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NORMAL GENERATION OF VECTOR BUNDLES OVER A CURVE

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Introduction

Many algebro-geometric calculations simplify when a particular multiplication map of global sections on vector bundles surjects. For example, a line bundle over a smooth variety X is normally generated iff L embeds X as a projectively normal variety. Equivalently, L is normally generated iff L is ample and for all $n \geq 1$ we have surjectivity for the multiplication map

$$S^n(H^0(X, L)) \to H^0(X, L^{\otimes n}).$$

This characterization means normal generation allows us to calculate dimensions of the spaces of quadrics, cubics, and so forth which vanish on X, using Riemann-Roch. So we want optimal numerical conditions forcing normal generation. Mumford [30] shows that if L_1 and L_2 are line bundles over a smooth curve C of genus g with $\deg(L_1) \geq 2g$ and $\deg(L_2) \geq 2g+1$, we have surjectivity of the multiplication map

$$\tau: H^0(C, L_1) \otimes H^0(C, L_2) \to H^0(C, L_1 \otimes L_2).$$

So L is normally generated whenever $\deg(L) \geq 2g+1$, which was first discovered by Castelnuovo [5], then rediscovered by Mattuck [27], and again by Mumford [30]. Examples show this result is optimal.

Recent work generalizes the "$2g+1$" theorem to higher syzygies. We recall notation introduced by Green and Lazarsfeld [16]. L has property N_0 iff L is normally generated. L has property N_1 iff L is normally presented, meaning L has property N_0 and the homogeneous ideal is generated by quadrics. L has property N_2 iff L has property N_1 and the relations among the quadrics are generated by the linear relations. L has property N_3 iff L has property N_2 and the relations among the relations are generated by the linear relations. And so on. Still working over a smooth curve C, Fujita [9] and Saint-Donat [36] built upon work of Mumford and showed L is normally presented provided $\deg(L) \geq 2g+2$.

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Generalizing these results, Green shows L has property N_p whenever $\deg(L) \geq 2g+1+p$.

Algebraic geometers have not yet generalized Green's Theorem, or even the special case of Castelnuovo's Theorem to higher dimensions. But the form such a generalization ought to take now seems clear. Mukai observes that a line bundle L over a curve C of genus g with $\deg(L) \geq 2g+1$ is of the form $L = \Omega_C \otimes A^{\otimes t}$, where $t \geq 3$ and A is ample. This leads him to conjecture that there should be some explicit number t_0, such that any line bundle B over a smooth surface S is normally generated if B is of the form $B = K_S \otimes S^{\otimes t}$, where $t \geq t_0$ and A is ample. Since recent work of Reider [35] shows $K_S \otimes A^{\otimes t}$ is very ample for $t \geq 4$, Mukai suggests $t_0 = 4$. We of course expect similar statements for higher syzygies.

Towards an ultimate goal of solving Mukai's conjecture, it seems worthwhile to consider special classes of varieties. We prove "Mukai" type results for ruled varieties with a curve as base. To do so, we first consider a simple and natural question. What are the optimal numerical conditions on vector bundles E and F over a smooth curve C of genus g which force surjectivity of the natural multiplication map

$$\tau: H^0(C, E) \otimes H^0(C, F) \to H^0(C, E \otimes F)?$$

We answer this question for $\mathrm{char}(k) = 0$ or $g \leq 1$, and now state our main result.

Theorem 1 ($\mathrm{char}(k) = 0$, *or* $g \leq 1$). *Let E and F be semistable vector bundles over a smooth projective curve C of genus g. If $\mu(E) \geq 2g$ and $\mu(F) > 2g$, then τ surjects.*

Examples show Theorem 1 is optimal. It generalizes Mumford's Theorem even if $\mathrm{char}(k) \neq 0$ and $g \geq 2$ because Theorem 1 remains true in this case provided either E or F is a line bundle. Furthermore, in Theorem 2.1, we generalize Theorem 1 to possibly unstable bundles by considering a numerical invariant which Mehta [28] defines in terms of a vector bundle's unique Harder-Narasimhan filtration [19]. Then in Theorem 4.1 we further generalize Theorem 1 to vector bundles over ruled varieties with a curve as base. The numerical invariants we consider in Theorem 4.1 basically involve stability of the push down bundle, and regularity of restriction to a fiber. The result is somewhat technical but allows us to prove the following theorems.

Theorem 2A ($\mathrm{char}(k) = 0$, *or* $g \leq 1$). *Let E be a rank-n vector bundle over a smooth projective curve C, and set $t_0 = 2n+1$. If A is an ample line bundle over X, then B is normally generated provided B is of* *the form* $B = K_X \otimes A^{\otimes t}$ *for some* $t \geq t_0$. *More generally, if* A_i *is an ample line bundle over* $X = \mathbf{P}(E)$ *for* $1 \leq i \leq t_0$, *then* $B = K_X \otimes A_1 \otimes \cdots \otimes A_{t_0}$ *is normally generated.*

Theorem 2B (char$(k) = 0$, *or* char$(k) = q > p+1$ *and* $g \leq 1$). *Let* E *be a rank-n vector bundle over a smooth projective curve* C. *Fix a positive integer* p *and set* $t_p = 2n + 2np$. *If* A *is an ample line bundle over* X, *then* B *has property* N_p *provided* B *is of the form* $B = K_X \otimes A^{\otimes t}$ *for some* $t \geq t_p$. *More generally, if* A_i *is an ample line bundle over* $X = \mathbf{P}(E)$ *for* $1 \leq i \leq t_p$, *then* $B = K_X \otimes A_1 \otimes \cdots \otimes A_{t_p}$ *has property* N_p.

We also use Theorem 4.1 to study Koszul rings on curves and ruled varieties over a curve. Our main result is Theorem 3. The first part of this result was conjectured by Kempf (cf. [23]), and then reproven by the same, subsequent to our work, using his results for Koszul rings on points in projective space and hyperplane sections [24]. Recall the definition of a Koszul ring (which is known by other names such as wonderful).

Definition. Let $S = k \oplus S_1 \oplus S_2 \oplus \cdots$ be a graded ring and k a field. S is a Koszul ring iff $\mathrm{Tor}_i^S(k, k)$ has pure degree i for all i.

Theorem 3. (1) *A linearly normal curve* C *of* deg $\geq 2g + 2$ *has a Koszul homogeneous coordinate ring.*

(2) (char$(k) = 0$, *or* $g \leq 1$) *Let* E *be a rank-n vector bundle over a smooth projective curve* C, *and set* $t_k = 4n$. *If* A *is an ample line bundle over* $X = \mathbf{P}(E)$, *then the homogeneous coordinate ring for the complete embedding determined by* $B = K_X \otimes A^{\otimes t}$ *is Koszul for* $t \geq t_k$. *Moreover, the above holds for* $B = K_X \otimes A_1 \otimes \cdots \otimes A_{t_k}$ *if* A_i *is ample for* $1 \leq i \leq t_k$.

When $n = 1$, Theorem 2A yields Castelnuovo's Theorem. Likewise, with $n = 1$ and $p = 1$, Theorem 2B gives a theorem of Fujita and Saint-Donat. Green's Theorem does not, however, follow from Theorem 2B. When $n \geq 2$ Theorems 2A or 2B may or may not be optimal. We do not know. But for all n we produce a ruled variety X of dimension n and a line bundle $B = K_X \otimes A^{\otimes(n+1)}$ (with A ample) which is not normally generated. In fact we produce examples were B is very ample, but not normally generated, and examples where B is not even very ample. We also produce line bundles of the form $B = K_X \otimes A^{\otimes(n+2)}$ (with A ample) which are not normally presented.

Kempf shows a variety with a Koszul homogeneous coordinate ring is projectively normal and defined by quadrics [22], so the above gives examples of line bundles of the form $B = K_X \otimes A^{\otimes(n+2)}$ which do not determine Koszul homogeneous coordinate rings. Thus at least part (1) of Theorem 3 is optimal.

We should point out that modifying Mukai's conjecture by considering very ample bundles instead of merely ample bundles makes the problem solvable for all varieties. Bertram, Ein, and Lazarsfeld [4] show that for a very ample bundle A over a smooth n-dimensional variety X, $K_X \otimes A^{\otimes n}$ is normally generated, provided it is very ample. (This was discovered independently by Andreatta, Ballico, and Sommese [1], [2].) Furthermore, Ein and Lazarfeld [6] show that $K_X \otimes A^{\otimes(n+1+p)}$ satisfies property N_p.

We approach vector bundle multiplication on a curve via the vector bundle M_E associated to a bundle E generated by global sections. To understand the map τ we need to understand cohomological properties of $M_E \otimes F$. So we consider stability properties of M_E. In §1 we prove Theorem 1.2 and Corollary 1.3, which relate stability properties of M_E to those of E under certain numerical conditions. This is useful because semistability is preserved under tensoring when $\text{char}(k) = 0$ or $g \leq 1$, and stability properties allow us to kill higher cohomology groups under certain, useful numerical conditions. In §2, we use that technique to prove Theorem 1 and related results. We then gather some familiar and elementary results on the regularity of vector bundles over ruled varieties in §3. With that done, we turn to §4 where we consider bundles V and W over $X = \mathbf{P}(E)$, and relate cohomological properties of $M_{\pi_* V} \otimes \pi_* W$ to those of $\pi_*(M_V \otimes W)$. This allows us to prove Theorem 4.1 which generalizes Theorem 1 to ruled varieties over a curve. Since property N_p is implied by surjectivity of a particular multiplication map ([17] and [25]), we derive Theorems 2A and B from Theorem 4.1 in §5. Then in §6 we give a reinterpretation due to Kempf [22] of Koszul rings, and then a further reinterpretation, suggested by Lazarsfeld, of Koszul rings in terms of vector bundle multiplication. Theorem 3 then follows from Theorem 4.1.

We owe special thanks to R. Lazarsfeld for introducing us to the subject of vector bundle multiplication and its applications, then guiding us through it. We also thank G. Kempf for introducing us to Koszul rings and related problems, and we are grateful to A. Bertram, D. Gieseker, and M. Green for useful discussions, comments, and encouragement.

0. Notation and conventions

C denotes a smooth projective curve of genus g over an algebraically closed field k.

All vector bundles are algebraic.

A vector bundle E over C has slope $\mu(E) = \deg(E)/\text{rank}(E)$.

A vector bundle E over C is semistable iff for every proper subbundle S, $\mu(S) \leq \mu(E)$. It is stable iff the inequality is strict.

The Harder-Narasimhan filtration of a vector bundle E over C is the unique filtration

$$\Sigma: 0 = E_0 \subset E_1 \subset \cdots \subset E_s = E,$$

such that E_i/E_{i-1} is semistable, and $\mu_i(E) = \mu(E_i/E_{i-1})$ is a strictly decreasing function of i.

If E is unstable (i.e., not semistable), then E_1 is called the maximal destabilizing subbundle.

If D_1 and D_2 are divisors on X, then $D_1 \equiv D_2$ means D_1 and D_2 are numerically equivalent.

1. The stability of M_E

If a vector bundle E over C is generated by $V \subseteq H^0(C, E)$, as in [17] the evaluation map α determines an exact sequence of bundles:

$$0 \to M_{V,E} \to V \otimes \mathcal{O}_C \xrightarrow{\alpha} E \to 0. \tag{1.1}$$

When $V = H^0(C, E)$ we write M_E. This section considers how stability properties of M_E relate to those of E. Our main result is:

1.2 Theorem. *Let E be a semistable vector bundle over C with $\mu(E) \geq 2g$. Then M_E is semistable, and $\mu(M_E) = -\mu(E)/(\mu(E) - g) \geq -2$. Furthermore, if E is stable and $\mu(E) \geq 2g$, then M_E is stable unless $\mu(E) = 2g$ and either C is hyperelliptic or $\Omega_C \hookrightarrow E$.*

To understand unstable bundles, we consider the unique Harder-Narasimhan filtration of E and its relation to the Harder-Narasimhan filtration of M_E. For this we need some invariants of a bundle we believe Mehta first defined [28].

Definition/Remark. Let $\Sigma: E_0 \subset E_1 \subset \cdots \subset E_s = E$ be the Harder-Narasimhan filtration of a vector bundle E over C. Define the following:

(1) $\mu^-(E) = \mu_s(E) = \mu(E_s/E_{s-1})$.
(2) $\mu^+(E) = \mu_1(E) = \mu(E_1)$.

Alternatively, we may state the definitions of μ^+ and μ^- as follows:

(1) $\mu^+ = \max\{\mu(S) | 0 \to S \to E\}$.
(2) $\mu^- = \min\{\mu(Q) | E \to Q \to 0\}$.

We have $\mu^+(E) \geq \mu(E) \geq \mu^-(E)$ with equality iff E is semistable, since the vector bundles S and Q in the above definitions need not be proper subbundles and quotient bundles.

For some applications, we want to consider only proper subbundles, so we have the following.

Definition. $\operatorname{prop}^+(E) = \sup_S\{\mu(S) | S \subsetneq E\}$.

1.3 Corollary. *Let C be an irrational curve and E a vector bundle over C. If E has Harder-Narasimhan filtration $\Sigma: 0 = E_0 \subset E_1 \subset \cdots \subset E_S = E$ and $\mu^-(E) \geq 2g$, then M_E has Harder-Narasimhan filtration*

$$\Sigma: 0 = M_{E_0} \subset M_{E_1} \subset \cdots \subset M_{E_S} = M_E$$

and $\mu^-(M_E) \geq -\mu^-(E)/(\mu^-(E) - g) \geq -2$.

On the rational curve, it is well known, and easy to show, that M_E is always a direct sum of $\mathcal{O}_C(-1)$'s. It also happens that on an elliptic curve, M_E is stable, semistable, or indecomposable, as E is stable, semistable, or indecomposable, and hence we can always determine the Harder-Narasimhan filtration of M_E from that of E if it is generated by global sections. When $g \geq 2$ and $\mu^-(E) < 2g$, on the other hand, determining the Harder-Narasimhan filtration of M_E becomes fairly subtle. In general, the best we can do is to bound the instability of M_E.

1.4 Proposition. *Let E be a vector bundle over C generated by global sections,*

$$\mu^+(M_E) \leq \max\left\{-2, \frac{-\mu^+(E)}{\mu^+(E) - g}\right\}.$$

Furthermore,

$$\operatorname{prop}^+(M_E) \leq \max\left\{-2, \frac{-\operatorname{prop}^+(E)}{\operatorname{prop}^+(E) - g}\right\},$$

and if $\operatorname{prop}^+(E) < 2g$, then $\operatorname{prop}^+(M_E) < -2$ unless C is hyperelliptic or $\Omega_C \hookrightarrow E$.

1.5 Proposition. *Let C be a curve of genus $g \geq 2$ and let E be a vector bundle over C. Suppose further that E has no trivial summands, and E is generated by global sections. If $\mu(E) < 2g$, then*

$$\mu^-(M_E) \geq \operatorname{rank}(E)(\mu^-(E) - 2g) - 2 + 2h^1(C, E).$$

1.6 Remark. The proof of Proposition 1.5 shows that optimality of Proposition 1.5 implies optimality of Proposition 1.4. The proofs of Theorem 1 and its generalizations show stability properties of M_E are intimately connected to surjectivity of multiplication maps. Consequently, Examples 2.6, where multiplication maps fail to surject, show Proposition 1.5 is optimal in general. However, assuming $E = L$ is a line bundle, we can improve our results by considering geometry of the curve. Unpublished work of the author shows numerology in Propositions 1.4 and 1.5 can be improved by considering the Clifford index of the curve. The author also shows that on a general curve, M_L is always semistable and typically stable.

We now prove the results already stated in this section. Theorem 1.2 and Proposition 1.5 will follow from Proposition 1.4. To prove 1.4, we generalize a familiar argument for showing M_L is stable for $\deg(L) \geq 2g+1$ (cf. [33]) to vector bundles of arbitrary rank. Doing so requires some technical lemmas.

1.7 Lemma. *If F is a vector bundle over C with $h^1(C, F) = 0$ and $V \subseteq H^0(C, F)$ generates F, then $\mu(M_{V,F}) \leq -\mu(F)/(\mu(F) - g)$ with equality iff $V = H^0(C, F)$.*

Proof of Lemma 1.7. Let $\operatorname{rank}(F) = n$. Then

$$\mu(M_{V,F}) = \frac{-\deg(F)}{\dim(V) - n} \leq \frac{-\deg(F)}{h^0(C, F) - n} = \mu(M_F).$$

Since $h^1(C, F) = 0$, Riemann-Roch implies $h^0(C, F) = n(\mu(F) - g + 1)$. We know $\deg(F) = n\mu(F)$, so we get $\mu(M_F) = -\mu(F)/(\mu(F) - g)$.

1.8 Corollary. *Let C be an irrational curve, and let E and F be vector bundles over C. Suppose further that E and F both have $h^1 = 0$, E is generated by global sections, and F is generated by $V \subseteq H^0(C, F)$. If $\mu(E) \geq \mu(F)$, then $\mu(M_E) \geq \mu(M_{V,F})$. Moreover, if the first inequality is strict, so is the second.*

Proof of Corollary 1.8. We have $\mu(M_{V,F}) \leq -\mu(F)/(\mu(F) - g)$ and $\mu(M_E) = -\mu(E)/(\mu(E) - g)$. But $f(x) = -x/(x-g)$ strictly increases for $x > g$.

1.9 Lemma. *Let C be an irrational curve and let E be a vector bundle over C. Suppose further that E has no trivial summands and is generated by global sections. If $N \subseteq M_E$ is a stable subbundle of maximal slope, there is a vector bundle F with no trivial summands and $\mu(F) \leq \mu^+(E)$, and a subspace $V \subseteq H^0(C, F)$ which generates F such that $N = M_{V,F}$. Furthermore, $\mu(F) = \mu^+(E)$ implies $F \subseteq E$.*

Proof of Lemma 1.9. Assume $N \subseteq M_E$ is a stable subbundle of maximal slope. Dualizing 1.1 gives

$$H^0(C, E)^* \otimes \mathscr{O}_C^* \to M_E^* \to 0.$$

So the image of $H^0(C, E)^*$ in $H^0(C, N^*)$, say V^*, generates N^*, and the kernel of the map $V^* \otimes \mathscr{O}_C^* \to N^*$ is a vector bundle F^* with no trivial summands. Dualizing, we have:

$$\begin{array}{ccccccccc} & & 0 & & 0 & & & & \\ & & \downarrow & & \downarrow & & & & \\ 0 & \to & N & \to & V \otimes \mathscr{O}_C & \to & F & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow \alpha & & \\ 0 & \to & M_E & \to & H^0(C, E) \otimes \mathscr{O}_C & \to & E & \to & 0 \end{array}$$

The map α is nonzero because $V \otimes \mathscr{O}_C \not\to M_E$. Denote the image of F in E by S, and note that $V \hookrightarrow H^0(C, S)$ because F has no trivial summands.

We claim $\deg(F) \leq \deg(S)$ with equality iff $F = S$. To see this, first note $N \hookrightarrow M_S$, because of the diagram:

$$\begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \\ & & N & & M_S & & \\ & & \downarrow & & \downarrow & & \\ 0 & \to & V \otimes \mathscr{O}_C & \to & H^0(C, S) \otimes \mathscr{O}_C & & \\ & & \downarrow & & \downarrow & & \\ & & F & \to & S & \to & 0 \\ & & \downarrow & & \downarrow & & \\ & & 0 & & 0. & & \end{array}$$

Since N is a subbundle of M_E of maximal slope and is contained in M_S, which is itself a subbundle of M_E, $\mu(N) \geq \mu(M_S)$. We know that

$$\mu(N) = \frac{-\deg(F)}{\operatorname{rank}(N)} \quad \text{and} \quad \mu(M_S) = \frac{-\deg(S)}{\operatorname{rank}(M_S)};$$

therefore, $\mu(N) \geq \mu(M_S)$ implies $\deg(F) \leq \deg(S)$ with equality iff $N = M_S$ because $\operatorname{rank}(N) \leq \operatorname{rank}(M_S)$. So $\mu(F) \leq \mu(S)$ with equality iff $N = M_S$ and $F = S$, because $\operatorname{rank}(F) \geq \operatorname{rank}(S)$. Since $S \subseteq E$, this implies $\mu(F) \leq \mu^+(E)$, with equality iff $F \subseteq E$. q.e.d.

If $h^1(C, F) = 0$, the bound on $\mu(F)$ bounds $\mu(M_{V,F})$ by Lemma 1.7 and Corollary 1.8. So we need to bound $\mu(M_{V,F})$ if $h^1(C, F) \neq 0$.

1.10 Lemma. *Let C be a curve. Let F be a vector bundle over C which has no trivial summands, and assume $h^1(C, F) \neq 0$. Suppose $V \subseteq H^0(C, F)$ generates F. If $N = M_{V,F}$ is stable, then $\mu(N) \leq -2$. Furthermore, $\mu(N) = -2$ implies that either C is hyperelliptic, or $F = \Omega_C$ and $N = M_{\Omega_C}$.*

1.11 Remark. From the classification of vector bundles over rational and elliptic curves ([18], [3]), we see that when C is rational or elliptic and E is a vector bundle over C generated by global sections with no trivial summands, $h^1(C, E) = 0$, so Lemma 1.10 is vacuously true for $g \leq 1$.

Proof of Lemma 1.10. Since $h^1(C, F) > 0$ and F is generated by global sections, Serre duality guarantees there is a nonzero map $F \to \Omega_C$ such that the image of V generically generates Ω_C. So we get:

$$\begin{array}{ccc} 0 & & 0 \\ \downarrow & & \downarrow \\ N & \xrightarrow{\alpha} & M_{\Omega_C} \\ \downarrow \beta & & \downarrow \\ V \otimes \mathscr{O}_C & \xrightarrow{\gamma} & H^0(C, \Omega_C) \otimes \mathscr{O}_C \\ \downarrow & & \downarrow \\ F & \longrightarrow & \Omega_C \\ \downarrow & & \downarrow \\ 0 & & 0 \end{array}$$

A simple diagram chase shows the map α exists. We claim $\alpha \neq 0$. To see this, simply observe $\gamma \circ \beta \neq 0$ because $\gamma \neq 0$ and β does not factor through $W \otimes \mathscr{O}_C$ for $W \subsetneq V$. Set $S = \operatorname{im}(\alpha)$. We see S is a quotient of N which sits as a subbundle of M_{Ω_C}. If C is nonhyperelliptic, M_{Ω_C} is stable by a theorem of Paranjape and Ramanan [33], while if C is hyperelliptic M_{Ω_C} is a direct sum of duals of the hyperelliptic bundle, and hence semistable. Therefore, $\mu(S) \leq -2$ with equality iff $S = M_{\Omega_C}$ or C is hyperelliptic. Since S is a quotient of N and N is stable, $\mu(N) \leq \mu(S) \leq -2$. Hence, $\mu(N) \leq -2$ with equality implying $\mu(N) = \mu(S) = -2$, and, by stability of N, that $N = S$. Because N must be a stable subbundle of M_{Ω_C} with $\mu = -2$, either $N = M_{\Omega_C}$ or C is hyperelliptic. In the former case, it is easily seen that $F = \Omega_C$.

Proof of Proposition 1.4. Suppose $N \hookrightarrow M_E$ is a stable bundle of maximal slope. By Lemma 1.9, $N = M_{V,F}$. If $\mu(N) \leq -2$ we are done, so we may assume $\mu(N) > -2$, and hence $h^1(C, F) = 0$ by Lemma 1.10. Lemma 1.7 now implies $\mu(N) \leq -\mu(F)/(\mu(F) - g)$. By Lemma 1.9, $\mu(F) \leq \mu^+(E)$, and so by Corollary 1.8, $\mu(N) \leq -\mu^+(E)/(\mu^+(E) - g)$.

The remaining statements follow similarly. q.e.d.

To prove Theorem 1.2, and many other results of this paper, we need to be able to show some vector bundle is generated by global sections and/or has no higher cohomology. Using μ^- this is simple. We have a simple but useful lemma. It belongs to folklore, but is presented here for nonspecialists.

1.12 Lemma. *Let E be a vector bundle over C.*

(1) *If $\mu^+(E) < 0$, then $h^0(C, E) = 0$.*

(2) *If $\mu^-(E) > 2g - 2$, then $h^1(C, E) = 0$.*

(3) *If $\mu^-(E) > 2g - 1$, then E is generated by global sections.*

(4) *If $\mu^-(E) > 2g$, then $\mathcal{O}_{\mathbf{P}(E)}(1)$ is very ample.*

Proof of Lemma 1.12. To prove (1), note that if $h^0(C, E) > 0$ then we have a map $\mathcal{O}_C \hookrightarrow E$, and hence $\mu^+(E) \geq 0$. To prove (2), use (1) and Serre duality. Finally, (3) and (4) follow from (2) and the fact that $\mu^-(E(-p)) = \mu^-(E) - 1$ for $p \in C$.

Proof of Theorem 1.2. If C is rational, M_E is a direct sum of $\mathcal{O}_C(-1)$'s, so we may assume C is irrational. Suppose E is semistable and $\mu(E) \geq 2g$. From Lemma 1.12(3), E is generated by global sections, and by Lemma 1.12(2), we have $h^1(C, E) = 0$, and hence Lemma 1.7 applies and shows $\mu(M_E) = -\mu(E)/(\mu(E) - g) \geq -2$. So by Proposition 1.4,

$$\mu^+(M_E) \leq \max\left\{-2, \frac{-\mu^+(E)}{\mu^+(E) - g}\right\} = \frac{-\mu(E)}{\mu(E) - g} = \mu(M_E).$$

The remaining statements follow similarly.

Proof of Corollary 1.3. Since $\mu^-(E) \geq 2g$, for $0 \leq i < s$ we have $\mu(E_{i+1}/E_i) \geq 2g$. Therefore by Lemma 1.12(2), $h^1(C, E_{i+1}/E_i) = 0$ and consequently $h^1(C, E_{i+1}) = 0$ for all i, and by changing index $h^1(C, E_i) = 0$ for all i. Likewise E_{i+1} (and hence E_{i+1}/E_i) is generated by global sections by Lemma 1.12(3), and by changing index E_i is generated by global sections. This gives us the following diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & M_{E_i} & \to & M_{E_{i+1}} & \to & M_{(E_{i+1}/E_i)} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & H^0(C, E_i) \otimes \mathcal{O}_C & \to & H^0(C, E_{i+1}) \otimes \mathcal{O}_C & \to & H^0(C, E_{i+1}/E_i) \otimes \mathcal{O}_C & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & E_i & \to & E_{i+1} & \to & E_{i+1}/E_i & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

The upper sequence shows us $M_{E_{i+1}}/M_{E_i} = M_{(E_{i+1}/E_i)}$. So Theorem 1.2 implies $M_{E_{i+1}}/M_{E_i}$ is semistable by the semistability of (E_{i+1}/E_i) and the fact that $\mu(E_{i+1}/E_i) \geq 2g$. Since $\mu(E_{i+1}/E_i)$ decreases with i, so does $\mu(M_{E_{i+1}}/M_{E_i})$ by Corollary 1.8. It follows that M_E has the stated Harder-Narasimhan filtration by uniqueness. Consequently, $\mu^-(M_E) = \mu(M_{E_s}/M_{E_{s-1}}) = -\mu^-(E)/(\mu^-(E) - g)$.

Sketch of proof of Proposition 1.5. First we consider the special case where $\mu^+(E) \leq 2g$. By Proposition 1.4, $\mu^+(M_E) \leq -2$. Let Q be a quotient bundle of M_E with minimal slope, and define N by

$$0 \to N \to M_E \to Q \to 0.$$

The idea is that since $\mu^+(M_E) \leq -2$ by Proposition 1.4, $\mu(N) \leq -2$ and $\deg(N) \leq -2\mathrm{rank}(N)$. Now this implies $\deg(Q) \geq -\deg(E) + 2\mathrm{rank}(N)$. Of course $\mathrm{rank}(N) = \mathrm{rank}(M_E) - \mathrm{rank}(Q)$, and likewise $\mathrm{rank}(M_E) = \mathrm{rank}(E)(\mu(E) - g) + h^1(C, E)$. So for any possible value of $\mathrm{rank}(Q)$, we can bound $\deg(Q)$, and hence $\mu(Q)$, from below. Direct calculation shows

$$\mu^-(M_E) \geq \mathrm{rank}(E)(\mu(E) - 2g) - 2 + 2h^1(C, E).$$

Equality holds iff $\mathrm{rank}(Q) = 1$ and $\mu(N) = -2$. So we need only observe $\mu(E) \geq \mu^-(E)$ to see the proposition holds if $\mu^+(E) \leq 2g$.

If $\mu^+(E) > 2g$, consider the Harder-Narasimhan filtration

$$\Sigma\colon 0 = E_0 \subset E_1 \subset \cdots \subset E_s = E.$$

Let n be the largest integer such that $\mu^-(E_n) > 2g$. We can obtain the Harder-Narasimhan filtration of M_{E_n} from that of E_n by Corollary 1.3, and show that $\mu^-(M_{E_n}) > -2$. Furthermore, since $h^1(C, E_n) = 0$ by Lemma 1.12(2), we get an exact sequence

$$0 \to M_{E_n} \to M_E \to M_{E/E_n} \to 0.$$

But the definition of n implies E/E_n has no subbundle N with $\mu(N) > 2g$, and so

$$\mu^-(M_{E/E_n}) \geq \mathrm{rank}(E/E_n)(\mu^-(E/E_n) - 2g) + 2h^1(C, E/E_n) - 2.$$

Since $\mathrm{rank}(E) > \mathrm{rank}(E/E_n)$, $\mu^-(E) = \mu^-(E/E_n)$, and $h^1(C, E) = h^1(C, E/E_n)$, the weaker inequality in the conclusion of Proposition 1.5 follows.

2. Surjectivity of the multiplication map on curves

In this section we prove Theorem 2.1, a generalization of Theorem 1, and a related result, Proposition 2.2. We use Corollary 1.3 and Lemma 2.5 which relates stability properties of a vector bundle $E \otimes F$ to those of E and F. Then we give examples which show Theorem 2.1 is optimal. This implies Theorem 1.2 is also optimal.

2.1 Theorem (char$(k) = 0$, *or* $g \leq 1$). *If E and F are vector bundles over C with*

(1) $\mu^-(E) \geq 2g$ *and*

(2) $\mu^-(F) > 2g$,

then $\tau: H^0(C, E) \otimes H^0(C, F) \to H^0(C, E \otimes F)$ *surjects.*

This generalizes Mumford's Theorem [30], and Examples 2.6 show it is optimal. Of course if E is generated by sections but $\mu^-(E) < 2g$, the map τ should still surject provided $\mu^-(F) \gg 2g$. For example, it has been shown by Green [13] and Eisenbud, Koh, and Stillman [7] that if L_1 and L_2 are line bundles generated by global sections and $\deg(L_1) + \deg(L_2) \geq 4g + 1$, then τ surjects. For bundles of arbitrary rank, we generalize the " $4g + 1$ " theorem. Examples 2.6 show our result is optimal.

2.2 Proposition (char$(k) = 0$, *or* $g \leq 1$). *Let E and F be vector bundles over C with E generated by global sections. If*

(1) $\mu^-(F) > 2g$, *and*

(2) $\mu^-(F) > 2g + \mathrm{rank}(E)(2g - \mu^-(E)) - 2h^1(C, E)$,

then $\tau: H^0(C, E) \otimes H^0(C, F) \to H^0(C, E \otimes F)$ *surjects.*

2.3 Remark. When char$(k) \neq 0$ and $g \geq 2$, Theorem 2.1 still holds when either E or F is a line bundle, and Proposition 2.2 still holds when F is a line bundle. The reason is that semistability is always preserved under tensoring by a line bundle. In other words, Lemma 2.5(2) still holds when the bundle E or F of Lemma 2.5(2) is of rank 1, and it is only when we apply Lemma 2.5(2) that we use the condition char$(k) = 0$ or $g \leq 1$.

2.4 Lemma. *For E and F vector bundles over C, and V an extension, we have*

$$0 \to E \to V \to F \to 0,$$

$$\mu^+(E) \leq \mu^+(V) \leq \max\{\mu^+(E), \mu^+(F)\},$$

and

$$\mu^-(F) \geq \mu^-(V) \geq \min\{\mu^-(E), \mu^-(F)\}.$$

Proof of Lemma 2.4. Since $S \subseteq E$ implies $S \subseteq V$, we have $\mu^+(V) \geq \mu^+(E)$. So suppose $S \subseteq V$ is a subbundle with $\mu(S) = \mu^+(V)$. Consider the following diagram:

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & S_E & \to & S & \to & S_F & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & E & \to & V & \to & F & \to & 0 \end{array}$$

We have $\mu(S_E) \leq \mu^+(E)$ and $\mu(S_F) \leq \mu^+(F)$ by definition of μ^+; hence,

$$\mu^+(V) = \mu(S) \leq \max\{\mu(S_E), \mu(S_F)\} \leq \max\{\mu^+(E), \mu^+(F)\}.$$

The result for μ^- follows similarly. q.e.d.

Lemma 1.12 allows us to use a lower bound on μ^- to kill off cohomology. However, many of the bundles we are concerned with are of the form $E\otimes F$, $S^n(E)$, and so on. It is known that if $\mathrm{char}(k) = 0$ or $g \leq 1$ the tensor product of semistable bundles is semistable ([18], [3], [32], [31], [20], [12]) (see also [26] for higher dimensions, [29] for an elementary proof, and [11] for examples where this fails in $\mathrm{char}(k) \neq 0$ when $g \geq 2$). This theorem enables us to calculate μ^- and μ^+ of vector bundles of the form $E \otimes F$, $S^n(E)$, and so on, if we know μ^+ and μ^- of E and F.

2.5 Lemma (*characteristic* 0 *or* $g \leq 1$). *If E and F are vector bundles over C, then the following hold*:

(1) $\mu^+(E \otimes F) = \mu^+(E) + \mu^+(F)$.
(2) $\mu^-(E \otimes F) = \mu^-(E) + \mu^-(F)$.
(3) $\mu^+(T^k(E)) = k\mu^+(E)$.
(4) $\mu^-(T^k(E)) = k\mu^-(E)$.
(5) $\mu^+(S^k(E)) = k\mu^+(E)$.
(6) $\mu^-(S^k(E)) = k\mu^-(E)$.
(7) $\mu^+(\bigwedge^k(E)) \leq k\mu^+(E)$.
(8) $\mu^0(\bigwedge^k(E)) \geq k\mu^-(E)$.

Proof of Lemma 2.5. If E and F are semistable, then (1) is well known, as we remarked above. So if one bundle, say E, is semistable, tensoring the Harder-Narasimhan filtration of F by E gives the Harder-Narasimhan filtration of $F \otimes E$, which implies (1) holds in this special case. If E is not semistable we can consider the Narder-Narasimhan filtration of E,

$$\Sigma: 0 = E_0 \subset E_1 \subset \cdots \subset E_s = E,$$

and tensor by F to get a filtration of $F \otimes E$,

$$\Sigma: 0 = F \otimes E_0 \subset F \otimes E_1 \subset \cdots \subset E_s = F \otimes E.$$

E_1 is semistable, so $\mu^+(E_1 \otimes F) = \mu^+(E_1) + \mu^+(F) = \mu^+(E) + \mu^+(F)$ since (1) holds in the special case where one bundle is semistable. To see $\mu^+(E_2 \otimes F) = \mu^+(E) + \mu^+(F)$, consider the exact sequence

$$0 \to E_1 \otimes F \to E_2 \otimes F \to (E_2/E_1) \otimes F \to 0.$$

Since (E_2/E_1) is semistable, we have $\mu^+((E_2/E_1)\otimes F) = \mu(E_2/E_1) + \mu^+(F)$ by the special case. Of course, by the definition of the Harder-Narasimhan filtration $\mu(E_2/E_1) < \mu(E_1)$. By Lemma 2.4,

$$\mu^+(E_1 \otimes F) \le \mu^+(E_2 \otimes F) \le \max\{\mu^+(E_1 \otimes F), \mu^+((E_2/E_1) \otimes F)\},$$

$$\mu(E_1) + \mu^+(F) \le \mu^+(E_2 \otimes F)$$

$$\le \max\{\mu(E_1) + \mu^+(F), \mu(E_2/E_1) + \mu^+(F)\},$$

$$\mu(E_1) + \mu^+(F) \le \mu^+(E_2 \otimes F) \le \mu(E_1) + \mu^+(F),$$

$$\mu^+(E) + \mu^+(F) \le \mu^+(E_2 \otimes F) \le \mu^+(E) + \mu^+(F).$$

Induction shows $\mu^+(E \otimes F) = \mu^+(E) + \mu^+(F)$.

Part (2) can be proved similarly, while (3) and (4) from (1) and (2). As for (5), since $S^k(E)$ is a subbundle of $T^k(E)$, it follows that $\mu^+(S^k(E)) \le k\mu^+(E)$ from (3). But we have some $N \subseteq E$ with $\mu(N) = \mu^+(E)$. Hence $S^k(N) \subseteq S^k(E)$, so $\mu^+(S^k(E)) \ge \mu(S^k(N)) = k\mu(N) = k\mu^+(E)$, and the result is obtained. Part (6) follows similarly, so do parts (7) and (8). q.e.d.

We can now prove the main result of this section.

Proof of Theorem 2.1. By Corollary 1.3, $\mu^-(M_E) \ge -2$. So by Lemma 2.5(2), $\mu^-(M_E \otimes F) > 2g - 2$; hence, $h^1(C, M_E \otimes F) = 0$ by Lemma 1.12(2). Tensoring sequence (1.1) by F and taking cohomology proves the theorem.

Sketch of proof of Proposition 2.2. Use Proposition 1.5 and mimic the proof of Theorem 2.1.

2.6 Examples. We now show Theorem 1 (and hence Theorem 1.2) is optimal by constructing semistable vector bundles E and F with $\mu(E) = \mu(F) = 2g$ such that the multiplication map τ fails to surject. In fact given any semistable vector bundle E with $\mu(E) = 2g$, we construct a corresponding F with the stated properties. Under further hypothesis, we construct a vector bundle G which is semistable and indecomposable, with $\mu(G) = 2g$ and $\mathscr{O}_{\mathbf{P}(G)}(1)$ very ample, but such that the tensor, symmetric, and antisymmetric multiplication maps all fail to surject.

Let E be a semistable vector bundle with $\mu(E) = 2g$. Now let $F = (M_E)^* \otimes \Omega_C$. By Theorem 1.2, F is semistable with $\mu(F) = 2g$. However, surjectivity fails for the multiplication map

$$\tau: H^0(C, E) \otimes H^0(C, F) \to H^0(C, E \otimes F), \tag{2.7}$$

because there is a surjective map

$$M_E \otimes F = M_E \otimes (M_E)^* \otimes \Omega_C \to \Omega_C$$

and hence $h^1(C, M_E \otimes F) \ne 0$. Furthermore, if we assume $g \ge 3$, C nonhyperelliptic, Ω_C does not embed in E, and E stable, then F is in fact stable by Theorem 1.2. Consider a line bundle L with $\deg = 2g$ and an indecomposable extension

$$0 \to L \to G \to E \oplus F \to 0.$$

G is semistable and indecomposable, and $\mu(G) = 2g$. Furthermore, $\mathscr{O}_{\mathbf{P}(G)}(1)$ is very ample, because E and F are stable with $\mu = 2g$ and the sequence is nonsplit, hence G has no line bundle quotients A with $\deg(A) \leq 2g$ (except E if E is a very ample line bundle). However, since surjectivity fails for (2.7) it also fails for the multiplication maps

$$S^2(H^0(C, G)) \to H^0(C, S^2(G)) \quad \text{and} \quad \bigwedge^2(H^0(C, G)) \to H^0(C, \bigwedge^2(G)).$$

To see Proposition 2.2 (and hence Proposition 1.5) is optimal, let L be a line bundle with $\deg(L) > 2g$, and $E = (M_L)^* \otimes \Omega_C$. Repeating the argument above we see that $H^0(C, L) \otimes H^0(C, E) \to H^0(C, E \otimes L)$ fails to surject. A simple calculation shows $\deg(L) = 2g + \operatorname{rank}(E)(2g - \mu(E))$. Furthermore, if $g \geq 3$ and C is not hyperelliptic, we can show every line subbundle $A \subset M_L$ has $\deg(A) \leq -3$, so every line bundle quotient $E \to B$ has $\deg(B) \geq 2g + 1$, and hence $\mathscr{O}_{\mathbf{P}(E)}(1)$ is very ample.

3. Regularity of vector bundles over ruled varieties

In order to extend Theorem 2.1 to ruled varieties with a curve as base, we present some basic facts about the regularity of a vector bundle V over $X = \mathbf{P}(E)$. Most of these results are familiar, and all are elementary, but are presented here for the reader's convenience.

Definition. Let E be a vector bundle over a scheme Y, and $X = \mathbf{P}(E) \xrightarrow{\pi} Y$. If $\mathscr{F}$ is a coherent sheaf over X, we say $\mathscr{F}$ is f π-regular when

$$R^i\pi_*(\mathscr{F}(f - i)) = 0 \quad \text{for } i > 0.$$

3.1 Lemma. *Let Y be a projective variety and E a rank n vector bundle over Y. If $X = \mathbf{P}(E) \xrightarrow{\pi} Y$ and V is a vector bundle over X, then the following are equivalent:*

(1) *V is v π-regular.*
(2) *For all $y \in Y$, $h^i(X_y, V_y(v - i)) = 0$ for $i > 0$.*
(3) *For $i \geq 0$, there exist vector bundles $\mathscr{T}_i(V)$ on Y and a canonical resolution of V:*

$$\cdots \to \pi^*\mathscr{T}_2(V)(-v - 2) \to \pi^*\mathscr{T}_1(V)(-v - 1) \to \pi^*\mathscr{T}_0(V) \to V \to 0.$$

Proof of Lemma 3.1. To see (1) implies (2) simply use the fact that V is v π-regular and the Base Change Theorem. Since $R^n\pi_*(V(v-n+1)) = 0$ (because the fiber has dimension $n-1$), the Base Change Theorem implies that if $R^{n-1}\pi_*(V(v-n+1)) = 0$ then $h^{n-1}(X_y, V_y(v-n+1)) = 0$ for all $y \in Y$. Assuming V is v π-regular and hence $(v+1)$ π-regular, $R^{n-1}\pi_*(V(v-n+2)) = 0$. So by the Base Change Theorem, if $R^{n-2}\pi_*(V(v-n+2)) = 0$, then $h^{n-2}(X_y, V(v-n+2)) = 0$ for all $y \in Y$. And so on.

That (2) implies (1) is obvious.

To show (1) implies (3) see [34, §8] or [10, V, §2].

To see (3) implies (1), break the long exact sequence of (3) into short exact sequences, and use the fact that $R^i\pi_*(\pi^*F(-v))$ is v π-regular for any vector bundle F over Y by the projection formula. Then simply calculate $R^i\pi_*(V(v-i))$ for $i > 0$.

3.2 Lemma. *Let E be a vector bundle over a projective variety Y and $X = \mathbf{P}(E) \xrightarrow{\pi} Y$. Suppose further that V and W are v and w π-regular vector bundles over X respectively.*

(1) *$V \otimes W$ is $(v+w)$ π-regular.*

(2) *If $v \leq 1$ then $h^i(X, V) = h^i(Y, \pi_*V)$.*

(3) *If $v \leq 0$ and $\widetilde{V} = \pi^*(\pi_*V)$, there is an exact sequence of vector bundles on X*

$$0 \to K_V \to \widetilde{V} \to V \to 0,$$

where K_V is 1 π-regular.

(4) *If v and $w \leq 0$, there is a surjective map*

$$\pi_*(V) \otimes \pi_*(W) \to \pi_*(V \otimes W).$$

(5) *If $v \leq 0$ and π_*V is generated by global sections, then V is also generated by global sections.*

(6) *If $v \leq 0$, and π_*V is generated by global sections, then M_V is 1 π-regular.*

(7) (char$(k) = 0$, *or* $g \leq 1$) *If $Y = C$ is a smooth curve, and v and $w \leq 0$, then*

$$\mu^-(\pi_*V \otimes W) \geq \mu^-(\pi_*V) + \mu^-(\pi_*W).$$

Proof of 3.2. (1) Use characterization (3) of Lemma 3.1. As in [14, proof of Lemma 1] from the resolutions of V and W we construct the necessary resolution of $V \otimes W$.

(2) This follows trivially form the definition of π-regular.

(3) Again we use characterization (3) of Lemma 3.1. Since $\mathcal{F}_0 = \widetilde{V}$ (see [34, §8] or [10, V, §2]), from the resolution of V we get the necessary resolution of K_V.

(4) Consider the exact sequence

$$0 \to K_V \otimes W \to \widetilde{V} \otimes W \to V \otimes W \to 0.$$

Since K_V is 1 π-regular by (3), $K_V \otimes W$ is 1 π-regular by (1), and hence the pushdown sequence is exact. The result follows.

(5) Assume $v \leq 0$ and $\pi_* V$ is generated by global sections, and hence $\widetilde{V}$ is generated by global sections. We have the following diagram:

$$\begin{array}{ccc} H^0(X, \widetilde{V}) \otimes \mathcal{O}_X & = & H^0(X, V) \otimes \mathcal{O}_X \\ \downarrow \beta & & \downarrow \gamma \\ \widetilde{V} & \xrightarrow{\alpha} & V \end{array}$$

Since α and β are surjective, so is γ.

(6) Consider the sequence

$$0 \to (M_V)_y \to H^0(X, V) \otimes \mathcal{O}_{X_y} \to V_y \to 0.$$

We claim that $H^0(X, V) = H^0(X_y, V_y) \oplus U_y$ for all $y \in Y$, where the map $H^0(X, V) \otimes \mathcal{O}_{X_y} \to V_y$ is the usual evaluation map, and the map $U_y \otimes \mathcal{O}_y \to V_y$ is the zero map. The claim implies $(M_V)_y = M_{V_y} \oplus (U_y \otimes \mathcal{O}_{X_y})$, and hence $(M_V)_y$ is 1-regular over X_y, implying M_V is 1 π-regular. The claim follows if the restriction map $H^0(X, V) \to H^0(X_y, V_y)$ surjects. However, since $H^0(X, V) = H^0(C, \pi_* V)$, this is equivalent to saying $\pi_* V$ is generated by global sections, which we assumed.

(7) From part (4) we have

$$\begin{aligned} \mu^-(\pi_* V \otimes W) &\geq \mu^-(\pi_* V \otimes \pi_* W) \\ &\geq \mu^-(\pi_* V) + \mu^-(\pi_* W) \quad \text{(by Lemma 2.5(2))}. \end{aligned}$$

4. Surjectivity of the multiplication map on ruled varieties

In this section we study vector bundle multiplication on ruled varieties with a curve as base. Our main result is the following generalization of Theorem 2.1:

4.1 Theorem (char$(k) = 0$, *or* $g \leq 1$). *Let* E *be a rank-*n *vector bundle over* C, *and let* $X = \mathbf{P}(E)$. *If* V *and* W *are* 0 π*-regular vector bundles over* X, *and* B_1 *and* B_2 *are* (-1) π*-regular line bundles over* X *such that*

(1) $\mu^-(\pi_* V) + \mu^-(\pi_* B_1) \geq 2g$ *and*

(2) $\mu^-(\pi_* W) + \mu^-(\pi_* B_2) > 2g$,

then $\tau: H^0(X, V \otimes B_1) \otimes H^0(X, W \otimes B_2) \to H^0(X, V \otimes B_1 \otimes W \otimes B_2)$ *surjects.*

Our approach to vector bundle multiplication on $\mathbf{P}(E)$ is to consider $\pi_* M_V \otimes W$. It is not generally the same as $\pi_* M_V \otimes \pi_* W$. It is, however, related. Our main goal is to calculate $\mu^-(\pi_* M_V \otimes W)$, and for this we need a definition.

Definition. Let E be a rank n vector bundle over C, and let $X = \mathbf{P}(E)$. For V and W vector bundles over X, set

$$\nu(V, W) = \min\left\{\mu^-\left(R^i\pi_*(V(-i-1)) \otimes \bigwedge^{i+2} E\right)\right\} + \mu^-(\pi_* W(-1)),$$

where $\mu^-(0) = +\infty$ and the minimum is taken over $0 \le i \le n-2$.

4.2 Proposition (char$(k) = 0$, *or* $g \le 1$). *Let* E *be a rank-n vector bundle over* C, *and let* $X = \mathbf{P}(E)$. *Suppose* V *is a* 0 π*-regular vector bundle over* X, *and* W *is a* (-1) π*-regular vector bundle over* X. *If* $\pi_* V$ *is generated by global sections, then*

$$\mu^-(\pi_*(M_V \otimes W)) \ge \min\{\mu^-(M_{\pi_* V}) + \mu^-(\pi_* W), \nu(V, W)\}.$$

We shall see in the proof that the above holds with $\nu(V, W)$ replaced by $\mu^-(\pi_* K_V \otimes W)$. However, we find it easier to calculate $\nu(V, W)$, and have the following relationship between the two.

4.3 Lemma (char$(k) = 0$, *or* $g \le 1$). *Under the hypothesis of Proposition* 4.2,

$$\mu^-(\pi_*(K_V \otimes W)) \ge \nu(V, W).$$

Proof of Lemma 4.3. Since $K_V \otimes W = K_V(1) \otimes W(-1)$, the (-1) π-regularity of W and Lemma 3.2(7) imply

$$\mu^-(\pi_* K_V \otimes W) \ge \mu^-(\pi_* K_V(1)) + \mu^-(\pi_* W(-1)).$$

Since $\pi_*(K_V(1))$ is the kernel of the map

$$\pi_* V \otimes E \to \pi_*(V(1)) \to 0,$$

and $\pi_*(K_{\mathscr{O}_X(1)} \otimes V)$ is the kernel of the same map, we have $\pi_*(K_V(1)) = \pi_*(K_{\mathscr{O}_X(1)} \otimes V)$. So we need only show

$$\mu^-(\pi_*(K_{\mathscr{O}_X(1)} \otimes V)) + \mu^-(\pi_* W(-1)) \ge \mu^-\left(\bigoplus_{i=0}^{n-2}\left(R^i\pi_*(V(-i-1)) \oplus \bigwedge^{i+2} E\right)\right) + \mu^-(\pi_* W(-1)),$$

or

$$\text{(4.4)} \qquad \mu^-(\pi_*(K_{\mathcal{O}_X(1)} \otimes V)) \geq \mu^-\left(\bigoplus_{i=0}^{n-2}\left(R^i\pi_*(V(-i-1)) \otimes \bigwedge^{i+2} E\right)\right).$$

So consider the relative Koszul complex:

$$0 \to \bigwedge^n \pi^* E(-n+1) \to \cdots \to \bigwedge^2 \pi^* E(-1) \to \pi^* E \to \mathcal{O}_X(1) \to 0.$$

Tensoring the long exact sequence by V, we get a short exact sequence:

$$0 \to K'_{\mathcal{O}_X(1)} \otimes V \to \bigwedge^2 \pi^* E \otimes V(-1) \to K_{\mathcal{O}_X(1)} \otimes V \to 0.$$

Now take the pushdown:

$$\bigwedge^2 E \otimes \pi_*(V(-1)) \to \pi_* K_{\mathcal{O}_X(1)} \otimes V \to R^1\pi_*(K'_{\mathcal{O}_X(1)} \otimes V) \to 0.$$

By Lemma 2.5,

$$\begin{aligned}&\mu^-(\pi_*(K_{\mathcal{O}_X(1)} \otimes V)) \\ &\qquad \geq \min\left\{\mu^-\left(\bigwedge^2 E \otimes \pi_*(V(-1))\right), \mu^-(R^1\pi_*(K'_{\mathcal{O}_X(1)} \otimes V))\right\}.\end{aligned}$$

Now we need to calculate $\mu^-(R^1\pi_*(K'_{\mathcal{O}_X(1)} \otimes V))$, so notice we have an exact sequence

$$0 \to K''_{\mathcal{O}_X(1)} \otimes V \to \bigwedge^3 \pi^* E \otimes V(-2) \to K'_{\mathcal{O}_X(1)} \otimes V \to 0.$$

Taking the pushdown, and arguing as before, we see

$$\begin{aligned}&\mu^-(R^1\pi_*(K'_{\mathcal{O}_X(1)} \otimes V)) \\ &\qquad \geq \min\left\{\mu^-\left(\bigwedge^3 E \otimes R^1\pi_*(V(-2))\right), \mu^-(R^2\pi_*(K''_{\mathcal{O}_X(1)} \otimes V))\right\}.\end{aligned}$$

Clearly (4.4) follows by repeating the argument.

Proof of Proposition 4.2. On X we have the following diagram

$$\begin{array}{ccccccccc} & & & & & & 0 & & \\ & & & & & & \downarrow & & \\ & & 0 & & & & K_V \otimes W & & \\ & & \downarrow & & & & \downarrow & & \\ 0 & \to & \pi^*(M_{\pi_* V}) \otimes W & \to & H^0(X, \tilde{V}) \otimes W & \to & \tilde{V} \otimes W & \to & 0 \\ & & \downarrow & & \| & & \downarrow & & \\ 0 & \to & M_V \otimes W & \to & H^0(X, V) \otimes W & \to & V \otimes W & \to & 0 \\ & & \downarrow & & & & \downarrow & & \\ & & K_V \otimes W & & & & 0 & & \\ & & \downarrow & & & & & & \\ & & 0 & & & & & & \end{array}$$

The horizontal exact sequences are the usual multiplication maps, and the right-hand vertical sequence is that of Lemma 3.2(4) tensored by W. Pushing down the left-hand vertical sequence give us the following:

$$0 \to M_{\pi_* V} \otimes \pi_* W \to \pi_* M_V \otimes W \to \pi_* K_V \otimes W \to 0.$$

By Lemma 2.4 we get

$$\begin{aligned} \mu^-(\pi_* M_V \otimes W) &\geq \min\{\mu^-(M_{\pi_* V} \otimes \pi_* W), \mu^-(\pi_* K_V \otimes W)\} \\ &\geq \min\{\mu^-(M_{\pi_* V}) + \mu^-(\pi_* W), \mu^-(\pi_* K_V \otimes W)\} \\ &\qquad\qquad \text{(by Lemma 2.5)} \\ &\geq \min\{\mu^-(M_{\pi_* V}) + \mu^-(\pi_* W), \nu(V, W)\} \\ &\qquad\qquad \text{(by Lemma 4.3).} \end{aligned}$$

4.5 Lemma (char$(k) = 0$, *or* $g \leq 1$). *Let* E *be a rank-n vector bundle over* C, *and let* $X = \mathbf{P}(E)$. *If* B *is a* (-1) π*-regular line bundle, then*

$$\mu^-(E) + \mu^-(\pi_* B(-1)) = \mu^-(\pi_* B).$$

Proof of Lemma 4.5. For some positive integer k and some line bundle L over C, $\pi_* B = S^k(E) \otimes L$ and $\pi_* B(-1) = S^{k-1}(E) \otimes L$. Now simply apply Lemma 2.5, and calculate μ^- for $\pi_* B$ and $\pi_* B(-1)$ to see that Lemma 4.5 holds.

4.6 Lemma (char$(k) = 0$, *or* $g \leq 1$). *Let* E *be a vector bundle over* C, *and let* $X = \mathbf{P}(E)$. *If* V *and* W *are* 0 π*-regular vector bundles over* X, *and* B_1 *and* B_2 *are* (-1) π*-regular line bundles over* X, *then*

$$\nu(V \otimes B_1, W \otimes B_2) \geq \mu^-(\pi_* V) + \mu^-(\pi_* B_1) + \mu^-(\pi_* W) + \mu^-(\pi_* B_2).$$

Proof of Lemma 4.6. Since $V \otimes B_1$ and $W \otimes B_2$ are (-1) π-regular, the higher direct images vanish, and hence

$$\nu(V \otimes B_1, W \otimes B_2) = \mu^-\left(\pi_*(V \otimes B_1(-1)) \otimes \bigwedge^2 E\right) + \mu^-(\pi_* W \otimes B_2(-1)).$$

Since $B_1(-1)$ and $B_2(-1)$ are 0 π-regular, Lemma 3.2(7) and Lemma 2.5(2) imply

$$\begin{aligned}\nu(V \otimes B_1, W \otimes B_1) &\geq \mu^-(\pi_* V) + \mu^-(\pi_* B_1(-1)) + \mu^2(\bigwedge^2 E) \\ &\quad + \mu^-(\pi_* W) + \mu^-(\pi_* B_2(-1)) \\ &\geq \mu^-(\pi_* V) + \mu^-(\pi_I W) + \mu^-(\pi_* B_1(-1)) \\ &\quad + \mu^-(\pi_* B_2(-1)) + 2\mu^-(E) \quad \text{(by Lemma 2.5(8))} \\ &\geq \mu^-(\pi_* V) + \mu^-(\pi_* W) + \mu^-(\pi_* B_1) \\ &\quad + \mu^-(\pi_* B_2) \quad \text{(by Lemma 4.5).}\end{aligned}$$

Proof of Theorem 4.1. By Lemma 3.2(7), $\mu^-(\pi_* V \otimes B_1) \geq \mu^-(\pi_* V) + \mu^-(\pi_* B_1) \geq 2g$, so $\pi_*(V \otimes B_1)$ is generated by global sections and $\mu^-(M_{\pi_*(V \otimes B_1)}) \geq -2$ by Corollary 1.3. Lemma 3.2(7) also implies $\mu^-(\pi_* W \otimes B_2) > 2g$. By Proposition 4.2,

$$\begin{aligned}&\mu^-(\pi_* M_{V \otimes B_1} \otimes W \otimes B_2) \\ &\qquad \geq \min\{\mu^-(M_{\pi_*(V \otimes B_1)}) + \mu^-(\pi_* W \otimes B_2), \nu(V \otimes B_1, W \otimes B_2)\}.\end{aligned}$$

From the above, $\mu^-(M_{\pi_*(V \otimes B_1)}) + \mu^-(\pi_* W \otimes B_2) > -2 + 2g = 2g - 2$, and by Lemma 4.6 we have

$$\begin{aligned}\nu(V \otimes B_1, W \otimes B_2) &\geq \mu^-(\pi_* V) + \mu^-(\pi_* W) + \mu^-(\pi_* B_1) + \mu^-(\pi_* B_2) \\ &> 4g > 2g - 2.\end{aligned}$$

So we conclude

$$\mu^-(\pi_* M_{V \otimes B_1} \otimes W \otimes B_2) > 2g - 2.$$

Lemma 1.12(2) implies $h^1(C, \pi_*(M_{V \otimes B_1} \otimes W \otimes B_2)) = 0$, and hence $h^1(X, M_{V \otimes B_1} \otimes W \otimes B_2) = 0$ by Lemma 3.2(2).

4.7 Examples. We can actually show surjectivity of the map

$$\tau: H^0(X, V \otimes B_1) \otimes H^0(X, W \otimes B_2) \to H^0(X, V \otimes B_1 \otimes W \otimes B_2)$$

whenever V and W are 0 π-regular, $\mu^-(\pi_* V) \geq 2g$, $\mu^-(\pi_* W) > 2g$, and $\mu^-(\pi_* K_V \otimes W) > 2g - 2$. We will show all three conditions are necessary,. but we do not know if they are independent. To see if 0-regularity is required, consider a bundle of the form $\mathcal{O}_X(1) \oplus \mathcal{O}_X(-1)$, and to see if the condition on μ^- of the pushdown of V and W is required, let $E = L \oplus L$, where $\deg(L) = 2g$ and L is not very ample. Then $\mu^-(\pi_*\mathcal{O}_X(1)) = 2g$ but $\mathcal{O}_X(1)$ is not very ample and hence not normally generated. (For a slightly more interesting example use the bundle G defined in Examples 2.6.) Finally, suppose V and W satisfy the conditions on regularity and on μ^- of the pushdown of V and W, but $\mu^-(\pi_*K_V \otimes W) \leq 2g-2$. Then for some semistable vector bundle Q over C with $\mu(Q) = \mu^-(\pi_*K_V \otimes W) \leq 2g-2$ we have a map

$$\pi_*K_V \otimes W \to Q \to 0.$$

Now let $W' = W \otimes \pi^*(Q^* \otimes \Omega_C)$. By the projection formula we see that V and W' satisfy the conditions on regularity and μ^- of the pushdown of V and W', but $\mu^-(\pi_*K_V \otimes W') = 2g-2$ and there is a map

$$\pi_*K_V \otimes W' \to \Omega_C \to 0.$$

Hence $h^1(C, \pi_*K_V \otimes W') \neq 0$, which implies that $h^1(X, M_V \otimes W') \neq 0$, and so the multiplication map fails to surject. Unfortunately, we do not know if any such V and W exist. This leaves the following problem.

4.8 Problem. Given 0 π-regular vector bundles V and W over X, can we find a nice lower bound on $\mu^-(\pi_*K_V \otimes W)$ in terms of $\mu^-(\pi_*V)$ and $\mu^-(\pi_*W)$? In particular, is it true that $\mu^-(\pi_*V) \geq 2g$ and $\mu^-(\pi_*W) > 2g$ imply $\mu^-(\pi_*K_V \otimes W) > 2g-2$?

A positive answer to the last part of Problem 4.8 implies that if V and W are 0 π-regular vector bundles over X, with $\mu^-(\pi_*V) \geq 2g$ and $\mu^-(\pi_*V) > 2g$, we have surjectivity of the multiplication map

$$\tau: H^0(X, V) \otimes H^0(X, W) \to H^0(X, V \otimes W).$$

This theorem—if true—would be optimal.

5. Syzygies of ruled varieties

We now use Theorem 4.1 to study syzygies of ruled varieties with a curve as base. In particular, we prove Theorems 2A and 2B, which follow from the more general Theorems 5.1A and 5.1B by Miyaoka's calculation of the ample cone of a ruled variety with a curve as base. We do not establish optimality, but we do show by example that $K_X \otimes A^{\otimes(n+1+p)}$ (A ample) does not imply property N_p for $p = 0$ or 1.

5.1A Theorem ($\mathrm{char}(k) = 0$, *or* $g \leq 1$). *Let* E *be a vector bundle over a smooth projective curve* C *of genus* g, *and let* $X = \mathbf{P}(E)$. *If* B *is a* (-1) π*-regular line bundle over* X *with* $\mu^-(\pi_* B) > 2g$, *then* B *is normally generated.*

5.1B Theorem ($\mathrm{char}(k) = 0$, *or* $\mathrm{char}(k) = q > p + 1$ *and* $g \leq 1$). *Fix an integer* $p > 0$. *Let* E *be a vector bundle over a smooth projective curve* C *of genus* g, *and let* $X = \mathbf{P}(E)$. *If* B *is a* $(-p-1)$ π*-regular line bundle over* X *with* $\mu^-(\pi_* B) \geq 2g + 2p$, *then* B *has property* N_p.

5.2 Remark. Actually, we need only assume B is $(-p)$ π-regular, but to keep the proofs simple, we prove the result as stated. Since a line bundle which determines a Koszul homogeneous coordinate ring satisfies property N_1, the forthcoming Theorem 6.1 implies Theorem 5.1B holds if $p = 1$ and B is (-1) π-regular. Our proof of Theorem 6.1 should indicate how Theorem 5.1B can be proved under the weaker hypothesis that B is $(-p)$ π-regular.

Before proving Theorem 5.1A and B, we show these results imply Theorems 2A and 2B. To do so, we need some lemmas, the first two of which are essentially due to Miyaoka [29, Theorem 5.1].

5.3 Lemma (*Miyaoka*; $\mathrm{char}(k) = 0$, *or* $g \leq 1$). *Let* E *be a vector bundle over* C, *and let* $X = \mathbf{P}(E)$. *If* A *is a line bundle over* X, *then* A *is ample iff*

(1) $\mathrm{rank}(\pi_* A) \geq \mathrm{rank}(E)$, *and*

(2) $\mu^-(\pi_* A) > 0$.

Alternatively, we have:

5.4 Lemma (*Miyaoka*; $\mathrm{char}(k) = 0$, *or* $g \leq 1$). *Let* E *be a vector bundle over* C, *and let* $X = \mathbf{P}(E)$. *If* $D_0 = \mathcal{O}_{\mathbf{P}(E)}(1)$ *and* f *is a fiber, then* $A \equiv aD_0 + bf$ *is ample iff*

(1) $a > 0$, *and*

(2) $b > a\mu^-(E)$.

Proof of Lemma 5.3. Condition (1) is equivalent to saying that $A|f$ is ample. Hence it is necessary, since ample bundles remain ample when restricted to a subvariety. So to prove the lemma, we assume condition (1) holds, and show that A is ample iff (2) holds. If E is semistable, this is a theorem of Miyaoka [29, Theorem 5.1]. We show this implies Lemma 5.3 for an arbitrary E.

Assume $\mu^-(\pi_* A) > 0$. By Lemma 2.5, this implies $\mu^-(\pi_* A^{\otimes k}) > 2g$ for $k \gg 0$. We claim this implies $A^{\otimes k}$ is very ample. If we let

$F = \pi_*(A^{\otimes k})$ and $Y = \mathbf{P}(F)$, then $X \subseteq Y$ by a fiberwise k-uple map. Set $B_Y = \mathscr{O}_{\mathbf{P}(F)}(1)$. We have $A^{\otimes k} = B_Y|X$. The lower bound on $\mu^-(F)$ implies B_Y is very ample by Lemma 1.12(4), and this shows $A^{\otimes k}$ is very ample.

For the only if direction, we restrict to a sub-projective bundle corresponding to a quotient bundle of minimal slope. Since a bundle of minimal slope is semistable we can apply Miyaoka's theorem. Assume A is ample, and let $E \to Q \to 0$ be a quotient bundle of minimal slope. We have $W = \mathbf{P}(Q) \subseteq \mathbf{P}(E) = X$. If $\pi_*(A) = S^k(E) \otimes L$, then $\pi_*(A|W) = S^k(Q) \otimes L$. If A is ample, so is $A|W$. Since Q is semistable, Miyaoka's Theorem implies $\mu(S^k(Q) \otimes L) > 0$. So we have $\mu^-(\pi_* A) = \mu^-(S^k(E) \otimes L) = \mu(S^k(Q) \otimes L) > 0$, and we are done.

Proof of Lemma 5.4. Simply note 5.3(1) is equivalent to 5.4(1), and 5.3(2) is equivalent to 5.4(2).

5.5 Lemma. *Let E be a rank-n vector bundle over a curve C of genus g, and let $X = \mathbf{P}(E)$. Fix an integer $t \geq n$. If A_i is ample for $1 \leq i \leq t$ and $B = K_X \otimes A_1 \otimes \cdots \otimes A_t$, then*

(1) *B is $(n-t)$ π-regular, and*

(2) *$\mu^-(\pi_* B) \geq 2g - 2 + t/n$.*

5.6 Remark. We actually prove that if E has Harder-Narisimhan filtration $\Sigma: 0 = E_0 \subset E_1 \subseteq \cdots \subseteq E_s = E$ and E_s/E_{s-1} has rank r and $\deg d$, then

$$\mu^-(\pi_* B) \geq 2g - 2 + \frac{t(r, d)}{r} + n(\mu(E) - \mu^-(E)).$$

One can show this result is optimal. So Lemma 5.5 is optimal iff E is stable, and $(d, n) = 1$.

Proof of Lemma 5.5. Assuming the notation of Lemma 5.4, $K_X \equiv -nD_0 + (2g - 2 + \deg(E))f$. Thus if $A_i = \mathscr{O}_X(Z_i)$ and $B = \mathscr{O}_X(D)$, then for some $a_i > 0$ and $b_i > a_i \mu^-(E)$,

$$D = K_X + \sum_1^t Z_i \equiv \left(-n + \sum_1^t a_i\right) D_0 + \left(2g - 2 + \deg(E) + \sum_1^t b_i\right) f.$$

Clearly $-n + \sum_1^t a_i \geq (t - n)$, and since $\mathscr{O}_X(D_0)$ is (-1) π-regular, B is $(n-t)$ π-regular. Now observe that $\deg(E) = n\mu(E) \geq n\mu^-(E)$, and $b_i = -a_i\mu^-(E) + (p_i/q_i)$ for some positive integers p_i and q_i with $q_i \leq n$. In fact, using the notation of Remark 5.6, we have $0 < q_i \leq r/(r, d) \leq n$.

Let $\alpha = -n + \sum_1^t a_i$, and $\beta = 2g - 2 + \deg(E) + \sum_1^t b_i$. Then for some line bundle L over C of $\deg = \beta$, we have $\pi_* B = S^\alpha(E) \otimes L$. Now we use Lemma 2.5 to calculate $\mu^-(\pi_* B)$:

$$\begin{aligned}\mu^-(\pi_* \mathscr{O}_X(D)) &= \alpha\mu^-(E) + \beta \\ &= \left(-n + \sum_1^t a_i\right)\mu^-(E) + 2g - 2 + n\mu(E) \\ &\quad + \sum_1^t \left(-a_i\mu^-(E) + \frac{p_i}{q_i}\right) \\ &= n(\mu(E) - \mu^-(E)) + 2g - 2 + \sum_1^t \frac{p_i}{q_i} \\ &\geq 2g - 2 + n(\mu(E) - \mu^-(E)) + t\left(\frac{(r, d)}{r}\right) \\ &\geq 2g - 2 + \frac{t}{n}. \quad \text{q.e.d.}\end{aligned}$$

Now we show first that Theorems 2A and 2B follow from Lemma 5.5 and Theorems 5.1A and 5.1B, and then we prove Theorems 5.1A and 5.1B.

Proof of Theorems 2A *and* 2B. Assume B satisfies the hypothesis of Theorem 2A. Plugging $t = 2n + 1$ into Lemma 5.5, we see that B is $(-n-1)$ π-regular and $\mu^-(\pi_* B) > 2g$. So by Theorem 5.1A, B is normally generated.

Now assume B satisfies the hypothesis of Theorem 2B. Plugging $t = 2n + 2np$ into Lemma 5.5, we see that B is $(-n-2p)$ π-regular and $\mu^-(\pi_* B) \geq 2g + 2p$. So by Theorem 5.1B, B has property N_p.

Proof of Theorem 5.1A. If we set $F = \pi_* B$, then by Lemma 1.12(4), $\mathscr{O}_{\mathbf{P}(F)}(1)$ is very ample. Hence B is very ample because $X \hookrightarrow \mathbf{P}(F)$ by a fiberwise d-uple map and $B = \mathscr{O}_{\mathbf{P}(F)}(1)|X$. So we need only show surjectivity for all n of the multiplication map

$$S^n(H^0(X, B)) \to H^0(X, B^{\otimes n}).$$

We only show surjectivity of

$$H^0(X, B) \otimes H^0(X, B) \to H^0(X, B^{\otimes 2}),$$

since the remainder of the proof is similar. We use Theorem 4.1. Let $V = W = \mathscr{O}_X$ and $B_1 = B_2 = B$. One easily checks that V, W, B_1, and B_2 satisfy the hypothesis of Theorem 4.1 and we are done.

Proof of Theorem 5.1B. Fix a positive integer p. By induction we may assume property N_{p-1} holds, and so we need only show

$$h^1(X, \bigwedge^{p+1}(M_B) \otimes B^{\otimes n}) = 0$$

for $n \geq 1$ ([17], [25]). Since $\operatorname{char}(k) = 0$ or $\operatorname{char}(k) = q > p+1$, we know $\bigwedge^{p+1} M_B$ is a direct summand of the tensor product $T^{(p+1)} M_B$, so we may instead show that for $n \geq 1$,

$$h^1(X, T^{(p+1)}(M_B) \otimes B^{\otimes n}) = 0.$$

We do only the case $n = 1$, since the remaining cases are similar. Set $U = T^{(p+1)}(M_B) \otimes B$. Since M_B is 1 π-regular by Lemma 3.2(6), and B is $(-p-1)$ π-regular by hypothesis, we add the regularities by Lemma 3.2(1) and see that U is a 0 π-regular vector bundle. So by Lemma 3.2(2) we need only show $h^1(C, \pi_* U) = 0$, and hence by Lemma 1.12(2) it suffices to show

$$\mu^-(\pi_* U) > 2g - 2. \tag{5.7}$$

To establish (5.7), notice $U = T^{(p+1)}(M_B(1)) \otimes B(-p-1)$. Therefore, since $M_B(1)$ and $B(-p-1)$ are 0 π-regular, repeated application of Lemma 3.2(7) shows that

$$\mu^-(\pi_* U) \geq (p+1)\mu^-(\pi_* M_B(1)) + \mu^-(\pi_* B(-p-1)),$$

so we need only show $(p+1)\mu^-(\pi_* M_B(1)) + \mu^-(\pi_* B(-p-1)) > 2g - 2$. First we calculate $\mu^-(\pi_* M_B(1))$. Setting $V = W = \mathcal{O}_X$, $B_1 = B$, and $B_2 = \mathcal{O}_X(1)$, and applying Lemma 4.6, we see that $\nu(B, \mathcal{O}_X(1)) \geq \mu^-(\pi_* B) + \mu^-(E)$. So Proposition 4.2 implies

$$\mu^-(\pi_* M_B(1)) \geq \min\{\mu^-(M_{\pi_* B}) + \mu^-(E), \nu(B, \mathcal{O}_X(1))\},$$

$$\begin{aligned} \mu^-(\pi_* M_B(1)) &\geq \min\{\mu^-(M_{\pi_* B}) + \mu^-(E), \mu^-(\pi_* B) + \mu^-(E)\} \\ &> -2 + \mu^-(E). \end{aligned}$$

The final inequality follows from Corollary 1.3, since $\mu^-(\pi_* B) > 2g$. We need only show

$$(p+1)(-2) + (p+1)\mu^-(E) + \mu^-(\pi_* B(-p-1)) \geq 2g - 2.$$

By repeated use of Lemma 4.5, this is equivalent to showing

$$(p+1)(-2) + \mu^-(\pi_* B) \geq 2g - 2,$$

which is equivalent to our assumption $\mu^-(\pi_* B) \geq 2g + 2p$.

5.8 Remark. Using Remark 5.6 to refine our proof of Theorem 2A shows that if E is semistable of rank n and degree d, with $(n, d) \neq 1$, the conclusion of 2A holds when the hypothesis holds with $(2n+1)$ replaced by $(n+1)$. Similarly, when E is unstable we can weaken the hypothesis, and there are of course analogous refinements of Theorem 2B. In 5.9A we show that the best possible result we can hope for is that on a variety X of dimension n, $K_X \otimes A^{\otimes(n+2)}$ (A ample) is normally generated. On a ruled surface, this fails iff there exists a rank-2 deg 1 stable vector bundle E, a positive integer m, and a line bundle L with $\mu(S^{2m}(E) \otimes L) = 2g$ such that surjectivity fails for the natural multiplication map

$$S^2 H^0(C, S^{2m}(E) \otimes L) \to H^0(C, S^{4m}(E) \otimes L^{\otimes 2}).$$

5.9A Example. We show by example that a line bundle $B = K_X \otimes A^{\otimes(n+1)}$ (A ample) need not be normally generated for any n, even if B is very ample. Let E be a stable deg 1 vector bundle with a line bundle quotient L with $\deg(L) = 1$. Observe $A = \mathscr{O}_{\mathbf{P}(E)}(1)$ is ample, and if $B = K_X \otimes A^{\otimes(n+1)}$, then $B = A \otimes \pi^*(\Omega_C \otimes \det(E))$. Hence $\pi_*(B)$ has a line bundle quotient $\Omega_C \otimes L \otimes \det(E)$, so there is a section $\Gamma \subseteq X$ with $\deg(\Gamma) = 2g$. If L and $\det(E)$ are chosen so that $L \otimes \det(E)$ is effective, B is not even very ample. In any case, if Γ is hyperelliptic, then B is not normally generated because there are no projectively normal hyperelliptic curves Γ with $\deg(\Gamma) = 2g$, and it is easily seen this implies B is not normally generated, as $h^1(C, \mathscr{I}_{\Gamma/X} \otimes B^{\otimes 2}) = 0$.

5.9B Example. Now we show a line bundle $B = K_X \otimes A^{\otimes(n+2)}$ (A ample) need not be normally presented for any n. Let X, E, and A be as in 5.9A, and consider $B = K_X \otimes A^{\otimes(n+2)}$. By a variation of the argument used in 5.9A, if we choose L and $\det(E)$ properly, X has a section Γ which B embeds with a trisecant line contained not in any fiber, and therefore not in X. Hence X is not defined by quadrics even set-theoretically!

6. Koszul rings

Now we use Theorem 4.1 to study Koszul homogeneous coordinate rings on ruled varieties with a curve as base. In particular, we prove Theorem 3 and a more general result, Theorem 6.1. That Theorem 3 follows from Theorem 6.1 is immediate from Lemma 5.5.

6.1 Theorem (char$(k) = 0$, *or* $g \leq 1$, *or* $X = C$). *Let* E *be a vector bundle over a curve* C *of genus* g, *and let* $X = \mathbf{P}(E)$. *If* B *is a* (-1)

π-regular line bundle over X, and $\mu^{-}(\pi_* B) \geq 2g+2$, then the homogeneous coordinate ring determined by B is Koszul.

Before proving Theorem 6.1, we show it implies Theorem 3.

Proof of Theorem 3. Lemma 5.5 shows the hypothesis of Theorem 6.1 implies that of Theorem 3. Plugging $t = 4n$ into Lemma 5.5, we get B is $(-3n)$ π-regular, and since $n \geq 1$, B is (-1) π-regular. Furthermore, $\mu^{-}(\pi_* B) \geq 2g - 2 + 4n/n = 2g+2$.

Proof of Theorem 6.1. We start with a reinterpretation due to Kempf [22] of a ring being Koszul. Let $S = \bigoplus S_n = \bigoplus H^0(X, B^{\otimes n})$. $\operatorname{Tor}_1^S(k, k)$ is purely of degree 1 iff S is generated by $S_1 = H^0(X, B)$. Now suppose S is indeed generated by S_1, and define R_n^1 by

$$0 \to R_n^1 \to S_1 \otimes S_n \to S_{(n+1)} \to 0.$$

$\operatorname{Tor}_2^A(k, k)$ is purely of degree 2 iff $R^1 = \bigoplus R_n^1$ is generated by R_1^1. Similarly, if we assume R^{m-1} is generated by $R_1^{(m-1)}$ and define R_n^m by

$$0 \to R_n^m \to R_1^{(m-1)} \otimes S_n \to R_{(n+1)}^{(m-1)} \to 0,$$

then $\operatorname{Tor}_{(m)}^S(k, k)$ is purely of degree m iff $R^m = \bigoplus R_n^m$ is generated by R_1^m.

Following an idea of Lazarsfeld, we rephrase this in terms of global sections of vector bundles. For R^1 we have

$$0 \to M_B \otimes B^{\otimes n} \to H^0(X, B) \otimes B^{\otimes n} \to B^{\otimes(n+1)} \to 0.$$

Taking global sections, we see $R^1 = \bigoplus H^0(X, M_B \otimes B^{\otimes n})$. Assuming R^1 is generated by R_1^1, we have

$$0 \to M_{M_B \otimes B} \otimes B^{\otimes n} \to H^0(X, M_B \otimes B) \otimes B^{\otimes n} \to M_B \otimes B^{\otimes(n+1)} \to 0.$$

So we see $R^2 = \bigoplus H^0(X, M_{(M_B \otimes B)} \otimes B^{\otimes n})$. Now the pattern is clear. Start by setting $M_n^0 = B^{\otimes n}$. Given a vector bundle $M_1^{(m-1)}$ which is generated by global sections, set

$$M_n^m = M_{M_1^{(m-1)}} \otimes B^{\otimes n}.$$

We get a sequence

$$0 \to M_n^m \to H^0(X, M_1^{(m-1)}) \otimes B^{\otimes n} \to M_{(n+1)}^{(m-1)} \to 0.$$

Taking global sections, we see $R^m = \bigoplus H^0(X, M_n^m)$, so showing R_1^m generates R^m is equivalent to showing surjectivity for $n \geq 1$ of the multiplication map

$$\tau: H^0(X, M_1^m) \otimes H^0(X, B^{\otimes n}) \to H^0(X, M_1^m \otimes B^{\otimes n}) = H^0(X, M_{(n+1)}^m). \tag{6.2}$$

Hence, the homogeneous coordinate ring is Koszul iff (6.2) surjects for $m, n \geq 0$.

Making the additional assumption that B is in fact (-2) π-regular, we now show (6.2) surjects. To do so, we use Theorem 4.1, and do only the case $n = 1$ as the remaining cases are similar. Set $V = M_0^m(1)$, $B_1 = B(-1)$, $W = \mathcal{O}_X$, and $B_2 = B$. We need to show these bundles satisfy the hypothesis of Theorem 4.1. Toward this end we have:

6.3 Claim (char$(k) = 0$, *or* $g \leq 1$, *or* $X = C$). Let E be a rank-n vector bundle over C, and let $X = \mathbf{P}(E)$. If B is a (-2) π-regular line bundle over X and $\mu^-(\pi_* B) \geq 2g + 2$, then for all m, $M_0^m(1)$ is in fact defined, is 0 π-regular, and $\mu^-(\pi_* M_0^m(1)) \geq -2 + \mu^-(E)$.

Now we assume Claim 6.3, and show the hypothesis of Theorem 4.1 is satisfied given our additional assumption that B is (-2) π-regular. By assumption $B = B_2$ is (-2) π-regular, and hence (-1) π-regular. Furthermore, $\mu^-(\pi_* B) \geq 2g + 2 > 2g$. Since $B_1 = B(-1)$ and B is (-2) π-regular, B_1 is (-1) π-regular. V is 0 π-regular by Claim 6.3, and therefore

$$\mu^-(\pi_* V) + \mu^-(\pi_* B_1) \geq -2 + \mu^-(E) + \mu^-(\pi_* B_1) \quad \text{(by Claim 6.3)},$$
$$\mu^-(\pi_* V) + \mu^-(\pi_* B_1) \geq -2 + \mu^-(E) + 2g + 2 - \mu^-(E) \quad \text{(by Lemma 4.6)}.$$

So 6.3 implies the hypothesis of Theorem 4.1 holds, and hence 6.2 surjects. (If $X = C$, $g \geq 2$, and char$(k) \neq 0$, Theorem 2.1 and Remark 2.3 imply (6.2) surjects.)

To finish the case where B is in fact (-2) π-regular, we need only prove Claim 6.3.

Proof of Claim 6.3. We use induction on m. Since $M_0^0 = \mathcal{O}_X$ and hence $M_0^0(1) = \mathcal{O}_X(1)$, the claim is obvious for $m = 0$. So we assume the claim is known for $(m-1)$. First we need to show M_n^m is defined. To do so we calculate the π-regularity and μ^- of the pushdown of $M_n^{(m-1)}$. By our inductive hypothesis, $M_0^{(m-1)}(1)$ is 0 π-regular, and hence by (-2) π-regularity of B we see that $M_1^{(m-1)} = M_0^{(m-1)}(1) \otimes B(-1)$ is (-1)

π-regular, and hence 0 π-regular.

$$\begin{aligned}\mu^-(\pi_* M_1^{(m-1)}) &= \mu^-(\pi_* M_0^{(m-1)}(1) \otimes B(-1)) \\ &\geq \mu^-(\pi_* M_0^{(m-1)}(1)) + \mu^-(\pi_* B(-1)) \quad \text{(by Lemma 3.2(7))} \\ &\geq -2 + \mu^-(E) + \mu^-(\pi_* B(-1)) \quad \text{(by inductive hypothesis)} \\ &\geq -2 + \mu^-(E) + \mu^-(\pi_* B) - \mu^-(E) \quad \text{(by Lemma 4.6)} \\ &\geq -2 + \mu^-(\pi_* B) \geq 2g.\end{aligned}$$

So by Lemma 1.12(3), $\pi_* M_1^{(m-1)}$ is generated by global sections. By Lemma 3.2(6), $M_0^m = M_{M_0^{(m-1)}}$ is 1 π-regular, and hence $M_0^m(1)$ is 0 π-regular. Furthermore, by Proposition 4.2,

$$\mu^-(\pi_* M_0^m(1)) \geq \min\{\mu^-(M_{\pi_* M_1^{(m-1)}}) + \mu^-(E), \nu(M_1^{(m-1)}, \mathscr{O}_X(1))\}.$$

Since $\mu^-(M_1^{(m-1)}) \geq 2g$, we have $\mu^-(M_{\pi_* M_1^{(m-1)}}) \geq -2$ by Corollary 1.3. So it is enough to show that $\nu(M_1^{(m-1)}, \mathscr{O}_X(1)) \geq -2 + \mu^-(E)$. Since $M_1^{(m-1)}$ is (-1) π-regular, $R^i \pi_*(M_1^{(m-1)}(-i-1)) = 0$ for $i > 0$, so we need only show $\mu^-(\pi_* M_1^{(m-1)}(-1) \bigotimes \bigwedge^2(E)) \geq -2 + \mu^-(E)$:

$$\begin{aligned}&\mu^-\left(\pi_* M_1^{m-1}(-1) \otimes \bigwedge^2 E\right) \\ &\quad = \mu^-\left(\pi_*(M_0^{m-1} \otimes B(-1)) \otimes \bigwedge^2 E\right) \\ &\quad = \mu^-\left(\pi_*(M_0^{m-1}(1) \otimes B(-2)) \otimes \bigwedge^2 E\right) \\ &\quad = \mu^-(\pi_*(M_0^{m-1}(1) \otimes B(-2))) + \mu^-\left(\bigwedge^2 E\right) \quad \text{(by Lemma 2.5(2))} \\ &\quad \geq \mu^-(\pi_* M_0^{m-1}(1)) + \mu^-(\pi_* B(-2)) + \mu^-\left(\bigwedge^2 E\right) \\ &\qquad\qquad \text{(by Lemma 3.2(7))} \\ &\quad \geq -2 + \mu^-(E) + \mu^-(\pi_* B(-2)) + \mu^-\left(\bigwedge^2 E\right) \\ &\qquad\qquad \text{(by inductive hypothesis)} \\ &\quad = -2 + \mu^-(E) + \mu^-(\pi_* B) - 2\mu^-(E) + \mu^-\left(\bigwedge^2 E\right) \quad \text{(by Lemma 4.5)}\end{aligned}$$

$$\geq -2 + \mu^-(E) + \mu^-(\pi_* B) - 2\mu^-(E) + 2\mu^-(E) \quad \text{(by Lemma 2.5(8))}$$
$$\geq -2 + \mu^-(E) + 2g + 2 > -2 + \mu^-(E),$$

which proves the claim. q.e.d.

Returning to the proof of Theorem 6.1, we may now assume B is (-1) π-regular, but not (-2) π-regular. Since $\mu^-(E)$ does not enter into the hypothesis of Theorem 6.1, by replacing E by $E \otimes L$ for a well-chosen line bundle L, we may assume $B = \mathscr{O}_X(1)$ and $\mu^-(E) \geq 2g + 2$. We claim it is enough to prove the following lemma.

6.4 Lemma (char$(k) = 0$, *or* $g \leq 1$). *Let E be a rank-n vector bundle over C, and let $X = \mathbf{P}(E)$. Assume further that $\mu^-(E) \geq 2g + 2$. Let V be a vector bundle over X which satisfies the following conditions:*

(1) *V is 0 π-regular.*
(2) $\mu^-(\pi_* V) \geq 2g$.
(3) $\nu(V, \mathscr{O}_X(1)) \geq 2g$.

If $U = M_V(1)$, then (1), (2), *and* (3) *hold with U substituted for V.*

Before proving Lemma 6.4, we show it implies (6.2) surjects if $B = \mathscr{O}_X(1)$. So assume the lemma, and set $V = B = \mathscr{O}_X(1)$. Clearly V satisfies (1) and (2). As for (3), we have $R^i\pi_*(\mathscr{O}_X(1 - 1 - i)) = 0$ for $i > 0$, so

$$\nu(\mathscr{O}_X(1), \mathscr{O}_X(1)) = \mu^-\left(\bigwedge^2 E\right) \geq 2\mu^-(E) \quad \text{(by Lemma 2.5(8))}$$
$$\geq 4g + 4 \geq 2g,$$

and hence V satisfies (3). This implies $M_1^1 = M_{\mathscr{O}_X(1)}(1)$ satisfies the conclusion, and hence the hypothesis of Lemma 6.4. By induction M_1^m satisfies the hypothesis of Lemma 6.4 for $m \geq 0$. 0 π-regularity shows $h^1(X, M_1^m) = h^1(C, \pi_* M_1^m)$ by Lemma 3.2(2). Since $\mu^-(\pi_* M_1^m) \geq 2g$. Lemma 1.12(2) yields $h^1(C, \pi_* M_1^m) = 0$, and (6.2) surjects for $n = 1$. The case $n \geq 2$ being similar, from Lemma 6.4 implies Theorem 6.1. We now prove the lemma.

Proof of Lemma 6.4. First we show that (1) holds for U. By (2), $\mu^-(\pi_* V) < 2g - 1$, so Lemma 1.12(3) implies $\pi_* V$ is generated by global sections. Since V is 0 π-regular by (1), from Lemma 3.2(6) it follows that M_V is 1 π-regular, so that $U = M_V(1)$ is 0 π-regular. Hence (1) holds for U.

Now we show that (2) holds for U. By Proposition 4.2,

$$\mu^-(\pi_* U) = \mu^-(\pi_* M_V(1)) \geq \min\{\mu^-(M_{\pi_* V}) + \mu^-(E), \nu(V, \mathcal{O}_X(1))\}.$$

By (3) we have $\nu(V, \mathcal{O}_X(1)) \geq 2g$, so we need only check that $\mu^-(M_{\pi_* V}) \times \mu^-(E) \geq 2g$. Since $\mu^-(\pi_* V) \geq 2g$, Corollary 1.3 implies $\mu^-(M_{\pi_* V}) \geq -2$, so $\mu^-(M_{\pi_* V}) + \mu^-(E) \geq -2 + 2g + 2 = 2g$, and (2) holds for U.

We finish by showing U satisfies (3). To do this, we use the sequence

$$0 \to U \to H^0(X, V) \otimes \mathcal{O}_X(1) \to V(1) \to 0.$$

We will bound $\nu(U, \mathcal{O}_X(1))$ by bounding $\mu^-(R^i\pi_*(U(-i-1)) \otimes \bigwedge^{i+2} E)$ for $0 \leq i \leq n-2$. We begin with $i = 0$. We need to show $\mu^-(\pi_* U(-1) \otimes \bigwedge^2 E) \geq 2g$. Since $\pi_* U(-1) = \pi_* M_V = M_{\pi_* V}$, it suffices to prove $\mu^-(M_{\pi_* V} \otimes \bigwedge^2 E) \geq 2g$. But we have

$$\begin{aligned}\mu^-\left(M_{\pi_* V} \otimes \bigwedge^2 E\right) &= \mu^-(M_{\pi_* V}) + \mu^-\left(\bigwedge^2 E\right) \quad \text{(by Lemma 2.5(2))}\\ &\geq \mu^-(M_{\pi_* V}) + 2\mu^-(E) \quad \text{(by Lemma 2.5(8))}\\ &\geq -2 + 2\mu^-(E) \quad \text{(by Corollary 1.3)}\\ &\geq -2 + 4g + 2 > 2g.\end{aligned}$$

So we need only do the cases where $i > 0$. Since $R^i\pi_*(\mathcal{O}_X(-i)) = 0$, we have a sequence

$$R^{i-1}\pi_*(V(-i)) \to R^i\pi_*(M_V(-i)) \to 0.$$

So we see

$$\begin{aligned}\mu^-\left(R^i\pi_*(M_V(-i)) \otimes \bigwedge^{i+2} E\right) &\geq \mu^-\left(R^{i-1}\pi_*(V(-i)) \otimes \bigwedge^{i+2} E\right)\\ &\qquad \text{(by Lemma 2.4)}\\ &\geq \mu^-\left(R^{i-1}\pi_*(V(-i)) \otimes \bigwedge^{i+1} E \otimes E\right)\end{aligned}$$

$$\geq \mu^{-}\left(R^{i-1}\pi_*(V(-i))\otimes \bigwedge^{i+1} E\right)\mu^{-}(E)$$

(by Lemma 2.5(2))

$$\geq \nu(V, \mathscr{O}_X(1)) + \mu^{-}(E) \geq 2g.$$

The second inequality follows from the surjective map $\bigwedge^{i+1} E \otimes E \to \bigwedge^{i+2} E$ and Lemma 2.4. This proves Lemma 6.4, and hence Theorem 6.1.

6.5 Problem. If $X = C$, can we use the geometry of the curve and its embedding to lower the necessary bound which forces the homogeneous coordinate ring to be Koszul? For example, it is known that on a nonhyperelliptic curve of genus $g \geq 4$, a line bundle of $\deg = 2g + 1$ which embeds C with no trisecant lines is normally presented. Is the homogeneous coordinate ring Koszul? Is it at least true that for a general curve of large genus a general line bundle of $\deg = 2g + 1$ determines a wonderful homogeneous coordinate ring? More generally, is it true that the geometry of a line bundle being normally presented is the same as that of the homogeneous coordinate ring being Koszul. If so, by analogy with a conjecture of Green and Lazarsfeld [16], we expect if L is a very ample line bundle which embeds C without a trisecant line, the homogeneous coordinate ring determined by L is Koszul provided

$$\deg(L) \geq 2g + 2 - 2h^1(C, L) - \mathrm{Cliff}(C).$$

One important case is known. Finkelberg and Vishik have shown that if $\mathrm{Cliff}(C) \geq 2$, then the canonical ring is Koszul [8].

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