s(M)_p$ as an s-representation. Then, by Lemma 5.2, there exists $g = \tau^\perp_c \in \Phi^*_p$, where c is a homotopically null loop at p such that

$dk|_{\nu_s(M)_p} = \tau_\gamma^\perp \circ \tau_c^\perp|_{\nu_s(M)_p}$. Hence $dk|_{\nu_s(M)_p} = \tau_{c*\gamma}^\perp|_{\nu_s(M)_p}$ which proves part (i). Since c is homotopically null, we have that $\tau_c^\perp|_{\nu_0(M)_p} = \mathrm{id}$. Assume now, in addition, that M^n is an irreducible full submanifold. Then, from Proposition 4.7, it follows that $dk|_{\nu_0(M)_p} = \tau_\gamma^\perp|_{\nu_0(M)_p} = \tau_{c*\gamma}^\perp|_{\nu_0(M)_p}$, which proves part (ii).

6. Homogeneity of the slices under the normal holonomy group. Proof of Theorem A

Let $M^n = K.v$ be a compact irreducible full submanifold of $\mathbb{R}^N$ $(n \geq 2)$. If $\xi \in \nu_0(M)_v$, then it is not difficult to see, from Proposition 4.7 and Theorem 5.1, that $M_\xi = K(v+\xi)$, where

$$M_\xi = \{c(1) + \tilde{\xi}(1) \text{ such that } c\colon [0, 1] \to M \text{ is piecewise } C^\infty,\ c(0) = v,$$
$$\text{and } \tilde{\xi} \text{ is } \nabla^\perp\text{-parallel along } c \text{ with } \tilde{\xi}(0) = \xi\}$$

(cf. [7]). If ξ is small, then $\dim(M_\xi) = \dim(M^n)$, and it is standard to show that $\operatorname{rank}(M_\xi) = \operatorname{rank}(M)$. Moreover, due to $(M_\xi)_{-\xi} = M$, M_ξ is also an irreducible full submanfiold of $\mathbb{R}^N$. If $\#(K_v\xi)$ is maximal, then $\nu_0(M_\xi)$ is globally flat (ξ small). So, by passing perhaps to a parallel orbit, we may assume that $\nu_0(M)$ is globally flat. Thus there exist globally defined autoparallel distributions $E_1, \cdots, E_g$ on M and different $\nabla^\perp$-parallel $n_1, \cdots, n_g \in C^\infty(M, \nu_0(M))$ such that $TM = E_1 \oplus \cdots \oplus E_g$ and $A_\xi X_i = \lambda_i(\xi) X_i = \langle n_i, \xi\rangle X_i$ if $\xi \in C^\infty(M, \nu_0(M))$, $X_i \in C^\infty(M, E_i)$, for $i \in I = \{1, \cdots, g\}$. Since M is contained in a sphere, $n_1, \cdots, n_g$ are all different from the zero section, namely, the position vector provides a parallel normal vector field to M, whose shape operator is minus the identity tensor. Hence we can find parallel $\xi_1, \cdots, \xi_g \in C^\infty(M, \nu_0(M))$ such that $\langle n_i, \xi_j\rangle = 1$ if and only if $i = j$ $(i, j = 1, \cdots, g)$.

Let $i \in I$ be fixed, and let M_{ξ_i} be the focal parallel manifold to M through $v + \xi_i(v)$. Then $M_{\xi_i} = K(v + \xi_i(v))$, and the map $\pi_i\colon M \to M_{\xi_i}$, $\pi(i(q)) = q + \xi_i(q)$ is a submersion. If $q \in M$, then $T_q\pi_i^{-1}(\pi_i(q)) = E_i(q) = \ker(\mathrm{Id} - A_{\xi_i})_q$, and $T_{\pi_i(q)} M_{\xi_i} = \sum_{j \neq i} E_j(q) = (E_i)_q^\perp$ (cf. [15], [7]). Let $S_i(q)$ be the connected component of $\pi_i^{-1}(\pi_i(q))$ containing q. Observe that $S_i(q)$ is also the integral manifold through q of the distribution $\ker(I - A_{\xi_i})$. Clearly $S_i(q) \subset \nu(M_{\xi_i})_{\pi_i(q)}$, and moreover, we have

Lemma 6.1. *For the notation and assumptions of this section, let $q \in M$ and let $K^q = \{k \in K\colon kS_i(q) = S_i(q)\}$. Then $(K^q)_0$ acts transitively on*

$S_i(q)$ and $(K^q)_0 = (K_{(q+\xi_i(q))})_0$, where $K_{(q+\xi_i(q))}$ is the isotropy subgroup of K at $q+\xi_i(q) \in M_{\xi_i}$, and $(\ )_0$ denotes connected component of the identity.

Proof. From the transitivity of K on M and the fact that K preserves the foliation E_i, it follows that $(K^q)_0$ acts transitively on $S_i(q)$. Since $\pi_i\colon M \to M_{\xi_i}$ is K-equivariant and $S_i(q) = (\pi_i^{-1}(\pi_i(q)))_q$, the lemma follows easily. q.e.d.

Let $q \in M$ be fixed; then $\xi_i(q) \in (\nu(M_{\xi_i}))_{\pi_i(q)}$. We denote, as in [7], by $\mathrm{Hol}_{-\xi_i(q)}(M_{\xi_i})$ the subset of the normal bundle of M_{ξ_i}, one gets by translating parallel $-\xi_i(q)$ along any piecewise differentiable curve in M_{ξ_i}. We have that $\overline{M}_i(q) = \mathrm{Hol}_{-\xi_i(q)}(M_{\xi_i})$ is always an immersed submanifold of $\nu(M_{\xi_i})$ (if M_{ξ_i} is simply connected it is actually embedded). If A^i is the shape operator of M_{ξ_i}, then $A^i_{-\xi_i(q)} = \tilde{A}_{-\xi_i(q)}(I - \tilde{A}_{\xi_i(q)})^{-1}$, where $\tilde{A}^i$ is the shape operator of M restricted to the foliation $(E_i)^{\perp}$. It is easy to check that 1 is not an eigenvalue of $A^i_{-\xi_i(q)}$. Thus $f_i = \exp_{\nu(M_{\xi_i})|\overline{M}_i(q)}$ is an immersion (cf. [7, Theorem B]). Let $c\colon [0, 1] \to M_{\xi_i}$ be a piecewise differentiable curve in M_{ξ_i} with $c(0) = \pi_i(q)$, and let $\tilde{c}\colon [0, 1] \to M$ be its horizontal lifting to M with $\tilde{c}(0) = q$ (i.e., $\pi_i \circ \tilde{c} = c$ and $\tilde{c}'(t) \in (E_i)^{\perp}_{\tilde{c}(t)}$ $\forall t \in [0, 1]$). We easily see, as in [7], that if $\eta(t)$ is a parallel normal vector field to M_{ξ_i} along c, then $\eta(t)$ may also be regarded as a parallel normal vector field to M along $\tilde{c}$. If $\eta(0) = -\xi_i(q)$, then we have $\eta(t) = -\xi_i(\tilde{c}(t))$ for all $t \in [0, 1]$. Thus

$$\begin{aligned} f_i(c(1), \eta(1)) &= c(1) - \xi_i(\tilde{c}(1)) = \pi_i(\tilde{c}(1)) - \xi_i(\tilde{c}(1)) \\ &= \tilde{c}(1) + \xi_i(\tilde{c}(1)) - \xi_i(\tilde{c}(1)) = \tilde{c}(1), \end{aligned}$$

which shows that $f_i(\overline{M}_i(q)) \subset M$ and that $\pi_i \circ f_i = \mathrm{pr}|_{\overline{M}_i(q)}$, where $\mathrm{pr}\colon \nu(M) \to M$ is the projection. It follows now immediately that f_i is 1-1.

From the above facts and [7, §2] we easily get

Lemma 6.2. *By the notation and assumptions of this section, we have, for all $i \in I$, $q \in M$,*

(i) $f_i\colon \overline{M}_i(q) \to \mathbb{R}^N$ *is a* 1-1 *immersion,*

(ii) $\pi_i(q) + \Phi_{\pi_i(q)}(-\xi_i(q)) \subset \pi_i^{-1}(\pi_i(q))$, *where* $\Phi_{\pi_i(q)}$ *denotes the normal holonomy group of* M_{ξ_i} *at* $\pi_i(q)$,

(iii) $T_q f_i(\overline{M}_i(q)) = (E_i)^{\perp}_q \oplus T_{-\xi_i(q)}\Phi^*_{\pi_i(q)}(-\xi_i(q))$, *where* Φ^* *denotes the restricted normal holonomy group.*

Let $i \in I$ be fixed and let, for $q \in M$,

$$\overline{E}_i(q) = T_{-\xi_i(p)}\Phi^*_{\pi_i(q)}(-\xi_i(q)) \subset E_i(q).$$

It is easy to see that for given $k \in K$, $\overline{E}_i(k.q) = k_*(\overline{E}_i(q))$ (recall that since M is irreducible, $k_*(\xi_i) = \xi_i$ due to Theorem 5.1). Thus $\overline{E}_i$ define a C^∞ distribution in M, which is K-invariant.

We have the following fundamental proposition.

Proposition 6.3. *By the notation and assumptions of this section, we have, for all* $q \in M$,

(i) $\overline{E}_i = E_i$,

(ii) $\Phi^*_{\pi_i(q)}(-\xi_i(q)) = (K_{\pi_i(q)})_0(-\xi_i(q)) = S_i(q)$,

(iii) $S_i(q)$ *is an orbit of an s-representation.*

Proof. By decomposition, we have $\nu(M_{\xi_i})_{\pi_i(q)} = \mathbf{V}_0 \oplus \mathbf{V}_s$, where $\mathbf{V}_0$ is the set of fixed points of $\Phi^*_{\pi_i(q)}$ and $\mathbf{V}_s = \mathbf{V}_0^\perp$. If $-\xi_i(q) = v_0 + v_s$, where $v_0 \in \mathbf{V}_0$, $v_s \in \mathbf{V}_s$, then due to Theorem 5.1(i) we have that $(K_{\pi_i(q)})_0 v_s \subset \Phi^*_{\pi_i(q)} v_s$. But Lemma 6.1 and Lemma 6.2(ii) imply that $\Phi^*_{\pi_i(q)}(v_0 + v_s) \subset (K_{\pi_i(q)})_0(v_0 + v_s)$, so that

$$(K_{\pi_i(q)})_0 v_s = \Phi^*_{\pi_i(q)} v_s$$

and

$$\Phi^*_{\pi_i(q)}(v_0 + v_s) = v_0 + (K_{\pi_i(q)})_0 v_s \subset (K_{\pi_i(q)})_0(v_0 + v_s). \tag{1}$$

Let now $\mathfrak{k}_0$ be the Lie algebra of $(K_{\pi_i(q)})_0$. Then

$$T_q(K_{\pi_i(q)})_0(-\xi_i(q)) = \{X.v_0 + X.v_s : X \in \mathfrak{k}_0\}.$$

Since (1) implies that

$$\{X.v_s : X \in \mathfrak{k}_0\} \subset T_q(K_{\pi_i(q)})_0(-\xi_i(q)),$$

we have

$$\{Xv_0 : X \in \mathfrak{k}_0\} \subset T_q(K_{\pi_i(q)})_0(-\xi_i(q)).$$

Hence

$$T_{-\xi_i(q)}(K_{\pi_i(q)})_0(-\xi_i(q)) = \{X.v_0 : X \in \mathfrak{k}_0\} \times \{X.v_s : X \in \mathfrak{k}_0\}. \tag{2}$$

Let now $\widetilde{K}_0 = \{k_{|\mathbf{V}_0} : k \in (K_{\pi_i(q)})_0\}$, $\widetilde{K}_s = \{k_{|\mathbf{V}_s} : k \in (K_{\pi_i(q)})_0\}$, and set $\widetilde{K} = K_0 \times K_s$. Then $(K_{\pi_i(q)})_0(-\xi_i(q)) \subset \widetilde{K}(-\xi_i(q))$. But, by (2), both orbits have the same dimension. Thus

$$(K_{\pi_i(q)})_0(-\xi_i(q)) = \widetilde{K}(-\xi_i(q)).$$

Since $S_i(q) = (K_{\pi_i(q)})_0(-\xi_i(q))$, $S_i(q)$ is a product of orbits. If we write orthogonally $E_i = \overline{E}_i \oplus \overline{E}'_i$, then $\overline{E}_i$ and $\overline{E}'_i$ are both autoparallel distributions of M. Let now

$$\mathfrak{H}_i = \bigoplus_{\substack{j\in I\\ j\neq i}} E_j \oplus \overline{E}_i.$$

Then $TM = \mathfrak{H}_i \oplus \mathfrak{H}_i^{\perp}$, where $\mathfrak{H}_i^{\perp} = \overline{E}'_i$. Observe that $\mathfrak{H}_i(q) = T_q f_i(\overline{M}_i(q))$. Since M is irreducible, the proof of this lemma is now a consequence of Lemma 1.1 and the following.

Lemma 6.4. (i) $\mathfrak{H}_i$ *and* $\mathfrak{H}_i^{\perp}$ *are both autoparallel distributions* (*and hence they are parallel*).

(ii) $\alpha(X, Y) = 0$ *if* $X \in C^\infty(M, \mathfrak{H}_i)$, $Y \in C^\infty(M, \mathfrak{H}_i^{\perp})$.

Proof. The fact that $\mathfrak{H}_i^{\perp}$ is autoparallel has been shown. With respect to $\mathfrak{H}_i$ we get, by Lemma 6.2(iii), that it is integrable. Let $q \in M$, and let $\xi \in C^\infty(M, \nu_0(M))$ be parallel such that $\lambda_1(\eta(q)), \cdots, \lambda_g(\eta(q))$ are all different, where $\lambda_1 = \langle n_1, \ \rangle, \cdots, \lambda_g = \langle n_g, \ \rangle$ are the different eigenvalues of the shape operator A restricted to $\nu_0(M)$. Let $X_i \in C^\infty(M, \overline{E}_i)$, $Y_j \in C^\infty(M, E_j)$ where $i \neq j$. A similar computation involving the Codazzi identity as in the proof of case (b) of Lemma 2.2 shows that

$$\begin{aligned}(\lambda_i(\eta) - \lambda_j(\eta))\nabla_{X_i} Y_j \\ = -X_i(\lambda_j(\eta))Y_j + Y_j(\lambda_i(\eta))X_i - \lambda_i(\eta)[X_i, Y_j] + A_\eta[X_i, Y_j]\end{aligned}$$

which implies that $\nabla_{X_i} Y_j \in C^\infty(M, \mathfrak{H}_i)$, since $\mathfrak{H}_i$ is integrable and clearly invariant under A_η due to the splitting of $S_i(q)$. In a similar way it is shown that $\nabla_{Z_k} Y_j \in C^\infty(M, \mathfrak{H}_i)$ if $Z_k \in C^\infty(M, E_k)$ for $k \neq i \neq j$. Since $E_1, \cdots, \widehat{E}_i, \cdots, E_g$ and $\overline{E}_i$ are all autoparallel, we can now conclude that $\mathfrak{H}_i$ is autoparallel. It is an easy fact that two complementary autoparallel distributions must be parallel. Hence we obtain part (i).

Let now $\eta \in C^\infty(M, \nu_0(M))$ and $\psi \in C^\infty(M, \nu(M))$. If $X, Y \in \mathfrak{X}(M)$, then $R^{\perp}(X, Y)\eta = 0$ (due to the theorem of Ambrose-Singer). Thus, by the Ricci identity, A_ψ commutes with A_η and hence preserves its eigenspaces at any point. Since η is arbitrary, $A_\psi E_j \subset E_j$ for all $j \in I$. Since $\mathfrak{H}_i^{\perp} \subset E_i$, by the Gauss equation we have $\alpha(X_j, Y) = 0$ for all $X_j \in C^\infty(M, E_j)$, $j \neq i$, and $Y \in C^\infty(M, \mathfrak{H}_i^{\perp})$. Let now $X_i \in C^\infty(M, \overline{E}_i)$ and let α^i be the shape operator of $S_i(q)$ as a submanifold of $\mathbb{R}^N$. Since $S_i(q)$ is a totally geodesic submanifold of M which is invariant under the shape operator A, $\alpha(X_i, Y) = \alpha^i(X_i, Y) = 0$ due to the splitting of $S_i(q)$ (see the proof of Proposition 6.3). Hence we obtain part (ii).

Remark 6.5. The fibers of the projection of an irreducible homogeneous submanifold of the Euclidean space to a parallel focal manifold are homogeneous under the normal holonomy group of the focal manifold (compare with the Homogeneous Slice Theorem of [7]).

Proof of Theorem A. Let M^n, $n \geq 2$, be a compact homogeneous irreducible full submanifold of $\mathbb{R}^N$ with $\mathrm{rank}(M) \geq 2$. Without loss of generality we may assume that $M = K.v$, where $v \in \mathbb{R}^N$, and K is a connected compact Lie subgroup of $SO(N)$. We may assume, perhaps by considering a parallel orbit, that $v_0(M)$ is globally flat (see the beginning of this section).We want to show first that M is a submanifold with constant principal curvatures (see [7]). We will use the notation of this section. Since $E_1, \cdots, E_g$ are invariant under the shape operator A of M, it suffices to prove, for any $i \in I$, that $A_{\eta(t)|E_i}$ has constant eigenvalues, where $\eta(t)$ is an arbitrary parallel normal vector field to M along some arbitrary C^∞ curve $c\colon [0, 1] \to M$. The property of having constant principal curvatures is equivalent to the fact that the higher order mean curvature tensors (in the symmetric tensor algebra of the normal bundle) be parallel (see [16]). Then we may assume that c is either vertical or horizontal with respect to $M \xrightarrow{\pi_i} M_{\xi_i}$ $(i = 1, \cdots, g)$.

Case a. c is vertical, i.e., $c\colon [0, 1] \to S_i(q)$ for some $q \in M$. Since $S_i(q)$ is a totally geodesic submanifold of M and E_i is invariant under the shape operator A, we get that $\eta(t)$ is also a parallel normal vector field to $S_i(q)$ along c and that $A_{\eta(t)|(E_i)_{c(t)}} = \tilde{A}_{\eta(t)}$, where $\tilde{A}$ is the shape operator of $S_i(q)$ as a submanifold of $\mathbb{R}^N$. Since $S_i(q)$ is a submanifold with constant principal curvatures (by Proposition 6.3(iii) and [14]), then $A_{\eta(t)|E_i}$ has constant eigenvalues.

Case b. c is horizontal with respect to $M \xrightarrow{\pi_i} M_{\xi_i}$. In a similar way as [7, p. 170] we can prove that $\eta(t)$ is also a parallel normal vector field to M_{ξ_i} along $\pi_i \circ c$. Let $\tau^\perp\colon \nu(M_{\xi_i})_{\pi_i(c(0))} \to \nu(M_{\xi_i})_{\pi_i(c(1))}$ be the $\nabla^\perp$-parallel transport along $\pi_i \circ c$. Then $\tau^\perp(S_i(c(0))) = S_i(c(1))$ (see Lemma 6.2 and Proposition 6.3) and $\tau^\perp(\eta(0)) = \eta(1)$. Since $\tau^\perp$ is an isometry, we get that

$$\tau^\perp \tilde{A}^0_{\eta(0)}(\tau^\perp)^{-1} = \tilde{A}^1_{\eta(1)},$$

where $\tilde{A}^0$, $\tilde{A}^1$ are the shape operators of $S_i(c(0))$ and $S_i(c(1))$ respectively. Thus $\tilde{A}^1_{\eta(1)}$ has the same eigenvalues as $\tilde{A}^0_{\eta(0)}$. Since $A_{\eta(\varepsilon)|E_i(c(\varepsilon))} = \tilde{A}^\varepsilon_{\eta(\varepsilon)}$ $(\varepsilon = 0, 1)$, we get (b).

Then we have shown that M has constant principal curvatures. Then, by [7], M is either isoparametric or a focal manifold of an isoparametric submanifold. If M is isoparametric, since it is homogeneous, M is an orbit of an s-representation (see [15]). If M is a focal manifold of an isoparametric submanifold, let us say $\widetilde{M}$, then $\operatorname{cod}(\widetilde{M}) \geq 3$ because $\operatorname{rank}(M) \geq 2$. By the remarkable result of [17] (see also [12]), $\widetilde{M}$ is an orbit of an s-representation. Hence M is also an orbit of an s-representation.

7. Some remarks

Remark 7.1. If M is an orbit of an irreducible s-representation which is not most singular, then $\operatorname{rank}(M) \geq 2$ (see [6]).

Remark 7.2. Theorem A is also true for a homogeneous submanifold of the sphere, which is not compact, i.e., the orbit of a Lie subgroup of $SO(N)$; the proof is essentially the same. Theorem C is also true if M is not compact.

Remark 7.3. It is an open problem to determine the orbits of compact Lie groups which are taut. It was not solved, even in the case of flat normal bundle. For this special case Theorem A provides an answer. More generally, if M is a homogeneous compact full submanifold of $\mathbb{R}^N$ with flat normal bundle, the following statements are equivalent:

(i) M is taut.
(ii) M is Dupin.
(iii) M is a submanifold with constant principal curvatures.
(iv) M is an orbit of an s-representation.
(v) The first normal space of M coincides with the normal space.

Are the following following statements equivalent for arbitrary compact orbits?

Theorem A could be included in a more general result as follows:

Conjecture. Let M^n, $n \geq 2$, be a homogeneous irreducible full submanifold of the sphere which is not an orbit of an s-representation. Then the normal holonomy group acts transitively on the unit sphere of the normal space.

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