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NODAL SETS OF EIGENFUNCTIONS ON RIEMANN SURFACES

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Abstract

On an n-dimensional smooth Riemannian manifold, the nodal set of an eigenfunction is its zero set. It has been a longstanding problem to estimate the $(n-1)$-dimensional Hausdorff measure of the nodal set in terms of the corresponding eigenvalue and the geometry of the manifold. In this paper, we give an upper bound on the length of the nodal set on a Riemann surface.

Defining the vanishing order of an eigenfunction at a point to be the order of the first nonzero term in its Taylor expansion at that point, we also give an upper bound on the sum of the vanishing orders over the points on the Riemann surface, where the eigenfunction and its gradient both vanish. This result sharpens a similar result by Donnelley and Fefferman.

1. Introduction

Let (M^n, g) be a connected, smooth, compact Riemannian manifold without boundary. Suppose that Δ is the Laplace-Beltrami operator on (M^n, g), and u is a real eigenfunction with corresponding eigenvalue λ, i.e., $\Delta u = -\lambda u$. The nodal set $\mathcal{N}$ of u is defined to be the set of points $x \in M$ where $u(x) = 0$.

Denote D to be the diameter of the manifold, and H to be the upper bound of the absolute value of the sectional curvature.

It is clear that outside of the singular set $\mathcal{S} = \{x \mid u(x) = 0, \nabla u(x) = 0\}$, $\mathcal{N}$ is a regular $(n-1)$-dimensional submanifold of M.

Yau conjectured in Problem 73 of [14] that

$$c_1\sqrt{\lambda} \le \mathcal{H}^{n-1}(\mathcal{N}) \le c_2\sqrt{\lambda},$$

and that the constants c_1 and c_2 depend only on the geometry of the manifold. Here $\mathcal{H}^{n-1}(\mathcal{N})$ is the $(n-1)$-dimensional Hausdorff measure of $\mathcal{N}$.

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In this paper, we will first prove an integral formula for the $(n-1)$-dimensional Hausdorff measure of the nodal set on an n-dimensional manifold.

Theorem 2.1. *Let* $q = |\nabla u|^2 + \lambda u^2/n$. *Then*

$$\mathscr{H}^{n-1}(\mathscr{N}) = \frac{1}{2}\int_M \frac{\Delta|u| + \lambda|u|}{\sqrt{q}}.$$

When M is a Riemann surface, i.e., $n = 2$, we can prove the following interesting results.

Theorem 4.2. *Suppose* B_R *is a geodesic ball of radius* R *in* M. *Then*

$$\mathscr{H}^1(\mathscr{N} \cap B_R) \leq R\sqrt{\lambda}(c_3 + c_4 R^2\sqrt{\lambda}),$$

where the constants c_3 *and* c_4 *depend only upon* H *and* D.

The optimal lower bound $\mathscr{H}^1(\mathscr{N}) \geq c_5\sqrt{\lambda}$ was obtained a while ago by Brüning in [3] and also by Yau.

A byproduct of this estimate is

Corollary 4.4. $|\ln q|_{\mathrm{BMO}} \leq c_6 \lambda^{3/4}$.

The singular set $\mathscr{S}$ consists of only isolated points. We have

Theorem 3.4. *Suppose that* $\mathscr{S} = \{p_i\}$ *and that* u *vanishes to order* $n_i + 1$ *at* p_i. *Then,*

$$\sum_{p_i \in B_R(x)} n_i \leq c_7 R^2 \lambda + c_8\sqrt{\lambda},$$

where c_7 *and* c_8 *depend only upon* H *and* D.

The global version of this is more interesting.

Corollary 3.6. *Using the same notation as above, we obtain*

$$\sum_i n_i \leq \frac{1}{4\pi}\left[\lambda \operatorname{vol}(M) - 2\int_M \min(K, 0)\right],$$

where K *is the Gaussian curvature of the surface.*

These results generalize and improve similar results in [8] by Donnelly and Fefferman.

For general n-dimensional manifolds, Hardt and Simon proved the following.

Theorem [12, Theorem 5.3]. *For any* n*-dimensional smooth manifold, one has, for large* λ,

$$\mathscr{H}^{n-1}(\mathscr{N}) \leq \exp(c_9\sqrt{\lambda}\ln\lambda),$$

where the constant c_9 *depends upon* H *and* D.

When (M, g) is real analytic, Donnelly and Fefferman proved

Theorem [7, Theorem 1.2]. $c_{10}\sqrt{\lambda} \le \mathscr{H}^{n-1}(\mathscr{N}) \le c_{11}\sqrt{\lambda}$.

However due to the nature of the proof, these constants are not controlled by the geometry of the manifold.

We used a rather different approach to the problem. Most of the methods used here can be applied with little modification to manifolds with boundary.

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2. Integral formula

In [1], Alt, Caffarelli, and Friedman proved certain regularity theorems for a free boundary problem. Inspired by their idea, we are going to write the Haussdorff measure of $\mathscr{N}$ in a singular integral.

Theorem 2.1. *Define* $q = |\nabla u|^2 + \lambda u^2/n$ *and let* $\Omega \subset M$ *be a domain with smooth boundary. Then*

$$\mathscr{H}^{n-1}(\mathscr{N} \cap \Omega) = \frac{1}{2}\int_\Omega \frac{\Delta|u| + \lambda|u|}{\sqrt{q}}. \tag{1}$$

Remark. 1. The integral on the right-hand side should be understood as

$$\lim_{\varepsilon\to 0} \frac{1}{2}\int_{\Omega-\mathscr{T}_\varepsilon} \frac{\Delta|u| + \lambda|u|}{\sqrt{q}},$$

where $\mathscr{T}_\varepsilon$ is a tubular neighborhood of the singular set $\mathscr{S} = \{x \mid u(x) = 0, \nabla u = 0\}$ in Ω. Because $1/\sqrt{q}$ is a regular function on $\Omega - \mathscr{T}_\varepsilon$, it makes sense to integrate the distribution $(\Delta|u| + \lambda|u|)/\sqrt{q}$ over $\Omega - \mathscr{T}_\varepsilon$.

2. λ/n can be replaced by any constant which is bounded from below by $c_1\lambda$. We chose the constant for the sake of the differential inequality which we will show later.

Proof. Theorem 1.7(ii) of [12] implies that $\dim_{\mathscr{H}}(\mathscr{S}) \le n-2$. If we can show

$$\mathscr{H}^{n-1}(\mathscr{N} \cap \Omega - \mathscr{T}_\varepsilon) = \frac{1}{2}\int_{\Omega-\mathscr{T}_\varepsilon} \frac{\Delta|u| + \lambda|u|}{\sqrt{q}},$$

then (1) follows.

Denoting ν to be the outward unit normal vector on $\mathscr{T}_\varepsilon$, by definition we get

$$\int_{\Omega-\mathcal{T}_\varepsilon} \frac{\Delta|u|+\lambda|u|}{\sqrt{q}} = \int_{\Omega-\mathcal{T}_\varepsilon} |u|\left[\Delta\left(\frac{1}{\sqrt{q}}\right)+\frac{\lambda}{\sqrt{q}}\right] + \int_{\partial\mathcal{T}_\varepsilon\cup\partial\Omega}\left[\frac{\partial|u|}{\partial\nu}\left(\frac{1}{\sqrt{q}}\right)-|u|\frac{\partial}{\partial\nu}\left(\frac{1}{\sqrt{q}}\right)\right].$$

Noticing that $\Delta|u|+\lambda|u|=0$ on $\{|u|\geq\delta\}-\mathcal{T}_\varepsilon$ and using the Green's identity, we obtain

$$\begin{aligned}\int_{\Omega-\mathcal{T}_\varepsilon} \frac{\Delta|u|+\lambda|u|}{\sqrt{q}} &= \lim_{\delta\to 0}\int_{\{|u|\geq\delta\}-\mathcal{T}_\varepsilon} |u|\left[\Delta\left(\frac{1}{\sqrt{q}}\right)+\frac{\lambda}{\sqrt{q}}\right] \\ &\quad + \int_{\partial\mathcal{T}_\varepsilon\cup\partial\Omega}\left[\frac{\partial|u|}{\partial\nu}\left(\frac{1}{\sqrt{q}}\right)-|u|\frac{\partial}{\partial\nu}\left(\frac{1}{\sqrt{q}}\right)\right] \\ &= \lim_{\delta\to 0}\int_{\{|u|=\delta\}-\mathcal{T}_\varepsilon}\left[\frac{|\nabla u|}{\sqrt{q}}+|u|\frac{\partial}{\partial\nu}\left(\frac{1}{\sqrt{q}}\right)\right] \\ &\quad + \lim_{\delta\to 0}\int_{\{|u|\leq\delta\}\cap\partial\mathcal{T}_\varepsilon}\left[\frac{\partial|u|}{\partial\nu}\left(\frac{1}{\sqrt{q}}\right)-|u|\frac{\partial}{\partial\nu}\left(\frac{1}{\sqrt{q}}\right)\right] \\ &= 2\mathcal{H}^{n-1}(\mathcal{N}\cap\Omega-\mathcal{T}_\varepsilon).\end{aligned}$$

The last equality is due to the fact that $|\nabla u|/\sqrt{q}\to 1$ on $\{|u|=\delta\}-\mathcal{T}_\varepsilon$ as $\delta\to 0$ and the fact that $\mathcal{N}\cap\Omega-\mathcal{T}_\varepsilon$ is an $(n-1)$-dimensional submanifold in Ω, so that

$$\lim_{\delta\to 0}\mathcal{H}^{n-1}(\{|u|=\delta\}-\mathcal{T}_\varepsilon) = 2\mathcal{H}^{n-1}(\mathcal{N}\cap\Omega-\mathcal{T}_\varepsilon).$$

Formally, we have

$$\begin{aligned}(2)\quad \int_\Omega \frac{\Delta|u|+\lambda|u|}{\sqrt{q}} &= \int_\Omega\left[-\nabla|u|\cdot\nabla\left(\frac{1}{\sqrt{q}}\right)+\frac{\lambda|u|}{\sqrt{q}}\right]+\int_{\partial\Omega}\frac{1}{\sqrt{q}}\frac{\partial|u|}{\partial\nu} \\ &\leq \frac{1}{2}\int_\Omega\left[\frac{\nabla|u|}{\sqrt{q}}\cdot\frac{\nabla q}{q}+\frac{\lambda|u|}{\sqrt{|\nabla u|^2+\lambda/n\lambda u^2}}\right]+\mathrm{vol}(\partial\Omega) \\ &\leq \frac{1}{2}\int_\Omega\left[\left|\frac{\nabla|u|}{\sqrt{q}}\right|\cdot|\nabla\ln q|+\frac{1}{\sqrt{\lambda/n}}\sqrt{\lambda}\right]+\mathrm{vol}(\partial\Omega) \\ (3)\quad &\leq \frac{1}{2}\int_\Omega|\nabla\ln q|+\sqrt{n\lambda}\,\mathrm{vol}(\Omega)+\mathrm{vol}(\partial\Omega).\end{aligned}$$

The last step is due to Kato's inequality $|\nabla|u||\leq|\nabla u|\leq\sqrt{q}$.

We still need to justify the first equality (2). Integration by parts gives

$$\int_{\Omega-\mathcal{T}_\varepsilon} \frac{\Delta|u|+\lambda|u|}{\sqrt{q}} = \int_{\partial\mathcal{T}_\varepsilon} \frac{1}{\sqrt{q}}\frac{\partial|u|}{\partial\nu} + \int_{\Omega-\mathcal{T}_\varepsilon}\left[-\nabla|u|\cdot\nabla\left(\frac{1}{\sqrt{q}}\right)+\frac{\lambda|u|}{\sqrt{q}}\right] + \int_{\partial\Omega}\frac{1}{\sqrt{q}}\frac{\partial|u|}{\partial\nu}.$$

The first integral on the right-hand side is

$$\leq \int_{\partial\mathcal{T}_\varepsilon}\left|\frac{\nabla|u|}{\sqrt{q}}\right| \leq \int_{\partial\mathcal{T}_\varepsilon} 1 = \mathcal{H}^{n-1}(\partial\mathcal{T}_\varepsilon). \tag{4}$$

The definition of the Minkowski content $\mathcal{M}$ and Theorem 3.2.39 in [10] show that

$$\begin{aligned}\mathcal{H}^{n-2+\delta}(\mathcal{S}) = \mathcal{M}^{n-2+\delta}(\mathcal{S}) &= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon^{2-\delta}}\mathcal{H}^n(\mathcal{T}_\varepsilon)\\ &= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon^{2-\delta}}\int_0^\varepsilon \mathcal{H}^{n-1}(\partial\mathcal{T}_\varepsilon)\,d\tau.\end{aligned}\tag{5}$$

Since $\mathcal{H}^{n-2+\delta}(\mathcal{S}) = 0$ for any $\delta > 0$, the vanishing of the last limit in (5) implies the existence of a sequence $\{\varepsilon_i\} \to 0$ as $i \to \infty$, such that $\mathcal{H}^{n-1}(\partial\mathcal{T}_{\varepsilon_i}) \to 0$ as $i \to \infty$. Letting $\varepsilon = \varepsilon_i \to 0$ in (4), we conclude that (2) is true. q.e.d.

We have thus established our main estimate of this section:

Theorem 2.2. *For any domain $\Omega \subset M$ which has a smooth boundary, we have*

$$2\mathcal{H}^{n-1}(\mathcal{N}\cap\Omega) \leq \frac{1}{2}\int_\Omega |\nabla \ln q| + \sqrt{n\lambda}\,\mathrm{vol}(\Omega) + \mathrm{vol}(\partial\Omega).$$

3. Vanishing order estimates

The goal of this section is to prove various vanishing order estimates in preparation for the next section. Some of the estimates are of independent interest. We assume λ to be large, as we can always change the constants in the estimates to make them true for smaller λ.

Donnelly and Fefferman proved the following vanishing order estimate. The same estimate was obtained later by Lin in [13] via a different approach.

Theorem [7, Theorem 4.2]. *There exists an $R_0 > 0$ such that for any $x \in M$ and $R < R_0$, one has*

$$\max_{B_{2R}(x)} |u| \leq e^{c_1\sqrt{\lambda}} \max_{B_R(x)} |u|,$$

where R_0 depends only upon H, and the constants c_0 and c_1 depend only upon H and D.

The same type of estimate holds for q.

Lemma 3.1. *There exists an $R_0 = R_0(H) > 0$ such that for any $x \in M$ and $R < R_0$,*

$$\max_{B_{2R}(x)} q \leq e^{c_2\sqrt{\lambda}} \max_{B_R(x)} q,$$

where the constants c_1 and c_2 depend only upon H and D.

In order to prove the lemma, we need to define the second-order frequency N_2 for a harmonic function and establish some of its properties.

Suppose h is a harmonic function. We define $N_2(R)$ of h to be

$$N_2(R) = R\int_{B_R} |\nabla^2 h|^2 \Big/ \int_{\partial B_R} |\nabla h|^2. \tag{6}$$

Theorem 3.2. *There is a positive constant K such that $e^{KR}N_2(R)$ is a monotone nondecreasing function of R.*

Proof. For simplicity, we only carry out the calculation on the Euclidean space, and without loss of generality assume that the ball is centered at the origin. In this case, we can choose $K = 0$:

$$\begin{aligned}\frac{d}{dR}\ln N_2(R) &= \frac{d}{dR}\ln R + \frac{d}{dR}\ln\int_{B_R}|\nabla^2 h|^2 - \frac{d}{dR}\ln\int_{\partial B_R}|\nabla h|^2 \\ &= \frac{1}{R} + \frac{\int_{\partial B_R}|\nabla^2 h|^2}{\int_{B_R}|\nabla^2 h|^2} - \frac{\frac{d}{dR}\int_{\partial B_R}|\nabla h|^2}{\int_{\partial B_R}|\nabla h|^2}.\end{aligned}$$

The following are some Rellich type identities for harmonic functions:

$$\begin{aligned}\int_{B_R}|\nabla^2 h|^2 &= \frac{1}{2}\int_{\partial B_R}\left|\left\langle \nabla|\nabla h|^2, \frac{x}{R}\right\rangle\right|^2, \\ \int_{\partial B_R}|\nabla^2 h|^2 &= \frac{n-2}{R}\int_{B_R}|\nabla^2 h|^2 + 2\int_{\partial B_R}\left|\left\langle \nabla^2 h, \frac{x}{R}\right\rangle\right|^2, \\ \frac{d}{dR}\int_{\partial B_R}|\nabla h|^2 &= \frac{n-1}{R}\int_{\partial B_R}|\nabla h|^2 + \int_{\partial B_R}\left|\left\langle \nabla|\nabla h|^2, \frac{x}{R}\right\rangle\right|^2.\end{aligned}$$

Putting together all of the above, we have

$$\begin{aligned}\frac{d}{dR}\ln N_2(R) = {}& 2\int_{\partial B_R}\left|\left\langle \nabla^2 h, \frac{x}{R}\right\rangle\right|^2 \Big/ \frac{1}{2}\int_{\partial B_R}\left|\left\langle \nabla|\nabla h|^2, \frac{x}{R}\right\rangle\right|^2 \\ & - \int_{\partial B_R}\left|\left\langle \nabla^2 h, \frac{x}{R}\right\rangle\right|^2 \Big/ \int_{\partial B_R}|\nabla h|^2.\end{aligned}$$

By elementary methods it can be shown that

$$4\int_{\partial B_R}\left|\left\langle\nabla^2 h, \frac{x}{R}\right\rangle\right|^2\int_{\partial B_R}|\nabla h|^2 \geq \left(\int_{\partial B_R}\left|\left\langle\nabla|\nabla h|^2, \frac{x}{R}\right\rangle\right|^2\right)^2.$$

Hence,

$$\frac{d}{dR}\ln N_2(R) \geq 0. \quad \text{q.e.d.}$$

If we note that

$$\frac{d}{dR}\ln\left[\frac{1}{R^{n-1}}\int_{\partial B_R}|\nabla h|^2\right] = \frac{2N_2(R)}{R},$$

then $N_2(R) \leq N$ implies

(7) $$\max_{B_{2R}}|\nabla h|^2 \leq e^{c_3 N}\max_{B_R}|\nabla h|^2.$$

We define a new metric $\widehat{ds^2} = dr^2 + r^2 ds^2$ on $(0, \infty)\times M$. Let $\hat{u} = r^\alpha u(x)$, where

$$\alpha = \tfrac{1}{2}\left[\sqrt{4\lambda + (n-1)^2} - (n-1)\right].$$

Under the new metric, it is easy to check that $\widehat{\Delta}\hat{u} = 0$.

Let the ball be centered on $(0, p)$ for some $p \in M$. Hence, $\widehat{B_2} = (0, 2)\times M$. Direct calculation shows that

$$|\widehat{\nabla}\hat{u}|^2 = r^{2\alpha-2}(|\nabla u|^2 + \alpha^2 u^2)$$

and

$$|\widehat{\nabla}^2\hat{u}|^2 = r^{2\alpha-4}[|\nabla^2 u|^2 + 2\alpha u\Delta u + 2(\alpha-1)^2|\nabla u|^2 + \alpha^2(\alpha^2 - 2\alpha + 2)u^2].$$

Substituting these into (6) and using the fact that $\Delta u = -\lambda u$, we have $N_2(2) \leq c_4\sqrt{\lambda}$.

Arguments similar to those in [13] yield that $N_2(R) \leq c_5\sqrt{\lambda}$ for any ball with center in $\widehat{B_1}$ and radius less than 1.

From (7) it follows that

$$\max_{\widehat{B_{2R}}}|\widehat{\nabla}\hat{u}|^2 \leq e^{c_6\sqrt{\lambda}}\max_{\widehat{B_R}}|\widehat{\nabla}\hat{u}|^2,$$

where $\widehat{B}$ stands for the corresponding ball in $(0, \infty)\times M$.

Getting rid of the hats, we get

$$\max_{B_{2R}}|\nabla u|^2 \leq e^{c_7\sqrt{\lambda}}\max_{B_R}(|\nabla u|^2 + c_8\lambda|u|^2).$$

We have thus finished the proof of Lemma 3.1. q.e.d.

For the remainder of this section, we assume that (M^2, g) is a smooth two-dimensional manifold (unless otherwise indicated) and K is its Gaussian curvature. It was proved in [5] that the singular set $\mathscr{S}$ consists of finitely many isolated points. Let $\mathscr{S} = \{p_1, p_2, \cdots, p_l\}$ and let k_i be the vanishing order of u at p_i. We are going to prove that the following inequality holds in the sense of distribution.

Theorem 3.3. *Using the notation defined above, we obtain*

$$\Delta \ln q \geq -\lambda + 2\min(K, 0) + 4\pi \sum_i (k_i - 1)\delta_{p_i}, \tag{8}$$

where δ_{p_i} is the Dirac function centered at p_i.

Proof. We are first going to prove by direct calculation that at the point where $q \neq 0$, we have

$$\Delta \ln q \geq -\lambda + 2\min(K, 0).$$

The standard Bochner formula gives

$$\begin{aligned}
\Delta \ln q &= \frac{1}{q}\Delta q - \frac{1}{q^2}|\nabla q|^2 \\
&= \frac{1}{q}[2|\nabla^2 u|^2 + 2\langle \nabla u, \nabla \Delta u\rangle + (2K+\lambda)|\nabla u|^2 - \lambda^2 u^2] \\
&\quad - \frac{1}{q^2}|2\langle \nabla^2 u, \nabla u\rangle + \lambda u \nabla u|^2 \\
&= \frac{1}{q}[2|\nabla^2 u|^2 - \lambda^2 u^2 + (2K-\lambda)|\nabla u|^2] \\
&\quad - \frac{1}{q^2}|2\langle \nabla^2 u, \nabla u\rangle + \lambda u \nabla u|^2,
\end{aligned}$$

where $\nabla^2 u$ denotes the Hessian of u. Choose a suitable orthogonal coordinate system at the point, such that

$$\nabla^2 u = \begin{bmatrix} u_{11} & 0 \\ 0 & u_{22} \end{bmatrix}.$$

The equation $\Delta u = -\lambda u$ implies that $u_{11} + u_{22} = -\lambda u$. Substituting $-u_{11} - u_{22}$ for λu and denoting

$$\nabla u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix},$$

we have

$$
\begin{aligned}
\Delta \ln q &= \frac{1}{q}[2(u_{11}^2 + u_{22}^2) - (u_{11} + u_{22})^2 + (2K - \lambda)|\nabla u|^2] \\
&\quad - \frac{1}{q^2}\left\| \begin{bmatrix} u_{11} - u_{22} & 0 \\ 0 & u_{22} - u_{11} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \right\|^2 \\
&= \frac{1}{q^2}[q(u_{11} - u_{22})^2 + q(2K - \lambda)|\nabla u|^2 - (u_{11} - u_{22})^2 |\nabla u|^2] \\
&= \frac{\lambda}{2}\frac{(u_{11} - u_{22})^2}{q^2} + (2K - \lambda)\frac{|\nabla u|^2}{q}.
\end{aligned} \tag{9}
$$

Noticing that the first term in (9) is nonnegative and that $|\nabla u|^2/q \leq 1$, we finally get

$$\Delta \ln q \geq -\lambda + 2\min(K, 0). \tag{10}$$

The next step is to deal with the singular set $\mathscr{S}$. Let p be a point in $\mathscr{S}$, and k be the vanishing order of u at p. Given a nonnegative test function $\phi \in C_0^\infty(U)$, where U is a neighborhood of p in M which intersects $\mathscr{S}$ only at p, we need to show that

$$\int_U \phi \Delta \ln q \geq \int_U \phi[-\lambda + 2\min(K, 0)] + 4\pi(k - 1)\phi(p). \tag{11}$$

Without loss of generality, we assume $p = 0$. We may choose polar coordinates (r, θ), under which u has the following expansions:

$$
\begin{aligned}
u &= ar^k \cos k\theta + O(r^{k+1}), \\
\nabla u &= \nabla(ar^k \cos k\theta) + O(r^k), \\
\nabla^2 u &= \nabla^2(ar^k \cos k\theta) + O(r^{k-1}).
\end{aligned}
$$

Elementary calculation yields

$$q = |\nabla u|^2 + \lambda u^2/2 = k^2 r^{2k-2} + O(r^{2k-1}),$$

$$\ln q = \ln k^2 + (2k - 2)\ln r + O(r), \tag{12}$$

$$\nabla \ln q = (2k - 2)/r + O(1). \tag{13}$$

We are now ready to prove (11). At first we compute:

$$
\begin{aligned}
\int_U \phi \Delta \ln q &= \int_U \Delta\phi \ln q = \lim_{\varepsilon \to 0} \int_{U - B_\varepsilon(0)} \Delta \phi \ln q \\
&= \lim_{\varepsilon \to 0}\left[\int_{\partial B_\varepsilon(0)} \frac{\partial \phi}{\partial \nu} \ln q - \int_{\partial B_\varepsilon(0)} \phi \frac{\partial}{\partial \nu} \ln q + \int_{U - B_\varepsilon(0)} \phi \Delta \ln q\right] \\
&= \lim_{\varepsilon \to 0}[I_1 - I_2 + I_3].
\end{aligned}
$$

By (12) and (13) we obtain, respectively,

$$\lim_{\varepsilon\to 0} I_1 = \lim_{\varepsilon\to 0}\int_{\partial B_\varepsilon(0)} \frac{\partial\phi}{\partial\nu}\ln q = 0, \tag{15}$$

$$\begin{aligned}\lim_{\varepsilon\to 0} I_2 &= \lim_{\varepsilon\to 0}\int_{\partial B_\varepsilon(0)} \phi\frac{\partial}{\partial\nu}\ln q \\ &= 2\pi(2k-2)\phi(0)\lim_{\varepsilon\to 0}\varepsilon\left[\frac{1}{\varepsilon}+O(1)\right] = 4\pi(k-1)\phi(0).\end{aligned} \tag{16}$$

(10) together with that $q \neq 0$ on $U - B_\varepsilon(0)$ implies

$$I_3 = \int_{U-B_\varepsilon(0)} \phi\Delta\ln q \geq \int_{U-B_\varepsilon(0)} \phi[-\lambda + 2\min(K, 0)]. \tag{17}$$

We conclude the proof of the theorem by combining (15), (16), and (17).

Theorem 3.4. *Suppose that $\mathscr{S} = \{p_i\}$ and that u vanishes to order $n_i + 1$ at p_i. Then*

$$\sum_{p_i\in B_R} n_i \leq c_9\sqrt{\lambda} + c_{10}R^2\lambda,$$

where c_9 and c_{10} depend only upon D and H.

Proof. Normalize q so that $\max_{B_{4R}(x)} q = 1$. Suppose that for some $p \in \overline{B_R(x)}$, $\ln q$ achieves its maximum in $B_R(x)$, i.e.,

$$\ln q(p) = \max_{B_R(x)} \ln q \geq -c_{11}\sqrt{\lambda}.$$

We have used Proposition 3.1 twice to get the last inequality. Define r to be the geodesic distance from p, and $k(r)$ to be an auxiliary function on $B_{3R}(p) \subset B_{4R}(x)$:

$$k(r) = \frac{1}{36\pi R^2}(r^2 - 9R^2) - \frac{1}{2\pi}\ln\left(\frac{r}{3R}\right).$$

Lemma 3.5. *For $R < C(H)$, $k(r)$ satisfies:*

(1) *$k(r) \geq 0$ on $B_{3R}(p)$ and $k(3R) = 0$,*

(2) *$k(r) \geq c_{12} > 0$ on $B_{2R}(p) \supset B_R(x)$, $\int_{B_{3R}(p)} k(r) \leq c_{13}R^2 < \infty$,*

(3) *$k'(r) \leq 0$ on $B_{3R}(p)$ and $k'(3R) = 0$,*

(4) *$\Delta k(r) \geq -\delta_p$.*

Proof. (1), (2), and (3) are trivial.

Remembering that $K \geq -H$, we use the comparison theorem for Laplace operators to obtain $\Delta r \leq \Delta_H r$ in the sense of distribution, where Δ_H denotes the Laplacian on the hyperbolic space form with constant curvature

$-H$. From (3) it follows that

$$\begin{aligned}\Delta k(r) &\geq \Delta_H k(r)\\ &= \frac{1}{18\pi R^2} - \delta_p + \sqrt{H}\left(\frac{r}{18\pi R^2} - \frac{1}{2\pi r}\right)\operatorname{cotanh}\left(\sqrt{H}r\right) + \frac{1}{2\pi r^2}.\end{aligned}$$

Choosing a small R, (4) becomes obvious. q.e.d.

Multiplying $k(r)$ on both sides of (8) and using the Green's identity, we get

$$\begin{aligned}&\int_{B_{3R}(p)} \ln q \Delta k(r)\\ &\geq \int_{B_{3R}(p)} k(r)\left[-\lambda + 2\min(K, 0) + 4\pi \sum_i (k_i - 1)\delta_{p_i}\right].\end{aligned} \tag{18}$$

By noticing that $\ln q \leq 0$ on $B_{3R}(p) \subset B_{4R}(x)$, the left-hand side of (18) can be estimated by

$$\leq \int_{B_{3R}(p)} -\ln q \delta_p \leq -\ln q(p) \leq c_{14}\sqrt{\lambda}. \tag{19}$$

The right-hand side of (18) is

$$\geq -c_{15}(\lambda + 2H)R^2 + 4\pi c_{16} \sum_{p_i \in B_R(x)} k_i - 1. \tag{20}$$

Comparing (19) with (20) and remembering that $n_i = k_i - 1$ complete the proof of Theorem 3.4. q.e.d.

It is actually easier to get the global version of this theorem. Integrating both sides of (8), we get the following.

Corollary 3.6. *On any Riemann surface,*

$$\sum_i n_i \leq \frac{1}{4\pi}\left[\lambda \operatorname{vol}(M) - 2\int_M \min(K, 0)\right]. \tag{21}$$

4. Nodal length estimate

The present section is devoted to a local estimate on the nodal length. In §1, we demonstrated that the nodal length within a geodesic ball $B_R(x)$ can be bounded by

$$c_1 \int_{B_R(x)} |\nabla \ln q| + c_2\sqrt{\lambda}R^{n-1}.$$

We are going to estimate the first term as follows.

Lemma 4.1. *Suppose that on $B_{4R}(x)$, f is nonnegative and*

$$\Delta f \leq h. \tag{22}$$

Then

$$R^{-1} \int_{B_R} |\nabla f| \leq c_3 \inf_{B_R(x)} f + c_4 R \|h\|_{L^2(B_{4R}(x))}. \tag{23}$$

Proof. This is the *weak Harnack* type inequality (8.76) in [11] with $q = 4$. For an alternative proof see [6]. q.e.d.

We have shown in §2 that on a two-dimensional manifold,

$$\Delta \ln q \geq -\lambda + 2\min(K, 0) + 4\pi(k_i - 1)\delta_{p_i}. \tag{24}$$

Normalize q so that $\max_{B_{4R}(x)} q = 1$ and let $f = -\ln q$. Then we have $f \geq 0$ on $B_{4R}(x)$ and $\inf_{B_R(x)} f \leq c_5\sqrt{\lambda}$. When λ is large, $h = c_6\lambda$ will satisfy (22). Substituting f and h in (23) and multiplying the resulting equation by R, we obtain

$$\int_{B_R} |\nabla \ln q| \leq c_7 R\sqrt{\lambda} + c_8 \lambda R^3. \tag{25}$$

We have thus proved

Theorem 4.2. *Suppose B_R is any geodesic ball of radius R in M. Then*

$$\mathscr{H}^1(\mathscr{N} \cap B_R) \leq R\sqrt{\lambda}(c_9 + c_{10} R^2 \sqrt{\lambda}), \tag{26}$$

where the constants c_9 and c_{10} depend only upon H and D.

We need the following lemma to extend (26) to a global estimate.

Lemma 4.3. *Any geodesic ball B_ρ of radius ρ can be covered by $c_{11}\rho/R^2$ many balls of smaller radius R, with the constant depending only upon H and D.*

Proof. Denote by $\mathscr{B}$ a maximum collection of disjoint balls in B_ρ with radius $R/2$. It is easy to see that $\{B_R(x) \mid B_{R/2}(x) \in \mathscr{B}\}$ covers B_ρ. We want to count the number of balls in $\mathscr{B}$. Suppose $B_{R/2}(x)$ has the smallest volume among all elements of $\mathscr{B}$. Let $V_H(r)$ be the volume of a ball with radius r in hyperbolic space form with constant curvature $-H$. Then by the Bishop-Gromov volume comparison theorem,

$$\operatorname{card}\mathscr{B} \leq \frac{\operatorname{vol}(B_\rho)}{\operatorname{vol}(B_{R/2}(x))} \leq \frac{V_{\sqrt{H}}(\rho)}{V_{\sqrt{H}}(R/2)} = \frac{\int_0^\rho \sinh(\sqrt{H}s)/\sqrt{H}}{\int_0^{R/2} \sinh(\sqrt{H}s)/\sqrt{H}} \leq c_{12}\rho/R^2,$$

where c_{12} depends only upon H and D. q.e.d.

Multiplying (25) by $c_{12}\rho/R^2$, we get

$$\int_{B_\rho} |\nabla \ln q| \le c_{13}\rho(\sqrt{\lambda}/R + c_{14}\lambda R^2).$$

We have the freedom to choose R, and the optimal choice is $R = \lambda^{-1/4}$:

$$\int_{B_\rho} |\nabla \ln q| \le c_{15}\rho\lambda^{3/4}.$$

Hence we have proved the following corollary.

Corollary 4.4. $|\ln q|_{\mathrm{BMO}} \le c_{16}\lambda^{3/4}$.

Remark. Following the steps by Chanillo and Muckenhoupt in [4], we were able to prove a BMO bound for $\ln q$ which is similar to theirs on n-dimensional manifolds. But this two-dimensional bound is sharper.

Taking ρ to be the diameter of the manifold, we obtain $B_\rho = M$. Hence we have our final theorem.

Theorem 4.5. *Let (M^2, g) be a smooth Riemann surface, and u an eigenfunction with corresponding eigenvalue λ. Then*

$$\mathscr{H}^1(\mathscr{N}) \le c_{17}\lambda^{3/4},$$

where the constant depends only upon D and H.

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