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THE STABILITY OF SOME EIGENVALUE ESTIMATES

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1. The Faber-Krahn inequality ([7], [8], [13]) states that among all bounded domains $\Omega \subseteq \mathbb{R}^n$ with the same volume the ball has the smallest first Dirichlet eigenvalue.

Also recently it has been proved [1] that the ratio $\lambda_2(\Omega)/\lambda_1(\Omega)$ of the first two Dirichlet eigenvalues of a normal bounded domain $\Omega \subseteq \mathbb{R}^n$ takes its maximum value if and only if Ω is a ball.

In this work we examine how stable these inequalities are. That means whether a bounded domain $\Omega \subseteq \mathbb{R}^n$ has to be near the ball in the sense of Hausdorff distance provided that one of the two quantities $\lambda_1(\Omega)|\Omega|^{-2/n}$ and $\lambda_2(\Omega)/\lambda_1(\Omega)$ is sufficiently near to the corresponding quantity for the ball, where $|\Omega|$ denotes the volume of Ω. We prove that this is true under the additional assumption that Ω is convex.

We prove the stability for the Faber-Krahn inequality for convex domains in §2, and for the inequality for the ratio of the first two Dirichlet eigenvalues for convex domains in §3. Actually an estimate for the Hausdorff distance of the domain and a ball can be derived in terms of how near one of the above quantities is to the corresponding quantity for the ball. In §4 we give an extension of the stability of the Faber-Krahn inequality for arbitrary bounded domains in $\mathbb{R}^2$.

Notation. For a bounded (normal) domain $\Omega \subseteq \mathbb{R}^n$, $\lambda_1(\Omega)$ and $\lambda_2(\Omega)$ denote the first two Dirichlet eigenvalues of Ω. For a measurable set $E \subseteq \mathbb{R}^n$, $|E|$ denotes its n-dimensional Lebesgue measure.

2. Theorem 2.1. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded convex domain such that*

$$\lambda_1(\Omega) \leq (1+\varepsilon)\lambda_1(D), \tag{2.1}$$

where $\varepsilon > 0$ is sufficiently small, and D is a ball with $|D| = |\Omega|$. Then there exist two balls $B_1, B_2 \subseteq \mathbb{R}^n$ such that $B_1 \subseteq \Omega \subseteq B_2$ and

$$|B_1| \geq (1 - C_n\varepsilon^{1/2n})|\Omega|, \qquad |\Omega| \geq (1 - C_n\varepsilon^{1/2n})|B_2|, \tag{2.2}$$

where $C_n > 0$ is a constant depending only on the dimension n.

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Proof. Assume $\Omega \subseteq \mathbb{R}^n$ satisfies the hypothesis of Theorem 2.1. Let $u_1 > 0$ be the first Dirichlet eigenfunction of Ω normalized so that $\int_\Omega u_1^2(x)\,dx = 1$. For $\delta > 0$ we define

$$\Omega_\delta = \{x \in \Omega : u_1(x) > \delta\}. \tag{2.3}$$

Since Ω is convex by [2], each Ω_δ is convex. We need:

Lemma 2.1. *For any δ such that $0 < \delta < \frac{1}{2}|\Omega|^{-1/2}$ we have*

$$|\Omega_\delta| \geq [1 - 2n\max(\delta|\Omega|^{1/2}, \varepsilon)]|\Omega|. \tag{2.4}$$

Proof of Lemma 2.1. Since the function $u_1 - \delta$ is C^2 and vanishes on $\partial\Omega_\delta$ by Rayleigh's theorem,

$$\lambda_1(\Omega_6) \leq \frac{\int_{\Omega_\delta} |\nabla(u_1(x) - \delta)|^2\,dx}{\int_{\Omega_\delta}(u_1(x) - \delta)^2\,dx}. \tag{2.5}$$

But

$$\begin{aligned}
\int_{\Omega_\delta} |\nabla(u_1(x) - \delta)|^2\,dx &= -\int_{\Omega_\delta} (u_1(x) - \delta)\Delta u_1(x)\,dx \\
&= \lambda_1(\Omega)\int_{\Omega_\delta} (u_1(x) - \delta)u_1(x)\,dx \\
&\leq \lambda_1(\Omega)\left(\int_{\Omega_\delta} (u_1(x) - \delta)^2\,dx\right)^{1/2}\left(\int_{\Omega_\delta} u_1^2(x)\,dx\right)^{1/2} \\
&\leq \lambda_1(\Omega)\left(\int_{\Omega_\delta} (u_1(x) - \delta)^2\,dx\right)^{1/2} \quad \text{since } \int_\Omega u_1^2(x)\,dx = 1.
\end{aligned}$$

Since $u_1 \leq \delta$ in $\Omega\backslash\Omega_\delta$ and $\delta|\Omega|^{1/2} < \frac{1}{2}$, by Minkowski's inequality we obtain

$$\begin{aligned}
\left(\int_{\Omega_\delta} (u_1(x) - \delta)^2\,dx\right)^{1/2} &\geq \left(\int_{\Omega_\delta} u_1^2(x)\,dx\right)^{1/2} - \left(\int_{\Omega_\delta} \delta^2\,dx\right)^{1/2} \\
&\geq \left(1 - \int_{\Omega\backslash\Omega_\delta} \delta^2\,dx\right)^{1/2} - \delta|\Omega| \geq 1 - 2\delta|\Omega|^{1/2}.
\end{aligned}$$

Thus (2.5) gives

$$\lambda_1(\Omega_\delta) \leq (1 - 2\delta|\Omega|^{1/2})^{-1}\lambda_1(\Omega). \tag{2.6}$$

If D_δ is a ball with $|D_\delta| = |\Omega_\delta|$, then by Faber-Krahn's inequality ([7], [8], [13]) (2.1) and (2.6) we have

$$\lambda_1(D_\delta) \leq \lambda_1(\Omega_\delta) \leq (1 - 2\delta|\Omega|^{1/2})^{-1}(1 + \varepsilon)\lambda_1(D). \tag{2.7}$$

Hence

$$\frac{|\Omega_\delta|}{|\Omega|} = \left[\frac{\lambda_1(D)}{\lambda_1(D_\delta)}\right]^{n/2} \geq \left(\frac{1-2\delta|\Omega|^{1/2}}{1+\varepsilon}\right)^{n/2}$$
$$\geq [1 - 2n\max(\delta|\Omega|^{1/2}, \varepsilon)] \quad \text{assuming } 0<\varepsilon<1. \quad \text{q.e.d.}$$

We may without loss of the generality assume that $|\Omega| = 1$. Let u_1^* defined on D be the decreasing spherical symmetrization of u_1. Let $\Gamma(t) = \{x \in \Omega : u_1(x) = t\}$, $\Gamma^*(t) = \{x \in D : u_1^*(x) = t\}$, $T = \sup_\Omega u_1$, and $\psi(t) = \int_{\Gamma(t)} \frac{1}{|\nabla u_1|} dH_{n-1}$ for $0<t<T$, where H_{n-1} denotes $(n-1)$-dimensional Hausdorff measure. Then

$$H_{n-1}(\Gamma(t))^2 \leq \psi(t) \int_{\Gamma(t)} |\nabla u_1|\, dH_{n-1},$$

and, by the isoperimetric inequality, $H_{n-1}(\Gamma^*(t)) \leq H_{n-1}(\Gamma(t))$. Thus as in the proof of Faber-Krahn's inequality we have

$$\begin{aligned}\lambda_1(\Omega) &= \int_\Omega |\nabla u_1(x)|^2\,dx = \int_0^T \int_{\Gamma(t)} |\nabla u_1|\,dH_{n-1}\,dt \\ &\geq \int_0^T H_{n-1}(\Gamma(t))^2 \frac{1}{\psi(t)}\,dt \geq \int_0^T H_{n-1}(\Gamma^*(t))^2 \frac{1}{\psi(t)}\,dt \\ &= \int_D |\nabla u_1^*(x)|^2\,dx = \lambda_1(D).\end{aligned}$$

Since $\lambda_1(\Omega) \leq (1+\varepsilon)\lambda_1(D)$,

$$\int_0^T [H_{n-1}(\Gamma(t))^2 - H_{n-1}(\Gamma^*(t))^2] \frac{1}{\psi(t)}\,dt \leq \lambda_1(D)\varepsilon. \tag{2.8}$$

Assuming $\varepsilon < 1/4$ we may take $\delta = \varepsilon^{1/2}$ in Lemma 2.1 and obtain

$$|\Omega \backslash \Omega_\delta| \leq 2n\varepsilon^{1/2}|\Omega| = 2n\varepsilon^{1/2}. \tag{2.9}$$

Thus by Cauchy-Schwarz's inequality we have

$$\begin{aligned}\varepsilon = \delta^2 &= \left(\int_0^\delta dt\right)^2 \leq \left(\int_0^\delta \psi(t)^{-1}\,dt\right)\left(\int_0^\delta \psi(t)\,dt\right) \\ &= \left(\int_0^\delta \psi(t)^{-1}\,dt\right)|\Omega\backslash\Omega_\delta| \leq 2\eta\varepsilon^{1/2}\int_0^\delta \psi(t)^{-1}\,dt,\end{aligned}$$

and therefore

$$\int_0^\delta \frac{1}{\psi(t)}\,dt \geq \frac{1}{2n}\varepsilon^{1/2}. \tag{2.10}$$

From (2.8) and (2.9) it follows that

$$\begin{aligned}&\inf_{0\le t\le\delta}[H_{n-1}(\Gamma(t))^2 - H_{n-1}(\Gamma^*(t))^2]\\ &\qquad\le 2n\varepsilon^{-1/2}\int_0^\delta [H_{n-1}(\Gamma(t))^2 - H_{n-1}(\Gamma^*(t))^2]\frac{1}{\psi(t)}\,dt\\ &\qquad\le 2n\lambda_1(D)\varepsilon^{1/2} = C'\varepsilon^{1/2},\end{aligned}$$

where C' depends only on n. Moreover $\Gamma(t)$ is the boundary of Ω_t, and $\Gamma^*(t)$ is the boundary of a ball with volume $|\Omega_t|$. Hence there exists a τ such that $0\le\tau\le\delta$. If w_n is the volume of the unit ball in $\mathbb{R}^n$, then

$$H_{n-1}(\partial\Omega_\tau)\le nw_n^{1/n}|\Omega_\tau|^{1-1/n} + C\varepsilon^{1/2}, \tag{2.11}$$

where ε is sufficiently small, and C depends only on n, since for ε small enough Lemma 2.1 implies that $|\Omega_\tau|\ge 1/2$. Let r be the radius of a disc with volume equal to $|\Omega_\tau|$, let ρ be the in radius of Ω_τ, and let B_1 be a ball of radius ρ with $B_1\subseteq\Omega_\tau$.

Since Ω_τ is convex, we have the following isoperimetric inequality [D, H, 0] of Bonnesen style:

$$\left(\frac{H_{n-1}(\partial\Omega_\tau)}{H_{n-1}(\partial B_1)}\right)^{n/n-1} - \frac{|\Omega_\tau|}{|B_1|}\ge\left[\left(\frac{H_{n-1}(\partial\Omega_\tau)}{H_{n-1}(\partial B_1)}\right)^{1/n-1} - 1\right]^n. \tag{2.12}$$

Using (2.11) and the isoperimetric inequality $H_{n-1}(\partial\Omega_\tau)\ge nw_n^{1/n}|\Omega_\tau|^{1-1/n}$ we obtain

$$(r-\rho)^n\le(r^{n-1}+C\varepsilon^{1/2})^{n/n-1} - r^n. \tag{2.13}$$

Since $1/2\le|\Omega_\tau|\le 1$ for sufficiently small ε, (2.13) implies

$$(r-\rho)\le C\varepsilon^{1/2n}, \tag{2.14}$$

where C depends only on n. Hence

$$|B_1|\ge(1-C'\varepsilon^{1/2n})|\Omega_\tau|\ge(1-C'\varepsilon^{1/2n})(1-2n\varepsilon^{1/2})|\Omega|,$$

where $\tau\le\delta=\varepsilon^{1/2}$, and C' denotes only on n. Also $B_1\subseteq\Omega_\tau\subseteq\Omega$. Since Ω is convex, the existence of B_2 follows from that of B_1.

3. Let τ_n denote the ratio $\lambda_2(D)/\lambda_1(D)$, where D is an n-dimensional ball.

Theorem 3.1. *Let $\Omega\subseteq\mathbb{R}^n$ be a bounded convex domain such that*

$$\frac{\lambda_2(\Omega)}{\lambda_1(\Omega)}\ge\tau_n-\varepsilon, \tag{3.1}$$

where $\varepsilon > 0$ *is sufficiently small. Then there exist two balls* $B_1, B_2 \subseteq \mathbb{R}^n$ *such that* $B_1 \subseteq \Omega \subseteq B_2$ *and*

$$|B_1| \geq (1 - C_n\varepsilon^{a_n})|\Omega|, \qquad |\Omega| \geq (1 - C_n\varepsilon^{a_n})|B_2|, \tag{3.2}$$

where $C_n > 0$ *and* $0 < a_n < 1$ *are constants depending only on* n.

For the proof we need the following:

Proposition 3.1. *Let* $\theta > 0$ *be given. Then there exists a constant* $C_{n,\theta} > 0$ *depending only on* n *and* θ *such that if* $\Omega \subseteq \mathbb{R}^n$ *is a bounded convex domain such that* $\lambda_2(\Omega) \geq (1+\theta)\lambda_1(\Omega)$, *then* $\lambda_1(\Omega) \leq C_{n,\theta}|\Omega|^{-2/n}$.

Before we can give the proof of the proposition we need the following lemmas:

Lemma 3.1. *Assume* Ω *is a domain, and* $u_1 > 0$ *is the first Dirichlet eigenfunction of* Ω *normalized so that* $\int_\Omega u_1^2(x)\,dx = 1$. *For* $0 < s < \sup_\Omega u_1$ *we define*

$$\Omega_s = \{x \in \Omega : u_1(x) > s\} \quad \text{and} \quad \kappa(s) = \frac{\lambda_1(\Omega_s) - \lambda_1(\Omega)}{\lambda_1(\Omega)}.$$

Then for all $0 < s < \sup_\Omega u_1$ *we have*

$$s^2|\Omega_s| \geq \left(\frac{\kappa(s)}{1+\kappa(s)}\right)^2 \int_{\Omega_s} u_1^2(x)\,dx. \tag{3.3}$$

Proof. We may assume that Ω_s is a normal domain. Since the function $u_1 - s$ vanishes on $\partial\Omega_s$,

$$\begin{aligned}\lambda_1(\Omega_s) &\leq \frac{\int_{\Omega_s} |\nabla(u_1(x) - s)|^2\,dx}{\int_{\Omega_s}(u_1(x) - s)^2\,dx} = \lambda_1(\Omega)\frac{\int_{\Omega_s} u_1(x)(u_1(x) - s)\,dx}{\int_{\Omega_s}(u_1(x) - s)^2\,dx} \\ &\leq \lambda_1(\Omega)\frac{(\int_{\Omega_s} u_1^2(x)\,dx)^{1/2}}{(\int_{\Omega_s}(u_1(x) - s)^2\,dx)^{1/2}}.\end{aligned}$$

Since $\lambda_1(\Omega_s) = (1 + \kappa(s))\lambda_1(\Omega)$, from the Minkowski's inequality

$$\left(\int_{\Omega_s}(u_1(x) - s)^2\,dx\right)^{1/2} \geq \left(\int_{\Omega_s} u_1^2(x)\,dx\right)^{1/2} - s|\Omega_s|^{1/2},$$

inequality (3.3) follows.

Lemma 3.2. *Let* $\Omega \subseteq \mathbb{R}^n$ *be a bounded convex domain with* $0 \in \overline{\Omega} \subseteq \Sigma_d$, *where* $\Sigma_d = \{(x_1, \cdots, x_n) \in \mathbb{R}^n : 0 \leq x_1 \leq d\}$. *For* $0 < \eta < d/6$ *we define* $\Omega^\eta = \Omega \cap \operatorname{int}\Sigma_{d-\eta}$. *Then*

$$\lambda_1(\Omega^\eta) \leq \left(1 + \frac{3\eta}{d}\right)\lambda_1(\Omega). \tag{3.4}$$

Proof. Since $0 \in \overline{\Omega} \subseteq \Sigma_d$ and Ω is convex, we conclude that

$$\left(1 - \frac{\eta}{d}\right)\Omega = \left(1 - \frac{\eta}{d}\right)\Omega + \frac{\eta}{d}0 \subseteq \Omega \cap \operatorname{int}\Sigma_{d-\eta} = \Omega^\eta,$$

so that

$$\lambda_1(\Omega^\eta) \leq \lambda_1\left(\left(1 - \frac{\eta}{d}\right)\Omega\right) = \left(1 - \frac{\eta}{d}\right)^{-2}\lambda_1(\Omega) \leq \left(1 + \frac{3\eta}{d}\right)\lambda_1(\Omega)$$

be the monotonicity of the first eigenvalue and the inequality $\eta/d < 1/6$.

Lemma 3.3. *If $\Omega \subseteq \mathbb{R}^n$ is a bounded normal domain, and Ω_1, Ω_2 are disjoint normal subdomains of Ω, then*

$$\lambda_2(\Omega) \leq \max\{\lambda_1(\Omega_1), \lambda_1(\Omega_2)\}. \tag{3.5}$$

Proof. This follows by a standard variational argument as in the proof of Courant's nodal domain theorem.

Proof of Proposition 3.1. Without loss of the generality we may assume that $\operatorname{diam}\Omega = 1$, $\Omega \subseteq \Sigma_1 = \{(x_1, \cdots, x_n) \in \mathbb{R}^n : 0 \leq x_1 \leq 1\}$, and 0, $(1, 0, \cdots, 0) \in \overline{\Omega}$. Let $u_1 > 0$ be the first Dirichlet eigenfunction of Ω normalized so that $\int_\Omega u_1^2(x)\,dx = 1$. Let $\alpha = \int_\Omega x_1 u_1^2(x)\,dx > 0$. Since $\int_\Omega (x_1 - \alpha)u_1^2(x)\,dx = 0$ by the Rayleigh-Ritz inequality for λ_2, we have

$$\lambda_2(\Omega) - \lambda_1(\Omega) \leq \frac{\int_\Omega |\nabla(x_1 - \alpha)|^2 u_1^2(x)\,dx}{\int_\Omega (x_1 - a)^2 u_1^2(x)\,dx} = \frac{1}{\int_\Omega (x_1 - \alpha)^2 u_1^2(x)\,dx}.$$

Using the assumption $\lambda_2(\Omega) \geq (1 + \theta)\lambda_1(\Omega)$ therefore yields

$$\lambda_1(\Omega) \leq \left(\theta \int_\Omega (x_1 - \alpha)^2 u_1^2(x)\,dx\right)^{-1}. \tag{3.6}$$

Since $0 < \alpha < \int_\Omega u_1^2(x)\,dx = 1$ and $0, (1, 0, \cdots, 0) \in \overline{\Omega}$, without loss of the generality we may assume that $\alpha \geq 1/2$. For $0 < s < \sup_\Omega u_1$ define $\Omega_s = \{x \in \Omega : u_1(x) > s\}$ and

$$\eta(s) = \inf\{\eta > 0 : \Omega_s \subseteq \{(x_1, \cdots, x_n) \in \mathbb{R}^n : \alpha - \eta \leq x_1 \leq \alpha + \eta\}\}.$$

By Lemma 3.2 we obtain, for $d = \alpha + \eta(s)$ and $\eta = 2\eta(s)$,

$$\lambda_1(\Omega^{\alpha - \eta(s)}) \leq (1 + 12\eta(s))\lambda_1(\Omega^{\alpha + \eta(s)}) \leq (1 + 12\eta(s))\lambda_1(\Omega_s),$$

as long as $0 < \eta(s) < 1/24$, since $\Omega_s \subseteq \Omega^{\alpha + \eta(s)}$ by the definition of $\eta(s)$. But $\Omega^{\alpha - \eta(s)}$ and Ω_s are disjoint normal subdomains of Ω; hence by Lemma 3.3 we have

$$\lambda_2(\Omega) \leq \max\{\lambda_1(\Omega_s), \lambda_1(\Omega^{\alpha - \eta(s)})\} \leq (1 + 12\eta(s))\lambda_1(\Omega_s)$$

if $\eta(s) < 1/24$. Since $(1+\theta)\lambda_1(\Omega) \leq \lambda_2(\Omega)$, using the notation of Lemma 3.1 gives that

$$1+\theta \leq (1+12\eta(s))(1+\kappa(s)) \quad \text{if } 0 < \eta(s) < \tfrac{1}{24}.$$

Thus there exists $c_1 > 0$ depending only on θ such that

$$\eta(s) \geq c_1 \quad \text{whenever } \kappa(s) \leq \tfrac{1}{2}\theta, \tag{3.7}$$

which implies, by the definition of $\eta(s)$,

$$\overline{\Omega}_s \cap \{(x_1, \cdots, x_n) \in \Sigma_1 : |x_1 - \alpha| \geq c_1\} \neq \varnothing, \quad \text{whenever } \kappa(s) \leq \tfrac{1}{2}\theta.$$

Since Ω is convex by [2], we conclude that each Ω_s is convex. Hence there exists $c_2 > 0$ depending only on n (in fact we may take $c_2 = 4^{-n}$) such that if

$$\Omega_s' = \{x \in \Omega_s : |x_j - \alpha| > \tfrac{1}{2}c_1\},$$

then

$$|\Omega_s'| \geq c_2|\Omega_s| \quad \text{whenever } \kappa(s) \leq \tfrac{1}{2}\theta. \tag{3.7$'$}$$

Now we have

$$\begin{aligned}\int_\Omega (x_1-\alpha)^2 u_1^2(x)\,dx &\geq \frac{c_1^2}{4}\int_{\Omega\cap\{x\in\mathbb{R}^n : |x_1-\alpha|>c_1/2\}} u_1^2(x)\,dx\\ &= \frac{c_1^2}{4}\int_{\Omega_0'} u_1^2(x)\,dx - \frac{c_1^2}{4}\int_0^{\sup_{\Omega_0'} u_1} 2t|\Omega_t'|\,dt\\ &\geq \frac{c_1^2}{4}\int_0^s 2tc_2|\Omega_t|\,dt = \frac{c_1^2 c_2}{4} I_s,\end{aligned}$$

whenever s is such that $0 < s \leq \sup_{\Omega_0'} u_1$, and $\kappa(s) \leq \theta/2$, where we have defined

$$I_s = \int_0^s 2t|\Omega_t|\,dt. \tag{3.8}$$

But $s > \sup_{\Omega_0'} u_1$ implies that $\eta(s) \leq c_1$, so that $\kappa(s) > \frac{1}{2}\theta$ by (3.7). Hence we have

$$\int_\Omega (x_1-\alpha)^2 u_1^2(x)\,dx \geq c_3 I_s \quad \text{whenever } \kappa(s) \leq \tfrac{1}{2}\theta, \tag{3.9}$$

where $c_3 > 0$ depends only on θ on n.

Since $\Omega_t \leq \Omega_{t'}$ for $t' < t$, $\kappa(s)$ is an increasing function. Since λ_1 is continuous under continuous deformations of the domain [5], $\kappa(s)$ is continuous on $(0, \sup_\Omega u_1)$. Moreover $\lim_{s\to 0^+} \kappa(s) = 0$ and $\lim_{s\to\sup_\Omega u_1} \kappa(s)$

$= +\infty$. Hence there exists $s_1 \in (0, \sup_\Omega u_1)$ such that $\kappa(s_1) = \theta/2$. Now we have

$$\begin{aligned}\int_{\Omega_{s_1}} u_1^2(x)\,dx &= \int_\Omega u_1^2(x)\,dx - \int_{\Omega\setminus\Omega_{s_1}} u_1^2(x)\,dx \\ &= 1 - \int_0^{s_1} 2t|\Omega_t \cap (\Omega\setminus\Omega_{s_1})|\,dt \geq 1 - \int_0^{s_1} 2t|\Omega_t|\,dt = 1 - I_{s_1},\end{aligned}$$

and also

$$I_{s_1} = \int_0^{s_1} 2t|\Omega_t|\,dt \geq \int_0^{s_1} 2t|\Omega_{s_1}|\,dt = s_1^2|\Omega_{s_1}|.$$

From Lemma 3.1 it follows that

$$I_{s_1} \geq s_1^2|\Omega_{s_1}| \geq \left(\frac{\kappa(s_1)}{1+\kappa(s_1)}\right)^2 \int_{\Omega_{s_1}} u_1^2(x)\,dx \geq \left(\frac{\theta}{2+\theta}\right)^2 (1 - I_{s_1}),$$

so that

$$I_{s_1} \geq \frac{\theta^2}{2\theta^2 + 4\theta + 4}. \tag{3.10}$$

Since $\kappa(s_1) = \theta/2$, by (3.9) we have

$$\int_\Omega (x_1 - \alpha)^2 u_1^2(x)\,dx \geq c_3 I_{s_1} \geq \frac{\theta^2 c_3}{2\theta^2 + 4\theta + 4}. \tag{3.11}$$

Hence using Lemma 3.6 we obtain

$$\lambda_1(\Omega) \leq \left(\omega \int_\Omega (x_1 - \alpha)^2 u_1^2(x)\,dx\right)^{-1} \leq C'_{n,\theta},$$

where $C'_{n,\theta}$ depends only on n and θ.

Finally, since $\operatorname{diam}\Omega = 1$, $|\Omega| \leq w_n =$ volume of the unit ball in $\mathbb{R}^n$ and therefore $|\Omega|^{-2/n} \geq w_n^{-2/n}$. Thus, taking $C_{n,\theta} = w_n^{2/n} C'_{n,\theta}$, we have

$$\lambda_1(\Omega) \leq C_{n,\theta}|\Omega|^{-2/n}. \tag{3.12}$$

Remark. For $n = 2$ one can also prove the proposition as follows: By dilating Ω one can show that there exist rectangles R_1, R_2 such that $R - 2 \subseteq \Omega \subseteq R_1$, R_1 has side lengths 1 and N, and R_2 has side lengths $1 - cN^{-2/3}$ and $2cN^{1/2}$ for some constant $c > 0$, where N is comparable to the ratio of the diameter to the inradius of Ω. Then the proposition follows by the monotonicity principle of the eigenvalues since $\lambda_2(R_2)/\lambda_1(R_1)$ is arbitrarily close to 1 if N is large enough [9].

Lemma 3.4. *If Ω is a bounded convex domain, and $u_1 > 0$ is a first Dirichlet eigenfunction of Ω, then*

$$|\nabla u_1| \leq \sqrt{\lambda_1(\Omega)} \sup_{\Omega} u_1. \tag{3.13}$$

Proof. If Ω is smooth and strictly convex, then by the same method as in [12], $|\nabla u_1|^2 + \lambda_1(\Omega) u_1^2$ assumes its maximum at an interior point where $|\nabla u_1|$ vanishes. Hence $|\nabla u_1|^2 + \lambda_1(\Omega) u_1^2 \leq \lambda_1 \sup_{\Omega} u_1^2$ and (3.13) follows. The general case follows by approximation.

Lemma 3.5. *Let $C > 0$. Then there exist $c_n > 0$ and β_n $(0 < \beta_n < 1)$ such that if $\Omega \subseteq \mathbb{R}^n$ is a bounded convex domain with $\lambda_1(\Omega) \leq C|\Omega|^{-2/n}$, and $u_1 > 0$ is the first Dirichlet eigenfunction of Ω normalized so that $\int_{\Omega} u_1^2(x)\,dx = 1$, then for any $\delta > 0$*

$$|\{x \in \Omega : u_1(x) > \delta\}| \geq (1 - C_n \delta^{\beta_n})|\Omega|, \tag{3.14}$$

where C_n and β_n depend only on the dimension n and on C.

Proof. We may assume that $|\Omega| = 1$. Let $p \in \Omega$ be the point with $u_1(p) = \sup_{\Omega} u_1$. Then $1 = \int_{\Omega} u_1^2(x)\,dx \leq u_1^2(p)|\Omega| = u_1^2(p)$, and therefore $u_1(p) \geq 1$. Since $\lambda_1(\Omega) \leq C$ by the assumption, Lemma 3.4 implies $|\nabla u_1| \leq \sqrt{\lambda_1(\Omega)} \sup_{\Omega} u_1 \leq C_n'$, where C_n' depends only on the dimension n and on C, and we have used the fact that $\|u_1\|_\infty^2 \leq C_n \lambda_1(\Omega)^{n/2}$, C_n depending only on n. Since $u_1 = 0$ on $\partial\Omega$, we have $\operatorname{dist}(p, \partial\Omega) \geq 1/C_n'$ and moreover there exists $\sigma > 0$ depending only on n and C such that the ball $B(p; \sigma)$ is contained in Ω and

$$u_1(x) \geq \tfrac{1}{2} \quad \text{for every } x \in B(p; \sigma). \tag{3.15}$$

Since Ω is convex, $|\Omega| = 1$, and $B(p; \sigma) \subseteq \Omega$, there exists a constant C_1 depending only on n and σ such that $H_{n-1}(\partial\Omega) \leq C_1$ and $\operatorname{diam}\Omega \leq C_1$, where H_{n-1} denotes $(n-1)$-dimensional Hausdorff measure. q.e.d.

We need the following lemma:

Lemma 3.6. *Let $\sigma_1 > 0$ be given. Then there exists a homogeneous harmonic polynomial P on $\mathbb{R}^n$ of degree N depending only on n and σ_1, whose restriction on S^{n-1} has a nodal domain Γ of diameter less than σ_1.*

Proof. We can construct P from a Legendre function having a sufficiently small first zero.

Now fix a polynomial P from Lemma 3.6 corresponding to $\sigma_1 = C_1^{-1}\sigma$. Then the degree N of P depends only on n and C. Let Γ be a nodal domain of $P|_{S^{n-1}}$ of diameter less than σ_1. Then we may assume $P > 0$ in the interior of Γ. Fix a point $\xi_0 \in \Gamma$ and let $c_0 = P(\xi_0) > 0$.

Let y be a point in Ω. We choose the coordinate axes so that we have: $0 \in \partial\Omega$, the points p, y, and 0 are on the same line, y is between p and 0, and $p = |p|\xi_0$. Since Γ has diameter less than $\sigma_1 = C_1^{-1}\sigma$ and $|p| \le \operatorname{diam}\Omega \le C_1$, the set $|p|\Gamma$ has diameter less than σ. Since it contains p, we have $|p|\Gamma \subseteq B(p; \sigma)$; hence, by (3.15),

$$u_1(|p|\xi) \ge \tfrac{1}{2} \quad \text{whenever } \xi \in \Gamma. \tag{3.16}$$

Define

$$V = \{x : 0 < |x| < |p|,\ x/|x| \in \Gamma\},$$

and $w = u_1 - lP$ on V, where $l = (2\sup_\Gamma |P|)^{-1}$. Then by (3.16) $w \ge 0$ on $|p|\Gamma$, and $w \ge 0$ on ∂V since P is zero on the boundary of Γ. Also $\Delta w = \Delta u_1 = -\lambda(\Omega)u_1 \le 0$ in V since P is harmonic. Hence, by the maximum principle, $w > 0$ in V. In particular, $u_1(y) > lP(|y|\xi_0) = c_0 l|y|^N$ since p, y, and 0 are on the same line. Since Ω is convex and $B(p; \sigma) \subseteq \Omega$, we have

$$\operatorname{dist}(y, \partial\Omega) \ge \frac{\sigma}{|p|} \ge \sigma_1|y|.$$

If we let $c_1 = c_0 l \sigma_1^N$, then

$$u_1(y) \ge c_1[\operatorname{dist}(y, \partial\Omega)]^N \quad \text{for all } y \in \Omega, \tag{3.17}$$

where $c_1 > 0$ and N depend only on n and C. Hence (3.14) follows from (3.17) with $\beta_n = N^{-1}$ and $C_n = c_1^{-1/n}C_1$, since $H_{n-1}(\partial\Omega) \le C_1$ and (3.17) implies

$$\{x \in \Omega : u_1(x) < \delta\} \subseteq \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) < (c_1^{-1}\delta)^{1/N}\}.$$

Proof of Theorem 3.1. Let $\lambda_1 = \lambda_1(\Omega)$, and let $\alpha = j_{n/2-1,1}$ and $\beta = j_{n/2-1}$ be the first positive zeros of the Bessel functions $J_{n/2-1}$ and $J_{n/2}$, respectively. Let Ω^* be the ball centered at 0 such that $|\Omega^*| = |\Omega|$, and let u_1^* defined on Ω^* be the spherical decreasing symmetrization of u_1, where $u_1 > 0$ is the first Dirichlet eigenfunction of Ω normalized so that $\int_\Omega u_1^2(x)\,dx = 1$. Also let $S_1 = \{x \in \mathbb{R}^n : |x| < \gamma^{-1}\}$ be the ball with $\lambda_1(S_1) = \lambda_1(\Omega)$, where $\gamma = \sqrt{\lambda_1}/\alpha$, and let z be the first Dirichlet eigenfunction of S_1 normalized so that $\int_{S_1} z^2(x)\,dx = 1$.

Assume now that Ω satisfies the hypothesis of Theorem 3.1. We may assume without loss of the generality that $|\Omega| = 1$. Since

$$\frac{\lambda_2(\Omega)}{\lambda_1(\Omega)} \ge \tau_n - \varepsilon = \frac{\beta^2}{\alpha^2} - \varepsilon,$$

Proposition 3.1 implies that $\lambda_1(\Omega) \le C_n$, where C_n depends only on n if ε is sufficiently small. Hence there exists a constant C_n' depending only on n such that

$$\gamma \le C_n' \quad \text{and} \quad |\nabla z| \le C_n'. \tag{3.18}$$

By Faber-Krahn's inequality we have $S_1 \subseteq \Omega^*$. Let

$$w(x) = \begin{cases} J_{n/2}(\beta x)/J_{n/2-1}(\alpha x), & 0 \le x < 1, \\ w(1) = \lim_{x\to 1-} w(x), & x \ge 1, \end{cases} \tag{3.19}$$

and

$$B(x) = w'(x) + (n-1)\frac{w(x)^2}{x^2}. \tag{3.20}$$

In [1] it is proved that w is increasing, B is decreasing, and moreover $\lim_{x\to 1-} w''(x) < 0$. Hence there exists C depending only on n such that

$$(1-x)^2 \le C(w(1) - w(x)) \quad \text{for } 0 < x \le 1. \tag{3.21}$$

By choosing the origin appropriately the following inequalities are proved in [1]:

$$\begin{aligned} \alpha^2\left(\frac{\lambda_2(\Omega)}{\lambda_1(\Omega)} - 1\right) &\le \frac{\int_\Omega B(\gamma r)u_1^2(x)\,dx}{\int_\Omega w(\gamma r)^2 u_1^2(x)\,dx} \le \frac{\int_{S_1} B(\gamma r)z^2(x)\,dx}{\int_{\Omega^*} w(\gamma r)^2 u_1^*(x)^2\,dx} \\ &= \frac{(\beta^2-\alpha^2)\int_{S_1} w(\gamma r)^2 z^2(x)\,dx}{\int_{\Omega^*} w(\gamma r)^2 u_1^*(x)^2\,dx} \le \beta^2 - \alpha^2 = \alpha^2(\tau_n - 1). \end{aligned}$$

Thus using the assumption $\lambda_2(\Omega)/\lambda_1(\Omega) \ge \tau_n - \varepsilon$ we obtain

$$\begin{aligned} 0 &\le \int_{\Omega^*} w(\gamma r)^2 u_1^*(x)^2\,dx - \int_{S_1} w(\gamma r)^2 z(x)^2\,dx \\ &\le C_1\varepsilon \int_{S_1} w(\gamma r)^2 z(x)^2\,dx \le C_1 w(1)^2\varepsilon, \end{aligned} \tag{3.22}$$

where C_1 depends only on n.

By Chiti's comparison result ([3], [4]) there exists r_1 with $0 \le r_1 \le 1/\gamma$ such that

$$\begin{cases} u_1^*(r) \le z(r) & \text{if } 0 \le r \le r_1, \\ u_1^*(r) \ge z(r) & \text{if } r_1 \le r \le 1/\gamma. \end{cases} \tag{3.23}$$

Hence, if r^* is the radius of Ω^*, then

$$\begin{aligned}
&\int_{\Omega^*} w(\gamma r)^2 u_1^*(x)^2\,dx - \int_{S_1} w(\gamma r)^2 z(x)^2\,dx \\
&\quad = \int_{0\le|x|\le r_1} w(\gamma r)^2 (u_1^*(x)^2 - z(x)^2)\,dx \\
&\qquad + \int_{r_1\le|x|\le 1/\gamma} w(\gamma r)^2 (u_1^*(x)^2 - z(x)^2)\,dx \\
&\qquad + \int_{1/\gamma\le|x|\le r^*} w(\gamma r)^2 u_1^*(x)^2\,dx \\
&\quad \ge w(\gamma r_1)^2 \int_{S_1} (u_1^*(x)^2 - z(x)^2)\,dx + w(1)^2 \int_{\Omega^*\backslash S_1} u_1^*(x)^2\,dx \\
&\quad = [w(1)^2 - w(\gamma r_1)^2] \int_{\Omega^*\backslash S_1} u_1^*(x)^2\,dx,
\end{aligned}$$

since $\int_{\Omega^*} u_1^*(x)^2\,dx = \int_\Omega u_1^2(x)\,dx = \int_{S_1} z^2(x)\,dx = 1$, and w is increasing. Also since u_1^* is decreasing, $\int_{\Omega^*\backslash S_1} u_1^*(x)^2\,dx \ge u_1(1/\gamma)^2|\Omega^*\backslash S_1|$. By (3.21) and $w(1) > 0$ we have $w(1)^2 - w(\gamma r_1)^2 \ge c(1-\gamma r_1)^2$, where c depends only on n. Hence (3.22) implies

$$(1-\gamma r_1)^2 u_1^*(1/\gamma)^2 |\Omega^*\backslash S_1| \le C\varepsilon, \tag{3.24}$$

where C depends only on n.

Moreover by (3.18), (3.23) and $z(1/\gamma) = 0$, we obtain $u_1^*(r_1) = z(r_1) = z(r_1) - z(1/\gamma) \le C(1-\gamma r_1)$, where C depends only on n. Thus from (3.24) and the fact that u_1^* is decreasing, it follows that

$$u_1^*(1/\gamma)^4 |\Omega^*\backslash S_1| \le C\varepsilon, \tag{3.25}$$

where C depends only on n. Let $\delta = u_1^*(1/\gamma)$. Therefore the definition of u_1^* and Lemma 3.5 yield

$$|S_1| = |\{x \in \Omega : u_1(x) > \delta\}| \ge (1 - C_n\delta^{\beta_n})|\Omega| = (1 - C_n\delta^{\beta_n})|\Omega^*|.$$

Since $|\Omega| = 1$, we have $|\Omega^*\backslash S_1| \le C_n u_1^*(1/\gamma)^{\beta_n}$ and, by (3.25),

$$|\Omega^*\backslash S_1| \le C\varepsilon^{\beta_n'}, \tag{3.26}$$

where $\beta_n' = (4\beta_n^{-1} + 1)^{-1}$, and C depends only on n.

But (3.26) implies $\lambda_1(\Omega) \le (1 + C'\varepsilon^{\beta_n'})\lambda_1(\Omega^*)$, where C' depends only on n. Hence Theorem 3.1 follows from Theorem 2.1 with $\alpha_n = \beta_n'/2n$.

4. Theorem 4.1. *Assume $\Omega \subseteq \mathbb{R}^2$ is a bounded domain such that $\lambda_1(\Omega) \le (1+\varepsilon)\lambda_1(D)$, where $\varepsilon > 0$ is sufficiently small, and D is a disc with $|D| = |\Omega|$. Then there exists a disc D_1 such that*

$$|\Omega \cap D_1| \ge (1 - C\varepsilon^{1/4})|\Omega \cup D_1|, \tag{4.1}$$

where C is a universal constant. Moreover, if Ω is simply connected, we may also assume that $D_1 \subseteq \Omega$.

Proof. Without loss of generality we may assume that Ω has smooth boundary and $|\Omega| = 1$. Let $u_1 > 0$ be the first Dirichlet eigenfunction of Ω normalized so that $\int_\Omega u_1^2(x)\,dx = 1$. As in §2 for $\delta > 0$ we define $\Omega_\delta = \{x \in \Omega : u_1(x) > \delta\}$. Then as in the proof of Theorem 2.1 there exists a τ such that $0 < \tau < \varepsilon^{1/2}$, Ω_τ is a disjoint union of a finite number of smooth connected domains U_j $(0 \le j \le m)$, and

$$L(\partial\Omega_\tau)^2 \le 4\pi|\Omega_\tau| + C_1\varepsilon^{1/2}, \tag{4.2}$$

where $L(\partial\Omega_\tau)$ denotes the length of $\partial\Omega_\tau$ and C_1 is a constant. Moreover by (2.6) (whose proof does not use convexity) we obtain

$$\lambda_1(\Omega_\tau) \le (1 - 2\varepsilon^{1/2})\lambda_1(\Omega). \tag{4.3}$$

Since $\lambda_1(\Omega_\tau) = \min_{0\le j\le m}\lambda_1(U_j)$, assuming $\lambda_1(\Omega_\tau) = \lambda_1(U_0)$ we have as in the proof of Lemma 2.1 that

$$|U_0| \ge (1 - 4\varepsilon^{1/2})|\Omega|, \tag{4.4}$$

if $\varepsilon > 0$ is sufficiently small. (4.2) and the isoperimetric inequality imply, respectively,

$$\left[\sum_{j=0}^m L(\partial U_j)\right]^2 \le 4\pi\sum_{j=0}^m |U_j| + C'\varepsilon^{1/2}$$

and $4\pi|U_j| \le L(\partial U_j)^2$ for $1 \le j \le m$. Hence

$$L(\partial U_0)^2 \le 4\pi|U_0| + C_1\varepsilon^{1/2}. \tag{4.5}$$

Let U be the convex hull of U_0, and let V be the union of U_0 and all the bounded components of $\mathbb{R}^2\backslash U_0$. Then $U_0 \subseteq V \subseteq U$, V is simply connected, and also $L(\partial U) \le L(\partial V) \le L(\partial U_0)$, since U_0 is a connected domain in $\mathbb{R}^2$ with smooth boundary. Let $\alpha = |U\backslash U_0|$. Then by the isoperimetric inequality and (4.5) we have

$$\begin{aligned} L(\partial U_0)^2 &\le 4\pi|U_0| + C_1\varepsilon^{1/2} = 4\pi|U| - 4\pi\alpha + C_1\varepsilon^{1/2} \\ &\le L(\partial U)^2 - 4\pi\alpha + C_1\varepsilon^{1/2} \le L(\partial U_0)^2 - 4\pi\alpha + C_1\varepsilon^{1/2}. \end{aligned}$$

Hence

$$|U \backslash U_0| \leq C_2 \varepsilon^{1/2} \tag{4.6}$$

for some constant C_2. If $x \in \partial V$ and $\operatorname{dist}(x, \partial U) = d$, then $L(\partial V) - L(\partial U) \geq 2d$. Since

$$(4\pi |U_0|)^{1/2} \leq (4\pi |U|)^{1/2} \leq L(\partial U) \leq L(\partial V) - 2d \leq L(\partial U_0) - 2d,$$

and (4.6) implies $|V| \leq |U_0| + C_2 \varepsilon^{1/2} \leq |\Omega| + C_2 \varepsilon^{1/2} = 1 + C_2 \varepsilon^{1/2}$, we have, by (4.5),

$$\sup_{x \in \partial V} \operatorname{dist}(x, \partial U) \leq C_3 \varepsilon^{1/2}. \tag{4.7}$$

From $L(\partial U) \leq L(\partial U_0)$, $|U| \geq |U_0|$ and (4.5) it follows that $L(\partial U)^2 \leq 4\pi |U| + C_1 \varepsilon^{1/2}$. Since U is convex, the Bonnesen-style isoperimetric inequality (2.12) implies as in the proof of Theorem 2.1 that there exists a disc $D(x_0; \rho)$ centered at $x_0 \in V$ and of radius ρ such that $D(x_0; \rho) \subseteq U$ and

$$|D(x_0; \rho)| \geq (1 - C_4 \varepsilon^{1/4}) |U|. \tag{4.8}$$

Let $D_1 = D(x_0; \rho - C_3 \varepsilon^{1/2})$. Then, by (4.7), and by (4.6) and (4.8), we have, respectively, $D_1 \subseteq V$ and

$$|U_0 \cap D_1| \geq (1 - C_5 \varepsilon^{1/4}) |U_0 \cup D_1|, \tag{4.9}$$

if $\varepsilon > 0$ is sufficiently small.

Thus using (4.4) and (4.9) we obtain (4.1) if $\varepsilon > 0$ is sufficiently small. Moreover, if Ω is simply connected, then $V \subseteq \Omega$ and hence $D_1 \subseteq \Omega$.

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