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THE RESOLVENT OF THE LAPLACIAN ON LOCALLY SYMMETRIC SPACES

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Introduction

Let X be an n-dimensional Riemannian symmetric space of strictly negative curvature. Such a space is described as follows. The identity component G of the group of isometries of X is a simple Lie group of rank 1 over the reals. The stability group K of any point $0 \in X$ is a maximal compact subgroup of G and $X = G/K$ with a Riemannian structure corresponding to a multiple of the Killing form of G. Let Δ denote the Laplace-Beltrami operator of X. If $T > 0$ and $x \in X$, let $B_T(x)$ be the metric ball in X of radius T and center x. Let ζ be the volume of the metric unit sphere in X. Then there is a number $h = h(X) > 0$ such that

$$\mathrm{Vol}(B_T(x)) \sim \zeta e^{hT}/h, \qquad T \to +\infty.$$

Here " $\sim$ " means that the limit of the ratio is 1. In the usual jargon of Lie theory, $h = 2\rho$. We use this as the definition since it gives a geometric interpretation of this important number and indicates that it has meaning for a more general class of spaces. It is convenient to write the eigenvalues of Δ in the form $\nu^2 - h^2/4$. In this paper we construct a meromorphic family $R_\nu(x, y)$ of smooth functions on $X \times X - \mathrm{diag}(X)$ such that

(1) $R_\nu(x, y)$ is holomorphic in ν for $\mathrm{Re}\,\nu \geq 0$.

(2) If $\mathrm{Re}\,\nu \geq 0$, then $R_\nu(x, y) \sim \delta(\nu)e^{-(\nu+h/2)d(x,y)}$, $d(x, y) \to \infty$.

(3) $R_\nu(x, y) \sim \zeta d(x, y)^{-n+2}|\log(d(x, y))|^{\delta_{2,n}}$ as $d(x, y) \to 0$. In particular, this implies that for fixed $x \in X$, $R_\nu(x, \cdot)$ is locally integrable on X.

(4) If $f \in C_c^\infty(X)$, $\mathrm{Re}\,\nu \geq 0$, then

$$\int_X R_\nu(x, y)(\Delta - \nu^2 + h^2/4)f(y)\,dV(y) = f(x).$$

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Property (4) will be proved for a larger class of functions in §2 (this is crucial to the applications). Property (3) implies that the left-hand side of the above formula makes sense. That such a family exists for $\operatorname{Re}\nu > h/2$ with slightly weaker estimates can be deduced by general methods of potential theory.

$\delta(\nu)$ can be expressed in terms of the Harish-Chandra c-function (see §1). Condition (2) implies that if $\operatorname{Re}\nu > h/2$, then $R_\nu(x, \cdot)$ is integrable. This combined with the growth of the volumes of the metric balls implies that if Γ is a discrete group of isometries of X such that $\Gamma\backslash X$ has finite volume, then for each x and $\operatorname{Re}\nu > h/2$,

$$\mathbf{P}_\nu(x, y) = \sum_{\gamma\in\Gamma} R_\nu(x, \gamma y)$$

is given by an absolutely convergent series for almost all y and defines an integrable function on $\Gamma\backslash X$. We prove (in §4) that $\mathbf{P}$ has a meromorphic continuation to all of $\mathbf{C}$ and that the poles for $\operatorname{Re}\nu \geq 0$, $\nu \neq 0$, are simple and are located at the ν such that $\nu^2 - h^2/4$ is an eigenvalue of Δ on $L^2(\Gamma\backslash X)$. The residues at these values are computed in terms of the corresponding eigenfunctions (Theorem 4.5). We also give a "functional equation," which in the special case when $\Gamma\backslash X$ is compact, says that $\mathbf{P}_\nu = \mathbf{P}_{-\nu}$ (Proposition 4.3, Theorem 4.5).

The implementation of the continuation involves the construction of a larger class of functions that are progressively less singular on the diagonal. Set $R_{1,\nu} = R_\nu$ and, for $\operatorname{Re}\nu > h/2$, set

$$R_{p+1,\nu}(x, y) = \int_X R_\nu(x, z) R_{p,\nu}(z, y)\, dV(z).$$

We show that if $p > n/4$, then $R_{p,\nu}(x, \cdot)$ is in L^1 and locally in L^2 (the precise results can be found in §3). We sum these functions over Γ as above to obtain

$$\mathbf{P}_{r,\nu}(x, y) = \sum_{\gamma\in\Gamma} R_{r,\nu}(x, \gamma y)$$

and get a holomorphic family of functions for $\operatorname{Re}\nu > h/2$. We show that if f is a function on $\Gamma\backslash X$ such that $\Delta^m f \in L^{2-\varepsilon}(\Gamma\backslash X)$ for $0 \leq m \leq N$ (sufficiently large) and some $\varepsilon > 0$ (possibly depending on f), then

$$\int_{\Gamma\backslash X} \mathbf{P}_{r,\nu}(x, y)(\Delta - \nu^2 + h^2/4)^r f(y)\, dV(y) = f(x)$$

for $\operatorname{Re}\nu > h$. This, combined with the fact that if $r > n/4$ and $\operatorname{Re}\nu > h$, then $\mathbf{P}_{r,\nu}(x, \cdot)$ is square integrable on $\Gamma\backslash X$, allows us to calculate the spectral decomposition of $\mathbf{P}_{r,\nu}(x, \cdot)$ using Langlands' decomposition of $L^2(\Gamma\backslash X)$. Since we have good estimates on these function, we are able to give results on the pointwise convergence of the spectral decomposition of functions with sufficiently many derivatives in $L^{2+\varepsilon}(\Gamma\backslash X)$ for some $\varepsilon > 0$ and the existence of an "Eisenstein transform" (see Theorems 4.2 and 4.7). Results of this nature have been proved in the case of a Fuchsian group acting on the upper half-plane in order to derive meromorphic continuations of various forms of "zeta functions."

As an application of these results we give, in §5, an asymptotic formula for the number of elements of $\Gamma\cdot x$ in a ball of X centered at y for $x, y \in X$, generalizing a result of Margulis [10, Theorem 2] (which applies to the case when $\Gamma\backslash X$ is compact and of constant negative curvature). Although our results only apply to the case of a locally symmetric space X of rank 1, the general formalism of §5 is meaningful in a larger context which we now describe. Let X be a simply connected Riemannian manifold of strictly negative curvature which has a compact Riemannian quotient. Let $B_T(x)$ be the metric ball of radius T and center x in X. Then [10] (cf. [9])

$$\lim_{T\to+\infty} \frac{\mathrm{Vol}(B_T(x))}{T} = h$$

with $h > 0$ depending only on X. h has been interpreted (in [9]) as the entropy of the geodesic flow on the sphere bundle of X. If Γ is a discrete group of isometries of X acting freely, properly discontinuously, and such that $\Gamma\backslash X$ has finite volume then we consider the series

$$L(x, y, s) = \sum_{\gamma\in\Gamma} e^{-(s+h/2)d(x,\gamma y)}$$

for $x, y \in X$. Then the series converges uniformly and absolutely for x, y fixed and $\mathrm{Re}\, s \geq h/2 + \varepsilon$, $\varepsilon > 0$. In §5 we give a conjecture about these functions which we prove for X a symmetric space. In particular, we prove that $L(x, y, s)$ has a meromorphic continuation in s to $\mathbf{C}$ and we relate the poles to the spectrum of the Laplacian. The proofs make essential use of the earlier results on the functions $\mathbf{P}_{r,\nu}$ and certain truncations which we used in the analysis of them. $L(x, y, s)$ has a simple pole at $s = 0$. We use our formula for the residue at 0 and a Tauberian theorem to derive the following asymptotic formula:

$$(\star) \qquad \sum_{\substack{\gamma\in\Gamma \\ B(x,\gamma y)\leq T}} 1 \sim \zeta e^{hT} / \mathrm{Vol}(\Gamma\backslash X), \qquad T \to +\infty.$$

In [10, Theorem 1.2] a similar result is given for general $\Gamma\backslash X$ which are compact with negative curvature (the right-hand side being of the form $C(\Gamma x, \Gamma y)e^{hT}$).

Results similar to Theorems 4.2 and 4.7 have been proved by Good [4] for the case when X is the upper half-plane. Note that in this case $n = 2$, so the smoothing which was necessitated by (3) above for large n is unnecessary.

Theorems 1.1 and 1.2 (which give the basic properties of the R_ν) are well known or at least easily derivable from the literature ([6], [3]). We have included proofs of these results using methods which might be extended to a more general class of spaces. It would be very interesting if there were analogous results to those in §§1 and 2 for the part of the "asymptotic expansion" of the zonal spherical function that decays at ∞. Calculations which we have done for complex groups indicate that the generalization will probably be very subtle.

The authors would like to dedicate $(\star)$ above to their long-time friend Manfredo Do Carmo in commemoration of his sixtieth birthday.

1. Zonal spherical functions

We begin this section by introducing notation which will be used throughout this paper. Let G be a connected, semisimple Lie group with maximal compact subgroup K. Let $G = NAK$ be an Iwasawa decomposition of G. We will assume $\dim A = 1$. As is customary, we will denote a Lie group by an upper case letter and its Lie algebra by the corresponding lower case german letter. Let H denote the (unique) element of $\mathfrak{a}$ such that the smallest eigenvalue of $\operatorname{ad} H_{|\mathfrak{n}}$ is 1. Then $\mathfrak{n} = \mathfrak{n}_1 \oplus \mathfrak{n}_2$ with $\operatorname{ad} H_{|\mathfrak{n}_j} = jI$. Set $p = \dim \mathfrak{n}_1$ and $q = \dim \mathfrak{n}_2$. Then $\dim G/K = n = p + q + 1$. We choose B to be the multiple of the Killing form of $\mathfrak{g}$ defined by $B(H, H) = 1$. If $\nu \in \mathfrak{a}_C^*$ (the complexified dual of $\mathfrak{a}$) and if $a \in A$, $a = \exp(tH)$, then we will use the notation $a^\nu = e^{t\nu(H)}$. We denote by λ the functional on $\mathfrak{a}$ defined by $\lambda(H) = 1$ (i.e., λ is the simple root) and by ρ the functional defined by $\rho(h) = \frac{1}{2}\operatorname{tr}(\operatorname{ad} h_{|\mathfrak{n}})$ for $h \in \mathfrak{a}$ (i.e., $\rho = (p + 2q)/2 \cdot \lambda$).

Set $A^+ = \{\exp(tH) | t > 0\}$. Then $G = K(\mathrm{Cl}(A^+))K$. If $a = \exp(tH)$, then we set

$$\gamma(a) = (e^t - e^{-t})^p (e^{2t} - e^{-2t})^q = 2^{p+q} (\sinh t)^p (\sinh 2t)^q.$$

On A we choose the measure $da = dt$, $a = \exp tH$. On K we use Haar measure normalized so that the total mass is 1. We normalize the invariant measure on G so that (if, say, $f \in C_c^\infty(G)$)

$$\int_G f(g)\,dg = \int_{K\times A^+\times K} \gamma(a) f(k_1 a k_2)\,dk_1\,da\,dk_2.$$

If U is an open subset of G with $KUK = U$, then we will use the notation $C^\infty(K\backslash U/K)$ for the space of all C^∞ functions on U such that $f(k_1uk_2) = f(u)$ for $u \in U$, $k_1, k_2 \in K$.

Let C denote the Casimir operator on G corresponding to B. It is standard that if $f \in C^\infty(K\backslash U/K)$, then

$$\text{(1)} \quad \mathrm{C}f(\exp tH) = \frac{d^2}{dt^2} f(\exp tH) + (p\coth t + 2q\coth 2t)\frac{d}{dt} f(\exp tH).$$

If $g \in G$ then we write $g = n(g)a(g)k(g)$ with $n(g) \in N$, $a(g) \in A$, and $k(g) \in K$. Let θ denote the Cartan involution of G corresponding to K. Set $M = \{k \in K \mid \mathrm{Ad}(k)H = H\}$. On M we use Haar measure normalized to have total mass 1. We set $\overline{N} = \theta(N)$. We normalize the invariant measure on $\overline{N}$ so that

$$\int_{\overline{N}\times M} a(\overline{n})^{2\rho} f(k(\overline{n})m)\,d\overline{n}\,dm = \int_K f(k)\,dk.$$

That is,

$$\text{(2)} \qquad \int_{\overline{N}} a(\overline{n})^{2\rho}\,d\overline{n} = 1.$$

The Harish-Chandra c-function is defined by the formula

$$\text{(3)} \qquad c(\nu) = \int_{\overline{N}} a(\overline{n})^{\nu+\rho}\,d\overline{n}.$$

Since $0 < a(\overline{n})^\mu \le 1$ for $\mu(H) \ge 0$, (2) implies that the integral defining (3) converges absolutely and uniformly in ν for $\operatorname{Re}\nu(H) \ge \rho(H)$. In fact, it is well known that the above integral is absolutely convergent for $\operatorname{Re}\nu(H) > 0$ and that $c(\nu)$ has a meromorphic continuation to $\mathfrak{a}_C^*$ (cf. [13, 8.10.16]).

If $\nu \in \mathfrak{a}_C^*$, then we set

$$\text{(4)} \qquad \varphi_\nu(g) = \int_K a(kg)^{\nu+\rho}\,dk.$$

Thus $\varphi_\nu \in C^\infty(K\backslash G/K)$ and

$$\text{(5)} \qquad C\varphi_\nu = (\nu(H)^2 - \rho(H)^2)\varphi_\nu.$$

As is well known

$$\text{(6)} \qquad \varphi_\nu = \varphi_{-\nu} \quad \text{and} \quad f = f(1)\varphi_\nu$$

if $f \in C^\infty(K\backslash G/K)$ and $Cf = (\nu(H)^2 - \rho(H)^2)f$.

If we replace G by $G-K$, then the above uniqueness is no longer true and there is another family of eigenfunctions for C. The following theorem summarizes the properties of these functions. Although most of the assertions about these functions can be deduced from the literature, we have opted to give complete proofs of the following two theorems since our interpretation of the expansion of the zonal spherical functions is somewhat different from that of the standard literature ([6], [3]).

Theorem 1.1. *If* $\nu \in \mathfrak{a}_C^*$, $\operatorname{Re}\nu(H) \geq 0$, *then there exists* $Q_\nu \in C^\infty(K\backslash(G-K)/K)$ *such that the following hold:*

(a) *The map* $\nu, g \mapsto Q_\nu(g)$ *is continuous on* $\{\nu | \operatorname{Re}\nu(H) \geq 0\} \times (G-K)$.

(b) $\nu \mapsto Q_\nu(g)$ *is holomorphic for* $\operatorname{Re}\nu(H) > 0$ *and has a meromorphic continuation to* $\mathfrak{a}_C^*$. *Furthermore,* $\varphi_\nu = c(-\nu)Q_\nu + c(\nu)Q_{-\nu}$ *on* $G-K$. *This last equation should be interpreted as an equality of meromorphic functions.*

(c) *There exists a constant* C_1 *such that*

$$|\gamma(\exp tH)^{1/2} Q_\nu(\exp tH) - e^{-\nu(H)t}| \leq C_1 t e^{-(2+\operatorname{Re}\nu(H))t}$$

for $t \geq 1$ *and* $\operatorname{Re}\nu \geq 0$.

(d) *There exists a function* $d(\nu)$ *such that*

$$Q_\nu(\exp tH) \sim d(\nu) t^{-p-q+1} |\log t|^{\delta_{p+q,1}} = d(\nu) t^{-n+2} |\log t|^{\delta_{n,2}} \quad \text{as } t \to 0^+.$$

$d(\nu)$ *is meromorphic in* ν *and can be calculated using* $(*)$ *in the proof of Lemma* 1.3.

(e) *If* $f \in C^\infty(K\backslash(G-K)/K)$ *and if* $Cf = (\nu(H)^2 - \rho(H)^2)f$ *with* $\operatorname{Re}\nu \geq 0$, *then* $f = aQ_\nu + b\varphi_\nu$.

Proof. Suppose that $f \in C^\infty(K\backslash(G-K)/K)$ and

$$Cf = (\nu(H)^2 - \rho(H)^2)f.$$

Then

$$\frac{d^2}{dt^2} f(\exp tH) + (p \coth t + 2q \coth 2t) \frac{d}{dt} f(\exp tH) = (\nu(H)^2 - \rho(H)^2) f(\exp tH)$$

by (1). In the classical theory of regular singularities (cf. [12, 5.4, 5.5]) this differential equation has the following equality as its indicial equation at $t = 0$:

$$s(s-1) + (p+q)s = 0.$$

The roots are $s = 0$ and $s = 1 - p - q$. This implies (cf. [12, 5.5]) that if $p + q > 1$, then the following limit exists:

(i) $$\lim_{t \to 0^+} t^{p+q-1} f(\exp tH).$$

If $p + q = 1$, then

$$f(\exp tH) \sim C(1 + |\log t|) \quad \text{as } t \to 0^+.$$

Once (a) and (b) have been proved, (i) combined with the above will imply (d).

We now construct Q_ν. We first note that (1) above can be written in the form

($\star$) $$\begin{aligned} &Cf(\exp tH) \\ &\quad = \gamma(\exp tH)^{-1/2} \frac{d^2}{dt^2} \gamma(\exp tH)^{1/2} f(\exp tH) - \psi(t) f(\exp tH) \end{aligned}$$

with $\psi(t) = (p \coth t + 2q \coth 2t)^2/4 - p \sinh t^{-2}/2 - 2q \sinh 2t^{-2}$. We set $\eta(t) = \psi(t) - (p + 2q)^2/4$. If $\operatorname{Im} z \geq 0$, then we consider the differential equation

$$-\varphi''(t) + \eta(t)\varphi(t) = z^2 \varphi(t)$$

on $(0, \infty)$.

Set $H_c^+ = \{z \in \mathbf{C} \mid \operatorname{Im} z \geq c\}$. Set $u(z) = (e^{2iz} - 1)/2iz$ for $z \in H_c^+$. Put $H^+ = H_0^+$. We note that

(ii) $$|u(z)| \leq \begin{cases} 1 & \text{for } z \in H^+, \\ e^{-2c} & \text{for } z \in H_c^+ \text{ and } c < 0. \end{cases}$$

We will now use the method of Appendix 8 in [13] (which is a variant of a technique in [2]) to construct a solution with the desired asymptotic properties. Some of the results in the proof of this theorem will be used in the proof of Theorem 1.2.

If f is a continuous function on $[a, \infty)$, then set $\|f\|_a = \sup_{t \geq a} |f(t)|$. Let $\mathscr{B}_a$ be the space of all continuous functions f on $[a, \infty)$ such that $\|f\|_a < \infty$. Then $\mathscr{B}_a$ is a Banach space under the norm $\|\cdot\|_a$. If $z \in H_c^+$, $f \in \mathscr{B}_a$, and $t \geq a$, then we set

$$L_{a,z} f(t) = \int_t^\infty u(z(s - t))(s - t)\eta(s) f(s)\, ds.$$

If we rewrite $\eta(t)$ in the form

(iii) $$\eta(t) = \frac{p^2/4 - p/2}{\sinh t^2} + \frac{q^2 - 2q}{\sinh 2t^2} + \frac{pq \cosh}{\sinh t \sinh 2t},$$

then it is easily seen that

(iv) $$|\eta(t)| \le C_1 e^{-2t} \quad \text{if } t \ge 1.$$

This implies that if $\operatorname{Im} z \ge 0$ and $t \ge a \ge 1$, then

$$|L_{a,z} f(t)| \le C_1 \int_t^\infty s e^{-2s}\, ds \cdot \|f\|_a = C_1(1+2t)e^{-2t}\|f\|_a/4.$$

Thus, in particular, $L_{a,z}$ is a uniformly bounded family of operators on $\mathscr{B}_a$, with $\|L_{a,z}\| \le C_1(1+2a)e^{-2a}/4$. If $z \in H_c^+$, $c \ge -1/2$, and $a \ge 1$, then one has

$$|L_{a,z} f(t)| \le C_1 \int_t^\infty s e^{-s}\, ds \cdot \|f\|_a = C_1(1+t)e^{-t}\|f\|_a.$$

One checks (as in [13, A.8.2.9]) that if $a \ge 1$ then $z \mapsto L_{a,z}$ is a continuous map of $H_{-1/2}^+$ into the bounded operators $L(\mathscr{B}_a, \mathscr{B}_a)$ on $\mathscr{B}_a$, which is holomorphic on $\{z \in \mathbf{C} \mid \operatorname{Im} z > -1/2\}$.

Fix $a_0 \ge 1$ such that $C_1(1+a_0)e^{-a_0} \le 1/2$. Fix $\mathscr{B} = \mathscr{B}_{a_0}$. Set $\|\cdot\| = \|\cdot\|_{a_0}$ and $L_z = L_{a_0,z}$. If $\operatorname{Im} z > -1/2$, then we put $h_z = (I - L_z)^{-1}1$ (1 is the constant function indentically equal to 1). Set $g_z(t) = e^{izt}h_z(t)$ for $t \ge a_0$. Then one checks that if $\operatorname{Im} z > -1/2$, then

(v) $$\left(-\frac{d^2}{dt^2} + \eta\right) g_z = z^2 g_z \quad \text{for } t \ge a_0.$$

Also

(vi) $$\begin{aligned} |g_z(t) - e^{izt}| &\le e^{-t \operatorname{Im} z} \int_t^\infty s|\eta(s)|\, ds \\ &\le C' e^{-(\operatorname{Im} z + 2)t}(1+t) \quad \text{if } \operatorname{Im} z \ge 0 \end{aligned}$$

for $t \ge a_0$, and

(vii) $$|g_z(t) - e^{izt}| \le e^{-t \operatorname{Im} z} \int_t^\infty s e^{s}|\eta(s)|\, ds \le C' e^{-(\operatorname{Im} z + 1)t}(1+t) \quad \text{if } \operatorname{Im} z \ge -1/2.$$

Since the only singularities of the differential equation in (iv) are at 0 and ∞ we see that g_z extends to a solution on $(0, \infty)$ for $\operatorname{Im} z \ge -1/2$.

Let $\nu \in \mathfrak{a}_C^*$ be such that $\operatorname{Re} \nu(H) \ge -1/2$. If $k_1, k_2 \in K$ and if $t \in \mathbf{R}$, $t > 0$, then we set $Q_\nu(k_1 \exp tH k_2) = \gamma(\exp tH)^{-1/2} g_{i\nu(H)}(t)$. Then $Q_\nu \in C^\infty(K\backslash(G-K)/K)$ and $CQ_\nu = (\nu(H)^2 - \rho(H)^2)Q_\nu$. Notice that Q_ν satisfies (a), (b) (except for the meromorphic continuation and the formula for φ_ν), of the theorem. Hence the remarks at the beginning of this proof imply that Q_ν satisfies (d). It is standard that (cf. [6])

$$\text{(viii)} \qquad \lim_{t\to+\infty} \gamma(\exp tH)^{1/2} e^{-\nu(H)t} \varphi_\nu(\exp tH) = c(\nu) \quad \text{for } \operatorname{Re}\nu(H) > 0.$$

(vi) and (vii) imply that if $|\operatorname{Re}\nu(H)| < 1/2$ and $\nu \neq 0$, then Q_ν and $Q_{-\nu}$ are linearly independent. Hence there exist holomorphic functions $a(\nu)$ and $b(\nu)$ on the punctured strip $|\operatorname{Re}\nu(H)| < 1/2$, $\nu \neq 0$, such that

$$\varphi_\nu = a(\nu)Q_\nu + b(\nu)Q_{-\nu}.$$

Since $\varphi_\nu = \varphi_{-\nu}$ it follows that $b(\nu) = a(-\nu)$. Also (vi), (vii), and (viii) imply that $b(\nu) = c(\nu)$ on the punctured strip. We have thus shown that

$$\text{(ix)} \quad \varphi_\nu = c(-\nu)Q_\nu + c(\nu)Q_{-\nu} \quad \text{on } G - K \text{ for } |\operatorname{Re}\nu(H)| < 1/2,\ \nu \neq 0.$$

We can thus implement the meromorphic continuation of Q_ν by observing that (ix) can be written in the form

$$Q_\nu = (\varphi_\nu - c(\nu)Q_{-\nu})c(-\nu)^{-1}.$$

This completes the proof of (b). Since (e) is now clear, the theorem follows.

Theorem 1.2. *There exists a family of rational functions* $a_k(\nu)$, $k = 1, 2, \cdots$, *on* $\mathfrak{a}_C^*$ *that are holomorphic for* $\operatorname{Re}\nu(H) \geq 0$ *with the following properties*:

(a) *The set* $\mathscr{S} = \{\nu | \nu$ *a pole of some* $a_k\}$ *is contained in* $\mathfrak{a}^*$ *and has no finite point of accumulation.*

(b) *If* $\nu \notin \mathscr{S}$, *then*

$$Q_\nu(\exp tH) = e^{-t(\nu+\rho)(H)} \left(1 + \sum_{k\geq 1} a_k(\nu)e^{-2kt} \right)$$

with the convergence uniform for $t \geq c$ *and* c *sufficiently large.*

(c) *Let* $c \leq 0$ *be given. Then there exist a nonzero polynomial* f_c *on* $\mathfrak{a}_C^*$ *and an integer* $d(c) \geq 0$ *such that for each* $\varepsilon > 0$

$$\left| f_c(\nu) \left(Q_\nu(\exp tH) - e^{-t(\nu+\rho)(H)} \left(1 + \sum_{1\leq k\leq N} a_k(\nu)e^{-2kt} \right) \right) \right| \leq (1 + |\nu(H)|)^{d(c)} C_{c,\varepsilon} e^{-t(\operatorname{Re}(\nu+\rho)(H)+2(N+1)-\varepsilon)}$$

for $t \geq 1$ *and* $\operatorname{Re}\nu(H) \geq c$. *There exists* $c_0 < 0$ *such that we may take* $f_{c_0}(\nu) \equiv 1$.

Proof. In light of formula (iii) in the proof of the previous theorem it is easily seen that

(i) $$\eta(t) = \sum_{j\geq 1} b_j e^{-2jt}$$

with convergence absolute and uniform on sets of the form $t \geq a > 0$.

We now consider the operators L_z in the proof of the previous theorem. We write $\eta(t) = \sum_{1\leq j\leq N} b_j e^{-2jt} + \eta_N(t) = \mu_N(t) + \eta_N(t)$. Then one sees easily that

(ii) $$|\eta_N(t)| \leq C_N e^{-2(N+1)t}$$

for $t \geq 1$.

This allows us to write L_z as $L_{z,N} + M_{z,N}$ with

$$L_{z,N}f(t) = \int_t^\infty u(z(s-t))(s-t)\mu_N(s)f(s)\,ds$$

and

$$M_{z,N}f(t) = \int_t^\infty u(z(s-t))(s-t)\eta_N(s)f(s)\,ds.$$

If we argue as in the preceding proof, we find that $M_{z,N}$ is holomorphic in z for $\operatorname{Im} z \geq N-1$ and

$$|M_{z,N}f(s)| \leq C_N(1+s)e^{-2Ns}\|f\|_t$$

for $s \geq t \geq 1$. On the other hand, if $f(t) = e^{-2kt}$ with $k = 0, 1, 2, \cdots$, then

$$L_{z,N}f(t) = -\frac{1}{4}\sum_{j=1}^{N} \frac{e^{-2(j+k)t}}{(j+k)(iz-j-k)} \cdot b_j.$$

Since $h_z = (I - L_z)^{-1}1 = 1 + L_z 1 + L_z^2 1 + \cdots$, we can use the above formulas to analyze the individual terms. The result now follows without any real difficulty. q.e.d.

The next result is the key to the rest of the results in this paper.

Lemma 1.3. $\lim_{t\to 0^+} \gamma(\exp tH)\frac{d}{dt}Q_\nu(\exp tH) = -2\nu(H)c(\nu)$.

Proof. The observations at the beginning of the proof of Theorem 1.2 imply that the limit on the left-hand side of the above formula exists for each ν for which Q_ν is defined and this limit is a meromorphic function of ν. We calculate this limit using an indirect method which will also be used in the next section. We note that on $G - K$ we have

$$(CQ_\nu)\varphi_\nu - Q_\nu(C\varphi_\nu) = 0.$$

That is,

$$(\gamma^{1/2}CQ_\nu)\gamma^{1/2}\varphi_\nu - \gamma^{1/2}Q_\nu(\gamma^{1/2}C\varphi_\nu) = 0.$$

We use formula $(\star)$ in the proof of Theorem 1.1 to rewrite this as

$$\begin{aligned}
0 = &\left\{\left(\frac{d^2}{dt^2} - \eta(t)\right)(\gamma(\exp tH)^{1/2}Q_\nu(\exp tH))\right\}\gamma(\exp tH)^{1/2}\varphi_\nu(\exp tH) \\
&-\gamma(\exp tH)^{1/2}Q_\nu(\exp tH)\left\{\left(\frac{d^2}{dt^2} - \eta(t)\right)(\gamma(\exp tH)^{1/2}\varphi_\nu(\exp tH))\right\} \\
= &\left(\frac{d^2}{dt^2}(\gamma(\exp tH)^{1/2}Q_\nu(\exp tH))\right)\gamma(\exp tH)^{1/2}\varphi_\nu(\exp tH) \\
&- \gamma(\exp tH)^{1/2}Q_\nu(\exp tH)\left(\frac{d^2}{dt^2}(\gamma(\exp tH)^{1/2}\varphi_\nu(\exp tH))\right) \\
= &\frac{d}{dt}\left\{\left(\frac{d}{dt}(\gamma(\exp tH)^{1/2}Q_\nu(\exp tH))\right)\gamma(\exp tH)^{1/2}\varphi_\nu(\exp tH)\right. \\
&\left. - \gamma(\exp tH)^{1/2}Q_\nu(\exp tH)\left(\frac{d}{dt}(\gamma(\exp tH)^{1/2}\varphi_\nu(\exp tH))\right)\right\}.
\end{aligned}$$

Thus

$$\begin{aligned}
&\left(\frac{d}{dt}(\gamma(\exp tH)^{1/2}Q_\nu(\exp tH))\right)\gamma(\exp tH)^{1/2}\varphi_\nu(\exp tH) \\
&\quad - \gamma(\exp tH)^{1/2}Q_\nu(\exp tH)\left(\frac{d}{dt}(\gamma(\exp tH)^{1/2}\varphi_\nu(\exp tH))\right) \\
&= \left(\frac{d}{dt}Q_\nu(\exp tH)\right)\gamma(\exp tH)\varphi_\nu(\exp tH) \\
&\quad - \gamma(\exp tH)Q_\nu(\exp tH)\left(\frac{d}{dt}\varphi_\nu(\exp tH)\right)
\end{aligned}$$

is constant as a function of t. We calculate this value (as a function of ν) for $\operatorname{Re}\nu(H) > 0$ by computing the limit in t at $+\infty$. We have that if $\operatorname{Re}\nu(H) > 0$, then as $t \to +\infty$

$$\begin{aligned}
\frac{d}{dt}(\gamma(\exp tH)^{1/2}Q_\nu(\exp tH)) &\sim -\nu(H)e^{-\nu(H)t}, \\
\gamma(\exp tH)^{1/2}Q_\nu(\exp tH) &\sim e^{-\nu(H)t}, \\
\frac{d}{dt}(\gamma(\exp tH)^{1/2}\varphi_\nu(\exp tH)) &\sim \nu(H)c(\nu)e^{\nu(H)t}, \\
\gamma(\exp tH)^{1/2}\varphi_\nu(\exp tH) &\sim c(\nu)e^{\nu(H)t}.
\end{aligned}$$

So the constant is given by $-2\nu(H)c(\nu)$.

We note that Theorem 1.1(d) implies that

$$\lim_{t\to 0^+} \gamma(\exp tH) Q_\nu(\exp tH) \left(\frac{d}{dt}\varphi_\nu(\exp tH)\right) = 0.$$

So

$$(*) \qquad \lim_{t\to 0^+} \gamma(\exp tH)\frac{d}{dt}Q_\nu(\exp tH) = -2\nu(H)c(\nu)$$

for $\operatorname{Re}\nu(H) > 0$. The result now follows by meromorphic continuation.

2. Analysis of the functions Q_ν

In this section we will study the functional analytic properties of the functions Q_ν which were constructed in the previous section. Recall that we used the notation $n = p + q + 1 = \dim G/K$.

If $1 \le s < \infty$, and f is a measurable function on G, then (as is usual) we say that $f \in L^s_{\mathrm{loc}}$ if

$$\int_U |f(g)|^s \, dg < \infty$$

for each open subset U of G with compact closure.

Lemma 2.1. *Let $\nu \in \mathfrak{a}^*_C$. If Q_ν is defined, then $Q_\nu \in L^1_{\mathrm{loc}}$. If $\operatorname{Re}\nu(H) > \rho(H)$, then $Q_\nu \in L^1(G)$.*

Proof. Let U be open in G with compact closure. Then there exist a and b, $0 \le a < b < \infty$, such that $U \subset K\exp([a, b]H)K$. Thus,

$$\int_U |Q_\nu(g)| \, dg \le \int_a^b \gamma(\exp tH)|Q_\nu(\exp tH)| \, dt.$$

If $0 < t \le b$, then

$$\gamma(\exp tH) \le C_b t^{n-1},$$

and Theorem 1.1(d) implies that

$$|Q_\nu(\exp tH)| \le C_{\nu,b} t^{-n+2} |\log t|^{\delta_{n,2}}$$

with $C_{\nu,b} < \infty$ where Q_ν is defined. Hence if $0 < t \le b$, then

$$\gamma(\exp tH)|Q_\nu(\exp tH)| \le C_b C_{\nu,b} t |\log t|^{\delta_{n,2}}.$$

This yields the first assertion.

We note that

$$\gamma(\exp tH) \le 2^{n-1} e^{2\rho(H)t}$$

for $t \geq 0$. Thus, Theorem 1.1(c) implies that if $\operatorname{Re}\nu(H) \geq 0$, then there exists $C' < \infty$ (depending on ν) such that

$$\gamma(\exp tH)|Q_\nu(\exp tH)| \leq C' e^{(\rho(H) - \operatorname{Re}\nu(H))t} \quad \text{for } t \geq 1.$$

If we set $U = K\exp([0, 1]H)K$, then

$$\int_G |Q_\nu(g)|\, dg = \int_U |Q_\nu(g)|\, dg + \int_1^\infty \gamma(\exp tH)|Q_\nu(\exp tH)|\, dt.$$

This combined with the first assertion implies the second. q.e.d.

We define the inner product $\langle\ ,\ \rangle$ on $\mathfrak{g}$ by $\langle X, Y\rangle = -B(X, \theta Y)$. If $g \in G$, then we set

$$\|g\| = \begin{cases} \|\operatorname{Ad}(g)\|_{HS} & \text{if } \mathfrak{n}_2 = 0, \\ \|\operatorname{Ad}(g)\|_{HS}^{1/2} & \text{if } \mathfrak{n}_2 \neq 0. \end{cases}$$

We note that if $g \in G$ and if $k_1, k_2 \in K$, then $\|g\| = \|g^{-1}\| = \|k_1 g k_2\|$. If $\mu \geq 0$, then we set $C_\mu^\infty(G)$ equal to the space of all functions $f \in C^\infty(G)$ such that

$$p_{X,\mu}(f) = \sup_{g\in G} \|g\|^{-\mu}|Xf(g)| < \infty$$

for all $X \in U(\mathfrak{g})$. We endow $C_\mu^\infty(G)$ with the topology induced by the seminorms $p_{X,\mu}$.

Lemma 2.2. *If $f \in C_\mu^\infty(G/K)$ and $\operatorname{Re}\nu(H) > \rho(H) + \mu$, then*

$$\int_G Q_\nu(x^{-1}y)(C - \nu(H)^2 + \rho(H)^2)f(y)\, dy = -2\nu(H)c(\nu)f(x).$$

In particular, if we set $\delta_1(f) = f(1)$, then $(C - \nu(H)^2 + \rho(H)^2)Q_\nu = -2\nu(H)c(\nu)\delta_1$ in the sense of distributions on $C_\mu^\infty(G/K)$.

Proof. If $f \in C_\mu^\infty(G)$, then

$$|f(xy)| \leq p_{1,\mu}(f)\|xy\|^\mu \leq \|x\|^\mu p_{1,\mu}(f)\|y\|^\mu.$$

Thus,

$$\begin{aligned} \int_G |Q_\nu(x^{-1}y)f(y)|\, dy &= \int_G |Q_\nu(y)f(xy)|\, dy \\ &\leq \|x\|^\mu p_{1,\mu}(f) \int_G |Q_\nu(y)|\, \|y\|^\mu\, dy \\ &= \|x\|^\mu p_{1,\mu}(f) \int_{A^+} \gamma(a)|Q_\nu(a)|\, \|a\|^\mu\, da. \end{aligned}$$

Since $\gamma(\exp tH)\|\exp tH\|^{\mu}|Q_\nu(\exp tH)| \le C_{\mu,\nu}e^{(\mu+\rho(H)-\operatorname{Re}\nu(H))t}$ for $t \ge 1$, and $\gamma(\exp tH)\|\exp tH\|^{\mu}|Q_\nu(\exp tH)| \le C'_{\mu\nu}t(1-\log t)^{\delta_{n,2}}$ for $0 \le t < 1$, we conclude that, if $\operatorname{Re}\nu(H) > \rho(H) + \mu$, then

$$\int_G |Q_\nu(x^{-1}y)f(y)|\,dy \le C''_{\mu,\nu}\|x\|^{\mu}p_{1,\mu}(f).$$

In light of the above calculations and inequalities it is enough to show that

$$(\star) \qquad \int_G Q_\nu(y)(C - \nu(H)^2 + \rho(H)^2)f(y)\,dy = -2\nu(H)c(\nu)f(1)$$

for $f \in C^\infty_\mu(G/K)$ and $\operatorname{Re}\nu(H) > \rho(H) + \mu$. We will assume that ν satisfies the inequality of the lemma throughout the rest of the proof. The formal aspects of the argument that follows are justified by the inequalities at the beginning of this proof.

For $f \in C^\infty_\mu(G/K)$ we set

$$f^0(y) = \int_K f(ky)\,dk.$$

Then the left-hand side of $(\star)$ is equal to

$$\begin{aligned}
&\int_{A^+} \gamma(a)Q_\nu(a)(C - \nu(H)^2 + \rho(H)^2)f^0(a)\,da \\
&= \lim_{r\to 0}\int_r^\infty \gamma(t)\{Q_\nu(t)Cf^0(\exp tH) - (CQ_\nu(t))f^0(\exp tH))\}\,dt \\
&= \lim_{r\to 0}\int_r^\infty \frac{d}{dt}\left[\gamma(t)^{1/2}(Q_\nu(t)\frac{d}{dt}(\gamma(t)^{1/2}f^0(\exp tH)) \right.\\
&\qquad \left. - \frac{d}{dt}(\gamma(t)^{1/2}Q_\nu(t))f^0(\exp tH)\right]\,dt,
\end{aligned}$$

where $\gamma(t) = \gamma(\exp tH)$, $Q_\nu(t) = Q_\nu(\exp tH)$. Here we have used $(\star)$ from the proof of Theorem 1.1. We conclude that $(\star)$ is equal to

$$\begin{aligned}
-\lim_{t\to 0^+}&\left[\gamma(t)^{1/2}\left(Q_\nu(t)\frac{d}{dt}(\gamma(t)^{1/2}f^0(\exp tH))\right)\right.\\
&\qquad \left. - \left(\frac{d}{dt}(\gamma(t)^{1/2}Q_\nu(t))f^0(\exp tH)\right)\right].
\end{aligned}$$

Theorem 1.1(d) implies that $\lim_{t\to 0^+}\gamma(t)Q_\nu(t) = 0$. Thus the above expression is equal to

$$\lim_{t\to 0^+} f^0(\exp tH)\gamma(t)\frac{d}{dt}Q_\nu(t),$$

which equals $-2\nu(H)c(\nu)f(1)$ by Lemma 1.3. The proof is now complete. q.e.d.

We note that if $n = 2$ or 3, then $Q_\nu \in L^2_{\text{loc}}$. Unfortunately (for our purposes), if $n > 3$, then Q_ν is not an element of L^2_{loc}. The rest of this section is devoted to the proof that certain convolution powers of Q_ν with itself are in L^2_{loc} (or better).

Lemma 2.3.

$$\int_K Q_\nu(xky)\,dk = \begin{cases} Q_\nu(x)\varphi_\nu(y) & \text{if } \|x\| > \|y\|, \\ \varphi_\nu(x)Q_\nu(y) & \text{if } \|x\| < \|y\|, \end{cases}$$

for all ν for which Q_ν is defined.

Proof. We prove the result for $\operatorname{Re}\nu(H) \geq 0$. The general assertion would then follow by analytic continuation. In this range the only singularities of $Q_\nu(x)$ are in K. We set

$$\beta_\nu(x, y) = \int_K Q_\nu(xky)\,dk.$$

Obviously, $\beta_\nu(x, y)$ is defined if $xKy \cap K$ is empty. If $xKy \cap K$ is nonempty, then $xK = Ky^{-1}$. So $\|x\| = \|y\|$. Thus,

(1) β_ν is defined and real analytic on the set $\{(x, y) \in G \times G \mid \|x\| \neq \|y\|\}$.

Let C_x (resp. C_y) denote the Casimir operator in the first (resp. second) variable in $G \times G$. It is clear that

$$\begin{aligned} C_x\beta_\nu(x, y) &= (\nu(H)^2 - \rho(H)^2)\beta_\nu(x, y), \\ C_y\beta_\nu(x, y) &= (\nu(H)^2 - \rho(H)^2)\beta_\nu(x, y) \end{aligned} \tag{2}$$

if $\|x\| \neq \|y\|$. We also observe that

$$\beta_\nu(k_1xk_2, k_3yk_4) = \beta_\nu(x, y) \quad \text{for } x, y \in G,\ k_i \in K,\ 1 \leq i \leq 4. \tag{3}$$

The lemma will follow from

(4) If U is a connected K-bi-invariant neighborhood of K in G and if $f \in C^\infty(K\backslash U/K)$ satisfies $Cf = (\nu(H)^2 - \rho(H)^2)f$ on U, then $f = (f(1)\varphi_\nu|_U$.

Indeed, set $u(t) = f(\exp tH)$. Then u satisfies the differential equation (1) in §1, where defined. We note that $U = \exp(JH)$, where J is an interval of the form $[0, b)$. Our assumption is that u extends to a C^∞ function on $(-b, b)$ if we set $u(-t) = u(t)$. But then u extends to a C^∞ function on $\mathbf{R}$. We set $g(k_1 \exp tHk_2) = u(t)$, $k_1, k_2 \in K$, $t \in \mathbf{R}$.

Then $g \in C^\infty(K\backslash G/K)$ and $Cg = (\nu(H)^2 - \rho(H)^2)g$, since $g_{|U} = f$. Condition (4) now follows from (6) in §1. q.e.d.

Let $x \in G - K$. Set $U_x = \{y \in G | \, \|y\| < \|x\|\}$. Then

$$C_y\beta_\nu(x, y) = (\nu(H)^2 - \rho(H)^2)\beta_\nu(x, y)$$

for $y \in U_x$. Since U_x is an open connected K-bi-invariant neighborhood of K, (4) implies that

$$\beta_\nu(x, y) = \beta_\nu(x, 1)\varphi_\nu(y).$$

But the formula for β_ν yields that $\beta_\nu(x, 1) = Q_\nu(x)$. Hence, $\beta_\nu(x, y) = Q_\nu(x)\varphi_\nu(y)$ for $\|x\| > \|y\|$. Similarly, if $y \in G - K$, then $\beta_\nu(x, y) = \varphi_\nu(x)Q_\nu(y)$ for $x \in U_y$.

Note. If $x \in K(\exp tH)K$, then

$$\|x\| = \begin{cases} (d + 2p\cosh 2t)^{1/2} & \text{if } q = 0, \\ (d + 2p\cosh 2t + 2q\cosh 4t)^{1/4} & \text{if } q > 0, \end{cases}$$

with $d = \dim(\mathfrak{m} + \mathfrak{a})$.

We can therefore rephrase the above lemma in the following way.

Lemma 2.3′.

$$\int_K Q_\nu((\exp tH)k(\exp sH))\, dk = \begin{cases} Q_\nu(\exp tH)\varphi_\nu(\exp sH) & \textit{if } |t| > |s|, \\ \varphi_\nu(\exp tH)Q_\nu(\exp sH) & \textit{if } |t| < |s|. \end{cases}$$

If $\operatorname{Re}\nu(H) > \rho(H)$ and $r = 1, 2, \cdots$, then we define $Q_{r,\nu}$ recursively as follows:

$$Q_{1,\nu} = Q_\nu, \qquad Q_{r+1,\nu} = Q_{1,\nu} \star Q_{r,\nu}.$$

Lemma 2.1 implies that $Q_{r,\nu} \in L^1(G)$ for all $r \geq 1$ and $\operatorname{Re}\nu(H) > \rho(H)$.

Lemma 2.4. *Let* $\operatorname{Re}\nu(H) > \rho(H)$. *There exist constants* $c_r(\nu)$, $c_r'(\nu)$, *and* $c_r''(\nu)$ *such that*

$$|Q_{r,\nu}(\exp tH)| \leq c_r(\nu)t^{r-1}\gamma(\exp tH)^{-1/2}e^{-\operatorname{Re}\nu(H)t} \quad \textit{for } t \geq 1,$$

$$|Q_{r,\nu}(\exp tH)| \leq \begin{cases} c_r'(\nu)(1 + t^{2r-1}\gamma(\exp tH)^{-1}) & \textit{if } r \neq n/2,\ 0 < |t| < 1, \\ c_r''(\nu)(1 + \log|t|) & \textit{if } r = n/2,\ 0 < |t| < 1. \end{cases}$$

In particular, if $r > n/2$ *then* $Q_{r,\nu}$ *is bounded in a neighborhood of* K.

Proof. If $r = 1$, then this lemma is just a restatement of (c) and (d) of Theorem 1.1. To prove the result for $r > 1$ we first calculate $Q_{r+1,\nu}(\exp tH)$:

$$\begin{aligned}
Q_{r+1,\nu}&(\exp tH)\\
&= \int_G Q_{r,\nu}(x)Q_\nu(x^{-1}\exp tH)\,dx\\
&= \int_{K\times A^+\times K} \gamma(a)Q_{r,\nu}(k_1ak_2)Q_\nu((k_1ak_2)^{-1}\exp tH)\,dk_1\,da\,dk_2\\
&= \int_{A^+\times K} \gamma(a)Q_{r,\nu}(a)Q_\nu(a^{-1}k\exp tH)\,da\,dk.
\end{aligned}$$

There exists $k_0 \in K$ such that $k_0ak_0^{-1} = a^{-1}$ for $a \in A$. Thus Lemma 2.3′ implies that

$$\begin{aligned}
Q_{r+1,\nu}(\exp tH) &= \int_0^\infty \gamma(s)Q_{r,\nu}(\exp sH)\int_K Q_\nu((\exp sH)k\exp tH)\,dk\,ds\\
&= \int_t^\infty \gamma(s)Q_{r,\nu}(\exp sH)Q_\nu(\exp sH)\,ds\cdot\varphi_\nu(\exp tH)\\
&\quad + \int_0^t \gamma(s)Q_{r,\nu}(\exp sH)\varphi_\nu(\exp sH)ds\cdot Q_\nu(\exp tH).
\end{aligned}$$

We will use this formula to prove the inequalities in the lemma by induction on r. We have already proved them for $r = 1$. So assume them for r. We note that (viii) in the proof of Theorem 1.1 implies that

$$|\varphi_\nu(\exp tH)| \le c''(\nu)\gamma(t)^{-1/2}e^{\mathrm{Re}\,\nu(H)t} \quad \text{for } t \ge 1, \tag{1}$$

if $\mathrm{Re}\,\nu(H) > 0$.

We first prove the inductive step for $t \ge 1$. We rewrite the identity above as

$$\begin{aligned}
Q_{r+1,\nu}(\exp tH) &= \int_t^\infty \gamma(s)Q_{r,\nu}(\exp sH)Q_\nu(\exp sH)\,ds\cdot\varphi_\nu(\exp tH)\\
&\quad + \int_1^t \gamma(s)Q_{r,\nu}(\exp sH)\varphi_\nu(\exp sH)\,ds\cdot Q_\nu(\exp tH)\\
&\quad + \int_0^1 \gamma(s)Q_{r,\nu}(\exp sH)\varphi_\nu(\exp sH)\,ds\cdot Q_\nu(\exp tH).
\end{aligned}\tag{2}$$

If we use the inductive hypothesis and (1) to estimate each term on the right-hand side of (2) we have

$$
\begin{aligned}
&|Q_{r+1,\nu}(\exp tH)| \\
&\quad \le c_r(\nu)c_1(\nu)c''(\nu)\int_t^\infty s^{r-1}e^{-2\operatorname{Re}\nu(H)s}\,ds\cdot\gamma(t)^{-1/2}e^{\operatorname{Re}\nu(H)t} \\
&\quad + c_r(\nu)c_1(\nu)c''(\nu)\int_1^t s^{r-1}\,ds\cdot\gamma(t)^{-1/2}e^{-\operatorname{Re}\nu(H)t} \\
&\quad + c'''(\nu)\gamma(t)^{-1/2}e^{-\operatorname{Re}\nu(H)t}
\end{aligned}
\tag{3}
$$

with

$$c'''(\nu)=c_1(\nu)\int_0^1\gamma(s)|Q_{r,\nu}(\exp sH)\varphi_\nu(\exp sH)|\,ds<\infty$$

by the inductive hypothesis.

Inequality (3) clearly implies the asserted inequality for $t\ge 1$. We are left with the inequalities for $0<t<1$. As in the previous step we start with the expression (2). We are still assuming the result for $r\ge 1$ and proving it for $r+1$. Set $D(\nu)=\sup_{0\le t\le 1}|\varphi_\nu(\exp tH)|$. Then, if $0<t<1$, equality (2) yields

$$
\begin{aligned}
|Q_{r+1,\nu}(\exp tH)| &\le c_r(\nu)c_1(\nu)D(\nu)\int_1^\infty s^{r-1}e^{-2\operatorname{Re}\nu(H)s}\,ds \\
&\quad + c_1'(\nu)D(\nu)\int_t^1 s|Q_{r,\nu}(\exp sH)|\,ds \\
&\quad + c_1'(\nu)D(\nu)\int_0^t \gamma(s)|Q_{r,\nu}(\exp sH)|\,ds\cdot t\gamma(t)^{-1} \\
&= A(t)+B(t)+C(t).
\end{aligned}
$$

We now look at the three possibilities: $r+1<n/2$, $r+1=n/2$, and $r+1>n/2$. If $r+1<n/2$, and we apply the inductive hypothesis, then

$$B(t)\le b_r(\nu)\int_t^1 s^{2r}\gamma(s)^{-1}\,ds\le b_{r+1}(\nu)(1+t^{2r+1}\gamma(t)^{-1}),$$

$$C(t)\le d_r(\nu)\int_0^t s^{2r-1}\,ds\cdot t\gamma(t)^{-1}\le d_{r+1}(\nu)t^{2r+1}\gamma(t)^{-1}$$

with $b_r(\nu)$ and $d_r(\nu)$ appropriate finite constants. This proves the second inequality in this case. We now look at $r+1=n/2$. Then in expression (4) we have

$$B(t)\le b_r(\nu)+b_r'(\nu)\int_t^1\frac{ds}{s}\le b_r(\nu)+b_r'(\nu)|\log t|,$$

$$C(t)\le d_r(\nu)\int_0^t s^{2r-1}\,ds\cdot t\gamma(t)^{-1}\le d_{r+1}(\nu)<\infty,$$

which completes the argument in this case.

If $r+1>n/2$ and $r\le n/2$, then either $r=n/2$ or $r=(n-1)/2=(p+q)/2$. If $r=n/2$, then the inductive hypothesis asserts that if $0<s<1$, then

$$|Q_{r,\nu}(\exp sH)|\le b(\nu)+d(\nu)|\log s|.$$

If $r=(p+q)/2$, then

$$|Q_{r,\nu}(\exp sH)|\le b(\nu)s^{2r-1-p-q}=b(\nu)s^{-1}.$$

Thus, in both cases,

$$|Q_{r,\nu}(\exp sH)|\le b'(\nu)s^{-1}$$

for $0<s<1$. So in this case we see that B and C are bounded for $0<t<1$, which implies the inequality. If $r>n/2$, then it is easily seen that $B(t)$ and $C(t)$ are bounded. The verification of the inductive step is now complete.

Corollary 2.5. *If $r>n/4$ and $\operatorname{Re}\nu(H)>\rho(H)$, then there exists $\varepsilon=\varepsilon_r>0$ such that $Q_{r,\nu}\in L^s(G)$ for $1\le s\le 2+\varepsilon$. If $r>n/2$, then $Q_{r,\nu}\in L^s(G)$ for $1\le s\le\infty$.*

Proof. Assume that $\operatorname{Re}\nu(H)>\rho(H)$. The preceding lemma implies that if U is a K-bi-invariant neighborhood of K, then $Q_{r,\nu}\in L^s(G-U)$ for all $1\le s\le\infty$. It also implies that if $r>n/2$, then $Q_{r,\nu}\in L^s(G)$ for all $1\le s\le\infty$. We must therefore show that if $r>n/4$, then $Q_{r,\nu}\in L^s_{\text{loc}}$ for $1\le s\le 2+\varepsilon$ for some $\varepsilon>0$. So assume that $r>n/4$. Lemma 2.4 yields that if $0<t\le 1$, then $|Q_{r,\nu}(\exp tH)|\le C(\delta,r,\nu)(1+t^{2r-n-\delta})$ for all $\delta>0$. Set $U=K\exp([0,1]H)K$. Then

$$(\star)\qquad \int_U |Q_{r,\nu}(g)|^{2+\varepsilon}\,dg=\int_0^1\gamma(t)|Q_{r,\nu}(\exp tH)|^{2+\varepsilon}\,dt \le C(\delta,r,\nu)\int_0^1\gamma(t)(1+t^{2r-n-\delta})^{2+\varepsilon}\,dt.$$

Choose $0<\delta<1$ such that $2r-\delta>n/2+\delta/2$. Then

$$(\star)\le 2^{p+2q}C(\delta,r,\nu)\int_0^1 t^{n-1}(1+t^{-n/2+\delta/2})^{2+\varepsilon}\,dt < c\int_0^1(1+t^{\delta-1-n\varepsilon/2})\,dt.$$

The last integral is finite if $\varepsilon<2\delta/n$. This completes the proof. q.e.d.

We now generalize the functional analytic interpretation of Q_ν to $Q_{r,\nu}$.

Lemma 2.6. *If $\mu \geq 0$ and $\operatorname{Re}\nu(H) > \rho(H) + \mu$, then*

$$\int_G Q_{r,\nu}(x^{-1}y)(C - \nu(H)^2 + \rho(H)^2)^r f(y)\,dy = (-2\nu(H)c(\nu))^r f(x)$$

for $f \in C_\mu^\infty(G/K)$.

Proof. It is enough to prove the lemma for $x = 1$, that is, to show that $(C - \nu(H)^2 + \rho(H)^2)^r Q_{r,\nu} = (-2\nu(H)c(\nu))^r \delta$ in the sense of distributions on $C_\mu^\infty(G/K)$. If $r = 1$ this is the assertion of Lemma 2.2. Now

$$\begin{aligned}
&(C - \nu(H)^2 + \rho(H)^2)^{r+1}(Q_{r,\nu} \star Q_{1,\nu}) \\
&\quad = (C - \nu(H)^2 + \rho(H)^2)^r Q_{r,\nu} \star (C - \nu(H)^2 + \rho(H)^2) Q_{1,\nu} \\
&\quad = -2\nu(H)c(\nu)(C - \nu(H)^2 + \rho(H)^2)^r Q_{r,\nu} \star \delta \\
&\quad = -2\nu(H)c(\nu)(C - \nu(H)^2 + \rho(H)^2)^r Q_{r,\nu}.
\end{aligned}$$

So the lemma follows from the obvious induction.

3. The functions $\mathbf{P}_{r,\nu}$

Before we introduce the functions of the title of this section, it will be necessary to give some results on convergence and regularity of certain series over discrete subgroups of G. These results are no doubt well known to the experts, however we have included proofs since there is no easily accessible reference to them which we could find.

The first result is quite general. Let G be a reductive Lie group and let φ be a continuous function on G such that

(i) $\qquad \varphi(x) \geq 1, \qquad x \in G.$

(ii) $\qquad \varphi(xy) \leq \varphi(x)\varphi(y), \qquad x, y \in G.$

(iii) $\qquad \int_G \varphi(g)^{-1}\,dg < \infty.$

Let Γ be a discrete subgroup of G.

Lemma 3.1. *If $t \geq 1$, then the series*

$$\sum_{\gamma \in \Gamma} \varphi(x^{-1}\gamma y)^{-1-t} = \Psi_t(x, y)$$

converges uniformly on compacta to a continuous function on $\Gamma\backslash G/K \times \Gamma\backslash G/K$. Furthermore, $\Psi_t(x, \cdot) \in L^\infty(\Gamma\backslash G/K)$ for all $t \geq 0$ and $\|\Psi_t(x, \cdot)\|_\infty \leq C_1\varphi(x)$ for $t \geq 0$.

Proof. Since $\varphi(x^{-1}y) \geq \varphi(y)\varphi(x)^{-1}$ and $\varphi(x)^t \geq \varphi(x)^s$ if $t \geq s$, it is clear that

$$\Psi_t(x, y) \leq \Psi_0(x, y) \leq \varphi(x)\Psi_0(1, y).$$

Thus, to prove the lemma, it is enough to show that $\Psi_0(1, y) = \Psi(y)$ defines a function in $L^\infty(\Gamma\backslash G)$. This is proved in the following (standard) way. Let U be an open neighborhood of 1 in G with compact closure such that $U\gamma \cap U\tau$ is empty for each pair γ, τ of distinct elements of Γ. Let $C_2 = \sup_{g\in U} \varphi(g)$. If $u \in U$ and $\gamma \in \Gamma$, then $\varphi(u\gamma g) \leq C_2\varphi(\gamma g)$. Thus

$$\int_U \varphi(u\gamma g)^{-1}\, du \geq C_2^{-1}\varphi(\gamma g)^{-1}\operatorname{vol}(U).$$

If we sum over γ and use the disjointness assumption, then we find that

$$C_2^{-1}\operatorname{vol}(U)\Psi(g) \leq \int_G \varphi(ug)^{-1}\, du = \int_G \varphi(u)^{-1}\, du.$$

This completes the proof. q.e.d.

We now assume that G is as in the previous sections. Let $\|\cdot\|$ be as in §2. If $g = k_1\exp(tH)k_2$, then $\|g\| \geq C_1 e^{|t|}$. Also $\gamma(t) \leq C_2 e^{2\rho(H)t}$ for $t \geq 0$. Thus,

$$\int_G \|g\|^{-2\rho(H)-t}\, dg \leq C_3 \int_0^\infty e^{2\rho(H)s} e^{-s(2\rho(H)+t)}\, ds = C_4/t.$$

This implies that the above lemma applies to $\varphi(g) = \|g\|^{2\rho(H)+\varepsilon}$ for any $\varepsilon > 0$.

We now begin the study of the series that are the subject of this paper. Let Γ be a discrete subgroup of G of cofinite volume such that, if it is not cocompact, then it satisfies Langlands' axioms [8]. If $\operatorname{Re}\nu(H) > \rho(H)$ and $r \geq 1$, then $Q_{r,\nu}(x^{-1}\cdot) \in L^1(G)$ for each $x \in G$. Thus Fubini's theorem implies that, for each $x \in G$, $\sum_{\gamma\in\Gamma} |Q_{r,\nu}(x^{-1}\gamma y)|$ converges for almost all $y \in G$. We set

$$\delta_r(\nu) = (-2\nu(H)c(\nu))^{-r}$$

and

$$\mathbf{P}_{r,\nu}(x, y) = \delta_r(\nu)\sum_{\gamma\in\Gamma} Q_{r,\nu}(x^{-1}\gamma y).$$

If $\operatorname{Re}\nu(H) > \rho(H)$, then $\mathbf{P}_{r,\nu}(x, \cdot) \in L^1(\Gamma\backslash G/K)$ for each $x \in G$.

We now introduce a simple "truncation" procedure to study the analytic properties of these functions. Let $u \in C^\infty(\mathbb{R})$ be such that $u(x) = u(-x)$, $u(x) = 0$ for $|x| < 1$, $u(x) = 1$ for $|x| > 2$, and $0 \leq u(x) \leq 1$ for all

$x \in \mathbb{R}$. We set $\beta(k_1 \exp tHk_2) = u(t)$ for $t \in \mathbb{R}$. Then $\beta \in C^\infty(K\backslash G/K)$. We set

$$\widetilde{\mathbf{P}}_{r,\nu}(x, y) = \delta_r(\nu) \sum_{\gamma \in \Gamma} \beta(x^{-1}\gamma y) Q_{r,\nu}(x^{-1}\gamma y).$$

Lemma 3.2. *Assume that* $\operatorname{Re}\nu(H) > \rho(H)$. *Then* $\widetilde{\mathbf{P}}_{r,\nu} \in C^\infty(\Gamma\backslash G/K) \times C^\infty(\Gamma\backslash G/K)$ *and it is holomorphic in* ν. *There exist constants* $C_{r,\nu,\varepsilon}$ *such that if* $\varepsilon > 0$, *then*

$$\|\widetilde{\mathbf{P}}_{r,\nu}(x, \cdot)\|_\infty \le C_{r,\nu,\varepsilon} \|x\|^{2\rho(H)+\varepsilon}.$$

As a function of ν, $\widetilde{\mathbf{P}}_{r,\nu}(x, y)$ *is holomorphic in this range. Finally, if* $1 < s < \infty$, *then the map* $x \mapsto \widetilde{\mathbf{P}}_{r,\nu}(x, \cdot)$ *is continuous from* G *to* $L^s(\Gamma\backslash G/K)$.

Proof. Lemma 2.4 implies that if $\operatorname{Re}\nu(H) > \rho(H)$, then

$$|\beta(x) Q_{r,\nu}(x)| \le C_{r,\operatorname{Re}\nu} \|x\|^{-\operatorname{Re}\nu(H) - \rho(H)} (1 + \log\|x\|)^{r-1}.$$

Suppose that $\operatorname{Re}\nu(H) > \rho(H) + 2\delta$ with $\delta > 0$. Then

$$|\delta_r(\nu)\beta(x) Q_{r,\nu}(x)| \le D_{r,\operatorname{Re}\nu} \|x\|^{-\delta - 2\rho(H)}$$

with $D_{r,\nu}$ continuous in the half-plane $\operatorname{Re}\nu(H) > \rho(H) + 2\delta$. This implies that if $\operatorname{Re}\nu(H) > \rho(H) + 2\delta$, then the series defining $\widetilde{\mathbf{P}}_{r,\nu}$ is dominated by

$$D_{r,\operatorname{Re}\nu} \sum_{\gamma \in \Gamma} \|x^{-1}\gamma y\|^{-\delta - 2\rho(H)}.$$

The convergence of the series defining $\widetilde{\mathbf{P}}_{r,\nu}$ and the L^∞ estimate in the lemma now follow from Lemma 3.1 and the observations preceding the statement of this lemma. This term-by-term domination also gives the last assertion of the lemma.

We now prove the regularity assertion. Let C be the Casimir operator of G. If $X_1, \cdots, X_n$ is a basis of $\mathfrak{g}$ and if the X^j are defined by the equation $B(X_i, X^j) = \delta_{ij}$, then $C = \sum_i X_i X^i$. Thus,

$$C(\beta Q_{r,\nu}) = (C\beta) Q_{r,\nu} + \sum_i (X_i\beta)(X^i Q_{r,\nu}) + \sum_i (X^i\beta)(X_i Q_{r,\nu}) + \beta C Q_{r,\nu}.$$

We set $Q_{0,\nu} = 0$ in $G - K$ (to be consistent, $Q_{0,\nu}$ should be defined to be δ_{1K}). Then on $G - K$ we have

$$(C - \nu(H)^2 + \rho(H)^2) Q_{r,\nu} = -2\nu(H) c(\nu) Q_{r-1,\nu}.$$

We note that the expressions $(C\beta)Q_{r,\nu}$, $(X_i\beta)(X^iQ_{r,\nu})$, and $(X^i\beta)\cdot(X_iQ_{r,\nu})$ are C^∞ with compact support. We therefore conclude that

$$(C-\nu(H)^2+\rho(H)^2)\widetilde{\mathbf{P}}_{r,\nu}(x,\cdot)=\widetilde{\mathbf{P}}_{r-1,\nu}(x,\cdot)+F_{r,\nu}(x,\cdot)$$

with $F_{r,\nu}(x,\cdot)$ a bounded element of $C^\infty(\Gamma\backslash G/K)$, $F_{r,\nu}\in C^\infty(\Gamma\backslash G/K\times \Gamma\backslash G/K)$, and $\widetilde{\mathbf{P}}_{0,\nu}=0$. Let C_1 (resp. C_2) denote C acting in the first (resp. second) factor of a function on $G\times G$. If we interchange the roles of x and y in the above discussion, then we find

$$(C_1+C_2-2(\nu(H)^2-\rho(H)^2))^{2r}\widetilde{\mathbf{P}}_{r,\nu}\in C^\infty(\Gamma\backslash G/K\times\Gamma\backslash G/K).$$

Elliptic regularity now implies that $\widetilde{\mathbf{P}}_{r,\nu}\in C^\infty(\Gamma\backslash G/K\times\Gamma\backslash G/K)$. q.e.d.

If $r>1$ (resp. $r=1$) and $\operatorname{Re}\nu(H)>\rho(H)$ (resp. Q_ν is defined), then we set

$$\begin{aligned}\mathbf{P}'_{r,\nu}(x,y)&=\mathbf{P}_{r,\nu}(x,y)-\widetilde{\mathbf{P}}_{r,\nu}(x,y)\\&=\delta_r(\nu)\sum_{\gamma\in\Gamma}(1-\beta(x^{-1}\gamma y))Q_{r,\nu}(x^{-1}\gamma y).\end{aligned}$$

The point here is that $Q_{1,\nu}=Q_\nu$ is meromorphic in ν for $\nu\in\mathfrak{a}_C^*$, but if $r>1$, then $Q_{r,\nu}$ has only been defined for $\operatorname{Re}\nu(H)>\rho(H)$.

Let p denote the canonical projection of G onto $\Gamma\backslash G/K$. If $p(x)\neq p(y)$, then the above sum is finite and the number of terms is dominated by a power of $\|x\|$. The following lemma is therefore straightforward.

Lemma 3.3. *If $r\geq 1$ (resp. $r=1$) and $\operatorname{Re}\nu(H)>\rho(H)$ (resp. Q_ν is defined), then $\mathbf{P}'_{r,\nu}$ is C^∞ on $(\Gamma\backslash G/K)\times(\Gamma\backslash G/K)-\operatorname{diag}(\Gamma\backslash G/K)$. If $r\geq 1$ (resp. $r=1$) and $p(x)\neq p(y)$, then $\nu\mapsto\mathbf{P}'_{r,\nu}(x,y)$ is holomorphic for $\operatorname{Re}\nu(H)>\rho(H)$ (resp. holomorphic for $\operatorname{Re}\nu(H)\geq 0$ and meromorphic on $\mathfrak{a}_C^*$).*

The following is the first of the main results of this paper.

Theorem 3.4. *If $\operatorname{Re}\nu(H)>\rho(H)$, then $\mathbf{P}_{r,\nu}$ is continuous on $(\Gamma\backslash G/K)\times(\Gamma\backslash G/K)-\operatorname{diag}(\Gamma\backslash G/K)$, and if $p(x)\neq p(y)$, then $\nu\mapsto\mathbf{P}_{r,\nu}(x,y)$ is holomorphic. If $r>n/4$, then there exists $\varepsilon>0$ such that $\mathbf{P}_{r,\nu}\in L^{2+\varepsilon}(\Gamma\backslash G/K)$ and, for each $\delta>0$,*

$$\|\mathbf{P}_{r,\nu}(x,\cdot)\|_2\leq C_{r,\nu,\delta}\|x\|^{2\rho(H)+\delta};$$

furthermore $x\mapsto\mathbf{P}_{r,\nu}(x,\cdot)$ is continuous from G to $L^{2+\varepsilon}(\Gamma\backslash G)$. Finally, if $r>n/2$, the $\mathbf{P}_{r,\nu}(x,\cdot)\in L^\infty(\Gamma\backslash G/K)$.

Proof. Lemmas 3.2 and 3.3 imply all of the assertions except for those concerning L^p. In light of Lemma 3.2 it is enough to prove all of these assertions for $\mathbf{P}'_{r,\nu}$. Since $\operatorname{supp}(1-\beta)$ is compact, there exists $C_1 > 0$ such that if $(1-\beta)(y) \neq 0$, then $\|y\| \le C_1$. Thus

$$\|y\| \le C_1\|x\| \quad \text{if } 1-\beta(x^{-1}y) \neq 0. \tag{1}$$

Suppose that $f \in L^s(G)$ with $1 < s \le \infty$, and $\operatorname{supp} f \subset D_R = \{x \in G \mid \|x\| \le R\}$ with $R < \infty$. We set

$$p_f(g) = \sum_{\gamma\in\Gamma} f(\gamma g).$$

If $\varphi \in L^v(\Gamma\backslash G)$ with $1/s + 1/v = 1$, then

$$\begin{aligned}\left|\int_{\Gamma\backslash G} p_f(g)\overline{\varphi(g)}\,dg\right| &\le \int_{\Gamma\backslash G}\sum_{\gamma\in\Gamma}|(\gamma g)\overline{\varphi(g)}|\,dg = \int_G |f(g)\overline{\varphi(g)}|\,dg \\ &= \int_{D_R}|f(g)\overline{\varphi(g)}|\,dg \le \|f\|_s\left(\int_{D_R}|\overline{\varphi(g)}|^v\,dg\right)^{1/v}\end{aligned}$$

by Hölder's inequality.

Consider the canonical map $\pi_R\colon D_R \to \Gamma\backslash G$. Then

$$|\pi_R^{-1}(x)| \le |\{\gamma\in\Gamma \mid \|\gamma\| \le R^2\}| \le C_2\operatorname{vol}(D_{R^2}).$$

Thus,

$$\int_{D_r}|\overline{\varphi(g)}|^v\,dg \le C_2\operatorname{vol}(D_{R^2})\|\varphi\|_v^v.$$

On the other hand,

$$\operatorname{vol}(D_{R^2}) = \int_{\|a\|\le R^2}\gamma(a)\,da \le C_3\int_0^{b+2\log R} e^{2\rho(H)t}\,dt \le C_4R^{4\rho(H)}.$$

Here the $b > 0$ which comes into the expression comes from the observation that there exists $1 \le C < \infty$ such that $C^{-1}e^t \le \|\exp tH\| \le Ce^t$ for $t \ge 0$.

We conclude that

$$\left(\int_{D_R}|\overline{\varphi(g)}|^v\,dg\right)^{1/v} \le C_5R^{4\rho(H)/v}\|\varphi\|_v,$$

which implies

$$\|p_f\|_s \le C_5R^{4\rho(h)/v}\|f\|_s. \tag{2}$$

Set $\varphi_{r,\nu} = (1-\beta)Q_{r,\nu}$. If $r > n/4$ and $\operatorname{Re}\nu(H) > \rho(H)$, then $\varphi_{r,\nu} \in L^{2+\varepsilon}(G)$ for some $\varepsilon > 0$ which is independent of ν (see Corollary 2.5). We apply the above material to $f_x(y) = \delta_r(\nu)\varphi_{r,\nu}(x^{-1}y)$, and note that $p_{f_x} = \mathbf{P}'_{r,\nu}(x,\cdot)$. So (2) above implies that $\mathbf{P}'_{r,\nu}(x,\cdot) \in L^{2+\varepsilon}(\Gamma\backslash G/K)$. Also, from (1) it follows that the "R" for f_x is $C_1\|x\|$. So (2) shows that if $1/v + 1/(2+\varepsilon) = 1$, then

$$\|\mathbf{P}'_{r,\nu}(x,\cdot)\|_{2+\varepsilon} \le C_{r,\nu}\|x\|^{4\rho(H)/v}.$$

Let ω be a compact subset of G. There exist constants $C_{\omega,r,\nu}$ depending only on r, ν and ω such that if $x, z \in \omega$, then

$$\begin{aligned}\|\mathbf{P}'_{r,\nu}(x,\cdot) - \mathbf{P}'_{r,\nu}(z,\cdot)\|_{2+\varepsilon} &= \|p_{f_x} - p_{f_z}\|_{2+\varepsilon} \\ &\le C_{\omega,r,\nu}\|L(x)\varphi_{r,\nu} - L(z)\varphi_{r,\nu}\|_{2+\varepsilon}\end{aligned}$$

with $L(x)f(y) = f(x^{-1}y)$. This implies that $x \mapsto \mathbf{P}'_{r,\nu}(x,\cdot)$ is continuous from G to $L^2(\Gamma\backslash G/K)$.

If $r > n/2$, then Corollary 2.5 combined with (2) yields that $\mathbf{P}'_{r,\nu}(x,\cdot) \in L^\infty(\Gamma\backslash G)$. The result now follows. q.e.d.

Let $L^{2^-}_\infty(\Gamma\backslash G/K)$ denote the space of all $f \in C^\infty(\Gamma\backslash G/K)$ such that $Xf \in L^{2-\varepsilon}(\Gamma\backslash G)$ for all $X \in U(\mathfrak{g})$ and all ε such that $2 > \varepsilon > 0$. (This makes sense since $\Gamma\backslash G$ has finite volume.)

Theorem 3.5. *Suppose that $r > n/4$ and $\operatorname{Re}\nu(H) > 2\rho(H)$. If $f \in L^{2^-}_\infty(\Gamma\backslash G/K)$, then*

$$\int_{\Gamma\backslash G} \mathbf{P}_{r,\nu}(x,y)(C - \nu(H)^2 + \rho(H)^2)^r f(y)\,dy = f(x).$$

Proof. Let $P = MAN$ be a percuspidal parabolic subgroup of G. Let $\mathcal{S} = \omega A_t^+ K$ be a Siegel set for P. Then our hypothesis implies that

$$|Xf(g)| \le C_{X,\varepsilon}\|g\|^{\rho(H)+\varepsilon}$$

for all $\varepsilon > 0$, $X \in U(\mathfrak{g})$, and $g \in \mathcal{S}$ (cf. [14, 5.A.3]). If $\Gamma\backslash G$ is not compact, then there exists a finite collection $\mathcal{S}_1, \cdots, \mathcal{S}_r$ of these sets such that $G = \bigcup_i \Gamma\mathcal{S}_i$. We therefore see that

$$|Xf(g)| \le C'_{X,\varepsilon}\|g\|^{\rho(H)+\varepsilon}$$

for all $\varepsilon > 0$, $X \in U(\mathfrak{g})$ and $g \in G$. Thus, $f \in C^\infty_{\rho(H)+\varepsilon}(G)$ (see Lemma

2.2) for all $\varepsilon > 0$. Put $u(g) = (C - \nu(H)^2 + \rho(H)^2)^r f$. Then

$$\begin{aligned}\int_{\Gamma\backslash G} \mathbf{P}_{r,\nu}(x, y)u(y)\,dy &= \delta_r(\nu)\int_{\Gamma\backslash G}\sum_{\gamma\in\Gamma} Q_{r,\nu}(x^{-1}\gamma y)u(y)\,dy \\ &= \delta_r(\nu)\int_{\Gamma\backslash G}\sum_{\gamma\in\Gamma} Q_{r,\nu}(x^{-1}\gamma y)u(\gamma y)\,dy \\ &= \delta_r(\nu)\int_G Q_{r,\nu}(x^{-1}y)u(y)\,dy.\end{aligned}$$

The result now follows from Lemma 2.6.

4. The meromorphic continuation of $\mathbf{P}_{r,\nu}$

To carry out this continuation we will calculate the spectral decomposition of the $\mathbf{P}_{r,\nu}$ for $r > n/4$ and $\operatorname{Re}\nu(H) > 2\rho(H)$. We therefore assume (until further notice) that the parameters satisfy these conditions. Let $Q = M_Q A_Q N_Q$ be a percuspidal parabolic subgroup. Let $E(Q, \mu, g)$ denote the right K-fixed Eisenstein series with respect to P. Here $\mu \in \mathfrak{a}_C^*$ will be identified with $\mu(H_Q) \in \mathbf{C}$ (notice that "the H_Q" is uniquely determined by Q and K). If $\operatorname{Re}\mu = 0$, then $E(Q, \mu, \cdot) \in L_\infty^{2^-}(\Gamma\backslash G/K)$ (cf. [11, A.2.3]). Theorem 3.5 now implies that if $\operatorname{Re}\nu(H) = 0$, then

$$\langle \mathbf{P}_{r,\nu}(x, \cdot), (C - \bar{\nu}(H)^2 + \rho(H)^2)^r E(Q, \mu)\rangle = \overline{E(Q, \mu, x)}.$$

Hence,

$$\langle \mathbf{P}_{r,\nu}(x, \cdot), E(Q, \mu)\rangle = \overline{E(Q, \mu, x)}/(\mu^2 - \nu(H)^2)^r.$$

Similarly, if $\varphi \in L_\infty^2(\Gamma\backslash G/K)$ and $C\varphi = (\mu^2 - \rho(H)^2)\varphi$, then Theorem 3.5 yields that

$$\langle \mathbf{P}_{r,\nu}(x, \cdot), \varphi\rangle = \overline{\varphi(x)}/(\mu^2 - \nu(H)^2)^r.$$

Let $\{\varphi_j\}$ be an orthonormal set of eigenfunctions of C in $L^2(\Gamma\backslash G/K)$ with $C\varphi_j = (\nu_j^2 - \rho(H)^2)\varphi_j$ such that if $\varphi \in L^2(\Gamma\backslash G/K)$ and φ is an eigenfunction of C, then φ is in the linear span of $\{\varphi_j\}$ (it is well known that such a sequence exists). Let $P_1, \cdots, P_m$ be a complete set of representatives for the Γ-conjugacy classes of percuspidal parabolic subgroups of G. Then Langlands' decomposition of L^2 as given in [11; Proposition A.2.3] now gives

Theorem 4.1. *If* $\operatorname{Re}\nu(H) > 2\rho(H)$ *and* $r > n/4$, *then as an element of* $L^2(\Gamma\backslash G/K)$

$$\mathbf{P}_{r,\nu}(x,\cdot) = \sum_j \frac{\overline{\varphi_j(x)}}{(\nu_j^2 - \nu(H)^2)^r}\varphi_j + (-1)^r \sum_i c_i \int_0^\infty \frac{\overline{E(P_i, i\mu, x)}}{(\mu^2 + \nu(H)^2)^r} E(P_i, i\mu)\, d\mu.$$

Theorem 4.2. *Let* $r > n/4$. *If* $x \in G$, *then*

$$\Psi_r(x) = \sum_j \frac{|\varphi_j(x)|^2}{(1 + |\nu_j|^2)^{2r}} + \sum_i c_i \int_0^\infty \frac{|E(P_i, i\mu, x)|^2}{(1 + |\mu|^2)^{2r}}\, d\mu < \infty.$$

The series and the integrals defining Ψ_r *converge uniformly on compacta of* $\Gamma\backslash G$ *and* $\Psi_r(x) \le C_\varepsilon \|x\|^{4\rho(H)+\varepsilon}$ *for each* $\varepsilon > 0$.

Proof. If $z \in \mathbf{C}$, then clearly

$$\frac{|w^2 \pm z^2|}{1 + |w|^2} \le (1 + |z|^2)$$

for all $w \in \mathbf{C}$. Hence,

$$\sum_{j \ge m} \frac{|\varphi_j(x)|^2}{(1 + |\nu_j|^2)^{2r}} + \sum_i c_i \int_{|\mu| \ge T} \frac{|E(P_i, i\mu, x)|^2}{(1 + |\mu|^2)^{2r}}\, d\mu \le (1 + |\nu(H)|^2)^{2r} \Psi_{r,m,T}(\nu, x)$$

with

$$\Psi_{r,m,T}(\nu, x) = \sum_{j \ge m} \frac{|\varphi_j(x)|^2}{|\nu_j^2 - \nu(H)^2|^{2r}} + \sum_i c_i \int_{|\mu| \ge T} \frac{|E(P_i, i\mu, x)|^2}{|\mu^2 + \nu(H)^2|^{2r}}.$$

In particular, this and Theorem 3.4 imply that

$$\Psi_r(x) \le (1 + |\nu(H)|^2)\|\mathbf{P}_{r,\nu}(x,\cdot)\|^2 \le C_\varepsilon \|x\|^{4\rho(H)+\varepsilon}$$

for each $\varepsilon > 0$. $\Psi_{r,0,0}(\nu,\cdot)$ is a continuous function for each $r > n/4$ and $\operatorname{Re}\nu(H) > 2\rho(H)$. Hence $\Psi_{r,m,T}(\nu,\cdot)$ is continuous for each r and ν as above. In particular, given $\varepsilon > 0$ and $x \in G$ there exist m_x, T_x, and U_x, a neighborhood of x in G, such that $\Psi_{r,m_x,T_x}(\nu, y) < \varepsilon$ for $y \in U_x$. Thus if ω is a compact subset of G, then the obvious covering argument implies that there exist n and S such that, if $x \in \omega$, then $\Psi_{r,m,T}(\nu, y) < \varepsilon$ for $m \ge n$ and $T \ge S$. This completes the proof.

Note. If $r > 2$, then one can prove that $|\Psi_r(x)| \leq C_\varepsilon \|x\|^{2\rho(H)+\varepsilon}$ for all $\varepsilon > 0$ using the above argument and Lemma 3.2.

Proposition 4.3. *Let $r > n/4$. Then $\mathbf{P}_{r,\nu}(x, \cdot)$ has a meromorphic continuation in ν, as a distribution, to $\mathfrak{a}_C^*$ such that the following hold:*

(1) *The poles for $\operatorname{Re}\nu(H) \geq 0$, $\nu \neq 0$, are of order r and are contained in the set of ν_j such that $\nu_j(H)^2 - \rho(H)^2$ is an eigenvalue of C on $L^2(\Gamma\backslash G/K)$.*

(2)

$$\mathbf{P}_{r,-\nu}(x, \cdot) = \mathbf{P}_{r,\nu}(x, \cdot) - \frac{2\pi}{(r-1)!} \frac{\partial^{r-1}}{\partial\mu^{r-1}}\bigg|_{\mu=\nu} \sum_j c_j \frac{E(P_j, -\mu, x)E(P_j, \mu, \cdot)}{(\mu(H) + \nu(H))^r} + \frac{2\pi}{(r-1)!} \frac{\partial^{r-1}}{\partial\mu^{r-1}}\bigg|_{\mu=-\nu} \sum_j c_j \frac{E(P_j, \mu, x)E(P_j, -\mu, \cdot)}{(\mu(H) - \nu(H))^r}.$$

(3) *If $\operatorname{Re}\nu(H) > 0$, then $\mathbf{P}_{r,\nu}(x, \cdot) \in L^2(\Gamma\backslash G/K)$ where defined.*

Proof. Theorem 4.1 and Theorem 4.2 imply that $\mathbf{P}_{r,\nu}(x, \cdot)$ has a meromorphic continuation (in L^2) to $\operatorname{Re}\nu(H) > 0$ with only possible poles at the ν_j with ν_j as in (1) (notice that we have identified ν_j with $\nu_j(H)$). If $\operatorname{Re}\nu(H) > 2\rho(H)$, then we set

$$\mathbf{P}_{r,\nu,d}(x, \cdot) = \sum_j \frac{\overline{\varphi_j(x)}}{(\nu_j^2 - \nu(H)^2)^r} \varphi_j$$

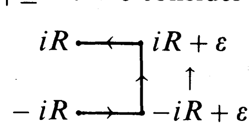
and

$$\mathbf{P}_{r,\nu,c}(x, \cdot) = (-1)^r \sum_i c_i \int_0^\infty \frac{\overline{E(P_i, i\mu, x)}}{(\mu^2 + \nu(H)^2)^r} E(P_i, i\mu)\, d\mu.$$

Clearly, $\mathbf{P}_{r,\nu,d}(x, \cdot)$ has a meromorphic continuation to all of $\mathfrak{a}_C^*$ with values in $L^2(\Gamma\backslash G/K)$. The possible poles of $\mathbf{P}_{r,\nu,d}(x, \cdot)$ for $\nu \neq 0$ are at the $\pm\nu_j$ and of order r. Furthermore, $\mathbf{P}_{r,\nu,d}(x, \cdot) = \mathbf{P}_{r,-\nu,d}(x, \cdot)$. To prove the theorem we must therefore analyze $\mathbf{P}_{r,\nu,c}(x, \cdot)$.

Let $R > 0$ and let $\varepsilon > 0$ be so small that $E(P_i, \nu)$ is holomorphic for $|\operatorname{Re}\nu(H)| \leq \varepsilon$ and $|\operatorname{Im}\nu(H)| \leq R$. We consider the curve $\mathscr{C}$:

$\mathscr{C}$



If $f \in C_c^\infty(\Gamma\backslash G/K)$, then we set

$$\alpha_j(\mu) = \alpha_{j,f}(\mu) = \int_{\Gamma\backslash G} E(P_j, \mu, g) f(g)\, dg.$$

If $0 < \operatorname{Re}\nu_0(H) < \varepsilon/2$ and $|\operatorname{Im}\nu_0(H)| < R/2$, then

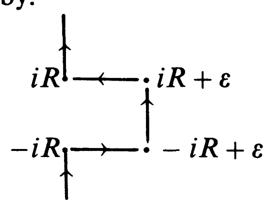
$$(-1)^r i \int_{-R}^{R} \frac{\overline{E(P_j, i\mu, x)}}{(\mu^2 + \nu_0(H)^2)^r} \alpha_j(i\mu)\, d\mu$$
$$= \int_{\mathscr{C}} \frac{E(P_j, -\mu, x)}{(\mu^2 - \nu_0(H)^2)^r} \alpha_j(\mu)\, d\mu - 2\pi i \operatorname{Res}_{\mu=\nu_0(H)} \frac{E(P_j, -\mu, x)}{(\mu^2 - \nu_0(H)^2)^r} \alpha_j(\mu).$$

By calculating the above residue in the obvious way we find that

$$\operatorname{Res}_{\mu=\nu_0(H)} \frac{E(P_j, -\mu, x)}{(\mu^2 - \nu_0(H)^2)^r} \alpha_j(\mu) = \frac{1}{(r-1)!} \left.\frac{\partial^{r-1}}{\partial\mu^{r-1}}\right|_{\mu=\nu_0} \frac{E(P_j, -\mu, x)\alpha_j(\mu)}{(\mu(H) + \nu_0))^r}.$$

Let $\mathscr{C}_1$ be the contour given by:

$\mathscr{C}_1$



Then for ν_0 as above we have

$$\int_{\Gamma\backslash G} \mathbf{P}_{r,\nu_0,c}(x, g) f(g)\, dg = -i \sum_j c_j \int_{\mathscr{C}_1} \frac{E(P_j, -\mu, x)}{(\mu^2 + \nu_0(H)^2)^r} \alpha_j(\mu)\, d\mu$$
$$- \frac{2\pi}{(r-1)!} \left.\frac{\partial^{r-1}}{\partial\mu^{r-1}}\right|_{\mu=\nu_0} \sum_j c_j \frac{E(P_j, -\mu, x)\alpha_j(\mu)}{(\mu(H) + \nu_0))^r}.$$

This implements the meromorphic continuation (as a distribution) of $\mathbf{P}_{r,\nu,c}(x, g)$ to the set $|\operatorname{Re}\nu(H)| < \varepsilon/2$, $|\operatorname{Im}\nu(H)| < R/2$. We therefore have a meromorphic continuation of $\mathbf{P}_{r,\nu,c}(x, g)$ to a neighborhood of $\operatorname{Re}\nu(H) \geq 0$ such that the only possible pole of $\mathbf{P}_{r,\nu_0,c}(x, g)$ is at 0 (the pole if it exists is of order $2r-1$). The asserted functional equation (where both sides make sense) is now clear and implements the meromorphic continuation to $\mathfrak{a}_C^*$. q.e.d.

We define $\mathbf{P}_{0,\nu} = \delta_1$ ($\delta_1(f) = f(1)$ for $f \in C_c^\infty(\Gamma\backslash G/K)$).

Lemma 4.4. *If* $\operatorname{Re}\nu(H) > \rho(H)$, *then* $(C - \nu(H)^2 + \rho(H)^2)\mathbf{P}_{r+1,\nu}(x, \cdot) = \mathbf{P}_{r,\nu}(x, \cdot)$ *(in the sense of distributions) for* $r \geq 0$.

Proof. If $f \in C_c^\infty(\Gamma\backslash G/K)$, then

$$\int_{\Gamma\backslash G} \mathbf{P}_{r+1,\nu}(x, g)(C - \nu(H)^2 + \rho(H)^2)f(g)\,dg$$
$$= \delta_{r+1}(\nu)\int_G Q_{r+1,\nu}(x^{-1}g)(C - \nu(H)^2 + \rho(H)^2)f(g)\,dg$$
$$= \delta_r(\nu)\int_G Q_{r,\nu}(x^{-1}g)f(g)\,dg = \int_{\Gamma\backslash G} \mathbf{P}_{r,\nu}(x, g)f(g)\,dg.$$

Theorem 4.5. *Let $\{\nu_j\}$ be as in Theorem 4.1. If $r \geq 1$, then $\mathbf{P}_{r,\nu}$ has a meromorphic continuation (in ν) to $\mathfrak{a}_\mathbf{C}^*$, in the sense of distributions. If $\operatorname{Re}\nu(H) \geq 0$, $\nu \neq 0$, and if ν is a pole of $\mathbf{P}_{r,\nu}(x, \cdot)$, then $\nu = \nu_j$ for some j and the pole is at most of order r and principal part at ν_j equal to that of*

$$\sum_j \frac{\overline{\varphi_j(x)}}{(\nu_j^2 - \nu(H)^2)^r}\varphi_j.$$

If 0 is a pole of $\mathbf{P}_{r,\nu}$, then it is a pole of order at most $2r$.

In light of the preceding two results this theorem is now clear.

The following result will be used in the next section.

Proposition 4.6. *$\widetilde{P}_{1,\nu}$ has a meromorphic continuation to $\mathfrak{a}_\mathbf{C}^*$ as an element of $C^\infty(\Gamma\backslash G \times \Gamma\backslash G)$. Furthermore, if $\operatorname{Re}\nu(H) \geq 0$, then the principal parts of $\widetilde{P}_{1,\nu}$ and of $\mathbf{P}_{1,\nu}$ (as functions of ν) are equal.*

Proof. Lemma 3.3 implies that $\mathbf{P}'_{1,\nu}$ is meromorphic in ν and holomorphic for $\operatorname{Re}\nu(H) \geq 0$. Thus $\widetilde{P}_{1,\nu} = \mathbf{P}_{1,\nu} - \mathbf{P}'_{1,\nu}$ is meromorphic in ν and has the same principal parts as $\mathbf{P}_{1,\nu}$ for $\operatorname{Re}\nu(H) \geq 0$. In the proof of Lemma 3.2 we have seen that the following equation holds in the sense of distributions for $\operatorname{Re}\nu(H) > \rho(H)$:

$$(\star) \qquad ((C_1 - \nu(H)^2 + \rho(H)^2) + (C_2 - \nu(H)^2 + \rho(H)^2))\widetilde{P}_{1,\nu} = F_\nu$$

with $F_\nu \in C^\infty(\Gamma\backslash G \times \Gamma\backslash G)$ and meromorphic in $\nu \in \mathfrak{a}_\mathbf{C}^*$. Thus $(\star)$ is true for all ν for which both sides of the equation are meaningful. Elliptic regularity now implies the proposition. q.e.d.

We conclude this section with an application of Theorem 4.2 to the pointwise convergence of the spectral decomposition of an element of $L^2(\Gamma\backslash G/K)$.

Theorem 4.7. *Let $r > n/4$ and assume that $f \in C^{2r}(\Gamma\backslash G/K)$ is such that $C^j f \in L^{2+\varepsilon}(\Gamma\backslash G)$ for $0 \leq j \leq r$ and for some $\varepsilon > 0$. Then*

$$f(x) = \sum_i \langle f, \varphi_i\rangle\varphi_i(x) + \sum_{j=1}^r c_j \int_0^\infty \langle f, E(P_j, i\mu)\rangle E(P_j, i\mu, x)\,d\mu$$

with the series and integrals converging uniformly on compacta of $\Gamma\backslash G/K$.

Proof. Since $C^j f \in L^{2+\varepsilon}(\Gamma\backslash G/K)$ for $0 \le j \le r$, the equation above holds in the sense of $L^2(\Gamma\backslash G)$ with f replaced by $C^j f$ for j in this range [11, Proposition A.2.3]. Then the right-hand side of the above equation is majorized termwise in absolute value by

$$(\star) \quad \sum_i \frac{|\langle (C-\lambda_\nu)^r f, \varphi_i\rangle| \, |\varphi_i(x)|}{|\nu_j^2 - \nu(H)^2|^r} + \sum_j c_j \int_0^\infty \frac{|\langle (C-\lambda_\nu)^r f, E(P_j, i\mu)\rangle| \, |E(P_j, i\mu, x)|}{|\mu^2 + \nu(H)^2|^r} \, d\mu,$$

where $\lambda_\nu = \nu(H)^2 - \rho(H)^2$. We apply the Cauchy-Schwarz inequality to this expression and find that

$$(\star) \le C\|(C-\lambda_\nu)f\|_2 \Psi_r(x)^{1/2}.$$

The result now follows if we argue as we did in the proof of Theorem 4.2.

5. A family of Dirichlet series associated with negatively curved manifolds

Let X be a complete simply connected Riemannian manifold which is the Riemannian covering of a compact Riemannian manifold. Let d denote the Riemannian distance function on X and let $B_T(x) = \{y \in X | d(x, y) < T\}$. Then according to [10, Remark 1]

$$(1) \qquad \lim_{T\to+\infty} \frac{\operatorname{vol}(B_T(x))}{T} = h,$$

with h independent of x. We note that if $X = G/K$ with the Riemannian structure corresponding to B, and ζ is the volume of the unit sphere in $\mathfrak{p}$, then

$$\int_X f(x)\, dV(x) = \zeta \int_K \int_0^\infty \gamma(t) f(k \exp tH)\, dt$$

for integrable f on X. From this it is easy to see that

$$\operatorname{Vol}(B_T(1K)) \sim \zeta e^{2\rho(H)T}/2\rho(H).$$

Thus in this case we have $E(x) \equiv \zeta$ and $h = 2\rho(H)$. Manning has interpreted h as the "topological entropy" of the geodesic flow.

Returning to the general situation, let Γ be a group of isometries of X acting freely and such that $\operatorname{Vol}(\Gamma\backslash X) < \infty$. If $x, y \in X$, we set

$$(2) \qquad L_\Gamma(x, y, s) = \sum_{\gamma\in\Gamma} e^{-(h/2+s)d(\gamma x, y)}.$$

Equation (1) above implies that the above series converges absolutely for $\operatorname{Re} s > h/2$ to a holomorphic function of s in this range. We now show how the results of this paper can be used to analyze these series if $X = G/K$ (as in the previous sections). Let Δ denote the Laplace-Beltrami operator of X. If $X = G/K$, then $\Delta f = Cf$ for $f \in C^\infty(G/K)$.

Theorem 5.1. *Let $X = G/K$ and let $\Gamma \subset G$ be a discrete torsion-free subgroup such that $\Gamma\backslash G$ has finite volume. Then $L_\Gamma(x, y, s)$ has a meromorphic continuation to $\mathbf{C}$ such that the poles in the range $\operatorname{Re} s \geq 0$, $s \neq 0$, are simple and at points of the form $\nu - 2j$ with $j = 0, 1, 2,$ and $\nu^2 - \rho(H)^2$ is an eigenvalue of Δ on $L^2(\Gamma\backslash X)$. $L_\Gamma(x, y, s)$ has a simple pole at $s = \rho(H)$ and*

$$\operatorname{Res}_{s=\rho(H)} L_\Gamma(x, y, s) = \zeta/\operatorname{vol}(M).$$

Furthermore, if $0 > \lambda_1 > \lambda_2 > \cdots$ are the eigenvalues of Δ on $L^2(\Gamma\backslash X)$ and if $\lambda_1 = s_1^2 - \rho(H)^2$ with $s_1 > \max\{\rho(H) - 2, 0\}$, then $L_\Gamma(x, y, s)$ is holomorphic for $\operatorname{Re} s > s_1$, $s \neq \rho(H)$, and $L_\Gamma(x, y, s)$ has at worst a simple pole at $s = s_1$ with residue

$$\zeta c(s_1\rho/\rho(H))^{-1} \sum_j \overline{\varphi_j(x)}\varphi_j(y)$$

with φ_j an orthonormal basis of the λ_1 eigenspace for Δ in $L^2(\Gamma\backslash X)$.

Proof. If $x, y \in X$, $x = gK$, $y = hK$, and $g^{-1}h = k_1 \exp tHk_2$, then $d(x, y) = |t|$. We write $\overline{x} = xK$ for $x \in G$. Then Theorem 1.2 can be rephrased as

$$Q_\nu(x^{-1}y) = e^{-(\nu+\rho)(H)d(\overline{x},\overline{y})}\left(1 + \sum_{k\geq 1} a_k(\nu)e^{2kd(\overline{x},\overline{y})}\right)$$

with $a_k(\nu)$ rational in ν and holomorphic for $\operatorname{Re}\nu(H) \geq 0$. Furthermore, if $d(x, y) \geq 1$ and $c \leq 0$ is given, then there exists a polynomial $f_c(\nu)$ of degree $\leq d(c)$ such that

$$(\star)\quad \left|f_c(\nu)\left\{Q_\nu(x^{-1}y) - e^{-(\nu+\rho)(H)d(\overline{x},\overline{y})}\left(1 + \sum_{k=1}^{N} a_k(\nu)e^{-2kd(\overline{x},\overline{y})}\right)\right\}\right| \leq C_{c,\varepsilon}(1 + |\nu|)^{d(c)}e^{-\{\operatorname{Re}(\nu+\rho)(H)+2N+2-\varepsilon\}d(\overline{x},\overline{y})}$$

for all $\varepsilon > 0$. We note that there exists $c_0 < 0$ such that f_{c_0} can be taken to be the constant polynomial 1 and $d(c_0) = 0$. Let β be as in the preceding sections. If $\operatorname{Re}\nu(H) > \rho(H)$, then

$$\begin{aligned}\widetilde{\mathbf{P}}_{1,\nu}(x, y) &= \delta_1(\nu)\sum_{\gamma\in\Gamma}\beta(x^{-1}\gamma y)Q_\nu(x^{-1}\gamma y)\\ &= \delta_1(\nu)L_\Gamma(\overline{x}, \overline{y}, \nu(H))\\ &\quad+\delta_1(\nu)\sum_{\gamma\in\Gamma}(\beta(x^{-1}\gamma y)-1)e^{-(\nu+\rho)(H)d(\overline{x}, \gamma\overline{y})}\\ &\quad+\delta_1(\nu)\sum_{\gamma\in\Gamma}\beta(x^{-1}\gamma y)\{Q_\nu(x^{-1}\gamma y)-e^{-(\nu+\rho)(H)d(\overline{x}, \gamma\overline{y})}\}.\end{aligned}$$

If ω is a compact subset of G, and $x, y \in \omega$, then the second sum is over a finite set depending only on ω. Thus the second term extends to a meromorphic function which is holomorphic wherever δ_1 is. $(\star)$ implies that the third sum is dominated by

$$C_{c,\varepsilon}\sum_{\gamma\in\Gamma}e^{-(\operatorname{Re}(\nu+\rho)(H)+2-\varepsilon)d(x, \gamma y)}$$

for all $\varepsilon > 0$. Therefore that the third term has a meromorphic continuation to $\operatorname{Re}\nu(H) > \rho(H) - 2$. This implements the meromorphic continuation of $L_\Gamma(\overline{x}, \overline{y}, s)$ to $\operatorname{Re}\nu(H) > \rho(H)-2$. Since $\delta_1(\nu)$ is holomorphic for $\operatorname{Re}\nu(H) \geq 0$, we see that the principal parts of $L_\Gamma(\overline{x}, \overline{y}, \nu(H))$ and $\delta_1(\nu)^{-1}\mathbf{P}_{1,\nu}(x, y)$ are the same for $\operatorname{Re}\nu(H) > \rho(H) - 2$. We now continue the continuation as above. We write $(a_0(\nu) \equiv 1)$

$$\begin{aligned}\mathbf{P}_{1,\nu}(x, y) &= \delta_1(\nu)L_\Gamma(\overline{x}, \overline{y}, \nu(H)) + \delta_1(\nu)\sum_{k=1}^{N}a_k(\nu)L_\Gamma(\overline{x}, \overline{y}, \nu(H)+2k)\\ &\quad+\delta_1(\nu)\sum_{\gamma\in\Gamma}(\beta(x^{-1}\gamma y)-1)\sum_{k=1}^{N}a_k(\nu)e^{-[(\nu+\rho)(H)+2k]d(\overline{x}, \gamma\overline{y})}\\ &\quad+\delta_1(\nu)\sum_{\gamma\in\Gamma}\beta(x^{-1}\gamma y)\\ &\qquad\times\left\{Q_\nu(x^{-1}\gamma y)-\sum_{k=1}^{N}a_k(\nu)e^{-[(\nu+\rho)(H)+2k]d(\overline{x}, \gamma\overline{y})}\right\}.\end{aligned}$$

The right-hand side of the above equation consists of four terms. The third term only involves finite sums (see the beginning of this proof) and thus has a meromorphic extension to $\mathfrak{a}_C^*$ with poles only at the ν where $\delta_1(\nu)$ or $a_k(\nu)$, $0 \leq k \leq N$, have poles. Hence this term is holomorphic for $\operatorname{Re}\nu(H) \geq 0$. If we argue as above, the fourth term is meromorphic for $\operatorname{Re}\nu(H) > \rho(H) - 2N - 2$ with poles at the ν for which $\delta_1(\nu)$ has a pole or some $a_k(\nu)$, $0 \le k \le N$, has a pole. We can therefore use the first two terms to see that since $L_\Gamma(x, y, s)$ continues to $\operatorname{Re} s > \rho(H) - 2$, it continues to $\operatorname{Re} s > \rho(H) - 4$, etc. The assertion about the pole structure is now clear. We are left with the calculation of the residue of $L_\Gamma(x, y, s)$ at $s = \rho(H)$.

The arguments above, combined with Theorem 4.5, imply that the principal part of $L_\Gamma(\overline{x}, \overline{y}, s)$ at $s = \rho(H)$ is equal to that of

$$\delta_1(\nu)^{-1}\zeta[\operatorname{vol}(\Gamma\backslash X)(\rho(H)^2 - \nu(H)^2)]^{-1}$$

since the space of square integrable eigenfunctions for the eigenvalue 0 is the space of constant functions. Thus

$$\operatorname{Res}_{s=\rho(H)} L_\Gamma(x, y, s) = -\delta_1(\rho(H))^{-1}\zeta[2\rho(H)\operatorname{vol}(\Gamma\backslash X)]^{-1}.$$

But, $\delta_1(\nu) = -(2\nu(H)c(\nu))^{-1}$ and with our normalization $c(\rho) = 1$. The last assertion follows from Theorem 4.5 and the argument which we just used to analyze the pole at $\rho(H)$.

Corollary 5.2 (notation as in Theorem 4.1). *Let* $x, y \in X$. *Then*

$$\sum_{\substack{\gamma\in\Gamma \\ d(\gamma x, y)\le T}} 1 \sim \zeta e^{2\rho(H)T} / \operatorname{Vol}(\Gamma\backslash X), \qquad T \to +\infty.$$

Proof. Fix x, y. We enumerate the elements of Γ as $\gamma_1, \gamma_2, \cdots$. Set $\mu_j = d(\gamma_j x, y)$ and $D(s) = L_\Gamma(x, y, s - \rho(H))$. Then

$$D(s) = \sum_{j=1}^{\infty} e^{-s\mu_j}.$$

Since the series defining $D(s)$ converges absolutely and uniformly on the strips $\operatorname{Re} s \ge 2\rho(H) + \varepsilon$, $\varepsilon > 0$, $D(s)$ has a meromorphic continuation to $\mathbf{C}$. If s_1 as in the preceding theorem exists, then set $t_1 = \rho(H) + s_1$, otherwise set $t_1 = \max\{2\rho(H) - 2, 0\}$. Thus $D(s)$ is holomorphic for $t_1 < \operatorname{Re} s$, $s \ne 2\rho(H)$. The Ikehara-Wiener theorem (cf. [1, p. 524]) therefore applies and implies that

$$\sum_{\mu_j \le T} 1 \sim (\operatorname{Res}_{s=2\rho(H)} D(s))e^{2\rho(H)T}, \qquad T \to +\infty.$$

The result now follows from the previous theorem.

Note. In [10] a general result of the above form for X of strictly negative curvature and $\Gamma\backslash X$ compact was announced. In the special case of constant negative curvature (i.e., G is locally isomorphic with $\mathrm{SO}(n, 1)$) the precise result as above is given for $\Gamma\backslash X$ compact. The finite volume version seems to be new. These results of Margulis, combined with ours above, suggest the following conjecture.

Conjecture 5.3. *Let X and Γ be as in the beginning of this section and assume that X has strictly negative curvature. Then $L_\Gamma(x, y, s)$ has a meromorphic continuation to* $\mathbf{C}$ *(perhaps only for* $\operatorname{Re} s > h/2-\varepsilon$ *for some* $\varepsilon > 0$) *and there exists an $\varepsilon > 0$ such that $s = h/2$ is the unique pole for* $\operatorname{Re} s > h/2 - \varepsilon$. *The residue at $s = h/2$ is* $C_X(\Gamma x, \Gamma y)/\operatorname{Vol}(\Gamma\backslash X)$.

One might be "brash" enough to augment this conjecture with an assertion generalizing that in Theorem 5.1 for the "next" eigenvalue of the Laplacian. We note that the above conjecture combined with the Ikehara-Wiener theorem yields a complete generalization of the above cited result of Margulis. In the context of the actual theorems, i.e., $X = G/K$ as above, we have a conjecture about the error term.

Conjecture 5.4. *Let t_1 be as in the proof of Corollary* 5.2. *Then*

$$\sum_{\substack{\gamma\in\Gamma \\ d(\gamma x, y)\leq T}} 1 - \zeta e^{2\rho(H)T}/\operatorname{Vol}(\Gamma\backslash X) = O(e^{t_1 T}) \quad \text{as } T \to +\infty.$$

We note that the above conjecture would follow from well-known results on Dirichlet series (cf. [7, Theorem 10.7g]) if we could show the following:

(i) $\lim_{s\to\infty} L_\Gamma(x, y, s) = 0$ for $\operatorname{Re} s > 0$.

(ii) There exist $0 < t_2 < t_1$ and $\tau_0 > 0$ such that

$$\mathrm{PV}\int_{-\infty}^{\infty} e^{i\lambda\tau} L_\Gamma(x, y, t_2 + i\lambda)\, d\lambda$$

converges uniformly for $\tau \geq \tau_0$.

Notice that (i) and (ii) would follow if we could prove the analogous results for $\mathbf{P}_{1,\nu}$.

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