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DERIVATIVES OF TOPOLOGICAL ENTROPY FOR ANOSOV AND GEODESIC FLOWS

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Abstract

In the first part of this article we calculate the first and second derivatives of the topological entropy for C^∞ perturbations of Anosov flows. Of particular interest is the appearance of the "variance" (familiar from Central Limit Theorems) in our formula for the second derivative. Our proof is based on the use of symbolic dynamics and thermodynamic methods developed in [17]. In the second part of this article we consider the special case of geodesic flows, and concentrate on finding a geometric interpretation of the formula. The third and final part of the paper deals with estimates on the variance term in the formula for the second derivative.

PART ONE. ANOSOV FLOWS

Let $\phi_t : M \to M$ be a C^∞ Anosov flow on a compact manifold. Such flows include, for example, geodesic flows associated to compact manifolds with negative sectional curvatures. In a recent article, Katok, Knieper, Weiss, and the present author shows that for C^∞ Anosov flows the *topological entropy* has a C^∞ dependence on C^∞ perturbations of the flow [17]. In the present note we shall add to these results by deriving explicit formulae for the first and second derivatives.

Consider a C^∞ family of Anosov flows $\lambda \mapsto \phi^{(\lambda)}$, $\lambda \in (-\varepsilon, \varepsilon)$. Denote by

$$\lambda \mapsto \alpha^{(\lambda)} = 1 + \lambda(D_0\alpha^{(\lambda)}) + (\lambda^2/2)(D_0^2\alpha^{(\lambda)}) + \cdots$$

the *velocity change* in the structural stability theorem (i.e., the velocity change in $\phi^{(0)}$ to make it topologically conjugate to $\phi^{(\lambda)}$). We denote by

$$\lambda \mapsto h^{(\lambda)} = 1 + \lambda(D_0h^{(\lambda)}) + (\lambda^2/2)(D_0^2h^{(\lambda)}) + \cdots$$

the topological entropy $h^{(\lambda)}$ of the flow $\phi^{(\lambda)}$, and then our main result is the following:

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Theorem 1. *The first derivative of* $h^{(\lambda)}$ *(at* $\lambda = 0$*) is*

$$D_0 h^{(\lambda)} = h^{(0)} \int (D_0 \alpha^{(\lambda)})\, dm,$$

and the second derivative of $h^{(\lambda)}$ *(at* $\lambda = 0$*) is*

$$D_0^2 h^{(\lambda)} = h^{(0)} \left\{ \mathrm{Var}\ (D_0 \alpha^{(\lambda)}) + \int (D_0^2 \alpha^{(\lambda)})\, dm \right.$$
$$\left. 2\left(\int D_0 \alpha^{(\lambda)}\, dm\right)^2 - 2\int (D_0 \alpha^{(\lambda)})^2\, dm \right\},$$

where m *is the maximal measure for* $\phi^{(0)}$*, and* Var *is the variance for* $\phi^{(0)}$.

We remark that the formula for the first derivative appears in the article [18], and we present an alternative derivation. However, the formula for the second derivative is much more subtle, and it is here that our approach shows its advantage. In particular, we are able to deal with the perplexing problem of differentiating the maximal measures, which our method shows gives rise to the variance term in the second derivative.

The variance is an important concept in the Central Limit Theorems for Anosov flows. It is particularly interesting that our formula for the second derivative should give rise to its appearance.

In §1 we begin with some preliminaries about Anosov flows, and in §2 we show how these lead to an easy derivation of the first derivative formula. In §3 we introduce the machinery of symbolic dynamics which we need to study the second derivative, and in §4 we explain the role of the variance. Finally, in §5 we complete our derivation of the second derivative formula.

1. Anosov flows

Let M be a compact Riemannian manifold, and let $\phi_t^{(0)} : M \to M$ be a C^∞ flow. We call the flow *Anosov* if there exists a continuous splitting $TM = E^0 \oplus E^s \oplus E^u$ such that

(i) E^0 is one-dimensional and tangent to the flow;

(ii) $\exists C$, $\lambda > 0$ such that $\|D\phi^{(0)}_{t|_{E^s}}\|$, $\|D\phi^{(0)}_{t|_{E^u}}\| \le Ce^{-\lambda t}$, for $t \ge 0$

(cf. [1]).

For an Anosov flow we can define the *topological entropy* $h(\phi)$ to be the topological entropy of the discrete time-one map (i.e., the entropy of the homeomorphism $\phi^{(0)}_{t=1} : M \to M$). By the variational principal, the topological entropy is precisely the supremum over the *measure theoretic entropies* $h(\phi^{(0)}, \nu)$, where ν is a $\phi^{(0)}$-invariant probability measure.

Anosov flows have two basic properties that we shall need to use:

(i) Openness of Anosov flows. Anosov flows are open in the space of C^∞ flows. i.e., if we perturb from an Anosov flow the resulting flow is still Anosov providing the perturbation was sufficiently small (in particular, if $\phi^{(0)}$ is Anosov and $\varepsilon > 0$ sufficiently small, then $\phi^{(\lambda)}$ is automatically Anosov for $\lambda \in (-\varepsilon, \varepsilon)$); and

(ii) Structural stability. We formulate this as follows:

Proposition 1 (Structural Stability). *If $\lambda \mapsto \phi^{(\lambda)}$, for $\lambda \in (-\varepsilon, \varepsilon)$, is a C^∞ family of Anosov flows, then there exist functions $\alpha^{(\lambda)} \in C^\alpha(M)$, $\Theta^{(\lambda)} \in C^\alpha(M, M)$ such that*

(i) $\alpha^{(0)} \equiv 1$; $\Theta^{(0)} \equiv I_M$,

(ii) $\Theta^{(\lambda)}$ *carries* $\phi^{(0)}$ *orbits to* $\phi^{(\lambda)}$ *orbits,*

(iii) $\alpha^{(\lambda)}$ *is a change of speed in* $\phi^{(0)}$ *to make* $\Theta^{(\lambda)}$ *a conjugacy,*

and, furthermore, the maps $\lambda \mapsto \alpha^{(\lambda)}$, $\Theta^{(\lambda)}$ *are* C^∞.

Remarks. (i) In Proposition 1, $\alpha > 0$ depends on the stable and unstable foliations for M, and $C^\alpha(M)$, $C^\alpha(M, M)$ are a Banach space and a Banach manifold, respectively. (Cf. [17] for details and definitions.)

(ii) It is understood that we may need to reduce the size of the interval $(-\varepsilon, \varepsilon)$ whenever necessary.

2. The first derivative

The formula for the first derivative of topological entropy of Anosov flows can be deduced by some simple arguments. We begin with an elementary result from calculus.

Elementary Calculus Lemma. *Let $A, B : (-1, 1) \to \mathbb{R}$ be two C^2 functions. If $A(\lambda) \geq B(\lambda)$, $-1 \leq \lambda \leq 1$, and $A(0) = B(0)$, then $D_0 A = D_0 B$ and $D_0^2 A \geq D_0^2 B$.*

The structural stability theorem tells us that $\phi^{(\lambda)}$ is topologically conjugate to the reparameterisation $\psi^{(\lambda)} = \alpha^{(\lambda)}\phi^{(0)}$ of the flow $\phi^{(0)}$ (cf. [18, Lemma 5]). In particular, we can assume that these two flows have the same topological entropy $h^{(\lambda)}$ (since topological entropy is a conjugacy invariant).

Let $\beta^{(\lambda)} = 1/\alpha^{(\lambda)}$, then the probability measure $m^{(\lambda)} \equiv (\beta^{(\lambda)} / \int \beta^{(\lambda)}\, dm)m$ is invariant under $\psi^{(\lambda)}$. By the variational principal we have $h^{(\lambda)} \geq h(\psi^{(\lambda)}, m^{(\lambda)})$.

We now recall the following formula relating the entropy of a reparameterised flow to the original flow.

Lemma 1 (Abramov).

$$h(\psi^{(\lambda)}, m^{(\lambda)}) = \frac{h(\phi^{(0)}, m)}{\int \beta^{(\lambda)}\, dm} = \frac{h^{(0)}}{\int \beta^{(\lambda)}\, dm}, \tag{2.1}$$

(*cf.* [27, §1] *for a convenient summary*).

We denote $A(\lambda) = h^{(\lambda)}$ and $B(\lambda) = h(\psi^{(\lambda)}, m^{(\lambda)})$ and apply the Elementary Calculus Lemma. To calculate $D_0 B$ and $D_0^2 B$ we can substitute the identity (2.1) into the expansion

$$\lambda \mapsto \beta^{(\lambda)} = 1 + \lambda(D_0\beta^{(\lambda)}) + (\lambda^2/2)(D_0^2\beta^{(\lambda)}) + \cdots$$

to get

$$\begin{aligned} B(\lambda) &= \frac{h^{(0)}}{1 + (\lambda \int D_0\beta^{(\lambda)}\, dm + (\lambda^2/2) \int D\beta^{(\lambda)}\, dm + \cdots)} \\ &= h^{(0)} \left(1 - \left(\int \lambda D_0\beta^{(\lambda)}\, dm + \frac{\lambda^2}{2} \int D_0^2\beta^{(\lambda)}\, dm + \cdots\right)\right. \\ &\qquad \left. + \left(\lambda \int D_0\beta^{(\lambda)}\, dm + \frac{\lambda^2}{2} \int D_0\beta^{(\lambda)}\, dm + \cdots\right)^2 - \cdots\right) \qquad (2.2) \\ &= h^{(0)} \left(1 - \lambda \int D_0\beta^{(\lambda)}\, dm\right. \\ &\qquad \left. + \frac{\lambda^2}{2}\left(2\left(\int D_0\beta^{(\lambda)}\, dm\right)^2 - \int D_0^2\beta^{(\lambda)}\, dm\right) + \cdots\right). \end{aligned}$$

From (2.2) we deduce that

$$\begin{aligned} D_0 B &= -\lambda \int D_0\beta^{(\lambda)}\, dm, \\ D_0^2 B &= 2\left(\int D_0\beta^{(\lambda)}\, dm\right)^2 - \int D_0^2\beta^{(\lambda)}\, dm. \end{aligned} \tag{2.3}$$

Lemma 2. *The derivatives of* $\alpha^{(\lambda)}$ *and* $\beta^{(\lambda)}$ *are related by*

$$\begin{aligned} D_0\beta^{(\lambda)} &= -D_0^2\alpha^{(\lambda)}, \\ D_0^2\beta^{(\lambda)} &= 2(D_0\alpha^{(\lambda)})^2 - D_0^2\alpha^{(\lambda)}. \end{aligned}$$

Proof. This is just the chain rule applied to $\beta^{(\lambda)} = 1/\alpha^{(\lambda)}$.

Using Lemma 2 we can rewrite the identities in (2.3) in terms of the derivatives $D_0\alpha^{(\lambda)}$ and $D_0^2\alpha^{(\lambda)}$. Then invoking the Elementary Calculus Lemma we get an expression for the first derivative of the topological entropy:

Proposition 2. *For a C^∞ family of Anosov flows $\lambda \mapsto \phi^{(\lambda)}$, $\lambda \in (-\varepsilon, \varepsilon)$, with topological entropies $h^{(\lambda)}$ the first derivative at $\lambda = 0$ is given by*

$$D_0 h^{(\lambda)} = h^{(0)} \int (D_0 \alpha^{(\lambda)})\, dm .$$

(*This result was derived in* [18] *by a completely different method.*)

The expression (2.3) and the Elementary Calculus Lemma also give a lower bound on the second derivative $D_0^2 h^{(\lambda)}$ of the form

(2.5)

$$\begin{aligned} D_0^2 h^{(\lambda)} &\geq h^{(0)} \left(2 \left(\int D_0 \beta^{(\lambda)}\, dm \right)^2 - \int D_0^2 \beta^{(\lambda)}\, dm \right) \\ &= h^{(0)} \left(2 \left(\int d_0 \alpha^{(\lambda)}\, dm \right)^2 + \int D_0^2 \alpha^{(\lambda)}\, dm - 2 \int (D_0 \alpha^{(\lambda)})^2\, dm \right), \end{aligned}$$

where we substitute using the identities from Lemma 2.

The reason for getting only a lower bound for $D_0^2 h^{(\lambda)}$ is that this simple ad hoc argument does not appreciate the way in which the maximal measure for $\phi^{(\lambda)}$ changes with λ. Because of this reason we have to use the more sophisticated method of symbolic dynamics, which we introduce in the next section.

3. Symbolic dynamics

Our approach to a more refined study of the second derivative of the topological entropy of Anosov flows will be based on the use of Markov sections and symbolic dynamics. We shall recall here some of the facts we shall need.

Let A be a $k \times k$ matrix with entries 0 or 1 and define a zero-dimensional compact space:

$$\Sigma_A = \left\{ x \in \prod_{-\infty}^{+\infty} \{1, \cdots, k\} | A(x_i, x_{i+1}) = 1, \; i \in \mathbb{Z} \right\}$$

with the metric

$$d(x, y) = \sum_{n=-\infty}^{+\infty} \delta(x_n, y_n) \cdot \left(\frac{1}{2}\right)^{|n|} .$$

For any $\alpha > 0$ the space $C^\infty(\Sigma_A)$ of α-Hölder continuous functions $f : \Sigma_A \to R$ is a Banach space with the norm

$$\|f\| = \|f\|_\infty + \sup \left\{ \frac{|f(x) - f(y)|}{d(x, y)^\alpha} : x \neq y \right\} .$$

We define a homeomorphism, called the *shift*, $\sigma : \Sigma_A \to \Sigma_A$ by $(\sigma x)_n = x_{n+1}$. Let $r : \Sigma_A \to \mathbb{R}^+$ be a strictly positive Hölder continuous function, and define a one-dimensional compact space:

$$\Sigma_A^r = \{(x, t) \in \Sigma_A \times \mathbb{R} | 0 \le t \le r(x)\}/(x, r(x)) \sim (\sigma x, 0).$$

We define a *suspended flow* $\sigma_t^r : \Sigma_A^r \to \Sigma_A^r$ by $\sigma_t^r(x, u) = (x, u+t)$, using the identifications as appropriate. Following Bowen or Ratner, we can associate to each Anosov flow $\phi^{(\lambda)}$:

(i) a suspended flow $\sigma_t^{r(\lambda)}$ (for functions $r(\lambda) : \Sigma_A \to \mathbb{R}^+$); and

(ii) a continuous map $\pi^{(\lambda)} : \Sigma_A \to M$,

such that $\phi_t^{(\lambda)} \circ \pi^{(\lambda)} = \pi^{(\lambda)} \circ \sigma_t^{r(\lambda)}$ and $\sigma^{r(\lambda)}$ has topological entropy $h^{(\lambda)}$ (cf. [25]). Furthermore, using structural stability, we can arrange the following:

(iii) the shift homeomorphism $\sigma : \Sigma_A \to \Sigma_A$ is independent of $\lambda \in (-\varepsilon, \varepsilon)$; and

(iv) $r(\lambda)(x) = \int_0^{r(0)} \beta^{(\lambda)} \circ \pi^{(0)}(x, t)\, dt \in C^\infty(\Sigma_A)$

(cf. [17] for more details).

By Proposition 1 we can deduce that $\lambda \mapsto r(\lambda) \in C^\alpha(\Sigma_A)$ is C^∞, and by (iv) we have the expansion

$$\begin{aligned} \lambda \mapsto r(\lambda) &= \int_0^{r(0)} \left(1 + \lambda(D_0\beta^{(\lambda)}) + \frac{\lambda^2}{2}(D_0^2\beta^{(\lambda)}) + \cdots\right) \circ \pi^{(0)}(x, t)\, dt \\ &= r(0)(x) + \lambda \int_0^{r(0)} (D_0\beta^{(\lambda)} \circ \pi^{(0)})(x, t)\, dt \\ &\quad + \frac{\lambda^2}{2} \int_0^{r(0)} (D_0^2\beta^{(\lambda)}) \circ \pi^{(0)}(x, t)\, dt + \cdots . \end{aligned} \tag{3.1}$$

The topological entropy $h^{(\lambda)}$ has the following useful characterization at the level of the symbolic dynamics.

Proposition 2. *Let $P : C^\alpha(\Sigma_A) \to \mathbb{R}$ denote the pressure function defined by*

$$P(g) = \sup\left\{h(\mu) + \int g\, d\mu | \mu \ \sigma\text{-invariant}\right\},$$

and $\mu = \mu_g$ the (unique) equilibrium state for g satisfying $P(g) = h(\mu_g) + \int g\, d\mu$. Then

(a) *$t = h^{(\lambda)}$ is the unique zero for $t \to P(-\operatorname{tr}(\lambda))$; and*

(b) *if $g = -h^{(\lambda)} r(\lambda)$, then $m = [\pi^{(\lambda)}]^*(\mu_g \times l / \int r^{(\lambda)}\, d\mu_g)$ is the unique measure of maximal entropy for the flow $\phi^{(\lambda)}$. (Cf. [28], for example).*

The importance of this characterization comes from the following results on the derivatives of the pressure function $P : C^\alpha(\Sigma_A) \to \mathbb{R}$.

Proposition 3. *$P : C^\alpha(\Sigma_A) \to \mathbb{R}$ is real analytic as a function on a Banach space, and the first and second derivatives at $f \in C^\alpha(\Sigma_A)$ are given by*

(a)

$$D_f P(g) = \int g \, d\mu_f ,$$

(b)

$$D_f^2 P(g_1, g_2) = \lim_{n\to+\infty} \frac{1}{n} \int \left(g_1^{(n)} - n\left(\int g_1 \, d\mu_f\right)\right) \times \left(g_2^{(n)} - n\left(\int g_2 \, d\mu_f\right)\right) d\mu_f ,$$

where we denote $g^{(n)}(x) = g(x) + g(\sigma x) + \cdots + g(\sigma^{n-1}x)$.
(*Cf.* [19], [29]).

Corollary 3.1. *$D_f[\int g \, d\mu_f] = D_f^2 P(f, g)$, for $g \in C^\alpha(\Sigma_A)$ fixed.*

Proof. This is immediate since $\int g \, d\mu_f = dP(f + tg)/dt|_{t=0}$.

Definition. We denote

$$\sigma_f^2(g) = \lim_{n\to+\infty} \frac{1}{n} \int \left(g^{(n)} - n\left(\int g \, d\mu_f\right)\right) \left(g^{(n)} - n\left(\int g \, d\mu_f\right)\right) d\mu_f ,$$

where $f, g \in C^\alpha(\Sigma_A)$.

It is sometimes also convenient to have the following well-known alternative expression for $\sigma_f^2(g)$.

Lemma 3. $\sigma_f^2(g) = \sum_{n=\infty}^{+\infty} \int (g\sigma^n - n(\int g \, d\mu))(g - n(\int g \, d\mu)) \, d\mu_f$.

Proof. Assume $\int g \, d\mu = 0$, without loss of generality. Then

$$\begin{aligned}
\sum_{n=-\infty}^{+\infty} \int g\sigma^n \cdot g \, d\mu_f &= \sum_{n=-\infty}^{+\infty} \left[\int g\sigma^n \left\{\frac{1}{N} \sum_{n=[-N/2]}^{[N/2]} g\sigma^n\right\} d\mu\right] \\
&\qquad\qquad \text{(for all } N \geq 0) \\
&= \lim_{N\to+\infty} \int \left[\sum_{n=[-N/2]}^{[N/2]} g\sigma^n\right] \left[\frac{1}{N} \sum_{n=[-N/2]}^{[N/2]} g\sigma^n\right] d\mu \\
&= \lim_{N\to+\infty} \frac{1}{N} \int \left[\sum_{n=0}^{N-1} g\sigma^n\right] \left[\sum_{n=0}^{N-1} g\sigma^n\right] d\mu \\
&= \sigma_f^2(g)
\end{aligned}$$

as required.

4. The variance for Anosov flows

In this section we shall recall the definition and basic properties of the variance. Let $\phi_t^{(0)} : M \to M$ be our initial Anosov flow with maximal measure m. Let $F : M \to \mathbb{R}$ be any Hölder continuous function with Hölder exponent $\alpha > 0$. Since the measure m is known to have Lebesgue spectrum, we can write

$$\rho_F(t) = \int F\phi_t^{(0)} \cdot F\, dm - \left(\int F\, dm\right)^2 = \int_{-\infty}^{+\infty} e^{it\lambda}\, d\sigma_F(\lambda)$$

(by Herglotz's theorem), where $\sigma_F(\lambda)$ is a smooth density on the real line.

Definition. We call the integral $\int_{-\infty}^{+\infty} \rho_F(t)\, dt$ the *variance* $\mathrm{Var}(F)$ (relative to m) of the function F.

For completeness, we shall also give two alternative expressions. We shall not need to make use of these other definitions in our treatment, but sometimes they are more convenient for verifying some particular property of the variance.

Proposition 4.
(i) *Assume* $\frac{d\sigma_F}{d\lambda}$ *is continuous at* $\lambda = 0$. *Then*

$$\mathrm{Var}(F) = \sigma_F(0),$$

(ii)

$$\mathrm{Var}(F) = \lim_{t\to+\infty} \int \frac{1}{t}\left\{\int_0^t F\phi_t^{(0)}(x)\, d\mu - t\left(\int F\, dm\right)\right\}^2 dm(x).$$

Proof. (i) Clearly,

$$\begin{aligned}\int_{-\infty}^{+\infty} \rho_F(t)\, dt &= \int_{-\infty}^{+\infty} \left\{\int_{-\infty}^{+\infty} e^{it\lambda}\, d\sigma_F(\lambda)\right\} dt \\ &= \int_{-\infty}^{+\infty} \left\{\int_{-\infty}^{+\infty} e^{it\lambda}\, dt\right\} d\sigma_F(\lambda) = \int_{-\infty}^{+\infty} \delta(\lambda)\, d\sigma_F(\lambda) \\ &= \sigma_F(0) = \mathrm{Var}(F),\end{aligned}$$

where $\delta(\lambda)$ denotes the Dirac density at zero, and the last inequality follows rigorously using Fejer kernels.

(ii) Without loss of generality we can assume $\int F\, dm = 0$. From part (i) we can write

$$\begin{aligned}\operatorname{Var}(F) &= \int_{-\infty}^{+\infty} \rho_F(t)\,dt \\ &= \int_{-\infty}^{+\infty} \left\{ \int F(\phi_t^{(0)}x)F(x)\,dm(x) \right\} dt \\ &= \int_{-\infty}^{+\infty} \left\{ \int F(\phi_t^{(0)}x)F(\phi_u^{(0)}x)\,dm(x) \right\} dt, \qquad \forall u \in \mathbb{R}.\end{aligned}$$

Thus

$$\begin{aligned}\operatorname{Var}(F) &= \int_{-\infty}^{+\infty} \left\{ \int F(\phi_t^{(0)}x) \left(\frac{1}{T} \int_{-T/2}^{T/2} F(\phi_u^{(0)}x)\,du \right) \right\} dt \\ &= \lim_{T\to+\infty} \left\{ \int_{-T/2}^{T/2} F(\phi_t^{(0)}x)\,dt \left(\frac{1}{T} \int_{-T/2}^{T/2} F(\phi_x^{(0)})\,du \right) \right\} dm(x) \\ &= \lim_{t\to+\infty} \int \frac{1}{t} \left\{ \int_0^t F(\phi_u^{(0)}x)\,du \right\}^2 dm(x).\end{aligned}$$

Remarks. (i) The variance $\operatorname{var}(F)$ has some simple but important properties which can be easily deduced from the alternative definitions contained in Proposition 4. For example

$\operatorname{Var}(F) \geq 0$ with equality if and only if there exists $U \in C^0(M)$ with

$$\int_0^t F(\phi_u^{(0)}x)\,du = t\left(\int F\,dm\right) + U(\phi_t^{(0)}x) - U(x), \qquad \text{for all } t \in \mathbb{R}.$$

Clearly, $\operatorname{Var}(F)$ is positive from Proposition 4(ii) since the integrand there is positive. If F satisfies the condition above, then by Proposition 4(ii) it is clear that $\operatorname{var}(F) = 0$. The converse can be deduced by a slightly indirect proof (for example, using symbolic dynamics, the strict convexity of pressure and Lemma 4, below).

(ii) The term "variance" is appropriate in the statistical sense since $\operatorname{Var}(F)$ arises in versions of the central limit theorem for Anosov and hyperbolic flows (cf. [9]).

We next want to interpret the variance of the function F at the level of the symbolic dynamics. The Hölder continuous function $F : M \to \mathbb{R}$ on the manifold M induces a Hölder continuous function $F \circ \pi : \Sigma_A^r \to \mathbb{R}$ on the symbolic space Σ_A^r. We can then define $f \in C^\alpha(\Sigma_A)$ by $f(x) = \int_0^{r(0)} (F \circ \pi)(x, t)\,dt$.

Lemma 4. (i) $\operatorname{Var}(F) = \operatorname{Var}(F \circ \pi)$.

(ii) $\operatorname{Var}(F) = \sigma_{-hr}^2(f)/\int f\,du_{-hr}$.

Proof. (i) This follows from the definition since $\pi : \Sigma_A^r \to M$ is an *isomorphism* between the flows $\sigma_t^r : \Sigma_A^r \to \Sigma_A^r$ and $\phi_t^{(0)} : M \to M$, relative to the measures of maximal entropy.

(ii) (Compare with [19, p. 11].) Without loss of generality we can assume $\int F\,dm = 0$. By Proposition 4(i) we can write

$$\begin{aligned}
\operatorname{Var}(F) &= \int_{-\infty}^{+\infty} \rho_F(t)\,dt \\
&= \int_{-\infty}^{+\infty} \int (F\circ\pi)(x\,,\,u)(F\circ\pi\circ\sigma_t^r)(x\,,\,u) \left(\frac{d\mu_{-hr}\times dl(u)}{\int r\,d\mu_{-hr}}\right) \\
&= \sum_{n=\infty}^{+\infty} \int \left\{\int_0^r (F\circ\pi)(x\,,\,u)\,du\right\} \\
&\qquad \cdot \left\{\int_0^{r\circ\sigma^n} (F\circ\pi)(\sigma^n x\,,\,v)\,dv\right\} \frac{d\mu_{-hr}(x)}{\int r\,d\mu_{-hr}}
\end{aligned}$$

(by substituting

$$\begin{aligned}
&(F\pi)s_t^r(x\,,\,u) \\
&\quad = \sum_{n=-\infty}^{+\infty} \int_0^{r\circ\sigma^n} (F\pi)(x\,,\,v)\delta(u+t-v-r^n(x))\,dv) \\
&\quad = \sum_{n=-\infty}^{+\infty} f(x)f(\sigma^n x)\frac{d\mu_{-hr}}{\int r\,d\mu_{-hr}}\,,
\end{aligned}$$

and the result follows by Lemma 3.

5. Derivatives of topological entropy

In this section we shall complete our derivation of the explicit formulae for the first and second derivatives of the topological entropy $\lambda \mapsto h^{(\lambda)}$ at $\lambda = 0$ described in the introduction. In principle, the higher derivatives can be similarly calculated using essentially the same method.

By Proposition 2(a) we can characterize $h^{(\lambda)}$ as the implicit solution in the real variable t to $P(-\operatorname{tr}(\lambda)) = 0$. Applying the implicit function theorem to the function in two variables

$$(\lambda\,,\,t) \mapsto P(-\operatorname{tr}(\lambda)) \ni \mathbb{R}\,, \qquad \text{where } (\lambda\,,\,t) \in (-\varepsilon\,,\,\varepsilon)\times\mathbb{R}$$

gives that the map $\lambda \mapsto h^{(\lambda)}$, $\lambda \in (-\varepsilon\,,\,\varepsilon)$ is C^∞. This was the basis of the argument in [20]. However, the Implicit Function Theorem also yields a formula for the first derivative:

$$\frac{dh^{(\lambda)}}{d\lambda} = -\frac{(dP/du)(-h^{(\lambda)}r^{(u)})|_{u=\lambda}}{(dP/dt)(-\operatorname{tr}^{(\lambda)})|_{t=h}(\lambda)}, \tag{5.1}$$

(compare with [19]).

We know the derivative of the pressure function by Proposition 3. In particular, we can explicitly calculate the numerator and denominator in (5.1) with the result that

$$\begin{aligned} \frac{dP}{du}(-h^{(\lambda)}r^{(u)})|_{u=\lambda} &= -h^{(\lambda)}\int (D_0^{12}r(\lambda))\,d\mu^{(\lambda)}, \\ \frac{dP}{dt}(-\operatorname{tr}(\lambda))|_{t=h} &= \int r(\lambda)\,d\mu^{(\lambda)}, \end{aligned} \tag{5.2}$$

where $\mu^{(\lambda)}$ denotes the unique equilibrium state for $-h^{(\lambda)}r(\lambda)$. Substituting the identities (5.2) into (5.1) gives the following expression for the first derivative of the topological entropy:

$$D_\lambda h^{(\lambda)} = -\frac{h^{(\lambda)}\int (D_0 r(\lambda))\,d\mu^{(\lambda)}}{\int r(\lambda)\,d\mu^{(\lambda)}}. \tag{5.3}$$

By Proposition 2(b) the measure $(\mu^{(\lambda)} \times l)/\int r(\lambda)\,d\mu^{(\lambda)}$ on the symbolic space Σ_A^r projects the measure of maximal entropy m for the flow $\phi^{(0)}$ on the manifold M, under the semi-conjugacy map $\pi : \Sigma_A^r \to M$.

Furthermore, using the identity (3.1) we can rewrite (5.3) as

$$D_0 h^{(\lambda)} = h^{(\lambda)}\int (D_0\alpha^{(\lambda)})\,dm. \tag{5.4}$$

Thus we have yet another derivation of the formula for the first derivative of the entropy given in Proposition 1. To check that the sign in (5.4) is correct, notice that *increasing* the velocity of the flow *increases* the topological entropy, which is consistent with (5.4).

To calculate the second derivative $D_0^2 h^{(\lambda)}$ we shall want to differentiate the first derivative formula (5.3). To simplify the calculations it is convenient to rewrite (5.3) as a logarithmic derivative:

$$D_\lambda(\log h^{(\lambda)}) = -\frac{\int (D_0 r(\lambda))\,d\mu^{(\lambda)}}{\int r(\lambda)\,d\mu^{(\lambda)}}. \tag{5.5}$$

(This simple device has the advantage that we cut down on the number of terms we have to differentiate.) We can now differentiate (5.5), making particular use of Corollary 3.1. Setting $\lambda = 0$ to get

$$D_0^2(\log h^{(\lambda)}) = \frac{-\int (D_0^2 r(\lambda))\, d\mu^{(0)} + \sigma^2(D_0 r(\lambda))}{\int r(\lambda)\, d\mu^{(0)}} \tag{5.6}$$
$$+ \frac{\int (D_0 r(\lambda))\, d\mu^{(0)}}{(\int r(\lambda)\, d\mu^{(0)})^2} \left\{ \int (D_0 r(\lambda))\, d\mu^{(0)} + \sigma^2(r^{(0)}) \right\}.$$

(We have also dropped the subscript on σ^2 to avoid the notation becoming too unwieldy.) Using the identity (3.1) and Lemma 2 we can rewrite (5.6) as

$$D_0^2(\log h^{(\lambda)}) = -\int (D_0^2 \beta^{(\lambda)})\, dm + \mathrm{Var}(D_0 \beta^{(\lambda)}) \tag{5.7}$$
$$+ \int D_0 \beta^{(\lambda)}\, dm \left\{ \int D_0 \beta^{(\lambda)}\, dm + \mathrm{Var}(\beta^{(0)}) \right\}.$$

By Proposition 1(i) we have that $\beta^{(0)} \equiv 1$. Furthermore, from the properties of the variance described in the remark in §3 we know that $\mathrm{Var}(1) = 0$, from which we deduce that the last term in (5.7) vanishes. In particular, we have that

$$D_0^2(\log h^{(\lambda)}) = \mathrm{Var}(D_0 \beta^{(\lambda)}) - \int (D_0^2 \beta^{(\lambda)})\, dm. \tag{5.8}$$

To convert (5.8) back into a formula for the second derivative of the topological entropy (rather than the logarithm of that quantity) we can use the simple identity

$$D_0^2(\log h^{(\lambda)}) = D_0^2 h^{(\lambda)} / h^{(0)} - (D_0 h^{(\lambda)} / h^{(0)})^2$$

and Lemma 2 to get the result stated in the introduction.

Theorem 1. *For a C^∞ family of Anosov flows $\lambda \mapsto \phi^{(\lambda)}$, $\lambda \in (-\varepsilon, \varepsilon)$, with topological entropies $h^{(\lambda)}$*

$$D_0^2 h^{(\lambda)} = h^{(0)} \left(\mathrm{Var}\ (D_0 \alpha^{(\lambda)}) + 2 \left(\int D_0 \alpha^{(\lambda)}\, dm \right)^2 \right.$$
$$\left. + \int D_0^2 \alpha^{(\lambda)}\, dm - 2 \int (D_0 \alpha^{(\lambda)})^2\, dm \right).$$

Remark. Note that if $D_0 \alpha^{(\lambda)} \equiv 0$, then $D_0^2 h^{(\lambda)} = h^{(0)} \int (D_0^2 \alpha^{(\lambda)})\, dm$, which is consistent with the first derivative formula. Furthermore, the last three terms on the right-hand side of the above equality are consistent with the ad hoc inequality which we derived in (2.5).

Corollary 1.1. *At critical points (i.e., whenever $D_0 h^{(\lambda)} = 0$) we have*

$$D_0^2 h^{(\lambda)} = h^{(0)} \left(\mathrm{Var}(D_0 \alpha^{(\lambda)}) + \int (D_0^2 \alpha^{(\lambda)})\, dm - 2 \int (D_0 \alpha^{(\lambda)})^2\, dm \right).$$

Proof. This follows from the identification of the second term on the right-hand side of the identity in Theorem 1 with $2(D_0 h^{(\lambda)})^2 / h^{(\lambda)}$ by using the identity (5.4).

PART TWO. GEODESIC FLOWS

> "Entropy?" repeated Mr. Tompkins. "I've heard that word before. One of my colleagues once gave a party, and after a few drinks, some chemistry students he'd invited started singing—Increases, decreases. Decreases, increases. What the hell do we care what entropy does?
> to the tune of Ach du lieber Augustine" (Mr. Tompkins in Wonderland—George Gamow)

Theorem 1 gave the formula for the second derivative of the topological entropy at the level of generality of Anosov flows. For such general Anosov flows there are no obvious constraints to place on the functions $D_0 \alpha^{(\lambda)}$ and $D_0^2 \alpha^{(\lambda)}$, and thus we can say no more. However, when we restrict to geodesic flows there is a need to reformulate this expression in geometric quantities, which we do in this part of the notes.

6. Riemannian metrics and topological entropy

Let V be a compact manifold of dimension n with a C^∞ metric g_0 with strictly negative sectional curvatures. As is well-known, any C^∞ Riemannian metric g on V can be interpreted as a C^∞ section $g \in \Gamma(V, \mathcal{S}^2)$, in the positive cone of the bundle of symmetric 2 tensors $\mathcal{S}^2$ over V. By choosing suitable C^∞ local coordinates we can always associate to such section a families of positive definite $n \times n$ matrices g_x associated to each fiber $T_x V$ over a point $x \in V$, on the manifold. Recall that there is a natural inner product on $\Gamma(V, \mathcal{S})$ defined on

$$\langle \sigma_1, \sigma_2 \rangle = \int_V \langle \sigma_{1,x}, \sigma_{2,x} \rangle_x d(\mathrm{Vol}_{g_0})(x),$$

where $\langle A, B \rangle_x = \mathrm{trace}(A \cdot B^T)$ is just the usual inner product on matrices. We let $\| \ \|_g$ denote the associated norm for the metric g, say.

Since we shall only be interested in metrics g close to g_0, it suffices to consider one parameter families given by perturbation series:

$$g_\lambda = g_0 + \lambda g^{(1)} + (\lambda^2/2) g^{(2)} + \cdots , \qquad \lambda \in (-\varepsilon, \varepsilon)$$

for $\varepsilon > 0$ sufficiently small, where $g^{(1)}, g^{(2)}, \cdots \in \Gamma(V, \mathcal{S}^2)$. (These again correspond to the sections in the bundle of symmetric 2 tensors, but no longer necessarily in the positive cone; cf. [18] for more details.)

Consider a fixed free homotopy class τ for the manifold V. We let $l_\lambda(c)$ denote the length of an arbitrary closed curve c in the class τ, with respect to the metric g_λ. For each metric g_λ the shortest closed curve in τ is well known to be a unique closed geodesic, which we denote by c_λ. Therefore,

(6.1)

$$\begin{aligned} l_\lambda(c_\lambda) \le l_\lambda(c_0) &= \int_{c_0} \|v\|_{g_\lambda} \\ &= \int_{c_0} \left(\|v\|_{g_0} + \lambda \left(\frac{d}{dv} \|v\|_{g_\lambda}|_{\lambda=0} \right) + \frac{\lambda^2}{2} \left(\frac{d^2}{d\lambda^2} \|v\|_{g_\lambda}|_{\lambda=0} \right) + \cdots \right) \end{aligned}$$

where the integral is over unit tangent vectors v to the closed geodesic c_0, for the unperturbed metric g_0 (i.e., $\|v\|^2_{g_0} = 1$). Consider the expansion

(6.2)

$$\begin{aligned} \|v\|_{g_\lambda} &= \left(1 + \left(\lambda g^{(1)}(v, v) + \frac{\lambda^2}{2} g^{(2)}(v, v) + \cdots \right) \right)^{1/2} \\ &= 1 + \frac{1}{2} \left(\lambda g^{(1)}(v, v) + \frac{\lambda^2}{2} g^{(2)}(v, v) + \cdots \right) \\ &\quad - \frac{1}{8} \left(\lambda g^{(1)}(v, v) + \frac{\lambda^2}{2} g^{(2)}(v, v) + \cdots \right)^2 + \cdots \\ &= 1 + \lambda \left(\frac{g^{(1)}(v, v)}{2} \right) + \frac{\lambda^2}{2} \left(\frac{g^{(2)}(v, v)}{2} - \frac{1}{4} (g^{(1)}(v, v))^2 \right) + \cdots . \end{aligned}$$

Substituting the expansion (6.2) into the inequality (6.1) we get the new inequality

$$\begin{aligned} l_\lambda(c_\lambda) \le l_0(c_0) &+ \lambda \int_{c_0} \frac{g_1(v, v)}{2} \\ &+ \frac{\lambda^2}{2} \int_{c_0} \left(\frac{g^{(2)}(v, v)}{2} - \frac{1}{4} (g^{(1)}(v, v))^2 \right) + \cdots . \end{aligned} \tag{6.3}$$

Since $\lambda \mapsto l_\lambda(c_\lambda)$ is smooth (for example, by a simple application of the implicit function theorem), we can apply the Elementary Calculus Lemma to the inequality (1.3). We recall the statement.

Elementary Calculus Lemma. *Let $A, B : (-1, 1) \to \mathbb{R}$ be two C^2 functions. If $A(\lambda) \geq B(\lambda)$, $-1 \leq \lambda \leq 1$, and $A(0) = B(0)$, then $D_0 A = D_0 B$ and $D_0^2 A \geq D_0^2 B$.*

Applying this lemma to both sides of the inequality (6.3) gives

$$(6.4)\qquad \begin{aligned} D_0(l_\lambda(c_\lambda)) &= \int_{c_0} \frac{g_1(v, v)}{2}, \\ D_0^2(l_\lambda(c_\lambda)) &\leq \int_{c_0} \left(\frac{g^{(2)}(v, v)}{2} - \frac{1}{4}(g^{(1)}(v, v))^2 \right). \end{aligned}$$

The geodesic flow is a special example of an Anosov flow [1]. Thus, in particular, it is structurally stable and we can deduce that the geodesic flow on (V, g_0) is conjugate to that on (V, g_λ), after a velocity change $\alpha^{(\lambda)} : M \to \mathbb{R}$ (on the unit tangent bundle $M = T_1 V$).

Writing $\beta^{(\lambda)} = 1/\alpha^{(\lambda)}$ we get another expression for the lengths of the closed geodesics:

$$(6.5)\qquad l_\lambda(c_\lambda) = \int_{c_0} \beta^{(\lambda)} = \int_{c_0} \left(1 + \lambda D_0 \beta^{(\lambda)} + \frac{\lambda^2}{2} D_0^2 \beta^{(\lambda)} + \cdots \right).$$

Recall that by Lemma 2 the derivatives of $\alpha^{(\lambda)}$ and $\beta^{(\lambda)}$ are related by

$$(6.6)\qquad \begin{aligned} D_0 \beta^{(\lambda)} &= -D_0 \alpha^{(\lambda)}, \\ D_0^2 \beta^{(\lambda)} &= 2(D_0 \alpha^{(\lambda)})^2 - D_0^2 \alpha^{(\lambda)}. \end{aligned}$$

(i) From (6.4), (6.5), (6.6), and the Elementary Calculus Lemma we can first deduce that the first derivatives are equal, i.e.,

$$(6.7)\qquad \int_{c_0} \frac{g_1(v, v)}{2} = D_0(l_\lambda(c_\lambda)) = \int_{c_0} (D_0 \beta^{(\lambda)}) = -\int_{c_0} (D_0 \alpha^{(\lambda)})$$

for *each* closed geodesic c_0. In particular, comparing these two expressions for the derivatives of lengths of closed geodesics we deduce that

$$\int_{c_0} (D_0 \alpha^{(\lambda)}) = \frac{1}{2} \int_{c_0} g_1(v, v) \quad \text{for } \textit{each} \text{ closed geodesic } c_0.$$

In particular, we can now apply the Livsic theorem [20] to deduce that the integrands are the same (up to coboundary).

Lemma 5. *The two functions $v \to g_1(v, v)/2$ and $v \to D_0 \alpha^{(\lambda)}$ differ by at most a coboundary.*

(ii) Secondly, we can also show, from (6.4), (6.5), (6.6), and the Elementary Calculus Lemma, that

$$\begin{aligned}\int_{c_0}\left(\frac{g^{(2)}(v\,,\,v)}{2}-\frac{1}{4}(g^{(1)}(v\,,\,v))^2\right) &\geq D_0^2(l_\lambda(c_\lambda))=\int_{c_0} D_0^2\beta^{(\lambda)}\\ &=\int_{c_0}(2(D_0\alpha^{(\lambda)})^2-D_0^2\alpha^{(\lambda)})\end{aligned}\tag{6.8}$$

for *each* closed orbit c_0 where we have used (6.6) to get the last line.

We recall the following result.

Proposition 5 (Sigmund). *The probability measures supported on closed orbits of Axiom* A *flows are dense* (*in the weak* topology*) *amongst all the flow invariant probability measures* [31].

Using Sigmund's result we can replace (6.8) by the inequality

$$\begin{aligned}&\int_M \frac{1}{2}(g^{(2)}(v\,,\,v)-\frac{1}{2}(g^{(1)}(v\,,\,v))^2)\,d\mu\\ &\qquad\geq\int_M D_0^2\beta^{(\lambda)}\,d\mu\\ &\qquad=\int_M(2(D_0\alpha^{(\lambda)})^2-D_0^2\alpha^{(\lambda)})\,d\mu\end{aligned}\tag{6.9}$$

for *any* flow invariant probability measure μ.

Finally, the following lemma will be of use to us later.

Lemma 6. *For any* $\varepsilon>0$ *we can choose a closed geodesic* c_0 *such that*

$$\left|\int_M D_0^2\beta^{(\lambda)}\,dm-\frac{1}{l_0(c_0)}D_0^2(l_\lambda(c_\lambda))\right|<\varepsilon\,.$$

Proof. By Sigmund's result we can choose for any $\varepsilon>0$ a closed geodesic c_0 such that

$$\left|\int_M D_0^2\beta^{(\lambda)}\,dm-\frac{1}{l_0(c_0)}\int_{c_0} D_0^2\beta^{(\lambda)}\right|<\varepsilon\,.$$

Furthermore, by the expansion (6.5) we see that $\int_{c_0} D_0^2\beta^{(\lambda)}=D_0^2(l_\lambda(c_\lambda))$, and the result follows.

7. The derivatives of topological entropy for geodesic flows

For the Riemannian metric g_0 on V we shall denote the incremental volume element on the manifold by just $d(\mathrm{Vol})$. We shall also use the same notation for the element of Liouville measure on the unit tangent bundle, the distinction hopefully being clear by the context.

For an arbitrary C^∞ Riemannian metric g on V we can denote by d the metric on V derived from this Riemannian metric, and let $(\widetilde{V}, \tilde{d})$ be the covering space for V with the metric $\tilde{d}$ induced from the lift on the metric d on V. The following gives a nice geometric definition of the topological entropy.

Definitions. (a) We define the (*topological*) *entropy* $h(g)$ of the Riemannian metric g to be

$$h(g) = \lim_{R\to+\infty} \frac{1}{R} \log(\mathrm{Vol}_{\tilde{d}}\{x \in \widetilde{V} | \tilde{d}(x, x_0) \le R\}).$$

(b) For any C^∞ function $F : SV \to \mathbb{R}$ with $\int_{SV} F d(\mathrm{Vol}) = 0$ we introduce the *autocorrelation function* $\rho(t) = \int_{SV} F\phi_t \cdot F d(\mathrm{Vol})$, $t \in \mathbb{R}$. We define the *variance* of the function F by $\mathrm{Var}(F) = \int_{-\infty}^{+\infty} \rho_F(t)\,dt \ge 0$.

Remark. The definition of the topological entropy is independent of the choice of point $x_0 \in \widetilde{V}$, and if we can consider the geodesic flow $\phi_t : SV \to SV$ on the unit tangent bundle SV of (V, g), then the quantity $h(g)$ is precisely the topological entropy of this flow, in the usual sense, cf. [22], [23].

The following result was given in [17].

Proposition 6. *The map $\lambda \to h(g_\lambda)$ is C^∞.*

By substituting the identity (6.7) into the expression for the first derivative of the topological entropy for geodesic flows we recover the following result due to Katok, Knieper, and Weiss [18].

Proposition 7. *For a C^∞ perturbation $\lambda \mapsto g^{(\lambda)}$, $\lambda \in (-\varepsilon, \varepsilon)$, we have that*

$$D_0 h(g_\lambda) = \frac{h(g_0)}{2} \int_M g^{(1)}(v, v)\,dm(v).$$

We would like to get an expression for the second derivative $D_0^2 h(g_\lambda)$. Using Lemma 5 to replace $D_0\alpha^{(\lambda)}$ by $-g_1(v, v)/2$ in two of the terms in the expression for the second derivative of the topological entropy we get the following formula.

Theorem 2. *The second derivative of the topological entropy is*

$$\frac{D_0^2 h(g_\lambda)}{h(g_0)} = \left(\mathrm{Var}\left(\frac{g_1(v, v)}{2}\right) + 2\left(\int_M \frac{g_1(v, v)}{2}\,dm\right)^2 \right.$$
$$\left. + \int_M D_0^2\alpha^{(\lambda)}\,dm - 2\int_M (D_0\alpha^{(\lambda)})^2\,dm\right),$$

where m is the measure of the maximal entropy for the geodesic flow, and $\mathrm{Var}(\cdot)$ *denotes the variance.*

Remark. If we write $D_0\alpha^{(\lambda)} = g_1(v, v)/2 + \widetilde{V}$, where $\widetilde{V}$ is the coboundary formed by differentiating V along the orbits of the flow, then

$$\int_M (D_0\alpha^{(\lambda)})^2\, dm = \left(\int_M \frac{g_1(v, v)}{2}\, dm\right)^2 - \int_M g_1(v, v)\cdot \widetilde{V}\, dm + \int_M (\widetilde{V})^2\, dm.$$

However, substituting this into the expression in Theorem 2 does not seem to lead to any further simplifications.

By means of the inequality (6.9) we can get a naive lower bound for the last two terms in the expression in Theorem 2. In particular, use of the inequality yields a lower bound on the second derivative of the topological entropy of the form

$$\begin{aligned} D_0^2 h(g_\lambda) = h(g_0) \Bigg(&\operatorname{Var}\left(\frac{g_1(v, v)}{2}\right) + 2\left(\int_M \frac{g_1(v, v)}{2}\, dm\right)^2 \\ &- \int_M \frac{g^{(2)}(v, v)}{2}\, dm + \frac{1}{4}\int_M (g^{(1)}(v, v))^2\, dm \Bigg). \end{aligned} \tag{7.1}$$

At *critical points* for the topological entropy (i.e., when $D_0 h(g_\lambda) = 0$) from Proposition 6, it follows that the second term on the right-hand side of this inequality vanishes, and (7.1) simplifies to

$$\begin{aligned} D_0^2 h(g_\lambda) = h(g_0) \Bigg(&\operatorname{Var}\left(\frac{g_1(v, v)}{2}\right) - \int_M \frac{g^{(2)}(v, v)}{2}\, dm \\ &+ \frac{1}{4}\int_M (g^{(1)}(v, v)\, dm)^2\, dm \Bigg). \end{aligned} \tag{7.2}$$

From (7.2) we can deduce the following simple result.

Corollary 2.1. *For first-order perturbations (i.e., perturbations with $g^{(2)} \equiv 0$) the topological entropy $\lambda \mapsto h^{(\lambda)}$ is (strictly) convex.*

Proof. By hypothesis, the second term on the right-hand side of (7.2) vanishes. Since the remaining two terms are positive, we deduce that $D_0^2 h(g_\lambda) \geq 0$. Furthermore, since the perturbation is nontrivial, $g^{(1)}(v, v)$ is not identically zero, and thus by (7.2),

$$D_0^2 h(g_\lambda) = \frac{h(g_0)}{4}\int_M (g^{(1)}(v, v))^2\, dm > 0.$$

8. Axiom-preserving perturbations

For geodesic flows and the special case of perturbations arising from changes in the Riemannian metric there is a very natural normalisation condition to introduce. Specifically, we can consider those perturbations $\lambda \to g_\lambda$ in the metric for which the perturbed metrics g_λ, $\lambda \in (-\varepsilon, \varepsilon)$, have the same volume as the unperturbed metric g_0. A refinement of this is to ask the perturbed metrics g_λ, $\lambda \in (-\varepsilon, \varepsilon)$ to have the same *volume form* as the unperturbed metric g_0.

These constraints introduce a relationship between the first- and second-order terms $g^{(1)}$ and $g^{(2)}$ in the expansion of the perturbation $\lambda \to g_\lambda$ which should be important in finding a lower bound for $D_0^2 h(g_\lambda)$ by estimating from below the right-hand side of (7.2). We summarize these in the following lemma.

Lemma 7. (a) *For volume-preserving deformations* g_λ *we have the following constraints on* $g^{(1)}$ *and* $g^{(2)}$:

(i)

$$\int_V \operatorname{tr}(g^{(1)})\, d(\mathrm{Vol}_v)(x) = 0,$$

and

(ii)

$$\int_V \operatorname{tr}(g^{(2)})\, d(\mathrm{Vol}_v) = \int_V [\operatorname{tr}(g^{(1)})^2 - \tfrac{1}{2}\operatorname{tr}^2(g^{(1)})]\, d(\mathrm{Vol}_v).$$

(b) *For volume-form-preserving perturbations* g_λ *we have the following constraints on* $g^{(1)}$ *and* $g^{(2)}$:

(i) $\operatorname{tr}(g^{(1)}) \equiv 0$, *and*

(ii) $\int_V \operatorname{tr}(g^{(2)})\, d(\mathrm{Vol}_v) = \int_V \operatorname{tr}(g^{(1)})^2\, d(\mathrm{Vol}_v)$.

(c) *The following simplifying identity always holds*:

$$\int_M g^{(i)}(v, v)\, dm(v) = \frac{1}{\dim V} \int_V \operatorname{tr}(g^{(i)})\, d(\mathrm{Vol}_v)(x).$$

Proof. For parts (a) and (b) cf. [3, pp. 65, 129]. For part (c) cf. [6].

Another interesting special case is that where we assume the unperturbed metric g_0 to be a locally symmetric metric. For a locally symmetric metric g_0 the measure of the maximal entropy m for the associated geodesic flow is known to be just the usual Liouville measure $dm = d\theta \times d(\mathrm{vol}_v)$, where $d\theta$ is the (normalized) Haar measure on the fibers of the unit tangent bundle (or sphere bundle) M over V, and $d(\mathrm{vol}_v)$ is the (normalized) volume on V (cf. [16]).

Furthermore, for locally symmetric metrics the value of $h(g_0)$ can be explicitly computed (cf. [16], for example). Assume for simplicity that g_0 corresponds to a compact manifold with maximal sectional curvature -1. There are only four possibilities for the universal covering space:

(i) Hyperbolic space over the real numbers, where $h(g_0) = n - 1$.

(ii) Hyperbolic space over the complex numbers, where $h(g_0) = 2n$.
(iii) Hyperbolic space over the quaternions, where $h(g_0) = 4n + 2$.
(iv) Hyperbolic space over the Cayley numbers, where $h(g_0) = 22$.

By comparing the formula for the first derivative of the topological entropy in Proposition 6 with part (a)(i) of Lemma 7 it is easy to see that for a volume preserving perturbation g_λ of a locally symmetric metric we have that $D_0 h^{(\lambda)} = 0$ since the maximal measure m is equal to the Liouville measure.

However, we have the following estimates for the second derivatives $D_0^2 h^{(\lambda)}$:

Theorem 3. *For a volume preserving perturbation g_λ of locally symmetric metrics we have the following*:

(i) *For any $\varepsilon > 0$ there exists a closed geodesic c_0 such that*

$$\left| D_0^2 h(g_\lambda) - h(g_0)\left(\mathrm{Var}\left(\frac{g_1(v, v)}{2}\right) - \frac{1}{l_0(c_0)} D_0^2(l_\lambda(c_\lambda))\right)\right| < \varepsilon,$$

(ii)

$$D_0^2 h(g_\lambda) = h(g_0)\left(\mathrm{Var}\left(\frac{g^{(1)}(v, v)}{2}\right) + \frac{1}{4}\int_M (g^{(1)}(v, v))^2 \, d(\mathrm{Vol})\right) + h(g_0)\left(\frac{1}{2}\frac{1}{\dim V}\int_V \left(\frac{1}{2}\mathrm{tr}^2(g^{(1)}) - \mathrm{tr}(g^{(1)})^2\right) d(\mathrm{Vol})\right).$$

Proof. (i) By comparing Proposition 6 and Lemma 7(a)(i), $\int_M g_1(v, v)\, dm = 0$, and the identity in Theorem 2 reduces to the expression

$$\begin{aligned} D_0^2 h(g_\lambda) &= h(g_0)\left(\mathrm{Var}\left(\frac{g_1(v, v)}{2}\right) + \int_M D_0^2 \alpha^{(\lambda)}\, dm - 2\int_M (D_0 \alpha^{(\lambda)})^2\, dm\right) \\ &= h(g_0)\left(\mathrm{Var}\left(\frac{g_1(v, v)}{2}\right) - \int_M D_0^2 \beta^{(\lambda)}\, dm\right) \end{aligned} \tag{8.1}$$

using the identity (6.6). We can then approximate the second term on the right-hand side of (8.1) by the integral around a closed orbit c_0.

(ii) By Part (c) of Lemma 7 we can write

$$\int_M g^{(2)}(v, v)\, d(\mathrm{Vol}) = \frac{1}{\dim V}\int_V \mathrm{tr}(g^{(2)})\, d(\mathrm{Vol}_v)(x),$$

and then by applying twice part (a)(ii) of Lemma 7 we get the identity

$$\int_M g^{(2)}(v, v)\, dm(v) = \frac{1}{\dim V}\int_V \left(\mathrm{tr}(g^{(1)})^2 - \frac{1}{2}\mathrm{tr}^2(g^{(1)})\right) d(\mathrm{Vol}_c). \tag{8.2}$$

Substituting (8.2) into the right-hand side of the inequality (7.2) completes the derivation.

Remark. In fact, it is easy to show that for any $\delta > 0$ there exists T_0 such that the number $\pi_0(T)$ of closed geodesics c_0 of length at most T, which can be used in Theorem 3(i) compared with the total number $\pi(T)$ of closed geodesics of length at most T, satisfies $|\pi_0(T)/\pi(T) - 1| < \delta$ for all $T > T_0$; cf. [18].

Corollary 3.1. *For any Teichmuller deformation g_λ of a Riemann surface and any $\varepsilon > 0$ there exists a closed geodesic c_0 such that*

$$\left|\mathrm{Var}\left(\frac{g_1(v, v)}{2}\right) - \frac{1}{l_0(c_0)} D_0^2(l_\lambda(c_\lambda))\right| < \varepsilon .$$

Proof. Since the topological entropy $h(g_\lambda)$ is constant under Teichmuller deformations, this follows immediately from the theorem.

Corollary 3.2. *In the case of surfaces (i.e., $\dim V = 2$) where $g^{(1)}$ has eigenvalues λ_1, λ_2 depending on $v \in V$, for a volume-preserving perturbation g_λ of locally symmetric metrics,*

$$\begin{aligned} D_0^2 h(g_\lambda) &= h(g_0)\left(\mathrm{Var}\left(\frac{g^{(1)}(v, v)}{2}\right) + \frac{1}{4}\int_M (g^{(1)}(v, v))^2 \, d(\mathrm{Vol})\right) \\ &\quad - h(g_0)\left(\frac{1}{4}\int_V (\lambda_1 - \lambda_2)^2 \, d(\mathrm{Vol})\right) . \end{aligned}$$

Proof. This follows immediately from the simple eigenvalue identity:

$$\tfrac{1}{2}\mathrm{tr}^2(g^{(1)}) - \mathrm{tr}(g^{(1)})^2 = \tfrac{1}{2}(\lambda_1 + \lambda_2)^2 - (\lambda_1^2 + \lambda_2^2) = -\tfrac{1}{2}(\lambda_1 - \lambda_2)^2 .$$

Observe that the first two terms on the right-hand side of the equality in Theorem 3(ii) are always positive. To understand the contribution from the last two terms (when $\dim V \geq 3$) we first recall some simple linear algebra.

Linear Algebra Lemma. *Let A be a $n \times n$ real matrix. Then $0 \leq \mathrm{tr}^2(A) \leq n\,\mathrm{tr}(A^2)$ where the upper bound is realized when A is just a scalar multiple of I, and the lower bound is realized when $\mathrm{tr}(A) = 0$.*

Proof. Let $\langle A, B\rangle = \mathrm{tr}(A \cdot B^T)$ denote the usual inner product on the space of $n \times n$ real matrices. Then the above inequalities are just a reformulation of the Cauchy-Schwartz inequality $|\langle A, 1\rangle| \leq \|A\| \cdot \|I\|$ with I being the identity matrix and observing that $\|I\|^2 = n$.

To apply this lemma, we set $A = g^{(1)}$ on each fiber to get $0 \leq \mathrm{tr}^2(g^{(1)}) \leq n^2\,\mathrm{tr}(g^{(1)})^2$. Then integrating over the manifold V yields

$$
\begin{aligned}
(8.3) \qquad -\int_V \operatorname{tr}(g^{(1)})^2 \, d(\mathrm{vol}) &\le \int_V \left(\frac{1}{2}\operatorname{tr}^2(g^{(1)}) - \operatorname{tr}(g^{(1)})^2\right) d(\mathrm{vol}) \\
&\le \frac{n-2}{2} \int_V \operatorname{tr}(g^{(1)})^2 \, d(\mathrm{vol}),
\end{aligned}
$$

the middle expression being precisely the second term on the right-hand side of the inequality in Theorem 3. Notice that the lower bound in (8.3) is always strictly negative, and the upper bound is always positive (and strictly positive if $\dim V \ge 3$).

The Linear Algebra Lemma also gives us the following trivial, yet useful, result.

Lemma 8. (a) *The second inequality in* (8.3) *is an equality iff* $g^{(1)} = f(x) \cdot I$ *for some smooth function* $f : V \to R$.

(b) *The first inequality in* (8.3) *is an equality iff* $\operatorname{tr}(g^{(1)}) \ne 0$.

9. Some special cases

In the case of conformal perturbations of a metric g_0 the perturbation must take the special form $g_\lambda = f_\lambda g_0$, where $f_\lambda : V \to \mathbb{R}^+$ is a C^∞ function with $f_0 \equiv 1$. If we expand $f_0 = 1 + \lambda f^{(1)} + \frac{\lambda^2}{2} f^{(2)} + \cdots$, then we can identify $g^{(1)} = f^{(1)} g_0$ and $g^{(2)} = f^{(2)} g_0$.

Under volume-preserving conformal perturbations it is already known that locally symmetric metrics are a minimum (cf. [14], [16]). However, Theorem 3 gives us an explicit lower bound on the second derivative of the topological entropy.

Proposition 8. *For a volume-preserving conformal perturbation* g_λ *of a locally symmetric metric* g_0,

$$
D_0^2 h(g_\lambda) = h(g_0)\left(\operatorname{Var}\left(\frac{g^{(1)}(v, v)}{2}\right) + \left(n + \frac{7}{4}\right)\int_V (f^{(1)})^2 \, d(\mathrm{Vol})\right) > 0.
$$

Proof. Since $g^{(1)} = f^{(1)} g_0$, the hypothesis of part (a) of Lemma 7 is satisfied. Substituting the corresponding identity into the inequality in Theorem 3 yields

$$
\begin{aligned}
(9.1) \qquad D_0^2 h(g_\lambda) = h(g_0)\Bigg(\operatorname{Var}\left(\frac{g^{(1)}(v, v)}{2}\right) &+ \frac{1}{4}\int_M (g^{(1)}(v, v))^2 \, d(\mathrm{Vol}) \\
&\frac{n-2}{\dim V}\int_V \operatorname{tr}(g^{(1)})^2 \, d(\mathrm{vol})\Bigg)
\end{aligned}
$$

Using the identification $g^{(1)} = f^{(1)} g_0$ we can write $g^{(1)}(v, v) = f^{(1)} \cdot \|v\|^2$ and $\operatorname{tr}(g^{(1)})^2 = n(f^{(1)})^2$, where $n = \dim V$ is the dimension of the manifold. Thus (9.1) becomes

$$\begin{aligned} D_0^2 h(g_\lambda) &= h(g_0) \left(\operatorname{Var}\left(\frac{g^{(1)}(v, v)}{2} \right) \right. \\ &\qquad \left. + \frac{1}{4} \int_V (f^{(1)})^2 \, d(\mathrm{Vol}) + (n-2) \int_V (f^{(1)})^2 \, d(\mathrm{vol}) \right) \\ &= h(g_0) \left(\operatorname{Var}\left(\frac{g^{(1)}(v, v)}{2} \right) + \left(n - \frac{7}{4} \right) \int_V (f^{(1)})^2 \, d(\mathrm{Vol}) \right), \end{aligned}$$

which completes the proof.

The case of volume-form-preserving perturbations is somewhat more subtle and complicated than that of volume-preserving conformal perturbations.

We begin by recalling Moser's remarkable result on the generality of volume-form-preserving perturbations: *Any Riemannian metric g' which has the same volume as g has in its diffeomorphism class a Riemannian metric with the same volume form as g.*

In the space of metrics with the same volume form there is a natural class of deformations of the special form $\lambda \to g_\lambda = g_0 e^{\lambda\sigma}$, where $\sigma \in \Gamma(V, \mathcal{S}^2)$ is a C^∞ section satisfying $\operatorname{tr}(\sigma) = 0$ (these are described as "geodesics" in [10] and [3]). These families form a particularly useful class of volume-form-preserving perturbations.

Example. On surfaces the condition $\operatorname{tr}\sigma = 0$ implies that $\sigma = \left[\begin{smallmatrix} a & b \\ b & -a \end{smallmatrix}\right]$. Then

$$\begin{aligned} g_\lambda = g_0 e^{\lambda\sigma} &= g_0(1 + \lambda\sigma + (\lambda^2/2)\sigma^2 + \cdots) \\ &= g_0 \left(1 + \lambda\sigma + \frac{\lambda^2}{2}(a^2 + b^2)1 + \right). \end{aligned}$$

For volume-form-preserving perturbations part (b) of Lemma 8 applies, and we cannot claim that the terms in the lower bound on the second derivative in Theorem 3(ii) is positive. Furthermore, Corollary 3.1 suggests that the term $D_0^2(l_\lambda(c_\lambda))/l_0(c_0)$ occurring in the estimate on the second derivative formula (in Theorem 3(i)) can be really positive. Thus to enhance our understanding, we have to know better the other term, involving the variance.

PART THREE. ESTIMATES ON THE VARIANCE

The term $\operatorname{Var}(g_1(v, v)/2)$ in the second derivative of the topological entropy is rather mysterious. In this part we begin an investigation of its properties.

10. A problem on Lebesgue spectrum

The variance has a very simple interpretation in ergodic theory. The geodesic flow ϕ is well-known to have continuous Lebesgue spectrum. This means that in applying Bochner's theorem we can write

$$\rho_F(t) = \int_{-\infty}^{+\infty} e^{i\lambda t}\, d\mu_F(\lambda),$$

where μ_F is a smooth measure on $\mathbb{R}$ with continuous density $\alpha_F \equiv d\mu_F/d\lambda : \mathbb{R} \to \mathbb{R}^+$. We recall that the value at $\lambda = 0$ is precisely the variance $\operatorname{Var}(F)$, as defined in §4.

This brings us to a simple question.

Question. Consider functions F satisfying $\int F\, d(\mathrm{Vol}) = 0$ and the normalization condition $\int |F|^2\, d(\mathrm{Vol}) = 1$, say. Given any $\varepsilon > 0$, can we construct functions F such that $\alpha_F(0) < \varepsilon$?

We have the following strong result in the case of functions F constant on fibers, and for the special case of geodesic flows associated to compact three-dimensional manifolds V with all sectional curvatures equal to -1.

Theorem 4. *Given $\varepsilon > 0$, $\operatorname{Var}(F) < \varepsilon$ for all but a finite-dimensional space in the smooth functions $F : V \to \mathbb{R}$ with zero integral.*

Corollary 4.1. *Given $\varepsilon > 0$, $0 \leq \operatorname{Var}(g^{(1)}(v, v)) < \varepsilon$ for all but a finite-dimensional space in the space of conformal volume preserving perturbations.*

Proof. This follows from the discussion in §9.

The proof of this theorem requires a little preparation from representation theory, and will be given in the next section.

11. Algebraic flows

The group $G = SL(2, \mathbb{C})$ has a natural action on the three-dimensional hyperbolic space $\mathbb{H}^3 = \operatorname{int}(D^3)$ corresponding to the action by linear fractional transformations on the Riemann sphere $\widehat{\mathbb{C}}$. There is then an explicit correspondence between G and the two-frame bundle $\mathscr{F}^2(\mathbb{H}^3)$ on

$\mathbb{H}^3$ (i.e., the bundle consisting of distinguished choice of unit tangent vectors and a subsequent choice of a pair orthonormal vectors). For example, we can fix (i) a reference point $x = (0, 0, 0) \in \mathbb{H}^3 = \operatorname{int}(D^3)$, and (ii) the obvious reference frame $F_x = (v_1, v_2, v_3)$ in $S_x\mathbb{H}^3 = \mathbb{R}^3$, and then identify each element $g \in G$ with the reference frame $F_y \in \mathscr{F}^2(\mathbb{H}^3)$ (above the point $y \in \mathbb{H}^3$) satisfying $gF_{y'} = F_x$.

Let $\Gamma \subset G$ be a discrete subgroup with compact quotient G/Γ, and consider the subgroups

$$B = \left\{ \begin{bmatrix} e^{-it} & 0 \\ 0 & e^{it} \end{bmatrix} \middle| \, 0 \le t \le 2\pi \right\} \quad \text{and} \quad g_t = \begin{bmatrix} e^{-t/2} & 0 \\ 0 & t^{t/2} \end{bmatrix}.$$

The flow $\phi_t : SV \to SV$, $Bg\Gamma \mapsto Bg_t g\Gamma$ on the double coset space $SV = B\backslash B/\Gamma$ corresponds to the geodesic flow on the unit tangent bundle of the compact three-dimensional manifold by $V = K\backslash G/\Gamma$, where

$$K = SU(2) = \left\{ -i \begin{bmatrix} -\overline{z}_2 & \overline{z}_1 \\ z_1 & z_2 \end{bmatrix} : z_1, z_2 \in \mathbb{C}, \ |z_1|^2 + |z_2|^2 = 1 \right\}$$

is the maximal compact subgroup of G (cf. [11], [8, p. 21], for example), and the induced metric g_0 on V is hyperbolic.

Remark. More specifically, the above correspondence between G and the two-frame bundle $\mathscr{F}^2(\mathbb{H}^3)$ induces a correspondence between G/Γ and the two-frame bundle $\mathscr{F}^2(V)$ over V. The maximal compact subgroup $K \subset G$ then acts by rotating the two-frames in each fiber and allowing the identification $V = K\backslash G/\Gamma$. The action of the smaller group $B \subset K$ corresponds to *fixing* the distinguished unit tangent vector in the two-frame and rotating the frame about this axis, which allows the identification $K\backslash G/\Gamma = SV$. Finally, since the one-parameter group $g_t : G/\Gamma \mapsto G/\Gamma$ defined by $g\Gamma \mapsto g_g g\Gamma$ corresponds to the *frame flow* and $g_t B = Bg_t$, the induced flow g_t on $B\backslash G/\Gamma$ is well defined and corresponds to the geodesic flow.

It is useful to recall two standard decompositions for these groups:

Lemma 9. (i) *We can write* G *as a product* $G = KAN$ *where*

$$A = \left\{ \begin{bmatrix} \lambda & 0 \\ 0 & 1/\lambda \end{bmatrix} : -\infty < \lambda < +\infty \right\} \quad \text{and} \quad N = \left\{ \begin{bmatrix} 1 & z \\ 0 & 1 \end{bmatrix} : z \in \mathbb{C} \right\},$$

(*i.e., the Iwasawa decomposition*).

(ii) *We can write* $K = B\Theta B$ *where*

$$\Theta = \left\{ \begin{bmatrix} \cos\theta & i\sin\theta \\ i\sin\theta & \cos\theta \end{bmatrix} : 0 \le \theta \le 2\pi \right\},$$

(*cf.* [13, *p.* 12]).

Remark. If we write $k \in K$ as

$$k = b_2\theta b_1 \begin{bmatrix} e^{-i\phi_2} & 0 \\ 0 & e^{i\phi_2} \end{bmatrix} \begin{bmatrix} \cos\theta & i\sin\theta \\ i\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} e^{-i\phi_1} & 0 \\ 0 & e^{i\phi_1} \end{bmatrix}$$
$$= \begin{bmatrix} \cos\theta e^{i(\phi_1+\phi_2)} & i\sin\theta e^{-i(\phi_2-\phi_1)} \\ i\sin\theta e^{i(\phi_2-\phi_1)} & \cos\theta e^{-i(\phi_1+\phi_2)} \end{bmatrix},$$

then the coefficients (ϕ_1, θ, ϕ_2) are usually called the *Euler angles* (cf. [13, p. 9–12]).

We want to make use of the very explicit knowledge of the irreducible unitary representations of $SL(2, \mathbb{C})$ (cf. [26], [2], [26], [34]) to study the variance and consequently the expression (2.3). (Compare with the work of Fomin and Gelfand on Lebesgue spectrum [11] and work of Moore on exponential decay of correlations [24].)

We recall a few elementary facts from the theory of unitary representations. To each $g \in G$ we can associate a unitary operator $U_g : L^2(M) \to L^2(M)$ on the Hilbert space of square integrable functions by $(U_g f)(x) = f(gx)$, $f \in L^2(m)$, $x \in M$. If $\mathcal{U} = \mathcal{U}(L^2(M))$ denotes the group of unitary operators from $L^2(M)$ to itself, then the map $g \mapsto U_g \in \mathcal{U}$, $g \in G$, is the *canonical representation* of G.

Let $L^2(M) \ominus \mathbb{C}$ denote the orthogonal complement of the constant functions $\mathbb{C}$. The following basic result appears in [12, p. 18].

Lemma 10. *There is an orthogonal splitting* $L^2(M) \ominus \mathbb{C} = \bigoplus_{i=0}^{+\infty} H_i$ (*of the orthogonal complement of the constant functions* $\mathbb{C}$) *into invariant Hilbert spaces* H_i *and on each of which the action* U_g, $g \in G$, *is irreducible.*

We now need to recall the different types of *irreducible* representation.

Proposition 9. *The irreducible unitary representations of* $SL(2, \mathbb{C})$ *are isomorphic to one of the following two types*:

(a) *Principal series,* $\rho \in \mathbb{R}$, $n \in \mathbb{Z}$. $H = L^2(\mathbb{C}, dx\,dy)$ *and*

$$U_g : H \to H,$$
$$U_g^{(\rho)} f(x) = f\left(\frac{ax+b}{cz+d}\right) |cz+d|^{n+i\rho-2}(cz+d)^{-n},$$

where

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL(2, \mathbb{C}).$$

In particular,

$$U_{g_t}^{(\rho)}(x) = f(e^{2t}x)e^{(-2+i\rho n)t}.$$

(b) *Complementary series,* $0 < \rho < 2$. $H \subseteq L^2(\mathbb{C}, dx\,dy)$ *is the completion of continuous functions of compact support using* $\langle f, g\rangle = \int \hat{f}(z)\hat{g}(z)|z|^{2\rho}\,dx\,dy$ *(where* $\hat{f}(z) = \int_{-\infty}^{+\infty} f(w)e^{izw}\,dw$ *is the usual Fourier transform on* $\mathbb{C}$*) and*

$$U_g : H \to H,$$
$$U_{g_t}^{(\rho)}\hat{f}(z) = \hat{f}\left(\frac{ax+b}{cz+d}\right)|cz+d|^{-2-\rho}, \quad \textit{where } \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL(2, \mathbb{C}).$$

In particular,

$$U_{g_t}^{(\rho)}\hat{f}(z) = \hat{f}(e^{2t}z)e^{(2-\rho)t},$$

(*cf.* [12], [6]).

Remark. We shall be particularly interested in principal series representations (with $n = 0$). A nice geometric description occurs in [21, pp. 114–115].

Combining this proposition with the preceding lemma we see that for each H_i in the decomposition $L^2(M) \ominus \mathbb{C} = \bigoplus_{-\infty}^{+\infty} H_i$ there corresponds an irreducible representation of one of the above forms and an isometry $V : H_t \to H$ satisfying $VU_g^{(\rho)} = UV$.

We recall some basic facts about these irreducible representations.

Lemma 11. (i) *The principal series with* $n = 0$ *and complementary series are the only irreducible representations containing spherical functions, i.e., vectors invariant under the action of the maximal compact subgroup* $K = SU(2)$. *When normalized these are, respectively,*

$$f(z) = \frac{1}{\sqrt{\pi}}(|z|^2+1)^{-(2+i\rho)/2} \quad \textit{and} \quad f(z) = \frac{1}{\sqrt{\pi}}(|z|^2+1)^{-(2+\rho)/2}.$$

(ii) *In the decomposition there are at most finitely many complementary series and infinitely many principal series for each* n*, although for fixed* n *and* $T > 0$ *there are only finitely many principal series with* $|\rho| < T$.

(iii) *Vectors invariant under the action of* B *are of the form* $f(re^{2\pi i\phi}) = f(r)$ *for the complementary series and* $f(re^{2\pi i\phi}) = f(r)e^{\pi in}$ *for the principal series* n *even.*

Proof. Part (i) is proved in (cf. [26, p. 386]). In fact, the vectors $F_i \in H_i$ whose images VF_i are spherical vectors are precisely the eigenvectors of the Laplace Beltrami operator on (V, g_0) with eigenvalues $-2(\rho^2+4)$, for the principal series with $n = 0$, or $-2(-\rho^2+4)$ for the complementary series.

For principal series part (ii) is proved in [12, p. 94] and for complementary series in [24, p. 178]. Part (iii) is proved in [11, p. 64].

12. Estimating the variance

We can now collect together the results in the preceding two sections to complete the proof of Theorem 4.

Proof of Theorem 4. Any C^∞ function $F : V \to \mathbb{R}$ with $\int F\, d(\mathrm{Vol}_g) = 0$ can be decomposed as $F = \int_{-\infty}^{+\infty} c_i F_i \in \bigoplus_{i=0}^{+\infty} H_i$ where F_i are the (normalized) eigenfunctions of the Laplacian. We want to use the representation theory to estimate the expressions $\mathrm{Var}(F_i)$ by making use of the identity $\rho(t) = \langle F_i \circ \phi_t, F_i\rangle = \langle U_{g_t}^{(\rho)}(VF_i), (VF_i)\rangle$, where (VF_i) are the spherical vectors in the corresponding irreducible representations.

(i) *Principal series with* $n = 0$. We begin by calculating $\mathrm{Var}(f)$ for principal series representations with $n = 0$. These can be realized as $H = L^2(\mathbb{C}, dx\,dy)$ and $U_{g_t}^{(\rho)} f(x) = f(e^{2t}x)e^{(-2+\rho)t}$ for $\rho \in \mathbb{R}$.

We can consider the (normalized) spherical vector

$$f(z) = \pi^{-1/2}(|z|^2 + 1)^{-(2+i\rho)/2} \in H$$

(cf. [3, p. 386]) and write its autocorrelation function

$$\begin{aligned}\rho(t) &= \langle U_{g_t}^{(\rho)} f, f\rangle \\ &= \frac{1}{\pi}\int_{-\infty}^{+\infty} ((|z|^2+1)^{-(2+i\rho)/2})(e^{(-2+i\rho)t}(e^{-4t}|z|^2+1)^{-(2-i\rho)/2})\,dz.\end{aligned}$$

This integral is explicitly evaluated in [3, pp. 386–387] with the solution that

$$\rho(t) = \frac{2}{\rho}\frac{\sin(\rho t)}{\sinh(2t)}. \tag{12.1}$$

If $VF_i = f$, then by the above correspondence the autocorrelation functions for f and F_i are the same, and $\mathrm{Var}(F_i) = \int_{-\infty}^{+\infty} \rho(t)\,dt$. With the formula in Proposition 3(ii) for $\rho(t)$ we have a standard integral:

$$\mathrm{Var}(F_i) = \frac{2}{\rho}\int_{-\infty}^{+\infty} \frac{\sin(\rho t)}{\sinh(2t)}\,dt = \frac{\pi}{\rho}\tanh\left(\frac{\rho\pi}{4}\right). \tag{12.2}$$

(ii) *Complementary series.* We can similarly calculate $\mathrm{Var}(F_i)$ for complementary series representations with $0 < \rho < 2$. An analogous estimate gives that

$$\rho(t) = \frac{2}{\rho}\frac{\sinh(\rho t)}{\sinh(2t)}, \tag{12.3}$$

(cf. [26, pp. 386–387]). As before, if $VG_i = f$, then $\mathrm{Var}(F_i) = \int_{-\infty}^{+\infty} \rho(t)\,dt$, and substituting (12.3) we have only to evaluate the following standard integral:

$$\mathrm{Var}(F_i) = \frac{2}{\rho}\int_{-\infty}^{+\infty} \frac{\sinh(\rho t)}{\sinh(2t)}\,dt = \frac{\pi}{\rho}\tan\left(\frac{\rho\pi}{4}\right). \tag{12.4}$$

We can patch together these estimates with the formula $\mathrm{Var}(F) = \sum_{i=0}^{+\infty} |c_i|^2 \mathrm{Var}(F_i)$ (cf. [31], although it follows directly from the definitions) and note that $\|F\|_2 = \sum_{i=0}^{+\infty} |c_i|^2$. Finally, we observe that

(a) for the principal series the map

$$\mathbb{R} - \{0\} \to (0, \pi^2/4),$$
$$\rho \to c_i(\rho) = (\pi/\rho)\tanh(\rho\pi/4)$$

is monotonically decreasing, and

(b) for the complementary series the map

$$(0, 2) \to (\pi^2/4, +\infty),$$
$$\rho \to c_i(\rho) = (\pi/\rho)\tan(\rho\pi/4)$$

is monotonically increasing.

Taking these observations together, allows us to conclude that the contribution to $\mathrm{Var}(F)$ to the second derivative of the topological entropy under conformal perturbations, although always positive, tends to zero.

These estimates are directly applicable to conformal perturbations, and help us to draw interesting conclusions about the second derivative of the topological entropy under conformal perturbations (cf. Corollary 4.1). In the remainder of this section we sketch how it is possible to construct a *volume-form-preserving* perturbation with the same normalisation and for which $\mathrm{Var}(g^{(1)}(v, v)) < \varepsilon$. We begin by recalling the following: If $\mathrm{trace}(g^{(1)}) \equiv 0$ then $g_\lambda = \exp(\lambda g^{(1)})$ is a volume-form-preserving perturbation (cf. [3], [11]). We shall assume that our perturbations will be shown to have the special form

$$g^{(1)} = g(x)\begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{2} \end{bmatrix}, \tag{12.5}$$

where $g : V \to \mathbb{R}$ is a C^∞ function, and the local coordinates are understood to come from the quotient of the AN translates of the fixed reference frame $F_x = (v_1, v_2, v_3)$. Since $\mathrm{trace}(g^{(1)}) \equiv 0$, any perturbation of this general form corresponds to the first-order term in a volume form preserving perturbation (cf. §9).

Rather than explicitly constructing $g^{(1)}$ we shall specify $g^{(1)}(v, v) \in H_i$ (for each H_i space corresponding to a principal series for $(\rho, 0)$) which must correspond to a section of the form (12.5). We can identify $S_x V \equiv S^2$, where $S_x V$ is the fiber above the reference point x with the natural coordinates (v_1, v_2, v_3) coming from the reference frame F_x. Then in standard spherical coordinates we can write $v_1 = \cos\theta$, $v_2 = \sin\theta\sin\theta$ and $v_3 = \sin\theta\cos\phi$.

Lemma 12. *The vector* $f(re^{2\pi i\phi}) = (r^2 - \frac{1}{2})(1 + r^2)^{-2-i\rho/2} \in L^2(\mathbb{C})$ *in a principal series with* $n = 0$ *corresponds to a* C^∞ *function* $(Vf) = g(x)(3\cos^2\theta - 1)/2$ *on* SV.

Proof. In [7, pp. 79–81], a simple method for constructing the identification $V : H_i \to H$ is described for the case of $SL(2, \mathbb{R})$, where the eigenvector of the Laplacian in H_i is mapped to the unique K-fixed vector in H, and then V is defined by comparing the respective G-orbits of this point. We shall employ a simple variant of this approach. We let $S_2 \subset L^2(\mathbb{C})$ be the (unique) five-dimensional K-invariant irreducible space with the basis $Y_2^j(\theta, \phi)\pi^{-1/2}(|z|^2 + 1)^{-(2+i\rho)/2}$, $-2 \le j \le 2$, where

(a) $Y_2^j(\theta, \phi)$ are the standard basis for the eigenspace of the spherical Laplacian on $\widehat{\mathbb{C}} \equiv S^2$ corresponding to the eigenvalue -2; cf. [13], [30], [33];

(b) (θ, ϕ) corresponds to $z \in \mathbb{C}$ by the stereographic projection so that, in particular, $\cos\theta = r(1 + r^2)^{-1/2}$.

Note: As a representation $k \to U_k \in \mathscr{U}(H_i)$ of the smaller compact group K, the principal series for $(\rho, 0)$ can be further decomposed into irreducible representations for K as $H_i = \bigoplus_{l=0}^{\infty} S_l$, where S_l is a $(2l+1)$-dimensional space (cf. [30, pp. 57–60]).

The compact group K acts on SV by rotating tangent vectors in each fiber. Thus by expanding the image $(Vf) \in H_i$ in terms of the spherical harmonics on each fiber $S_y V \equiv S^2$, $y \in V$, and a comparison of dimensions of the orbits, we can deduce that $(Vf)|S_y V$ is in the (five-dimensional) eigenspace for the spherical Laplacian with eigenvalue -2 *up to a scalar multiple and an additive constant.* Moreover, we observe that the function $(\cos^2\theta - 1)/2$ on S^2 can be written, up to an additive constant, as $Y_2^0(\theta, \phi)$ which is a *zonal* spherical function, i.e., independent of ϕ cf. [33]. Thus, we conclude that the image Vf can be written as $\frac{1}{2}g(x)(\cos^2\theta - 1)$ (cf. [15, p. 19]).

If we fix a fiber $S_y V$, then the function $(Vf)|S_y V$ is a C^∞ (even real analytic) zonal spherical function. By the correspondence $U_g(Vf) = V(U_g f)$ we translate tangent vectors into this fiber and deduce that the function $g(x)$ is C^∞ using the function $Y_2^j(\theta, \phi)\pi^{-1/2}(|z|^2+1)^{-(2+i\rho)/2}$ and the action of AN are both C^∞.

By means of the identification $C^\infty(SV) \equiv \Gamma(V, C^\infty(S^2))$ we can write

$$g^{(1)}(v, v) = g(x)\frac{3\cos^2\theta - 1}{2} = g(x)\left(v_1^2 \frac{(v_2^2+v_3^2)}{2}\right).$$

In particular, $g^{(1)}$ indeed takes the form (12.6). The following is the promised estimate on the variance for *certain* volume-form-preserving perturbations (compare with the Appendix, [26] and [25]).

Proposition 10. *For representations in the principal series with* $n = 0$ *we have that* $\operatorname{Var}(f)/\int |f|^2\, d(\mathrm{Vol}) = O(1/|\rho|)$ *as* $|\rho| \to \infty$.

Proof. The proof just consists of some new explicit estimates. We want to find an upper bound on $\operatorname{Var}(f)$ by making use of the identity $\operatorname{Var}(f) = \int_{-\infty}^{+\infty} \rho(t)\, dt$. We observe that

$$\int |f|^2\, d(\mathrm{Vol}) = \|f\|_2^2 = \pi \int_0^\infty \frac{(2r^2-1)^2}{(1+r^2)^4} r\, dr = \frac{\pi}{4}$$

is actually *independent* of ρ, and therefore we need not concern ourselves further with the exact normalization constants. By definition, the autocorrelation function takes the form

$$\begin{aligned}\rho(t) &= \langle U_{g_t}^{(\rho)} f, f\rangle \\ &= 2\pi \int_0^{+\infty} \left(\left(r^2 - \frac{1}{2}\right)(r^2+1)^{-2-i\rho/2}\right) \\ &\quad \cdot \left(e^{(-2+i\rho)t}\left(e^{(-2+i\rho)t}(e^{-4t}r^2 - \frac{1}{2}\right)(e^{-4t}r^2+1)^{-2+i\rho/2}\right) r\, dr.\end{aligned}$$

We can explicitly integrate this expression after making some changes of variables. We begin by setting $\xi = r^2$, and then we get

$$\begin{aligned}\rho(t) = \pi \int_0^{+\infty} &\left(\left(\xi - \frac{1}{2}\right)(\xi+1)^{-2-i\rho/2}\right) \\ &\cdot \left(e^{-(-2+i\rho)t}\left(e^{-4t}\xi - \frac{1}{2}\right)(e^{-4t}\xi+1)^{-2+i\rho/2}\right) d\xi.\end{aligned}$$

Substituting $x = (1-e^{-4t})/(1+\xi)$ in the above equation and using $1-x = (e^{-4t}+\xi)/(1+\xi)$ and $d\xi/dx = (1+\xi)^2/(1-e^{-4t}))$ we obtain

$$\rho(t) = \frac{e^{-(6-3i\rho/2)t}}{(1-e^{-4t})^3}\pi \int_0^{1-e^{-4t}} (1-x)^{-2-i\rho/2}\left\{(1-e^{-4t}) - \frac{3x}{2}\right\} \cdot \left\{(1-e^{-4t})e^{-4t} - x\left(e^{-4t} - \frac{1}{2}\right)\right\} dx. \tag{12.7}$$

To estimate this integral we consider terms of the general form

$$\int_0^{1-e^{4t}} x^m(1-x)^{-2-i\rho/2}\, dx = \int_{e^{-4t}}^1 (1-w)^m w^{-2-i\rho/2}\, dw, \quad m = 0, 1, 2, \tag{12.8}$$

by the change of variables $w = 1-x$. But we can easily evaluate this last integral as

$$\begin{aligned}\int_{e^{-4t}}^1 (1-w)^m w^{-2-i\rho/2}\, dw &= \sum_{k=0}^m \binom{m}{k} \int_{e^{-4t}}^1 w^{k-2-i\rho/2}\, dw \\ &= \sum_{k=0}^m \binom{m}{k} \left\{\frac{1-e^{-4t(k-1+i\rho/2)}}{k-1-i\rho/2}\right\}.\end{aligned} \tag{12.9}$$

Finally, we see that the expression in (12.8) is of order $O(e^{4t}/|\rho|)$ the most significant contribution in t coming from the term $k = 0$. Substituting (12.8) and (12.9) into (12.7) we conclude that $|\rho(t)| \le O(e^{-2|t|}/|\rho|)$. Integrating this over $t \in \mathbb{R}$ yields $\mathrm{Var}(f) = \int_{-\infty}^{+\infty} \rho(t)\, dt \le \int_{-\infty}^{+\infty} |\rho(t)|\, dt \le \frac{C'}{|\rho|}$.

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