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FULL HOLOMORPHIC MAPS FROM THE RIEMANN SPHERE TO COMPLEX PROJECTIVE SPACES

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Abstract

In this paper we study the topology of $\mathrm{FRat}_k(\mathbf{CP}^n)$, the space of full, based, holomorphic maps of degree k from S^2 to $\mathbf{CP}^n$, that is, those based holomorphic maps whose image does not lie in any proper projective subspace of $\mathbf{CP}^n$. We prove that the natural inclusion of $\mathrm{FRat}_k(\mathbf{CP}^n)$ into $\mathrm{Rat}_k(\mathbf{CP}^n)$, the space of all based holomorphic maps, is a homotopy equivalence through dimension $2(k-n)$. We compute $H_*(\mathrm{FRat}_k(\mathbf{CP}^2))$ completely and obtain partial results for $H_*(\mathrm{FRat}_k(\mathbf{CP}^n))$ for $n > 2$.

1. Introduction

Much attention has been devoted recently to studying the topology of spaces of holomorphic maps from the Riemann sphere to various complex manifolds. Segal [14] obtained the first results here, for maps into complex projective space, by showing that the natural inclusion into the space of continuous maps, given by forgetting the complex structure, is a homotopy equivalence through a range. More recently, F. Cohen, R. Cohen, Mann, and Milgram [3] determined the stable homotopy type of the space of based holomorphic maps to complex projective spaces, and Mann and Milgram [9], [10] continued the program to include maps to complex Grassmannians and to complex flag manifolds respectively. In each case these spaces are the minimal sets of an energy functional defined on the space of smooth maps. Except in the case of self-maps of S^2 these functionals have nonminimal critical sets, these are the harmonic maps, and it is natural to attempt to extend the program of study mentioned above to include harmonic maps.

A map of S^2 into complex projective space is said to be full if its image does not lie in any proper, projective subspace. In [5] Din and Zakrzewski describe how to construct harmonic maps from S^2 to $\mathbf{CP}^n$

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out of full holomorphic maps. Eells and Wood [7] give a rigorous proof that the full harmonic maps are in one-to-one correspondence with the set of pairs, (f, r), where $f: S^2 \to \mathbf{CP}^n$ is full holomorphic and r is an integer satisfying $0 \leq r \leq n$. The construction is straightforward. Let z be a complex coordinate in a neighborhood $U \subset S^2$. We can lift f, locally, to a map $\tilde{f}: U \to \mathbf{C}^{n+1}$. If f is full, then $\tilde{f}$ and its first n derivatives will be linearly independent, except perhaps at a finite number of singular points. For each $z \in \mathbf{C}$ the component of $\partial^r \tilde{f}$ perpendicular to $\mathrm{span}\{\tilde{f}, \partial \tilde{f}, \cdots, \partial^{r-1} \tilde{f}\}$ defines a line in $\mathbf{C}^{n+1}$. This line does not depend on the choice of the lift $\tilde{f}$, and we can extend over the singular points to obtain a map $\phi: S^2 \to \mathbf{CP}^n$. From the fact that f is holomorphic and that the standard projection $\mathbf{C}^{n\pm 1}\backslash\{0\} \to \mathbf{CP}^n$ is a Riemannian submersion it follows that ϕ is harmonic. The interested reader is encouraged to see [7] and [8] for more details.

The original motivation for this paper was a desire to study the topology of the space of harmonic maps. The correspondence described above relates harmonic maps to full holomorphic maps, and so we were led to study the space of full maps. In this paper we give the results of that study. It is important to note, however, that the construction described above is discontinuous, so more work is needed before the results in this paper can be applied to the problem of harmonic maps.

To give some idea of the difficulties that remain we consider the simplest case of maps to $\mathbf{CP}^2$. In this case the correspondence reduces to a one-to-one correspondence between full holomorphic maps and nonminimal harmonic maps, that is, harmonic maps that are neither holomorphic nor antiholomorphic. Given a full holomorphic map $f: S^2 \to \mathbf{CP}^2$ there will be a finite number of points $x \in S^2$ such that, for any local lift $\tilde{f}$, $\tilde{f}$ and $\partial \tilde{f}$ will not be linearly independent at x. There is a multiplicity associated to this singularity and the number of such points, counting multiplicities, is called the ramification index of f. If we fix the degree of f, then the degree and energy of the corresponding harmonic map ϕ are strictly decreasing functions of the ramification index. However it is easy to construct a sequence of full holomorphic maps $\{f_i\}$ which are unramified but converge to a function which is ramified. Since harmonic maps with different degrees or energies cannot lie in the same connected component, the corresponding sequence of harmonic maps $\{\phi_i\}$ does not converge to the correct value, and the correspondence is discontinuous. However if we restrict to holomorphic maps of a fixed ramification and degree, the correspondence is potentially much more useful and we hope to report on this in a future paper.

We denote by $\mathrm{Rat}_k(\mathbf{CP}^n)$ the space of based holomorphic maps $f: S^2 \to \mathbf{CP}^n$ of topological degree k. Let $\mathrm{FRat}_k(\mathbf{CP}^n)$ be the subspace of $\mathrm{Rat}_k(\mathbf{CP}^n)$ consisting of those maps whose image is not contained in any proper projective subspace of $\mathbf{CP}^n$. Any $f \in \mathrm{Rat}_k(\mathbf{CP}^n)$ can be given, in homogeneous coordinates, by

$$f(z) = [p_0(z), \cdots, p_n(z)], \tag{1.1}$$

where the p_i are polynomials in the complex coordinate $z \in \mathbf{C}$ satisfying certain conditions. f will be in $\mathrm{FRat}_k(\mathbf{CP}^n)$ exactly when the p_i are linearly independent in $\mathbf{C}[z]$.

The space $\mathrm{Rat}_k(\mathbf{CP}^n)$ has been studied extensively in [14], [2] and [3]. In [3] the stable homotopy types of thee spaces are described. The program has been continued in [9], [10] and [1] extending the previous results to holomorphic maps from S^2 to complex Grassmannians and flag manifolds, and finally to the moduli space of instantons. In this paper we show that the topology of $\mathrm{FRat}_k(\mathbf{CP}^n)$ is closely related to that of $\mathrm{Rat}_k(\mathbf{CP}^n)$. However, the techniques used to study $\mathrm{FRat}_k(\mathbf{CP}^n)$ are closer in spirit to those used to study maps to flag manifolds and instantons than those for $\mathrm{Rat}_k(\mathbf{CP}^n)$. In all of these cases the holomorphic mapping space under consideration includes naturally into a loop space, and this inclusion is a homotopy equivalence through a range. The structure of the results obtained in this paper for $\mathrm{FRat}_k(\mathbf{CP}^n)$ more closely resembles the cases for maps to flags and for instantons in that the inclusion has a nontrivial kernel in homology above the range of stability. This is stated more precisely in Theorems D and E and in Corollary F.

The main technical tool that we make use of in this paper is a stratification

$$\mathrm{Rat}_k(\mathbf{CP}^n) = \bigcup_{l=0}^{n} X_l.$$

The sets X_l are composed of all the maps for which the complex dimension of the span of the polynomial factors in (1.1) is $n - l + 1$. Thus, $X_0 = \mathrm{FRat}_k(\mathbf{CP}^n)$. The description (1.1) gives $\mathrm{Rat}_k(\mathbf{CP}^n)$ the structure of a complex manifold, and the sets X_l are submanifolds. There is a Leray spectral sequence associated to this stratification with E^1 term given as a direct sum of the homology groups of the sets X_l and converging to the homology of $\mathrm{Rat}_k(\mathbf{CP}^n)$. Since the homology of $\mathrm{FRat}_k(\mathbf{CP}^n)$ corresponds to the fiber terms $E^l_{0,t}$, we can, in principle, use this spectral sequence to study $\mathrm{FRat}_k(\mathbf{CP}^n)$. More precisely there is a surjection

$$H_t(\mathrm{FRat}_k(\mathbf{CP}^n)) \cong E^1_{0,t} \to E^\infty_{0,t},$$

which corresponds to the homomorphism

$$i(k, n)_*: H_t(\mathrm{FRat}_k(\mathbf{CP}^n)) \to H_t(\mathrm{Rat}_k(\mathbf{CP}^n))$$

induced in homology by the inclusion

$$(1.2) \qquad i(k, n): \mathrm{FRat}_k(\mathbf{CP}^n) \hookrightarrow \mathrm{Rat}_k(\mathbf{CP}^n).$$

In low dimensions we use this spectral sequence to show that $i(k, n)_*$ is an isomorphism and we prove

Theorem A. *$i(k, n)$ is a homotopy equivalence through dimension $2(k-n)$.*

In [14] Segal gave a homotopy inclusion

$$(1.3) \qquad i_k: \mathrm{Rat}_k(\mathbf{CP}^n) \to \mathrm{Rat}_{k+1}(\mathbf{CP}^n),$$

and showed that it was a homotopy equivalence through dimension $q = k(2n-1)$. Thus, if we use i_k to form the direct limit $\mathrm{Rat}_\infty(\mathbf{CP}^n) = \lim_{k\to\infty} \mathrm{Rat}_k(\mathbf{CP}^n)$, we have the stable result that $\mathrm{Rat}_\infty(\mathbf{CP}^n) \simeq \Omega^2_0\mathbf{CP}^n \simeq \Omega^2 S^{2n+1}$. In this paper we show that i_k restricts to a map

$$\bar{i}_k: \mathrm{FRat}_k(\mathbf{CP}^n) \hookrightarrow \mathrm{FRat}_{k+1}(\mathbf{CP}^n),$$

so that the following diagram commutes:

$$(1.4) \qquad \begin{array}{ccc} \mathrm{FRat}_k(\mathbf{CP}^n) & \xrightarrow[\bar{i}_k]{} & \mathrm{FRat}_{k+1}(\mathbf{CP}^n) \\ \Big\downarrow{\scriptstyle i(k,n)} & & \Big\downarrow{\scriptstyle i(k+1,n)} \\ \mathrm{Rat}_k(\mathbf{CP}^n) & \xrightarrow[i_k]{} & \mathrm{Rat}_{k+1}(\mathbf{CP}^n). \end{array}$$

Thus, if we form the direct limit $\mathrm{FRat}_\infty(\mathbf{CP}^n) = \lim_{k\to\infty} \mathrm{FRat}_k(\mathbf{CP}^n)$, then the following stable result follows from Theorem A.

Theorem B. *The map $\bar{i}_k$ is a homotopy equivalence through dimension $2(k-n)$ and thus*

$$\mathrm{FRat}_\infty(\mathbf{CP}^n) \simeq \Omega^2_0\mathbf{CP}^n \simeq \Omega^2 S^{2n+1}.$$

In many ways the most interesting part of the homology of $\mathrm{FRat}_k(\mathbf{CP}^n)$ is the classes in dimensions above the range of stability. To study these homology classes we construct homology classes in $H_*(\mathrm{FRat}_k(\mathbf{CP}^n))$ whose image in $H_*(\mathrm{Rat}_k(\mathbf{CP}^n))$ we can identify. To be more precise we need to recall some facts about the spaces $\mathrm{Rat}_k(\mathbf{CP}^n)$.

In [3] it was shown that there is a stable homotopy equivalence

$$\text{Rat}_k(\mathbf{CP}^n) \simeq_s \bigvee_{j=1}^{k} D_j(S^{2n-1}), \tag{1.5}$$

where $D_j(X)$ is the twisted smash product $F(\mathbf{C}, j)_+ \wedge_{\Sigma_j} X^{(j)}$. The homology of these spaces is well understood, and (1.5) implies that there is an isomorphism

$$\tilde{H}_*(\text{Rat}_k(\mathbf{CP}^n); A) \simeq \coprod_{j=1}^{k} \tilde{H}_*(D_j(S^{2n-1}); A) \tag{1.6}$$

for any coefficient module A. This makes $H_*(\text{Rat}_k(\mathbf{CP}^n))$ a bigraded module. Graded first by homological dimension and second by the index j in (1.6).

F. Cohen, in an unpublished manuscript, described an algebraic structure on the homology of $\text{Rat}_k(\mathbf{CP}^n)$ which corresponds to the Dyer-Lashoff operations on $H_*(\Omega^2\mathbf{CP}^2)$. In particular he constructed a homology product compatible with the loop sum product on $H_*(\Omega^2\mathbf{CP}^n)$. This gives $H_*(\text{Rat}_k(\mathbf{CP}^n))$ the structure of a bigraded algebra. Boyer and Mann [2] used this structure to obtain a lower bound for the image of the map in homology induced by the forgetful map $\text{Rat}_k(\mathbf{CP}^n) \to \Omega^2\mathbf{CP}^n$. In this paper we show that this homology product, which we denote by $*$, can be modified to give $H_*(\text{FRat}_k(\mathbf{CP}^n))$ the structure of a module over $H_*(\text{Rat}_k(\mathbf{CP}^n))$. $\text{Rat}_1(\mathbf{CP}^n) \simeq S^{2n-1}$ and there is a homomorphism

$$H_*(\text{Rat}_k(\mathbf{CP}^n)) \to H_*(\text{Rat}_{k+1}(\mathbf{CP}^n))$$

given by mapping any homology class x to the product $e_1 * x$ where $e_1 \in H_{2n-1}(\text{Rat}_1(\mathbf{CP}^n))$ is a generator. Using this structure we prove

Theorem C. *For any coefficient module A the image of $i(k, n)_*$ contains the submodule*

$$\coprod_{j=1}^{k-n} \tilde{H}_*(D_j(S^{2n-1}); A) \subset H_*(\text{Rat}_k(\mathbf{CP}^n); A).$$

Furthermore the image of $i(k, n)_*$ contains the submodule

$$ne_1 * \coprod_{j=1}^{k-n} \tilde{H}_*(D_j(S^{2n-1}); A) \subset \coprod_{j=1}^{k-n+1} \tilde{H}_*(D_j(S^{2n-1}); A) \subset H_*(\text{Rat}_k(\mathbf{CP}^n); A).$$

Finally, we return to the stratification described above for the special case $n = 2$. In this case there are only two strata, and the Leray spectral sequence reduces to the long exact sequence in homology of the pair $(\mathrm{Rat}_k(\mathbf{CP}^2), \mathrm{FRat}_k(\mathbf{CP}^2))$. The following theorem completely describes the image of $i(k, 2)_*$.

Theorem D. *For any coefficient module A the image of $i(k, 2)_*$ is*

$$2e_1 * \widetilde{H}_*(D_{k-2}(S^3); A) \oplus \coprod_{j=1}^{k-2} \widetilde{H}_*(D_j(S^3); A) \subset H_*(\mathrm{Rat}_k(\mathbf{CP}^2)).$$

Next we calculate the kernel of $i(k, 2)_*$. For coefficients in the field $\mathbf{Z}/p$ the kernel can be described explicitly in terms of the homology of $\mathrm{Rat}_k(\mathbf{CP}^1)$ using the homology product described above. There is a homomorphism

$$\theta : H_*(\mathrm{Rat}_k(\mathbf{CP}^1); \mathbf{Z}/p) \to H_*(\mathrm{Rat}_k(\mathbf{CP}^1); \mathbf{Z}/p),$$

which is given by $\theta(x) = (k - l)e_1 * x$ for x a homogeneous element in the summand

$$\widetilde{H}_*(D_l(S^1); \mathbf{Z}/p) \subset H_*(\mathrm{Rat}_k(\mathbf{CP}^1); \mathbf{Z}/p).$$

Now we consider two submodules of $H_*(\mathrm{Rat}_k(\mathbf{CP}^1); \mathbf{Z}/p)$. Let $C(l)_*$ be the cokernel of θ in $\widetilde{H}_*(D_l(S^1); \mathbf{Z}/p)$ and let $K(l)_*$ be the kernel of the restriction of θ to $\widetilde{H}_*(D_l(S^1); \mathbf{Z}/p)$. Then the kernel of $i(k, 2)_*$ is isomorphic to the direct sum of two graded modules, $\widetilde{C}_*$ and $\widetilde{K}_*$, defined as follows:

$$(1.7) \qquad \widetilde{C}_q = \coprod_{\substack{0 \le l < k \\ l \ne k-1}} C(l)_{q-2k+3}, \qquad \widetilde{K}_q = \coprod_{0 \le l \le k-1} K(l)_{q-2k+5}.$$

Thus, we have the following.

Theorem E. *$H_*(\mathrm{FRat}_k(\mathbf{CP}^2); \mathbf{Z}/p)$ is isomorphic to the direct sum of the image of $i(k, n)_*$ described in Theorem D and the modules $\widetilde{C}_*$ and $\widetilde{K}_*$.*

Over the rationals the situation is much simpler. $D_1(S^{2n-1}) \cong S^{2n-1}$, and $H_*(D_l(S^{2n-1}); \mathbf{Q})$ is trivial for $l > 1$. So we have

Corollary F. *With coefficients $\mathbf{Q}$,*

$$H_*(\mathrm{FRat}_k(\mathbf{CP}^2); \mathbf{Q}) \cong H_*(S^3 \times S^{2k-3}).$$

Theorem A and Corollary F show that $\mathrm{FRat}_k(\mathbf{CP}^2)$ has unstable rational homology; that is, for every k there are rational homology classes in $\mathrm{FRat}_k(\mathbf{CP}^2)$ that are in the kernel of $i(k, 2)_*$.

This paper is organized as follows: In §2 we recall the results of [3] to describe $H_*(\mathrm{Rat}_k(\mathbf{CP}^n))$. In §3 we describe the stratification of $\mathrm{Rat}_k(\mathbf{CP}^n)$ by rank and prove Theorem A. In §4 we define the homology product and prove Theorems B and C. Finally, in §5, we consider the case $n = 2$ and prove Theorems D and E except for the proof of two lemmas which we give in §6.

2. Preliminaries

Let $\mathrm{Rat}_k(\mathbf{CP}^n)$ be the space of holomorphic maps $f: S^2 \to \mathbf{CP}^n$ of degree k which satisfy $f(\infty) = [1, \cdots, 1]$. As stated in §1 any such map can be written uniquely, in homogeneous coordinates, in the form

$$f(z) = [p_0 0(z), \cdots, p_n(z)]. \tag{2.1}$$

This will define a map into $\mathbf{CP}^n$ as long as all $n+1$ polynomials have no common root. f will have degree k and satisfy the basing condition if the p_i are all monic and have degree k. Let $\mathrm{FRat}_k(\mathbf{CP}^n)$ be the subspace of all such maps with the additional condition that the polynomials p_i be linearly independent in $\mathbf{C}[z]$. Since the polynomials all have degree k, there are no full maps of degree less than n.

$\mathrm{Rat}_k(\mathbf{CP}^n)$ and $\mathrm{Frat}_k(\mathbf{CP}^n)$ inherit their topologies as subspaces of $\Omega^2\mathbf{CP}^n$. However there is a simpler way to describe the topology of these spaces. For any space X we can form the k-fold symmetric product

$$SP^k(X) = X^k/\Sigma_k,$$

where Σ_k is the symmetric group on k letters. If we restrict our attention to the configuration space $F(X, k) \subset X^k$ of ordered k-tuples of distinct points, then this is a free Σ_k-space, and the quotient $DP^k(x) = F(X, k)/\Sigma_k$ is called the deleted symmetric product. We will only be interested in the case where $X = C$. $SP^k(\mathbf{C})$ is homeomorphic to the space of monic, complex polynomials of degree k. The homeomorphism maps a polynomial to the unordered k-tuple of its roots. Using (2.1) we may identify $\mathrm{Rat}_k(\mathbf{CP}^n)$ and $\mathrm{FRat}_k(\mathbf{CP}^n)$ with open subsets of the $n+1$-fold product $SP^k(\mathbf{C}) \times \cdots \times SP^k(\mathbf{C})$.

We can stratify $SP^k(\mathbf{C})$ by the multiplicity of the points in each unordered k-tuple. The generic stratum, consisting of k-tuples of distinct points, is $DP^k(\mathbf{C})$. This induces a stratification of the product $SP^k(\mathbf{C}) \times \cdots \times SP^k(\mathbf{C})$ by crossing with the strata in the first factor and this, in turn, induces a stratification of $\mathrm{Rat}_k(\mathbf{CP}^n)$ and $\mathrm{FRat}_k(\mathbf{CP}^n)$. The generic stratum is composed of maps which have no repeated roots in the first polynomial factor. The generic strata play a very important role in analyzing these spaces.

Let $Y_0(k, n) \subset \mathrm{Rat}_k(\mathbf{CP}^n)$ be the generic stratum. In [3] it was proved that there is homotopy equivalence

$$Y_0 \simeq F(\mathbf{C}, k) \times_{\Sigma_k} (S^{2n-1})^k, \tag{2.2}$$

where Σ_k acts by permutation. The inclusion $Y_0(k, n) \hookrightarrow \mathrm{Rat}_k(\mathbf{CP}^n)$ induces a surjection in homology. In order to be more precise we need to recall the May-Milgram model for $\Omega^2\Sigma^2 X$.

Definition 2.3. Let X be a connected CW complex. Then define

$$J_2(X) = \coprod_{j=1}^{\infty} F(\mathbf{C}, j) \times_{\Sigma^j} X^j/\{\text{equivalence}\},$$

where the equivalence is given by

$$\langle z_1, \cdots, z_j; x_1, \cdots, x_j\rangle \sim \langle z_1, \cdots, \hat{z}_i, \cdots, z_j; x_1, \cdots, \hat{x}_i, \cdots, x_j\rangle$$

when $x_i = *$ the basepoint in X. We also define

$$J_2^k(X) = \coprod_{j=1}^{k} F(\mathbf{C}, j) \times_{\Sigma^j} X^j/\{\text{equivalence}\}.$$

These subspaces give a natural filtration of $J_2(X)$.

$J_2(X)$ is homotopy equivalent to $\Omega^2\Sigma^2 X$ [11], [12]. By looping the Hopf fibration we have that $\Omega^2_k\mathbf{CP}^n \simeq \Omega^2 S^{2n+1} \simeq \Omega^2\Sigma^2 S^{2n-1}$ so we can use $J_2(S^{2n-1})$ as a model for $\Omega^2_k\mathbf{CP}^n$. The following diagram commutes up to homotopy:

$$\begin{array}{ccccc} Y_0 & \longrightarrow & \mathrm{Rat}_k(\mathbf{CP}^n) & \longrightarrow & \Omega^2_k\mathbf{CP}^n \\ \downarrow \simeq & & & & \downarrow \simeq \\ F(\mathbf{C}, k) \times_{\Sigma^k} (S^{2n-1})^k & \longrightarrow & J_2^k(S^{2n-1}) & \longrightarrow & > J_2(S^{2n-1}). \end{array} \tag{2.4}$$

The image in $H_*(J_2(S^{2n-1}))$ of the lower map is well understood, and this diagram implies that $H_*(\mathrm{Rat}_k(\mathbf{CP}^n))$ is at least this large. The main result of [3] is that the homomorphism $H_*(\mathrm{Rat}_k(\mathbf{CP}^n)) \to H_*(\Omega^2_k\mathbf{CP}^n)$ induced by the natural inclusion is an injection. Thus, the inclusion $Y_0 \to \mathrm{Rat}_k(\mathbf{CP}^n)$ induces a surjection in homology. An important consequence of this is that we can find representatives for any homology class in $\mathrm{Rat}_k(\mathbf{CP}^n)$ in the homology of $F(\mathbf{C}, k) \times_{\Sigma_k} (S^{2n-1})^k$. $\mathrm{FRat}_k(\mathbf{CP}^n)$

also includes naturally in $\Omega^2\mathbf{CP}^n$ and this factors through $\mathrm{Rat}_k(\mathbf{CP}^n)$. As it turns out the corresponding homomorphism $H_*(\mathrm{FRat}_k(\mathbf{CP}^n)) \to H_*(\Omega^2_k\mathbf{CP}^n)$ has a nontrivial kernel, and this greatly complicates the calculations in this case.

Let us consider the simplest nontrivial case, $\mathrm{FRat}_n(\mathbf{CP}^n)$. In §3 we show that $\mathrm{FRat}_n(\mathbf{CP}^n)$ is homotopy equivalent to $\mathrm{U}(n)$, the unitary group. $H_*(\mathrm{U}(n))$ is an exterior algebra on odd-dimensional generators $u_1, \cdots, u_{2n-1}$. We will show that $i(n, n)_*$ maps u_{2n-1} onto n times the class $e_1 \in H_{2n-1}(\mathrm{Rat}_n(\mathbf{CP}^n))$. However $H_q(\mathrm{Rat}_n(\mathbf{CP}^n)) = 0$ for $0 < q < 2n-1$, so all the other generators must be mapped to zero. Thus, even in this first case $i(k, n)_*$ has a large kernel.

Returning to $J_2(S^{2n-1})$ we recall the following theorem of Snaith.

Theorem 2.5 [15]. *For any connected CW complex X the natural projection*

$$F(\mathbf{C}, k) \times_{\Sigma_k} X^k \to J_2^k(X) \to J_2^k(X)/J_2^{k-1}(X)$$

has a stable section, which induces a stable splitting

$$J_2(X) \simeq_s \bigvee_0^\infty (J_2^k(X)/J_2^{k-1}(X)),$$

and thus we may write

$$\tilde{H}_q(J_2(X)) \simeq \coprod_{k=1}^\infty \tilde{H}_q(J_2^k(X), J_2^{k-1}(X)). \tag{2.6}$$

There is a filtration-preserving product

$$J_2(X) \times J_2(X) \to J_2(X)$$

given on representatives by

(2.7)
$$\begin{aligned}(\langle z_1, \cdots, z_k; x_1, \cdots, x_k\rangle, \langle z_1', \cdots, z_l'; x_1', \cdots, x_l'\rangle) \\ \mapsto \langle \phi z_1, \cdots, \phi z_k, \phi' z_1', \cdots, \phi' z_l'; x_1, \cdots, x_k, x_1', \cdots, x_l'\rangle,\end{aligned}$$

where ϕ and ϕ' are homeomorphisms of $\mathbf{C}$ onto disjoint open disks $D, D' \subset \mathbf{C}$. This gives $H_*(J_2(X))$ the structure of a bigraded algebra. The bigrading is by q and k in (2.6). The calculation of the homology of $J_2(S^{2n-1})$ is given in [12]. At the prime 2

$$H_*(J_2(S^{2n-1}); \mathbf{Z}/2) = \mathbf{Z}/2[e_1, q_1, \cdots, q_i, \cdots], \tag{2.8}$$

a polynomial algebra. And at odd primes

(2.9)
$$H_*(J_2(S^{2n-1}); \mathbf{Z}/p) = E(e_1, q_1, \cdots, q_i, \cdots) \otimes \mathbf{Z}/p[f_1, \cdots, f_i, \cdots],$$

the tensor product of an exterior algebra and a polynomial algebra. The bidegrees of the generators are $\deg(e_1) = (2n-1, 1)$, $\deg(q_i) = (2np^i - 1, p)$ and $\deg(f_i) = (2np^i - 2, p)$.

We filter $H_*(J_2(S^{2n-1}))$ by the second grading degree by letting $F_l(H_*(J_2(S^{2n-1})))$ be the submodule of elements of second grading degree $j \leq l$. Then $H_*(\mathrm{Rat}_k(\mathbf{CP}^n))$ is mapped isomorphically onto $F_k(H_*(J_2(S^{2n-1})))$. This gives $H_*(\mathrm{Rat}_K(\mathbf{CP}^n))$ the structure of a graded algebra truncated by equating elements of second grading degree more than k to zero. Since $J_2^k(S^{2n-1})/J_2^{k-1}(S^{2n-1}) = D_k(S^{2n-1})$, using (2.6), we obtain the descriptions given in (1.6).

3. The co-rank stratification of $\mathrm{Rat}_k(\mathbf{CP}^n)$

In this section we construct a stratification of $\mathrm{Rat}_k(\mathbf{CP}^n)$ in which $\mathrm{FRat}_k(\mathbf{CP}^n)$ is the generic stratum, and we construct explicit geometric models for the lower strata in terms of the spaces of full maps into lower-dimensional projective spaces. By examining the Leray spectral sequence associated to this stratification we obtain information about the homomorphism $i(k, n)_*$. In the case $n = 2$, when there are only two strata, $H_*(\mathrm{FRat}_k(\mathbf{CP}^2))$ can be completely determined and this is done in §5.

The stratification that we construct for $\mathrm{Rat}_k(\mathbf{CP}^n)$ satisfies certain conditions which allow us to identify the E^1 term of the associated Leray spectral sequence. This same technique is used in [9], [10] and [1]. We give an exposition following [1].

Definition 3.1 [1]. Let M be a smooth manifold, and suppose there is a decomposition

$$M = \bigcup_{\alpha \in \mathscr{A}} S_\alpha,$$

where the sets S_α are pairwise disjoint. This will be called a L-stratification of M if the following properties are satisfied:

(L.1) The index set $\mathscr{A}$ is finite with a given well ordering $\leq$.

(L.2) The sets $F_\alpha = \bigcup_{\beta \leq \alpha} S_\beta$ are open and dense in M.

(L.3) S_α is a submanifold of F_α with orientable normal bundle.

Now, given an L-stratification of a manifold M, let (ν, ν_0) be the normal bundle pair of $S_\alpha \hookrightarrow F_\alpha$. By excision we have an isomorphism of relative homology groups, $H_q(F_\alpha, F_{\alpha-1}) \cong H_q(\nu, \nu_0)$. But, since ν is orientable, the Thom Isomorphism Theorem implies that $H_q(\nu, \nu_0) \cong H_{q-c}(S_\alpha)$ where c is the codimension of S_α in F_α or, equivalently, in M.

Associated to the filtration $\{F_\alpha\}$ we have the Leray spectral sequence with $E^1 = H_{s+t}(F_s, F_{s-1})$ converging to $H_{s+t}(M)$. Thus, if we can compute the homology groups $H_q(S_\alpha)$, then the Leray spectral sequence will give us information about $H_q(M)$.

The stratification we are concerned with here catalogues the degree to which the polynomial factors of a map fail to be linearly independent. Recall that for $[p_0, \cdots, p_n]$ to be an element of $\mathrm{FRat}_k(\mathbf{CP}^n)$ we require that the polynomials $p_0, \cdots, p_n$ be linearly independent. Since the p_i are all monic and of degree k, this is equivalent to requiring that the polynomials $p_1 - p_0, \cdots, p_n - p_0$ be linearly independent. We can define an embedding

$$(3.2) \qquad e(k, n): \mathrm{Rat}_k(\mathbf{CP}^n) \hookrightarrow \mathbf{C}^k \times \mathrm{Mat}_{n,k}(\mathbf{C})$$

as an open submanifold as follows. Let

$$p_j(z) = z^k + a_{1,j} z^{k-1} + \cdots + a_{k,j}.$$

Then $e(k, n)([p_0, \cdots, p_n]) = (v, A)$ where the components of v are the coefficients $a_{i,0}$ and the entries in A are $a_{i,j} - a_{i,0}$. Thus, the columns of A are given by the coefficients of the polynomials $p_i - p_0$. The map will be full if and only if A has rank n. Also note that we cannot have $A = 0$, the zero matrix, since this would imply that all the p_i were equal to p_0 and so the polynomials would not give a map into $\mathbf{CP}^n$. We will stratify $\mathrm{Rat}_k(\mathbf{CP}^n)$ by the rank of these matrices. First let us consider the corresponding stratification of $\mathrm{Mat}_{n,k}(\mathbf{C})$.

Definition 3.3. Given $0 \le l \le n$ we define subsets $\mathscr{M}_l$ of $\mathrm{Mat}_{n,k}(\mathbf{C})$ as follows:

$$\mathscr{M}_l = \{A \in \mathrm{Mat}_{n,k}(\mathbf{C}) | A \text{ has rank } l\}.$$

Given any matrix $A \in \mathrm{Mat}_{n,k}(\mathbf{C})$ or rank l we can always factor A as CB where B is $l \times n$ and C is $k \times l$ and both have (full) rank l. The choice of B and C is unique up to a $\mathrm{GL}(\mathbf{C}^l)$-action. The space of full rank $l \times n$ matrices is the Steifel manifold of l-frames in $\mathbf{C}^n$. For the purposes of this paper we do not require that the l-frames be orthonormal. It is not difficult to see that $\mathscr{M}_l$ is a complex submanifold of $\mathrm{Mat}_{n,k}(\mathbf{C})$ of complex dimension $l(n - l + k)$. From the discussion above we have that

$$M_l \cong \mathbf{V}_l(\mathbf{C}^n) \times_{\mathrm{GL}(\mathbf{C}^l)} \mathbf{V}_l(\mathbf{C}^k).$$

Definition 3.4. For $0 \le l < n$, using the embedding (3.2) let

$$X_l(k, n) = \mathrm{Rat}_k(\mathbf{CP}^n) \cap (\mathbf{C}^k \times \mathscr{M}_{n-l}).$$

We will simplify this to X_l when there is no risk of confusion.

Note that elements of $X_0(k, n)$ have full rank so $X_0(k, n) = \mathrm{FRat}_k(\mathbf{CP}^n)$. When $k = n$, $\mathscr{M}_n \subset \mathrm{Mat}_{n,n}(\mathbf{C})$ is the set of invertible matrices. In this case $\mathrm{FRat}_n(\mathbf{CP}^n) \cong \mathbf{C}^n \times \mathscr{M}_n$ and we have proved

Corollary 3.5. $\mathrm{FRat}_n(\mathbf{CP}^n)$ *is homotopy equivalent to* $\mathrm{U}(n)$.

We would now like to find explicit models for the other strata.

Lemma 3.6. $\mathrm{Rat}_k(\mathbf{CP}^n)$ *is a left* $\mathrm{GL}(\mathbf{C}^n)$*-space. The sets* X_l *are invariant under this action. In particular,* $\mathrm{FRat}_k(\mathbf{CP}^n)$ *is a* $\mathrm{GL}(\mathbf{C}^n)$*-space.*

Proof. First embed $\mathrm{GL}(\mathbf{C}^n) \hookrightarrow \mathrm{GL}(\mathbf{C}^{n+1})$ as a subgroup of elements that leave the vector $(1, \cdots, 1)$ fixed; these also act on $\mathbf{CP}^n$ and leave our chosen base point fixed. The action on maps $S^2 \to \mathbf{CP}^n$ is now given by composition. There is an alternative description of this action. Given $[p_0, \cdots, p_n] \in \mathrm{Rat}_k(\mathbf{CP}^n)$ we form the $(n+1)\times(k+1)$ matrix of the coefficients of the polynomials. $\mathrm{GL}(\mathbf{C}^{n+1})$ acts by left multiplication on such a matrix and, using the embedding above, $\mathrm{GL}(\mathbf{C}^n)$ acts on $\mathrm{Rat}_k(\mathbf{CP}^n)$. A simple calculation shows that these two actions are the same. This second description of the action makes it clear that the rank condition which defines the sets X_l is preserved. Since $X_0 = \mathrm{FRat}_k(\mathbf{CP}^n)$, this defines a $\mathrm{GL}(\mathbf{C}^n)$ action on the full maps. Note that the embedding $e(k, n)$ and matrix multiplication give an action which corresponds to a particular choice of the embedding of $\mathrm{GL}(\mathbf{C}^n) \hookrightarrow \mathrm{GL}(\mathbf{C}^{n+1})$. q.e.d.

We can now state and prove the main technical result of this section.

Theorem 3.7. *The sets* X_l *are complex submanifolds of* $\mathrm{Rat}_k(\mathbf{CP}^n)$ *of complex dimension* $k + (n-l)(k+l)$. *What is more we can give a precise description of the strata*:

$$X_l \cong \mathbf{V}_{n-l}(\mathbf{C}^n) \times_{\mathrm{GL}(\mathbf{C}^{n-l})} \mathrm{FRat}_k(\mathbf{CP}^{n-l}).$$

Once again by $\mathbf{V}_{n-l}(\mathbf{C}^n)$ *we mean the manifold of all* $(n-l)$*-frames in* $\mathbf{C}^n$. *These sets form an* L*-stratification of* $\mathrm{Rat}_k(\mathbf{CP}^n)$.

Proof. As noted above the image of $\mathrm{Rat}_k(\mathbf{CP}^n)$ under the embedding $e(k, n)$ must miss the stratum $\mathbf{C}^k \times \mathscr{M}_0$ completely since $\mathscr{M}_0$ contains only the zero matrix. Thus, the union of the sets X_l does fill out $\mathrm{Rat}_k(\mathbf{CP}^n)$. We may take $\mathrm{Rat}_k(\mathbf{CP}^n)$ to be an open submanifold of $\mathbf{C}^k \times \mathrm{Mat}_{n,k}(\mathbf{C})$. Since the $\mathscr{M}_l$ are complex submanifolds, their normal bundles are orientable. Thus we do indeed have an L-stratification, and it only remains to verify the description of X_l.

Suppose the image of a map $f: S^2 \to \mathbf{CP}^n$ is contained in a proper projective subspace $V \subset \mathbf{CP}^n$. Then we can factor $f = \phi \circ g$ where

$\phi: \mathbf{CP}^{n-l} \to \mathbf{CP}^n$ is a projective linear embedding and $g \in \mathrm{FRat}_k(\mathbf{CP}^{n-l})$. The possible embeddings ϕ which fix the base point are catalogued by points in $\mathbf{V}_{n-l}(\mathbf{C}^n)$, and the choice is determined up to a $\mathrm{GL}(\mathbf{C}^{n-l})$-action. q.e.d.

With Theorem 3.7 we are now in a position to examine the Leray spectral sequence associated to this L-stratification. This spectral sequence converges to the homology of $\mathrm{Rat}_k(\mathbf{CP}^n)$, and by inspecting this spectral sequence we see that all the homology in dimension $2(k-n)$ and less comes from the generic stratum, $\mathrm{FRat}_k(\mathbf{CP}^n)$. From this observation it is easy to prove Theorem A.

Proof of Theorem. First note that from Corollary 3.5 it follows that for $k = n$ the theorem implies only that $\mathrm{FRat}_n(\mathbf{CP}^n)$ is path connected. Now the real codimension of X_l in $\mathrm{Rat}_k(\mathbf{CP}^n)$ is $2l(k-n+1)$ so the strata which form the complement of $\mathrm{FRat}_k(\mathbf{CP}^n)$ in $\mathrm{Rat}_k(\mathbf{CP}^n)$ all have codimension greater than or equal to that of X_1, which is $2(k-n+1)$. For $k > n$, $2(k-n+1) \geq 4$. It was shown in [3] that $\mathrm{Rat}_k(\mathbf{CP}^n)$ is simply connected for $n > 1$ so, by general position, $\mathrm{FRat}_k(\mathbf{CP}^n)$ is also simply connected.

To complete the proof we will show that $i(k, n)$ is a homology equivalence through dimension $2(k-n)$ by examining the Leray spectral sequence associated to the stratification 3.4. The E^1 term of this spectral sequence is, following the discussion at the beginning of this section,

$$E^1_{s,t} = H_{s+t-2s(k-n+1)}(X_s; A),$$

and it converges to a filtration of $H_*(\mathrm{Rat}_k(\mathbf{CP}^n); A)$. For $s > 0$ there are no nonzero terms of total degree less than $l(2(k-n)+1)$. This implies that for $m \leq 2(k-n)+1$ we have

$$H_t(\mathrm{FRat}_k(\mathbf{CP}^n); A) = E^1_{0,t} \to E^\infty_{0,t} \cong H_t(\mathrm{Rat}_k(\mathbf{CP}^n); A), \tag{3.10}$$

where the middle map is onto. This composition is the homomorphism $i(k, n)_*$. For $i(k, n)_*$ to have a kernel there would have to be a nonzero differential, but differentials reduce total degree by exactly one so the first possible nonzero differential is when $t = 2(k-n)+1$. Thus, for t less than this $i(k, n)_*$ is an isomorphism. q.e.d.

With this theorem we have only started to extract the information that this stratification and the associated spectral sequence contain. For $t \geq 2(k-n)+1$ we have to adjust (3.10) to take account of terms from lower strata which survive to E^∞ and of nonzero differentials. In the next section we shall determine the image of $i(k, n)_*$ and then return to this spectral sequence.

4. The image of $i(k, n)_*$

In this section we show that the homotopy inclusion

$$i_k: \mathrm{Rat}_k(\mathbf{CP}^n) \hookrightarrow \mathrm{Rat}_{k+1}(\mathbf{CP}^n)$$

defined by Segal in [14] restricts to give a homotopy inclusion

$$\bar{i}_k: \mathrm{FRat}_k(\mathbf{CP}^n) \hookrightarrow \mathrm{FRat}_{k+1}(\mathbf{CP}^n).$$

We then prove that, following [2], we can define a homology product which gives $H_*(\mathrm{FRat}_*(\mathbf{CP}^n)))$ the structure of a module over $H_*(\mathrm{Rat}_*(\mathbf{CP}^n))$. Using this structure we are able to construct nontrivial classes in $H_*(\mathrm{FRat}_k(\mathbf{CP}^n))$ which are mapped by $i(k, n)_*$ isomorphically into $H_*(\mathrm{Rat}_k(\mathbf{CP}^n))$ and fill out a particular piece of the image of $i(k, n)_*$.

Definition 4.1. Let $B_k \in \mathbf{C}$ be the disk of radius $k - \frac{1}{2}$ centered at the origin and let

$$\mathrm{rat}_k(\mathbf{CP}^n) = \{[p_0, \cdots, p_n] \in \mathrm{Rat}_k(\mathbf{CP}^n) | \text{the roots of all } p_i \text{ lie in } B_k\}.$$

Recall from §2 we defined the generic set $Y_0(k, n) \subset \mathrm{Rat}_k(\mathbf{CP}^n)$. Let $Z_0(k, n) = Y_0(k, n) \cap \mathrm{FRat}_k(\mathbf{CP}^n)$ be the corresponding set of full maps.

Lemma 4.2. $\mathrm{Rat}_k(\mathbf{CP}^n)$ *is homotopy equivalent to* $\mathrm{rat}_k(\mathbf{CP}^n)$. *Furthermore the spaces* $\mathrm{FRat}_k(\mathbf{CP}^n)$, Y_0, *and* Z_0 *are homotopy equivalent to their respective intersections with* $\mathrm{rat}_k(\mathbf{CP}^n)$.

Proof. Given $f = [p_0, \cdots, p_n] \in \mathrm{Rat}_k(\mathbf{CP}^n)$ let $\{z_1, \cdots, z_i, \cdots\}$ be the set of zeros of all the polynomials p_i. Let

$$M = \max\{1, |z_1|, \cdots, |z_i|, \cdots\}.$$

M depends continuously on f. Now we define a function g which assigns to a polynomial $p(z) \cong z^k + a_1 z^{k-1} + \cdots + a_k$ the polynomial

$$g(p)(z) = z^k + \frac{a_1}{M} z^{k-1} + \cdots + \frac{a_k}{M^k}.$$

Now if $p(z) = (z - x_1) \cdots (z - x_k)$, then

$$g(p)(z) = \left(z - \frac{x_1}{M}\right) \cdots \left(z - \frac{x_k}{M}\right),$$

so the multiplicities of the roots are preserved and g preserves the generic strata. The assignment $\bar{g}: [p_0, \cdots, p_n] \mapsto [g(p_0), \cdots, g(p_n)]$ is a deformation retract of $\mathrm{Rat}_k(\mathbf{CP}^n)$ onto $\mathrm{rat}_k(\mathbf{CP}^n)$. Since the map g can be expressed, in terms of the vector of coefficients $(1, a_1, \cdots, a_k)$, as multiplication by the matrix $\mathrm{diag}(1, M^{-1}, \cdots, M^{-k})$, which is nonsingular, g preserves linear independence. q.e.d.

Now in the vector space $\mathbf{C}[z]$ multiplication by a linear factor $(z-k)$ is a linear injection. Thus, if

$$[p_0, \cdots, p_n] \in \mathrm{rat}_k(\mathbf{CP}^n)$$

and the p_i are linearly independent, then the polynomials $q_i(z) = p_i(z)\cdot(z-k)$ will also be linearly independent. Since linear independence is an open condition, there is an $\varepsilon_0 > 0$ such that if we let $a_0 = k+\varepsilon$ and $a_i = k$ for $i \geq 1$, then the polynomials $q_i(z) = p_i(z)(z-a_i)$ are linearly independent for $0 < \varepsilon < \varepsilon_0$. Furthermore we can choose ε_0 to be a continuous function of $p_0, \cdots, p_n$. Since the q_i have no roots in common, $[q_0, \cdots, q_n] \in \mathrm{FRat}_k(\mathbf{CP}^n)$ and this defines $\bar{i}_k$.

Consider the commutative diagram

$$\begin{array}{ccc} \mathrm{FRat}_k(\mathbf{CP}^n) & \xrightarrow{\bar{i}_k} & \mathrm{FRat}_{k+1}(\mathbf{CP}^n) \\ \Big\downarrow{\scriptstyle i(k,n)} & & \Big\downarrow{\scriptstyle i(k+1,n)} \\ \mathrm{Rat}_k(\mathbf{CP}^n) & \xrightarrow{i_k} & \mathrm{Rat}_{k+1}(\mathbf{CP}^n). \end{array}$$

By Theorem A the vertical arrows are homotopy equivalences through a range that grows with k, and in [14] it is proved that the same is true of the bottom map. So the top map is a homotopy equivalence through a range that grows with k, and this proves Theorem B.

Now we turn to the question of constructing homology classes. As mentioned in §1 F. Cohen constructed a map

$$\mathrm{Rat}_k(\mathbf{CP}^n) \times \mathrm{Rat}_l(\mathbf{CP}^n) \to \mathrm{Rat}_{k+l}(\mathbf{CP}^n). \tag{4.3}$$

This maps induces a pairing in homology

$$H_*(\mathrm{Rat}_k(\mathbf{CP}^n)) \otimes H_*(\mathrm{Rat}_l(\mathbf{CP}^n)) \xrightarrow{*} H_*(\mathrm{Rat}_{k+l}(\mathbf{CP}^n)). \tag{4.4}$$

This pairing corresponds to the algebraic structures in (2.7) and (2.8). The map (4.3) is given explicitly in terms of the zeros of the polynomial factors comprising the holomorphic map. This description is not well suited for keeping track of the linear independence of the polynomials. A result of this is that (4.3) does not restrict to give a suitable product for the full maps. However it is possible to construct such a products if we restrict to the generic sets $Y_0(k, n)$ and $Z_0(k, n)$.

In §2 we noted that

$$Y_0(k, n) \simeq F(\mathbf{C}, k) \times_{\Sigma_k} (S^{2n-1})^k. \tag{4.5}$$

In [3] a particular homotopy equivalence is given in terms of the roots of the polynomials. There is a second equivalence which is better suited to our purposes. This map is a generalization of the pole and principal part description for rational maps. The assignment

$$[p_0, \cdots, p_n] \mapsto \langle x_1, \cdots, x_k; (p_1(x_1), \cdots, p_n(x_1)), \cdots, (p_1(x_k), \cdots, p_n(x_k))\rangle$$

defines a homeomorphism

$$Y_0(k, n) \cong F(\mathbf{C}, k) \times_{\Sigma_k} (\mathbf{C}^n \backslash \{0\})^k.$$

This gives another homotopy equivalence of the form (4.5) which we denote by h.

Now we turn to the generic stratum in $\mathrm{FRat}_k(\mathbf{CP}^n)$. Let

$$\begin{aligned} Z_0(k, n) &= \{[p_0, \cdots, p_n] \in \mathrm{FRat}_k(\mathbf{CP}^n) | p_0 \text{ has no multiple roots}\} \\ &= Y_0(k, n) \cap \mathrm{FRat}_k(\mathbf{CP}^n). \end{aligned}$$

To find a model for $Z_0(k, n)$ we need to restrict the fibre in (4.5) to reflect the linear independence condition on the polynomials. Given $[p_0, \cdots, p_n] \in Z_0(k, n)$ let $x_1, \cdots, x_k \in \mathbf{C}$ be the distinct roots of p_0. Suppose there exist $c_1, \cdots, c_n \in \mathbf{C}$ such that, for $i = 1, \cdots, n$, we have

$$c_1 p_1(x_i) + \cdots + c_n p_n(x_i) = 0.$$

Then, since $p_0(x_i) = 0$,

$$c_1(p_1(x_i) - p_0(x_i)) + \cdots + c_n(p_n(x_i) - p_0(x_i)) = 0.$$

The p_i are all monic and of degree k, so the left-hand side is a polynomial of degree less than k which vanishes at k distinct points, so it must vanish identically. Since the p_i are linearly independent, we may conclude that the constants c_i are all zero. The converse is also true and we have

Lemma 4.6. *$[p_0, \cdots, p_n] \in Z_0(k, n)$ if and only if the matrix $(p_i(x_j))$, $i = 1, \cdots, n$ and $j = 1, \cdots, k$, has rank n.*

Corollary 4.7. *The map h restricts to give a homotopy equivalence*

$$\overline{h}: Z_0(k, n) \simeq F(\mathbf{C}, k) \times_{\Sigma_k} \{(u_1, \cdots, u_k) \in (S^{2n-1})^k | u_i \text{ span } \mathbf{C}^n\}.$$

The spaces $F(\mathbf{C}, k) \times_{\Sigma_k} (S^{2n-1})^k$ are the basic building blocks in the May-Milgram model for $\Omega^2 S^{2n-1}$ and we recall the product (2.7):

$$F(\mathbf{C}, k) \times_{\Sigma_k} (S^{2n-1})^k \times F(\mathbf{C}, l) \times_{\Sigma_l} (S^{2n-1})^l \xrightarrow{*} F(\mathbf{C}, k+l) \times_{\Sigma_{k+l}} (S^{2n-1})^{k+l}$$

given by

$$(4.8) \quad \begin{aligned} &\langle x_1, \cdots, x_k; u_1, \cdots, u_k\rangle * \langle y_1, \cdots, y_l; v_1, \cdots, v_l\rangle \\ &= \langle \phi x_1, \cdots, \phi x_k, \phi' y_1, \cdots, \phi' y_l; u_1, \cdots, u_k, v_1, \cdots, v_l\rangle, \end{aligned}$$

where ϕ and ϕ' are again homeomorphisms of $\mathbf{C}$ onto two disjoint disks.

Now if the vectors $\{u_1, \cdots, u_k\}$ span $\mathbf{C}^n$, then so will $\{u_1, \cdots, u_k, v_1, \cdots, v_k\}$. Thus if we use the homotopy equivalences h and $\overline{h}$, then (4.8) gives us a map

$$Z_0(k, n) \times Y_0(l, n) \to Z_0(k+l, n). \tag{4.9}$$

Hence we obtain homology pairings

$$H_*(Z_0(k, n)) \otimes H_*(Y_0(k, n)) \to H_*(Z_0(k, n)).$$

Since the inclusion of $Y_0(k, n)$ into $\mathrm{Rat}_k(\mathbf{CP}^n)$ induces a surjection in homology, we actually have a pairing

$$H_*(Z_0(k, n)) \otimes H_*(\mathrm{Rat}_k(\mathbf{CP}^n)) \to H_*(\mathrm{FRat}_k(\mathbf{CP}^n)).$$

This pairing is compatible with the pairing (4.4) in the sense that the following diagram commutes up to homotopy:

$$\begin{array}{ccc} Z_0(k, n) \times Y_0(l, n) & \longrightarrow & Z_0(k+l, n) \\ \downarrow & & \downarrow \\ \mathrm{Rat}_k(\mathbf{CP}^n) \times \mathrm{Rat}_k(\mathbf{CP}^n) & \longrightarrow & \mathrm{Rat}_k(\mathbf{CP}^n) \end{array}$$

where the top map is the map (4.9), and the lower map is the map of (4.3). We will denote the pairing induced by (4.9) by $*$ as well. As a consequence of this we have the following.

Lemma 4.10. *Let $z \in H_*(\mathrm{FRat}_k(\mathbf{CP}^n))$ be in the image of the homomorphism*

$$H_*(Z_0(k, n)) \to H_*(\mathrm{FRat}_k(\mathbf{CP}^n))$$

induced by the inclusion of the generic subset. Then for any $y \in H_*(\mathrm{Rat}_l(\mathbf{CP}^n))$

$$i(k+l, n)_*(z * y) = (i(k, n)_*(z)) * y$$

in $H_*(\mathrm{Rat}_{k+l}(\mathbf{CP}^n))$.

Giyen z and y as in Lemma 4.10 we can identify their product in terms of its image under the homomorphism $i(k, n)_*$. What we need to get the process off the ground are candidates for the class $z \in H_*(\mathrm{FRat}_k(\mathbf{CP}^n))$. To take full advantage of Lemma 4.10 we need to know the image $i(k, n)_*(z) \in H_*(\mathrm{Rat}_k(\mathbf{CP}^n))$. By Corollary 3.5, $\mathrm{FRat}_n(\mathbf{CP}^n)$ is path connected. Choose a generator $[n] \in H_0(\mathrm{FRat}_n(\mathbf{CP}^n); \mathbf{Z})$. Let A be any coefficient module. Then, for any $y \in H_*(\mathrm{Rat}_k(\mathbf{CP}^n); A)$, $[n] * y$ is a nontrivial class in $H_*(\mathrm{FRat}_{k+n}(\mathbf{CP}^n); A)$. This follows from the fact that we may take $[n]$ as a generator in $H_0(\mathrm{Rat}_n(\mathbf{CP}^n); \mathbf{Z})$ as well, and the assignment $y \mapsto [n] * y$ is an injection

$$H_*(\mathrm{Rat}_k(\mathbf{CP}^n)) \to H_*(\mathrm{Rat}_{k+n}(\mathbf{CP}^n)),$$

which corresponds to the n-fold iteration of Segal's map. Thus, we have proved

Theorem 4.11. *For any coefficients* A, $H_*((\mathrm{FRat}_k(\mathbf{CP}^2); A)$ *contains a submodule which is mapped isomorphically by* $i(k, n)_*$ *onto*

$$[n] * H_*(\mathrm{Rat}_{k-n}(\mathbf{CP}^n); A) \subset H_*(\mathrm{Rat}_k(\mathbf{CP}^n); A).$$

We can make more use of Lemma 3.5. If we can find nontrivial classes in the image of

$$i(n, n)_*: H_*(U(n)) \to H_*(\mathrm{Rat}_n(\mathbf{CP}^n)),$$

then the same argument will detect more classes in the image of $i(k, n)_*$. As noted in §2 the first possible nontrivial map $H_q(U(n)) \to H_q(\mathrm{Rat}_n(\mathbf{CP}^n))$ for $q > 0$ is when $q = 2n - 1$. We have the generator $e_1 \in H_{2n-1}(\mathrm{Rat}_k(\mathbf{CP}^n); \mathbf{Z})$. Letting $u_{2n-1} \in H_*(U(n); \mathbf{Z})$ be the generator in dimension $2n-1$ we show

Lemma 4.12. $i(n, n)_*(u_{2n-1}) = ne_1$.

Proof. We may think of elements of $\mathrm{U}(n)$ as $n \times n$ complex matrices whose columns are orthonormal vectors in $\mathbf{C}^n$. Thus $U(n)$ is naturally a subset of $(S^{2n-1})^n$. The image of u_{2n-1} under this inclusion is the diagonal class $\Delta \in H_{2n-1}((S_{2n-1})^n; \mathbf{Z})$ which is the image of the generator in $H_{2n-1}(S^{2n-1}; \mathbf{Z})$ under the diagonal map $S^{2n-1} \to (S^{2n-1})^n$. Now the composition

$$S^{2n-1} \xrightarrow{j} (S^{2n-1})^n \hookrightarrow F(\mathbf{C}, k) \times_{\Sigma_n} (S^{2n-1})^n \to \mathrm{Rat}_n(\mathbf{CP}^n),$$

where j is the inclusion of any factor, maps the generator in $H_{2n-1}(S^{2n-1}; \mathbf{Z})$ to e_1 [2]. Thus, Δ is mapped to $ne_1 \in H_*(\mathrm{Rat}_n(\mathbf{CP}^n); \mathbf{Z})$. Finally we have the following commutative diagram:

$$\begin{array}{ccc}
\mathrm{U}(n) & \longrightarrow & (S^{2n-1})^n \\
\downarrow & & \downarrow \\
F(\mathbf{C}, k) \times_{\Sigma_n} \mathrm{U}(n) & & F(\mathbf{C}, k) \times_{\Sigma_k} (S^{2n-1})^n \\
\downarrow & & \downarrow \\
\mathrm{FRat}_n(\mathbf{CP}^n) & \xrightarrow{i(n,n)_*} & \mathrm{Rat}_n(\mathbf{CP}^n).
\end{array}$$

The left-hand vertical composition is the homotopy equivalence of Corollary 3.9, and this completes the proof. q.e.d.

Lemma 4.12 implies the following theorem which, together with Theorem 4.11, proves Theorem C.

Theorem 4.13. *For any coefficients* A, *the image*

$$i(k, n)_*(H_*(\mathrm{FRat}_k(\mathbf{CP}^n); A)) \subset H_*(\mathrm{Rat}_k(\mathbf{CP}^n); A)$$

contains $ne_1 * H_*(\mathrm{Rat}_{k-n}(\mathbf{CP}^n); A)$.

We note that $H_*(\mathrm{FRat}_k(\mathbf{CP}^n))$ has additional algebraic structure. We can define maps

$$\Psi_p: F(\mathbf{C}, p) \times_{\Sigma_p} (Z_0(k, n))^p \to Z_0(pk, n),$$

which give rise to Dyer-Lashoff operations which are compatible with those defined on $\mathrm{Rat}(\mathbf{CP}^n)$ in [2]. However, the homology classes which we can construct with Ψ_p appear first, in lower degree components, using the constructions above.

5. The Homology of $\mathrm{FRat}_k(\mathbf{CP}^2)$

In this section we restrict our attention to the case of full holomorphic maps into $\mathbf{CP}^2$. The geometric description of the stratification of $\mathrm{Rat}_k(\mathbf{CP}^n)$ in Theorem 3.7 is given inductively in terms of the spaces $\mathrm{FRat}_k(\mathbf{CP}^l)$ for $l < n$, so this is the core calculation. Since there are only two strata in this case, the Leray spectral sequence reduces to the long exact sequence of the pair $(\mathrm{Rat}_k(\mathbf{CP}^2), \mathrm{FRat}_k(\mathbf{CP}^2))$:

$$(5.1)\quad \begin{aligned} \cdots \to H_q(\mathrm{FRat}_k(\mathbf{CP}^2)) &\xrightarrow{i(k,2))_*} H_q(\mathrm{Rat}_k(\mathbf{CP}^2)) \\ &\xrightarrow{j} H_q(\mathrm{Rat}_k(\mathbf{CP}^2), \mathrm{FRat}_k(\mathbf{CP}^2)) \xrightarrow{\partial} H_{q-1}(\mathrm{FRat}_k(\mathbf{CP}^2)) \to \cdots \end{aligned}$$

We first use the Thom isomorphism

$$(5.2)\qquad H_q(\mathrm{Rat}_k(\mathbf{CP}^2), \mathrm{FRat}_k(\mathbf{CP}^2)) \cong H_{q-2k+2}(X_1(2, k))$$

to identify this term in (5.1). From §3 we know that X_1 is the total space of a fibre bundle. The differentials in the Serre spectral sequence can be deduced form calculations in [13]. Thus we calculate the homology groups (5.2). Next we examine the homomorphism

$$J: H_q(\mathrm{Rat}_k(\mathbf{CP}^2)) \to H_{q-2k+2}(X_1),$$

which is the composition of the Thom isomorphism with the homomorphism j of (5.1). We show that in this case Theorems 4.11 and 4.13 describe the image of $i_*(k, n)$ completely, and we compute $H_*(\mathrm{FRat}_k(\mathbf{CP}^2), \mathbf{Z}/p)$ explicitly.

From Theorem 3.7 it follows that

$$X_1 \simeq S^3 \times_{S^1} \mathrm{Rat}_k(\mathbf{CP}^1).$$

In [13] Milgram determines all the differentials in the Serre spectral sequence for the fibration

$$\mathrm{Rat}_k(\mathbf{CP}^1) \to \mathrm{Rat}_k(\mathbf{CP}^1) \times_{S^1} E_{S^1} \to \mathbf{CP}^\infty, \tag{5.3}$$

where the S^1 action is the same as in Lemma 3.6. Thus X_1 is the restriction of (5.3) to $\mathbf{CP}^1 \subset \mathbf{CP}^\infty$. The Serre spectral sequence for the fibration

$$\mathrm{Rat}_k(\mathbf{CP}^1) \to X_1 \to \mathbf{CP}^1,$$

has

$$\begin{aligned} E_2^{s,t} &= H^s(\mathbf{CP}^1; H^t(\mathrm{Rat}_k(\mathbf{CP}^1); A)) \\ &= H^s(\mathbf{CP}^1; \mathbf{Z}) \otimes H^t(\mathrm{Rat}_k(\mathbf{CP}^1); A). \end{aligned}$$

So the only nonzero terms are

$$E_2^{0,t} = H^t(\mathrm{Rat}_k(\mathbf{CP}^1); A)$$

and

$$E_2^{2,t} = b \otimes H^t(\mathrm{Rat}_k(\mathbf{CP}^1); A),$$

where b is a generator of $H^2(\mathbf{CP}^1; \mathbf{Z})$. The cohomology of the rational maps is given in [13]. As we saw in §2 it is a specific submodule of the cohomology of the loop space $\Omega^2 S^3$ which is easily described in terms of the May-Milgram model.

$$H^*(J_2(S^1); \mathbf{Z}/2) = E(e_1^*, \cdots, (e_1^{2^i})^*, \cdots, q_1^*, \cdots, (q_1^{2^i})^*, \cdots),$$

an exterior algebra on generators dual to those in (2.7). For p an odd prime we have

$$H^*(J_2(S^1); \mathbf{Z}/p) = E(e_1^*, q_1^*, \cdots, q_i^*, \cdots) \otimes V,$$

where V is a polynomial algebra on generators $(f_i^{p^r})^*$ truncated by the relations

$$((f_i^{p^r})^*)^p = 0$$

for all i and r. Once again the generators are the duals of those in (2.8). In both cases the bidegrees of the generators are the same as those of their duals, and the cohomology of $\mathrm{Rat}_k(\mathbf{CP}^n)$ is the submodule generated by elements of second grading k or less. The results of [13] imply that the differential

$$d^2 : H^t(\mathrm{Rat}_k(\mathbf{CP}^1); \mathbf{Z}) \to b \otimes H^{t-1}(\mathrm{Rat}_k(\mathbf{CP}^1); \mathbf{Z}/p)$$

is given by $d^2((e_1^*)^n) = (k-n+1)b(e_1^*)^{n-1}$, and d^2 is zero on all other generators. From this we can compute $h_*(X_1; \mathbf{Z}/p)$ explicitly.

For example

$$H_*(X_1(2,2); \mathbf{Z}/2) = \begin{cases} \text{Generator} & \text{Dimension} \\ 1 & 0 \\ e_1 & 1 \\ b & 2 \\ q_1 & 3 \\ be_1^2 & 4 \\ bq_1 & 5. \end{cases}$$

In this case we already know that

$$H_*(\mathrm{FRat}_2(\mathbf{CP}^2); \mathbf{Z}/2) = H_*(U(2); \mathbf{Z}/2).$$

Moreover from §4 it follows that $i_*(2,2)$ maps the generator in $H_3(\mathrm{FRat}_2(\mathbf{CP}^2); \mathbf{Z}/2)$ to zero, and exactness determines the remaining maps in the long exact sequence (5.1).

To describe the results we recall [4] that $D_j(S^3) \cong \Sigma^{2j} D_j(S^1)$. Using (1.6) we can form a direct sum of suspension isomorphisms to define a 1-1 correspondence between the homology groups of $\mathrm{Rat}_k(\mathbf{CP}^1)$ and those of $\mathrm{Rat}_k(\mathbf{CP}^2)$. This correspondence does not preserve homological degree but rather shifts it by an amount which depends on the second grading degree. The correspondence is implicit in (2.8) and (2.9) since the algebras $H_*(J_2(S^1))$ and $h_*(J_2(S^3))$ are identical except for the first grading degree of the generators. Given a monomial $x \in H_q(\mathrm{Rat}_k(\mathbf{CP}^1))$ with second grading degree j we will denote by $\tilde{x}$ the corresponding monomial in $H_{q-2j}(\mathrm{Rat}_k(\mathbf{CP}^2))$.

Using this notation we have $J(\tilde{e}_1^2) = be_1^2$, $J(\tilde{q}_1) = bq_1$ and $J(\tilde{e}_1) = e_1$. From this first example we can predict the general result.

Lemma 5.4. *Let*

$$x^* \in H^*(\mathrm{Rat}_k(\mathbf{CP}^1); \mathbf{Z}/p)$$

be a monomial with second grading degree k. *Then* bx^* *survives to* E_∞ *in the spectral sequence for* $H^*(X_1;\mathbf{Z}/p)$, *and* J *maps* $\tilde{x}\in H_*(\mathrm{Rat}_k(\mathbf{CP}^2);\mathbf{Z}/p)$ *to* $bx\in H_*(X_1;\mathbf{Z}/p)$.

Lemma 5.5. *Let*

$$y^*\in H^*(\mathrm{Rat}_k(\mathbf{CP}^1);\mathbf{Z}/p)$$

be a monomial with second grading degree $k-1$ *such that* h^* *survives to* E_∞ *in the spectral sequence for* $H^*(X_1;\mathbf{Z}/p)$. *Then* J *maps* $\tilde{y}\in H_*(\mathrm{Rat}_k(\mathbf{CP}^2);\mathbf{Z}/p)$ *to* $y\in H_*(X_1;\mathbf{Z}/p)$.

Since Theorems 4.11 and 4.13 set a lower bound on the kernel of J, these two lemmas completely describe this map. By exactness the remaining generators in $H_*(X_1;\mathbf{Z}/p)$ must be mapped by the Thom isomorphism and the connecting homomorphism ∂ of (5.1) to nontrivial classes in $H_*(\mathrm{FRat}_k(\mathbf{CP}^2))$. This proves Theorem E.

Proofs. (All homology with coefficients in $\mathbf{Z}/p$). First we note that we can embed

$$i_1:\mathrm{Rat}_k(\mathbf{CP}^1)\hookrightarrow \mathrm{Rat}_k(\mathbf{CP}^2)$$

by composition with a linear embedding $\mathbf{CP}^1\hookrightarrow\mathbf{CP}^2$. The embedding i_1 factors as the composition of the inclusion of the fibre

$$i_2:\mathrm{Rat}_k(\mathbf{CP}^1)\hookrightarrow X_1$$

followed by the inclusion $X_1\hookrightarrow\mathrm{Rat}_k(\mathbf{CP}^2)$. It follows from the naturality of the Thom isomorphism [6] that the following diagram commutes:

(5.6)

$$\begin{array}{ccc}
H_q(\mathrm{Rat}_k(\mathbf{CP}^2)) & \xrightarrow{j} & H_q(\mathrm{Rat}_k(\mathbf{CP}^2),\mathrm{FRat}_k(\mathbf{CP}^2)) \\
\downarrow{\scriptstyle j_1} & & \downarrow{\scriptstyle\cong} \\
H_q(\mathrm{Rat}_k(\mathbf{CP}^2),\mathrm{Rat}_k(\mathbf{CP}^2)\backslash\mathrm{Rat}_k(\mathbf{CP}^1)) & & H_{q+2k-2}(X_1) \\
\downarrow{\scriptstyle\cong} & & \downarrow{\scriptstyle j_2} \\
H_{q+2k}(\mathrm{Rat}_k(\mathbf{CP}^1)) & \xleftarrow{\cong} & H_{q-2k+2}(X_1,X_1\backslash\mathrm{Rat}_k(\mathbf{CP}^1))
\end{array}$$

Here the isomorphisms are the appropriate Thom isomorphisms, and the maps j, j_1 and j_2 are from exact sequences of pairs. Denote by J_1 the composition of j_1 with the following Thom isomorphism.

Lemma 5.7. *The kernel of*

$$J_1:H_*(\mathrm{Rat}_k(\mathbf{CP}^2))\to H_*(\mathrm{Rat}_k(\mathbf{CP}^1))$$

is $H_*(\mathrm{Rat}_{k-1}(\mathbf{CP}^2))$. *Moreover*

$$H_*(\mathrm{Rat}_k(\mathbf{CP}^2)/\mathrm{Rat}_{k-1}(\mathbf{CP}^2)) \cong H_*(D_k(S^3)),$$

and the map induced on quotients by J_1 *is the isomorphism*

$$H_q(D_k(S^3)) = H_q(\Sigma^{2k} D_k(S^1)) \cong H_{q-2k}(D_k(S^1)).$$

Since the image of the map

$$\begin{aligned} H^q(\mathrm{Rat}_k(\mathbf{CP}^1)) &\cong H^{q+2}(X_1, X_1 \backslash \mathrm{Rat}_k(\mathbf{CP}^1)) \\ &\cong H^{q+2}(X_1, X_1 \backslash \mathrm{Rat}_k(\mathbf{CP}^1)) \xrightarrow{j_2^*} H^{q+2}(X_1) \end{aligned}$$

corresponds to those classes in

$$E_2^{2,*} \cong b \otimes H^*(\mathrm{Rat}_k(\mathbf{CP}^1)),$$

which survive to E_∞. Hence Lemma 5.4 is proved.

Lemma 5.5 is a result of the following.

Lemma 5.8. *The following diagram commutes:*

$$\begin{array}{ccc} H_q(\mathrm{Rat}_{k-1}(\mathbf{CP}^2)) & \xrightarrow{j_1} & H_{q-2k+2}(\mathrm{Rat}_{k-1}(\mathbf{P}^1)) \\ \downarrow{\scriptstyle (i_{k-1})_*} & & \downarrow{\scriptstyle (i'_{k-1})_*} \\ H_q(\mathrm{Rat}_k(\mathbf{CP}^2)) & & H_{q-2k+2}(\mathrm{Rat}_k(\mathbf{CP}^1)) \\ \downarrow{\scriptstyle J} & & \downarrow{\scriptstyle (i_2)_*} \\ H_{q-2k+2}(X_1) & \xrightarrow{=} & H_{q-2k+2}(X_1). \end{array}$$

Here i_{i-1} *and* i'_{k-1} *are the appropriate Segal inclusions.*

Indeed the image of $(i_2)_*$ is precisely that part of

$$E^2_{0,*} \cong H_*(\mathrm{Rat}_k(\mathbf{CP}^1)),$$

which survives to E^∞.

Proofs for Lemmas 5.7 and 5.8 are given in the next section.

Since the inclusion $Y_0(k, 1) \hookrightarrow \mathrm{Rat}_k(\mathbf{CP}^1)$ induces a surjection in homology, it follows that the generic set $Y_0(k, 2) \cap X_1$ carries all the homology of X_1. Applying the 5-lemma to the long exact sequence of the pair $(Y_0(k, 2), Z_0(k, 2))$ and the sequence (5.1) we see that all the homology of $\mathrm{FRat}_k(\mathbf{CP}^2)$ is carried on the generic set $Z_0(k, 2)$. Using this the first hypothesis in Lemma 4.10 is unnecessary in this case.

Finally we note that the rational homology of $\mathrm{FRat}_k(\mathbf{CP}^2)$ is significantly simpler. $H_*(D_k(S^1); \mathbf{Q}) = 0$ for $k > 1$ so

$$H_*(\mathrm{Rat}_k(\mathbf{CP}^1); \mathbf{Q}) \cong H_*(\mathrm{Rat}_1(\mathbf{CP}^1); \mathbf{Q}) \cong H_*(S^1; \mathbf{Q}).$$

X_1 fibers over S^2 with the same twisting as the Hopf bundle, and $H_*(X_1; \mathbf{Q}) \cong H_*(S^3; \mathbf{Q})$. Thus, over $\mathbf{Q}$, $i_*(k, 2)$ is a surjection and

$$H_q(\mathbf{FRat}_k(\mathbf{CP}^2); \mathbf{Q}) \cong H_q(\mathrm{Rat}_k(\mathbf{CP}^2); \mathbf{Q}) \oplus H_{q-2k+3}(S^3; \mathbf{Q}).$$

This proves Corollary F.

6. The proofs of Lemmas 5.7 and 5.8

To prove Lemma 5.7 we need to compute the homomorphism

$$J_1: H_q(\mathrm{Rat}_k(\mathbf{CP}^2)) \to H_{q-2k}(\mathrm{Rat}_k(\mathbf{CP}^1)).$$

The embedding

$$i_1: \mathrm{Rat}_k(\mathbf{CP}^1) \hookrightarrow \mathrm{Rat}_k(\mathbf{CP}^2)$$

can be chosen to leave the first polynomial factor unchanged and so restrict to a map

$$Y_0(k, 1) \hookrightarrow Y_0(k, 2). \tag{6.1}$$

By naturality of the Thom isomorphism we have the following commutative diagram:

$$\begin{array}{ccc} H_q(\mathrm{Rat}_k(\mathbf{CP}^2)) & \xrightarrow{J_1} & H_{q-2k}(\mathrm{Rat}_k(\mathbf{CP}^1)) \\ \downarrow & & \downarrow \\ H_q(Y_0(k, 2)) & \xrightarrow{J_1'} & H_{q-2k}(Y_0(k, 1)) \end{array} \tag{6.2}$$

where J_1' is a homomorphism constructed by applying the Thom isomorphism theorem to the embedding (6.1). Since the vertical maps in (6.2) are surjections, it suffices to compute J_1'.

Using the homotopy equivalence of §4

$$\bar{h}: Y_0(k, n) \simeq F(\mathbf{C}, k) \times_{\Sigma_k} (S^{2n-1})^k,$$

$i_1|_{Y_0(k,1)}$ can be written

$$F(\mathbf{C}, k) \times_{\Sigma_k} (S^1)^k \xrightarrow{\mathrm{id} \times i_3^k} F(\mathbf{C}, k) \times_{\Sigma_k} (S^3)^k,$$

where $i_3: S^1 \hookrightarrow S^3$ is the standard embedding. $\mathrm{id} \times i_3^k$ is a map of fiber bundles covering the identity on the base. The map J_1' induces a map of Serre spectral sequences. For the fibration

$$F(\mathbf{C}, k) \times_{\Sigma_k} (S^{2n-1})^k \to DP^k(\mathbf{C})$$

the E^2 term is

$$E^2_{s,t} = H_s(DP^k(\mathbf{C}); H_t((S^{2n-1})^k)).$$

The map of spectral sequences induced by J_1' is given on E^2 by $\mathrm{id} \otimes J-3$ where

$$J_3: H_1((S^3)^k) \to H_{q-2k}((S^1)^k)$$

is the map arising form the embedding i_3 through the Thom isomorphism.

Segal's inclusion i_{k-1} when restricted to the generic sets induces the inclusion

$$F(\mathbf{C}, k-1) \times_{\Sigma_{k-1}} (S^3)^{k-1} \hookrightarrow F(\mathbf{C}, k) \times_{\Sigma_k} (S^3)^k,$$

given by the product of the identity on the first factor and the inclusion

$$(6.3) \qquad (S^3)^{k-1} \hookrightarrow \{\mathrm{pt}\} \times (S^3)^{k-1} \subset (S^3)^k$$

on the second. This is also a fiber bundle map which covers the identity, and to show that $H_*(\mathrm{Rat}_{k-1}(\mathbf{CP}^3))$ is the kernel of J_1 it suffices to show that the following composition is trivial

$$(6.4) \qquad H_q((S^3)^{k-1}) \to H_q((S^3)^k) \xrightarrow{J_3} H_{q-2k}((S^1)^k),$$

where the first map is induced by the inclusion (6.3). We will prove

Lemma 6.5. *J_3 is an isomorphism in the top dimension $q = 3k$ and zero otherwise.*

Proof. J_3 is the composition

$$H_q((S^3)^k) \xrightarrow{j_3} H_q((S^3)^k, (S^3)^k \backslash (S^1)^k) \xrightarrow{\cong} H_{q-2k}((S^1)^k).$$

The first part of the lemma follows immediately since $H_{3k}((S^3)^k \backslash (S^1)^k) = 0$. Next note [6] that in cohomology the Thom isomorphism is an isomorphism $H_*((S^3)^k)$-modules, that is, if

$$\Phi^*: H^q((S^1)^k) \to H^{q+2k}((S^3)^k \backslash (S^1)^k)$$

is the Thom isomorphism in cohomology, $y \in H^*((S^3)^k)$ and $x \in H^*((S^1)^k)$, then

$$\Phi^*(x \cup i_3^*(y)) = \Phi^*(x) \cup y.$$

Since $i_3^*(y) = 0$ for any $y \in H^q((S^3)^k)$, $q > 0$, we have $\Phi^*(x) \cup y = 0$ and thus $j_3^* \circ \Phi^*(x) \cup y = 0$. The lemma follows by duality. q.e.d.

To complete the proof of Lemma 5.7 recall that

$$D_k(S^3) = (F(\mathbf{C}, k) \times_{\Sigma_k} (S^3)^k)/(F(\mathbf{C}, k-1) \times_{\Sigma_{k-1}} (S^3)^{k-1}).$$

Since $\mathrm{id} \times i_3^k$ maps

$$F(\mathbf{C}, k-1) \times_{\Sigma_{k-1}} (S^1)^{k-1} \to F(\mathbf{C}, k-1) \times_{\Sigma_{k-1}} (S^3)^{k-1},$$

it induces a map

$$D_k(S^1) \to D_k(S^3).$$

Although $D_k(S^3)$ is not a manifold, it is the Thom space of the vector bundle

$$F(\mathbf{C}, k) \times_{\Sigma_k} (\mathbf{R}^3)^k,$$

and by excision we have

$$\begin{aligned} &H_*(D_k(S^3), D_k(S^3)\backslash D_k(S^1)) \\ &\quad \cong H_*(F(\mathbf{C}, k) \times_{\Sigma_k} (\mathbf{R}^3)^k, F(\mathbf{C}, k) \times_{\Sigma_k} (\mathbf{R}^3)^k \backslash F(\mathbf{C}, k) \times_{\Sigma_k} \mathbf{R}^k). \end{aligned}$$

By results of [4], $F(\mathbf{C}, k) \times_{\Sigma_k} \mathbf{R}^k$ has a trivial normal bundle in $F(\mathbf{C}, k) \times_{\Sigma_k} (\mathbf{R}^3)^k$ and so

$$\widetilde{H}_q(D_k(S^3), D_k(S^3)\backslash D_k(S^1)) \cong \widetilde{H}_{q-2k}(D_k(S^1)).$$

Thus we obtain a commutative diagram

$$\begin{array}{ccc} \widetilde{H}_q(F(\mathbf{C}, k) \times_{\Sigma_k} (S^3)^k) & \longrightarrow & \widetilde{H}_q(D_k(S^3)) \\ \downarrow J_3' & & \downarrow \sigma \\ \widetilde{H}_{q-2k}(F(\mathbf{C}, k) \times_{\Sigma_k} (S^2)^k) & \longrightarrow & \widetilde{H}_{q-2k}(D_k(S^1)) \end{array}$$

where σ is the suspension isomorphism. Hence the proof of Lemma 5.7 is complete.

Since the $\mathrm{GL}(\mathbf{C}^n)$-action described in Lemma 3.6 leaves the first polynomial factor unchanged, it restricts to an action on the generic set. From this it is easy to show that

$$\begin{aligned} X_1 \cap Y_0(k, 2) &\cong S^3 \times_{S_1} Y_0(k, 1) \\ &\simeq S^3 \times_{S^1} (F(\mathbf{C}, k) \times_{\Sigma_k} (S^1)^k). \end{aligned}$$

The S^1 and Σ_k actions commute so

$$X_1 \cap Y_0(k, 2) \simeq F(\mathbf{C}, k) \times_{\Sigma_k} (S^3 \times_{S_1} (S^1)^k).$$

Furthermore the assignment

$$\langle u; \lambda_1, \cdots, \lambda_k\rangle \mapsto (\lambda_1 u, \lambda_1^{-1}\lambda_2, \cdots, \lambda_1^{-1}\lambda_k)$$

defines a homeomorphism from the twisted product $S^3 \times_{S^1} (S^1)^k$ to the Cartesian product $S^3 \times (S^1)^{k-1}$. Using the same argument as in the previous proof we see that to prove that the diagram in Lemma 5.8 commutes it suffices to prove the analogous diagram of generic sets commutes. This diagram is made up of the homology groups of fiber-bundles over $DP^k(\mathbf{C})$, and the maps are all determined by maps of the homology groups of the fibres by a spectral sequence argument. As a result of this it suffices to prove that the following diagram commutes:

$$\begin{array}{ccc} H_q((S^3)^{k-1}) & \xrightarrow{J_3} & H_{q-2k+2}((S^1)^{k-1} \\ \downarrow & & \downarrow \\ H_q((S^3)^k) & & H_{q-2k+2}((S^1)^k) \\ \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle (i_4)_*} \\ H_q((S^3)^k, (S^3)^k \backslash S^3 \times (S^1)^{k-1}) & \xrightarrow{\Phi} & H_{q-2k+2}(S^3 \times (S^1)^{k-1}) \end{array} \tag{6.6}$$

where i_4 is the inclusion of the fiber. There is an embedding

$$i_5 \colon S^3 \times (S^1)^{k-1} \hookrightarrow (S^3)^k$$

given by

$$i_5(u, \lambda_2, \cdots, \lambda_k) = (u, \lambda_2 u, \cdots, \lambda_k u).$$

This embedding gives rise to the Thom isomorphism Φ.

We have already computed the map J_3. We have the following isomorphisms

$$H^*((S^3)^k) = E(a_1, \cdots, a_k),$$
$$H^*((S^1)^k) = E(b_1, \cdots, b_k),$$
$$H^*(S^3 \times (S^1)^{k-1}) = E(a, \overline{b}_2, \cdots, \overline{b}_k).$$

In each case the right-hand side is an exterior algebra on generators in the following dimensions: a and a_i, for $i = 1, \cdots, k$, are in dimension 3, and b_i, for $i = 1, \cdots, k$, and $\overline{b}_i$, for $i = 2, \cdots, k$ are in dimension 1.

These isomorphisms are determined by choosing orientations for S^1 and S^3, and we assume that the maps which induce the homomorphisms in (6.6) are all orientation preserving. Thus we will complete the proof by proving the following:

$$(6.7)\qquad \begin{aligned} (\Phi \circ j)^*(a \cdot \bar{b}_2 \cdots \bar{b}_k) &= a_1 \cdots a_k, \\ (\Phi \circ j)^*(\bar{b}_2 \cdots \bar{b}_k) &= \sum_i (-1)^{i+1} a_1 \cdots \hat{a}_i \cdots a_k, \end{aligned}$$

and $(\Phi \circ j)^*$ is zero on all other generators. Here, as usual, the hat indicates omission. Also

$$(6.8)\qquad (i_4)_*((b_2 \cdots b_k)^*) = (\bar{b}_2 \cdots \bar{b}_k)^*.$$

As in the proof of Lemma 6.5 the first equation follows since $H^*(S^3 \times (S^1)^{k-1} \backslash (S^1)^k) = 0$. Also as in that proof of the map, $(\Phi \circ j)^*$ is a map of $H^*((S^3)^k)$ algebras. Now $(i_5)^*(a_i) = a$ for any $i = 1, \cdots, k$. This means that, for any $i = 1, \cdots, k$,

$$a_i(\Phi \circ j)^*(\bar{b}_2 \cdots \bar{b}_k) = (\Phi \circ j)^*(a\bar{b}_2 \cdots \bar{b}_k) = a_1 \cdots a_k,$$

which proves the second equation of (6.7) and completes the description of $(\Phi \circ j)^*$. Equation (6.8) is clear from the definition of i_4.

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