J. DIFFERENTIAL GEOMETRY
39 (1994) 659-680

CONJUGACY AND RIGIDITY FOR MANIFOLDS WITH A PARALLEL VECTOR FIELD

CHRISTOPHER B. CROKE & BRUCE KLEINER

Abstract

The main theorem in this paper is that any compact Riemannian manifold with geodesic flow isomorphic to the geodesic flow of a local Riemannian product $M = (X \times R)/\Gamma$ is isometric to M.

1. Introduction

In this paper we consider the question: Which compact Riemannian manifolds M are determined uniquely by their geodesic flows? To formulate this precisely we need a few definitions. If M and N are complete Riemannian manifolds, then their geodesic flows are C^0 conjugate if there is a homeomorphism $F: SM \to SN$ from the unit sphere bundle SM to the unit sphere bundle SN which commutes with the geodesic flows: $F \circ g_M^t = g_N^t \circ F$ for all $t \in R$ where g^t is the geodesic flow after time t. If $0 \le r \le \infty$, and F can be chosen to be a C^r diffeomorphism, then M and N have C^r conjugate geodesic flows. A complete Riemannian manifold M is C^r conjugacy rigid if any Riemannian manifold N whose geodesic flow is C^r conjugate to the geodesic flow of M is isometric to M. A more precise formulation of our question then is: Which compact Riemannian manifolds M are C^r conjugacy rigid?

It was pointed out by Weinstein (see [2, §4F]) that the geodesic flow of a Zoll surface is C^∞ conjugate to the geodesic flow of a round sphere. Using a variation of this idea we show in §6 that on any smooth manifold there are infinite-dimensional families of pairwise nonisometric metrics with mutually C^∞ conjugate geodesic flows. In particular, any Riemannian manifold containing an open subset isometric to a neighborhood of an equator $S^{n-1}(1) \subset S^n(1)$ is not conjugacy rigid.

On the other hand, surfaces of nonpositive curvature are C^0 conjugacy rigid (see [4] for the C^1 case, and [6] for the C^0 case). When both M and

Received October 6, 1992, and, in revised form, August 9, 1993. The first author was supported in part by NSF grant #DMS90-01707 and #D MS85-05550 while at MSRI. The second author was supported by an NSF Postdoctoral Fellowship.

N have negative curvature this question is closely related to the question of whether M and N must be isometric if they have the same marked length spectrum (see [18], [4], [6]). In [5] it was shown that if the geodesic flow of a compact n-dimensional Riemannian manifold N is C^0 conjugate to the geodesic flow of a flat manifold M, and $\mathrm{vol}(N) = \mathrm{vol}(M)$, then N is isometric to M. Finally, we mention that RP^n with its standard metric is C^0 conjugacy rigid as follows from the Blaschke conjecture for spheres (proved for $n = 2$ in [9] and for general n in [2, Appendix D]); this implies that spherical space forms S^n/Γ, where $\Gamma \subset O(n+1)$ has even order, are also C^0 conjugacy rigid.

The rigidity result in this paper concerns a special class of compact Riemannian manifolds which includes Riemannian products $X \times S^1$ for any compact Riemannian manifold X, nonproduct flat tori T^k, and Riemannian products $X \times T^k$ where T^k is nonproduct flat torus. These are manifolds that have a parallel vector field. By the de Rham splitting theorem (see [14, p. 187]) a complete Riemannian manifold M^n has a nontrivial parallel vector field if and only if it is of the form $(X \times R)/\Gamma$ where $\Gamma \subset \mathrm{Isom}(X) \times \mathrm{Isom}^+(R) \subset \mathrm{Isom}(X \times R)$, X is a simply connected complete Riemannian manifold, $X \times R$ has the product metric, and $\mathrm{Isom}^+(R)$ are the orientation preserving isometries. Our main result is that compact Riemannian manifolds with a parallel vector field are determined by their geodesic flows.

Theorem 1.1. *Let M^n be a compact Riemannian manifold with a nontrivial parallel vector field. Then M^n is C^1 conjugacy rigid.*

An ingredient in the proof of the above theorem which is of independent interest is

Proposition 1.2. *If M and N are compact Riemannian manifolds with C^1 conjugate geodesic flows, then they have the same volume.*

It is easy to construct C^∞ self-conjugacies of the geodesic flow of a round sphere that do not preserve the Liouville measure. Hence although C^1 conjugacies do not have to be measure preserving, Proposition 1.2 states that the total measure must be preserved. In particular the result of [5] thus implies that flat manifolds are C^1 conjugacy rigid.

In §3 we study conjugacies between manifolds both of which are nontrivial Riemannian products. The ideas in this section also have interesting applications in the setting of manifolds of nonpositive curvature, which we will pursue in a future paper with Patrick Eberlein.

Outline of the proof of Theorem 1.1. Suppose $F: SM \to SN$ is a C^1 conjugacy of the geodesic flow of M^n to the geodesic flow of N^n. If vol_{SM}

and vol_{SN} are the Liouville volume forms on SM and SN respectively, then $F^*\mathrm{Vol}_{SN} = \phi\mathrm{vol}_{SM}$ for some $\phi \in C^0(SM)$. Our first step is to prove that if M has a unit parallel field S, then $\phi = 1$ on the "vertical" (i.e., $\{S(m)|m \in M\}$) directions in M (Proposition 2.4). We then exploit an extremal property of the Jacobi equation along the vertical geodesics in M^n to show that $\mathrm{vol}(N^n) \geq \mathrm{vol}(M^n)$ with equality holding only if $F(S)$ is a parallel vector field on N^n. By Proposition 1.2 we have $\mathrm{vol}(M^n) = \mathrm{vol}(N^n)$, and so $F(S)$ is a parallel field in our situation (Proposition 5.3). To complete the proof, we use the behavior of nearly vertical geodesics to see that M and N are isometric (Corollary 3.3).

Remark. The conjugacy problem formulated above is very closely related to the boundary rigidity problem discussed in [3]. A Riemannian manifold $(M, \partial M, g_0)$ with boundary ∂M is called boundary rigid if every $(N, \partial M, g_1)$ (with diffeomorphic boundary) with $d_{g_0}(p, q) = d_{g_1}(p, q)$ for every $p, q \in \partial M$ must be isometric to M (d_g represents the distance in M between boundary points).

Theorem 1.1, along with the arguments in §§5 and 7 of [3], yields the fact that SGM subdomains of compact Riemannian manifolds with a parallel field are boundary rigid. The condition SGM is a condition on d_{g_0} (see [3, §1]) which states, loosely speaking, that every geodesic segment whose interior lies in the interior of M is the unique geodesic between its endpoints. We mention that Viktor Schroeder had independently noticed that the arguments of [3] show that SGM subdomains of a manifold which is a product with an interval are boundary rigid.

The other known examples of boundary rigid manifolds are subdomains M of an open hemisphere of a round sphere (see [17] and [11, §5.5B]), compact M^n that can be isometrically immersed in R^n (see [11, §5.5B], [17], and [3]), and any SGM surface (two-dimensional) of nonpositive curvature [4].

We would like to thank the referee for his suggestions which made this paper more readable.

2. The behavior of volume under C^1 conjugacy

In this section we will often be dealing with differential forms θ with continuous coefficients whose exterior derivatives (defined weakly by Stokes' theorem: $\int_c d\theta = \int_{\partial c} \theta$ for every smooth chain c) are also differential forms with continuous coefficients. We denote the space of such forms on a smooth manifold M by $\Omega^*_{C^0}(M)$, and note that $\Omega^*_{C^0}(M)$ is closed under exterior differentiation, pullback by C^1 maps, and wedge products.

Lemma 2.1. *Let M^{2n-1} be a compact manifold, and let X be a smooth vector field on M. Suppose $\theta_0, \theta_1 \in \Omega^1_{C^0}(M)$ are both invariant under the flow of X and satisfy $\theta_i(X) \equiv 1$, $i = 1, 2$. Then*

$$\int_{M^{2n-1}} \theta_0 \wedge (d\theta_0)^{n-1} = \int_{M^{2n-1}} \theta_1 \wedge (d\theta_1)^{n-1}.$$

Proof. By Cartan's identity we have $i_X d\theta_i = L_X\theta_i - d i_X\theta_i = 0$. (Note that each of the terms in the identity is well defined since $d\theta_i \in \Omega^2_{C^0}(M)$, θ_i is invariant under the flow of X, and $i_X\theta_i \equiv 1$; moreover the identity holds as one can see by integrating both sides over arbitrary 1-chains.) Let $\theta_t = (1-t)\theta_0 + t\theta_1$, and $\dot\theta_t = d\theta_t/dt$. Since $i_X\dot\theta_t = 0$, we have

$$i_X(\dot\theta_t \wedge (d\theta_t)^{n-1}) = (i_X\dot\theta_t) \wedge (d\theta_t)^{n-1} - (n-1)\dot\theta_t \wedge (i_X d\theta_t) \wedge (d\theta_t)^{n-2} = 0.$$

Hence $\dot\theta_t \wedge (d\theta_t)^{n-1} = 0$ and

$$\begin{aligned} &\frac{d}{dt}\int_{M^{2n-1}} \theta_t \wedge (d\theta_t)^{n-1} \\ &\quad = \int_{M^{2n-1}} \dot\theta_t \wedge (d\theta_t)^{n-1} + (n-1)\int_{M^{2n-1}} \theta_t \wedge (d\dot\theta_t) \wedge (d\theta_t)^{n-2} \\ &\quad = (n-1)\int_{M^{2n-1}} \left[-d(\theta_t \wedge \dot\theta_t \wedge (d\theta_t)^{n-2}) + d\theta_t \wedge \dot\theta_t \wedge (d\theta_t)^{n-}\right], \end{aligned}$$

which is 0 since M^{2n-1} is closed. Therefore

$$\int_{M^{2n-1}} \theta_0 \wedge (d\theta_0)^{n-1} = \int_{M^{2n-1}} \theta_1 \wedge (d\theta_1)^{n-1}.$$

q.e.d.

We recall that the canonical contact form $\theta \in \Omega^1(SM)$ is given by

$$\theta(v)(\xi) = \langle \pi_*\xi, v\rangle,$$

where $v \in SM$, $\xi \in T_v(SM)$, and $\pi: SM \to M$ is the bundle projection. For a discussion of the following facts, we refer the reader to [2, Chapter 1]:

1. θ is invariant under the geodesic flow;
2. $\mathrm{vol}_{SM} = \frac{1}{(n-1)!}\theta \wedge (d\theta)^{n-1}$ where vol_{SM} is the canonical volume form on SM;
3. $\int_{SM} \mathrm{vol}_{SM} = \mathrm{vol}(S^{n-1}) \int_M \mathrm{vol}_M = \mathrm{vol}(S^{n-1})\mathrm{vol}(M)$ where $\mathrm{vol}(S^{n-1})$ is the volume of the standard $(n-1)$-sphere.

As a consequence of the preceding lemma, we get Proposition 1.2 of the introduction.

Proof of Proposition 1.2. If $F: SM \to SN$ is a C^1 conjugacy of geodesic flows, and θ_0, θ_1 are the contact forms on SM and SN respectively, then θ_0 and $F^*\theta_1 \in \Omega^1_{C^0}(M)$ both satisfy the conditions of the previous lemma. Hence $\text{volume}(SM) = \text{volume}(SN)$ and so M and N have the same volume. q.e.d.

If we have a Riemannian product $M_1 \times \cdots \times M_k$, then we define $\alpha = (\alpha_1, \cdots, \alpha_k): S(M_1 \times \cdots \times M_k) \to R^k$ by $\alpha_i(v) = |\pi_{i*}(v)|$ where $\pi_i: M_1 \times \cdots \times M_k \to M_i$ is the projection onto the ith factor. If $M = (M_1 \times \cdots \times M_k)/\Gamma$ where $\Gamma \subseteq \text{Isom}(M_1) \times \cdots \times \text{Isom}(M_k) \subseteq \text{Isom}(M_1 \times \cdots \times M_k)$, then α descends to a map defined on SM, which we also denote by α.

Proposition 2.2. *Let M be a compact Riemannian manifold of the form $(M_1 \times \cdots \times M_k)/\Gamma$ where $\Gamma \subseteq \text{Isom}(M_1) \times \cdots \times \text{Isom}(M_k)$. If $F: SM \to SN$ is a C^1 conjugacy for some compact Riemannian manifold N, then*

$$\int_{\alpha^{-1}(\Omega)} \text{vol}_{SM} = \int_{F(\alpha^{-1}(\Omega))} \text{vol}_{SN}$$

for every domain $\Omega \subseteq (R^+)^k$, where vol_{SM} and vol_{SN} are the Liouville volume forms on SM and SN respectively.

Remark. The proof shows in fact that

$$\int_U \text{vol}_{SM} = \int_{F(U)} \text{vol}_{SN}$$

for U a connected component of $\alpha^{-1}(\Omega)$.

Proof. It suffices to consider compact domains $\Omega \subset (0, \infty) \times \cdots \times (0, \infty) \subset R^k$ with smooth boundary. Let θ_0 be the canonical contact form on SM and let θ_1 be the pullback of the contact form on SN by the conjugacy F. We want to show that

$$\int_{\alpha^{-1}(\Omega)} \theta_0 \wedge (d\theta_0)^{n-1} = \int_{\alpha^{-1}(\Omega)} \theta_1 \wedge (d\theta_1)^{n-1}.$$

Let $\theta_t = (1-t)\theta_0 + t\theta_1$, and $\dot{\theta}_t = d\theta_t/dt$. Differentiating we get

$$\begin{aligned}
&\frac{d}{dt}\int_{\alpha^{-1}(\Omega)} \theta_t \wedge (d\theta_t)^{n-1} \\
&= \int_{\alpha^{-1}(\Omega)} (\dot{\theta}_t \wedge (d\theta_t)^{n-1} + (n-1)\theta_t \wedge d\dot{\theta}_t \wedge (d\theta_t)^{n-2}) \\
&= \int_{\alpha^{-1}(\Omega)} (n(\dot{\theta}_t \wedge (d\theta_t)^{n-1}) - (n-1)d(\theta_t \wedge \dot{\theta}_t \wedge (d\theta_t)^{n-2})) \\
&= \int_{\alpha^{-1}(\Omega)} (n(\dot{\theta}_t \wedge (d\theta_t)^{n-1})) - (n-1)\int_{\partial(\alpha^{-1}(\Omega))} \theta_t \wedge \dot{\theta}_t \wedge (d\theta_t)^{n-2}.
\end{aligned}$$

We will show that both integrands are zero for small t. If X is the vector field generating the geodesic flow, then $i_X\dot{\theta}_t = 0$ and $i_X d\theta_t = 0$, so we have $i_X(\dot{\theta}_t \wedge (d\theta_t)^{n-1}) = 0$ and hence $\dot{\theta}_t \wedge (d\theta_t)^{n-1} = 0$. For small t, $d\theta_t$ is nondegenerate when restricted to $\operatorname{Ker}(\theta_t)$, so for small t there exists a unique continuous vector field Y_t satisfying $\theta_t(Y_t) = 0$ and $i_{Y_t} d\theta_t = \dot{\theta}_t$, since the map $Y_t \mapsto i_{Y_t} d\theta_t$ is a linear isomorphism from $\operatorname{Ker}(\theta_t)$ to the annihilator of X. The uniqueness of Y_t together with the fact the $\dot{\theta}_t$ and $d\theta_t$ are invariant under the geodesic flow of M implies that Y_t is invariant under the geodesic flow as well. The next lemma guarantees that Y_t is tangent to the level sets of α; in particular Y_t is tangent to $\partial(\alpha^{-1}(\Omega))$. This implies that the form $\theta_t \wedge \dot{\theta}_t \wedge (d\theta_t)^{n-2}$ restricts to zero on $\partial(\alpha^{-1}(\Omega))$ since for every $x \in \partial(\alpha^{-1}(\Omega))$ either $Y_t(x) = 0$, giving $\dot{\theta}_t(x) = (i_{Y_t} d\theta_t)(x) = 0$, or $Y_t(x) \neq 0$ and $i_{Y_t}(\theta_t \wedge \dot{\theta}_t \wedge (d\theta_t)^{n-2}) = 0$. We complete the proof of the proposition by noting that $\operatorname{Vol}_t = \int_{\alpha^{-1}(\Omega)} \theta_t \wedge (d\theta_t)^{n-1}$ is a polynomial in t since $\theta_t = (1-t)\theta_0 + t\theta_1$; therefore Vol_t must be constant since it is constant for small t. q.e.d.

Lemma 2.3. *Let M be a compact Riemannian manifold of the form $(M_1 \times \cdots \times M_k)/\Gamma$ where $\Gamma \subset \operatorname{Isom}(M_1) \times \cdots \times \operatorname{Isom}(M_k)$. Then any continuous vector field Y on SM which is invariant under the geodesic flow must be tangent to the fibers of $\alpha\colon SM \to R^k$.*

Proof. The geodesic flow shears the fibers of α, forcing invariant vector fields to be tangent to the fibers. To convey the idea of the proof in a simplified setting we give an analog for the lemma in the $k = 1$ case.

Model. Let N be a compact Riemannian manifold, and let Z be a continuous vector field on TN which is invariant under the geodesic flow of N. Then $\alpha_*(Z) = 0$ where $\alpha\colon TN \to R$ is defined as before, $\alpha(v) = |v|$.

Proof of model. Let X and ξ be the geodesic spray and the homothetic vector field on TN, respectively. Let $T_0N = \{v \in TN \,|\, |v| \neq 0\}$, and let C be the distribution on T_0N whose restriction to each sphere bundle $\alpha^{-1}(r)$ is the canonical contact distribution (i.e., the kernel of the contact form θ). If $\Phi^t\colon TN \to TN$ is the geodesic flow after time t, then $\Phi^t_*(\xi) = \xi + tX$, $\Phi^t_*(C) = C$, and we have a direct sum decomposition $TT_0N = RX \oplus R\xi \otimes C$. Now consider the projection of Z to the Φ^t invariant subbundle $D = RX \oplus R\xi$. This is invariant under the geodesic flow, and so it must lie in RX since otherwise the shear $\Phi^t_*(\xi) = \xi + tX$ will produce arbitrarily long vectors. Hence Z lies in $RX \oplus C$, and $\alpha_*(Z) = 0$.

Proof of the lemma. Let $T_0M = \{v \in TM \| |v| \neq 0\}$. We extend Y to a flow invariant vector field on T_0M using the fiber homothety (i.e., if $S_\lambda: TM \to TM$ denotes scalar multiplication by λ, then $Y_v = S_{|v|*}Y_{v/|v|}$), and denote the resulting vector field by the same name. For $i = 1, \cdots, k$ let $T_0M_i = \{v \in TM_i \| |v| \neq 0\}$, $\overline{X}_i$ be the geodesic spray on TM_i, and $\overline{\xi}_i$ be the homothetic vector field on TM_i, and let $\overline{C}_i$ be the distribution on T_0M_i whose restriction to $\alpha_i^{-1}(r)$ is the canonical contact distribution. If Φ_i^t is the geodesic flow on TM_i, then $\Phi_{i*}^t(\overline{\xi}_i) = \overline{\xi}_i + t\overline{X}_i$, $\overline{C}_i$ is invariant under $\overline{\Phi}_i^t$, and

$$TT_0M_i = R\overline{X}_i \oplus R\overline{\xi}_i \oplus \overline{C}_i.$$

We now let X_i, ξ_i, and C_i be the corresponding objects on $T_0M_1 \times \cdots \times T_0M_k \subseteq TM_1 \times \cdots \times TM_k \simeq T(M_1 \times \cdots \times M_k)$. If Φ^t is the geodesic flow of $T(M_1 \times \cdots \times M_k)$, and $D_i = RX_i \oplus R\xi_i$, then we have a Φ^t invariant decomposition

$$T(T_0M_1 \times \cdots \times T_0M_k) = D_1 \oplus C_1 \oplus \cdots \oplus D_k \oplus C_k,$$

and $\Phi_*^t(\xi_i) = \xi_i + tX_i$. Now observe that our flow invariant vector field Y lifts to a Φ^t invariant vector field on $T_0M_1 \times \cdots \times T_0M_k$, and so we may project this to each subbundle D_i to get Φ^t invariant sections Y_i. Now as in the model, Φ^t shears ξ_i into $\xi_i + tX_i$ and Y_i actually sits in RX_i; otherwise, by flowing Y_i under Φ^t we would get arbitrarily long vectors. But this implies that Y is a section of $RX_1 \oplus C_1 \oplus \cdots \oplus RX_k \oplus C_k$, so it is tangent to the fibers of α.

Proposition 2.4. *When $M = (X \times R)/\Gamma$ for $\Gamma \subseteq \mathrm{Isom}(X) \times \mathrm{Isom}^+(R)$ is compact, and $F: SM \to SN$ is a C^1 conjugacy of geodesic flows, then $F^*\mathrm{vol}_{SN} = \phi\mathrm{vol}_{SM}$ where $\phi(v) = 1$ for every vertical vector $v \in SM$ (i.e., every $v \in SM$ tangent to the local R factor).*

Proof. Since vol_{SM}, vol_{SN} are flow invariant and F is a C^1 conjugacy, $\phi: SM \to R$ is a continuous, flow invariant function. We will first show that $\phi|_{\alpha_1^{-1}(0)}$ is constant on each of the two component (U^+ the 'upward' component' and U^- the 'downward' component) of $\alpha_1^{-1}(0)$. Pulling ϕ back to $S(X \times R)$ we get a continuous flow invariant function $\hat{\phi}$, and it is enough to see that $\hat{\phi}|_{\alpha_1^{-1}(0)}$ is constant on each component ($\widehat{U}^+$ and $\widehat{U}^-$). Pick $x_1, x_2 \in X$, and find a sequence of geodesics $\gamma_n: R \to X \times R$ such that $\pi_1(\gamma_n(0)) = x_1$, $\pi_1(\gamma_n(t_n)) = x_2$ and $\alpha_1(\gamma_n'(0)) \to 0$, $\alpha_1(\gamma_n'(t_n)) \to 0$. Assume that $\gamma_n'(0)$ converges to a vector in $\widehat{U}^+$. Then the uniform continuity of $\widehat{\phi}$ implies that

$$\hat{\phi}|_{\widehat{U}^+}(x_1 \times R) = lim_{n\to\infty}\hat{\phi}(\gamma_n'(0)) = \lim_{n\to\infty} \hat{\phi}(\gamma_n'(t_n)) = \hat{\phi}|_{\widehat{U}^+}(x_2 \times R),$$

so that $\hat{\phi}|_{\widehat{U}^+} \equiv \phi_0^+$ for some $\phi_0^+ \in R$. Similarly for $\widehat{U}^-$.

The proposition now follows from Proposition 2.2 by integrating $\phi\mathrm{vol}_{SM}$ over each component of $\alpha_1^{-1}([0, \varepsilon))$ and letting $\varepsilon \to 0$. More precisely, let U_ε^+ be the 'upward' component of $\alpha_1^{-1}([0, \varepsilon))$ and apply Proposition 2.2 and the remark following it to get for every $\varepsilon > 0$,

$$\begin{aligned}\int_{U_\varepsilon^+} \mathrm{vol}_{SM} &= \int_{F(U_\varepsilon^+)} \mathrm{vol}_{SN} = \int_{U_\varepsilon^+} F^*(\mathrm{vol}_{SN}) \\ &= \int_{U_\varepsilon^+} \phi\mathrm{vol}_{SM} = \phi_0 \int_{U_\varepsilon^+} \mathrm{vol}_{SM} + \int_{U_\varepsilon^+} (\phi - \phi_0)\mathrm{vol}_{SM}.\end{aligned}$$

Dividing by $\int_{U_\varepsilon^+} \mathrm{vol}_{SM}$ and letting $\varepsilon \to 0$ we deduce that $\phi|_{U^+} \equiv \phi_0^+ = 1$.

3. Conjugacy rigidity for products

By a uniform conjugacy $F: SM \to SN$ between the unit sphere bundles of complete Riemannian manifolds M and N we mean a C^0 conjugacy where both F and F^{-1} are uniformly continuous.

Lemma 3.1. *Let $F: SM \to SN$ be a uniform conjugacy between the unit sphere bundles of complete Riemannian manifolds M and N. Then there is a $D > 0$ such that for all $v, w \in SM$:*

$$-2D \le d_M(\pi_M(v), \pi_M(w)) - d_N(\pi_N F(v), \pi_N F(w)) \le 2D.$$

Proof. The uniform continuity of the conjugacy implies that F (resp. F^{-1}) carries fibers of π_M (resp. π_N) to sets of uniformly bounded diameter D in SN (resp. SM). For $v, w \in SM$, let $\tau: [0, l] \to M$ be a minimizing geodesic from $\pi_M(v)$ to $\pi_M(w)$. Then both $d_N(\pi_N F(v),$ $\pi_N F(\tau'(l)))$ and $d_N(\pi_N F(w), \pi_N F(\tau'(l)))$ are bounded by D and

$$\begin{aligned}d_N(\pi_N F(\tau'(0)), \pi_N F(\tau'(l))) &= d_N(\pi_N F(\tau'(0)), \tau_N(g^l(F(\tau'(0))))) \\ &\le l = d_M(\pi_M(v), \pi_M(w)).\end{aligned}$$

Hence the first inequality in the lemma follows from the triangle inequality. The second follows from the same argument applied to F^{-1}.

Proposition 3.2. *For $i = 1, 2$ let $M_i = X_i \times R$ be complete Riemannian manifolds, let $\pi_2: M_i \to R$ be projection onto the second factor, and let dx be the standard 1-form on R. Let $U_i = \{v \in SM_i | \alpha_1(v) = 0, dx(\pi_{2*}(v)) > 0\}$; U_i is the set of "upward pointing" elements of SM_i.*

Suppose $F: SM_1 \to SM_2$ is a uniform conjugacy of geodesic flows satisfying $F(U_1) = U_2$, and define $G: M_1 \to M_2$ by $G(\pi(u)) = \pi(F(u))$ for every $u \in U_1$. Then G is an isometry.

Proof. Our first step is to show that F preserves α. For $x_i \in M_i$ let $S_i(x_i) \in U_i$ be the "vertical" vector at x_i. Pick $x_i \in M_i$ such that $F(S_1(x_1)) = S_2(x_2)$. Let $h_i: M_i \to R$ be the Busemann functions coming from the vertical geodesics γ_i (i.e., $\gamma_i'(0) = S_i(x_i)$ and $h_i(y) = \lim_{t\to\infty} d(\gamma_i(t), y) - t$). Let $H_i: SM_i \to R$ be the induced map (i.e., $H_i = h_i \circ \pi_{M_i}$). Lemma 3.1 allows us to conclude that $H_2 \circ F - H_1$ is bounded. This forces F to preserve α_2 since for every $v \in SM_i$, $\alpha_2(v)$ is determined by the growth rate of h_i along the geodesic $t \mapsto \exp(tv)$. Thus F preserves α.

Let $\overline{G}: X_1 \to X_2$ be the map induced by $G: M_1 \to M_2$, i.e., $\overline{G}(x) = \pi_1(G(x, t))$ for all $(x, t) \in X_1 \times R = M_1$. We first show that $\overline{G}$ has Lipschitz constant ≤ 1. Pick $x, y \in X_1$ and a minimizing geodesic segment $\sigma: [0, l] \to X_1$ from x to y. Define $\eta_\lambda: [0, \sqrt{l^2+\lambda^2}] \to X_1 \times R = M_1$ by

$$\eta_\lambda(t) = \left(\sigma\left(\frac{tl}{\sqrt{l^2+\lambda^2}}\right), \frac{t\lambda}{\sqrt{l^2+\lambda^2}}\right);$$

so $\text{length}(\sigma) = \alpha_1(\eta_\lambda'(t))\text{length}(\eta_\lambda)$. Let $\overline{\eta}_\lambda: [0, \sqrt{l^2+\lambda^2}] \to X_2 \times R$ be the geodesic in M_2 corresponding to η_λ: $\overline{\eta_\lambda}(t) = \pi_{m_2} \circ F \circ \eta_\lambda'(t)$. Let $\overline{\sigma}_\lambda = \pi_1 \circ \overline{\eta}_\lambda$. Then

$$\begin{aligned}\text{length}(\overline{\sigma}_\lambda) &= \alpha_1(\overline{\eta}_\lambda'(t))\text{length}(\overline{\eta}_\lambda)\\ &= \alpha_1(\eta_\lambda'(t))\text{length}(\eta_\lambda) = \text{length}(\sigma).\end{aligned}$$

As λ goes to ∞, $d(\overline{\eta}_\lambda'(0),\ F(S_1(x, 0)))$ and $d(\overline{\eta}_\lambda'(\sqrt{l^2+\lambda^2}),$ $F(S_1(y, \lambda)))$ go to 0. Thus $\overline{\sigma}_\lambda(0) \to \overline{G}(x)$ and $\overline{\sigma}_\lambda(\sqrt{l^2+\lambda^2}) \to \overline{G}(y)$, from which we see that $d(\overline{G}(x), \overline{G}(y)) \leq d(x, y)$ for all $x, y \in X_1$. Applying the same reasoning to $\overline{G}^{-1}$ we conclude that $\overline{G}: X_1 \to X_2$ is an isometry.

To complete the proof that G is an isometry we will show that for every $t \in R$, the set $G(X_1 \times \{t\})$ lies at a constant height in $X_2 \times R$. Choose a geodesic $\gamma: R \to M_1$ of the form $\gamma_1(t) = (x_1, t)$ for some $x_1 \in X_1$, and let $\gamma_2: R \to M_2$ be the corresponding geodesic in M_2. We may assume that $\gamma_2(0) = (x_2, 0)$ for some $x_2 \in X_2$ since otherwise we may arrange this by composing F with the differential of a translation. Pick $m_1 \in X_1 \times \{0\}$, and let $\eta_\lambda: [0, l_\lambda] \to M_1$ be a minimizing geodesic segment from m_1 to $\gamma_1(\lambda)$. If $\overline{\eta}_\lambda$ is the corresponding family of segments in M_2, then $\overline{\eta}_\lambda(0) \to G(m_1)$ and $d(\overline{\eta}_\lambda(l_\lambda), \gamma_2(\lambda)) \to 0$ as $\lambda \to \infty$ by the uniform continuity of F. Hence

$$\limsup_{\lambda\to\infty}(d(G(m_1), \gamma_2(\gamma)) - \operatorname{length}(\eta_\lambda)) = \limsup_{\lambda\to\infty}(d(G(m_1), \gamma_2(\lambda)) - \lambda) \le 0.$$

This forces $\pi_2(G(m_1)) \in [0, \infty)$. Letting $\lambda \to -\infty$ instead of $+\infty$ we get $\pi_2(G(m_1)) \in (-\infty, 0]$. Hence $G(X_1 \times \{0\}) \subseteq X_2 \times \{0\}$. q.e.d.

Corollary 3.3. *Let M_i be compact manifolds with unit parallel fields S_i. If there is a C^0 conjugacy $F: SM_1 \to SM_2$ such that $F(S_1(M_1)) = S_2(M_2)$, then M_1 is isometric to M_2.*

Proof. If $n \ge 3$, then we can lift F to a conjugacy $\widetilde{F}: S\widetilde{M}_1 \to S\widetilde{M}_2$, and apply the preceding proposition to get a Γ equivariant isometry $\widetilde{G}: \widetilde{M}_1 \to \widetilde{M}_2$. This descends to the desired isometry $G: M_1 \to M_2$.

The case $n = 2$ (which concerns only flat tori and Klein bottles) was done in [4]. q.e.d.

Similar arguments give results for more general product manifolds.

Proposition 3.4. *Let $M = M_1 \times M_2$ and $N = N_1 \times N_2$ be Riemannian products such that* $\dim(M_i) \neq 0$. *If there is a uniform conjugacy $F: SM \to SN$ which preserves α, then M is isometric to N.*

Proof. The proof is similar to the proof of Proposition 3.2. We will show that M_1 is isometric to N_1; the proof that M_2 is isometric to N_2 is the same. Let τ be any geodesic in M_2, then there is a unique unit vector field W along $M_1 \times \tau(0)$ such that $\pi_{1*}(W) = 0$ and $\pi_{2*}(W) = \tau'(0)$. We define a map $G: M_1 \to N_1$ by $G(x) = \pi_1(F(W(x, \tau(0))))$ which we will show is an isometry. Let $x, y \in M_1$ and let $\sigma: [0, l] \to M_1$ be a minimizing geodesic segment from x to y. Define $\eta_\lambda: [0, \sqrt{l^2+\lambda^2}] \to M = M_1 \times M_2$ by

$$\eta_\lambda(t) = \left(\sigma\left(\frac{tl}{\sqrt{l^2+\lambda^2}}\right), \tau\left(\frac{t\lambda}{\sqrt{l^2+\lambda^2}}\right)\right).$$

Let $\overline{\eta}_\lambda: [0, \sqrt{l^2+\lambda^2}] \to N$ be the geodesic corresponding to η_λ: $\overline{\eta}_\lambda(t) = \pi_N \circ F \circ \eta'_\lambda(t)$. Let $\overline{\sigma}_\lambda = \pi_1 \circ \overline{\eta}_\lambda$. Then as in Proposition 3.2 we see that $\operatorname{length}(\overline{\sigma}_\lambda) = \operatorname{length}(\sigma)$ and further that as $\lambda \to \infty$, $\overline{\sigma}_\lambda$ converges to a geodesic segment from $G(x)$ to $G(y)$, so G is distance nonincreasing. Reversing the roles of M and N we see that G is an isometry and the proposition follows. q.e.d.

Although it is easy to find conjugacies of products where α is not preserved (for example, take F to be the identity on $X_1 \times X_2 \times X_3$ where

$M_1 = X_1$ while $N_1 = X_1 \times X_2$) in most cases one expects α to be preserved. For example, if the M_i's are manifolds all of whose geodesics are closed of the same period, then any conjugacy to a nontrivial product manifold must preserve α. Another important special case where Proposition 3.4 can be applied is the case where N (and sometimes M) is assumed to have nonpositive curvature. We will study these cases in a future paper with Patrick Eberlein.

4. Jacobi tensors and conjugacy

We begin by studying the image of minimizing geodesics under C^0 conjugacies. A line in a Riemannian manifold is a geodesic which is the minimizing geodesic between any pair of points on it. A geodesic γ is said to be recurrent if there exists an increasing sequence $t_i \to \infty$ such that $\gamma'(t_i) \to \gamma'(0)$. For a geodesic γ let $\tilde{\gamma}$ be a lift of γ to the universal cover $\widetilde{M}$ of M. Then we define $\operatorname{excess}(\gamma) = \operatorname{excess}(\tilde{\gamma}) = \lim_{t\to\infty} t - d_{\widetilde{M}}(\tilde{\gamma}(0), \tilde{\gamma}(t))$, which is possibly infinite. In fact, we have

Lemma 4.1. *Each recurrent geodesic γ on a complete Riemannian manifold M either has* $\operatorname{excess}(\gamma) = \infty$ *or lifts to a line in the universal cover $\widetilde{M}$.*

Proof. We will show that $\operatorname{excess}(\gamma)$ is 0 or ∞. Thus if $\operatorname{excess}(\gamma) < \infty$ then $\operatorname{excess}(\gamma) = 0$. This clearly shows that $\tilde{\gamma}$ is a ray (i.e., $\tilde{\gamma}$ minimizes from $\tilde{\gamma}(0)$ to $\tilde{\gamma}(t)$ for all $t > 0$). But for all $s > 0$, $\gamma_s(t) = \gamma(t-s)$ is also recurrent and hence has $\operatorname{excess}(\gamma_s) = 0$. Thus $\tilde{\gamma}$ is a line.

Assume $\operatorname{excess}(\gamma) > 0$. Then there is $E > 0$ and T such that for all $t > T$, $d_{\widetilde{M}}(\tilde{\gamma}(0), \tilde{\gamma}(t)) \le t - E$. Let $\varepsilon = E/4$. By the recurrence property of γ we can choose $t_i > T$ such that $d(\gamma(0), \gamma(t_0)) < \varepsilon$, $t_i + t_0 < t_{i+1}$, and some lift $\tilde{\gamma}_i$ of γ to $\widetilde{M}$ has $d_{\widetilde{M}}(\tilde{\gamma}_i(t_i), \tilde{\gamma}(0)) < \varepsilon$ and $d_{\widetilde{M}}(\tilde{\gamma}_i(t_i + t_0), \tilde{\gamma}(t_0)) < \varepsilon$. Let $\tilde{\tau}$ be a minimizing geodesic segment in $\widetilde{M}$ from $\tilde{\gamma}(0)$ to $\tilde{\gamma}(t_0)$, and τ its projection to M. We know that the length L of τ satisfies $L \le t_0 - E$. By the above we can choose lifts $\tilde{\tau}_i$ of τ such that $d_{\widetilde{M}}(\tilde{\gamma}(t_i), \tilde{\tau}_i(0)) < \varepsilon$ and $d_{\widetilde{M}}(\tilde{\gamma}(t_i + t_0), \tilde{\tau}_i(L)) < \varepsilon$. Hence

$$d_{\widetilde{M}}(\tilde{\gamma}(t_i), \tilde{\gamma}(t_i + t_0)) < L + 2\varepsilon < t_0 - E/2.$$

Thus $\operatorname{excess}(\gamma) = \infty$. q.e.d.

For compact (or finite volume) M the recurrent geodesics are dense in SM (see [15], Theorem 2.3]). Hence Lemma 4.1 has an interesting consequence:

Corollary 4.2. *If M is a compact manifold without conjugate points, and N is a Riemannian manifold whose geodesic flow is C^0 conjugate to that of M, then N also has no conjugate points.*

Proof. We first lift the conjugacy F to a uniform conjugacy between the universal covers $F: S\widetilde{M} \to S\widetilde{N}$. For $n \geq 3$ it is clear that such a lift exists. For surfaces of genus greater than one this can also be done (see [4]). For surfaces of genus one M is flat by Hopf's theorem [12], and it was shown in [4] that N must be flat. Hence we can assume we have such a lift.

Now Lemma 3.1 tells us that for any geodesic γ, $\operatorname{excess} F(\gamma) = \operatorname{excess} F(\tilde{\gamma})$ is bounded. Now if γ is recurrent, then $F(\gamma)$ is also recurrent, and hence $F(\tilde{\gamma})$ is a line. Thus N has no conjugate points since the recurrent geodesics are dense in SM. q.e.d.

Let $F: SM \to SN$ be a C^1 conjugacy. The space of Jacobi fields Ψ along a geodesic γ splits as $\Psi = \Psi^\perp + \Psi^t + \Psi^b$, where $\Psi^\perp$ consists of those Jacobi fields that are perpendicular to γ, Ψ^t is spanned by γ', and Ψ^b is spanned by $t\gamma'$. The Jacobi fields in $\Psi^\perp + \Psi^t$ are the ones that come from variations of geodesics that are parameterized by arc length.

For a Jacobi field j in $\Psi^\perp + \Psi^t$ we define a vector field Tj along $T\gamma$ in SM as the variation field of the variation $T\gamma_s$, where γ_s is a variation of parametrized geodesics whose variation field is j. Tj is determined by the fact that $\pi_{M^*}(Tj) = j$ and that the vertical (with respect to the usual connection) component $v(Tj)$ of Tj is equal to j', the covariation derivative of j with respect to γ' when $v(Tj)$ and j' are thought of as tangent vectors perpendicular to γ'. As is easy to see, the Tj's are precisely those vector fields along $T\gamma$ that are invariant under Dg^t. Since F takes parametrized geodesics to parametrized geodesics and is differentiable, it induces a map Φ from $\Psi^\perp + \Psi^t$ to the corresponding Jacobi fields along the image geodesic. More precisely we see $F_*(Tj) = T\Phi(j)$, and hence $\pi_{N^*}(F_*(Tj)) = \Phi(j)$. In particular Φ is a linear isomorphism. For $j \in \Psi^\perp + \Psi^t$ we can write $\Phi(j) = \Phi(j)^\perp + c(j)F(T\gamma)$ where $c(j)$ is a constant. It is easy to see that $\Phi(j)^\perp$ is zero if and only if $j \in \Psi^t$.

Similar arguments apply to Jacobi tensors. These are tensors fields J of type $(1, 1)$ along a geodesic γ, which when applied to parallel vector fields P yield a Jacobi field $J(P)$. J is said to be perpendicular if $J(P)$ is perpendicular to γ whenever P is perpendicular to γ. We will let $J^\perp$ be the corresponding perpendicular Jacobi tensor, i.e., for P perpendicular to γ, $J^\perp(P)$ is the perpendicular part of $J(P)$. If we fix a parallel orthonormal basis for $\gamma'^\perp$ we can think of perpendicular Jacobi tensors as matrices whose column vectors represent Jacobi fields with respect to this basis; they are the solutions to $J''(t)+R(t)J(t)=0$ where $R(t)$ is the curvature transformation (see [8] or [2, p. 239]). Along a geodesic $\gamma_v(t)$ there are two perpendicular Jacobi tensors of particular interest to us. They are called $I_v(t)$ and $J_v(t)$ and are determined by $I_v(0)=\mathrm{Id}$, $I_v'(0)=0$, $J_v(0)=0$, $J_v'(0)=\mathrm{Id}$. Let $A_v(t)=\Phi(I_v)(t)$ and $B_v(t)=\Phi(J_v)(t)$ be the corresponding (not necessarily perpendicular) Jacobi tensors along $\overline{\gamma}_v$. Strictly speaking in order to define the tensors $A_v(t)$ and $B_v(t)$ we need to choose an isomorphism from $\gamma_v'^{\perp}(0)$ to $\overline{\gamma}_v'^{\perp}(0)$. It is of course sufficient to choose parallel orthonormal bases along γ_v and $\overline{\gamma_v}$. All equations will be written with respect to such fixed choices. Of course determinants and norms are well defined independent of such choices and vary continuously.

A nonsingular perpendicular Jacobi tensor J is Lagrangian if $J(t)'J(t)^{-1}$ is symmetric. If $A_v(t)$ is a nonsingular Lagrangian Jacobi tensor, we define a new Lagrangian Jacobi tensor $Z_v(t)$ by

$$Z_v(t)=A_v(t)\int_0^t A_v^{-1}(x)A_v^{-1*}(x)\,dx. \tag{4.1}$$

Note that $Z_v(0)=0$ and $Z_v'(0)=A_v^{-1*}(0)$, which is nonsingular. In which case, for dimension reasons, we can write our perpendicular Jacobi tensor $B_v^{\perp}$ uniquely as

$$B_v^{\perp}(t)=Z_v(t)C_v+A_v(t)D_v, \tag{4.2}$$

where C_v and D_v are constant matrices.

Lemma 4.3. *Let M be a compact Riemannian manifold, and $F: SM\to SN$ a C^1 conjugacy. Let $v\in SM$ be such that $A_v(t)=A_v^{\perp}(t)$ is nonsingular Lagrangian. Then* $\det(C_v)=\phi(v)$ *where C_v is defined by equation* (4.2), *and $\phi(v)$ is defined by* $F^*\mathrm{vol}_{SN}=\phi\mathrm{vol}_{SM}$.

Proof. Fix $v\in SM$. Then DF_v is a linear isomorphism from T_vSM to $T_{F(v)}SN$ which takes $X_M(v)$ to $X_N(v)$, X_M and X_N are the vector fields generating the respective geodesic flows. Let $DF_v^{\perp}$ be the linear isomorphism from $X_M^{\perp}(v)$ to $X_N^{\perp}(F(v))$ gotten by restricting DF to $X_M^{\perp}$ followed by projection to $X_N^{\perp}$. Then $\phi(v)=\det(DF(v))=\det(DF^{\perp}(v))$. Since the Liouville measure is the Riemannian measure with respect to the canonical Riemannian metric on SM and, we have fixed parallel orthonormal frames along γ_v (resp. $\overline{\gamma}_v$), there is a canonical choice of orthonormal bases for $X_M^{\perp}(v)$ (resp. $X_N^{\perp}(F(v))$) which consists of $TJ_i(0)$ where J_i is a Jacobi field with $J_i'(0)=0$, and $J_i(0)$ a basis vector or of the form $J_i(0)=0$, and $J_i'(0)$ is a basis vector. With respect to these bases it is easy to see that

$$DF_v^{\perp} = \begin{pmatrix} A_v(0) & B_v^{\perp}(0) \\ A_v'(0) & B_v^{\perp\prime}(0) \end{pmatrix} = \begin{pmatrix} A_v(0) & Z_v(0)C_v + A_v(0)D_v \\ A_v'(0) & Z_v'(0)C_v + A_v'(0)D_v \end{pmatrix}.$$

Since $Z_v(0) = 0$ and $Z_v'(0) = A_v^{-1*}(0)$, we see that

$$\phi(v) = \det(DF_v^{\perp}) = \det \begin{pmatrix} A_v(0) & 0 \\ A_v'(0) & A_v^{-1*}(0) \end{pmatrix} \begin{pmatrix} I & D_v \\ 0 & C_v \end{pmatrix} = \det(C_v).$$

5. Rigidity for manifolds with a parallel field

In this section we will complete the proof of Theorem 1.1.

Let S be a parallel field of unit length on a compact Riemannian manifold M. For a vertical vector v (i.e., $v = S(x)$) let γ_v be the corresponding geodesic (i.e., $\gamma_v'(o) = v$), and $\overline{\gamma}_v$ the image geodesic in N. The flow transformations of S are isometries, while the integral curves of S are the geodesics γ_v tangent to the local Euclidean de Rham factor. Since S induces a measure preserving flow in M, by the Poincaré recurrence theorem [15, Theorem 2.3] there is a dense set of γ_v which are recurrent and therefore Lemmas 4.1 and 3.1 (as in the proof of Corollary 4.2) tell us that for all v, lifts of γ_v and $\overline{\gamma}_v$ are lines in the universal covers. Hence by Theorem 1 of [7] no bounded Jacobi fields ever vanish along these geodesics. Because of the local product structure, if P is a parallel unit vector field along γ_v, then $i_v(t) = P(t)$, $j_v(t) = t \cdot P(t)$ and $q_v^s(t) = si_v(t) + j_v(t)$ are Jacobi fields along γ_v where $q_v^s(t)$ is just $j_{g^{-s}v}(t+s)$ (note that the parameter shift has $\gamma_{g^{-s}v}(t+s) = \gamma_v(t)$). Since Ti_v is a bounded vector field along $T\gamma$, we have $T\Phi(i_v) = F_*(Tiv)$ is bounded, and therefore $\Phi(i_v)$ is bounded and, hence by the above, never vanishes. Also $\Phi(i_{g^{-s}v})(s+t) = \Phi(i_v)(t)$ since both sides are the image of P under F. We also have

$$\begin{aligned} &s\Phi(i_v)^{\perp}(t) + \Phi(j_v)^{\perp}(t) + (sc(i_v) + c(j_v))\overline{\gamma}_v'(t) \\ &\quad = s\Phi(i_v)(t) + \Phi(j_v)(t) = \Phi(q_v^s)(t) = \Phi(j_{g^{-s}v})(t+s) \\ &\quad = \Phi(j_{g^{-s}v})^{\perp}(t+s) + c(j_{g^{-s}v})\overline{\gamma}_v'(t). \end{aligned}$$

As w varies over SM and P varies over parallel unit vector fields along γ_w, $c(j_w)$ varies continuously and hence is bounded. Therefore $c(i_v) = 0$, $\Phi(i_v) = \Phi(i_v)^{\perp}$, and $\Phi(j_{g^{-s}v})^{\perp}(t+s) = \Phi(j_v)^{\perp}(t) + s\Phi(i_v)^{\perp}(t)$. In particular F maps parallel perpendicular Jacobi fields P along γ_v into perpendicular Jacobi fields along $\overline{\gamma}_v$ that never vanish.

If $v = S(x)$ for a parallel vector field S, then using the fact that the metric is a product metric along γ_v, it is easy to see that $I_v(t) = \mathrm{Id}$ and $J_v(t) = t \cdot \mathrm{Id}$ with respect to a parallel orthonormal frame. The above arguments show that

$$(5.1) \qquad A_{g^{-s}v}(t+s) = A_v(t), \qquad B^{\perp}_{g^{-s}v}(t+s) = sA^{\perp}_v(t) + B^{\perp}_v(t),$$

and $A_v(t)$ is a nonsingular bounded perpendicular Jacobi tensor.

Lemma 5.1. *Let M be a compact Riemannian manifold with a unit parallel field S and let $F: SM \to SN$ be a C^1 conjugacy. Then the map $G = \pi_N \circ F \circ S$ is a C^1 diffeomorphism from M to N. Further if $V \in T_xM$ is perpendicular to $S(x)$, then $DG(V)$ is perpendicular to $DG(S(x)) = F(S(x))$. Hence we can define a vector field $F(S)$ on N by $F(S)(G(x)) = F(S(x))$.*

Proof. The action of DG on $S(x)^{\perp}$ is encoded in the tensor $A_{S(x)}(t)$ (i.e., $A_{S(x)}(t)$ represents DG with respect to our fixed choice of bases). Thus DG takes $S(x)^{\perp}$ to $DG(S(x))^{\perp}$ and is nonsingular, so $G: M \to N$ is a covering map. If $n \geq 3$, then π_M and π_N induce isomorphisms of fundamental groups (since the fibers are simply connected), so G is a composition of maps which are isomorphisms on π_1, and therefore G is a diffeomorphism in this case. The fact that $DG(S(x)) = F(S(x))$ follows from the fact that G takes $\gamma_{S(x)}$ to $F(\gamma_{S(x)})$.

If $n = 2$, then M is a flat torus (or Klein bottle), and results in [4] show that N must then be isometric to M. Although there are many self-conjugacies of the geodesic flow of a flat 2-torus, they all satisfy the lemma. q.e.d.

We note that if $v = S(x)$ for some $x \in M$, then $A_v(t)$ is a nonsingular Lagrangian tensor (i.e., $A'_v(t)A_v^{-1}(t)$ is symmetric). This follows since $A'_v(t)A_v^{-1}(t)$ is the second fundamental form of the image under G of a local horizontal slice of M (recall M is locally a product). (Although in general the image under G of a smooth submanifold needs only be C^1, in this case the normal is the image of $S(x)$ and hence is differentiable.)

Lemma 5.2. *If M is a compact manifold with a parallel unit field S, and $F: SM \to SN$ is a C^1 conjugacy, then for all vertical v (i.e., $v = S(m)$ for some $m \in M$) we have*

$$\lim_{s\to\infty} \det^{-1/2}\left[\frac{1}{s}\int_{-s}^{0} A_v^{-1}(x)A_v^{-1*}(x)\,dx\right] = 1.$$

Furthermore the convergence is uniform in v.

Proof. For fixed t, (4.2) and (5.1) yield

(5.2) $$Z_{g^{-s}v}(t+s)C_{g^{-s}v}+A_{g^{-s}v}(t+s)D_{g^{-s}v}=Z_v(t)C_v+A_v(t)D_v+sA_v(t).$$

Now $A_{g^{-s}v}(t+s)=A_v(t)$ (Equation (5.1)) and

$$\begin{aligned}Z_{g^{-s}v}(t+s)&=A_v(t)\left\{\int_{-s}^{t}A_v^{-1}(x)A_v^{-1*}(x)\,dx\right\}\\&=A_v(t)\left\{\int_{-s}^{0}A_v^{-1}(x)A_v^{-1*}(x)\,dx\right\}\\&\quad+A_v(t)\left\{\int_{0}^{t}A_v^{-1}(x)A_v^{-1*}(x)\,dx\right\}\\&=A_v(t)\left\{\int_{-s}^{0}A_v^{-1}(x)A_v^{-1*}(x)\,dx\right\}+Z_v(t).\end{aligned}$$

Thus (5.2) yields

$$\begin{aligned}A_v(t)\left\{\int_{-s}^{0}A_v^{-1}(x)A_v^{-1*}(x)\,dx\right\}C_{g^{-s}v}+Z_v(t)C_{g^{-s}v}+A_v(t)D_{g^{-s}v}\\=Z_v(t)C_v+A_v(t)D_v+sA_v(t).\end{aligned}$$

Since the norms of C_v and D_v vary continuously with vertical v, we see that the norms are uniformly bounded. Thus on each side of the equation there is only one term which is not uniformly bounded in s. Thus dividing by s and letting s go to ∞ we see that uniformly in v,

$$\lim_{s\to\infty}\frac{1}{s}A_v(t)\left\{\int_{-s}^{0}A_v^{-1}(x)A_v^{-1*}(x)\,dx\right\}C_{g^{-s}v}=A_v(t).$$

Thus the nonsingularity of $A_v(t)$ and the fact that $\det(C_w)=1$ for all vertical w (by Lemma 4.3 and Proposition 2.4) allow us to conclude that

$$\lim_{s\to\infty}\det{}^{-1/2}\left[\frac{1}{s}\int_{-s}^{0}A_v^{-1}(x)A_v^{-1*}(x)\,dx\right]=1.\quad \text{q.e.d.}$$

In the next proof we will use the strict convexity of $\Xi=\det^{-1/2}$ on the space, $\mathcal{SA}$, of positive definite selfadjoint endomorphisms (see [1, 11.8.9.5]), which is an open, convex subset of the linear space of self-adjoint endomorphisms. For fixed S_0 we define the linear part, L_{S_0}, of Ξ by $l_{S_0}(S)=D\Xi_{S_0}(S-S_0)+\Xi(S_0)$ and the remainder, R_{S_0}, of Ξ by $\Xi(S)=L_{S_0}(S)+R_{S_0}(S)$. The strict convexity of Ξ implies that R_{S_0} is nonnegative and strictly convex.

Let S be a measurable map from a space P, with probability measure dp, to $\mathcal{SA}$. Integrating $\Xi = L_{S_0} + R_{S_0}$; with $S_0 = \int_P S(p)\,dp$ we see

$$\int_P \Xi(S(p))\,dp = \Xi\left(\int_P S(p)\,dp\right) + \int_P R_{\int_P S(p)\,dp}(S(p))\,dp. \tag{5.3}$$

Proposition 5.3. *If M is a compact Riemannian manifold with a unit parallel field S, and $F\colon SM \to SM$ is a C^1 conjugacy, then $F(S)$ is a parallel vector field on N.*

Proof. We will show that along any vertical geodesic $\overline{\gamma}_v$, that $A_v(t)$ is constant. This will imply that the Jacobi fields coming from variations of integral curves of $F(S)$ are parallel. Now if V is a (locally defined) vector field invariant under the flow of $F(S)$, then $V|_{\overline{\gamma}_0}$ is such a Jacobi field along $\overline{\gamma}_v$ so

$$\nabla_V F(S) = \nabla_{F(S)} V - [F(S), V] = \nabla_{F(S)} V = 0.$$

Hence $F(S)$ is parallel.

On M the vertical flow $S_t(m)$ is measure preserving, hence, using (5.3) yields

$$\begin{aligned}
\mathrm{Vol}(M) = \mathrm{Vol}(N) &= \int_M \det(DG(m))\,dm \\
&= \frac{1}{s}\int_{-s}^{0}\int_M \det(DG(S_t(m)))\,dm\,dt \\
&= \int_M \frac{1}{s}\int_{-s}^{0} \det(A_{S(m)}(t))\,dt\,dm \\
&= \int_M \det{}^{-1/2}\left[\frac{1}{s}\int_{-s}^{0} A_{S(m)}^{-1}(t)A_{S(m)}^{-1*}(t)\,dt\right] dm \\
&\quad + \int_M \frac{1}{s}\int_{-s}^{0} R_{Q(v,s)}[A_{S(m)}^{-1}(t)A_{S(m)}^{-1*}(t)]\,dt\,dm,
\end{aligned}$$

where $Q(v, s) = (1/s)\int_{-s}^{0} A_{S(m)}^{-1}(t)A_{S(m)}^{-1*}(t)\,dt$. But letting s go to infinity Lemma 5.2 tells us that

$$\lim_{s\to\infty}\int_M \frac{1}{s}\int_{-s}^{0} R_{Q(v,s)}[A_{S(m)}^{-1}(t)A_{S(m)}^{-1*}(t)]\,dt\,dm = 0.$$

Now let $H(v, a, Q) = \int_a^{a+1} R_Q[A_v^{-1}(t)A_v^{-1*}(t)]\,dt$. By change of variable, $H(v, a, Q) = H(g^a v, 0, Q)$. Hence if we let

$$H(v, a) = \inf\{H(v, a, Q) | Q \in \mathcal{SA}\} \geq 0,$$

then $H(v, a) = H(g^a v, 0)$. We note that $H(v, a) = 0$ if and only if $A_v^{-1}(t)A_v^{-1*}(t)$ is constant for $t \in [a, a+1]$. We now have for N_s the greatest integer less than $s \geq 0$:

$$\begin{aligned}\int_M \frac{1}{s}\int_{-s}^{0} & R_{Q(v,s)}[A_{S(m)}^{-1}(t)A_{S(m)}^{-1*}(t)]\,dt\,dm \\ &\geq \int_M \frac{1}{s}\sum_{a=-N_s}^{-1} H(S(m), a)\,dm = \frac{1}{s}\sum_{a=-N_s}^{-1}\int_M H(g^a S(m), 0)\,dm \\ &= \frac{1}{s}\sum_{a=-N_s}^{-1}\int_M H(S(m), 0)\,dm = \frac{N_s}{s}\int_M H(S(m), 0)\,dm,\end{aligned}$$

where we have used the facts that dm is invariant under the flow S_t induced by S, and that $g^a S(m) = S(S_a m)$. Thus letting s go to ∞ we see that $\int_M H(S(m))\,dm = 0$, and hence $H(S(m), 0) = 0$ for almost all m, and $A_v^{-1}(t)A_v^{-1*}(t)$ is independent of t. (Use continuity to get all from almost all.)

Now differentiating the fact that $A_v^*(t)A_v(t)$ is a constant, we obtain that $A_v'(t) = -A_v^{-1*}(t)A_v'^*(t)A_v(t)$, while the fact that $A_v(t)$ is Lagrangian implies that $A_v'(t) = A_v^{-1*}(t)A_v'^*(t)A_v(t)$. Thus $A_v'(t) = 0$ and hence $A_v(t)$ is independent of t. q.e.d.

Proof of Theorem 1.1. The theorem follows immediately from Proposition 5.3 and Corollary 3.3.

6. Examples

In this section we first give a criterion for two surfaces of revolution (with boundary) to have conjugate geodesic flows, and then we apply this criterion to show that any smooth manifold M^n admits highly nonrigid metrics. We construct such examples by gluing in different surfaces of revolution which are isometric near their boundaries. Furthermore, these conjugacies can be arranged to preserve the contact forms (see §2 for definition). Let $M^2(f, a)$ be the surface of revolution obtained by revolving about the x-axis the graph of the smooth positive function $f(x)$ defined on an interval $[0, a]$. More generally let $M^n(f, a)$ be the corresponding n-dimensional manifold of revolution.

Let Ω be the collection of pairs (f, a) of smooth positive functions f defined on some interval $[0, a]$ such that $f(a) = f(0)$, f has a maximum at some $c \in (0, a)$, $f'(x) > 0$ for $x \in [0, c)$, and $f'(x) < 0$ for

$x \in (c, a]$. For $y \in (f(0), f(c))$ there are two points $x_-^y \in (0, c)$ and $x_+^y \in (c, a)$ in $f^{-1}(y)$. Thus $f \in \Omega$ gives rise to a function

$$DL_f(y) = \frac{ds}{dy}(x_-^y) - \frac{ds}{dy}(x_+^y) \quad \text{for } y \in (f(0), f(c)),$$

where $s(x)$ is the arc length along the graph of $y = f(x)$. Notice that ds/dy is positive at x_-^y, negative at x_+^y, and larger than 1 in absolute value at both points. We will say $(g, b) \in \Omega$ is compatible with $(f, a) \in \Omega$ if g agrees with f in a small neighborhood of 0 and such that $DL_f = DL_g$; hence in particular f and g have the same maximum and minimum values, and for small values of $t \geq 0$ we have $f(t) = g(t)$ and $f(a-t) = g(b-t)$. In this case, we see that there are canonical isometries I_0 and I_a mapping a neighborhood of each of the two boundary components of $M^n(f, a)$ to the corresponding neighborhood of $M^n(g, b)$. We note that for a given (f, a) there are lots of compatible (g, b). In fact we can choose for the increasing part of g any smooth function increasing from $f(0)$ to $f(c)$ as long as $ds/dy < DL_f(y) - 1$ (where ds/dy refers to the graph of g) and then define the decreasing part of g by $DL_g = DL_f$ as long as we get smoothness at the maximum.

Lemma 6.1. *Let (f, a) and (g, b) be two compatible elements of Ω. Then there is a smooth contact diffeomorphism F between the unit sphere bundles $SM^n(f, a)$ and $SM^n(g, b)$, which is equal to DI_0 or DI_a near the two boundary components, and $F \circ g^t = g^t \circ F$ wherever they are defined.*

Let M^n be a compact Riemannian manifold without boundary such that some $M^n(f, a)$ for $(f, a) \in \Omega$ is a subdomain of M. Then for any compactible $(g, b) \in \Omega$ we can construct a new smooth Riemannian manifold N^n by replacing $M^n(f, a)$ with $M^n(g, b)$. The lemma tells us that the geodesic flows of M and N are C^∞ contact conjugate, where the conjugacy is the identity outside $SM(f, a)$. Thus for any such M^n there is a large family of nonisometric N^n with geodesic flow C^∞ contact conjugate to the geodesic flow of M^n. Also it is easy to see that on any differentiable manifold there are metrics such as above by, for example, choosing the metric in an open ball to be a metric ball of radius $\frac{3}{4}\pi$ with constant curvature 1.

Proof of lemma. We will only give the proof in the case of two dimensions, as the extension to higher dimensions is straightforward.

For our surface $M^2(f, a)$ (resp. $M^2(g, b)$) we will use coordinates (x, θ) for $x \in [0, a]$ (resp. $[0, b]$) so the metric is $(f'^2(x) + 1)\, dx^2 + (f^2(x))\, d\theta^2$. For a unit vector V on M^2 we will let $\phi(V)$ be the angle which V makes with the curve $x = \text{const.}$ so that Clairaut's relation along a unit speed geodesic $\gamma(t)$ states that $\cos(\phi(\gamma'(t)))f(x(\gamma(t)))$ is constant.

For a unit vector V in $M^2(f, a)$ with base at (x_0, θ_0) such that $x_0 \in [0, c]$ and $\langle V, d/dx\rangle \geq 0$ (or $x_0 \in [c, a]$ and $\langle V, d/dx\rangle \leq 0$) then the geodesic $\gamma_V(t)$ determined by V will have an increasing (or decreasing) x coordinate for $t \in [0, l_0]$ until the first value l_0 of t such that $f(\gamma_V(l_0)) = f(x_0)$. Now if W is a similar unit vector in $M^2(g, b)$ at (x_1, θ_1) where $\phi(V) = \phi(W)$ and $f(x) = g(x_1)$, then we shall show

Claim. $l_0 = l_1$ and $\theta(\gamma_V(l_0)) - \theta_0 = \theta(\gamma_W(l_0)) - \theta_1$.

To see the claim it is sufficient (by integration over y) to see that for every $y \in [f(x_0), f(c)]$ we have $(d\theta/dy)(x_-^y) - (d\theta/dy)(x_+^y)$ and $(dt/dy)(x_-^y) - (dt/dy)(x_+^y)$ are the same for γ_V and γ_W. By Clairaut's relation for any y, $d\theta/dt$ and ds/dt are the same for γ_V and γ_W at points with the same y value. Now

$$\frac{d\theta}{dy} = \frac{d\theta}{dt}\frac{dt}{ds}\frac{ds}{dy},$$

hence combining the results at x_-^y and x_+^y and using the fact that (f, a) and (g, b) are compatible we get the result we need. Similarly using

$$\frac{dt}{dy} = \frac{dt}{ds}\frac{ds}{dy}$$

we get the other equality we need, and the claim follows.

We divide the geodesics and hence the unit sphere bundle into two sets: the interior ones (S_iM) are those unit vectors tangent to geodesics that never intersect the boundary, and the exterior ones (S_eM) whose geodesics do intersect the boundary. Any vector $V \in S_eM^2(f, a)$ can be uniquely written as $g^t(V_0)$ where V_0 is a unit vector at a point $(0, \theta)$ (note that t will be negative if V_0 points outside $M^2(f, a)$, and for the special case where V_0 is tangent to the boundary take the smallest t in absolute value). Then let V_1 in $SM^2(g, b)$ be the corresponding unit vector at $(0, \theta)$, and the claim tells us that $F(V) = g^t(V_1)$ is well defined, maps $S_eM^2(f, a)$ smoothly onto $S_eM^2(g, b)$, and is precisely DI_0 or DI_1 near the boundaries. For $V \in S_iM^2(f, a)$ there is a unit vector V_0 with $g^t(V_0) = V$, $\phi(V) = 0$ (or π), and the base point (x_0, θ_0) has $x_0 \leq c$. We will let V_1 be the corresponding unit vector in $SM^2(g, b)$ at (x_1, θ_0) where $f(x_0) = g(x_1)$, and define $F(V) = g^t(V_1)$. Although there is some ambiguity in the choice of V_0 (and t), the claim tells us that the map F is well defined. Further it is clear that F is C^∞ and commutes with the geodesic flow. (Note that the geodesic flow can take vectors near $\overline{S_iM} \cap S_eM$ to vectors in neighborhoods of the boundary where F is just DI_0 and hence is smooth.)

The only question left is whether the map F above preserves the contact form. Since the contact form is preserved by isometries and geodesic flows, this is clear for $V \in S_eM^2(f, a)$ and for V near the boundary of $S_iM^2(f, a)$. Let γ be the geodesic in $M^2(f, a)$ with $\gamma(0) = (x_0, 0)$ and $\gamma'(0)$ tangent to the curve $x = x_0$. Let γ_1 be the image geodesic. By the S^1 symmetry and the invariance of the contact form under the geodesic flows, it is sufficient to show that the contact form is preserved at $\gamma'(0)$. To do this we need to find two independent Jacobi fields in $\Psi^{\perp}(\gamma)$ that are mapped to $\Psi^{\perp}(\gamma_1)$. One such Jacobi field is the variation field through the geodesics γ_t where $\gamma_t(0) = (x_0 + t, 0)$ and $\phi(\gamma_t'(0)) = 0$. This variation (reps. the image variation) will consist of geodesics whose initial tangent vectors are perpendicular to the geodesic $\theta = 0$, and hence the variation Jacobi field will have derivative 0 at $t = 0$ and therefore remain perpendicular to γ (resp. γ_1). To find the other Jacobi field we use the S^1 symmetry of the two spaces. The Killing field $d/d\theta$ is a Jacobi field along both γ and γ_1, and F takes one to the other. The tangential components are easily seen to be $|(d/d\theta)(x_0, 0)|_f \gamma'(t)$ and $|(d/d\theta)(x_1, 0)|_g \gamma_1'(t)$. Since $f(x_0) = g(x_1)$, the tangential components of $d/d\theta$ are taken to each other and hence so are the perpendicular components $(d/d\theta)^{\perp}$. Except along the geodesic $x = c$ and its image (which can be handled by continuity), $(d/d\theta)^{\perp}$ is a nontrivial Jacobi field which is independent from the other Jacobi field we considered since $(d/d\theta)^{\perp}(0) = 0$, and thus F is contact.

References

[1] M. Berger, *Géométrie*, 5 vol., CEDIC-Nathan, Paris, 1976.
[2] A. Besse, *Manifolds all of whose geodesics are closed*, Ergebnisse Math. u.i. Grenzgeb., no. 93, Springer, Berlin, 1978.
[3] C. Croke, *Rigidity and the distance between boundary points*, J. Differential Geometry **33** (1991) 445–464 .
[4] ——, *Rigidity for surfaces of non-positive curvature*, Comm. Math. Helv. **65** (1990) 150–169 .
[5] ——, *Volumes of balls in manifolds without conjugate points*, Internat. J. Math. **3** (1992) 455-467 .
[6] C. Croke, A. Fathi & J. Feldman, *The marked length-spectrum of a surface of nonpositive curvature*, Topology **31** (1992) 847–855 .
[7] J. Eschenburg & J. O'Sullivan, *Growth of Jacobi fields and divergence of geodesics*, Math. Z. **150** (1976) 221–237 .
[8] L. Green, *A theorem of E. Hopf*, Michigan J. Math. **5** (1958) 31–34 .
[9] ——, *Auf wiedersehenflächen*, Ann. of Math. (2) **78** (1963) 289–299 .

[10] D. Gromoll & J. Wolf, *Some relations between the metric structure and the algebraic structure of the fundamental group in manifolds of non-positive curvature*, Bull. Amer. Math. Soc. **77** (1971) 545–552 .

[11] M. Gromov, *Filling Riemannian manifolds*, J. Differential Geometry **18** (1983) 1–147 .

[12] E. Hopf, *Closed surfaces without conjugate points*, Proc. Nat. Acad. Sci. U.S.A. **34** (1948) 47–51 .

[13] W. Klingenberg, *Riemannian manifolds with geodesic flow of Anosov type*, Ann. of Math. (2) **99** (1974) 1–13 .

[14] S. Kobayashi & K. Nomizu, *Foundations of differential geometry*, Vol. 1, Interscience, New York, 1963.

[15] R. Mañé, *Ergodic theory and differentiable dynamics*, Ergebnisse Math. u. i. Grenzbeg., 3 folg, band 8, Springer, Berlin, 1987.

[16] ——, *On a theorem of Klingenberg*, Collection: Dynamical Systems and Bifurcation Theory (Rio de Janeiro, 1985), Pitman Res. Notes Math. Ser. Vol. 160 , Pitman, London, 319–345.

[17] R. Michel, *Sur la rigidité imposée par la longuer des géodésiques*, Invent. Math. **65** (1981) 71–83 .

[18] J.-P. Otal, *Le spectre marqué des longueurs des surfaces à courbure négative*, Ann. of Math. (2) **131** (1990) 151–162 .

UNIVERSITY OF PENNSYLVANIA