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RATIONAL CONNECTEDNESS AND BOUNDEDNESS OF FANO MANIFOLDS

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0. Introduction

Fano manifolds are, by definition, smooth projective varieties with ample first Chern class (= anticanonical class). They are of special interest from the viewpoint of classification theory via minimal models; in fact, a principal goal of the minimal model program is to decompose a general algebraic variety into the Fano-like part and the minimal part (cf. [5], [11]).

Two-dimensional Fano manifolds are usually called Del Pezzo surfaces. Their classification into ten families of rational surfaces is an immediate consequence of Castelnuovo's criteria for rationality and minimality and Enriques' theory of adjunction. However, the systematic study of Fano manifolds initiated by G. Fano has revealed that their structure is not so simple in higher dimensions. For instance, the list of Fano 3-folds consists of 104 deformation classes, many of which are not rational [4], [12]. In dimension ≥ 4, their complete classification is thus virtually impossible and we should rather be concerned with vague but more accessible questions:

Question 1. Does the set of n-dimensional Fano manifolds form a bounded family?

Question 2. What can be said about geometric properties shared by the Fano manifolds in common?

The aim of this paper is to answer these questions: rational connectedness and boundedness.

A variety X is said to be *rationally connected* if two general points can be joined by an irreducible rational curve on X. Rational connectedness is a birational and deformation invariant, thus fitting well into the classification theory [8]. Roughly speaking, it is a crude generalization of unirationality, which is often too subtle to deal with in a general framework. Through crude, this generalization is natural enough to yield most of the geometric properties known for unirational varieties such as

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1-connectedness, vanishing of global differential forms, etc. For further implications of rational connectedness, the reader is referred to [8].

Theorem 0.1. *A Fano manifold X over an algebraically closed field of characteristic zero is rationally connected. More precisely, any two general points x and y can be joined by an irreducible rational curve C such that $(C, -K_X) \leq c(\dim X)$, where $c(n)$ is an effectively computable function in n, a positive integer.*

As was pointed out by G. Fano (cf. [7]), we can derive from (0.1) an effective bound of the degree of a Fano n-fold:

Theorem 0.2. *For a Fano manifold X of dimension n over an algebraically closed field of characteristic zero, the degree of X with respect to the anticanonical divisor is bounded, i.e.,*

$$c_1(X)^n \leq c(n)^n,$$

where the function $c(n)$ is the same one as in (0.1). *In particular, by a theorem of Kollár and Matsusaka* [6], *the n-dimensional Fano manifolds form a bounded family*; *i.e., they are* (*noneffectively*) *parametrized by a quasiprojective scheme.*

This article is the third part of our joint program. In the first part [7], Fano manifolds with Picard number one are discussed. In this specific case, we can choose a rather small number $\frac{1}{4}(n+2)^2$ instead of $c(n)$. For general Fano manifolds with Picard number ≥ 2 treated here, our bounding function $c(n)$ grows very rapidly as follows: $c(n) = O(n^{2^n n})$. For the explicit definition of $c(n)$, see §4 below.

The second part [8] is devoted to the general theory of rationally connected varieties, including a numerical characterization of rationally connected 3-folds. Note that "maximal rationally connected fibrations" constructed there will provide a shortcut toward the rational connectedness of Fano manifolds, except that we cannot keep track of the degrees of rational curves by this method.

In general classification theory, we have to deal with $\mathbb{Q}$-Fano varieties as well. In the fourth part (in preparation), we will prove the rational connectedness and the boundedness of $\mathbb{Q}$-Fano 3-folds with only terminal singularities.

In this paper, all schemes are defined over an algebraically closed field with uncountable elements.

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In his new papers [1], [2], F. Campana proves that two points on a Fano manifold can be joined by a chain of rational curves (cf. Theorem 3.3 below). His idea is basically the same as ours in §§1–3.

1. Free rational curves and the associated rational fibration

In this section, we recall and strengthen some results in [7], [8], to which the reader is referred for more detail.

Let T_X denote the tangent sheaf of a smooth variety X.

An irreducible rational curve C on smooth projective X is said to be *free* if T_X restricted to C is semipositive. Let $f\colon \mathbb{P}^1 \to X$ be the composite of the normalization and the embedding of C (in what follows, f is called the normalization of C for simplicity). C is free if and only if f^*T_X is generated by global sections.

Let $\mathscr{C} = \{C_s\}_{s\in S}$ be a flat family of curves on X. $\mathscr{C}$ is called a *covering family of irreducible rational curves* if

(1) the parameter space S is an irreducible variety,

(2) every member C_s is an irreducible rational curve, and

(3) $\bigcup_{s\in S} C_s$ contains an open dense subset of X.

A variety X is *uniruled* if and only if there exists a covering family of irreducible rational curves on X.

Proposition 1.1. *Let X be a smooth projective variety over an algebraically closed field with uncountably many elements. Then we have the following*:

(1) *Let Σ be the set of points $\in X$ which are contained in a noncovering family of rational curves. Then Σ is a union of countably many proper closed subsets of X. In other words, every rational curve passing through $x \in X\backslash\Sigma$ belongs to a covering family.*

(2) *If the ground field is of characteristic zero, then a general member of a covering family of irreducible rational curves is free.*

The proof is easy. Note, however, that the condition on the characteristic is indeed necessary in order to show the freeness of general rational curves (2) above; except this statement, all the arguments in this paper are characteristic-free. A direct consequence of (1.1) is

Corollary 1.2. *Let $C_s \subset X$ be a family of irreducible rational curves on a smooth projective variety X. Suppose that C_s has a reducible limit C_∞*

which contains a point $\in X\backslash\Sigma$. *Then* $\{C_s\}$ *is a subfamily of a covering family whose generic member is free.*

Let $\mathscr{C} = \{(x, C_s); x \in C_s\}$ be a family of free rational curves parametrized by an irreducible variety S. Since $\mathscr{C}$ is a locally closed subset of $X \times S$, we have natural projections $\mathrm{pr}_X : \mathscr{C} \to X$ and $\mathrm{pr}_S : \mathscr{C} \to S$. From our assumption, we see that pr_X and pr_S are both flat (if the family is free, pr_X is smooth).

Take a general point $x \in X$. We define $V^k(x) = V^k(x, \mathscr{C})$ inductively:

$$V^0(x) = \{x\},$$
$$V^{k+1}(x) = \mathrm{pr}_X \mathrm{pr}_S^{-1} \mathrm{pr}_S \mathrm{pr}_X^{-1}(V^k(x)).$$

$V^k(x)$ is the set of the points on X which can be joined with x by k or less rational curves in $\mathscr{C}$.

Let cl[$*$] denote the closure of a subset $*$.

Lemma 1.3. $\mathrm{cl}[V^k(x)]$ *is an increasing sequence of closed subsets, which is stable for* $k \geq \max_n \dim V^n(x)$; *furthermore we have* $\dim V^{k+1}(x) > \dim V^k(x)$ *unless* $\mathrm{cl}[V^{k+1}(x)] = \mathrm{cl}\, V^k(x)]$.

Proof. Take an irreducible component V_i^k of $V^k = V^k(x)$. $\mathrm{pr}_X^{-1}(V_i^k)$ is a union of irreducible components W_{ij}^k. pr_X is flat so that each W_{ij}^k is surjectively mapped onto V_i^k. Since pr_s is flat with irreducible fibres, $\widetilde{W}_{ij}^k = \mathrm{pr}_s^{-1}\mathrm{pr}_s(W_{ij}^k)$ is again irreducible. Hence either $\mathrm{cl}[\widetilde{W}_{ij}^k] = \mathrm{cl}[W_{ij}^k]$ or $\mathrm{cl}[\widetilde{W}_{ij}^k] \supset \mathrm{cl}[W_{ij}^k]$ has bigger dimension. Thus $\mathrm{cl}[\mathrm{pr}_X(\widetilde{W}_{ij}^k)]$ is equal to either $\mathrm{cl}[V_i^k]$ or an irreducible variety which contains V_i^k as a proper subvariety. In particular, $\mathrm{cl}[\mathrm{pr}_X \mathrm{pr}_S^{-1} \mathrm{pr}_S \mathrm{pr}_X^{-1}(V_i^k)]$ has irreducible components of dimension greater than $\dim V_i^k$ unless it is equal to $\mathrm{cl}[V_i^k]$. This means that V^k is decomposed into $V_{\mathrm{stable}}^k \cup V_{\mathrm{unstable}}^k$ such that

$$\mathrm{cl}[\mathrm{pr}_X \mathrm{pr}_S^{-1} \mathrm{pr}_S \mathrm{pr}_X^{-1}(V_{\mathrm{stable}}^k)] = \mathrm{cl}[V_{\mathrm{stable}}^k],$$

while $\mathrm{cl}[V_{\mathrm{unstable}}^k]$ consists of irreducible components which is a proper subvariety of components of $\mathrm{cl}[V^{k+1}]$. Let $m(k)$ be the minimum of the dimensions of the components of V_{unstable}^k. We infer that $m(k+1) \geq m(k)+1$ if $V_{\mathrm{unstable}}^{k=1}$ is nonempty. Thus if V_{unstable}^k is nonempty, then every component of V_{unstable}^k has dimension $\geq k$, and $k < \max_i \dim V^i$.

The argument above is used in [7] in order to show that $\bigcup V^k(x)$ is dense in X with Picard number one.

It is clear that $V(x) = \bigcup V^k(x) = V^{\dim V(x)}(x)$ is the set of points on X which can be joined with x by finitely many rational curves in $\mathscr{C}$. Hence we infer

$$V(x) \cap V(y) \text{ is nonempty iff } V(x) = V(y).$$

This gives rise to a rational map $\pi_{\mathscr{C}}: X \dashrightarrow \mathrm{Hilb}(X)$, where $\pi_{\mathscr{C}}(x)$ is defined to be $\mathrm{cl}[V(x)]$, the closure of $V(x)$.

Call this rational map $\pi_{\mathscr{C}}: X \dashrightarrow \mathrm{Im}\, \pi_{\mathscr{C}} \subset \mathrm{Hilb}(X)$ the *rationally connected fibration associated with* $\mathscr{C}$, a covering family of free rational curves.

Lemma 1.4. *Let x_1 and x_2 be two points on a "fibre" $\mathrm{cl}[V(x)]$ of $\pi_{\mathscr{C}}$, the rationally connected fibration associated with a covering family of free rational curves. Then there exists a chain of rational curves $\Gamma_1, \cdots, \Gamma_d$ which connects x_1 and x_2 such that*

$$\sum_i (\Gamma_i, -K_X) \leq \dim V(x)(C_s, -K_X),$$

where C_s is a member of $\mathscr{C}$, parametrized by S.

Proof. If x_1 and x_2 sit in $V(x)$, then $V(x) = V(x_i)$, so that they are connected by $\dim V(x)$ or less rational curves C_i' in $\mathscr{C}$. By specializing x_i's, we get the assertion.

2. Construction of horizontal rational curves

Let X be a Fano manifold and let $\pi: U \to Z$ be a surjective proper morphism from an open dense subset U onto a quasiprojective variety Z. The purpose of this section is to construct horizontal rational curves on X meeting the fibre over a given point $z \in Z$. The technique used here follows [10] and [9], and all the results stated in this section hold in arbitrary characteristics.

Theorem 2.1. *Let X be a Fano manifold. Suppose that there is a nonempty open subset U of X, a smooth quasiprojective variety Z and a proper surjective morphism $\pi: U \to Z$. Suppose that Z is of positive dimension and let z be a general point on Z. Then there exists a rational curve C on X with the following two properties:*

(a) *C meets $\pi^{-1}(z)$;*

(b) *C is not contained in $\pi^{-1}(z)$.*

(Call C a horizontal rational curve meeting $\pi^{-1}(z)$.) Furthermore, we can choose C to satisfy

(c) $(C, -K_X) \leq \dim X + 1$.

For our proof of Theorem 2.1, we define the notion of *relative deformations.* F. Campana [2] proved (2.1) without the explicit usage of this notion, but we introduce it here in view of interesting byproducts (2.8) and (2.9) below.

Definition. Let X, Y and Z be irreducible schemes. Let $U \subset X$ be an open dense subset and $\pi: U \to Z$ a (possibly nonproper) morphism. Let $f: Y \to X$ be a morphism, with $f(Y)$ meeting U. By a *relative deformation over* Z of f, parametrized by a connected punctured scheme (S, o) and equipped with a base subscheme $B \subset Y$, we mean a morphism

$$F = \{f_s\}: Y \times S \to X$$

which satisfies the following three conditions:

(1) $f_o = f$;

(2) $F|_{B\times S} = f(\mathrm{pr}_Y)|_{B\times S}$;

(3) $\pi f_s = \pi f$ for every $s \in S$, when viewed as a rational map.

Let $\mathrm{Hom}_Z(f; B)(S, o)$ denote the set of relative deformations of f parametrized by (S, o). Then $\mathrm{Hom}_Z(f; B)$ turns out to be a contravariant functor of the category of connected punctured schemes to the category of sets. When Y and X are both projective, it is a locally closed subfunctor of the Hilbert functor of $Y \times X$ and hence representable by a quasiprojective scheme $\mathrm{Hom}_Z(f; B)$, the *universal relative deformation.*

From now on, we assume that X, Y *and* Z *are all smooth and projective, and that* $f(B) \subset X$ *is contained in* U, *on which* π *is a morphism.*

The differential $d\pi: T_U \to \pi^* T_Z$ of π gives rise to a subsheaf $f^* T_{X/Z} \subset f^* T_X$ which is defined by

$$\Gamma(V, f^* T_{X/Z}) = \{\eta \in \Gamma(V, f^* T_X);\, d\pi(\eta|_{f^{-1}(U)\cap V}) = 0\}.$$

Thus the Zariski tangent space of $\mathrm{Hom}_Z(Y, X; f, B)$ at f is a subspace of $H^0(Y, \mathcal{I}_B f^* T_{X/Z})$, provided $f(Y) \cap U \neq \varnothing$. When π is a smooth morphism on an open subset containing $f(Y)$, it is well known that the Zariski tangent space is identical with $H^0(Y, \mathcal{I}_B f^* T_{X/Z})$, and that the obstruction lies in $H^1(Y, \mathcal{I}_B f^* T_{X/Z})$; hence in this case $\dim \mathrm{Hom}_Z(Y, X; f, B)$ at f is bounded from below by

$$\dim H^0(Y, \mathcal{I}_B f^* T_{X/Z}) - \dim H^1(Y, \mathcal{I}_B f^* T_{X/Z}).$$

Theorem 2.2. *Suppose that the source variety* Y *is a smooth projective curve of genus* ≥ 1 *and that* $\dim \mathrm{Hom}_Z(Y, Z; f, B) > 0$. *Take any curve in* $\mathrm{Hom}_Z(Y, X; f, B)$ *and let* Δ *denote its smooth compactification. Then the naturally induced rational map* $F: Y \times \Delta \dashrightarrow X$ *is never a* *morphism provided B is nonempty. Let $\overline{F}: W \to X$ be a resolution of the indeterminacy of F. Then every exceptional curve on W (with respect to the blowing up $W \to Y \times \Delta$) is mapped by $\overline{F}$ to a fibre of $\pi: X \dashrightarrow Z$. Furthermore, some irreducible component E of the exceptional divisor satisfies:*

(1) *$\overline{F}(E)$ is not a point and contains some point of $f(B)$,*

(2) *$(E, \overline{F}^*H) \leq 2(\deg f^*H)/\deg B$,*

where H is a fixed ample divisor on X.

Proof. We have only to prove that the image of an exceptional curve is contained in a fibre of π (the rest is found in [9]). Replacing W, we may assume that $\pi\overline{F}: W \dashrightarrow Z$ is again a morphism. Take a suitable ample divisor D on Z. Since $\pi\overline{F}$ is induced by a trivial deformation of πf, $\pi F(\{y\} \times \Delta)$ is a point for a general point $y \in Y$. Thus $\pi\overline{F}^*D$ is a nef divisor on S which intersects trivially with the pullback of $\Delta_y = \{y\} \times \Delta$. It follows that $\pi\overline{F}^*D$ is algebraically equivalent to a multiple of Δ_y, so that $\pi\overline{F}^*D$ intersects trivially with the exceptional divisors. This completes the proofs.

Corollary 2.3. *Suppose that Y is a curve of genus ≥ 1 and that Z has positive dimension. If $f: Y \to X$ has a nontrivial relative deformation over Z with base points B, then we have a new morphism $f': Y \to X$ such that the following hold.*

(1) *Either $\pi f = \pi f'$ or $f'(Y)$ is contained in the locus of indeterminacy of $\pi: X \to Z$.*

(2) $\deg f'_*(Y) < \deg f_*(Y)$.

(3) *$\pi f'(Y)$ contains a point of $\pi f(B)$.*

Proof. Take the strict transform Y_q of $Y \times \{q\}$ which contains a point of indeterminacy of F. Define f' to be the restriction of $\overline{F}$ to Y_q.

Corollary 2.4. *Assume that there is an open subset $U \subset X$ such that $\pi|_U$ is a proper morphism over an open subset of Z. Let $f: Y \to X$ be a morphism of a curve of genus ≥ 1. If Z is not a single point, then there exists a morphism $f': Y \to X$ such that the following hold:*

(1) $\pi f' = \pi f$,

(2) $\mathrm{Hom}_Z(Y, Z; f', B)$ *is a zero-dimensional scheme, whenever B is nonempty.*

Proof. Clear by Corollary 2.3.

The above general results are applied to Fano manifolds via the following:

Lemma 2.5. *Let $\pi: X \dashrightarrow Z$ be a dominant rational map between projective varieties. Let H and D be ample divisors on X and Z, respectively. Then there is a constant α which depends only on $\pi: X \dashrightarrow Z$, H*

and D such that

$$\deg f^*H \geq \alpha \deg(\pi f)^*D$$

for any smooth projective curve Y and any morphism $f: Y \to X$ whose image meets U, the domain on which π is defined.

Proof. Let $\rho: \overline{X} \to X$ be a blowing-up such that $\pi\rho$ is a morphism. Then there exists a constant a and an effective divisor E supported by the exceptional locus of ρ such that $a\rho^*H - E$ is ample on $\overline{X}$. Hence there is a constant b such that $b(a\rho^*H - E) - (\pi\rho)^*D$ is ample. On the other hand, since $f(Y)$ is not contained in the locus of indeterminacy of π, f naturally induces a morphism $\overline{f}: Y \to \overline{X}$. $\overline{f}(Y)$ is of course not contained in the effective divisor E, so that $\deg f^*(E) \geq 0$. Thus we get

$$\begin{aligned} ab \deg f^*H - \deg(\pi f)^*D &= ab \deg \overline{f}^* \rho^* H - \deg \overline{f}^*(\pi\rho)^*D \\ &\geq b \deg \overline{f}^*(a\rho^*H - E) - \deg \overline{f}^*(\pi\rho)^*D \\ &\deg \overline{f}^*\{b(a\rho^*H - E) - (\pi\rho)^*D\} \geq 0. \end{aligned}$$

Now put $\alpha = 1/(ab)$.

Corollary 2.6. *Suppose that X is a Fano manifold. Let $\pi: X \dashrightarrow Z$ be a dominant rational map and Y a nonsingular projective curve of genus g. Assume that π is a proper morphism over an open dense subset of Z. Let $B \subset Y$ be a closed subscheme of degree $b > 0$. Then there exists a constant α depending only on (X, Z, π) such that if a morphism $f: Y \to X$ satisfies*

(1) *$f(Y)$ is not contained in the locus of indeterminacy of π, and*

(2) *$\deg \pi f > \alpha(b + g)$,*

then $\dim_f \operatorname{Hom}(Y, X; B, f(B)) > 0$. *Here* $\operatorname{Hom}(Y, Z; B, f(B))$ *stands for the set of morphisms: $Y \to X$ which map B to $f(B)$.*

Proof. f has a nontrivial deformation with base points B if

$$\chi(Y, \mathscr{I}_B f^* T_X) = \deg f^*(-K_X) - \dim X(g + b - 1)$$

is positive. Then the assertion is clear by Lemma 2.5.

Step I *of the proof of Theorem* 2.1: *Case* char $= p > 0$.

Let $\pi: X \dashrightarrow Z$ be a dominant rational map as in (2.1). Take any point z on Z and a smooth complete intersection curve $Y_0 \subset X$ which intersects $\pi^{-1}(z)$. Fix a point P_0 from $Y_0 \cap \pi^{-1}(z)$. Choose a Frobenius k-morphism $f: Y \to Y_0 \subset X$ such that $\deg \pi f > \alpha(g+1)$, where g is the genus of Y_0. Corollary 2.4 implies that we can replace f by $f': Y \to X$ such that

(1) $\deg \pi f' = \deg \pi f > \alpha(g + 1)$ and

(2) f' has no relative deformation over Z with base point $P = f^{-1}(P_0)$.

The first condition assures the existence of absolute deformation by (2.6). Then there exists a rational curve passing through P_0 which is the image of an exceptional divisor on a blowing-up of $Y \times \Gamma$. On the other hand, since the deformation of f' induces a nontrivial deformation of $\pi f'$, some of the exceptional divisors are mapped to rational curves on Z. We can easily check that one of them passes through z. Once we find a horizontal rational curve meeting $\pi^{-1}(z)$, it is a routine work to let it split into a union of rational curves with degree $\leq (n+1)$ such that one of them meets $\pi^{-1}(z)$.

Step 2: *Lifting to characteristic zero* (when the original variety is defined over a field of characteristic zero).

This is clear since our horizontal rational curve meeting $\pi^{-1}(z)$ has bounded degree $\leq n+1$ with respect to the ample divisor $-K_X$.

This completes the proof of Theorem 2.1.

As an easy application of our argument above, we have:

Corollary 2.8 (char ≥ 0). *Let* $\pi: X \to Z$ *be a surjective smooth morphism between smooth projective varieties. If* $\dim Z > 0$, *then* $-K_{X/Y}$ *cannot be ample.*

Corollary 2.9 (char ≥ 0). *Let* $\pi: X \to Z$ *be a surjective smooth morphism between smooth projective varieties. If* X *is a Fano manifold, then so is* Z.

Proof. Let H be an ample divisor on Z, and a a positive constant such that $-K_X - a\pi^*H$ is nef. Then we prove that $-K_Z - aH$ is nef. Let $f: C \to Z$ be a nonconstant morphism from a smooth projective curve C. Let X_C denote the fibre product $X \times_Z C$, and let π_C and g be the projections from X_C to C and to X, respectively. $g^*(-K_X)$ is ample while $-K_{X_C/C}$ is not. Hence, by Kleiman's criterion, for any positive real number ε, we can find an irreducible curve D in X_C such that

$$(*) \qquad (D, -K_{X_C/C}) < \varepsilon(D, -g^*K_X).$$

Noting that $-K_{X_C/C} = g^*\pi^*K_Z - g^*K_X$, we get

$$(**) \qquad (D, -g^*\pi^*K_Z) > (1-\varepsilon)(D, -g^*K_X) \geq (1-\varepsilon)(D, ag^*\pi^*H).$$

Condition $(*)$ implies that D is not contained in a fibre, so we can divide the both sides of $(**)$ by the mapping degree of $D \to C$, to conclude

$$\deg f^*(-K_Z) > (1-\varepsilon)a \deg f^*H.$$

Since $\varepsilon > 0$ and $f: C \to Z$ are arbitrary, $-K_Z - aH$ is nef.

Remark 2.10. The above smoothness condition on π may look too restrictive, but a nonsingular image of a Fano manifold by a flat morphism is not always Fano. An example (a conic bundle) is found in [14]. This corollary was proved by [13] in a special case where $\pi: X \to Z$ is a $\mathbb{P}^n$-bundle.

3. Connecting two points by a chain of rational curves

We prove here that two points on a Fano manifold can be connected by a chain of rational curves whose total degree is bounded from above. This result is again characteristic-free.

Let $C_1, \cdots, C_r$ be a chain of curves (or simply a union of curves) on a polarized manifold (X, H). The *total degree* of this chain (or union) is understood to be $\sum(C_i, H)$. When X is Fano, H will be the anticanonical divisor $-K_X$ unless otherwise mentioned.

Lemma 3.1. *Let X be a Fano manifold, and $\pi: X \dashrightarrow Z$ a dominant rational map onto a smooth projective variety Z. Let $q: \widehat{X} \to X$ be a resolution of the indeterminacy of π, and $\hat{\pi}: \widehat{X} \to Z$ the naturally induced surjective morphism. Then there exists a family of irreducible rational curves $\widehat{\mathscr{C}}$ on $\widehat{X}$ which has the following two properties*:

(a) $(\widehat{C}, -q^*K_X) \leq \dim X + 1$ *for* $\widehat{C} \in \widehat{\mathscr{C}}$.

(b) $\hat{\pi}_*\widehat{\mathscr{C}}$ *is a covering family of irreducible rational curves on* Z.

Proof. If $\pi: X \dashrightarrow Z$ is a proper morphism over an open dense subset of Z, this is a restatement of Theorem 2.1. Now suppose that π has big indeterminacy; i.e., an exceptional divisor $E \subset \widehat{X}$ is surjectively mapped onto Z. E is a (proper image of a) ruled variety with fibres $F \simeq \mathbb{P}^r$ which are nontrivially mapped to Z. Define $\widehat{\mathscr{C}}$ to be the family of lines in the fibres F. In this case, $(\widehat{C}, -q^*K_X) = 0$.

Lemma 3.2. *Let the assumption and the notation be the same as in* (3.1). *Assume that two points on a fibre $\hat{\pi}^{-1}(z)$ are connected with each other by a chain of rational curvers $\Gamma_1, \cdots, \Gamma_k$ such that $\sum(\Gamma_i, -q^*K_X) \leq a$. Let $\hat{x}_1$ and $\hat{x}_2$ be two points on $\widehat{X}$. If $\hat{\pi}(x_1)$ and $\hat{\pi}(x_2)$ can be connected by b rational curves in $\hat{\pi}_*\widehat{\mathscr{C}}$, then $\hat{x}_1$ and $\hat{x}_2$ can be connected by rational curves R_i on $\widehat{X}$ such that $\Sigma(R_i, -q^*K_X) \leq (b+1)a + b(\dim X + 1)$. In particular, $x_i = q(\hat{x}_i)$, $i = 1, 2$, can be connected by a chain of rational curves of total degree $\leq (b+1)a + b(\dim X + 1)$.*

The proof is easy and left to the reader.

Theorem 3.3 (char ≥ 0). *Two arbitrary points on an n-dimensional Fano manifold can be joined by a chain of rational curves of total degree $\leq (2^n - 1)(n+1)$.*

Proof. We present here a proof in characteristic zero. In positive characteristics some additional work is required.

Fix a covering family $\mathscr{C}_0$ of irreducible rational curves of degree $\leq n+1$ on $X = X_0$, which induces a dominant rational map $p_1: X \dashrightarrow X_1$, where X_1 is a suitable smooth model of $\pi_{\mathscr{C}_0}(X)$. Let X_{01} be a modification of X which gives a resolution of p_1, and let $\widetilde{\mathscr{C}}_{01}$ be a family of irreducible rational curves on X_{01} as in (3.1). $\widetilde{\mathscr{C}}_{01}$ induces $\widetilde{\mathscr{C}}_1$ and $\mathscr{C}_1$, families of irreducible rational curves on X and X_1, respectively. $\mathscr{C}_1$ is a covering family which induces $\pi_{\mathscr{C}_1}: X_1 \dashrightarrow \pi_{\mathscr{C}_1}(X_1)$. Define X_2 as a smooth model of $\pi_{\mathscr{C}_1}(X_1)$, and $p_2: X \dashrightarrow X_2$ to be the composite of p_1 and $\pi_{\mathscr{C}_2}$. Then, replacing $p: X \dashrightarrow Z$ by $p_2: X \dashrightarrow X_2$, we can find a family of irreducible rational curves $\widetilde{\mathscr{C}}_{02}, \widetilde{\mathscr{C}}_2, \mathscr{C}_2$ on X_{02}, X, X_2 in the same manner as above (here X_{02} is the resolution of the indeterminacy of p_2). We can inductively define families $\widetilde{\mathscr{C}}_k$ of irreducible rational curves of degree $\leq n+1$, which induces a covering family $\mathscr{C}_k$ of rational curves on X_k via rational maps $p_k: X \dashrightarrow X_k$. Eventually we arrive at r such that X_r is a single point. Let n_i denote the dimension of X_i. Then we have $n = n_0 > n_1 > \cdots > n_r = 0$. Now it suffices to show by induction that two points x_1 and x_2 on a fibre $p_k^{-1}(z)$ can be joined by a chain of rational curves of total degree $\leq (2^{n-n_k} - 1)(n+1)$. If $k = 1$, this follows from the trivial inequality

$$\dim V(x, \mathscr{C}_0) = n - n_1 \leq 2^{n-n_1} - 1.$$

Assume that the above statement is true up to k. Then (3.2) states that two points in $p_{k+1}^{-1}(z)$ can be connected by rational curves of total degree less than or equal to

$$\begin{aligned}(n_k - n_{k+1}) + 1)(2^{n-n_k} - 1)(n+1) + (n_{k+1} - n_k)(n+1) \\ \leq 2^{n_k - n_{k+1}}(2^{n-n_k} - 1)(n+1) + 2(^{n_k - n_{k+1}} - 1)(n+1) \\ = (2^{n-n_{k+1}} - 1)(n+1).\end{aligned}$$

This completes the proof. q.e.d.

In characteristic zero, it is known that the following three conditions are equivalent [8, Theorem 2.1]:

(a) Two general points on X can be joined by an irreducible rational curve; i.e., X is rationally connected in our sense.

(b) Two arbitrary points on X can be joined by a finite chain of rational curves.

(c) Two general points on X can be joined by a finite chain of rational curves.

In [2], rational connectedness means the property (b) or (c) (the equivalence of these two is immediate and true in any characteristic).

4. Gluing rational curves to a single rational curve

In this section, we show that two or more rational curves on a given manifold can be fused to a single rational curve under a certain condition. The material is found in [8], but we include the proof here to specify an effective bound of the degree of curves.

Let I be a finite set, and let P_i, $i \in I$, be distinct points on $D = \mathbb{P}^1$. Let E_I denote the curve $\bigcup_{i\in I} D_i \cup D$, where $D_i = \mathbb{P}^1$, and the point ∞ on D_i is identified with P_i. E_I is a connected curve with $|I|$ ordinary double points and has an $|I|$-dimensional versal deformation $\mathscr{E}_I \to (S_I, o)$, where S_I is a smooth scheme of dimension $|I|$.

Given two smooth projective schemes X and Y together with closed subsets A and B of X and Y, respectively, let $\mathrm{Hom}(Y, X; B, A)$ denote the set of morphisms $f\colon Y \to X$ such that $f(B) \subset A$. $\mathrm{Hom}(Y, X; B, A)$ is a locally closed subset of $\mathrm{Hilb}(Y \times X)$, when morphisms are considered as subvarieties given by the graphs.

The following lemma was implicitly proven in [10].

Lemma 4.1. *Let $C \subset X$ be a rational curve on a nonsingular projective variety X. Let $f\colon \mathbb{P}^1 \to X$ be the normalization of C. If the inequality*

$$\dim_f \mathrm{Hom}(\mathbb{P}^1, X; \{0, \infty\}, \{f(0), f(\infty)\}) \geq 2$$

holds, then we can find a rational curve C' such that

(1) *C' passes through a prescribed point x on C,*

(2) *C is algebraically equivalent to $C' +$ (nonzero effective curve).*

The following theorem is given in [8, Corollary 1.6] in a more general situation.

Theorem 4.2 (Gluing lemma). *Let C_i, $i \in I$, be free rational curves on a smooth projective variety X, and $C = f(\mathbb{P}^1)$ a rational curve on X, with $\mathrm{Hom}(\mathbb{P}^1, X; \{0, \infty\}, \{f(0), f(\infty)\})$ being one-dimensional. Assume that C meets each C_i at Q_i and fix a point $Q \in C - \bigcup Q_i$. If $|I| \geq \dim X + 1 + (C, K_X)$, then there exist a subset $J \subset I$, a one-dimensional subscheme T of S_J and a morphism $T \times_{S_J} \mathscr{E}_J \to X \times T$, which induces a one-parameter family of curves containing Q such that*

(a) *it extends the natural morphism $f_J\colon E_J \to C \cup \bigcup_{i\in J} C_i$,*

(b) *a general member of this one-parameter family is an irreducible rational curve.*

Proof. The Zariski tangent space of $\mathrm{Hom}_{S_I}(\mathscr{E}_I, X \times S_I)$ at f_I is $\simeq H^0(E_I, T_{S_I} \oplus f_I^* T_X)$, so that $\dim_{f_I} \mathrm{Hom} \geq \chi(f_I^* T_X) + |I|$. One base condition imposed by Q decreases the dimension up to $\dim X$, so that $\dim_F \mathrm{Hom}_{S_I}(\ ; Q) \geq (-K_X)(C + \Sigma C_i) + |I|$. On the other hand, $\dim_{f_I} \mathrm{Hom}(E_I, X; Q)$ is, by an easy calculation, bounded by

$$(-K_X)\left(\sum_{i \in I} C_i\right) + \dim \mathrm{Hom}(E_0, X; \{\text{one point}\}, \{Q\})$$
$$\leq (-K_X)\left(\sum C_i\right) + 1 + (\dim X).$$

Thus, if $|I| \geq \dim X + 1 - (-K_X)C$, then $\mathrm{Hom}(E_I, X; \{\text{point}\}, \{Q\})$ is a proper subset of $\mathrm{Hom}_{S_I}(\mathscr{E}_I, X \times S_I)$, hence the assertion. q.e.d.

A similar but easier argument shows

Proposition 4.2bis. *Let C_1 and C_2 be free rational curves on X, a smooth projective variety. Assume that they meet each other at P and fix a point Q on $C_2 \backslash P$. Then the natural map $E_{\{1,2\}} \to C_1 \cup C_2$ extends to $\mathscr{E}_{\{1,2\}} \to X$ such that the image of every curve passes through Q.*

Theorem 4.3 (char $= 0$). *Let $x \in X$ be a sufficiently general point of a Fano manifold X (more precisely, $x \in X \backslash \Delta$, where Δ is the countable union of subschemes covered by the noncovering families of irreducible rational curves). Let $\mathrm{cl}[V(x)]$ be the fibre containing x of the rationally connected fibration $\pi_{\mathscr{C}}$ associated with $\mathscr{C}$, a family of free rational curves. Let y be an arbitrary point on $\mathrm{cl}[V(x)]$. Then there exists a one-parameter flat family $\{\Gamma_t\}$ of degree*

$$(\Gamma_t, -K_X) \leq a(n, e, d),$$

such that (1) *$\Gamma_t (t \neq 0)$ is an irreducible rational curve containing y, and* (2) *Γ_0 is a connected union of rational curves containing the prescribed general point x. Here $a(n, e, d)$ is a function in $n = \dim X$, $d = \dim V(x)$, $e = (C_s, -K_X)$, $C_s \in \mathscr{C}$; $a(n, e, d)$ grows like $n^{ed} e$.*

Proof. Let $C_1, \cdots, C_m$ be a sequence of rational curves which connects x and y. Choose a sequence $x_0, x_1, \cdots, x_m$ such that $x_0 = x$, $x_i \in C_i \cap C_{i+1}$, $x_m = y$. Assume that this sequence is *maximal* in the sense that no component C_i $(i \geq 1)$ deforms to $C_i^+ \cup C_i^-$ with $x_i \in C_i^+$, $x_{i+1} \in C_i^-$. This condition implies that $\dim_f \mathrm{Hom}(\mathbb{P}^1, X; \{0, \infty\}, \{x_i, x_{i+1}\}) = 1$ for $i \geq 1$. Since $x \in C_1$ is very general, there exist free rational curves D_i in $\mathscr{C}$ passing through general points on C_1. Thus there are distinct points $p-1, \cdots, p_k$ on C_1 and free rational curves $D_1, \cdots, D_k$ passing through them. Then the gluing lemma asserts that $C_1 \cup D_1 \cup \cdots \cup D_k$

can be deformed to a single rational curve $C_1' \supset \{x_1\}$. We can choose $k \leq n + 1 + (C_1, K_X)$ so that

$$\begin{aligned}(C_1', -K_X) &= k(D_1, -K_X) + (C_1, -K_X) \\ &\leq (n + 1 + C_1K_X)e + (C_1, -K_X) \\ &\leq ne + (C_1, -K_X).\end{aligned}$$

Since C_1' belongs to a covering family with base point x_1, we may assume that C_1' is free by (1.2); thus there are infinitely many points on C_2 through which pass free rational curves D_1^i in the deformation class of C_1'. If we repeat the above argument by replacing D_i by D_1^i and C_1 by C_2, we can perform the same procedure to get an irreducible curve C_2' connecting x_2 with a point $\in X$ whose specialization is x; reiterating things in this way, we eventually come across a one-parameter family of irreducible rational curves $C_{m,t}$ joining y to x_t whose limiting point is x. The degree of C_i' is estimated inductively by

$$\begin{aligned}(C_{k+1}', -K_X) &\leq n(C_k', -K_X) + (C_{k+1}, -K_X) \\ &\leq n(C_k', -K_X) + (n+1) \\ &\leq n^k\left(n + 1 + \frac{n+1}{n-1}\right) - \frac{n+1}{n-1}.\end{aligned}$$

On the other hand, since the total degree of the chain is bounded by ed, the length m of the chain is bounded by ed. This completes the proof. q.e.d.

The same method, together with the specialization argument, yields the following.

Theorem 4.4 (char $= 0$). *Let x be a sufficiently general point on a Fano manifold X of dimension n. Assume that a point y on X can be joined with x by a sequence of rational curves of total degree $\leq N$. Then x and y can be joined by a single free rational curve of degree $\leq en^N(1 + \frac{1}{n} + \frac{n+1}{n(n-1)})$, where e is the degree of a free rational curve passing through x.*

Combining (4.4) with (3.3), we get Theorem 0.1, with the function $c(n) = (n+1)n^{(2^n-1)(n+1)}\{1 + \frac{1}{n} + \frac{n+1}{n(n-1)}\}$. Fano's beautiful idea to deduce the bound of the degree of a Fano n-fold in terms of $c(n)$ is to use the asymptotic Riemann-Roch theorem for $n(-K_X - rP)$, where n is a sufficiently large and divisible integer, r a positive rational number and P a general point on X; for details, see [7]. Finally, the result that the bound of the degree of c_1 implies the boundedness of Fano n-folds is a special case of a theorem of Kollár and Matsusaka [6].

Remark 4.5. Our estimate of the minimum degree of rational curves connecting two points is, though indeed effective, very far from being sharp. This is a consequence of theoretical possibility of the existence of rational curves whose deformations are highly obstructed. When every rational curve is free, that is, when X has semipositive tangent bundle, two arbitrary points are connected to each other by an irreducible curve C of degree $\leq (\frac{1}{4})(n+2)^2$, so that

$$c_1^n(X) \leq \left(\frac{n+2}{2}\right)^{2n}$$

in such a case. It seems possible that the bounding function $c(n)$ could eventually be linear in n.

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