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CALCULUS OF VARIATIONS VIA THE GRIFFITHS FORMALISM

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Abstract

For general variational problems with one independent variable, we discuss Griffiths' construction of the Euler-Lagrange system on an affine subbundle of T^*M where M is the phase space of the variational problem. We show that, subject to some regularity assumptions, the Griffiths criterion gives necessary and sufficient conditions for the associated functional to be stationary.

Introduction

In the calculus of variations, it is of fundamental importance to find the extremals of a given functional

$$\Phi(\gamma) = \int_\gamma \phi,$$

where γ is a curve in M, the phase space of the variational problem, and ϕ is a one-form on M. Some well-known examples of variational problems include the action functional associated with mechanical systems and the arclength functional.

In this paper, we study variational problems arising from functionals whose domain of definition consists of integral curves of an exterior differential system. In [17], based on the pioneering work of Cartan [11], Griffiths gave a construction of the Euler-Lagrange system for such functionals and showed how to extend the rich geometric structures that are familiar in the case of classical mechanics to this general setting.

Griffiths' book contains a wealth of examples from mechanics and geometry which indicate the scope of applications of this generalization of the classical variational problem. However, the Euler-Lagrange system derived by Griffiths was arrived at only by heuristic reasoning. In particular, the fundamental unsolved problem, which we shall refer to as the *Griffiths Problem,* is whether or not, in general, the Euler-Lagrange system gives *necessary* as well as sufficient conditions for stationary values of Φ.

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The sufficiency of the Griffiths criterion was first established by Bryant [4, Proposition 3, p. 69], while the necessity of the criterion constitutes the main result of this paper. The key ingredient in the resolution of the Griffiths Problem is the notion of *regularity* of integral curves of a differential system I. Intuitively, an integral curve of I is said to be *regular* if it admits "enough" compactly supported variations as integrals of I. As will be shown in the following, for bracket generating differential systems (that is, systems whose derived flag terminates at (0)), the generic integral curve is regular.

The main result of this paper states that regular extremals of Φ are necessarily integrals of the Euler-Lagrange system obtained via the Griffiths formalism. This result is sharp in the sense that nonregular extremals exist which do not satisfy the corresponding Euler-Lagrange system. An explicit example of such a variational problem is constructed in §2. The existence of nonregular extremals has important consequences in the study of geodesics in sub-Riemannian geometry. In particular, they provide counterexamples to the often-stated assertion that "minimizing paths in sub-Riemannian manifolds satisfy the geodesic equations" (see, for instance, [22]).

The paper is organized as follows. In §1 the setup of the general variational problem and its corresponding Euler-Lagrange system are presented. §2 deals with the variational equations of integral curves of a differential system and the concept of regularity for such curves. The proof of the main result, the necessity of the Griffiths criterion, is then presented in §3 while in §4, we apply the Griffiths formalism to investigate a number of geometrically interesting variational problems. There are three appendices. The first is concerned with the calculus of variations for closed curves. In the second appendix, we give a construction of the *holonomy map* associated with integral curves of an exterior differential system. The third appendix deals with a generalization of the well-known lemma of du Bois-Reymond which is needed in the proof of the main result.

At this time, we would like to emphasize that although the subject matter of this paper is intimately related to the much-studied classical problem of Lagrange in the calculus of variations, the approach taken here is fundamentally different from that of classical papers in this area (see, for instance, [3] and references therein) and, being coordinate free, is decisively more geometric. Our guiding philosophy is that the theory of exterior differential systems [7], coupled with the method of moving frames [12], constitutes a computationally effective and theoretically natural setting for studying variational problems arising from geometry. This philosophy, introduced in Griffiths' book *Exterior differential systems and the calculus of variations* is perhaps not as widely appreciated as it should be, especially as regards the study of geometrically constrained variational problems. Our purpose here is to present a somewhat updated and hopefully clear exposition of a portion of the Griffiths philosophy, together with some applications to geometry.

1. The general variational problem

Let $I \subset T^*M$ be a rank-p subbundle on a smooth m-dimensional manifold M, and let $\phi \in \Omega^1(M)$ be a one-form on M. In this paper we shall restrict our consideration to smooth integral curves of I. Hence we fix an interval $(a, b) \subset \mathbb{R}$ and consider the space of smooth immersions

$$\mathscr{V}(I) = \{\gamma : (a, b) \to M \mid \gamma^*(I) = 0\},$$

which are integral curves of I. We shall use the Whitney C^∞-topology whenever we need a topology on $\mathscr{V}(I)$. Furthermore, we shall identify integral curves γ which differ only by reparametrization. Thus, if $s : (a, b) \to (a, b)$ is a smooth orientation-preserving diffeomorphism, then we identify $\gamma \circ s$ with γ.

By a *variational problem,* henceforth denoted by the triple (M, I, ϕ), we shall mean the study of the functional

$$\Phi : \mathscr{V}(I) \to \mathbb{R}$$

given by

$$\Phi(\gamma) = \int_\gamma \phi, \tag{1.1}$$

where γ is a typical integral curve of the differential system I on M. With the C^∞-topology on $\mathscr{V}(I)$, the functional Φ is everywhere smooth in its domain of definition.

It is important to note that this variational problem is a proper generalization of the *classical variational problem* where one studies a functional $\mathscr{L}$ on the space of smooth maps $x : (a, b) \to \mathbb{R}^n$ defined by

$$\mathscr{L}(x) = \int_a^b L(t, x(t), x'(t), \cdots, x^{(k)}(t))\, dt, \tag{1.2}$$

and L (the *Lagrangian*) is a smooth real-valued function on $(a, b) \times \mathbb{R}^{(k+1)n}$. Identifying $(a, b) \times \mathbb{R}^{(k+1)n}$ with $J^k((a, b), \mathbb{R}^n)$, it is easy to show that this classical variational problem is equivalent to studying the functional

$$\mathscr{L}(\gamma) = \int_{\gamma} \phi,$$

where $\gamma : (a, b) \to J^k((a, b), \mathbb{R}^n)$ is an integral curve of the canonical contact system on $J^k((a, b), \mathbb{R}^n)$, and $\phi = L dt$ is the standard Lagrangian one-form on $J^k((a, b), \mathbb{R}^n)$. Examples of "nonclassical" variational problems can be found in Bryant [4] and Griffiths [17], and also in §4 of this paper.

The basic problem in the calculus of variations is to describe the critical points of the functional Φ, that is, to determine the Euler-Lagrange equations of Φ. This can be described as follows:

Let $T_\gamma \mathscr{V}(I)$ denote the "tangent space" (see §2 for a precise definition) to $\mathscr{V}(I)$ at γ and consider the differential of the functional (1.1) as a map

$$\delta\Phi(\gamma) : T_\gamma \mathscr{V}(I) \to \mathbb{R}$$

given by

$$\delta\Phi(\gamma)(v) = \frac{d}{ds}\left(\int_{\gamma_s} \phi\right)\Bigg|_{s=0},$$

where $\gamma_s \in \mathscr{V}(I)$ is any *compactly supported* variation of γ with $\gamma_0 = \gamma$ and v is the associated infinitesimal variation defined along γ corresponding to the deformation $s \to \gamma_s$. Here, by "compactly supported variation of γ" we mean as usual a one-parameter family of smooth immersions $\gamma_s : (a, b) \to M$ with $\gamma_0 = \gamma$ defined by

$$\gamma_s(t) = \Gamma(t, s)$$

for a smooth map $\Gamma : (a, b) \times (-\varepsilon, \varepsilon) \to M$ which, outside of a compact set $K \subset (a, b)$, satisfies

$$\Gamma(t, s) = \Gamma(t, 0) \quad \forall\, t \in (a, b) \setminus K,$$

or equivalently

$$\gamma_s(t) = \gamma(t) \quad \forall\, t \in (a, b) \setminus K.$$

The associated compactly supported infinitesimal variation $v = \frac{\partial \Gamma}{\partial s}(t, 0)$ is called a *variational vector field.*

In terms of these, the Euler-Lagrange equations are the conditions that

$$\delta\Phi(\gamma)(v) = 0 \quad \forall\, v \in T_\gamma \mathscr{V}(I). \tag{1.3}$$

Integral curves γ satisfying this equation are called *extremals* of Φ.

Remark. In the classical approach to the calculus of variations, instead of compactly supported variations, it is usually assumed that variations have fixed endpoints. Of course, since the class of compact variations is a proper subset of fixed endpoints variations, all stationary points of Φ under fixed endpoints variations are necessarily stationary points under compact variations. However, the converse is not always true.

For geometric problems, it is natural to study stationary points under compact variations since, being geometric, the variational problem (M, I, ϕ) should be independent of the prolongation class of the system I.

We now proceed to consider the instructive special case where $I = \{0\}$. In this case, the extremal problem is easily solved:

Proposition 1. *If $\gamma : (a, b) \to M$ is an extremal of the functional*

$$\Phi(\gamma) = \int_\gamma \phi$$

when one considers compactly supported variations, then for all $t \in (a, b)$,

$$v \lrcorner d\phi|_{\gamma(t)} = 0, \tag{1.4}$$

where $v \in C^\infty(TM)$ is any compactly supported vector field along γ. Conversely, if γ satisfies this condition, then $\gamma : (a, b) \to M$ is an extremal.

Proof. Let $\Gamma : (a, b) \times (-\varepsilon, \varepsilon) \to M$ be an arbitrary compactly supported variation of γ and let $\gamma_s(t) = \Gamma(t, s)$ for $|s| < \varepsilon$. A straightforward computation gives

$$\frac{d}{ds}\left(\int_{\gamma_s} \phi\right)\bigg|_{s=0} = \int_a^b v \lrcorner d\phi,$$

where $v = \frac{\partial \Gamma}{\partial s}(t, 0)$ is the variational vector field associated with $\Gamma(t, s)$. Since Γ is an arbitrary compactly supported variation and v can be any compactly supported vector field along γ, we see that Proposition 1 follows.

Remark. Proposition 1 above allows us to associate to a variational problem $(M, \{0\}, \phi)$ a Pfaffian system $\tilde{I}$ on M. From (1.4), this system is simply the *Cartan system* of the two-form $d\phi$ on M:

$$\mathscr{C}(d\phi) = \{v \lrcorner d\phi \mid v \in C_\circ^\infty(TM)\},$$

where $C_0^\infty(TM)$ denotes the space of smooth compactly supported vector fields on M. Thus extremals of $(M, \{0\}, \phi)$ are *characteristic curves* of the two-form $d\phi$ on M. Following Griffiths, we shall call the differential system $\tilde{I}$ on M generated by $\mathscr{C}(d\phi)$ the *Euler-Lagrange system* of the variational problem $(M, \{0\}, \phi)$.

In the case $\operatorname{rank}(I) > 0$, Bryant [4, Proposition 3] and Griffiths [17, pp. 77–83] generalized the above proposition, giving a simple construction of extremals of an arbitrary variational problem. Let us proceed to give a brief outline of this construction.

To each variational problem (M, I, ϕ), we associate the affine sub-bundle $Z = I + \phi$ of the cotangent bundle T^*M. This means that for each $x \in M$, we view $Z_x = I_x + \phi_x$ as an affine subspace of T^*_xM. Note that Z determines I since for each $x \in M$, I_x is the vector subspace of T^*_xM parallel to Z_x. However Z only determines ϕ modulo I since $I + \phi = I + \phi + \eta$ if and only if $\eta \in C^\infty(I)$. Now recall that on T^*M, there is a canonical one-form σ defined by $\sigma(v) = \xi(\pi_*(v))$ for $\xi \in T^*M$, where $v \in T_\xi(T^*M)$ and $\pi : T^*M \to M$ is the canonical projection [23]. We denote the restriction of σ to the affine sub-bundle Z by ζ.

Locally Z can be identified with an $(m+p)$-dimensional product manifold $Z_U \cong U \times \mathbb{R}^p$ for an open set $U \subset M$. This is given by identifying the pair $(x, \lambda) \in U \times \mathbb{R}^p$ with the one-form

$$\phi_x + \lambda\theta_x \in T^*_xM.$$

Here we view $\lambda = (\lambda_1, \cdots, \lambda_p)$ as a row-vector and $\theta = {}^t(\theta^1, \cdots, \theta^p)$ as a column-vector, where $\{\theta^\alpha\}$ is a basis for the sections of I over U. Under this identification, the canonical one-form ζ on Z takes the form $\zeta = \phi + \lambda\theta$.

At this point, we note that each variational problem (M, I, ϕ) determines a canonical *associated variational problem* $(Z, \{0\}, \zeta)$. The importance of $(Z, \{0\}, \zeta)$ for the variational problem (M, I, ϕ) comes from the following well-known result (see Bryant [4]).

Theorem 2. *Let (M, I, ϕ) be a smooth variational problem and let $(Z, \{0\}, \zeta)$ be the associated variational problem. Then the projection $\pi : Z \to M$ maps extremals of $(Z, \{0\}, \zeta)$ to extremals of (M, I, ϕ).*

Proof. Let $\tilde{\gamma} : (a, b) \to Z$ be a smooth map and suppose that $\tilde{\gamma}$ is an extremal of $(Z, \{0\}, \zeta)$. Let $\gamma = \pi \circ \tilde{\gamma}$, where $\pi : Z \to M$ is the projection map. By construction,

$$d\zeta = d\phi + d\lambda \wedge \theta + \lambda d\theta.$$

Contracting $d\zeta$ with vertical vectors $\partial/\partial\lambda$ on Z, we see that $\gamma(t)$ is an integral curve of I on M.

Now suppose that $\Gamma : (a, b) \times (-\varepsilon, \varepsilon) \to M$ is a compactly supported variation of γ through integrals of I. Let $\widetilde{\Gamma} : (a, b) \times (-\varepsilon, \varepsilon) \to Z$ be any lifting of Γ with $\widetilde{\Gamma}(t, 0) = \tilde{\gamma}(t)$. For each $(t, s) \in (a, b) \times (-\varepsilon, \varepsilon)$, the definition of Z implies that $\widetilde{\Gamma}(t, s) - \phi|_{\Gamma(t,s)} \in I|_{\Gamma(t,s)}$. Since, for each fixed s, $\gamma_s(t) = \Gamma(t, s)$ is an integral of I, we have

$$\widetilde{\Gamma}(t, s)[\gamma_s'(t)] = \phi[\gamma_s'(t)].$$

By the definition of ζ, we have that $\zeta[\tilde{\gamma}_s'(t)] = \widetilde{\Gamma}(t, s)[\gamma_s'(t)]$, so for each s, we must have $\int_a^b \tilde{\gamma}_s^*(\zeta) = \int_a^b \gamma_s^*(\phi)$. Since $\tilde{\gamma}_0 = \tilde{\gamma}$ is an extremal of $(Z, \{0\}, \zeta)$, it follows that $\gamma = \pi \circ \tilde{\gamma}$ is an extremal of (M, I, ϕ).

Remarks. 1. Associated with a variational problem (M, I, ϕ) is the Euler-Lagrange system $\widetilde{I}$ on Z obtained by applying Proposition 1 to the corresponding variational problem $(Z, \{0\}, \zeta)$. By the preceding theorem, integral curves of this system give rise to extremals of (M, I, ϕ).

In most applications however, one is interested in finding extremals of the variational problem satisfying some transversality conditions. This naturally arises when the basic differential system I admits an independence condition, say ω, a one-form on M. Integrals of (I, ω) are integrals $\gamma \in \mathscr{V}(I)$ satisfying $\gamma^*\omega \neq 0$. The corresponding Euler-Lagrange system, generated by the involutive prolongation of the Cartan system $(\mathscr{C}(d\zeta), \widehat{\omega})$ on Z, is a Pfaffian system $(\widetilde{I}, \widetilde{\omega})$ on the associated momentum space $\widetilde{M} \subset Z$ (see Griffiths [17, pp. 78–83] for the details of this construction). Here $\hat{\omega}$ and $\tilde{\omega}$ are respectively one-forms on Z and $\widetilde{M}$ obtained by pulling back ω on M.

2. In the classical literature on the calculus of variations, attention is focused almost exclusively on what is called the Lagrange problem [3]. In our language, this corresponds to taking the differential system I to be the restriction of the canonical contact system on $J^1(\mathbb{R}, \mathbb{R}^n)$ and ϕ to be the restriction of the standard Lagrangian one-form on $J^1(\mathbb{R}, \mathbb{R}^n)$ to a submanifold M of $J^1(\mathbb{R}, \mathbb{R}^n)$. This is of course a special case of the general variational problem studied in this paper. Applying the above construction, one easily recovers the classical Euler-Lagrange equations obtained via the Lagrange multiplier rule.

3. The one-form ζ on Z plays the role of a generalized *Cartan form* in the sense that the projection of the characteristic curves of $d\zeta$ to M are extremals of the variational problem. In the classical case (1.2), ζ is precisely the Cartan form which yields much essential information about the variational problem [11]. In particular, ζ occurs as the integrand of the Hilbert invariant integral which plays a crucial role in the proofs of various sufficiency theorems [18].

Theorem 2 above allows us to find extremals of (M, I, ϕ) by integrating the corresponding Euler-Lagrange system $\widetilde{I}$. However, we have not shown that all the extremals of (M, I, ϕ) arise this way for a general variational problem. Thus we are naturally led to the following.

Griffiths Problem. *Does every extremal of a variational problem* (M, I, ϕ) *arise from the projection of an extremal of* $(Z, \{0\}, \zeta)$?

Clearly, every extremal γ of (M, I, ϕ), being an integral of I, has a lifting to a curve $\tilde{\gamma} \subset Z$. The fundamental question is: Does there exist a lifting $\tilde{\gamma}$ which is an integral curve of the Euler-Lagrange system $\tilde{I}$ on Z?

In the following sections, we shall attempt a resolution of this problem. We shall show that, subject to some regularity conditions on $\mathscr{V}(I)$, every extremal of a variational problem (M, I, ϕ) arises from the projection of an integral curve of the associated Euler-Lagrange system $\tilde{I}$ on Z. In actual examples (see, for instance, §4), these regularity conditions are either identically satisfied or correspond to some natural geometric conditions on the class of integral curves of the differential system I. We begin with the following simple observation.

Let γ be an integral curve of I which admits a compactly supported variation, denoted by $\Gamma : (a, b) \times (-\varepsilon, \varepsilon) \to M$, and let $\gamma_s(t) = \Gamma(t, s)$ for $|s| < \varepsilon$. As before, a straightforward computation gives

$$\frac{d}{ds}\left(\int_{\gamma_s} \phi\right)\Bigg|_{s=0} = \int_a^b v \lrcorner d\phi,$$

where v is the variational vector field associated with $\Gamma(t, s)$. Hence, the condition that the integral curve $\gamma : (a, b) \to M$ be an extremal is that

$$\int_a^b v \lrcorner d\phi = 0 \tag{1.5}$$

for all variational vector fields of $\gamma \in \mathscr{V}(I)$. However, except for the case $I = \{0\}$, very little is known about the space of variational vector fields of an integral curve γ. Thus, as a first step towards solving the Griffiths Problem, we proceed in the following section to undertake a detailed study of these vector fields.

2. Regularity and the variational equations

The difficulty in deriving the correct Euler-Lagrange equations from the integral relation (1.5) is that only certain transverse vector fields to $\gamma \subset M$ represent infinitesimal variations of γ as an integral curve of I. In particular, there may be no such compactly supported variational vector fields along γ. In this section, we shall present a *regularity criterion* under which we can establish the existence of such variational vector fields.

We begin with the derivation of the *variational equations* for an integral curve γ of I. These equations may be viewed geometrically as describing "to first order" the quantity $T_\gamma \mathscr{V}(I)$, the space of smooth variational vector fields of γ. We shall show that for the subset of *regular* integral curves (to be defined below) $\mathscr{V}_R(I) \subset \mathscr{V}(I)$, $T_\gamma \mathscr{V}_R(I)$ coincides exactly with the null space of the *variational operator*.

A map $\widehat{\Gamma} : (-\varepsilon, \varepsilon) \to \mathscr{V}(I)$ is said to be a smooth compactly supported variation of γ as an integral curve of I if the map

$$\Gamma : (a, b) \times (-\varepsilon, \varepsilon) \to M$$

defined by $\Gamma(t, s) = \widehat{\Gamma}(s)(t)$ is smooth, compactly supported in (a, b), and such that if we let

$$\gamma_s : (a, b) \to M, \qquad s \in (-\varepsilon, \varepsilon),$$

be the restriction of Γ to $(a, b) \times \{s\} \cong (a, b)$, then $\gamma_0 = \gamma$ and $\gamma_s^*(I) = 0$. Here, "compactly supported in (a, b)" means that $\Gamma(t, s)$ coincides with $\Gamma(t, 0)$ outside of a compact subset of (a, b). In terms of these, the space of variational vector fields can be described as

$$T_\gamma \mathscr{V}(I) = \left\{ \frac{\partial \Gamma}{\partial s}(t, 0) \mid \widehat{\Gamma} : (-\varepsilon, \varepsilon) \to \mathscr{V}(I) \text{ is a compact variation of } \gamma \right\}.$$

As the notation suggests, for $\mathscr{V}(I)$ a differentiable manifold in a neighborhood of the integral curve γ, $T_\gamma \mathscr{V}(I)$ is the tangent space to $\mathscr{V}(I)$ at γ.

Although intuitive, the above description is highly unsatisfactory since a direct computation of $T_\gamma \mathscr{V}(I)$ requires a detailed knowledge of the space of solutions of the differential system I. What is needed is a description of $T_\gamma \mathscr{V}(I)$ which depends only on the system I.

In [17, p. 44], Griffiths derived the variational equations for an arbitrary integral curve γ of a differential system I. We proceed to give a construction of these equations.

We associate to a rank p differential system I on M^{n+p} a "horizontal" distribution given by

$$I^\perp = \{v \in TM \mid v \lrcorner I = 0\}.$$

Furthermore, on each integral curve $\gamma : (a, b) \to M$ of the system I, there are naturally defined vector bundles which we denote by

$$T_\gamma = \gamma^*(TM), \qquad I_\gamma^* = \gamma^*(TM/I^\perp).$$

Geometrically, T_γ is the space of smooth tangent vector fields to M defined along γ, while I_γ^* can be thought of as the associated "vertical" bundle. Clearly T_γ and I_γ^* are respectively rank $n+p$ and rank p vector bundles over γ. Associated with these vector bundles are mappings

$$0 \to I_\gamma^\perp \xrightarrow{i} T_\gamma \xrightarrow{\pi_V} I_\gamma^* \to 0,$$

where i denotes the inclusion map into T_γ while π_V is the projection to I_γ^*. Note that there is no canonical splitting of this sequence.

In what follows, we find it convenient to make a choice of splitting $s : I_\gamma^* \to T_\gamma$ satisfying $\pi_V \circ s = \mathbf{1}$ (the identity map on I_γ^*), in which case

$$T_\gamma \cong I_\gamma^\perp \oplus s(I_\gamma^*).$$

This then enables us to introduce the projection operator

$$\pi_H : T_\gamma \to I_\gamma^\perp$$

which depends on the splitting s.

In terms of the above, Griffiths [17] deduced that the variational equations of $\gamma \in \mathscr{V}(I)$ take the form

$$\mathscr{D}_\gamma(v) = 0, \tag{2.1}$$

where $\mathscr{D}_\gamma : C^\infty(T_\gamma) \to \Omega^1(I_\gamma^*)$ is a linear differential operator defined by

$$\mathscr{D}_\gamma(v) = e_\alpha \otimes (v \,\lrcorner\, d\theta^\alpha + d(v \,\lrcorner\, \theta^\alpha))\big|_\gamma. \tag{2.2}$$

Here $\{\theta^\alpha\}$ is a local coframe field of I on U, an open subset of M containing γ, while $\{e_\alpha\}$ is a frame field of I_γ^* dual to $\{\theta^\alpha\}$ along γ. Hence, with respect to a fixed frame field, the variational equations (2.1) are

$$(v \,\lrcorner\, d\theta^\alpha + d(v \,\lrcorner\, \theta^\alpha))\big|_\gamma = 0. \tag{2.3}$$

These variational equations are of course canonically associated with each integral curve of I. In fact, given a differential system I on M, there is a differential system $I^\natural$ on TM with the property that all integral curves $\hat{\gamma} : (a, b) \to TM$ of $I^\natural$ which are transverse to the fibers of the projection $\pi : TM \to M$ are solutions of the variational equations of $\gamma = \pi(\hat{\gamma}) \in \mathscr{V}(I)$. The system $I^\natural$ is constructed as follows:

Let $I \subset \Omega^*(M)$ be a differential system on M, and let $I^\natural \subset \Omega^*(TM)$ be the system on TM generated by $\pi^*(I) \cup \pi^\natural(I)$. Here $\pi^*(I)$ means the pullback of the system I to TM while $\pi^\natural : \Omega^k(M) \to \Omega^k(TM)$ denotes the operator

$$\pi^\natural = d \circ \pi^\flat + \pi^\flat \circ d,$$

where $\pi^\flat : \Omega^k(M) \to \Omega^{k-1}(TM)$ is defined to be

$$\pi^\flat(\theta)(v) = \pi^*[v \lrcorner \theta|_{\pi(v)}]$$

for any vector v on M.

By construction, an integral curve $\hat{\gamma}$ of $I^\natural$ on TM projects down to M to be an integral curve γ of I, and in addition satisfies

$$\hat{\gamma}^*\pi^\natural(I) = 0.$$

As can readily be checked, this last equation is equivalent to (2.3). Henceforth, we shall call $I^\natural$ the variational system associated with I.

The importance of the variational equations arises from the fact that they allow us to obtain an almost explicit description of $T_\gamma\mathscr{V}(I)$. Geometrically, solutions of these equations are the possible infinitesimal variations of γ as an integral curve of I. In particular, $\ker\mathscr{D}_\gamma$ contains the set of variational vector fields of γ. We now proceed to show that, subject to a maximal rank condition, every compactly supported solution $v \in \ker\mathscr{D}_\gamma$ of the variational equation is a variational vector field of the integral curve γ associated to an actual variation $\gamma_s \in \mathscr{V}(I)$.

To accomplish this, we need a more explicit description of the variational equations of $\gamma \in \mathscr{V}(I)$. Let U be an open subset of M and let $\gamma : (a, b) \to U$ be an integral curve of I. Choose a coframing $\{\theta^\alpha, \eta^i\}$ of U so that I is generated by $\{\theta^\alpha\}$ on U. We then have the structure equations

$$d\theta^\alpha = -\phi^\alpha_\beta \wedge \theta^\beta + \tfrac{1}{2}c^\alpha_{jk}\eta^j \wedge \eta^k, \tag{2.4}$$

where $c^\alpha_{jk} + c^\alpha_{kj} = 0$. Note that these equations uniquely determine the one-forms ϕ^α_β mod I.

This choice of coframing induces a splitting $T_\gamma \cong I^\perp_\gamma \oplus s(I^*_\gamma)$ along γ whereby $I^\perp_\gamma$ and I^*_γ are spanned by $\{e_i\}$ and $\{e_\alpha\}$, dual frame fields to $\{\eta^i\}$ and $\{\theta^\alpha\}$. With respect to this splitting, $v \in C^\infty(T_\gamma)$ can be expressed as

$$v = e_i w^i + s(e_\alpha)u^\alpha,$$

or equivalently

$$v = e_H w + s(e_V)u,$$

where $u = (u^\alpha) \in C^\infty((a, b), \mathbb{R}^p)$ and $w = (w^i) \in C^\infty((a, b), \mathbb{R}^n)$ are smooth vector-valued functions, while $e_H = (e_i)$ and $e_V = (e_\alpha)$ are regarded as row-vectors. If $v \in C^\infty(T_\gamma)$ is a variational vector field, then $\pi_H(v) = e_H w \in C^\infty(I^\perp_\gamma)$ manifests itself as the infinitesimal horizontal variation, while $\pi_V(v) = e_V u \in C^\infty(I_\gamma^*)$ generates the corresponding vertical variation.

Now, let $\gamma : (a, b) \to U$ be an integral curve of I. Clearly, we have

$$\gamma^*\theta^\alpha = 0, \quad \gamma^*\eta^i = n^i(t)dt, \quad \gamma^*\phi^\alpha_\beta = f^\alpha_\beta(t)dt, \quad \gamma^* c^\alpha_{jk} = c^\alpha_{jk}(t),$$

for some smooth functions $n^i(t)$, $f^\alpha_\beta(t)$, and $c^\alpha_{jk}(t)$ on (a, b), and a parametrization t of γ. To simplify notation, we let

$$n = (n^i), \quad f = (f^\alpha_\beta), \quad c = (c^\alpha_i),$$

where $c^\alpha_i = c^\alpha_{ji} n^j$. In terms of these, the variational operator (2.2) of γ takes the form

$$\mathscr{D}_\gamma(v) = e_V \otimes [du + (fu - cw)dt],$$

in which case the corresponding variational equations are

$$\dot{u} + fu = cw,$$

where $\dot{}$ denotes the derivative with respect to the parameter t.

We seek to describe compactly supported solutions of these variational equations. Hence, we construct a mapping

$$J_\gamma : C^\infty_\circ(I_\gamma^\perp) \to \ker \mathscr{D}_\gamma$$

whereby $\sigma = e_H w \in C^\infty_\circ(I_\gamma^\perp)$, a section of $C^\infty(I_\gamma^\perp)$ with compact support in (a, b), determines $\hat{\sigma} = e_V u_\sigma$, a section of $C^\infty(I_\gamma^*)$, by the requirement that u_σ satisfies the variational equation

$$\dot{u}_\sigma + fu_\sigma = cw \tag{2.5a}$$

with initial condition

$$u_\sigma(a) = 0. \tag{2.5b}$$

Note that this is an under-determined system of first-order linear ordinary differential equations for u_σ. Solutions to these variational equations always exist and depend on the $\mathbb{R}^n$-valued function $w \in C^\infty_\circ((a, b), \mathbb{R}^n)$ of the parameter t. The p constants of integration are determined by the initial condition (2.5b).

Explicitly, the J_γ-map is given by

$$J_\gamma(\sigma) = \sigma + s(\hat{\sigma}).$$

We seek conditions under which $J_\gamma(\sigma)$ has compact support in (a, b). Notice that since $\sigma \in C^\infty_\circ(I_\gamma^\perp)$, the horizontal component of $J_\gamma(\sigma)$ trivially has compact support in (a, b). Hence it suffices to consider $\pi_V \circ$

$J_\gamma(\sigma)$, the vertical component of the J_γ-map. By construction $J_\gamma(\sigma) \in \ker_{-}\mathscr{D}_\gamma$, sections of $\ker\mathscr{D}_\gamma$ vanishing in a neighbourhood of $t = a$. Thus we are led to consider the mapping

$$\mathscr{H}_\gamma : C^\infty_\circ(I^\perp_\gamma) \to I^*_\gamma|_{t=b}$$

given by

$$\mathscr{H}_\gamma(\sigma) = \pi_V(J_\gamma(\sigma))|_{t=b}, \tag{2.6}$$

where $I^*_\gamma|_{t=b} \cong \mathbb{R}^p$ denotes the fiber of the vector bundle I^*_γ at the point $\gamma|_{t=b}$. Note that $J_\gamma(\sigma) \in \ker_\circ\mathscr{D}_\gamma$, sections of $\ker\mathscr{D}_\gamma$ with compact support in (a, b), if and only if $\mathscr{H}_\gamma(\sigma) = 0$.

We call $\mathscr{H}_\gamma$ the *holonomy map* associated with the integral curve $\gamma \in \mathscr{V}(I)$. It will be constructed in a more precise fashion in Appendix B. We now introduce the following.

Definition. An integral curve $\gamma \in \mathscr{V}(I)$ is said to be *regular* if its holonomy map $\mathscr{H}_\gamma$ is surjective.

As we will show below, the holonomy map plays a crucial role in the study of $T_\gamma\mathscr{V}(I)$. In the case $\mathscr{H}_\gamma$ is a surjective map, we shall give an explicit description of $T_\gamma\mathscr{V}(I)$. Henceforth, we denote the space of regular integral curves of the differential system I by $\mathscr{V}_R(I)$.

At this point we note that although the holonomy map $\mathscr{H}_\gamma : C^\infty_\circ(I^\perp_\gamma) \to \mathbb{R}^p$ as constructed above depends on the choice of splitting of T_γ, the notion of surjectivity of $\mathscr{H}_\gamma$ is independent of this splitting. Indeed, as we will show below, the rank of $\mathscr{H}_\gamma$ is well defined.

This being so, we now fix a splitting of T_γ. With respect to this splitting, we seek a basis of sections of I such that $\mathscr{H}_\gamma$ takes a particularly simple form. Hence let

$$\tilde\theta^\alpha = \tilde h^\alpha_\beta \theta^\beta$$

be a change of basis of the sections of I, where $\tilde h = (\tilde h^\alpha_\beta)$ is any smooth extension to U of the map $h : \mathbb{R} \to \mathrm{GL}(p, \mathbb{R})$ defined along the integral curve $\gamma \subset U$ by the relation

$$h^{-1}\dot h = f. \tag{2.7}$$

With respect to this change of basis, the corresponding structure equations become

$$d\tilde\theta^\alpha = -\tilde\phi^\alpha_\beta \wedge \tilde\theta^\beta + \tfrac{1}{2}\tilde c^\alpha_{ij}\eta^i \wedge \eta^j,$$

where, in view of (2.7), we have $\gamma^*(\tilde\phi^\alpha_\beta) = 0$. A basis of sections of I satisfying this last relation is called a *parallel basis* (see Appendix B for the motivation for this terminology).

In this parallel basis, the variational equations (2.5a) take the form

$$\dot{\tilde{u}}_\sigma = \tilde{c}w, \qquad \tilde{c} = hc, \tag{2.8a}$$

while the initial condition (2.5b) becomes

$$\tilde{u}_\sigma(a) = 0. \tag{2.8b}$$

Henceforth, we shall denote by H the matrix c computed with respect to a parallel basis, that is, $H = hc$. Since the solution of (2.7) is unique up to a left multiplication by a nonsingular constant matrix, the H-matrix is likewise defined.

Integrating (2.8), we have that

$$\tilde{u}_\sigma(t) = \int_a^t H(\tau)w(\tau)\, d\tau.$$

This is of course just the explicit representation of the $(\pi_V \circ J_\gamma)$-map. Hence, with respect to a parallel basis of sections of I, the holonomy map can be expressed as

$$\widetilde{\mathscr{H}}_\gamma : C_\circ^\infty((a, b), \mathbb{R}^n) \to \mathbb{R}^p,$$

where

$$\widetilde{\mathscr{H}}_\gamma(w) = \int_a^b H(\tau)w(\tau)\, d\tau. \tag{2.9}$$

Armed with these concepts, we now proceed to consider compactly supported variations of an integral curve $\gamma \in \mathscr{V}(I)$. The fundamental result is the following theorem of Bryant [5].

Theorem 3. *If γ is a regular integral curve of I, then every compactly supported solution of the variational equation $v \in \ker_\circ \mathscr{D}_\gamma$ is a variational vector field.*

Proof. Without loss of generality, we shall restrict our consideration to an imbedded regular integral curve γ of I. This being so, we now choose flow-box coordinates so that γ is a straight line in M. In a rectangular neighborhood of γ, we can view M^{n+p} as a product manifold

$$M \cong N^n \times P^p,$$

where $\gamma \subset N^n$ and $I_\gamma^\perp = T_\gamma N^n$. Locally, we think of N^n and P^p as vector spaces $\mathbb{R}^n$ and $\mathbb{R}^p$ respectively.

Now, since γ is regular, its holonomy map $\mathscr{H}_\gamma : C_\circ^\infty(I_\gamma^\perp) \to \mathbb{R}^p$ is surjective. Without loss of generality, we may assume that $\mathscr{H}_\gamma$ is defined with respect to a parallel basis. Hence, we can find horizontal vectors $\sigma_\alpha =$

$e_H w_\alpha \in C_\circ^\infty(I_\gamma^\perp)$ so that $\{\widetilde{\mathscr{H}}_\gamma(w_\alpha)\}$ forms a basis of $\mathbb{R}^p$. Furthermore, let $\sigma = e_H w$ be the projection of $v \in \ker_\circ \mathscr{D}_\gamma$ to $C_\circ^\infty(I_\gamma^\perp)$. We then have

$$\widetilde{\mathscr{H}}_\gamma(w) = 0. \tag{2.10}$$

Now, let us construct a compactly supported deformation of γ in N^n depending on the $(p+1)$ parameters (s, s_α),

$$\Gamma(t, s, s_\alpha) = \gamma(t) + sw(t) + \sum s_\alpha w_\alpha(t).$$

For fixed values of (s, s_α), sufficiently small, let $\widetilde{\Gamma}(t, s, s_\alpha)$ be the unique lifting of $\Gamma(t, s, s_\alpha)$ to $M = N^n \times P^p$ obtained by solving the system of ordinary differential equations corresponding to the differential system I. In a neighborhood of $(s, s_\alpha) = 0$, it can be shown that

$$\widetilde{\Gamma}(t, s, s_\alpha) = \gamma(t) + sv(t) + \sum s_\alpha v_\alpha(t) + O(s^2, s_\alpha^2), \tag{2.11}$$

where $v, v_\alpha \in \ker \mathscr{D}_\gamma$ are the liftings of w, w_α to M. Note that by construction

$$\widetilde{\Gamma}(t, 0, 0) = \gamma(t). \tag{2.12}$$

We now proceed to consider the mapping $\Pi : \mathbb{R}^{p+1} \to P^p$ given by

$$\Pi(s, s_\alpha) = \widetilde{\Gamma}(t, s, s_\alpha)|_{t=b}.$$

From equation (2.12), we necessarily have that $\Pi(0, 0) = 0$, while from (2.10) and (2.11) we obtain the relations

$$\frac{\partial \Pi}{\partial s}(0, 0) = 0, \qquad \frac{\partial \Pi}{\partial s_\alpha}(0, 0) = \widetilde{\mathscr{H}}_\gamma(w_\alpha).$$

Applying the implicit function theorem, for s sufficiently small, we are led to conclude that there exists a smooth curve $s_\alpha = \tau_\alpha(s)$ such that $\Pi(s, \tau_\alpha(s)) = 0$. Differentiating this last equation, we obtain the relation

$$\frac{\partial \Pi}{\partial s} + \sum \frac{\partial \Pi}{\partial s_\alpha}\tau'_\alpha = 0,$$

which, when restricted to $(s, s_\alpha) = 0$, gives $\tau'_\alpha(0) = 0$. Thus we have constructed a compactly supported variation $\widehat{\Gamma}(t, s) = \widetilde{\Gamma}(t, s, \tau_\alpha(s))$ of γ satisfying $\frac{\partial \widehat{\Gamma}}{\partial s}(t, 0) = v(t)$.

Remarks. 1. Theorem 3 enables us to conclude that

$$T_\gamma \mathscr{V}_R(I) \cong \ker_\circ \mathscr{D}_\gamma.$$

As will be shown in the next section, this explicit description of the space of variational vector fields of a regular integral curve of the system I is the key to establishing the validity of the Griffiths formalism for variational problems arising from functionals whose domain of definition consists of regular integral curves of an exterior differential system.

2. Since M is an $(n+p)$-dimensional manifold and I is a rank p sub-bundle, the general integral curve of I depends locally on n functions of one variable. However, if we identify integrals that differ only by reparametrization, then the general *geometric integral curve* depends locally on $n-1$ functions of one variable.

Similarly, the space of compactly supported variational vector fields of a *regular* integral curve γ also depends upon n functions of a single variable. Furthermore, if we identify variational vector fields that differ by a tangent vector of γ then we again have that $T_\gamma \mathscr{V}_R(I)$ depends on $n-1$ functions of one variable.

Theorem 3 provides us with a set of sufficient (though not necessary) conditions under which we can conclude the existence of compactly supported variations of $\gamma \in \mathscr{V}(I)$. However, to apply the theorem, we need to establish the regularity of $\gamma \in \mathscr{V}(I)$. We now proceed to provide a regularity test for any integral curve of I.

We begin by considering the subspace

$$\mathscr{S}_H = \text{ linear span}\{H(t)\xi \quad \forall\, t \in (a, b),\ \ \xi \in \mathbb{R}^n\} \subset \mathbb{R}^p,$$

where $H(t)$ is a $(p \times n)$-matrix of functions on (a, b). Geometrically, if one regards the columns of the H-matrix as curves in $\mathbb{R}^p$, the subspace $\mathscr{S}_H$ measures the extent to which these curves "fill out" $\mathbb{R}^p$. This leads us to the following.

Definition. The $p \times n$ H-matrix is said to be k-linearly full if $\dim \mathscr{S}_H = k$. When H is p-linearly full, it is said to be linearly full in $\mathbb{R}^p$.

Note that H is k-linearly full if and only if there exist $p-k$ linearly independent vectors $\lambda^{(a)} \in (\mathbb{R}^p)^*$ such that

$$\lambda^{(a)} H(t) = 0 \tag{2.13}$$

and any vector $\mu \in (\mathbb{R}^p)^*$ satisfying (2.13) is a linear combination of $\lambda^{(a)}$.

The crucial observation is the following.

Proposition 4. rank $\mathscr{H}_\gamma = \dim \mathscr{S}_H$.

Proof. Suppose that $\dim \mathscr{S}_H < k$. This implies that there exist at least $p-k+1$ linearly independent vectors $\lambda^{(a)} \in (\mathbb{R}^p)^*$ such that

$$\lambda^{(a)} H(t) = 0 \quad \forall\, t \in (a, b).$$

Hence rank $\mathscr{H}_\gamma < k$.

Conversely, suppose that $\dim \mathscr{S}_H = k$. Then we can find a set of k column vectors

$$H_\mu = H(t_\mu)\xi_\mu, \qquad \xi_\mu \in (\mathbb{R}^n)^*,$$

where $t_\mu \in (a, b)$ are k distinct t-values such that $\{H_\mu\}$ are linearly independent vectors in $\mathbb{R}^p$. Now, for each μ, introduce a "bump function" $\phi_\mu(t)$ centered at t_μ and set $w_\mu(t) = \xi_\mu \phi_\mu(t)$. Taking the limit as $\phi_\mu(t) \to \delta_\mu(t)$, where $\delta_\mu(t)$ denotes the "delta function" centered at t_μ, we have that $\mathscr{H}_\gamma(w_\mu) \to H_\mu$. Hence we conclude that $\mathscr{H}_\gamma$ has rank k. q.e.d.

As an immediate consequence of Proposition 4, we have

Corollary 5. *An integral curve $\gamma \in \mathscr{V}(I)$ of a rank p differential system I is regular if and only if, with respect to a parallel basis, its H-matrix is linearly full in $\mathbb{R}^p$.*

Although the above corollary gives necessary and sufficient conditions under which we can establish the regularity of integral curves of I, a direct application requires the generally impossible task of explicitly integrating the system of ordinary differential equations (2.7).

We now present an effective computational test for regularity.

Theorem 6. *An integral curve $\gamma \in \mathscr{V}(I)$ is nonregular if and only if there exists a lifting to $Z_0^* = Z_0 \backslash \{0\}$ to be an integral curve of the differential system*

$$\widehat{I}_0 = \{v \lrcorner d\zeta_0 \mid \forall\, v \in C^\infty(TZ_0^*)\},$$

where $Z_0 = I \subset T^*M$, viewed as a submanifold of T^*M, and ζ_0 is the restriction of the canonical 1-form on T^*M to Z_0.

Proof. Let U be an open subset of M and let $\gamma : (a, b) \to U$ be an integral curve of the differential system I. Choose a coframing $\{\theta^\alpha, \eta^i\}$ of U so that I is generated by $\{\theta^\alpha\}$ on U with structure equations given by

$$d\theta^\alpha = -\phi^\alpha_\beta \wedge \theta^\beta + \tfrac{1}{2} c^\alpha_{ij} \eta^i \wedge \eta^j.$$

Locally Z_0^* can be identified with

$$Z_0^*|_U \cong U \times (\mathbb{R}^p)^*$$

for an open set $U \subset M$. This is given by identifying the pair $(x, \lambda) \in U \times (\mathbb{R}^p)^*$ with the 1-form $\lambda_\alpha \theta^\alpha_x \in T^*_x M \setminus \{0\}$. Under this identification, the canonical 1-form ζ_0 on Z_0^* takes the form $\zeta_0 = \lambda_\alpha \theta^\alpha$. Differentiating, we obtain

$$d\zeta_0 = (d\lambda_\alpha - \lambda_\beta \phi^\beta_\alpha) \wedge \theta^\alpha + \tfrac{1}{2} \lambda_\alpha c^\alpha_{ij} \eta^i \wedge \eta^j.$$

Taking $(\eta^i, \theta^\alpha, d\lambda_\alpha)$ as a coframing on Z_0^*, the system $\widehat{I}_0$ is generated by

$$\widehat{I}_0 = \begin{cases} \dfrac{\partial}{\partial \lambda_\alpha} \lrcorner d\zeta_0 = \theta^\alpha, \\ \dfrac{\partial}{\partial \theta^\alpha} \lrcorner d\zeta_0 = d\lambda_\alpha - \lambda_\beta \phi_\alpha^\beta, \\ \dfrac{\partial}{\partial \eta^k} \lrcorner d\zeta_0 = \lambda_\alpha c_{ik}^\alpha \eta^i. \end{cases}$$

Thus, integral curves of $\widehat{I}_0$, in addition to being an integral of I, satisfy

$$\dot{\lambda}_\alpha = \lambda_\beta f_\alpha^\beta, \tag{2.14a}$$

$$\lambda_\alpha c_k^\alpha = 0. \tag{2.14b}$$

Now, let γ be a nonregular integral curve of I. By Corollary 5, there exists a constant vector $(\mu_\alpha) \in (\mathbb{R}^p)^*$ such that

$$\mu_\alpha h_\beta^\alpha c_k^\beta = 0,$$

where $\dot{h}_\beta^\alpha = h_\gamma^\alpha f_\beta^\gamma$. Thus γ lifts to be an integral curve of $\widehat{I}_0$ since $\lambda_\alpha(t) = \mu_\beta h_\alpha^\beta(t)$ satisfies (2.14).

Conversely, let $\hat{\gamma} \in \mathscr{V}(\widehat{I}_0)$ and let γ be its projection to M. To establish the nonregularity of γ, it suffices to show that every solution of (2.14a) is of the form

$$\lambda_\alpha(t) = \mu_\beta h_\alpha^\beta(t),$$

where $\mu_\alpha = \lambda_\alpha(0)$ and $\dot{h}_\gamma^\alpha = h_\beta^\alpha f_\gamma^\beta$ with $h_\beta^\alpha(0) = \delta_\beta^\alpha$.

To proof this last assertion, we let $\lambda_\alpha(t)$ be any solution of (2.14a) and let

$$\phi_\alpha(t) = \lambda_\alpha(t) - \mu_\beta h_\alpha^\beta(t).$$

Computing, we obtain

$$\dot{\phi}_\alpha(t) = \phi_\beta(t) f_\alpha^\beta(t)$$

with $\phi_\alpha(0) = \lambda_\alpha(0) - \mu_\alpha = 0$. Thus by uniqueness, $\phi_\alpha(t) \equiv 0$. q.e.d.

A direct consequence of (2.14) is the following.

Corollary 7. *The generic integral curve of a bracket generating differential system is regular.*

However, it is not true that every integral curve of a bracket generating differential system admits compactly supported variations as we now illustrate.

The example we shall consider is a rank 3 system in $\mathbb{R}^5$, with coordinates (x, y, z, p, q), generated by

$$I = \begin{cases} \theta^1 = dy - p\,dx, \\ \theta^2 = dp - q\,dx, \\ \theta^3 = dz - q^2 dx. \end{cases} \tag{2.15}$$

This system arises from the differential equation

$$\frac{dz}{dx} = \left(\frac{d^2y}{dx^2}\right)^2$$

studied by Hilbert [19] in his investigations into the foundations of the calculus of variations. The Pfaffian system I was also studied by Cartan ([9], [10]) who showed that it is invariant under the exceptional simple Lie group G_2.

To obtain a coframing of $\mathbb{R}^5$, we augment $\{\theta^\alpha\}$ with the one-forms $\eta^0 = dx$ and $\eta^1 = dq$. The structure equations of I take the form

$$d\begin{bmatrix}\theta^1\\ \theta^2\\ \theta^3\end{bmatrix} = -\begin{bmatrix}0 & -\eta^0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix} \wedge \begin{bmatrix}\theta^1\\ \theta^2\\ \theta^3\end{bmatrix} + \begin{bmatrix}0\\ \eta^0\wedge\eta^1\\ 2q\eta^0\wedge\eta^1\end{bmatrix}.$$

We can now apply Theorem 6 to deduce that $\gamma \in \mathscr{V}(I, dx)$ is a regular integral curve if and only if $q'' \neq 0$.

If $q'' = 0$, we have that

$$y(x) = a_0 + a_1 x + \tfrac{1}{2}a_2 x^2 + \tfrac{1}{6}a_3 x^3, \tag{2.16a}$$

and hence

$$z(x) = b_0 + (a_2)^2 x + a_2 a_3 x^2 + \tfrac{1}{3}(a_3)^2 x^3, \tag{2.16b}$$

for constants $b_0, a_i \in \mathbb{R}$. This gives a 5-parameter family of nonregular integral curves of I which we shall henceforth refer to as $\mathscr{R} \subset \mathscr{V}(I, dx)$.

Regarding the set $\mathscr{R}$, we have the following result due to Bryant [5].

Proposition 8. *Every integral curve $\gamma \in \mathscr{R}$ is rigid in the C^∞-topology on $\mathscr{V}(I)$.*

Proof. Along an integral curve $\gamma \in \mathscr{V}(I, dx)$, we have

$$y(x) = f(x), \qquad p(x) = f'(x), \qquad q(x) = f''(x),$$
$$z(x) = z(0) + \int_0^x (f''(\sigma))^2\, d\sigma$$

for $f \in C^\infty((a, b), \mathbb{R})$. Hence, a compact variation of $\gamma : (a, b) \to \mathbb{R}^5$ must have the form

$$\begin{aligned} y(x, s) &= f(x) + s\psi(x) + s^2\phi(x, s), \\ p(x, s) &= f'(x) + s\psi'(x) + s^2\phi_x(x, s), \\ q(x, s) &= f''(x) + s\psi''(x) + s^2\phi_{xx}(x, s), \\ z(x, s) &= \zeta(s) + \int_0^x (f''(\sigma) + s\psi''(\sigma) + s^2\phi_{\sigma\sigma}(\sigma, s))^2\, d\sigma, \end{aligned}$$

where $\psi(x) = \phi(x, s) \equiv 0$ for $x \in (a, b) \setminus [\alpha, \beta]$ and $\zeta(0) = z(0)$. Clearly $y(x, s), p(x, s)$, and $q(x, s)$ are compact variations of $y(x)$, $p(x)$, and $q(x)$ respectively. Furthermore, if we demand that $z(x, s)$ be a compact variation of $z(x)$, that is,

$$z(x, s) = z(x) \quad \forall\, x \in (a, b) \setminus [\alpha, \beta],$$

then

$$\int_\alpha^\beta (f''(\sigma) + s\psi''(\sigma) + s^2\phi_{\sigma\sigma}(\sigma, s))^2\, d\sigma = \int_\alpha^\beta (f''(\sigma))^2\, d\sigma.$$

This in turn implies that

$$\int_\alpha^\beta f''(\sigma)\psi''(\sigma)\, d\sigma = 0,$$

and that

$$\int_\alpha^\beta (\psi''(\sigma))^2 + 2f''(\sigma)\phi_{\sigma\sigma}(\sigma, 0)\, d\sigma = 0.$$

Integrating by parts, we obtain

$$\int_\alpha^\beta (\psi''(\sigma))^2 + 2f^{(4)}(\sigma)\phi(\sigma, 0)\, d\sigma = 0.$$

However, by assumption $q''(\sigma) = f^{(4)}(\sigma) = 0$ and hence $\psi''(\sigma) = 0$, which in turn implies that $\psi(\sigma) = 0$. Thus, there are no compactly supported variations of γ—that is, γ is rigid.

Remark. The above proof only establishes that $\gamma \in \mathscr{R}$ is infinitesimally rigid in $\mathscr{V}(I)$. Robert Bryant and the author have shown that $\gamma \in \mathscr{R}$ is indeed rigid in the C^∞-topology. In fact, it can be shown that the generic rank 3 system on M^5 studied by Cartan in [9] always admits a 5-parameter family of rigid integral curves!

Finally, we conclude this section by showing that the notion of regularity of an integral curve $\gamma \in \mathscr{V}(I)$ is "geometric" in the sense that it is independent of the choice of splitting

$$T_\gamma \cong I_\gamma^\perp \oplus s(I_\gamma^*).$$

Indeed, we shall show that

Proposition 9. *The rank of the holonomy map* $\mathscr{H}_\gamma : C^\infty_\circ(I^\perp_\gamma) \to I^*_\gamma|_{t=b}$ *is well defined, independent of the choice of splitting of* T_γ.

Proof. A splitting of T_γ is simply a choice of basis $\{e_\alpha\}$ of I^*_γ, or equivalently, a choice of basis $\{\eta^i\}$ of T^*M/I. Hence, a change of splitting is simply another choice

$$\tilde{\eta}^i = \eta^i + S^i_\alpha \theta^\alpha \quad \Leftrightarrow \quad \eta^i = \tilde{\eta}^i - S^i_\alpha \theta^\alpha, \tag{2.17}$$

where $\{\theta^\alpha\}$ is a basis of sections of I and S^i_α are smooth functions on M. From (2.4), we have the structure equations

$$d\theta^\alpha = -\phi^\alpha_\beta \wedge \theta^\beta + \tfrac{1}{2} c^\alpha_{jk} \eta^j \wedge \eta^k,$$

where $c^\alpha_{jk} + c^\alpha_{kj} = 0$. Without loss of generality, we can choose $\{\theta^\alpha\}$ to be a parallel basis, that is, $\gamma^*(\phi^\alpha_\beta) = 0$. In this case, the holonomy map is

$$\mathscr{H}_\gamma(w) = \int_a^b c(\tau) w(\tau)\, d\tau,$$

where $w \in C^\infty((a, b), \mathbb{R}^p)$.

After a change of splitting (2.17), the corresponding holonomy map takes the form

$$\widetilde{\mathscr{H}}_\gamma(w) = \int_a^b h(\tau) c(\tau) w(\tau)\, d\tau,$$

where $h(t)$ satisfies the equation

$$\dot{h}(t) = h(t) c(t) S(t). \tag{2.18}$$

We now proceed to show that rank $\mathscr{H}_\gamma$ = rank $\widetilde{\mathscr{H}}_\gamma$. By Proposition 4, this is equivalent to showing that $c(t)$ is k-linearly full if and only if $h(t)c(t)$ is k-linearly full for $h(t)$ satisfying (2.18). This follows from the observation that for each $\lambda \in (\mathbb{R}^p)^*$ satisfying $\lambda c(t) = 0$, $\forall\, t \in (a, b)$, we can construct a corresponding constant vector $\mu = \lambda h^{-1}(t) \in (\mathbb{R}^p)^*$ satisfying

$$\mu h(t) c(t) = 0 \quad \forall\, t \in (a, b).$$

3. The main result

The results of the previous section provide us with a good description of the space of variational vector fields of an integral curve γ. We now apply these results to partially solve the Griffiths Problem.

Theorem 10. *Let (M, I, ϕ) be a smooth variational problem and let $(Z, \{0\}, \zeta)$ be the associated variational problem. Then every extremal of (M, I, ϕ) which is a regular integral of I has a unique lifting to be an extremal of $(Z, \{0\}, \zeta)$.*

Proof. Let U be an open subset of M and let $\gamma : (a, b) \to U$ be a regular integral curve of the differential system I. Choose a coframing $\{\theta^\alpha, \eta^i\}$ of U so that I is generated by $\{\theta^\alpha\}$ on U with structure equations given by

$$d\theta^\alpha = -\phi^\alpha_\beta \wedge \theta^\beta + \tfrac{1}{2}c^\alpha_{ij}\eta^i \wedge \eta^j,$$

where $c^\alpha_{ij} = c^\alpha_{[ij]}$ are functions on U, skew-symmetric in $[ij]$, and ϕ^α_β are one-forms on U which vanish along the integral curve γ,

$$\gamma^*(\phi^\alpha_\beta) = 0. \tag{3.1}$$

The existence of such a coframing was established in §2. There, it was shown that such a coframing induced a splitting $T_\gamma \cong I^\perp_\gamma \oplus s(I^*_\gamma)$ along γ whereby $I^\perp_\gamma$ and I^*_γ are spanned by $\{e_i\}$ and $\{e_\alpha\}$, dual frame fields to $\{\eta^i\}$ and $\{\theta^\alpha\}$. With respect to this splitting, every section $v \in C^\infty(T_\gamma)$ can be expressed as

$$v = e_i w^i + s(e_\alpha)u^\alpha$$

for some smooth functions w^i and u^α on γ. If in addition $v \in \ker \mathscr{D}_\gamma$, then the corresponding variational equations take the form

$$\dot{u}^\alpha = c^\alpha_i w^i. \tag{3.2}$$

Now, since γ is a regular integral curve, it follows from Theorem 2 that $v \in \ker_\circ \mathscr{D}_\gamma$ is a compactly supported variational vector field of γ. Hence, by (1.5), the extremal condition can be expressed as

$$\int_a^b v \lrcorner d\phi = 0 \quad \forall\, v \in \ker_\circ \mathscr{D}_\gamma. \tag{3.3}$$

Here ϕ is the Lagrangian one-form on U associated with the variational problem, and so

$$d\phi \equiv A_{i\alpha}\eta^i \wedge \theta^\alpha + \tfrac{1}{2}B_{ij}\eta^i \wedge \eta^j \mod \{\theta^\alpha \wedge \theta^\beta\} \tag{3.4}$$

for some smooth functions $A_{i\alpha}$ and $B_{[ij]}$ on U. (3.3) now evaluates to give

$$\int_a^b (A_{i\alpha}u^\alpha + B_{ik}w^k)n^i\, dt = 0.$$

Integrating by parts, and taking into account the variational equations (3.2), we get

$$\int_a^b \Lambda_k(t) w^k(t)\, dt = 0, \tag{3.5}$$

where $\Lambda_k = \partial_t^{-1}[A_{i\alpha} n^i] c_k^\alpha - B_{ik} n^i$ and ∂_t^{-1} is the antiderivative operator defined by

$$\partial_t^{-1}[\sigma] = \int_a^t \sigma(\tau)\, d\tau.$$

Note that (3.5) contains no boundary terms because $v \in \ker_\circ \mathscr{D}_\gamma$.

Now, from the form of (3.2), the corresponding holonomy condition $\widetilde{\mathscr{H}}_\gamma(w) = 0$ is given by

$$\int_a^b c_k^\alpha(t) w^k(t)\, dt = 0. \tag{3.6}$$

Thus by Lemma C.1, we have that the extremals of the variational problem (M, I, ϕ) satisfy the equations

$$\partial_t^{-1}[A_{i\alpha} n^i] c_k^\alpha - B_{ik} n^i = k_\alpha c_k^\alpha \tag{3.7}$$

for some constants k_α. These are the *Euler-Lagrange* equations of the variational problem (M, I, ϕ).

To complete the proof, it suffices to show that every integral of (3.7) lifts to be an integral curve of the Euler-Lagrange system on Z.

On $Z \cong M \times \mathbb{R}^p$, we have the canonical one-form $\zeta = \phi + \lambda_\alpha \theta^\alpha$. Differentiating, we obtain

$$d\zeta \equiv (d\lambda_\alpha + A_{i\alpha}\eta^i - \lambda_\beta \phi_\alpha^\beta) \wedge \theta^\alpha + \tfrac{1}{2}(\lambda_\alpha c_{ij}^\alpha + B_{ij})\eta^i \wedge \eta^j \mod \{\theta^\alpha \wedge \theta^\beta\}.$$

Taking $(\eta^i, \theta^\alpha, d\lambda_\alpha)$ as a coframing on Z, the Euler-Lagrange system on Z is generated by

$$\tilde{I} = \begin{cases} \dfrac{\partial}{\partial \lambda_\alpha} \lrcorner\, d\zeta = \theta^\alpha, \\ \dfrac{\partial}{\partial \theta^\alpha} \lrcorner\, d\zeta = d\lambda_\alpha + A_{i\alpha}\eta^i - \lambda_\beta \phi_\alpha^\beta, \\ \dfrac{\partial}{\partial \eta^k} \lrcorner\, d\zeta = (B_{ik} + \lambda_\alpha c_{ik}^\alpha)\eta^i. \end{cases} \tag{3.8}$$

To construct a lifting of an integral curve γ of the Euler-Lagrange equations (3.7) we merely let

$$\lambda_\alpha = k_\alpha - \partial_t^{-1}[A_{i\alpha} n^i]. \tag{3.9}$$

If $\tilde{\gamma}$ is the lifted curve, then $\tilde{\gamma}$ is an integral of θ^α by construction. Similarly, in view of (3.1) and (3.9) we have that

$$\tilde{\gamma}^*(d\lambda_\alpha + A_{i\alpha}\eta^i - \lambda_\beta\phi_\alpha^\beta) = 0,$$

while

$$\tilde{\gamma}^*[(B_{ik} + \lambda_\alpha c_{ik}^\alpha)\eta^i] = 0$$

follows from equation (3.7) and relation (3.9).

Finally, if $\tilde{\gamma}$ and $\hat{\gamma}$ are two different liftings of γ on M to Z, then the differences of their Euler-Lagrange equations on M give

$$(\tilde{k}_\alpha - \hat{k}_\alpha)c_k^\alpha = 0.$$

However, since γ is regular, (c_k^α) is linearly full in $\mathbb{R}^p$ and so $\tilde{k}_\alpha - \hat{k}_\alpha = 0$. Hence the lifting is unique.

Remarks. 1. The above theorem establishes the validity of the Griffiths formalism for variational problems given by functionals whose domain of definition consists of *regular* integrals of an exterior differential system I. However, integral curves of a differential system need not always be regular. In fact, it may happen that the system I admits no regular curves, which happens for differential systems with a nontrivial derived system. In such cases, however, we may still be able to apply the Griffiths formalism to study the associated variational problem by restricting to the leaves of the foliation generated by the derived system of I.

For a general differential system, the geometry (and topology) of the space of solutions $\mathscr{V}(I)$ of I could be very complicated. For instance, it could happen that some subsets of $\mathscr{V}(I)$, isolated or otherwise, fail to be regular, as is the case for the 5-parameter family $\mathscr{R}$ of rigid integral curves of the Hilbert system (2.15) studied in the previous section. For such nonregular curves, the Griffiths criterion is not universally valid—that is, not all nonregular extremals are obtainable via the Griffiths formalism. This follows from the fact that the exceptional integral curves $\gamma \in \mathscr{R}$, being rigid, are trivially stationary points of any variational functional $\Phi : \mathscr{V}(I) \to \mathbb{R}$ whose domain of definition consists of integral curves of the Hilbert system. However, an easy computation shows that these rigid curves are generally not solutions of the associated Euler-Lagrange system.

The existence of these nonregular extremals provide counterexamples to the often stated assertion in sub-Riemannian geometry that "every minimizing curve satisfies the geodesic equations" (see, for instance, [22]).

2. As the readers are no doubt aware, closely related to the Griffiths formalism is the classical Lagrange multiplier rule for finding extremals of variational problems. In [3], Bliss showed that for the classical Lagrange

problem on $J^1(\mathbb{R}, \mathbb{R}^n)$, viewed as a fiber bundle over $\mathbb{R} \times \mathbb{R}^n$, with admissible variations being curves in $\mathbb{R} \times \mathbb{R}^n$ joining two fixed points, every "normal" extremal of the variational problem is obtainable via the Lagrange multiplier rule.

For this classical case, it is not hard to establish that the condition of "normality" of an integral curve coincides with the notion of "regularity" introduced in §2. Thus, as an immediate consequence of Theorem 10, we have the following strengthening of the classical result.

Corollary 11. *Every regular extremal of the classical Lagrange problem is an integral curve of the associated Euler-Lagrange system.*

To conclude this section, we like to emphasize that irrespective of whether Griffiths' criterion is universally valid, from the discussions given in §2, we have that a "generic" integral curve of a Pfaffian differential system (those with a trivial derived system) will be regular. Hence, by restricting our consideration to the set of regular curves, we can study the associated variational problem via the Griffiths formalism.

4. Examples

In this section, we shall investigate a number of geometric variational problems. These can be described in a unified manner as seeking to minimize the arclength functional

$$\Phi(\gamma) = \int_\gamma ds \tag{4.1}$$

subject to some differential geometric constraints.

The first is a generalized *Delaunay problem* which asks for the shortest curve γ in Σ^3, a three-dimensional space form, satisfying the nonholonomic constraint

$$\kappa = \kappa_0 \qquad \text{or} \qquad \tau = \tau_0, \tag{4.2a}$$

where κ and τ denote the curvature and torsion of γ respectively. This problem was much studied classically (see, for instance, Carathéodory [8, p. 373]) for a detailed discussion of curves in Euclidean 3-space.

The second and third variational problems we shall consider are the isoperimetric problems of Pappus and Poincaré respectively. These consist of finding the curve γ of shortest length among all smooth closed curves bounding a connected region $\Omega \subset \Sigma^2$ of a Riemannian surface, satisfying the integral constraint

$$\int_\Omega dA = A_0 \tag{4.2b}$$

in the case of the Pappus problem, and

$$\int_{\Omega} K\,dA = K_0 \tag{4.2c}$$

in the case of the Poincaré problem. Here dA denotes the area form of the surface Σ while K is the corresponding curvature.

1. The Delaunay problem. We approach the problem via the method of moving frames. Hence let $(x; e_i)$, $x \in \Sigma^3$, be an orthonormal frame of Σ^3, a three-dimensional space form with constant sectional curvature c. Denote by $\mathscr{F}(\Sigma^3)$ the bundle of orthonormal frames of Σ^3. On $\mathscr{F}(\Sigma^3)$ we have the equations

$$d(x, e_1, e_2, e_3) = (x, e_1, e_2, e_3)\begin{bmatrix} 0 & -c\omega^1 & -c\omega^2 & -c\omega^3 \\ \omega^1 & 0 & \omega_2^1 & \omega_3^1 \\ \omega^2 & \omega_1^2 & 0 & \omega_3^2 \\ \omega^3 & \omega_1^3 & \omega_2^3 & 0 \end{bmatrix}, \tag{4.3a}$$

where $\{\omega^i, \omega_j^i\}$ is a coframing of $\mathscr{F}(\Sigma^3)$ with $\omega_j^i + \omega_i^j = 0$. Here $\{\omega^i\}$ is an orthonormal coframe dual to $\{e_i\}$ while $\{\omega_j^i\}$ are the connection forms on $\mathscr{F}(\Sigma^3)$.

Taking exterior derivatives of (4.3a), it follows that the structure equations of Cartan are given by

$$d\omega^i = -\omega_j^i \wedge \omega^j, \qquad d\omega_j^i = -\omega_k^i \wedge \omega_j^k + c\omega^i \wedge \omega^j. \tag{4.3b}$$

By scaling, the value of the constant c can be taken to be $1, 0$, or -1 depending on whether the space form Σ^3 is the 3-sphere $\mathbb{S}^3$, euclidean 3-space $\mathbb{E}^3$, or hyperbolic 3-space $\mathbb{H}^3$.

Now, let $x : \mathbb{R} \to \Sigma^3$ be an immersed curve and let $\gamma : \mathbb{R} \to \mathscr{F}(\Sigma^3)$ be the framing of $x(t)$ given by

$$\gamma(t) = (x(t), e_1(t), e_2(t), e_3(t)).$$

We call such a framed curve a *Frénet curve* if, in addition, it is an integral curve of the differential system generated by $\{\omega^2, \omega^3, \omega_1^3\}$. Hence, for a Frenet curve γ, we have that

$$\begin{aligned} &d(x(t), e_1(t), e_2(t), e_3(t)) \\ &\quad = (x(t), e_1(t), e_2(t), e_3(t))\begin{bmatrix} 0 & -c\gamma^*\omega^1 & 0 & 0 \\ \gamma^*\omega^1 & 0 & \gamma^*\omega_2^1 & 0 \\ 0 & \gamma^*\omega_1^2 & 0 & \gamma^*\omega_3^2 \\ 0 & 0 & \gamma^*\omega_2^3 & 0 \end{bmatrix}. \end{aligned}$$

Geometrically, this corresponds to the property that $e_1(t)$, $e_2(t)$, and $e_3(t)$ are respectively the unit tangent, normal, and binormal vectors forming an oriented orthonormal frame of $T_{x(t)}\Sigma^3$. Hence $(e_1(t), e_2(t), e_3(t))$ is the classical Frénet frame on the space form Σ^3.

In what follows, we parametrize γ by arclength s. For Frénet curves, we define the curvature $\kappa(s)$ and torsion $\tau(s)$ by

$$\gamma^*\omega_1^2 = \kappa(s)ds, \qquad \gamma^*\omega_2^3 = \tau(s)ds.$$

◇ We first consider curves $\gamma \subset \Sigma^3$ of *constant curvature* κ_0. Following Griffiths [17], we set up the differential system I on $\mathscr{F}(\Sigma^3) \times \mathbb{R}$, where $\mathbb{R}$ has coordinate τ. The generators of I are

$$\begin{gathered}\theta^1 = \omega^2, \qquad \theta^2 = \omega^3, \qquad \theta^3 = \omega_1^3, \\ \theta^4 = \omega_1^2 - \kappa_0\omega^1, \quad \theta^5 = \omega_2^3 - \tau\omega^1,\end{gathered}$$

for some constant κ_0. To complete the coframing on $\mathscr{F}(\Sigma^3) \times \mathbb{R}$, we add the one-forms

$$\eta^0 = \omega^1, \qquad \eta^1 = d\tau - \kappa_0\tau\theta^1 - \kappa_0\theta^3.$$

From (4.3), the structure equations of I can be shown to take the form

$$d\begin{bmatrix}\theta^1\\ \theta^2\\ \theta^3\\ \theta^4\\ \theta^5\end{bmatrix} \equiv -\begin{bmatrix} 0 & -\tau\eta^0 & 0 & -\eta^0 & 0 \\ \tau\eta^0 & 0 & -\eta^0 & 0 & 0 \\ 0 & c\eta^0 & 0 & \tau\eta^0 & -\kappa_0\eta^0 \\ (c+\kappa_0^2)\eta^0 & 0 & -\tau\eta^0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \wedge \begin{bmatrix}\theta^1\\ \theta^2\\ \theta^3\\ \theta^4\\ \theta^5\end{bmatrix} + \begin{bmatrix} 0\\ 0\\ 0\\ 0\\ \eta^0\wedge\eta^1\end{bmatrix},$$

where $\equiv$ denotes congruence modulo $\{\theta^\alpha \wedge \theta^\beta\}$.

Applying Theorem 6, one can easily verify that $\gamma \in \mathscr{V}(I, ds)$ is a regular integral of I if and only if

$$-2\tau\ddot{\tau} + 3(\dot{\tau})^2 + 4\tau^2(c + \kappa_0^2 - \tau^2) \neq 0.$$

For these regular integrals, we can apply the Griffiths formalism to study the associated variational problem.

The Lagrangian one-form is $\phi = \eta^0$ and hence, on $Z \cong [\mathscr{F}(\Sigma^3) \times \mathbb{R}] \times \mathbb{R}^5$, the canonical one-form takes the form

$$\zeta = \eta^0 + \lambda_\alpha\theta^\alpha.$$

We take $(\eta^i, \theta^\alpha, d\lambda_\alpha)$ as a coframe on Z and compute the Cartan system $\mathscr{C}(d\zeta)$ by contracting $d\zeta$ with $\{\partial/\partial\lambda_\alpha, \partial/\partial\theta^\alpha, \partial/\partial\eta^i\}$ respectively. Thus the Euler-Lagrange system on Z is generated by

$$
\begin{aligned}
&\frac{\partial}{\partial\lambda_\alpha} \lrcorner\, d\zeta = \theta^\alpha = 0, \\
&\frac{\partial}{\partial\theta^1} \lrcorner\, d\zeta = -d\lambda_1 - [\kappa_0 - \lambda_2\tau - \lambda_4(c+\kappa_0^2)]\eta^0 = 0, \\
&\frac{\partial}{\partial\theta^2} \lrcorner\, d\zeta = -d\lambda_2 - (\lambda_1\tau - \lambda_3 c)\eta^0 = 0, \\
&\frac{\partial}{\partial\theta^3} \lrcorner\, d\zeta = -d\lambda_3 - (\lambda_2 + \lambda_4\tau)\eta^0 = 0, \\
&\frac{\partial}{\partial\theta^4} \lrcorner\, d\zeta = -d\lambda_4 - (\lambda_1 - \lambda_3\tau)\eta^0 = 0, \\
&\frac{\partial}{\partial\theta^5} \lrcorner\, d\zeta = -d\lambda_5 - \lambda_3\kappa_0\eta^0 = 0, \\
&\frac{\partial}{\partial\eta^1} \lrcorner\, d\zeta = -\lambda_5\eta^0 = 0.
\end{aligned} \tag{4.4}
$$

Subject to the independence condition $\gamma^*(\eta^0) \neq 0$, these equations give

$$\lambda_5 = 0, \qquad \lambda_3\kappa_0 = 0. \tag{4.5}$$

Since we are considering regular curves, $\kappa_0 \neq 0$, we get $\lambda_3 = 0$, and hence

$$\lambda_2 + \lambda_4\tau = 0. \tag{4.6}$$

The remaining equations now take the form

$$
\begin{aligned}
&\text{(a)} \quad \dot\lambda_1 = -[\kappa_0 - \lambda_2\tau - \lambda_4(c+\kappa_0^2)], \\
&\text{(b)} \quad \dot\lambda_2 = -\lambda_1\tau, \\
&\text{(c)} \quad \dot\lambda_4 = -\lambda_1.
\end{aligned} \tag{4.7}
$$

(4.6) together with (b), (c) in (4.7) gives

$$2\frac{d\lambda_4}{\lambda_4} + \frac{d\tau}{\tau} = 0.$$

This yields the first integral

$$\lambda_4^2\tau = c_1 \in \mathbb{R}^* \tag{4.8}$$

on solutions to the Euler-Lagrange system. We note that the curve $\gamma \subset \Sigma^3$ is uniquely determined, up to rigid motions, by knowing the constant c_1

and the function $\lambda_4(s)$—this is because we assume κ_0 is a known constant and $\tau(s) = c_1/\lambda_4^2(s)$.

The remaining equations give

$$\ddot{\lambda}_4 = [\kappa_0 - (c+\kappa_0^2)\lambda_4 + c_1^2\lambda_4^{-3}].$$

This equation has a first integral given by

$$\dot{\lambda}_4^2 - [2\kappa_0\lambda_4 - (c+\kappa_0^2)\lambda_4^2 - c_1^2\lambda_4^{-2}] = c_2, \tag{4.9}$$

for some constant c_2. Hence, the phase portrait of the solution curves to the Euler-Lagrange system associated with the functional (4.1) with constant κ_0 is given in the $(\lambda_4, \dot{\lambda}_4)$-plane by a three-parameter family of algebraic curves. For general values of c_1, c_2, and κ_0, these are elliptic curves.

◇ Let us now consider curves $\gamma \subset \Sigma^3$ of *constant torsion* τ_0. As before, we set up the differential system I on $\mathscr{F}(\Sigma^3) \times \mathbb{R}$ with generators

$$\theta^1 = \omega^2, \qquad \theta^2 = \omega^3, \qquad \theta^3 = \omega_1^3,$$
$$\theta^4 = \omega_1^2 - \kappa\omega^1, \quad \theta^5 = \omega_2^3 - \tau_0\omega^1,$$

where τ_0 is now a fixed constant. Furthermore, to complete the coframing on $\mathscr{F}(\Sigma^3) \times \mathbb{R}$, we add the one-forms

$$\eta^0 = \omega^1, \qquad \eta^1 = d\kappa - (c+\kappa^2)\theta^1 + \tau_0\theta^3.$$

A short calculation shows that the structure equations of I can be written as

$$d\begin{bmatrix}\theta^1\\ \theta^2\\ \theta^3\\ \theta^4\\ \theta^5\end{bmatrix} \equiv -\begin{bmatrix} 0 & -\tau_0\eta^0 & 0 & -\eta^0 & 0\\ \tau_0\eta^0 & 0 & -\eta^0 & 0 & 0\\ 0 & c\eta^0 & 0 & \tau_0\eta^0 & -\kappa\eta^0\\ 0 & 0 & 0 & 0 & 0\\ \kappa\tau_0\eta^0 & 0 & \kappa\eta^0 & 0 & 0\end{bmatrix} \wedge \begin{bmatrix}\theta^1\\ \theta^2\\ \theta^3\\ \theta^4\\ \theta^5\end{bmatrix} + \begin{bmatrix}0\\ 0\\ 0\\ \eta^0\wedge\eta^1\\ 0\end{bmatrix}.$$

Applying Theorem 6, we are led to conclude that $\gamma \in \mathscr{V}(I, ds)$ is regular if and only if $\tau_0(c-\tau_0^2) \neq 0$ and $\kappa \not\equiv 0$. Hence, for these regular curves, we can again apply the Griffiths formalism to study the associated variational problem.

As before, the Lagrangian one-form for the variational problem is $\phi = \eta^0$ and hence, on $Z \cong [\mathscr{F}(\Sigma^3) \times \mathbb{R}] \times \mathbb{R}^5$, the canonical one-form is

$$\zeta = \eta^0 + \lambda_\alpha\theta^\alpha.$$

Taking $(\eta^i, \theta^\alpha, d\lambda_\alpha)$ as a coframe on Z, the Euler-Lagrange system on Z is generated by

$$
(4.10)\qquad
\begin{aligned}
&\frac{\partial}{\partial\lambda_\alpha} \lrcorner d\zeta = \theta^\alpha = 0,\\
&\frac{\partial}{\partial\theta^1} \lrcorner d\zeta = -d\lambda_1 - (\kappa - \lambda_2\tau_0 - \lambda_5\kappa\tau_0)\eta^0 = 0,\\
&\frac{\partial}{\partial\theta^2} \lrcorner d\zeta = -d\lambda_2 - (\lambda_1\tau_0 - \lambda_3 c)\eta^0 = 0,\\
&\frac{\partial}{\partial\theta^3} \lrcorner d\zeta = -d\lambda_3 - (\lambda_2 - \lambda_5\kappa)\eta^0 = 0,\\
&\frac{\partial}{\partial\theta^4} \lrcorner d\zeta = -d\lambda_4 - (\lambda_1 - \lambda_3\tau_0)\eta^0 = 0,\\
&\frac{\partial}{\partial\theta^5} \lrcorner d\zeta = -d\lambda_5 - \lambda_3\kappa\eta^0 = 0,\\
&\frac{\partial}{\partial\eta^1} \lrcorner d\zeta = -\lambda_4\eta^0 = 0.
\end{aligned}
$$

Along solution curves, (4.10) gives

$$(4.11)\qquad \lambda_4 = 0, \qquad \lambda_1 - \lambda_3\tau_0 = 0$$

and hence, combining the second and fourth equations of (4.10), we get

$$(4.12)\qquad \kappa = 2\tau_0\lambda_2.$$

The remaining equations of (4.10) are

$$
(4.13)\qquad
\begin{aligned}
\dot\lambda_2 &= -\lambda_3(\tau_0^2 - c),\\
\dot\lambda_3 &= -\lambda_2(1 - 2\tau_0\lambda_5),\\
\dot\lambda_5 &= -2\tau_0\lambda_2\lambda_3.
\end{aligned}
$$

At this stage, we find it convenient to introduce the following change of variables:

$$\mu_1 = \lambda_2, \quad \mu_2 = 2\tau_0\lambda_3, \quad \mu_3 = 1 - 2\tau_0\lambda_5.$$

In terms of these, system (4.13) becomes

$$
(4.14)\qquad
\begin{aligned}
&\text{(a)}\quad \dot\mu_1 = -(\tau_0^2 - c)\mu_2/2\tau_0,\\
&\text{(b)}\quad \dot\mu_2 = -2\tau_0\mu_1\mu_3,\\
&\text{(c)}\quad \dot\mu_3 = 2\tau_0\mu_1\mu_2.
\end{aligned}
$$

Combining (b) and (c) in (4.14), we obtain the first integral given by

$$(4.15)\qquad \mu_2^2 + \mu_3^2 = c_1^2.$$

Now, letting $\mu_2 = c_1 \cos\theta$ and $\mu_3 = c_1 \sin\theta$, and differentiating, we obtain the following identities:

$$\mu_2[\dot{\theta} - 2\tau_0\mu_1] = 0, \qquad \mu_3[\dot{\theta} - 2\tau_0\mu_1] = 0.$$

Hence, there are two cases to consider:

I: $\mu_2 = \mu_3 = 0$. In this case, we have

$$\lambda_1 = 0, \quad \lambda_2 = \lambda, \quad \lambda_3 = 0, \quad \lambda_4 = 0, \quad \lambda_5 = (2\tau_0)^{-1},$$

where λ is an arbitrary constant. From (4.12) we have that the curvature satisfies $\kappa = 2\lambda\tau_0$. For this class of solutions, κ and τ_0 are both nonzero constants, and hence the extremal curves are helices.

II: $\mu_2 \neq 0$ or $\mu_3 \neq 0$. In this case, we have $\dot{\theta} = 2\tau_0\mu_1$ which, when differentiated, gives

$$\ddot{\theta} = -(\tau_0^2 - c)c_1 \cos\theta.$$

This equation has a first integral given by

$$\dot{\theta}^2 + 2(\tau_0^2 - c)c_1 \sin\theta = c_2. \tag{4.16}$$

For general values of the constants c_1, c_2, and τ_0, the phase portraits in the $(\theta, \dot{\theta})$-plane are transcendental curves.

Remark. The above analyses enable us to conclude that the generalized Delaunay problem is completely integrable in the sense of being able to determine the invariants of the problem up to a single quadrature. However, there is much more to be done before we can deduce the global qualitative behavior of the resulting extremals $\gamma \subset \Sigma^3$. In particular, it is not known whether simple closed extremal curves exist in either of the above two cases.

2. The Pappus problem. Let (Σ, ds^2) be a Riemannian surface and let $\mathscr{F}(\Sigma)$ be the associated orthonormal frame bundle with the usual coframing $\{\omega^1, \omega^2, \rho\}$. The Cartan structure equations take the form

$$d\omega^1 = \rho \wedge \omega^2, \quad d\omega^2 = -\rho \wedge \omega^1, \quad d\rho = -K\omega^1 \wedge \omega^2,$$

where K is the Gauss curvature of the surface Σ.

Let Ω be a connected region in Σ bounded by a smooth closed curve γ. For such a region, it is not too difficult to show that the area constraint (4.2b) is equivalent to the integral constraint

$$\int_\gamma \alpha = A_1, \tag{4.2b$'$}$$

where α is a one-form on $\mathscr{F}(\Sigma)$ satisfying $d\alpha = \omega^1 \wedge \omega^2$ and $A_1 = A_0 + C$ is some prescribed constant. Here A_0 denotes the area of the region Ω while C is a universal constant depending on the homotopy class of γ.

Hence the Pappus problem is equivalent to finding critical values of the arclength functional (4.1) subject to the integral constraint (4.2b$'$).

Now, curves $\gamma \subset \mathscr{F}(\Sigma)$ satisfying the constraint (4.2b$'$) can easily be shown to be integrals of the system I on $\mathscr{F}(\Sigma) \times \mathbb{R}^2$ with generators

$$\theta^1 = \omega^2, \quad \theta^2 = \rho - \kappa\omega^1, \quad \theta^3 = dz - \alpha.$$

To complete the coframing on $\mathscr{F}(\Sigma) \times \mathbb{R}^2$, we add the one-forms

$$\eta^0 = \omega^1, \qquad \eta^1 = d\kappa - (K + \kappa^2)\theta^1.$$

The structure equations of I are given by

$$d\begin{bmatrix}\theta^1\\ \theta^2\\ \theta^3\end{bmatrix} \equiv -\begin{bmatrix}0 & -\eta^0 & 0\\ 0 & 0 & 0\\ \eta^0 & 0 & 0\end{bmatrix} \wedge \begin{bmatrix}\theta^1\\ \theta^2\\ \theta^3\end{bmatrix} + \begin{bmatrix}0\\ \eta^0 \wedge \eta^1\\ 0\end{bmatrix} \quad \text{mod } \{\theta^\alpha \wedge \theta^\beta\}.$$

Applying Theorem 6, it is easy to verify that $\gamma \in \mathscr{V}(I, ds)$ is always regular, and hence we can apply the Griffiths formalism to study the associated variational problem.

As before, on $Z \cong [\mathscr{F}(\Sigma) \times \mathbb{R}^2] \times \mathbb{R}^3$, the canonical one-form is

$$\zeta = \eta^0 + \lambda_\alpha \theta^\alpha.$$

Computing the Cartan system $\mathscr{C}(d\zeta)$, we have that the Euler-Lagrange system on Z is generated by

$$\begin{aligned}
&\frac{\partial}{\partial \lambda_\alpha} \lrcorner\, d\zeta = \theta^\alpha = 0,\\
&\frac{\partial}{\partial \theta^1} \lrcorner\, d\zeta = -d\lambda_1 - (\kappa + \lambda_3)\eta^0 = 0,\\
&\frac{\partial}{\partial \theta^2} \lrcorner\, d\zeta = -d\lambda_2 - \lambda_1\eta^0 = 0,\\
&\frac{\partial}{\partial \theta^3} \lrcorner\, d\zeta = -d\lambda_3 = 0,\\
&\frac{\partial}{\partial \eta^1} \lrcorner\, d\zeta = -\lambda_2\eta^0 = 0.
\end{aligned} \tag{4.17}$$

Subject to the independence condition $\gamma^*(\eta^0) \neq 0$, these equations give

$$\lambda_1 = \lambda_2 = 0 \quad \text{and} \quad \kappa = \lambda_3 = \text{constant}. \tag{4.18}$$

Remark. In the special case of a planar surface, it is known classically that extremals of the Pappus functional are circles. From (4.18), one observes that extremals of the Pappus functional on an arbitrary surface Σ can also be simply characterized as curves of *constant curvature*. However, it has not been established that such extremal curves always exist. An interesting problem is to find differential geometric conditions on Σ under which existence of closed curves of constant geodesic curvature can be established.

Of particular interest is the study of the gradient of the arclength functional (4.1) subject to the area-constraint (4.2b). Using this technique, Grayson [16] was able to establish the existence of a closed geodesic on Riemannian surfaces. Similar results should hold for curves of constant curvature. See Gage [15] for the special case of planar curves.

3. The Poincaré problem. As with the previous problem, let (Σ, ds^2) be a Riemannian surface with structure equations

$$d\omega^1 = \rho \wedge \omega^2, \quad d\omega^2 = -\rho \wedge \omega^1, \quad d\rho = -K\omega^1 \wedge \omega^2.$$

If we let Ω be the simply connected region in Σ bounded by a smooth closed curve γ, then by the Gauss-Bonnet theorem, we have

$$\int_\Omega K\,dA + \int_\gamma \kappa\,ds = 2\pi, \tag{4.19}$$

in which case the integral constraint (4.2c) becomes

$$\int_\gamma \kappa\,ds = K_1 \tag{4.2c$'$}$$

for some constant $K_1 = 2\pi - K_0$.

Now curves $\gamma \subset \mathscr{F}(\Sigma)$ satisfying the constraint (4.2c$'$) are integrals of the system I on $\mathscr{F}(\Sigma) \times \mathbb{R}^2$ with generators

$$\theta^1 = \omega^2, \quad \theta^2 = \rho - \kappa\omega^1, \quad \theta^3 = dz - \kappa\omega^1.$$

We complete the coframing on $\mathscr{F}(\Sigma) \times \mathbb{R}^2$ by adding the one-forms

$$\eta^0 = \omega^1, \qquad \eta^1 = d\kappa - (K + \kappa^2)\theta^1.$$

The structure equations of I are given by

$$d\begin{bmatrix}\theta^1\\ \theta^2\\ \theta^3\end{bmatrix} \equiv -\begin{bmatrix}0 & -\eta^0 & 0\\ 0 & 0 & 0\\ -K\eta^0 & 0 & 0\end{bmatrix} \wedge \begin{bmatrix}\theta^1\\ \theta^2\\ \theta^3\end{bmatrix} + \begin{bmatrix}0\\ \eta^0 \wedge \eta^1\\ \eta^0 \wedge \eta^1\end{bmatrix} \quad \text{mod } \{\theta^\alpha \wedge \theta^\beta\}.$$

Applying Theorem 6, we conclude that $\gamma \in \mathscr{V}(I, ds)$ is regular if and only if $K|_\gamma \neq 0$. Hence, by restricting to nonflat portions of the surface Σ, we can apply the Griffiths formalism to study the associated variational problem.

Proceeding as before, we find that the Euler-Lagrange system on $Z \cong [\mathcal{F}(\Sigma) \times \mathbb{R}^2] \times \mathbb{R}^3$ is generated by

$$
\begin{aligned}
&\frac{\partial}{\partial \lambda_\alpha} \lrcorner\, d\zeta = \theta^\alpha = 0, \\
&\frac{\partial}{\partial \theta^1} \lrcorner\, d\zeta = -d\lambda_1 - (\kappa + \lambda_3 K)\eta^0 = 0, \\
&\frac{\partial}{\partial \theta^2} \lrcorner\, d\zeta = -d\lambda_2 - \lambda_1 \eta^0 = 0, \\
&\frac{\partial}{\partial \theta^3} \lrcorner\, d\zeta = -d\lambda_3 = 0, \\
&\frac{\partial}{\partial \eta^1} \lrcorner\, d\zeta = -(\lambda_2 + \lambda_3)\eta^0 = 0.
\end{aligned} \tag{4.20}
$$

Subject to the independence condition $\gamma^*(\eta^0) \neq 0$, these generate the relations

$$\lambda_1 = 0, \qquad \lambda_2 = -\lambda_3 = \lambda = \text{constant}, \tag{4.21}$$

and

$$\kappa = \lambda K. \tag{4.22}$$

In this general setting, not much is known about the existence of closed extremals for this isoperimetric problem. However, in the special case of closed curves γ on a convex surface Σ that bisect the integral curvature of Σ

$$\int_\Omega K\, dA = 2\pi, \tag{4.23}$$

extremals of the above isoperimetric problem are necessarily geodesics [21]. This result follows from the Gauss-Bonnet formula (4.19), taking into account equation (4.22) and the constraint (4.23), which gives

$$\lambda \int_\gamma K\, ds = 0.$$

Since $K > 0$ by hypothesis, we necessarily have $\lambda = 0$ which then implies from equation (4.22) that $\kappa = 0$. Thus γ is a geodesic on Σ. The converse follows trivially from the Gauss-Bonnet formula (4.19).

Remark. The problem of finding closed geodesics on a convex surface Σ using techniques from the calculus of variations was initiated by Poincaré in [21]. There, it was suggested that solutions could be found by solving the above isoperimetric problem. Recently, this approach of finding closed geodesics has been reexamined by a number of authors.

In particular, Berger and Bombieri [2] and Allard [1], using the so-called direct methods in the calculus of variations, were able to establish the existence of a simple closed geodesic on Σ. We refer the reader to [13], [16], and [20] for more sophisticated methods of finding closed geodesics on Σ.

Appendix A. Calculus of variations and closed extremals

For some variational problems (M, I, ϕ), for instance the isoperimetric problems of Pappus and Poincaré studied in §4, it is natural to restrict the domain of definition of the variational functional

$$\Phi(\gamma) = \int_\gamma \phi$$

to the class of periodic or closed integral curves of the differential system I, which we denote by

$$\mathscr{V}^c(I) = \{\gamma : S^1 \to M \mid \gamma^*(I) = 0\}.$$

As before, we make $\mathscr{V}^c(I)$ into a topological space by endowing it with the C^∞-topology. Here, we are making the a priori assumption that the system I admits closed integral curves. It is not clear to what extent this assumption is justified. To the best of this author's knowledge, there is no existence theorem for closed integral curves of a "general" exterior differential system. However, for the special cases of the contact system and the geometric systems considered in §4, it is known that closed integral curves exist in abundance.

Assuming that $\mathscr{V}^c(I)$ is nonempty, our objective now is to describe the critical points of the functional $\Phi : \mathscr{V}^c(I) \to \mathbb{R}$. We seek conditions on the closed integral curves under which we may apply the Griffiths formalism to study the extremals of the variational problem. As in §2, we are led to consider $T_\gamma\mathscr{V}^c(I)$—the space of variational vector fields to $\gamma \in \mathscr{V}^c(I)$—and the variational equations of γ,

$$\mathscr{D}_\gamma(v) = 0,$$

where $\mathscr{D}_\gamma : C^\infty(T_\gamma) \to \Omega^1(I^*_\gamma)$ is the variational operator.

At this stage we notice that since S^1 is compact, every variation of $\gamma \in \mathscr{V}^c(I)$ trivially has compact support. However, it is not true that every $v \in \ker\mathscr{D}_\gamma$ is a variational vector field. For instance, take I to be the geodesic equations on a "barbell" surface and let $\gamma_\circ$ be the lone closed geodesic in the neck of the surface. The variational equations in this case are just the Jacobi equations. However, $v \in \ker\mathscr{D}_{\gamma_\circ}$ cannot be realized by a variation of $\gamma_\circ$ through closed geodesics since there are no other closed geodesics in a neighborhood of $\gamma_\circ$.

Hence, as before, we need to introduce the notion of regularity for closed integral curves of a differential system I. In what follows, we say that an integral curve $\gamma \in \mathscr{V}^c(I)$ is regular if there exists an open interval $(a, b) \subset S^1$ such that the holonomy map of γ restricted to (a, b) is surjective. An application of Theorem 3 then allow us to conclude that there exist variations of $\gamma \in \mathscr{V}^c(I)$ through closed integral curves of I, in which case Theorem 10 can be applied to study closed extremals of the variational problem (M, I, ϕ).

However, the regularity of closed integral curves of I is not sufficient to guarantee the existence of closed extremals of the variational problem (M, I, ϕ). It is a problem of fundamental importance to establish necessary and sufficient conditions on the manifold M, the differential system I, and the functional Φ under which one can conclude such an existence.

Appendix B. The holonomy map

In §2 we constructed natural vector bundles on an integral curve $\gamma : (a, b) \to M$ of a sub-bundle $I \subset T^*M$, with mappings

$$0 \to I_\gamma^\perp \xrightarrow{i} T_\gamma \xrightarrow{\pi_V} I_\gamma^* \to 0.$$

In addition, we constructed the variational operator

$$\mathscr{D}_\gamma : C^\infty(T_\gamma) \to \Omega^1(I_\gamma^*),$$

which satisfies

$$\mathscr{D}_\gamma(fv) = df \otimes \pi_\nu(v) + f\mathscr{D}_\gamma(v),$$

where f is a smooth function on (a, b). Furthermore, we chose a splitting of T_γ given by $s : I_\gamma^* \to T_\gamma$ satisfying $\pi_\nu \circ s = \mathbf{1}$,

$$T_\gamma \cong I_\gamma^\perp \oplus s(I_\gamma^*).$$

This splitting allows us to define a connection on I_γ^*,

$$\nabla^s : C^\infty(I_\gamma^*) \to \Omega^1(I_\gamma^*),$$

given by

$$\nabla^s(\nu) = \mathscr{D}_\gamma(s(\nu)).$$

Clearly

$$\nabla^s(f\nu) = df \otimes \nu + f\nabla^s(\nu)$$

and so ∇^s is a connection on I_γ^*, depending on the splitting s.

At this stage we note that I_γ^* is a trivial vector bundle of rank p,

$$I_\gamma^* = (a, b) \times \mathbb{R}^p,$$

and so ∇^s is a globally flat connection on I_γ^*. Hence there exists a parallel frame field on (a, b) with respect to which ∇^s is the trivial connection on I_γ^* given by a differential of maps.

Let us now introduce the complex $\mathscr{E}^s = \mathscr{E}^s(I_\gamma^*)$ given by

$$0 \to C_\circ^\infty(I_\gamma^*) \xrightarrow{\nabla^s} \Omega_\circ^1(I_\gamma^*) \to 0.$$

This complex is not exact, and one can easily show that $H^1(\mathscr{E}^s) \cong \mathbb{R}^p$. Hence we obtain the exact sequence

$$0 \to C_\circ^\infty(I_\gamma^*) \xrightarrow{\nabla^s} \Omega_\circ^1(I_\gamma^*) \xrightarrow{\pi^s} H^1(\mathscr{E}^s) \to 0,$$

where π^s denotes the map to the cohomology of $\mathscr{E}^s$.

This then leads us to the following definition of the holonomy map:

$$\mathscr{H}^s : C_\circ^\infty(I_\gamma^\perp) \to H^1(\mathscr{E}^s)$$

with $\mathscr{H}^s(\sigma) = -\pi^s(\mathscr{D}_\gamma(\sigma))$. It is now easy to show that $\mathscr{H}^s$ coincides with the holonomy map constructed in §2.

Appendix C. The du Bois-Reymond lemma

In this appendix we state and prove a result that is needed in the derivation of the Euler-Lagrange equations given in §3.

Lemma C.1. *Let $H(\tau)$ be a $(p \times n)$-matrix of smooth functions with the property that the map*

$$\mathscr{H} : C_\circ^\infty((a, b), \mathbb{R}^n) \to \mathbb{R}^p$$

given by

$$\mathscr{H}(w) = \int_a^b H(\tau) w(\tau)\, d\tau$$

is surjective. Furthermore, let ${}^t\Lambda(\tau)$ be a smooth $\mathbb{R}^n$-valued function satisfying

$$\int_a^b \Lambda(\tau) w(\tau)\, d\tau = 0$$

for every $w \in C_\circ^\infty((a, b), \mathbb{R}^n)$ satisfying $\mathscr{H}(w) = 0$. Then $\Lambda(\tau)$ is necessarily of the form $\Lambda(\tau) = kH(\tau)$ for some constant vector ${}^tk \in \mathbb{R}^p$.

Proof. We shall use essentially the same idea as in the proof of the classical du Bois-Reymond lemma. Hence we construct the vector-valued function

$$\zeta_{\circ}(\tau) = \phi^2(\tau)(\Lambda(\tau) - kH(\tau)),$$

where ${}^t k \in (\mathbb{R}^p)^*$ is as yet an undetermined constant vector, and ϕ is any nonzero real-valued function with support in (a, b). Furthermore, we demand that

$$\int_a^b H(\tau)\, {}^t\zeta_{\circ}(\tau)\, d\tau = 0,$$

which expands to give

$$\int_a^b H(\tau)\, {}^t\Lambda(\tau)\phi^2(\tau)\, d\tau - \left[\int_a^b H(\tau)\, {}^t H(\tau)\phi^2(\tau)\, d\tau\right] {}^t k = 0. \tag{C.1}$$

The assumption that $\mathscr{H}$ is a surjective map implies that the $(p \times p)$-matrix

$$\left[\int_a^b H(\tau)\, {}^t H(\tau)\phi^2(\tau)\, d\tau\right]$$

is of full rank. Hence (C.1) is a system of linear algebraic equations which can be solved uniquely for k.

We now take for $w(\tau)$ the $\mathbb{R}^n$-valued function ${}^t\zeta_{\circ}(\tau)$ with k as determined above. In terms of these, we have that

$$\begin{aligned}\int_a^b [\Lambda(\tau) - kH(\tau)]\, {}^t[\Lambda(\tau) - kH(\tau)]\phi^2(\tau)\, d\tau \\ = \int_a^b \Lambda(\tau)w(\tau)\, d\tau - k\int_a^b H(\tau)w(\tau)\, d\tau.\end{aligned}$$

The right side of this last relation is zero by hypothesis while the left side is everywhere ≥ 0. Hence, we necessarily have

$$\Lambda(\tau) - kH(\tau) = 0.$$

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