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CHERN CLASSES OF VECTOR BUNDLES WITH HOLOMORPHIC CONNECTIONS ON A COMPLETE SMOOTH COMPLEX VARIETY

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Introduction

Let X be a complete smooth variety over the complex field $\mathbf{C}$, X_{an} the associated complex manifold, and $\mathscr{E}$ a holomorphic vector bundle (locally free sheaf) on X_{an} with a holomorphic connection $\nabla: \mathscr{E} \to \mathscr{E} \otimes_{\mathscr{O}_{X_{an}}} \Omega^1_{X_{an}}$, where $\Omega^1_{X_{an}}$ is the sheaf of holomorphic 1-forms on X_{an}. It is well known that $\mathscr{E}$ has vanishing Chern classes in $H(X_{an,\mathbf{Q}})$, so that the integral Chern classes are torsion.

Recall that the ith Deligne complex $\mathscr{D}(i) = \mathscr{D}(i)X_{an}$ is defined by

$$0 \to \mathbf{Z}(i) \to \mathscr{O}_{X_{an}} \xrightarrow{d} \Omega^1_{X_{an}} \xrightarrow{d} \cdots \xrightarrow{d} \Omega^{i-1}_{X_{an}} \to 0,$$

where $\mathbf{Z}(i)$ is the subsheaf of abelian groups of the constant sheaf $\mathbf{C}$ on X_{an} generated by $(2\pi\sqrt{-1})^i\mathbf{Z}$. The Deligne-Beilinson cohomology group (see [4] and the references given there) $H^j_{\mathscr{D}}(X_{an}, i)$ is defined to be the jth hypercohomology of $\mathscr{D}(i)$. Then there is an exact sequence

$$0 \to J^i(X) \to H^{2i}_{\mathscr{D}}(X_{an}, i) \xrightarrow{\rho} Hg^i(X_{an}) \to 0,$$

where $Hg^i(X_{an}) \subset H^{2i}(X, \mathbf{Z}(i))$ is the subspace of classes of Hodge type (i, i) (i.e., which maps to $F^iH^{2i}(X_{an}, \mathbf{C})$ in $H^{2i}(X_{an}, \mathbf{C})$, where F denotes the Hodge filtration), and $J^i(X)$ is the ith *intermediate Jacobian* of X, defined by

$$J^i(X) = H^{2i-1}(X_{an}, \mathbf{C})/\{\mathrm{im}H^{2i-1}(X_{an}, \mathbf{Z}(i)) + F^iH^{2i-1}(X_{an}, \mathbf{C})\}.$$

The topological Chern class $c_i(\mathscr{E}) \in Hg^i(X_{an}) \subset H^{2i}(X_{an}, \mathbf{Z}(i))$ is the image under ρ of the "refined" Chern class with values in Deligne-Beilinson cohomology,

$$c_i^{\mathscr{D}}(\mathscr{E}) \in H^{2i}_{\mathscr{D}}(X_{an}, i).$$

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If $\mathscr{E}$ has a connection ∇, then $c_i(\mathscr{E})$ is the torsion, so for some integer $N > 0$, $Nc_i(\mathscr{E}) \in J^i(X)$.

For $i = 1$, $J^1(X) = \mathrm{Pic}^0(X)$, the Picard variety of X, and it is a consequence of Hodge theory and GAGA that every element of $\mathrm{Pic}^0(X)$ is the class of an invertible sheaf $\mathscr{L}$ with an integrable connection.

For $i = 2$, Bloch [2] shows that the elements of $H^4_{\mathscr{D}}(X_{an}, 2)$, which are second Chern classes $c_2^{\mathscr{D}}(\mathscr{E})$ for locally free $\mathscr{E}$ with an integrable connection, form a countable set. More precisely, he defines a countable subgroup $\Delta \subset C$ using the *dilogarithm* function, and shows that

$$Nc_2^{\mathscr{D}}(\mathscr{E}) \in \mathrm{im}(H^3(X_{an}, \Delta) \to H^3(X_{an}, \mathbf{C}) \to J^2(X)),$$

where N is the exponent of $c_2(\mathscr{E})$ in $H^4_{\mathscr{D}}(X_{an}, \mathbf{Z}(2))$. He also comments on the relationships between his results and a conjecture of Cheeger and Simons, in the light of which he conjectures that $c_i^{\mathscr{D}}(\mathscr{E})$ is the *torsion* for all $i > 1$ for any locally free sheaf $\mathscr{E}$ with integrable connection.

Our aim in this note is to prove the following result.

Theorem. *Let X be a smooth complete variety over $\mathbf{C}$. Then for any $i > 1$, the set*

$$\{c_i^{\mathscr{D}}(\varepsilon) \in H^{2i}_{\mathscr{D}}(X_{an}, i) | \mathscr{E} \text{ has a holomorphic connection}\}$$

is countable.

Note that we do not require the connections to be integrable.

1. Proof of the Theorem

We begin by noting that by GAGA,

(i) if $\mathscr{E}$ is a locally free $\mathscr{O}_{X_{an}}$-module of finite rank (i.e., a holomorphic vector bundle), then there is a locally free $\mathscr{O}_X$-module $\mathscr{E}_0$, unique up to isomorphism, such that ε is the associated analytic sheaf;

(ii) if $\mathscr{E}$, $\mathscr{E}_0$ are as in (i), and ∇ is a holomorphic connection on $\mathscr{E}$, then there is an algebraic connection ∇_0 on $\mathscr{E}_0$, unique up to isomorphism, such that the associated analytic connection on $(\mathscr{E}_0)_{an} \simeq \mathscr{E}$ is ∇.

One way to see (ii) is as follows: if X is any smooth algebraic variety, and $\mathscr{F}$ a locally free $\mathscr{O}_X$-module of finite rank, then consider the sheaf of algebraic 1-jets of the locally free $\mathscr{O}_X$-module $\mathscr{F}$, defined by

$$\mathscr{J}^1(\mathscr{F}) = p^2_*(p_1^*\mathscr{F} \otimes_{\mathscr{O}_{X\times X}} \mathscr{O}_{X\times X}/\mathscr{I}^2_\Delta),$$

where $\mathscr{I}_\Delta$ is the ideal sheaf of the diagonal on $X \times X$, and $p_i : X \times X \to X$

are the projections. The natural exact sequence (the *jet sequence*)

$$(*) \qquad 0 \to \mathcal{F} \otimes_{\mathcal{O}_X} \Omega^1_{X/\mathbf{C}} \to \mathcal{J}^1(\mathcal{F}) \to \mathcal{F} \to 0 \quad \ldots$$

obtained as p_{2*} of the sequence by tensoring

$$0 \to \mathcal{I}_\Delta/\mathcal{I}_\Delta^2 \to \mathcal{O}_{X\times X}/\mathcal{I}_\Delta^2 \to \mathcal{O}_\Delta \to 0$$

with $p_1^*\mathcal{F}$, yields an extension class

$$A(\mathcal{F}) \in \mathrm{Ext}^1_X(\mathcal{F}, \mathcal{F} \otimes_{\mathcal{O}_X} \Omega^1_{X/\mathbf{C}}) \simeq H^1(X, \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{F}) \otimes_{\mathcal{O}_X} \Omega^1_{X/\mathbf{C}}),$$

the *Atiyah class* of $\mathcal{F}$, whose vanishing is a necessary and sufficient condition for $\mathcal{F}$ to have an algebraic connection (see [1]). In fact connections on $\mathcal{F}$ are naturally in bijection with splittings of the jet sequence.

There is a corresponding Atiyah class for the existence of a holomorphic connection on $\mathcal{F}_{an}$, which lies in

$$H^1(X_{an}, \mathcal{E}nd_{\mathcal{O}_{X_{an}}}(\mathcal{F}_{an}) \otimes_{\mathcal{O}_{X_{an}}} \Omega^1_{X_{an}})$$

where $\Omega^1_{X_{an}}$ is the sheaf of holomorphic 1-forms on X_{an}. Further, the jet sequence for $\mathcal{F}_{an}$ is the sequence of holomorphic sheaves associated to the algebraic jet sequence, so $A(\mathcal{F}) \mapsto A(\mathcal{F}_{an})$ under the natural map on cohomology groups. By GAGA, if X is complete, then the map on cohomology is an isomorphism, and therefore in this case, if $A(\mathcal{F}_{an})$ vanishes, so does $A(\mathcal{F})$.

Hence, in (ii), we see that $\mathcal{E}_0$ has some algebraic connection ∇'. Now $\sigma = \nabla'_{an} - \nabla$ is a holomorphic section

$$\sigma \in H^0(X_{an}, \mathcal{E}nd_{\mathcal{O}_{X_{an}}}(\mathcal{E}) \otimes_{\mathcal{O}_{X_{an}}} \Omega^1_{X_{an}}).$$

Again by GAGA, any holomorphic section σ as above is of the form $\sigma = \tau_{an}$, where τ is an algebraic section

$$\tau \in H^0(X, \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E}_0) \otimes_{\mathcal{O}_X} \Omega^1_{X/C});$$

now $\nabla_0 = \nabla' - \tau$ is an algebraic connection on $\mathcal{E}_0$ such that $(\nabla_0)_{an} = \nabla$.

The Atiyah class $A(\mathcal{F})$ is also related to the topological Chern classes $c_i(\mathcal{F}_{an}) \in Hg^i(X_{an}) \subset H^{2i}(X_{an}, \mathbf{Z}(i))$, as follows (see [1]–this relationship will be exploited in the proof of the Theorem). If X is any smooth algebraic variety over $\mathbf{C}$, and $\mathcal{F}$ is locally free of finite rank on X, then the exterior product of differentials and composition of endomorphisms induces a map of sheaves

$$(\mathcal{E}nd_{\mathcal{O}_X}(\mathcal{F}) \otimes_{\mathcal{O}_X} \Omega^1_{X/\mathbf{C}})^{\otimes i} \to \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{F}) \otimes_{\mathcal{O}_X} \Omega^i_{X/\mathbf{C}},$$

and hence a map on cohomology

$$\psi_i : H^1(X, \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{F}) \otimes_{\mathcal{O}_X} \Omega^1_{X/\mathbf{C}})^{\otimes i} \to H^i(X, \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{F}) \otimes_{\mathcal{O}_X} \Omega^i_{X/\mathbf{C}}).$$

Let

$$\begin{aligned} M_i(\mathcal{F}) &= \psi_i(A(\mathcal{F})^{\otimes i}) = \psi_i(A(\mathcal{F}) \otimes \cdots \otimes A(\mathcal{F})) \\ &\in H^i(X, \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{F}) \otimes_{\mathcal{O}_X} \Omega^i_{X/\mathbf{C}}), \end{aligned}$$

and let

$$N_i(\mathcal{F}) = \mathrm{tr}(M_i(\mathcal{F})) \in H^i(X, \Omega^i_{X/\mathbf{C}}),$$

where 'tr' is the map on cohomology induced by the trace on the sheaf of endomorphisms. The classes $N_i(\mathcal{F})$ are the *Newton classes* of $\mathcal{F}$, where $N_i(\mathcal{F})$ is a polynomial (with integral coefficients) in the Chern classes

$$c_j(\mathcal{F}) \in H^j(X, \Omega^j_{X/\mathbf{C}})$$

for $j \leq i$, such that $c_i(\mathcal{F})$ has a nonzero coefficient (in terms of the splitting principle, the Newton class N_i is the sum of the i th powers of the 'Chern roots'). Conversely, the i th Chern class is a polynomial (with rational coefficients) in the Newton class $N_j(\mathcal{F})$ for $j \leq i$.

In particular, if the Atiyah class $A(\mathcal{F})$ vanishes (i.e., if $\mathcal{F}$ has an algebraic connection), the Chern classes with values in $H^i(X, \Omega^i_{X/\mathbf{C}})$ vanish.

If X is smooth and complete over $\mathbf{C}$, the topological Chern class $c_i(\mathcal{F}_{an})$ is compatible with the Chern class $c_i(\mathcal{F}) \in H^i(X, \Omega^i_{X/\mathbf{C}}$ in the following way: Hodge theory and GAGA yield maps (the latter two are isomorphisms)

$$\begin{aligned} Hg^i(X_{an}) \to F^i H^{2i}(X, \mathbf{C}) \cap \overline{F}^i H^{2i}(X, \mathbf{C}) &\xrightarrow{\cong} H^i(X_{an}, \Omega^i_{X_{an}}) \\ &\xrightarrow{\cong} H^i(X, \Omega^i_{X/\mathbf{C}}) \end{aligned}$$

under which $c_i(\mathcal{F}_{an})$ maps to $c_i(\mathcal{F})$. Hence for smooth and complete X, $c_i(\mathcal{F}) = 0 \Leftrightarrow c_i(\mathcal{F}_{an})_{\mathbf{Q}} = 0$, where $c_i(\mathcal{F}_{an}) \mapsto c_i(\mathcal{F}_{an})_{\mathbf{Q}} \in H^i_g(X_{an}) \otimes \mathbf{Q} \subset H^{2i}(X_{an}, \mathbf{Q}(i))$.

More generally, if k is a field, $f : X \to S$ a smooth morphism of smooth k-varieties, and $\mathcal{F}$ a locally free $\mathcal{O}_X$-module of finite rank, then one has the notion of an algebraic connection on $\mathcal{F}$ relative to S, which is a map of sheaves

$$\mathcal{F} \to \mathcal{F} \otimes_{\mathcal{O}_X} \Omega^1_{X/S}$$

satisfying the Leibniz rule. There is a Atiyah class

$$A_S(\mathcal{F}) \in H^1(X, \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{F}) \otimes_{\mathcal{O}_X} \Omega^1_{X/S}),$$

constructed using the ideal of the diagonal in $X \times_S X$, whose vanishing is equivalent to the existence of an algebraic connection on $\mathcal{F}$ relative to S. This is compatible with the 'global' Atiyah class $A_k(X)$ (the obstruction to the existence of a connection relative to $\operatorname{Spec} k$), in the sense that $A(\mathcal{F}) \mapsto A_S(\mathcal{F})$ under the map induced by the sheaf map

$$\Omega^1_{X/k} \to \Omega^1_{X/S}.$$

Further, it is compatible with base change $S' \to S$, where S' is a smooth k-variety.

The following result is the main step in the proof.

Proposition 1. (*Rigidity*). *Let X be a smooth complete variety over $\mathbf{C}$, and Y a smooth connected variety over $\mathbf{C}$. Let $\mathcal{E}$ be a locally free $\mathcal{O}_{X\times Y}$-module of finite rank on $X \times Y$ which has a connection relative to the projection $p_2 \colon X \times Y \to Y$. Then for any $i > 1$, the mapping*

$$c(i) : Y \to H^{2i}_{\mathcal{D}}(X_{an}, i), \qquad y \mapsto c_i^{\mathcal{D}}((\mathcal{E} \otimes \mathbf{C}(y))_{an}),$$

is constant.

Proof. To simplify the notation, we drop the subscript '*an*.' Since any two points of Y lie on the image of a morphism from a connected smooth affine curve, we are reduced to the case where Y is an affine curve.

The map $c(i)$ has the following alternative description. One has the 'algebraic' Chern class $c_i^{CH}(\epsilon) = \xi_i \in CH^i(X \times Y)$, the Chow group of codimension i algebraic cycles on $X \times Y$ (see [5]), for example. The Chern class $c_i^{CH}(\mathcal{E} \otimes \mathbf{C}(y)) \in CH^i(X)$ is the image of ξ_i under the natural map

$$i_y^* \colon CH^i(X \times Y) \to CH^i(X),$$

where $i_y \colon X \to Y \times Y$ is $i_y(x) = (x, y)$. The map $c(i)$ is then given by

$$c(i)(y) = Cl_{\mathcal{D}}(i_y^*, \xi_i),$$

where

$$Cl_{\mathcal{D}} \colon CH^i(X) \to H^{2i}_{\mathcal{D}}(X, i)$$

is the cycle class map with values in Deligne-Beilinson cohomology. If we fix a base point $y_0 \in Y$, then the algebraic cycle $i_y^*(\xi_i) - i_{y_0}^*(\xi_i)$ is (co)homologous to 0 on X, and

$$c(i)(y) - c(i)(y_0) = Cl_{\mathcal{D}}(i_y^*(\xi_i)) - Cl_{\mathcal{D}}(i_y^*(\xi_i)) \in J^i(X),$$

the ith intermediate Jacobian of X; one property of the cycle class in Deligne-Beilinson cohomology is that this element of $J^i(X)$ is the image of $i_y^*(\xi_i) - i_{y_0}^*(\xi_i)$ under the *Abel-Jacobi mapping.*

Let $\overline{Y}$ be the projective smooth curve associated to Y, and let

$$\overline{\xi}_i \in CH^i(X \times \overline{Y})$$

be a preimage of ξ_i under the restriction map

$$CH^i(X \times \overline{Y}) \to CH^i(X \times Y).$$

Choose an algebraic cycle $\sum_j n_j Z_j$ representing ξ_i, and take $\overline{\xi}_i$ to be the class of $\Sigma_j n_j \overline{Z}_j$, where $\overline{Z}_j$ is the Zariski closure Z_j. Then the Abel-Jacobi map gives a map from zero cycles of degree 0 on $\overline{Y}$ to $J^i(X)$, by

$$\theta : \sum_j ((y_j) - (y_0)) \mapsto Cl_{\mathscr{D}} \left(\sum_j (i_y^*(\overline{\xi}_i) - i_{y_0}^*(\overline{\xi}_i)) \right) \in J^i(X),$$

whose value on $(y) - (y_0)$ is $c(i)(y) - c(i)(y_0)$ for $y \in Y$. The mapping θ clearly factors through the Jacobian of $\overline{Y}$, since $Cl_{\mathscr{D}}$ is well defined on rational equivalence classes, and so there is an induced mapping

$$[\overline{\xi}_i] : J(\overline{Y}) \to J^i(X).$$

We are reduced to proving this map is *constant.*

The mapping $[\overline{\xi}_i] : J(\overline{Y}) \to J^i(X)$ induced by the class $\overline{\xi}_i \in CH^i(X \times Y)$ is related to the topological cycle class of $\overline{\xi}_i$ in $H^{2i}(X \times \overline{Y}, \mathbf{Z}(i))$ in the following way (see Part One of the article [3] of Clemens and Griffiths). There is a Künneth component $\eta_i \in H^{2i-1}(X, \mathbf{Z}(i)) \otimes H^1(\overline{Y}, \mathbf{Z})$ of this topological cycle class (this Künneth component in fact depends only on ξ_i); its image in $H^{2i}(X \times \overline{Y}, \mathbf{C})$ lies in $F^i \cap \overline{F}^i$, where F^i is the Hodge filtration on $H^{2i}(X \times \overline{Y}, \mathbf{C})$. Under the isomorphism (Hodge theory)

$$F^i \cap \overline{F}^i \simeq H^i(X \times \overline{Y}, \Omega^i_{X \times \overline{Y}/\mathbf{C}}),$$

η_i is mapped to an element in the subspace

$$H^i(X, \Omega^{i-1}_{X/\mathbf{C}}) \otimes_{\mathbf{C}} H^0(\overline{Y}, \Omega^1_{\overline{Y}}) \oplus H^{i-1}(X, \Omega^i_{X/\mathbf{C}}) \otimes_{\mathbf{C}} H^1(\overline{Y}, \mathscr{O}_{\overline{Y}}),$$

and these two summands are the complex conjugates of each other. Hence we may write image $(\eta_i) = \mu_i + \overline{\mu}_i$ with

$$\mu_i \in H^i(X, \Omega^{i-1}_{X/\mathbf{C}}) \otimes_{\mathbf{C}} H^0(\overline{Y}, \Omega^1_{\overline{Y}}) \simeq \mathrm{Hom}_{\mathbf{C}}(H^1(\overline{Y}, \mathscr{O}_{\overline{Y}}), H^i(X, \Omega^{i-1}_{X/\mathbf{C}})),$$

and $\overline{\mu}_i$ is the complex conjugate of μ_i, since their sum is a real cohomology class. Similarly we may regard η_i as an element of

$$\mathrm{Hom}_{\mathbf{Z}}(H^1(\overline{Y}, \mathbf{Z}(1)), H^{2i-1}(X, \mathbf{Z}(i))).$$

This homomorphism is the mapping on lattices inducing the Abel-Jacobi map $[\bar{\xi}_i] : J(\overline{Y}) \to J^{2i-1}(X)$; the mapping μ_i, composed with the inclusion

$$H^i(X, \Omega^{i-1}_{X/\mathbf{C}}) \simeq F^{i-1} \cap \overline{F}^i \hookrightarrow H^{2i-1}(X, \mathbf{C})/F^i H^{2i}(X, \mathbf{C}),$$

is the corresponding map of $\mathbf{C}$-vector spaces.

The upshot of this is that we are reduced to proving that $\mu_i = 0$.

Since $A_Y(\mathscr{E}) = 0$,

$$A(\mathscr{E}) \in \ker(H^1(X \times Y, \mathscr{E}nd(\mathscr{E}) \otimes_{\mathscr{O}_{X\times Y}} \Omega^1_{X\times Y/\mathbf{C}}) \to H^1(X \times Y, \mathscr{E}nd(\mathscr{E}) \otimes_{\mathscr{O}_{X\times Y}} \Omega^1_{X\times Y/Y})).$$

Now the natural map

$$\Omega^1_{X\times Y/\mathbf{C}} \to \Omega^1_{X\times Y/Y}$$

induces an isomorphism

$$p_1^*\Omega^1_{X/\mathbf{C}} \simeq \Omega^1_{X\times Y/Y},$$

and similarly there is an isomorphism

$$p_2^*\Omega^1_{Y/\mathbf{C}} \simeq \Omega^1_{X\times Y/Y}.$$

This leads to a direct sum decomposition

$$\Omega^1_{X\times Y} \simeq p_1^*\Omega^1_{X/\mathbf{C}} \oplus p_2^*\Omega^1_{Y/\mathbf{C}};$$

there is a similar decomposition on $X \times \overline{Y}$. This yields a direct sum decomposition

$$\begin{aligned} &H^1(X \times Y, \mathscr{E}nd(\mathscr{E}) \otimes_{\mathscr{O}_{X\times Y}} \Omega^1_{X\times Y/\mathbf{C}}) \\ &\quad = H^1(X \times Y, \mathscr{E}nd(\mathscr{E}) \otimes_{\mathscr{O}_{X\times Y}} p_1^*\Omega^1_{X/\mathbf{C}}) \\ &\quad\quad \oplus H^1(X \times Y, \mathscr{E}nd(\mathscr{E}) \otimes_{\mathscr{O}_{X\times Y}} p_2^*\Omega^1_{Y/\mathbf{C}}) \end{aligned}$$

such that the Atiyah class $A(\mathscr{E})$ has components $A_Y(\mathscr{E}) = 0$ and $A_X(\mathscr{E})$ in the respective summands. Hence $A(\mathscr{E}) = A_X(\mathscr{E})$ lies in the subgroup

$$H^1(X \times Y, \mathscr{E}nd(\mathscr{E}) \otimes_{\mathscr{O}_{X\times Y}} p_2^*\Omega^1_{Y/\mathbf{C}}).$$

Since Y is a curve, $\Omega^i_{Y/\mathbf{C}} = 0$ for $i > 1$. Thus

$$M_i(\mathscr{E}) \in H^i(X \times Y, \mathscr{E}nd(\mathscr{E}) \otimes_{\mathscr{O}_{X\times Y}} \Omega^i_{X\times Y/\mathbf{C}})$$

lies in the subspace

$$H^i(X \times Y, \mathscr{E}nd(\mathscr{E}) \otimes_{\mathscr{O}_{X\times Y}} p_2^*\Omega^i_{Y/\mathbf{C}}) = 0 \quad \text{for } i > 1.$$

Hence the Newton classes $N_i(\mathscr{E})$ vanish for $i > 1$. This implies that the Chern class

$$c_i(\mathscr{E}) \in H^i(X \times Y, \Omega^i_{X\times Y/\mathbf{C}})$$

is a rational multiple of $N_1(\mathscr{E})^i$, where

$$N_1(\mathscr{E}) \in H^1(X \times Y, \Omega^1_{X\times Y/\mathbf{C}}).$$

But again $N_1(\mathscr{E})$ lies in the subspace

$$H^1(X \times Y, p_2^*\Omega^1_{Y/\mathbf{C}}),$$

so

$$N_1(\mathscr{E})^i \in H^i(X \times Y, p_2^*\Omega^i_{Y/\mathbf{C}}) = 0 \quad \text{for } i > 1.$$

We observe that the restriction map

$$H^i(X \times \overline{Y}, \Omega^i_{X\times\overline{Y}/\mathbf{C}}) \to H^i(X \times Y, \Omega^i_{X\times Y/\mathbf{C}})$$

respects the decompositions

$$\begin{aligned} H^i(X \times \overline{Y}, \Omega^i_{X\times\overline{Y}/\mathbf{C}}) &= H^i(X \times \overline{Y}, p_1^*\Omega^i_{X/\mathbf{C}}) \\ &\quad \oplus H^i(X \times \overline{Y}, p_1^*\Omega^{i-1}_{X/\mathbf{C}} \otimes_{\mathscr{O}_{X\times\overline{Y}}} p_2^*\Omega^1_{\overline{Y}/\mathbf{C}}), \\ H^i(X \times Y, \Omega^i_{X\times Y/\mathbf{C}}) &= H^i(X \times \overline{Y}, p_1^*\Omega^i_{X/\mathbf{C}}) \\ &\quad \oplus H^i(X \times Y, p_1^*\Omega^{i-1}_{X/\mathbf{C}} \otimes_{\mathscr{O}_{X\times Y}} p_2^*\Omega^1_{Y/\mathbf{C}}). \end{aligned}$$

The summands

$$H^i(X \times \overline{Y}, p_1^*\Omega^i_{X/\mathbf{C}}), \;\; H^i(X \times \overline{Y}, p_1^*\Omega^{i-1}_{X/\mathbf{C}} \otimes_{\mathscr{O}_{X\times\overline{Y}}} p_2^*\Omega^1_{\overline{Y}/\mathbf{C}})$$

further decompose respectively as

$$H^i(X, \Omega^i_{X/\mathbf{C}}) \otimes H^0(\overline{Y}, \mathscr{O}_{\overline{Y}}) \oplus H^{i-1}(X, \Omega^i_{X/\mathbf{C}}) \otimes H^1(\overline{Y}, \mathscr{O}_{\overline{Y}}),$$

$$H^i(X, \Omega^{i-1}_{X/\mathbf{C}} \otimes H^0(\overline{Y}, \Omega^1_{\overline{Y}/\mathbf{C}}) \oplus H^{i-1}(X, \Omega^{i-1}_{X/\mathbf{C}}) \otimes H^1(\overline{Y}, \Omega^1_{\overline{Y}}).$$

Thus

$$Cl(\overline{\xi}_i) \in H^i(X \times Y, \Omega^i_{X\times Y/\mathbf{C}})$$

is a sum of four components, two of which are μ_i and $\overline{\mu}_i$; in particular μ_i is the component in

$$H^i(X, \Omega^{i-1}_{X/\mathbf{C}}) \otimes H^0(\overline{Y}, \Omega^1_{\overline{Y}/\mathbf{C}}).$$

The restriction map

$$H^i(X \times \overline{Y}, p_1^*\Omega^{i-1}_{X/\mathbf{C}} \otimes_{\mathscr{O}_{X\times\overline{Y}}} p_2^*\Omega^1_{\overline{Y}/\mathbf{C}}) \to H^i(X \times Y, p_1^*\Omega^{i-1}_{X/\mathbf{C}} \otimes p_2^*\Omega^1_{Y/\mathbf{C}})$$

is injective on the summand

$$H^i(X, \Omega^{i-1}_{X/\mathbf{C}}) \otimes H^0(\overline{Y}, \Omega^1_{\overline{Y}/\mathbf{C}})$$

of the domain, and vanishes on the other summand. Since the restriction of $Cl(\overline{\xi}_i)$ vanishes for $i > 1$, μ_i restricts to 0 on $X \times Y$, i.e., $\mu_i = 0$, as desired. q.e.d.

We now give a short alternative proof using the construction of the Deligne-Beilinson cohomology $H^j_{\mathscr{D}}(i)$ of open smooth complex varieties (see [4]) but not using [3].

Alternative proof. As before, we reduce our proof to the case where Y is a smooth, connected affine curve, so that the Chern classes $c_i(\mathscr{E}) \in H^i(X \times Y, \Omega^i_{X\times Y})$ vanish for $i \geq 2$. There is an exact sequence

$$\begin{aligned} 0 \to H^{2i-1}(X \times Y, \mathbf{C}/Z(i))/F^iH^{2i-1}(X \times Y) \\ \to H^{2i}_{\mathscr{D}}(X \times Y, i) \to F^i_{\mathbf{Z}(i)}H^{2i}(X \times Y) \to 0, \end{aligned}$$

where

$$F^j_{\mathbf{Z}(l)}H^i := \{\omega \in F^jH^i \text{ such that image of } \omega \text{ vanishes in } H^i(C/\mathbf{Z}(l))\}.$$

One has an exact sequence

$$0 \to F^{i+1}_{\mathbf{Z}(i)}H^{2i}(X \times Y) \to F^i_{\mathbf{Z}(i)}H^{2i}(X \times Y) \to H^i(X \times \overline{Y}, \Omega^i_{X\times\overline{Y}}(\log D)),$$

where $\overline{Y}$ is as above, $D = X \times \{\infty\}$ with $\{\infty\} := \overline{Y} - Y$, and

$$\begin{aligned} &H^i(X \times \overline{Y}, \Omega^i_{X\times\overline{Y}}(\log D)) \\ &= H^i(X, \Omega^i_X) \otimes H^0(\overline{Y}, \mathscr{O}_{\overline{Y}}) \oplus H^i(X, \Omega^{i-1}_X) \otimes H^0(\overline{Y}, \Omega^1_{\overline{Y}}(\log\{\infty\})) \\ &\quad \oplus H^{i-1}(X, \Omega^i_X) \otimes H^1(\overline{Y}, \mathscr{O}_{\overline{Y}}). \end{aligned}$$

(Since Y is an affine curve, $H^1(\overline{Y}, \Omega^1_{\overline{Y}}(\log\{\infty\})) = 0$.) In this decomposition the image of $c_i^{\mathscr{D}}(\mathscr{E}) \in H^{2i}_{\mathscr{D}}(X\times Y, i)$ is written as $a_{i,i}+a_{i-1,i}+a_{i,i-1}$. From the vanishing image of $c_i^{\mathscr{D}}(\mathscr{E})$ in

$$\begin{aligned} H^i(X \times Y, \Omega^i_{X\times Y}) = H^i(X, \Omega^i_X) \otimes H^0(Y, \mathscr{O}_Y) \\ \oplus H^i(X, \Omega^{i-1}_X) \otimes H^0(Y, \Omega^1_Y), \end{aligned}$$

and from the injectivities of $H^0(\overline{Y}, \mathscr{O}_{\overline{Y}})$ and $H^0(\overline{Y}, \Omega^1_{\overline{Y}}(\log\{\infty\}))$ respectively in $H^0(Y, \mathscr{O}_Y)$ and $H^0(Y, \Omega^1_Y)$, it follows that $a_{i,i} = a_{i-1,i} = 0$.

As $a_{i-1,i} = a_{i,i-1}$, where the dual space to $H^1(\overline{Y}, \mathcal{O}_{\overline{Y}})$ is $H^0(\overline{Y}, \Omega^1_{\overline{Y}})$ in $H^0(\overline{Y}, \Omega^1_{\overline{Y}}(\log\{\infty\}))$, one obtains that $c_i^{\mathcal{D}}(\mathcal{E})$ maps to $F^{i+1}_{\mathbf{Z}(i)}H^{2i}(X \times Y)$. As $F^{i+1}_{\mathbf{Z}(i)}H^{2i}(X \times \overline{Y}) = 0$, one has $F^{i+1}_{\mathbf{Z}(i)}H^{2i}(X \times Y) \hookrightarrow F^{i}_{\mathbf{Z}(i-1)}H^{2i-1}(X \times \{\infty\}) = 0$ via the Gysin sequence. Therefore $c_i^{\mathcal{D}}(\mathcal{E})$ comes from a class $\gamma_i \in H^{2i-1}(X \times Y, \mathbf{C}/Z(i))$, with $\gamma_i = \alpha_i + \beta_i$, $\alpha_i \in H^{2i-1}(X, \mathbf{C}/Z(i)) \otimes H^0(Y, \mathbf{Z})$, $\beta_i \in H^{2i-2}(X, \mathbf{C}/Z(i)) \otimes H^1(Y, \mathbf{Z})$, and one has

$$\begin{aligned} c_i^{\mathcal{D}}(\mathcal{E}|_{X\times\{y\}}) &= c_i^{\mathcal{D}}(\mathcal{E})|_{X\times\{y\}} \text{ via the morphism} \\ &\quad H^{2i}_{\mathcal{D}}(X \times Y, i) \to H^{2i}_{\mathcal{D}}(X \times \{y\}, i) \\ &= \text{image } (\gamma_i|X \times \{y\}) \text{ via the morphism} \\ &\quad H^{2i-1}(X \times Y, C/\mathbf{Z}(i)) \to H^{2i}_{\mathcal{D}}(X \times Y, i) \\ &= \text{image}(\alpha_i|X \times \{y\}) \text{ via the morphism} \\ &\quad H^{2i-1}(X \times \{y\}, \mathbf{C}/\mathbf{Z}(i)) \to H^{2i}_{\mathcal{D}}(X \times \{y\}, i). \end{aligned}$$

The class of α_i is constant as desired. q.e.d.

The proof of the Theorem is now completed by a routine argument. Let $k \subset \mathbf{C}$ be a countable algebraically closed field of definition for X. Let X_0 be a model of X over k, i.e., a smooth complete k-variety with $X_0 \times_{\operatorname{Spec} k} \operatorname{Spec} \mathbf{C} = X$. First note that, up to isomorphism, there is only a countable number of locally free $\mathcal{O}_{X_0}$-modules $\mathcal{E}_0$ which have an algebraic connection over k. This is because there are in fact only countably many locally free $\mathcal{O}_{X_0}$-modules up to isomorphism over k (cover X_0 by infinitely many affine $\operatorname{Spec} A_i$; there are only countably many projective A_i-modules up to isomorphism for each i, and only countably many possibilities for transition matrices).

Each locally free sheaf $\mathcal{E}_0$ defined over k and carrying a connection yields a locally free $\mathcal{O}_X$-module $\mathcal{E}$ by extension of scalars. Clearly there are only countably many classes $c_i^{\mathcal{D}}(\mathcal{E})$ with $\mathcal{E}$ of this special form.

By the rigidity result, it then suffices to prove that if $\mathcal{F}$ is any locally free $\mathcal{O}_X$-module with a connection, there exist the following:

(i) a connected smooth variety Y_0 defined over k, and the corresponding complex variety $Y = (Y_0)_{\mathbf{C}}$,

(ii) a locally free $\mathcal{O}_{X_0 \times Y_0}$-module $\mathcal{G}_0$ with a connection ∇_0 relative to Y_0, and the corresponding objects $\mathcal{G}$, ∇ over $\mathbf{C}$, and

(iii) a closed point $y \in Y$ such that $(\mathcal{G}, \nabla) \otimes \mathbf{C}(y)$ is isomorphic to the given locally free sheaf $\mathcal{F}$ with its given connection.

Given this data, the Chern class $c_i^{\mathscr{D}}(\mathscr{F})$ equals $c_i^{\mathscr{D}}(\mathscr{E})$, where $y_0 \in Y_0$ is a closed point, regarded in a natural way as a $\mathbf{C}$-point of Y, and $\mathscr{E} = \mathscr{G} \otimes \mathbf{C}(y_0)$. Then $\mathscr{E}_0 = \mathscr{G}_0 \otimes k(y_0)$ is a locally free sheaf defined over k with a connection, and $(\mathscr{E}_0)_{\mathbf{C}} = \mathscr{E}$. This would prove the Theorem.

To make the claimed construction, note that $\mathscr{F}$ and its given connection are defined over a finitely generated k-subalgebra K of $\mathbf{C}$. Let $\mathscr{F}_K$ and ∇_K be corresponding objects over K. Let Y_0 be a smooth k-variety with function field K; then X_K is the generic fiber of the proper and smooth morphism $X_0 \times_k Y_0 \to Y_0$. By replacing Y_0 by an open subset if necessary, we may further assume that there exists a locally free sheaf $\mathscr{G}_0$ on $X_0 \times Y_0$, with a connection relative to Y_0, whose restriction to the generic fiber over Y_0 is $\mathscr{F}_K$, and with the connection ∇_K (to verify that the connection extends to an open set, one may think of it as a splitting of the jet sequence (*)). The given embedding $K \subset \mathbf{C}$ determines a closed point $y \in Y$, such that y maps to the generic point of Y_0. If $(\mathscr{G}, \nabla)$ is the locally free sheaf with a connection relative to Y obtained on $X \times Y$, then $(\mathscr{G}, \nabla) \otimes \mathbf{C}(y) \simeq (\mathscr{F}_K, \nabla_K) \otimes_K \mathbf{C}$, which by choice is the sheaf $\mathscr{F}$ with its given connection. Hence the proof is complete.

References

[1] M. F. Atiyah, *Complex analytic connections in fibre bundles*, Trans. Amer. Math. Soc. **85** (1957) 181–207.

[2] S. Bloch, *Applications of the dilogarithm function in algebraic K-theory and algebraic geometry*, Proc. Internat. Sympos. Algebraic Geometry (Kyoto, 1977), Kinokuniya, Tokyo, 1978.

[3] H. Clemens & P. A. Griffiths, *The intermediate Jacobian of the cubic threefold* Ann. of Math. **95** (1972) 281–356.

[4] H. Esnault & E. Viehweg, *Deligne-Beilinson cohomology*, Beilinson's conjectures on special values of L-functions, (M. Rapaport, P. Schneider & N. Schappacher, eds.), Academic Press, New York, 1988.

[5] W. Fulton, *Intersection theory*, Ergebnisse Math. Grenzgeb. (3), B. 2, Springer, Berlin, 1984.

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