

WEIL-PETERSSON VOLUMES

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Abstract

An explicit method of integration of top-dimensional differential forms over the moduli spaces of punctured Riemann surfaces is presented. This method is applied to the computation of Weil-Petersson volumes of moduli spaces, and we find that the volume for the twice-punctured torus is $\pi^4/8$ and that, for large g , the volume for the once-punctured surface of genus g is at least $g^{-2}c^{-2g}(2g)!$, where $c < .15$ is a constant independent of g . Our methods depend upon a certain bundle (introduced earlier) over the classical Teichmüller space of punctured surfaces, and some of our computations rely on standard techniques from quantum field theory.

1. Introduction

In [11] and [13] we proposed an explicit method of integration of top-dimensional differential forms over the moduli space \mathcal{M}_g^s of the surface F_g^s of genus g with $s > 0$ punctures, and one purpose of this paper is to present a complete exposition of this integration scheme. We also gave in [11] an expression of the Weil-Petersson Kähler two-form in coordinates reasonably well suited to our method of integration and put forward the computation of the various Weil-Petersson volumes μ_g^s of \mathcal{M}_g^s as test cases for the utility of our techniques.

We have had some success on these test cases, and this is also reported herein. Specifically, we have computed that the Weil-Petersson volume μ_1^2 is $\pi^4/8$. Furthermore, we derive an asymptotic expression for μ_g^1 to the effect that, for large g , $\mu_g^1 > g^{-2}c^{-2g}(2g)!$, where $c < .15$ is a constant independent of g . This latter estimate has found physical significance in [15] and agrees with predications from two-dimensional quantum gravity (see [18]).

Little is known beyond these new results about Weil-Petersson volumes. (*Added in proof:* There has been much progress on this recently; see [19])

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and [25].) Scott Wolpert proved [23] that $\mu_1^1 = \pi^2/6$ (and we give a quick proof of this herein) and also [22] that μ_g^s is a rational multiple of $\pi^{6g-6+2s}$. On the other hand, recent developments in two-dimensional quantum gravity suggest a general method of computing integrals of monomials of certain “visible” Chern classes over the Deligne-Mumford compactification of moduli space (see [18]). These Chern classes are closely related to the Miller-Morita-Mumford classes (see [7]–[9]), and this ties in with the computation here (over uncompactified moduli space) in that the first Miller-Morita-Mumford class is (pointwise) a constant multiple of the Weil-Petersson Kähler two-form (see [24]). We expect that the techniques of [18] (perhaps in concert with our approach) will lead to many further explicit computations, and we will comment further on this below.

The basic approach (from [10]) of the integration scheme is to pass to a certain principal \mathbb{R}_+^s -bundle $\widetilde{\mathcal{F}}_g^s$ over the usual Teichmüller space \mathcal{F}_g^s of F_g^s . (This bundle comes equipped with a canonical section.) The action of the mapping class group MC_g^s of F_g^s on \mathcal{F}_g^s lifts to an action on $\widetilde{\mathcal{F}}_g^s$. Furthermore, $\widetilde{\mathcal{F}}_g^s$ is endowed with good global coordinates as well as an MC_g^s -invariant cell decomposition, which is analogous to the Harer-Mumford-Thurston decomposition [2] of \mathcal{F}_g^s . The decomposition of $\widetilde{\mathcal{F}}_g^s$ is well suited to the matrix-model techniques of quantum field theory, and certain cohomology invariants of \mathcal{M}_g^s have been computed by exploiting this (see [3] and [12]).

Our method of integration is as follows. Given a top-dimensional differential form ω on \mathcal{M}_g^s , we pull the form back to $\widetilde{\mathcal{F}}_g^s$ in coordinates, integrate over (the intersection of the canonical section with) one cell in each MC_g^s -orbit of top-dimensional cell in the decomposition, and weight the contribution to $\int_{\mathcal{M}_g^s} \omega$ by the inverse of the order of the MC_g^s -isotropy group of the cell. In this way, we solve the problem of specifying a fundamental domain for the action of MC_g^s on $\widetilde{\mathcal{F}}_g^s$.

One requirement for this procedure is that the form ω pull back to something reasonable (i.e., computable) in the natural coordinates on $\widetilde{\mathcal{F}}_g^s$. The Weil-Petersson Kähler two-form pulls back nicely, and the volume form is in principle easily derived. For completeness and because of the reliance of our other results on this formula, the derivation of the expression of the Weil-Petersson Kähler two-form in our basic coordinates (only sketched in [11]) is given in Appendix A.

Another requirement is that the various domains of integration corresponding to the top-dimensional cells in $\widetilde{\mathcal{F}}_g^s$ must be tractable, and

unfortunately, they are quite bad in our basic coordinates. New “simplicial coordinates” are introduced which render each domain of integration a straight simplex, but we cannot in general compute our basic coordinates from these simplicial coordinates. Indeed, this leads to a family of arithmetic problems (which amount to inverting a collection of coupled quadric equations), one problem for each trivalent graph. (A general theorem from [10] shows that each such system is invertible, but the proof is nonconstructive.) We have solved roughly ten of these arithmetic problems using symbolic manipulation on the computer, and the results are most remarkable. Whereas one would expect iterated square-roots in the solution, in fact, the solution is essentially rational. Furthermore, the numerators and denominators in the rational expressions derived factor into products which arise from certain closed edge-paths on the underlying graph.

Though there are five MC_1^2 -orbits of top-dimensional cells in $\tilde{\mathcal{F}}_1^2$, it turns out that only one is relevant for the computation of μ_1^2 , and we solve the arithmetic problem corresponding to the associated graph herein. This leads to an expression of μ_1^2 as the integral of a certain rational function over a four-dimensional simplex. Using some computational techniques from quantum field theory, we evaluate this integral in closed form.

Our asymptotic estimates on μ_g^1 involve two ingredients: a lower bound to the volume of a single top-dimensional cell in $\tilde{\mathcal{F}}_g^1$ and an asymptotic formula for the number of such cells in $\tilde{\mathcal{F}}_g^1$. The latter estimate follows from standard matrix-model calculations. For the former estimate, we avoid the arithmetic problems mentioned above by a simple but effective estimate on the associated Jacobian determinant. Of course, we must also take a large exterior power of the Kähler two-form to find the volume form, and it turns out that the volume form admits a simple expression which is entirely independent of the choice of basis for the coordinate system. (In contrast, the two-form does depend on the choice of basis.)

We finally discuss the prospects for further applications of these techniques. The unique obstruction in principle to computing any particular Weil-Petersson volume is the solution of some finite subcollection of the arithmetic problems discussed before; we think, in any case, that these arithmetic problems warrant further study in their own right. Furthermore, assuming a satisfactory solution to the arithmetic problems, it is not, we think, unrealistic to expect to apply the matrix-model in conjunction with the integration scheme and compute, as it were, all of the μ_g^s at once in a generating function. Such an approach would presumably tie in with the developments in two-dimensional quantum gravity mentioned above.

More generally, we mention that a recent geometric interpretation of the simplicial coordinates has led to explicit differential forms in these coordinates which represent Witten's visible classes on a certain compactification of moduli space; the computation (using the techniques of [12]) of a generating function for the integrals of monomials in these forms looks promising. (Further computations or a further analysis of the compactification is required, though, to give these integrals geometric and/or physical significance.) This material will be taken up elsewhere. (*Added in proof*: There has been dramatic progress on this recently; see [5] and [20].)

This paper is organized as follows. §2 establishes some notations and conventions. §3 quickly treats most of the relevant background material from [10] and [12] as well as introducing new related material; in particular, §3.4 develops the simplicial coordinates, and §3.5 is dedicated to the arithmetic problems. §4 treats the surface F_1^1 , and §5 treats the surface F_1^2 ; indeed, we derive the expression for μ_1^2 as the integral of a rational function over a simplex in §5.1 and evaluate this integral in §5.2. In §6.1 the Weil-Petersson volume form is computed for any once-punctured surface, and in §6.2 we pull together various results to give our asymptotic estimate on μ_g^1 . Finally, Appendix A gives the derivation of the Weil-Petersson two-form in our coordinates (the starting point is Wolpert's formula in terms of the hyperbolic geometry of the underlying surface), and Appendix B derives a standard matrix-model estimate (explained to us by Steve Shenker) on the number of MC_g^1 -orbits of top-dimensional cells in \mathcal{F}_g^1 .

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2. Notation and conventions

We establish here some definitions which will hold throughout this paper.

Suppose that $g \geq 0$ and $s \geq 1$ are integers satisfying the condition that $2g - 2 + s > 0$, and let F_g^s denote a fixed smooth surface of genus

g with s unlabeled punctures; thus, F_g^s has negative Euler characteristic and at least one puncture. Let \mathcal{M}_g^s and \mathcal{T}_g^s denote the moduli space and Teichmüller space, respectively, of surfaces diffeomorphic to F_g^s . The mapping class group of F_g^s is denoted MC_g^s , so MC_g^s acts on \mathcal{T}_g^s with quotient \mathcal{M}_g^s . By a “differential form” on \mathcal{M}_g^s we mean a differential form on \mathcal{T}_g^s which is invariant under the action of MC_g^s .

In [10], we introduced a principal \mathbb{R}_+^s -bundle $\widetilde{\mathcal{T}}_g^s$ over \mathcal{T}_g^s , called the “decorated Teichmüller space” of F_g^s , which is defined as follows. The fiber over a point of \mathcal{T}_g^s is the collection of all (not necessarily simple or disjoint) s -tuples of horocycles about the punctures of F_g^s , one horocycle about each puncture; the specification of these distinguished horocycles is called a “decoration” on the underlying marked conformal surface, and the distinguished horocycles are called the “decorated” horocycles. The ordered s -tuple of hyperbolic lengths of the horocycles give coordinates on the fibers. Of course, since \mathcal{T}_g^s is homeomorphic to an open ball of dimension $6g - 6 + 2s$, it follows that $\widetilde{\mathcal{T}}_g^s$ is homeomorphic to an open ball of dimension $6g - 6 + 3s$. Furthermore, the action of MC_g^s on \mathcal{T}_g^s lifts to an action of MC_g^s on $\widetilde{\mathcal{T}}_g^s$ in the natural way, where this action permutes the decorated horocycles and their lengths.

Suppose that G is a one-dimensional CW complex. A “hook” of G is simply an edge of the first barycentric subdivision of G . Thus, an edge e of G gives rise to exactly two hooks, and we usually denote a hook of G contained in e by \hat{e} . We say that a hook \hat{e} is “incident” on a vertex v of G if v lies in the closure of \hat{e} . The “valence” of a vertex v of G is the number of distinct hooks incident on it. A “loop” is an edge e of G so that both of the hooks of G contained in e are incident on a common vertex of G .

We define a “graph” to be a one-dimensional CW complex so that each vertex has valence at least three. In particular, we emphasize that G may have loops, and an edge of G is not necessarily determined by its endpoints. We say simply that the graph G is “trivalent” if each of its vertices has valence three. Finally, given an edge e of G connecting distinct endpoints, we may collapse e to a point to produce another graph, which is said to arise from G by “contracting” the edge e .

3. Integration over moduli space

Suppose that ω is a top-dimensional differential form on \mathcal{M}_g^s . We describe in this section a method of computing $\int_{\mathcal{M}_g^s} \omega$. In practice, the

method depends on a specification of coordinates on \mathcal{F}_g^s , and we will specialize our method to particular coordinatizations in subsequent sections.

In order to formulate our method, we first briefly recall some constructions from [10] and [12].

3.1. Fatgraphs. Further details on most of the material in this subsection can be found in [12, §1].

A “fatgraph” G is a graph (in the sense of §2) together with a cyclic order on the hooks about each vertex. Given a graph, the specification of cyclic order on the hooks about each vertex is called a “fattening” of the graph. A morphism of fatgraphs is a morphism between the underlying graphs which respects cyclic orders, and we let $[G]$ denote the isomorphism class and $\text{Aut}(G)$ denote the automorphism group of G .

Notice that if $\phi \in \text{Aut}(G)$ fixes an oriented edge e of G , then ϕ must also fix the edge clockwise from the terminal point of e ; continuing in this way, one concludes that if $\phi \in \text{Aut}(G)$ fixes an oriented edge of a connected fatgraph G , then ϕ is the identity.

Associated with the fatgraph G is a punctured surface $F(G)$ which contains G as its spine. To construct $F(G)$, embed G in \mathbb{R}^3 in such a way that a neighborhood of the vertices lies in some oriented plane $\Pi \subset \mathbb{R}^3$ and the cyclic orders on the hooks about the vertices of G agree with the counter-clockwise sense in Π . Choose a neighborhood U in Π of the set of vertices, and extend U to a surface $F'(G)$ with boundary embedded in \mathbb{R}^3 by adjoining one band to U for each component of $G - U$ in such a way that $G \subset F'(G)$ and each band respects the orientation on Π . Finally, define an abstract punctured surface $F(G) \supset F'(G) \supset G$ by adjoining one punctured disk to each boundary component of $F'(G)$. Let $s(G)$ denote the number of punctures of $F(G)$, or, in other words, the number of boundary components of $F'(G)$.

A “marking” on G is defined to be a marking (in the usual sense of Riemann surfaces) on the surface $F(G)$. (Thus, a marking on G can be regarded as an equivalence class of homeomorphisms $F \rightarrow F(G)$ from a base surface F , where two such homeomorphisms h_1, h_2 are regarded as equivalent if there is some homeomorphism $g: F(G) \rightarrow F(G)$ isotopic to the identity so that $g \circ h_1$ is isotopic to h_2 .) Contraction of edges (which are not allowed to be loops) defines a partial order on

$$\mathcal{G}_g^s = \{\text{marked isomorphism classes of fatgraphs } G : F(G) \text{ is homeomorphic to } F_g^s\},$$

and we have

Theorem 3.1.1 [10, Theorem 5.5] and [12, §1]. *The geometric realization of the poset \mathcal{G}_g^s is naturally isomorphic to the dual of an MC_g^s -invariant cell decomposition of $\tilde{\mathcal{F}}_g^s$. $\text{Aut}(G)$ is naturally isomorphic to the isotropy subgroup in MC_g^s of any cell in the decomposition corresponding to G .*

Remark. This is the analogue on $\tilde{\mathcal{F}}_g^s$ of the Harer-Mumford-Thurston cell decomposition [2] on \mathcal{F}_g^s .

An unmarked fatgraph G can be compactly described by a pair of permutations as follows. Suppose that G has $v_k \geq 0$ k -valent vertices for $k \geq 3$, and let K denote the largest valence of the vertices of G . Thus, G has $N = \sum_{K \geq k \geq 3} kv_k \equiv 0 \pmod{2}$ (2) hooks. Let σ be a permutation on N letters of type $\{3^{v_3}4^{v_4}\dots K^{v_K}\}$, and label the hooks of G by elements of $\{1, 2, \dots, N\}$ in the natural way so that each cycle of σ corresponds to the hooks incident on some vertex. Thus, σ permutes the labels on hooks incident on a vertex. We define another permutation τ on N letters of type $\{2^{N/2}\}$, where τ has one transposition for each edge of G , and each transposition permutes the pair of labels associated with the hooks of a single edge of G . The pair of permutations (σ, τ) uniquely determines the fatgraph G .

Using this formalism, it is straightforward to enumerate the set $\{[G] : G \in \mathcal{G}_g^s\}$ on the computer, at least for $2g - 2 + s$ reasonably small. For later use, Table 1 (next page) records the results of this enumeration for the surface F_1^2 ; we computed with Think C (version 2) [16] on a MacII. (We emphasize that the relevant data from this table can easily be checked by hand; cf. the remark following Corollary 3.3.2.) Each permutation is given as a tuple whose i th entry is the image of i under the permutation. The permutation σ is followed by a collection of associated permutations τ , and the number in brackets following each permutation τ is the order of the corresponding fatgraph automorphism group.

To give some examples and for later application, we draw the five trivalent fatgraphs of Table 1 in Figure 1 (p. 567). (The letters next to the edges will be explained and used later.)

To close this section, we give a construction (from [14]) of certain trivalent fatgraphs G so that $F(G)$ is homeomorphic to F_g^1 . These fatgraphs are the starting point for the computation of Weil-Petersson volume forms for once-punctured surfaces; in effect, we perform explicit computations (in §6.1) for the fatgraphs constructed here and use the fact (proved here) that these fatgraphs correspond to once-punctured surfaces.

For $\sigma = (2, 3, 4, 5, 6, 1)$, there are three fatgraphs:

$$(2, 1, 5, 6, 3, 4)[1] \quad (3, 5, 1, 6, 2, 4)[2] \quad (4, 5, 6, 1, 2, 3)[6]$$

For $\sigma = (2, 3, 4, 1, 6, 7, 8, 5)$, there are four fatgraphs:

$$(2, 1, 5, 7, 3, 8, 4, 6)[1] \quad (3, 5, 1, 7, 2, 8, 4, 6)[4] \\ (5, 6, 7, 8, 1, 2, 3, 4)[8] \quad (5, 6, 8, 7, 1, 2, 4, 3)[2]$$

For $\sigma = (2, 3, 4, 5, 1, 7, 8, 6)$, there are four fatgraphs:

$$(2, 1, 6, 7, 8, 3, 4, 5)[1] \quad (3, 4, 1, 2, 6, 5, 8, 7)[1] \\ (3, 6, 1, 7, 8, 2, 4, 5)[1] \quad (3, 6, 1, 8, 7, 2, 5, 4)[1]$$

For $\sigma = (2, 3, 4, 1, 6, 7, 5, 9, 10, 8)$, there are eight fatgraphs:

$$(2, 1, 5, 8, 3, 9, 10, 4, 6, 7)[1] \quad (3, 5, 1, 6, 2, 4, 8, 7, 10, 9)[1] \\ (3, 5, 1, 8, 2, 9, 10, 4, 6, 7)[2] \quad (3, 5, 1, 8, 2, 10, 9, 4, 7, 6)[2] \\ (5, 6, 7, 8, 1, 2, 3, 4, 10, 9)[1] \quad (5, 6, 8, 9, 1, 2, 10, 3, 4, 7)[2] \\ (5, 6, 8, 10, 1, 2, 9, 3, 7, 4)[1] \quad (5, 8, 6, 9, 1, 3, 10, 2, 4, 7)[1]$$

For $\sigma = (2, 3, 1, 5, 6, 4, 8, 9, 7, 11, 12, 10)$, there are five fatgraphs:

$$(2, 1, 4, 3, 7, 10, 5, 11, 12, 6, 8, 9)[1] \quad (4, 5, 7, 1, 2, 10, 3, 11, 12, 6, 8, 9)[4] \\ (4, 5, 7, 1, 2, 10, 3, 12, 11, 6, 9, 8)[2] \quad (4, 7, 10, 1, 8, 11, 2, 5, 12, 3, 6, 9)[4] \\ (4, 7, 10, 1, 8, 12, 2, 5, 11, 3, 9, 6)[3]$$

TABLE 1. The fatgraph complex for F_1^2

Consider the CW complex illustrated in Figure 2 (p. 568), together with the indicated cyclic orders on the hooks about the trivalent vertices, and adopt the indicated notation for the hooks corresponding to the univalent vertices. We describe a collection of fatgraphs containing this CW complex as follows. Connect the hook e_{n+1} to some hook e'_{i_1} among $\{e'_1, e'_2, \dots, e'_n\}$, and then connect the hook e_1 to some hook e'_{i_2} among $\{e'_1, e'_2, \dots, e'_{n+1}\} - \{e'_{i_1}\}$. Continue in this way, inductively connecting the hook e_j to some hook $e'_{i_{j+1}}$ among $\{e'_1, e'_2, \dots, e'_{n+1}\} - \{e'_{i_1}, e'_{i_2}, \dots, e'_{i_j}\}$ for $j = 1, 2, \dots, n$. Let $\mathcal{G}(n)$ denote the family of fatgraphs so constructed for $n \geq 1$. By construction, each $G \in \mathcal{G}(n)$ has a unique cycle of length two.

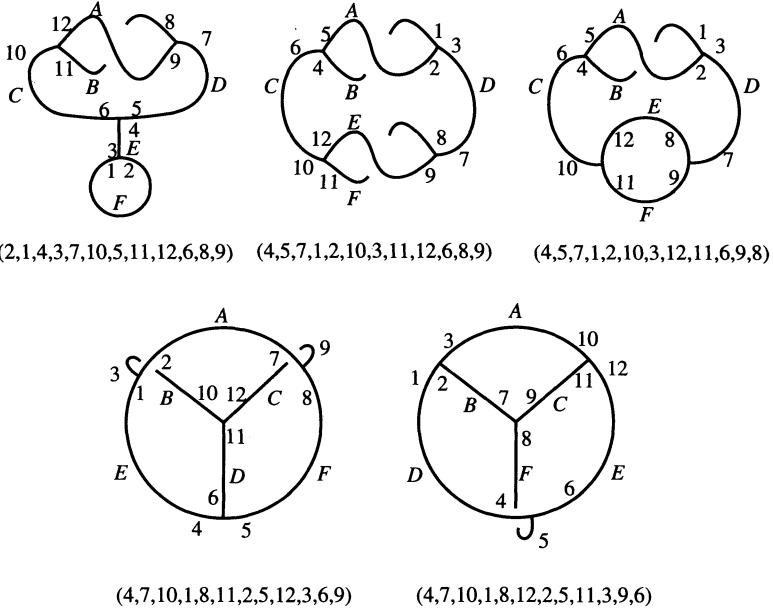


FIGURE 1

Proposition 3.1.2. *If $G \in \mathcal{G}(n)$, then $F(G)$ is homeomorphic to F_{n+1}^1 , and $\#\text{Aut}(G) \leq 2$.*

Proof. Suppose that $G \in \mathcal{G}(n)$. Inspection of Figure 2 shows that $s(G)$ agrees with $s(G')$, where $G' \subset G$ and $G - G'$ consists of those edges disjoint from the hooks e'_1 and e'_{n+1} . Further inspection shows that $s(G) = s(G') = 1$, so

$$1 - 2g = \chi(F(G)) = \chi(G) = -2n - 1,$$

whence $F(G)$ is indeed homeomorphic to F_{n+1}^1 . If $\phi \in \text{Aut}(G)$, then ϕ must fix the unique cycle in G of length two. Since a fatgraph automorphism fixing an oriented edge must be trivial, we conclude that $\#\text{Aut}(G) \leq 2$, as was claimed.

Remark. In fact, arguing as above, one can show that $\#\{[G] \cap \mathcal{G}(n)\} \leq 2$ for each $G \in \mathcal{G}(n)$, so

$$\#\{[G] : G \in \mathcal{G}_g^1\} \geq \frac{g-1}{2}(g-1)!.$$

A much better estimate on this quantity will be derived in Appendix B.

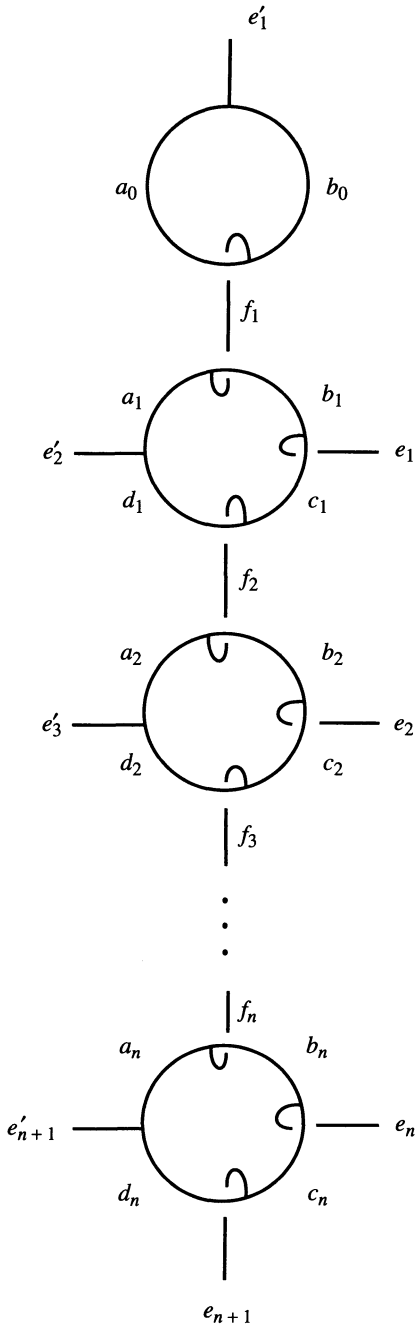


FIGURE 2

3.2. The integration scheme. Consider the functions $\rho_i: \widetilde{\mathcal{F}}_g^s \rightarrow \mathbb{R}_+$, where $\rho_i(\tilde{\Gamma})$ is the hyperbolic length of the i th decorated horocycle with respect to the decorated conformal type $\tilde{\Gamma} \in \widetilde{\mathcal{F}}_g^s$ for $i = 1, \dots, s$. Level sets of the function $\rho = \times_{i=1}^s \rho_i: \widetilde{\mathcal{F}}_g^s \rightarrow \mathbb{R}_+^s$ define a codimension- s foliation of $\widetilde{\mathcal{F}}_g^s$, and the action of MC_g^s on $\widetilde{\mathcal{F}}_g^s$ obviously preserves this foliation. The leaf corresponding to $\rho = \times_{i=1}^s 1$ is invariant under the MC_g^s -action and provides a canonical section $S: \widetilde{\mathcal{F}}_g^s \rightarrow \widetilde{\mathcal{F}}_g^s$ for the decorated bundle $\widetilde{\mathcal{F}}_g^s \rightarrow \mathcal{M}_g^s$.

Our general integration scheme is provided by

Theorem 3.2.1. *If ω is a top-dimensional differential form on \mathcal{M}_g^s , then*

$$\int_{\mathcal{M}_g^s} \omega = \sum_{(*)} \frac{1}{\text{Aut}(G)} \int_{\mathcal{D}(G)} \phi^*(\omega),$$

where the index set $(*)$ in the sum is the set of all (unmarked) isomorphism classes $[G]$ of trivalent fatgraphs so that $F(G)$ is homeomorphic to F_g^s , $\phi^*(\omega)$ denotes the pull-back of ω by the forgetful map $\phi: \widetilde{\mathcal{F}}_g^s \rightarrow \mathcal{M}_g^s$, and $\mathcal{D}(G) \subset \widetilde{\mathcal{F}}_g^s$ is the intersection of the image of the section $S: \mathcal{F}_g^s \rightarrow \widetilde{\mathcal{F}}_g^s$ with the interior of any cell in the cell-decomposition of $\widetilde{\mathcal{F}}_g^s$ which corresponds to G .

Remark. Let us emphasize that the index set $(*)$ in the sum is finite, though the number of summands grows super-exponentially with the topological type of the surface (cf. the remark following Proposition 3.1.2 and Theorem B in Appendix B). Furthermore, our computer enumeration of fatgraph complexes (mentioned before) is of obvious relevance to the implementation of the integration scheme.

In order to prove the theorem, we require the technical

Lemma 3.2.2. *The image of the section S meets transversely each face of the cell-decomposition of $\widetilde{\mathcal{F}}_g^s$. Furthermore, for each trivalent fatgraph G , the projection $\mathcal{D}(G) \rightarrow \phi(\mathcal{D}(G))$ is $\#\text{Aut}(G)$ -to-one except over a locus in \mathcal{M}_g^s of zero measure.*

The proof of the lemma is postponed until the next subsection, where we introduce coordinates on $\widetilde{\mathcal{F}}_g^s$. Assuming the lemma for the time being, we can give the

Proof of Theorem 3.2.1. Let $\mathcal{F} \subset S(\mathcal{F}_g^s) \subset \widetilde{\mathcal{F}}_g^s$ be a fundamental domain for the action of MC_g^s on $S(\mathcal{F}_g^s)$. Of course, $\int_{\mathcal{M}_g^s} \omega = \int_{\mathcal{F}} \phi^*(\omega)$ by naturality of integration. Since \mathcal{F} meets the cells of the

cell-decomposition of $\widetilde{\mathcal{F}}_g^s$ transversely (by the first part of the technical lemma), we conclude that $\bigcup_{(*)} MC_g^s(\mathcal{D}(G))$ has full measure in $S(\mathcal{F}_g^s)$ (where the index set $(*)$ has the same meaning as above) and find

$$\int_{\mathcal{F}} \phi^*(\omega) = \sum_{(*)} \int_{\mathcal{F} \cap MC_g^s(\mathcal{D}(G))} \phi^*(\omega).$$

The second assertion in the technical lemma guarantees that

$$\int_{\mathcal{F} \cap MC_g^s(\mathcal{D}(G))} \phi^*(\omega) = \frac{1}{\#\text{Aut}(G)} \int_{\mathcal{D}(G)} \phi^*(\omega),$$

proving the theorem. q.e.d.

We remark that the successful application of our integration scheme depends on finding coordinates on $\widetilde{\mathcal{F}}_g^s$ with two attributes: the form ω of interest must pull-back to something reasonable in the coordinates, and the region $\mathcal{D}(G)$ of integration must admit a tractable expression in the coordinates. In the next subsection, we will describe coordinates with the first attribute for the Weil-Petersen volume forms.

3.3. Coordinates on decorated Teichmüller space. We will require several coordinatizations of $\widetilde{\mathcal{F}}_g^s$ in the sequel. To begin this subsection we quickly recall that the relevant constructions; full details can be found in [9, §3]. The basic coordinates (with respect to which our other coordinates are defined) are called “ λ -lengths” and defined as follows. Fix some $\tilde{\Gamma} \in \widetilde{\mathcal{F}}_g^s$ and suppose that c is (the isotopy class of) a simple arc in F_g^s connecting (not necessarily distinct) punctures; let us call such an arc c an “ideal arc” in F_g^s . The Fuchsian group underlying $\tilde{\Gamma}$ determines an identification of the universal cover of F_g^s with the Poincaré disk \mathbb{D} , and we may choose a lift \tilde{c} to \mathbb{D} of a geodesic representative for c . Insofar as c connects punctures of F_g^s , $\tilde{\Gamma}$ determines a horocycle centered at each of the ideal endpoints of \tilde{c} . Let δ be the (signed) hyperbolic length along \tilde{c} between these horocycles, taken with a positive sign if and only if the horocycles are disjoint. Finally, the λ -length of c with respect to $\tilde{\Gamma}$ is defined to be

$$\lambda(c; \tilde{\Gamma}) = \sqrt{2e^\delta}.$$

Now, let Δ be (the isotopy class of) a collection of disjointly embedded ideal arcs so that each component of $F_g^s - \Delta$ is an ideal triangle in F_g^s ; let us call such a collection Δ an “ideal triangulation” of F_g^s .

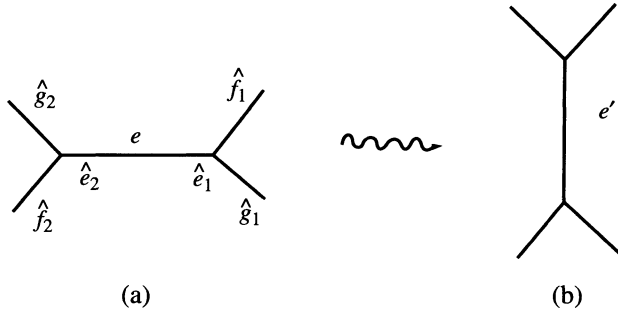


FIGURE 3

The Poincaré dual of an ideal triangulation Δ of F_g^s is a fatgraph G embedded as a spine of F_g^s , where the cyclic order on the hooks about a vertex is induced by the orientation of the surface. By definition, $F(G)$ is homeomorphic to F_g^s , and a marking on F_g^s induces a marking on G and conversely. It is convenient to translate λ -lengths into the setting of fatgraphs, where the λ -length $\lambda(e; \tilde{\Gamma})$ of an edge e of G is defined to be the λ -length of its dual ideal arc. If \hat{e} is a hook of G , we may write $\lambda(\hat{e}; \tilde{\Gamma})$ for the λ -length of the edge of G containing \hat{e} , and if the decorated conformal type $\tilde{\Gamma}$ is fixed or understood and x is an ideal arc, an edge of G , or a hook of G , we may write simply $\lambda(x) = \lambda(x; \tilde{\Gamma})$.

Our basic coordinatization of $\tilde{\mathcal{F}}_g^s$ is described in

Theorem 3.3.1 [10, Theorem 3.1]. *Fix a marked trivalent fatgraph G so that $F(G)$ is homeomorphic to F_g^s . The assignment of λ -lengths to the edges of G establishes a homeomorphism between $\tilde{\mathcal{F}}_g^s$ and the collection of all \mathbb{R}_+ -valued functions defined on the set of edges of G .*

For later application, we briefly discuss the dependence of λ -length coordinates on the choice of trivalent fatgraph G . To this end, suppose that \hat{e}_1 and \hat{e}_2 are the hooks of some edge e of G with distinct endpoints v_1 and v_2 , and let $\hat{e}_i, \hat{f}_i, \hat{g}_i$ be the hooks of G incident on v_i occurring in this cyclic order for $i = 1, 2$ (see Figure 3(a)). We may alter G by a “Whitehead move” along e to produce another trivalent fatgraph G' as in Figure 3(b). Identify the edges of G' with the edges of G in the natural way, where the edge of G' arising from e is denoted e' as in Figure 3(b). Of course, a marking on G gives rise to a marking on G' .

Corollary 3.3.2. *Fix g and s . The collection of finite compositions of Whitehead moves act transitively on the collection of marked trivalent fatgraphs in \mathcal{E}_g^s .*

Proof. Since $\tilde{\mathcal{F}}_g^s$ is connected and two top-dimensional cells in the cell-decomposition of $\tilde{\mathcal{F}}_g^s$ differ by a Whitehead move (cf. [10, Proposition 7.1]), this follows easily from Theorem 3.1.1.

Remark. One can use this result to check that Table 1 accurately enumerates the trivalent fatgraphs associated to F_1^2 ; simply check that the asserted family is closed under Whitehead moves.

The effect of a Whitehead move on λ -lengths is given by

Lemma 3.3.3 (Lemma A.1a of Appendix A). *With the notation above, the λ -length of each edge of G' agrees with the λ -length of its corresponding edge of G , except that*

$$\lambda(e') = \frac{\lambda(\hat{f}_1)\lambda(\hat{f}_2) + \lambda(\hat{g}_1)\lambda(\hat{g}_2)}{\lambda(e)}.$$

We call such an algebraic transformation of λ -lengths a “Ptolemy transformation.” (See Appendix A for the explanation of this terminology.)

Corollary 3.3.2 and Lemma 3.3.3 together give

Theorem 3.3.4 [10, Proposition 7.3]. *Fix a marked trivalent fatgraph G with $F(G)$ homeomorphic to F_g^s . The action of MC_g^s on λ -lengths with respect to G is given by compositions of Ptolemy transformations. In particular, the action of MC_g^s is by tuples of rational maps.*

Proof. In light of the previous two results, the theorem follows immediately from naturality of λ -lengths under the action of MC_g^s .

Remark. In fact, one can give explicit formulas for this representation of MC_g^s [10, §7 and the Addendum], but we will not need this material here.

Closely related to these coordinates on $\tilde{\mathcal{F}}_g^s$ are certain parameters, called “ h -lengths,” defined as follows. Fix $\tilde{\Gamma} \in \tilde{\mathcal{F}}_g^s$ and some ideal triangulation Δ of F_g^s . There is a unique family $\Delta^\#$ of ideal arcs representing Δ which are geodesic for the Poincaré metric underlying $\tilde{\Gamma}$. If h is a horocycle in F_g^s corresponding to the decoration of $\tilde{\Gamma}$, then $\Delta^\#$ decomposes h into a collection of arcs, and we refer to each such arc as a “(Δ)-horocyclic segment.” Notice that there is a natural bijective correspondence between the horocyclic segments and the hooks of the corresponding fatgraph, where we assign to a horocyclic segment the hook opposite it at the vertex.

Suppose that Δ is an ideal triangulation of F_g^s with corresponding trivalent fatgraph G . We define a “sector” of G to be the region between two consecutive hooks of G which are incident on a common vertex. One identifies each sector with its corresponding Δ -horocyclic segment, which,

in turn, is identified (as above) with a hook of G . If \hat{e} is a hook of G incident on the vertex v , then we shall denote the sector opposite to \hat{e} at v by (v, \hat{e}) . In particular, there are exactly three sectors associated to each vertex of G , so if G has N edges, then G has $2N$ sectors.

We define the h -length $\alpha(v, \hat{e}; \tilde{\Gamma})$ of (v, \hat{e}) with respect to $\tilde{\Gamma}$ to be one-half the Poincaré length of the corresponding horocyclic segment; when the decorated conformal type $\tilde{\Gamma}$ is fixed or understood, we write simply $\alpha(v, \hat{e}) = \alpha(v, \hat{e}; \tilde{\Gamma})$.

Now, suppose that $\tilde{\Gamma} \in \tilde{\mathcal{F}}_g^s$, and let (v, \hat{e}) , (v, \hat{f}) , and (v, \hat{g}) be three distinct sectors of G . An elementary computation in hyperbolic geometry [10, Proposition 2.8] shows that the h -lengths of the sectors are given by

$$\alpha(v, \hat{e}) = \frac{\lambda(\hat{e})}{\lambda(\hat{f})\lambda(\hat{g})}, \quad \alpha(v, \hat{f}) = \frac{\lambda(\hat{f})}{\lambda(\hat{e})\lambda(\hat{g})}, \quad \alpha(v, \hat{g}) = \frac{\lambda(\hat{g})}{\lambda(\hat{e})\lambda(\hat{f})}.$$

Thus, for instance, $\lambda(e) = [\alpha(v, \hat{f})\alpha(v, \hat{g})]^{-1/2}$, and it follows from Theorem 3.3.1 that the collection of h -lengths of sectors of a marked trivalent fatgraph G uniquely determines the corresponding decorated conformal type. Moreover, if \hat{e}_1, \hat{e}_2 are distinct hooks of a common edge e of G , and $(v_1, \hat{e}_1), (v_1, \hat{f}_1), (v_1, \hat{g}_1)$ are distinct sectors and $(v_2, \hat{e}_2), (v_2, \hat{f}_2), (v_2, \hat{g}_2)$ are distinct sectors (perhaps with $v_1 = v_2$), then the quadric “coupling equation”

$$\alpha(v_1, \hat{f}_1)\alpha(v_1, \hat{g}_1) = \alpha(v_2, \hat{f}_2)\alpha(v_2, \hat{g}_2)$$

holds among the h -lengths. We summarize with

Corollary 3.3.5 [10, Proposition 3.5]. *Fix a marked trivalent fatgraph G so that $F(G)$ is homeomorphic to F_g^s . The assignment of h -lengths to the sectors of G establishes a homeomorphism between $\tilde{\mathcal{F}}_g^s$ and the collection of \mathbb{R}_+ -valued functions defined on the collection of sectors of G which satisfy the coupling equations.*

Thus, we may identify $\tilde{\mathcal{F}}_g^s$ with the quadric variety determined by the coupling equations.

We next explain the description of the regions $\mathcal{D}(G)$ of integration of our integration scheme in λ -lengths and h -lengths, and we begin with a discussion of the section $\mathcal{S}: \mathcal{T}_g^s \rightarrow \tilde{\mathcal{F}}_g^s$.

Suppose that G is a trivalent fatgraph and recall the surface $F'(G)$ (which arose in the construction of the punctured surface $F(G)$ from the fatgraph G) with $s = s(G)$ boundary components, say $\{\partial_i\}_1^s$. Each boundary component ∂_i gives rise to a closed edge-path on G in the

natural way, and if a closed edge-path consecutively traverses hooks \hat{f} and \hat{g} incident on the vertex v with incident hooks \hat{e} , \hat{f} , and \hat{g} , then we say that the edge-path “traverses” the sector (v, \hat{e}) . By definition of h -lengths, if $\tilde{\Gamma} \in \tilde{\mathcal{F}}_g^s$, we find that the hyperbolic length of the horocycle in F_g^s corresponding to ∂_i is

$$\rho_i(\tilde{\Gamma}) = 2 \sum \alpha(v, \hat{e}; \tilde{\Gamma}),$$

where the sum is over all sectors (v, \hat{e}) of G traversed by the closed edge-path corresponding to the boundary component ∂_i of $F'(G)$. Thus, the canonical section S is affine in the h -length parameters.

In order to finally describe $\mathcal{D}(G)$, we briefly recall [9, §§4 and 5] the idea of the proof of Theorem 3.1.1. Affine duality establishes a homeomorphism between the (open) forward light-cont L^+ in Minkowski three space \mathbb{M} and the bundle of horocycles over the circle at infinity in the hyperbolic plane (as in the Appendix). Using this identification, a point $\tilde{\Gamma}$ of $\tilde{\mathcal{F}}_g^s$ gives rise to a group $\Gamma < \text{SO}^+(1, 2)$ as well as a discrete set $\mathcal{B} \subset L^+$ which arises from the decoration. The closed convex hull H of \mathcal{B} in \mathbb{M} is a Γ -invariant convex body in \mathbb{M} , and extremal edges of H give rise in the natural way to a collection of geodesics $\Delta(\tilde{\Gamma}) \subset F_g^s$. The collection $\Delta(\tilde{\Gamma})$ is generically an ideal triangulation of F_g^s , and the (open top-dimensional) cell in $\tilde{\mathcal{F}}_g^s$ corresponding to an arbitrary ideal triangulation Δ of F_g^s is

$$\mathcal{E}(\Delta) = \{\tilde{\Gamma} \in \tilde{\mathcal{F}}_g^s : \Delta(\tilde{\Gamma}) \text{ is isotopic to } \Delta\}.$$

The condition that $\tilde{\Gamma} \in \tilde{\mathcal{F}}_g^s$ lie in $\mathcal{E}(\Delta)$ for some ideal triangulation Δ is therefore the requirement that the lift to \mathbb{M} of each ideal arc (as a segment connecting distinct points of L^+) in Δ be extremal in the convex hull H . Each ideal arc in Δ separates two ideal triangles in F_g^s , and it is in fact sufficient that the lift to \mathbb{M} of each ideal arc in Δ be extremal in the hull of the lift to \mathbb{M} of these two ideal triangles. Membership in $\mathcal{E}(\Delta)$ is therefore guaranteed by a collection of coupled conditions, one such “face condition” for each ideal arc in Δ , where the face condition corresponding to an ideal arc in Δ is the constraint that the associated simplex in \mathbb{M} have positive volume. Thus, a point $\tilde{\Gamma} \in \tilde{\mathcal{F}}_g^s$ lies in the cell corresponding to G if and only if each face condition holds for the λ -lengths (with respect to $\tilde{\Gamma}$) of edges of G .

In fact, each face condition can be expressed in terms of λ -lengths as follows. Let G be a trivalent fatgraph with distinct hooks \hat{e}_1, \hat{e}_2 of a

common edge e of G , and let $(v_1, \hat{e}_1), (v_1, \hat{f}_1), (v_1, \hat{g}_1)$ be distinct sectors and $(v_2, \hat{e}_2), (v_2, \hat{f}_2), (v_2, \hat{g}_2)$ be distinct sectors (perhaps with $v_1 = v_2$); assume moreover that the hooks incident on vertex v_i occur in the cyclic order $\hat{e}_i, \hat{f}_i, \hat{g}_i$ for $i = 1, 2$. The face condition corresponding to the edge e of G is

$$\begin{aligned} X_e &= X_e(\tilde{\Gamma}) = \sum_{i=1}^2 \frac{\lambda^2(\hat{f}_i) + \lambda^2(\hat{g}_i) - \lambda^2(\hat{e}_i)}{\lambda(\hat{e}_i)\lambda(\hat{f}_i)\lambda(\hat{g}_i)} \\ &= \sum_{i=1}^2 \alpha(\hat{f}_i) + \alpha(\hat{g}_i) - \alpha(\hat{e}_i) > 0, \end{aligned}$$

so the face conditions are linear in h -lengths. One imagines $\mathcal{D}(G)$ in h -length space, therefore, as an affine slice of the intersection of a quadric variety with a finite collection of half-spaces.

Proof of Lemma 3.2.2. Since the coordinate transformations on λ -lengths associated with changes of underlying marked fatgraphs are diffeomorphisms (indeed, they are rational; cf. the last assertion of Theorem 3.4.4), we may choose any convenient fatgraph to prove transversality. Let G be a trivalent fatgraph with $F(G)$ homeomorphic to F_g^s , where G has $s - 1$ loops (i.e., cycles of length one), say $\{l_i\}_1^{s-1}$, where l_i has endpoint v_i for $i = 1, \dots, s - 1$. There is a distinct puncture of F_g^s associated with each l_i , and we let ρ_i be the hyperbolic length of the corresponding horocycle, ρ_s the hyperbolic length of the remaining horocycle of F_g^s , $\hat{t}_i \notin l_i$ the hook of G incident on v_i , and t_i the edge of G containing \hat{t}_i for $i = 1, \dots, s - 1$.

Now, identify $\tilde{\mathcal{F}}_g^s$ with the quadric determined by the coupling equations and let $\{\alpha_i\}_1^{2N}$ denote the various h -lengths, where G has N edges. We may assume that $\alpha_i = \alpha(v_i, \hat{t}_i)$, so that $\rho_i = 2\alpha_i$ for $i = 1, \dots, s - 1$, and $\rho_s = 2 \sum_s^{2N} \alpha_i$. As observed before, the face condition associated to an edge e of G is determined by the positivity of a certain function X_e , which is linear in the h -lengths, and membership in a face of codimension- p in the cell-decomposition of $\tilde{\mathcal{F}}_g^s$ corresponds to the vanishing of p distinct functions among $\{X_e : e \text{ is an edge of } G\}$.

We prove transversality by showing that the gradients

$$\{\nabla \rho_i : i = 1, \dots, s\} \cup \{\nabla X_e : e \text{ lies in a proper subset of the set of edges of } G\}$$

are linearly independent in \mathbb{R}^{2N} . In fact, the only functions among $\{X_e : e \text{ is an edge of } G\}$ with nontrivial dependence on α_i are $X_{t_i} = 2\alpha_i$ and

X_{t_i} for $i = 1, \dots, s - 1$, so we may project out the subspace spanned by $\{\nabla\rho_i : i = 1, \dots, s - 1\}$.

Now, suppose that $v \notin \{v_i\}_1^{s-1}$ is a vertex of G , say with incident hooks $\hat{e}, \hat{f}, \hat{g}$ contained, respectively, in the edges e, f, g . Only $\nabla X_e, \nabla X_f,$ and ∇X_g among $\{\nabla X_x : x \text{ is an edge of } G\}$ have a nonzero projection into the subspace Σ spanned by the vectors $\nabla\alpha(v, \hat{e}), \nabla\alpha(v, \hat{f}),$ and $\nabla\alpha(v, \hat{g})$ (taken, in this order, as a basis for Σ), and these projections are given by $(-1, 1, 1), (1, -1, 1),$ and $(1, 1, -1)$, respectively; of course, $\frac{1}{2}\nabla\rho_s$ has projection $(1, 1, 1)$ into Σ . Thus, if the projection into Σ of $a\nabla X_e + b\nabla X_f + c\nabla X_g + d\nabla\rho_s$ vanishes, then we must have

$$0 = b + c + d - a = a + c + d - b = a + b + d - c,$$

so $a = b = c = -d$.

Finally, a face of the cell-decomposition of $\tilde{\mathcal{F}}_g^s$ is determined by the vanishing of a *proper* collection of functions among $\{X_x : x \text{ is an edge of } G\}$, and the first assertion of the lemma follows easily.

For the second assertion, suppose that $\tilde{\Gamma} \in \mathcal{D}(G)$, say with the corresponding λ -length coordinates $\lambda: \{\text{edges of } G\} \rightarrow \mathbb{R}_+$. Each $\psi \in \text{Aut}(G)$ induces a permutation π_ψ on $\{\text{edges of } G\}$, and $MC_g^s(\tilde{\Gamma}) \cap \mathcal{D}(G)$ corresponds to the assignments

$$\{\lambda \circ \pi_\psi : \psi \in \text{Aut}(G)\}$$

of λ -lengths. Thus, the projection $\mathcal{D}(G) \rightarrow \phi(\mathcal{D}(G))$ is $\#\text{Aut}(G)$ -to-one unless there is a nontrivial $\psi \in \text{Aut}(G)$ so that $\lambda \circ \pi_\psi = \lambda$. If there is an unoriented edge e of G which is not fixed by π_ψ , then this imposes the restriction that $\lambda(e) = \lambda(\pi_\psi(e))$, so λ lies in a hyperplane and $\phi(\tilde{\Gamma})$ lies in a locus of measure zero in $\phi(\mathcal{D}(G))$, as asserted. If every unoriented edge of G is fixed by π_ψ , where $\psi \in \text{Aut}(G)$ is nontrivial, then an elementary argument (which we omit) shows that $g = s = 1$ and ψ is the hyperelliptic involution; in this case, simply repeat the argument above on the quotient of the decorated bundle by the hyperelliptic involution.

Remark. Notice that we have in fact proved that each level set $\{\rho_i(\tilde{\Gamma}) = r_i \text{ for } i = 1, \dots, s\}$ for any $r_1, \dots, r_s \in \mathbb{R}_+$ meets transversely each cell in the cell-decomposition of $\tilde{\mathcal{F}}_g^s$.

To close this subsection, we recall the expression in λ -lengths for the pullback of the Kähler two-form of the Weil-Petersson metric on \mathcal{F}_g^s .

Theorem 3.3.6 (Theorem A.2 of Appendix A). *Fix a trivalent fatgraph G so that $F(G)$ is homeomorphic to F_g^s . The pullback to $\tilde{\mathcal{F}}_g^s$ of the*

Weil-Petersson Kähler two-form on \mathcal{F}_g^s is given by

$$-2 \sum d \log \lambda(\hat{e}) \wedge d \log \lambda(\hat{f}) + d \log \lambda(\hat{f}) \wedge d \log \lambda(\hat{g}) + d \log \lambda(\hat{g}) \wedge d \log \lambda(\hat{e}),$$

where the sum is over all vertices of G with incident hooks in the order $\hat{e}, \hat{f}, \hat{g}$ (where the order is consistent with the fattening).

Thus, λ -lengths satisfy only one of the restrictions mentioned before on a suitable coordinatization of $\widetilde{\mathcal{F}}_g^s$; namely, the form of interest (essentially an exterior power of the two-form above) is reasonable in λ -lengths. To render the integration scheme usable for computing Weil-Petersson volumes, we will introduce certain further coordinates on each region $\mathcal{D}(G)$ in the next subsection.

3.4. Coordinates on regions of integration. Fix a marked trivalent fatgraph G , where $F(G)$ is homeomorphic to F_g^s , so that G has $N = 6g - 6 + 3s$ edges. We refer to each function

$$X_e: \widetilde{\mathcal{F}}_g^s \rightarrow \mathbb{R}$$

defined in the previous subsection for each edge e of G as a “simplicial coordinate” on G . It is convenient to combine these simplicial coordinates into a vector $\vec{X}(\tilde{\Gamma}) \in \mathbb{R}^N$ whose entries are indexed by the edges of G . Let us emphasize that whereas λ -lengths are the restriction of a pairing $\{\text{ideal arcs in } F_g^s\} \times \widetilde{\mathcal{F}}_g^s \rightarrow \mathbb{R}_+$, the simplicial coordinates depend as well on the underlying fatgraph G .

Recall the following theorem.

Theorem 3.4.1 [10, Theorem 5.4]. *If $\vec{X} \in \mathbb{R}_+^N$ is a vector whose entries are indexed by the edges of G , then there is a unique $\tilde{\Gamma} \in \widetilde{\mathcal{F}}_g^s$ so that $\vec{X}(\tilde{\Gamma}) = \vec{X}$.*

Remark. The proof involves an energy functional on \mathbb{R}_+^{2N} (whose gradient limits on the variety determined by the coupling equation) and is unfortunately nonconstructive. Actually, there is a stronger version of the theorem which allows simplicial coordinates to vanish on an appropriate collection of edges of G , but we will not require this stronger result (since we are interested here in integrating only top-dimensional forms over \mathcal{M}_g^s).

It follows that simplicial coordinates establish a homeomorphism between the interior of any cell in the decomposition of $\widetilde{\mathcal{F}}_g^s$ corresponding to G and the collection of \mathbb{R}_+ -valued functions defined on the set of edges of G . It is natural to consider the expression in simplicial coordinates for the restriction of the section $S: \mathcal{F}_g^s \rightarrow \widetilde{\mathcal{F}}_g^s$, and we have

Lemma 3.4.2. *If \mathcal{P} is a closed edge-path on G , then we have the equality*

$$2 \sum \alpha(v, \hat{e}) = \frac{1}{2} \sum X_f + X_g,$$

where each sum is over all sectors (v, \hat{e}) traversed by \mathcal{P} (counted with multiplicity), $(v, \hat{e}), (v, \hat{f}), (v, \hat{g})$ are the sectors of G incident on v , and f (and g , respectively) is the edge of G containing \hat{f} (and \hat{g}).

Proof. Consider a (not necessarily closed) edge-path on G of length two through the (not necessarily distinct) vertices v_1, v, v_2 (in this order), where $\hat{e}'_i, \hat{f}'_i, \hat{g}'_i$ are the hooks incident on v_i and \hat{e}_i, \hat{e}'_i are the hooks of a common edge e_i of G for $i = 1, 2$, and $\hat{e}_1, \hat{f}, \hat{e}_2$ are the hooks incident on v . Of course,

$$\begin{aligned} X_{e_1} + X_{e_2} &= \alpha(\hat{f}) + \alpha(\hat{e}_1) - \alpha(\hat{e}_2) + \alpha(\hat{f}) - \alpha(\hat{e}_1) \\ &\quad + \sum_{i=1}^2 \alpha(\hat{f}'_i) + \alpha(\hat{g}'_i) - \alpha(\hat{e}'_i) \\ &= 2\alpha(\hat{f}) + \sum_{i=1}^2 \alpha(\hat{f}'_i) + \alpha(\hat{g}'_i) - \alpha(\hat{e}'_i), \end{aligned}$$

and the result follows easily. **q.e.d.**

In case \mathcal{P} is a closed edge-path on G and $\tilde{\Gamma} \in \tilde{\mathcal{F}}_g^s$, we define $X(\mathcal{P}; \tilde{\Gamma}) = X(\mathcal{P}) = 2 \sum \alpha(v, \hat{e})$, where the sum is over all sectors (v, \hat{e}) (counted with unsigned multiplicity) traversed by \mathcal{P} , and we have

Corollary 3.4.3. *Fix a marked trivalent fatgraph G . In simplicial coordinates on $\mathcal{D}(G)$, the region $\mathcal{D}(G)$ of integration is given by the straight simplex*

$$\{\vec{X} : X_e > 0 \text{ for each edge } e \text{ of } G \text{ and } X(\mathcal{P}_i) = 1 \text{ for } i = 1, \dots, s\},$$

where \mathcal{P}_i is the closed edge-path on G corresponding to the i th puncture of F_g^s .

Proof. The previous lemma gives that $\rho_i(\tilde{\Gamma}) = X(\mathcal{P}_i; \tilde{\Gamma})$ for $i = 1, \dots, s$ and each $\tilde{\Gamma} \in \tilde{\mathcal{F}}_g^s$; the result then follows immediately from Theorem 3.4.1. **q.e.d.**

Thus, simplicial coordinates satisfy the second provision of our integration scheme since the regions of integration are pleasant in these coordinates. Clearly, if we could compute λ -lengths in terms of simplicial coordinates on G for each trivalent fatgraph G , then the integration scheme would provide an effective means of computing Weil-Petersson volumes. It is to this purpose that the next subsection is dedicated; we will achieve only limited success but sufficient for our applications below.

3.5. The arithmetic problems. Our considerations of the previous subsections have led to a collection of arithmetic problems (to be restated below for the convenience of the reader) which must be solved for the systematic application of our integration scheme to the computation of Weil-Petersson volumes. Indeed, there is one such problem to be solved for each trivalent fatgraph; namely, we must compute λ -lengths from simplicial coordinates. The fattening of the graph evidently plays no role in the arithmetic problems, and we must solve the following

Arithmetic problem. Suppose that G is a trivalent graph, say with edges $\{e_i\}_{i=1}^N$, and associate to the edge e_i a real-valued variable λ_i (the “ λ -length”) for $i = 1, \dots, N$. Suppose that \hat{e}_i^1 and \hat{e}_i^2 are the hooks of G contained in e_i and incident on the (not necessarily distinct) vertices v and w , respectively. Suppose that $\hat{e}_j, \hat{e}_k, \hat{e}_l^1$ (and $\hat{e}_l, \hat{e}_m, \hat{e}_l^2$, respectively) are the hooks of G incident on v (and w) and that $\hat{e}_j, \hat{e}_k, \hat{e}_l, \hat{e}_m$ are contained in the (not necessarily distinct) edges e_j, e_k, e_l, e_m , respectively. Define the “simplicial coordinate”

$$X_i = \frac{\lambda_j^2 + \lambda_k^2 - \lambda_i^2}{\lambda_i \lambda_j \lambda_k} + \frac{\lambda_l^2 + \lambda_m^2 - \lambda_i^2}{\lambda_i \lambda_l \lambda_m}.$$

Associate to the edge e_i of G is simplicial coordinate X_i in this way for $i = 1, \dots, N$, and denote this system of coupled nonlinear equations by $(*_G)$.

We ask for a solution to $(*_G)$ in the following sense: suppose that the $\lambda_i > 0$ for $i = 1, \dots, N$ are chosen in a range so that $X_i > 0$ for $i = 1, \dots, N$; we must express $\lambda_j = \lambda_j(X_1, \dots, X_N)$ as a function of $\{X_i\}_{i=1}^N$ for each $j = 1, \dots, N$.

Remark. For applications to once-punctured surfaces, one needs only compute the volume form $\prod_{i=1}^N d \log \lambda_i$ in terms of $\{X_i\}_{i=1}^N$ (see §6.2).

It follows from Theorem 3.4.1 that the system $(*_G)$ admits a unique solution for each trivalent graph G . For later application, we next give some examples.

Example 3.5.1. Suppose that G is the fatgraph whose corresponding permutations are

$$\sigma = (2, 3, 1, 5, 6, 4) \quad \text{and} \quad \tau = (4, 5, 6, 1, 2, 3)$$

(in the notation of §3.1), and let a, b, c denote the λ -lengths of G with corresponding simplicial coordinates A, B, C , respectively. Thus, we have

$$A = 2 \frac{b^2 + c^2 - a^2}{abc}, \quad B = 2 \frac{a^2 + c^2 - b^2}{abc}, \quad C = 2 \frac{a^2 + b^2 - c^2}{abc}$$

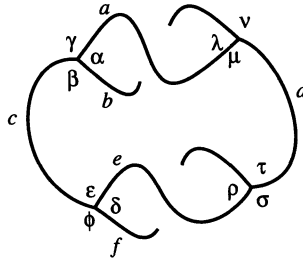


FIGURE 4

so

$$a^2 = \frac{16}{(A+B)(A+C)}, \quad b^2 = \frac{16}{(A+B)(B+C)}, \quad c^2 = \frac{16}{(A+C)(B+C)},$$

solving the system $(*_G)$.

Example 3.5.2. Let G denote the fatgraph illustrated in Figure 4, and adopt the notation indicated there for the edges of G . (The Greek letters in the figure denote the associated h -lengths and will be used below.) We abuse notation slightly and let a, b, \dots, f denote the λ -lengths of the corresponding edges and let A, B, \dots, F , respectively, denote the simplicial coordinates of the edges a, b, \dots, f . As before, associated with a closed edge-path of G , we consider the sum of the simplicial coordinates (counted with unsigned multiplicity) of the edges traversed, and define the quantities

- (1)
$$Q_1 = A + B = \frac{2(c+d)}{ab},$$
- (2)
$$Q_2 = B + C + D + E = 2 \left(\frac{f}{e} + \frac{a}{b} \right) \left(\frac{1}{c} + \frac{1}{d} \right),$$
- (3)
$$Q_3 = A + C + D + E = 2 \left(\frac{f}{e} + \frac{b}{a} \right) \left(\frac{1}{c} + \frac{1}{d} \right),$$
- (4)
$$Q_4 = B + C + D + F = 2 \left(\frac{e}{f} + \frac{a}{b} \right) \left(\frac{1}{c} + \frac{1}{d} \right),$$
- (5)
$$Q_5 = A + C + D + F = 2 \left(\frac{e}{f} + \frac{b}{a} \right) \left(\frac{1}{c} + \frac{1}{d} \right),$$

$$(6) \quad Q_6 = E + F = \frac{2(c+d)}{ef},$$

$$(7) \quad Q_7 = Q_3 + Q_4,$$

$$(8) \quad Q_8 = 2C + Q_1 + Q_6 = d \left(\frac{1}{ab} + \frac{1}{ef} \right) + \frac{1}{c} \left(\frac{a}{b} + \frac{b}{a} + \frac{e}{f} + \frac{f}{e} \right),$$

$$(9) \quad Q_9 = 2D + Q_1 + Q_6 = c \left(\frac{1}{ab} + \frac{1}{ef} \right) + \frac{1}{d} \left(\frac{a}{b} + \frac{b}{a} + \frac{e}{f} + \frac{f}{e} \right).$$

We cannot give any motivation for the particular choice of edge-paths above other than to say that they arise naturally in our computations. It follows easily from (2) and (5) (and (3) and (4), respectively) that

$$Q_2 = \frac{af}{be} Q_5 \quad \text{and} \quad Q_4 = \frac{ae}{bf} Q_3,$$

so

$$\left(\frac{a}{b} \right)^2 = \frac{Q_1 Q_4}{Q_3 Q_5} \quad \text{and} \quad \left(\frac{f}{e} \right)^2 = \frac{Q_2 Q_3}{Q_4 Q_5}.$$

Substituting these into (6) gives

$$\left(\frac{e}{a} \right)^2 = \frac{Q_1 Q_5}{Q_2 Q_6},$$

so

$$\left(\frac{f}{a} \right)^2 = \left(\frac{e}{a} \right)^2 \left(\frac{f}{e} \right)^2 = \frac{Q_1 Q_3}{Q_4 Q_6}.$$

From (1) and (2), then, we have

$$\left(\frac{c+d}{cd} \right)^2 = \left(\frac{\frac{b}{a} Q_1}{2 \frac{e}{a} \frac{d}{a}} \right)^2 = \left(\frac{Q_2}{2 \left(\frac{e}{b} + \frac{f}{e} \right)} \right)^2,$$

so

$$\left(\frac{cd}{a^2} \right)^2 = \left(\frac{\left(\frac{e}{b} + \frac{f}{e} \right) \frac{b}{a} Q_1}{Q_2} \right)^2 = \left(\frac{(1 + Q_3/Q_4) Q_1}{Q_2} \right)^2;$$

that is,

$$\frac{cd}{a^2} = \frac{Q_1 Q_7}{Q_2 Q_4}.$$

Now, from (8) and (9), $dQ_9 = cQ_8$, and therefore

$$\left(\frac{d}{a}\right)^2 = \frac{Q_1 Q_7 Q_8}{Q_2 Q_4 Q_9} \quad \text{and} \quad \left(\frac{c}{a}\right)^2 = \frac{Q_1 Q_7 Q_9}{Q_2 Q_4 Q_8}.$$

Finally, using (1) again, we have

$$a^2 Q_1^2 = \frac{4\left(\frac{c}{a} + \frac{d}{a}\right)^2}{\left(\frac{b}{a}\right)^2} = \frac{4Q_1 Q_7 (Q_8 + Q_9)^2}{Q_3 Q_5 Q_8 Q_9}.$$

Thus, if

$$K = \frac{4Q_7 (Q_8 + Q_9)^2}{Q_8 Q_9},$$

then

$$\begin{aligned} a^2 &= \frac{K}{Q_1 Q_3 Q_5}, & b^2 &= \frac{K}{Q_1 Q_2 Q_4}, \\ c^2 &= \frac{K Q_7 Q_9}{Q_2 Q_3 Q_4 Q_5 Q_8}, & d^2 &= \frac{K Q_7 Q_8}{Q_2 Q_3 Q_4 Q_5 Q_9}, \\ e^2 &= \frac{K}{Q_2 Q_3 Q_6}, & f^2 &= \frac{K}{Q_4 Q_5 Q_6}, \end{aligned}$$

solving the system $(*_G)$.

Remark. In fact, this computation was first performed on the computer using [6] on a MacII. Indeed, many trivalent graphs have been similarly handled, and the phenomena of the examples above are typical. Specifically, for each graph G , there is a collection $\{P_j\}_{j=1}^M$ of closed edge-paths on G so that the associated “ h -lengths” (derived from the λ -lengths as before) are rational functions of $\{X(P_j)\}_{j=1}^M$, where $X(P)$ denotes the sum of the simplicial coordinates of edges traversed by P as in §3.4. It is tempting to conjecture that this is always the case; notice that this conjecture implies, in particular, that each Weil-Petersson volume is in integral of a certain rational function over a straight simplex.

Crucial for our asymptotic estimates of Weil-Petersson volumes is

Proposition 3.5.3. Fix a trivalent fatgraph G with edges $\{e_i\}_{i=1}^N$, let λ_i and X_i , respectively, denote the λ -length and simplicial coordinate of e_i for $i = 1, \dots, N$, and let ρ denote the sum of all h -lengths of sectors of G . Then $X_i = -\partial \rho / \partial \log \lambda_i$, and $|\det H| \leq (3\rho/N)^N$, where H is the hessian of ρ as a function of $\{\log \lambda_i\}_1^N$.

Proof. If we adopt the notation in the statement of the Arithmetic Problem for the edges of G near e_i , then

$$\frac{\partial \rho}{\partial \log \lambda_i} = \frac{\partial}{\partial \log \lambda_i} \left[\frac{\lambda_j^2 + \lambda_k^2 + \lambda_i^2}{\lambda_i \lambda_j \lambda_k} + \frac{\lambda_l^2 + \lambda_m^2 + \lambda_i^2}{\lambda_i \lambda_l \lambda_m} \right] = -X_i,$$

proving the first part.

For the second part, we suppose first that G has no loops, and define a matrix $A^{N \times 2N}$ as follows. The rows of A are indexed by the edges, and the columns of A are indexed by the sectors of G . If the l th sector of G lies between the hooks \hat{e}_j and \hat{e}_k and is opposite the hook \hat{e}_i , where $\hat{e}_i, \hat{e}_j, \hat{e}_k$ are contained in the (distinct) edges e_i, e_j, e_k of G , respectively, then we define

$$1 = A_{il} = -A_{jl} = -A_{kl} \quad \text{and} \quad A_{ml} = 0 \quad \text{if } m \notin \{i, j, k\}.$$

In case G has loops and $j = k$, then $A_{il} = 1$ and $A_{jl} = -2$, while if $i = j$ (for instance), then $A_{il} = 0$ and $A_{kl} = -1$.

As an example, consider the fatgraph G in Figure 4, where the λ -lengths and h -lengths are as indicated and will be taken in their respective alphabetic orders. With this notation, the matrix A associated to G is given by

$$A = \begin{pmatrix} -1 & +1 & -1 & 0 & 0 & 0 & -1 & -1 & +1 & 0 & 0 & 0 \\ -1 & -1 & +1 & 0 & 0 & 0 & -1 & +1 & -1 & 0 & 0 & 0 \\ +1 & -1 & -1 & +1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & +1 & -1 & -1 & +1 & -1 & -1 \\ 0 & 0 & 0 & -1 & -1 & +1 & 0 & 0 & 0 & -1 & -1 & +1 \\ 0 & 0 & 0 & -1 & +1 & -1 & 0 & 0 & 0 & -1 & +1 & -1 \end{pmatrix}.$$

If we adopt the notation $\text{LOG}(x_i)_{i=1}^N = (\log x_i)_{i=1}^N$ and $\text{EXP}(x_i)_{i=1}^N = (\exp x_i)_{i=1}^N$, then by definition

$$(X_i)_{i=1}^N = -A \text{EXP} A^t \text{LOG}(\lambda_i)_{i=1}^N,$$

where the superscript t denotes the transpose, so by the chain rule

$$H = ADA^t,$$

D being the diagonal matrix with entries given by the h -lengths. In particular, if X_i is given linearly in h -lengths by

$$X_i = \frac{\lambda_j}{\lambda_i \lambda_k} + \frac{\lambda_k}{\lambda_i \lambda_j} - \frac{\lambda_i}{\lambda_j \lambda_k} + \frac{\lambda_l}{\lambda_i \lambda_m} + \frac{\lambda_m}{\lambda_i \lambda_l} - \frac{\lambda_i}{\lambda_l \lambda_m},$$

as above, then one finds that

$$H_{ii} = \frac{\lambda_j}{\lambda_i \lambda_k} + \frac{\lambda_k}{\lambda_i \lambda_j} + \frac{\lambda_i}{\lambda_j \lambda_k} + \frac{\lambda_l}{\lambda_i \lambda_m} + \frac{\lambda_m}{\lambda_i \lambda_l} + \frac{\lambda_i}{\lambda_l \lambda_m}.$$

Finally, since the determinant is the product and the trace the sum of the eigenvalues, by the dominance of geometric over arithmetic mean, we have

$$|\det H| \leq \left(\frac{\text{tr } H}{N} \right)^N = \left(\frac{3\rho}{N} \right)^N,$$

as was asserted.

4. The once-punctured torus

For completeness and to explain a small subtlety, we include here the computation of the Weil-Petersson volume μ_1^1 of \mathcal{M}_1^1 .

It is an easy exercise (using Corollary 3.3.2) to check that there is a unique isomorphism class of unmarked trivalent fatgraph G whose associated surface is homeomorphic to F_1^1 ; namely, the fatgraph G of Example 3.5.1.

The subtlety here arises from the hyperelliptic involution, which is manifested as a fatgraph automorphism $\iota \in \text{Aut}(G)$ which leaves invariant each (unoriented) edge of G . Of course, according to Theorem 3.3.1, if we specify a marking on G , then tuples of λ -lengths on the edges of G give coordinates on \mathcal{F}_1^1 . On the other hand, ordered triples of positive real numbers determine only a point of \mathcal{F}_1^1/ι .

Remark. This observation corrects our discussion of the representation of MC_1^1 (cf. Theorem 3.3.4) in [10, §7]. More generally, suppose that G is a trivalent fatgraph whose associated surface is homeomorphic to F_g^s with $gs \neq 1$. Given an ordered triple of edges of G , there is at most one vertex of G on which the edges are incident. Thus, aside from $F_g^s = F_1^1$, we may identify \mathcal{F}_g^s with $\mathbb{R}_+^{6g-6+3s}$, and the corresponding action of MC_g^s on $\mathbb{R}_+^{6g-6+3s}$ gives a faithful representation.

4.1. Computation for the once-punctured torus. Let G be the fatgraph of Example 3.5.1, and adopt the notation there for the λ -lengths and simplicial coordinates of the edges of G ; we assume, moreover, that the fattening on G is compatible with the ordering a, b, c on the λ -lengths.

By Theorem 3.3.6, the Weil-Petersson Kähler two-form (which is the volume form in this case) pulls back to

$$\begin{aligned} \omega &= 4(d \log a \wedge d \log b + d \log b \wedge d \log c + d \log c \wedge d \log a) \\ &= \frac{2(C dA \wedge dB + A dB \wedge dC + B dC \wedge dA)}{(B + C)(C + A)(A + B)}, \end{aligned}$$

where the second equality follows from Example 3.5.1. Of course, the horocycle has hyperbolic length $2(A + B + C)$ by Lemma 3.4.2, so the canonical section $S: \mathcal{T}_1^1 \rightarrow \widetilde{\mathcal{T}}_1^1$ is determined by the condition that $C = \frac{1}{2} - A - B$. Thus, we find that the volume form is given by

$$\omega = \frac{dA \wedge dB}{(A + B)(\frac{1}{2} - A)(\frac{1}{2} - B)}.$$

One checks easily that $\# \text{Aut}(G) = 6$, so it follows from Theorem 3.2.1 that

$$\mu_1^1 = \frac{1}{6} \int_{\mathcal{D}(G)} \omega.$$

In light of the remarks above, we find from Corollary 3.4.3 that $\mathcal{D}(G)/i$ is coordinatized by

$$\mathcal{D} = \{A > 0, B > 0 : A + B < \frac{1}{2}\},$$

so

$$\begin{aligned} \mu_1^1 &= 2 \frac{1}{6} \int_{\mathcal{D}} \omega = \frac{1}{3} \int_{\mathcal{D}} \frac{dA \wedge dB}{(A + B)(\frac{1}{2} - A)(\frac{1}{2} - B)} \\ &= \frac{1}{3} \int_0^{1/2} dA \int_0^{1/2-A} dB \frac{\partial}{\partial B} \log \left(\frac{A + B}{\frac{1}{2} - b} \right) \\ &= -\frac{4}{3} \int_0^1 \frac{dx \log x}{1 - x^2} = \frac{\pi^2}{6}, \end{aligned}$$

and we have

Theorem 4.1.1. *The Weil-Petersson volume of \mathcal{M}_1^1 is $\pi^2/6$.*

Remark. As in [22], it follows that the Weil-Petersson volume of \mathcal{M}_0^4 is $\pi^2/3$.

5. The twice-punctured torus

In this section, we compute the Weil-Petersson volume μ_1^2 of the moduli space of the twice-punctured torus. In the first part we derive an expression for μ_1^2 as an explicit integral over a four-dimensional simplex, and in the second part we evaluate this integral in closed form.

5.1. Derivation of the integral expression. To begin, we show that only one of the trivalent fatgraphs in Table 1 contributes to the integral. To this end, recall that we have illustrated these fatgraphs in Figure 1, and adopt the notation there, where an upper case letter next to an edge denotes the corresponding simplicial coordinate. In light of Lemma 3.4.2, we can compute the hyperbolic length of each of the horocycles in terms of the simplicial coordinates, and these expressions are given in Table 2.

The Permutation τ	The hyperbolic lengths of horocycles
(2, 1, 4, 3, 7, 10, 5, 11, 12, 6, 8, 9)	$F, F + 2(A + B + C + D + E)$
(4, 5, 7, 1, 2, 10, 3, 11, 12, 6, 8, 9)	$2C + A + B + E + F, 2D + A + B + E + F$
(4, 5, 7, 1, 2, 10, 3, 12, 11, 6, 9, 8)	$E + F, E + F + 2(A + B + C + D)$
(4, 7, 10, 1, 8, 11, 2, 5, 12, 3, 6, 9)	$2(A + D) + B + C + E + F, B + C + E + F$
(4, 7, 10, 1, 8, 12, 2, 5, 11, 3, 9, 6)	$A + B + C, A + B + C + 2(D + E + F)$

TABLE 2. Hyperbolic lengths of horocycles on F_1^2

It follows by inspection of the expressions in Table 2 that if the two horocycles are required to have the same hyperbolic length, then some simplicial coordinate must vanish *except* in the case of the fatgraph corresponding to $\tau = (4, 5, 7, 1, 2, 10, 3, 11, 12, 6, 8, 9)$. For the remainder of this section, we shall let G denote this fatgraph and adopt simplicial coordinates on the edges of G as indicated in Figure 4.

We summarize with

Corollary 5.1.1. *If ω is a four-form on the moduli space \mathcal{M}_1^2 , then*

$$\int_{\mathcal{M}_1^2} \omega = \frac{1}{4} \int_{\mathcal{D}(G)} \phi^*(\omega)$$

(where, recall, $\phi: \widetilde{\mathcal{F}}_1^2 \rightarrow \mathcal{M}_1^2$ is the natural map). Furthermore, with respect to the simplicial coordinates A, B, E, F on the edges of G , the domain of integration is given by the unit simplex

$$\mathcal{D}(G) = \{A, B, E, F > 0 : A + B + E + F < 1\}.$$

Proof. It follows from the discussion given above that the image of the canonical section $S: \mathcal{F}_1^2 \rightarrow \widetilde{\mathcal{F}}_1^2$ is contained in the closure of $MC_1^2(\mathcal{D}(G))$, and the expression for the integral follows directly from Corollary 3.4.3 since $\#\text{Aut}(G) = 4$ from Table 1. Furthermore, the canonical section S

is given in simplicial coordinates by the conditions

$$C = D = \frac{1}{2}(1 - A - B - E - F),$$

so we may take the simplicial coordinates A, B, E, F as coordinates on $\mathcal{D}(G)$ subject only to the stated restrictions. q.e.d.

As before, we let a, b, \dots, f , respectively, denote the λ -length of the edge of G with associated simplicial coordinate A, B, \dots, F . Furthermore, let $\tilde{a} = d \log a, \tilde{b} = d \log b, \dots, \tilde{f} = d \log f$. For convenience, we will often drop the symbol \wedge in an exterior product and write, for instance, simply $\tilde{a}\tilde{b}$ for $\tilde{a} \wedge \tilde{b}$.

According to Theorem 3.3.6, the Weil-Petersson Kähler two-form pulls back to

$$\begin{aligned} \omega_1 &= -2(\tilde{a}\tilde{c} + \tilde{c}\tilde{b} + \tilde{b}\tilde{a} + \tilde{a}\tilde{d} + \tilde{d}\tilde{b} + \tilde{e}\tilde{c} + \tilde{c}\tilde{f} + \tilde{f}\tilde{e} + \tilde{f}\tilde{e} + \tilde{e}\tilde{d} + \tilde{d}\tilde{f}) \\ &= -2[2(\tilde{f}\tilde{e} + \tilde{b}\tilde{a}) + (\tilde{c} + \tilde{d})(\tilde{b} + \tilde{f} - \tilde{a} - \tilde{e})], \end{aligned}$$

so the Weil-Petersson volume form pulls back to

$$\begin{aligned} \omega &= \frac{1}{2}\omega_1^2 = 8(\tilde{f}\tilde{e} + \tilde{b}\tilde{a})[\tilde{f}\tilde{e} + \tilde{b}\tilde{a} + (\tilde{c} + \tilde{d})(\tilde{b} + \tilde{f} - \tilde{a} - \tilde{e})] \\ &= 8[(\tilde{f} - \tilde{b})(\tilde{e} - \tilde{b})(\tilde{a} - \tilde{b})(\tilde{c} - \tilde{b}) + (\tilde{f} - \tilde{b})(\tilde{e} - \tilde{b})(\tilde{a} - \tilde{b})(\tilde{d} - \tilde{b})] \\ &= \frac{1}{2}d \log \left(\frac{f}{b}\right)^2 \wedge d \log \left(\frac{e}{b}\right)^2 \wedge d \log \left(\frac{a}{b}\right)^2 \wedge d \log \left[\left(\frac{c}{b}\right)^2 \left(\frac{d}{b}\right)^2\right], \end{aligned}$$

where the third equality is easily verified directly.

Recall the expressions $\{Q_j\}_1^9$ which arose in the solution of $(*_G)$ in Example 3.5.2, and adopt the convention, as above, that $\tilde{Q}_j = d \log Q_j$ for $j = 1, \dots, 9$. Using the solution to $(*_G)$, we can express ω in terms of $\{Q_j\}$ as

$$\begin{aligned} \omega &= (\tilde{Q}_1 + \tilde{Q}_2 - \tilde{Q}_5 - \tilde{Q}_6)(\tilde{Q}_1 + \tilde{Q}_4 - \tilde{Q}_3 - \tilde{Q}_6) \\ &\quad \times (\tilde{Q}_2 + \tilde{Q}_4 - \tilde{Q}_3 - \tilde{Q}_5)(\tilde{Q}_1 + \tilde{Q}_7 - \tilde{Q}_3 - \tilde{Q}_5). \end{aligned}$$

Finally, on the image of the section S , we find

$$\begin{aligned} Q_1 &= A + B, & Q_2 &= 1 - A - F, \\ Q_3 &= 1 - B - F, & Q_4 &= 1 - A - E, \\ Q_5 &= 1 - B - E, & Q_6 &= E + F, \\ Q_7 &= 2 - A - B - E - F, & Q_8 &= Q_9 = 1, \end{aligned}$$

and we set $q_i = Q_i^{-1}$ for $i = 1, \dots, 9$ for convenience. A short computation gives that

$$\omega = (\det M) dA \wedge dB \wedge dE \wedge dF,$$

where

$$\begin{aligned}
 M &= \begin{pmatrix} +q_1 - q_2 & +q_1 + q_5 & +q_5 - q_6 & -q_2 - q_6 \\ +q_1 - q_4 & +q_1 + q_3 & -q_4 - q_6 & +q_3 - q_6 \\ -q_2 - q_4 & +q_3 + q_5 & +q_5 - q_4 & +q_3 - q_2 \\ +q_1 - q_7 & q_1 + q_3 + q_5 - q_7 & +q_5 - q_7 & +q_3 - q_7 \end{pmatrix} \\
 &= \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} q_3 q_4 & 0 & 0 & 0 \\ 0 & q_2 q_5 & 0 & 0 \\ 0 & 0 & q_1 q_6 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
 &\quad \times \begin{pmatrix} -Q_3 & +Q_4 & -Q_3 & +Q_4 \\ -Q_5 & +Q_2 & +Q_2 & -Q_5 \\ +Q_6 & +Q_6 & -Q_1 & -Q_1 \\ q_1 - q_7 & q_1 + q_3 + q_5 - q_7 & q_5 - q_7 & q_3 - q_7 \end{pmatrix}.
 \end{aligned}$$

Routine matrix manipulations (using the fact that $Q_7 = Q_3 + Q_4 = Q_2 + Q_5$) allow the final computation of $\det M$, and one finds that the Weil-Petersson volume form pulls back to

$$\omega = \frac{8Q_7}{Q_1 Q_2 Q_3 Q_4 Q_5 Q_6} dA \wedge dB \wedge dE \wedge dF.$$

Our main result for this section then follows from Corollary 5.1.1:

Proposition 5.1.2. *The Weil-Petersson volume μ_1^2 of \mathcal{M}_1^2 is*

$$\mu_1^2 = \int_{\Delta} \frac{1 - \sum_{i=1}^4 x_i}{(x_1 + x_2)(x_3 + x_4)(\frac{1}{2} - x_1 - x_3)(\frac{1}{2} - x_1 - x_4)(\frac{1}{2} - x_2 - x_3)(\frac{1}{2} - x_2 - x_4)},$$

where \int_{Δ} denotes the integral over the simplex $\{x_i > 0 \text{ for } i = 1, 2, 3, 4 : \sum_{i=1}^4 x_i < \frac{1}{2}\}$ with respect to the Euclidean volume element.

Proof. This expression arises from that derived above by a homothetic change of variables.

Remark. The integrand admits an eight-fold symmetry group whereas $\#\text{Aut}(G) = 4$; the ‘‘extra’’ symmetry arises from a change of orientation on F_1^2 .

5.2. Evaluation of the integral. This subsection is dedicated to the computation of μ_1^2 in closed form, and our starting point is the integral expression in Proposition 5.1.2. Our general computational scheme is in the category of folklore in quantum field theory (and was learned by the author in bits and pieces from Itzhak Bars, Eric D’Hoker, and H. S. La).

Recall that if $x > 0$, then $\frac{1}{x} = \int_0^{\infty} d\sigma \exp\{-\sigma x\}$, and σ is called the ‘‘Schwinger parameter’’ associated to $\frac{1}{x}$. Introducing Schwinger parameters $\alpha, \beta, \dots, \phi$, respectively, for the factors $(x_1 + x_2), (x_3 + x_4), \dots$,

$(\frac{1}{2} - x_2 - x_4)$ in the denominator of the integrand in Proposition 5.1.2, we find

$$\mu_1^2 = \int_{\Delta} \int_{\mathcal{O}} \left(1 - \sum_{i=1}^4 x_i \right) \exp \left(-\frac{h}{2} + \sum_{i=1}^4 h_i x_i \right),$$

where

$$\begin{aligned} h &= \gamma + \delta + \epsilon + \phi, & h_1 &= \gamma + \delta - \alpha, \\ h_2 &= \epsilon + \phi - \alpha, & h_3 &= \gamma + \epsilon - \beta, & h_4 &= \delta + \phi - \beta, \end{aligned}$$

and $\int_{\mathcal{O}}$ denotes the integral over the orthant

$$\mathcal{O} = \mathcal{O}(\alpha, \beta, \dots, \phi) = \{0 < \alpha < \infty, 0 < \beta < \infty, \dots, 0 < \phi < \infty\}$$

with respect to $d\alpha d\beta \dots d\phi$. Since the integrand is nonnegative, convergence is equivalent to absolute convergence (and we henceforth omit such routine justifications), so

$$\mu_1^2 = \int_{\mathcal{O}} \exp \left(-\frac{h}{2} \right) \int_{\Delta} \left(1 - \sum_{i=1}^4 x_i \right) \exp \left(\sum_{i=1}^4 h_i x_i \right).$$

Now, we apply the standard formula

$$\begin{aligned} &\int_{\Delta(s)} f \left(\sum_{i=1}^n y_i \right) \exp \left(\sum_{i=1}^n k_i y_i \right) \prod_{i=1}^n dy_i \\ &= \int_0^s d\lambda f(\lambda) \int_{\Gamma} \frac{dz \exp(\lambda z)}{2\pi\sqrt{-1}} \prod_{i=1}^n (z - k_i)^{-1}, \end{aligned}$$

where $\Delta(s)$ denotes the s -simplex $\{y_i > 0 \text{ for } i = 1, \dots, n : \sum_{i=1}^n y_i < s\}$, and Γ is any contour $\{t + \sqrt{-1}\xi : \xi \in \mathbb{R}\}$ so that $t > \max\{k_i\}_1^n$. Thus,

$$\mu_1^2 = \int_{\mathcal{O}} \exp \left(-\frac{h}{2} \right) \int_0^{1/2} d\lambda (1 - \lambda) \int_{\Gamma} \frac{dz \exp(\lambda z)}{2\pi\sqrt{-1}} \prod_{i=1}^4 (z - h_i)^{-1},$$

and we may take the contour $\Gamma = \{z = t + \sqrt{-1}\xi : \xi \in \mathbb{R}\}$, where $t > h > \max\{h_i\}_1^4$. Since the contour is chosen independently of the variables α

and β , we find

$$\begin{aligned} \mu_1^2 &= \int_0^{1/2} d\lambda(1-\lambda) \int_{\mathcal{O}(\gamma, \delta, \epsilon, \phi)} \exp\left(-\frac{h}{2}\right) \int_{\Gamma} \frac{dz \exp(\lambda z)}{2\pi\sqrt{-1}} \\ &\quad \times \int_{\mathcal{O}(\alpha, \beta)} \frac{1}{(z+\alpha-\gamma-\delta)(z+\alpha-\epsilon-\phi)(z+\beta-\gamma-\epsilon)(z+\beta-\delta-\phi)} \\ &= \int_0^{1/2} d\lambda(1-\lambda) \int_{\mathcal{O}} \frac{\exp(-h/2)}{(\gamma+\delta-\epsilon-\phi)(\gamma+\epsilon-\delta-\phi)} \\ &\quad \times \int_{\Gamma} \frac{dz \exp(\lambda z)}{2\pi\sqrt{-1}} \log \frac{z-\epsilon-\phi}{z-\gamma-\delta} \log \frac{z-\delta-\phi}{z-\gamma-\epsilon}, \end{aligned}$$

where \log denotes the principal value of logarithm with branch cut the negative real axis. To justify this, we must check that the arguments of \log avoid the branch cut; we compute

$$\operatorname{Re} \left(\log \frac{z-\epsilon-\phi}{z-\gamma-\delta} \right) = \frac{(t-\epsilon-\phi)(t-\gamma-\delta) + \xi^2}{(t-\gamma-\delta)^2 + \xi^2} > 0$$

since $t > h$, and similarly for $(z-\delta-\phi)/(z-\gamma-\epsilon)$.

Furthermore, we claim that

$$\log \frac{z-\epsilon-\phi}{z-\gamma-\delta} = 2 \sum_{k=1}^{\infty} \frac{1}{2k-1} \left(\frac{\gamma+\delta-\epsilon-\phi}{2z-h} \right)^{2k-1},$$

and the convergence is absolute. To see this, we observe that

$$\begin{aligned} \log \frac{1+\eta}{1-\eta} &= 2 \sum_{k=1}^{\infty} \frac{\eta^{2k-1}}{2k-1} \quad \text{for } |\eta|^2 < 1, \\ \zeta &= \frac{1+(\zeta-1)/(\zeta+1)}{1-(\zeta-1)/(\zeta+1)}, \end{aligned}$$

so the claim follows provided

$$\begin{aligned} 1 &> \left| \frac{(z-\epsilon-\phi)/(z-\gamma-\delta) - 1}{(z-\epsilon-\phi)/(z-\gamma-\delta) + 1} \right|^2 = \left| \frac{\gamma+\delta-\epsilon-\phi}{2z-h} \right|^2 \\ &= \frac{(\gamma+\delta-\epsilon-\phi)^2}{4\xi^2 + (2t-h)^2}, \end{aligned}$$

and this holds because of the condition that $t > h$. There is a similar

expression for $\log(z - \delta - \phi)/(z - \gamma - \epsilon)$, and we find

$$\begin{aligned} \mu_1^2 &= 4 \int_0^{1/2} d\lambda(1 - \lambda) \\ &\times \int_{\mathcal{O}} \exp\left(-\frac{h}{2}\right) \sum_{k,l=1}^{\infty} \frac{(\gamma + \delta - \epsilon - \phi)^{2k-2}(\gamma + \epsilon - \delta - \phi)^{2l-2}}{(2k - 1)(2l - 1)} \\ &\times \int_{\Gamma} \frac{dz}{2\pi\sqrt{-1}} \frac{\exp(\lambda z)}{(2z - h)^{2k+2l-2}}. \end{aligned}$$

A standard argument with contours (see, for instance, the discussion of the Cauchy discontinuous factor in [17]) shows

$$\begin{aligned} \int_{\Gamma} \frac{dz}{2\pi\sqrt{-1}} \frac{\exp(\lambda z)}{(2z - h)^{2k+2l-2}} &= \frac{\exp(\lambda h/2)}{2^{2k+2l-2}} \operatorname{Res}_{z=h/2} \left[\frac{\exp\{\lambda(z - h/2)\}}{(z - h/2)^{2k+2l-2}} \right] \\ &= \frac{\lambda^{2k+2l-3} \exp(\lambda h/2)}{2^{2k+2l-2} (2k + 2l - 3)!}, \end{aligned}$$

where Res denotes the residue. We conclude that

$$\mu_1^2 = 4 \int_0^{1/2} d\lambda(1 - \lambda) \sum_{k,l=1}^{\infty} \frac{\lambda^{2k+2l-3} [X_{2k-2, 2l-2}((1 - \lambda)/2)]}{2^{2k+2l-2} (2k - 1)(2l - 1)(2k + 2l - 3)!},$$

where

$$X_{i,j} = X_{i,j}(\tau) = \int_{\mathcal{O}(\gamma, \delta, \epsilon, \phi)} [\exp(-\tau h)] (\gamma + \delta - \epsilon - \phi)^i (\gamma + \epsilon - \delta - \phi)^j$$

for $i, j \geq 0$.

To compute $X_{i,j}$, we define the quantities

$$\begin{aligned} Y_{i,j} &= Y_{i,j}(\tau) \\ &= \int_{\mathcal{O}(\delta, \epsilon, \phi)} \{\exp[-\tau(\delta + \epsilon + \phi)]\} (\delta - \epsilon - \phi)^i (\epsilon - \delta - \phi)^j, \\ Z_{i,j} &= Z_{i,j}(\tau) = \int_{\mathcal{O}(\delta, \epsilon)} \{\exp[-\tau(\delta + \epsilon)]\} (-1)^j (\delta - \epsilon)^{i+j}. \end{aligned}$$

Integrating by parts, we find

$$\begin{aligned} X_{i,j} &= \frac{1}{\tau} (Y_{i,j} + iX_{i-1,j} + jX_{i,j-1}), \\ Y_{i,j} &= \frac{1}{\tau} (Z_{i,j} - iY_{i-1,j} - jY_{i,j-1}), \end{aligned}$$

where any expression with a negative subscript is taken to vanish. Of course, the standard formula

$$\int_{\mathcal{O}(y_1, \dots, y_n)} y_1^{p_1-1} \dots y_n^{p_n-1} \exp\left(-\tau \sum_{i=1}^n y_i\right) = \frac{\Gamma(p_1) \dots \Gamma(p_n)}{\tau^{-(p_1+\dots+p_n)}}$$

(where $\Gamma(\cdot)$ denotes the gamma function) allows us to compute

$$Z_{i,j} = \begin{cases} \frac{(-1)^i (i+j)!}{\tau^{i+j+2}}, & \text{if } i+j \text{ is even;} \\ 0, & \text{otherwise.} \end{cases}$$

Straightforward inductive arguments show that $Y_{i,j} = Y_{j,i}$ and $X_{i,j} = X_{j,i}$, and furthermore

$$Y_{i,j} = \begin{cases} 0, & \text{if } i \text{ is odd and } j \text{ is odd;} \\ \frac{-(i+j+1)!}{2(j+1)\tau^{i+j+3}}, & \text{if } i \text{ is odd and } j \text{ is even;} \\ \frac{(i+j+2)!}{2(i+1)(j+1)\tau^{i+j+3}}, & \text{if } i \text{ is even and } j \text{ is even,} \end{cases}$$

and so

$$X_{i,j} = \begin{cases} \frac{(i+j+2)!}{2(i+1)(j+1)\tau^{i+j+4}}, & \text{if } i \text{ is even and } j \text{ is even;} \\ 0, & \text{otherwise.} \end{cases}$$

Substituting this into our previous expression for μ_1^2 , we find that

$$\mu_1^2 = 8 \sum_{k,l=1}^{\infty} \frac{2k+2l-2}{(2k-1)^2(2l-1)^2} \int_0^{1/2} d\lambda \lambda^{2k+2l-3} (1-\lambda)^{1-2k-2l}.$$

Of course,

$$\int_0^{1/2} d\lambda \lambda^{2k+2l-3} (1-\lambda)^{1-2k-2l} = {}_2F_1\left(2k+2l-2, 2k+2l-1; 2k+2l-1; \frac{1}{2}\right) = \frac{1}{2k+2l-2},$$

where ${}_2F_1$ denotes the hypergeometric function. Thus,

$$\begin{aligned} \mu_1^2 &= 8 \sum_{k,l=1}^{\infty} \frac{1}{(2k-1)^2(2l-1)^2} = 8 \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} \sum_{l=1}^{\infty} \frac{1}{(2l-1)^2} \\ &= 8 \frac{\pi^2}{8} \frac{\pi^2}{8} = \frac{\pi^4}{8}, \end{aligned}$$

and we have shown

Theorem 5.2.1. *The Weil-Petersson volume of \mathcal{M}_1^2 is $\pi^4/8$.*

6. Asymptotic divergence

We shall show in this section that the Weil-Petersson volume μ_g^1 of \mathcal{M}_g^1 grows at least as fast as $g^{-2}c^{-2g}(2g)!$ for some constant $c < .15$ independent of g . The estimate is derived by bounding below the volume of a single cell in the decomposition of $\widehat{\mathcal{F}}_g^1$ together with a matrix-model estimate (given in Appendix B) on the number of such cells. The next section is dedicated to computing the Weil-Petersson volume form from the Kähler two-form, and some work is involved. The final section pulls together the results necessary for our estimate on μ_g^1 . We remark parenthetically that our lower bound on the volume of single cell in fact diverges with g .

6.1. Computation of volume forms for once-punctured surfaces. To begin, we compute the volume form explicitly (in terms of λ -lengths with respect to $G \in \mathcal{G}(n)$), and a compact expression is given which is moreover independent of the particular element $G \in \mathcal{G}(n)$. Using this computation, we then give the expression for the volume form in λ -lengths with respect to any fatgraph whose associated surface is once-punctured.

We adopt the notation of Figure 2 for the edges of $G \in \mathcal{G}(n)$, setting $f_{n+1} = e_{n+1}$ for convenience, and, as before, identify an edge with its λ -length, set $\tilde{x} = d \log x$ for each edge x of G , and often suppress the symbol \wedge in an exterior product. By Theorem 3.3.6, the Weil-Petersson Kähler two-form pulls back to

$$\omega' = -2 \left(\omega'_0 + \sum_{j=1}^n \omega_j \right),$$

where

$$\begin{aligned} \omega_0 &= \omega'_0 = \tilde{e}'_1 \tilde{a}_0 + 2\tilde{a}_0 \tilde{b}_0 + \tilde{b}_0 \tilde{e}'_1 + \tilde{f}'_1 \tilde{a}_0 + \tilde{b}_0 \tilde{f}'_1 \\ &= [\tilde{e}'_1 + \tilde{f}'_1 - (\tilde{a}_0 + \tilde{b}_0)](\tilde{a}_0 - \tilde{b}_0) \end{aligned}$$

and

$$\begin{aligned} \omega_j &= \tilde{b}_j \tilde{a}_j + \tilde{a}_j \tilde{f}_j + \tilde{f}_j \tilde{b}_j + \tilde{a}_j \tilde{e}'_{j+1} + \tilde{e}'_{j+1} \tilde{d}_j + \tilde{d}_j \tilde{a}_j \\ &\quad + \tilde{c}_j \tilde{f}_{j+1} + \tilde{f}_{j+1} \tilde{d}_j + \tilde{d}_j \tilde{c}_j + \tilde{b}_j \tilde{e}_j + \tilde{e}_j \tilde{c}_j + \tilde{c}_j \tilde{b}_j \\ &= \xi_j + \eta_j + \zeta_j, \end{aligned}$$

with

$$\begin{aligned} \xi_j &= \frac{1}{2}[\tilde{e}'_{j+1} + \tilde{f}_j - \tilde{e}_j + \tilde{f}_{j+1} - 2\tilde{d}_j][\tilde{d}_j - \tilde{a}_j + \tilde{b}_j - \tilde{c}_j], \\ \eta_j &= \frac{1}{2}[\tilde{e}'_{j+1} - \tilde{f}_j + \tilde{e}_j + \tilde{f}_{j+1} - 2\tilde{d}_j][\tilde{d}_j - \tilde{a}_j - \tilde{b}_j + \tilde{c}_j], \\ \zeta_j &= [\tilde{c}_j - \tilde{a}_j][\tilde{f}_{j+1} - \tilde{f}_j + \tilde{b}_j - \tilde{d}_j] \end{aligned}$$

for $j = 1, \dots, n$, where one checks these equalities of two-forms directly. Furthermore, one computes directly that

$$\omega'_j = \xi_j \eta_j \zeta_j = (\tilde{e}'_j - \tilde{f}_j)(\tilde{f}_{j+1} - \tilde{f}_j)\{2\tilde{a}_j \tilde{b}_j \tilde{c}_j \tilde{d}_j + \psi_j\},$$

where

$$\psi_j = (\tilde{e}'_{j+1} + \tilde{f}_{j+1})(\tilde{b}_j - \tilde{c}_j)(\tilde{d}_j - \tilde{a}_j)(\tilde{c}_j - \tilde{a}_j).$$

Our overall goal in this subsection is to compute $\omega = (\omega')^{3n+1} / (3n + 1)!$, and insofar as

$$\omega' = \omega_0 + \sum_{j=1}^n (\xi_j + \eta_j + \zeta_j)$$

is a sum of $(3n + 1)$ two-forms, we are led to consider a monomial

$$\omega_0^{\epsilon_0} \xi_1^{\epsilon_1} \eta_1^{\epsilon_2} \zeta_1^{\epsilon_3} \dots \zeta_n^{\epsilon_{3n}}$$

in the $(6n + 3)$ variables $\{a_0, \dots, f_n\}$. Notice that none of ξ_j, η_j, ζ_j contain any terms of degree greater than one in $\tilde{a}_j, \tilde{b}_j, \tilde{c}_j, \tilde{d}_j$, while none of ξ_i, η_i, ζ_i for $i \neq j$ have any dependence on these variables (and neither does ω_0). Thus, a nonvanishing monomial of degree $(6n + 2)$ in the $(6n + 3)$ variables must have $\epsilon_i = 1$ for all i , and we conclude that

$$\omega = \frac{(\omega')^{3n+1}}{(3n + 1)!} = (-2)^{3n+1} \omega'_0 \prod_{j=1}^n (\xi_j \eta_j \zeta_j) = (-2)^{3n+1} \prod_{j=0}^n \omega'_j,$$

where $\prod_{j=1}^n \theta_j$ denotes the exterior product $\theta_1 \wedge \dots \wedge \theta_n$ of forms $\{\theta_j\}_{j=1}^n$ in this order. Substituting in the expressions computed above for ω'_j , we find that

$$\begin{aligned} \omega &= (-2)^{3n+1} \omega'_0 \prod_{j=1}^n \{(\tilde{e}'_j - \tilde{f}_j)(\tilde{f}_{j+1} - \tilde{f}_j)\} \\ &\quad \times \left[2^n \prod_{j=1}^n \{\tilde{a}_j \tilde{b}_j \tilde{c}_j \tilde{d}_j\} + 2^{n-1} \sum_{j=1}^n \psi_j \tilde{a}_1 \tilde{b}_1 \tilde{c}_1 \tilde{d}_1 \dots \widehat{\tilde{a}_j \tilde{b}_j \tilde{c}_j \tilde{d}_j} \dots \tilde{a}_n \tilde{b}_n \tilde{c}_n \tilde{d}_n \right], \end{aligned}$$

where $\hat{}$ denotes the omission.

We define

$$\begin{aligned} \chi &= \prod_{j=1}^n (\tilde{e}'_j - \tilde{f}_j)(\tilde{f}_{j+1} - \tilde{f}_j), \\ V_1 &= 2\omega'_0 \chi \prod_{j=1}^n (\tilde{a}_j \tilde{b}_j \tilde{c}_j \tilde{d}_j), \\ V_2 &= \omega'_0 \chi \sum_{j=1}^n \psi_j \{ \widehat{\tilde{a}_1 \tilde{b}_1 \tilde{c}_1 \tilde{d}_1 \dots \tilde{a}_j \tilde{b}_j \tilde{c}_j \tilde{d}_j \dots \tilde{a}_n \tilde{b}_n \tilde{c}_n \tilde{d}_n} \}, \end{aligned}$$

so

$$2^{-4n} (-1)^{n+1} \omega = V_1 + V_2;$$

we will compute the quantities V_1 and V_2 separately below.

First, let us compute

$$\begin{aligned} (-1)^{n(n-1)/2} \chi &= \prod_{j=1}^n (\tilde{f}_j - \tilde{f}_{j+1}) \prod_{k=1}^n (\tilde{e}_k - \tilde{f}_k) \\ &= \left[\sum_{j=0}^n (-1)^j \tilde{f}_1 \dots \hat{\tilde{f}}_{n-j} \dots \tilde{f}_n \right] (\tilde{f}_n - \tilde{e}_{n+1}) \prod_{k=1}^n (\tilde{e}_k - \tilde{f}_k) \\ &= \sum_{j=0}^{n-1} (-1)^j \tilde{f}_1 \dots \hat{\tilde{f}}_{n-j} \dots \tilde{f}_n \\ &\quad \times \left[(-1)^n \prod_{k=1}^n \tilde{e}_k + (-1)^{n-1} \tilde{e}_1 \dots \tilde{e}_{n-j-1} \tilde{f}_{n-j} \tilde{e}_{n-j+1} \dots \tilde{e}_n \right] (\tilde{f}_n - \tilde{e}_{n+1}) \\ &= F \left[\sum_{j=0}^n (-1)^j \tilde{e}_1 \dots \hat{\tilde{e}}_{n+1-j} \dots \tilde{e}_{n+1} \right] - \left[\sum_{j=0}^n \tilde{f}_1 \dots \hat{\tilde{f}}_{n-j} \dots \tilde{f}_n \right] E, \end{aligned}$$

where $F = \prod_{j=1}^n \tilde{f}_k$ and $E = \prod_{j=1}^{n+1} \tilde{e}_j$.

One now easily computes V_1 as

$$\begin{aligned} V_1 &= 2 \prod_{j=1}^n \tilde{a}_j \tilde{b}_j \tilde{c}_j \tilde{d}_j [2\tilde{a}_0 \tilde{b}_0 + (\tilde{b}_0 - \tilde{a}_0)(\tilde{f}_1 + \tilde{e}'_1)] \chi \\ &= 4 \prod_{j=1}^n \tilde{a}_j \tilde{b}_j \tilde{c}_j \tilde{d}_j [\tilde{a}_0 \tilde{b}_0 \chi + (-1)^{n(n-1)/2} (\tilde{b}_0 - \tilde{a}_0) FE]. \end{aligned}$$

In order to compute V_2 , we must introduce the three-forms

$$\phi_j = (\tilde{b}_j - \tilde{c}_j)(\tilde{d}_j - \tilde{a}_j)(\tilde{c}_j - \tilde{a}_j)$$

for $j = 1, \dots, n$, and define the function $\tau: \{2, \dots, n+1\} \rightarrow \{1, \dots, n\}$, where $\tau(l) = j \Leftrightarrow e'_{j+1} = e_l$. Thus,

$$\begin{aligned} (-1)^{n(n-1)/2} V_2 &= 2\tilde{a}_0\tilde{b}_0 \left\{ F \left[\sum_{j=0}^n (-1)^j \tilde{e}_1 \dots \hat{\tilde{e}}_{n+1-j} \dots \tilde{e}_{n+1} \right] \left(\sum_{k=1}^n \tilde{e}'_{k+1} \phi_k \right) \right. \\ &\quad + F \left[\sum_{j=0}^n (-1)^j \tilde{e}_1 \dots \hat{\tilde{e}}_{n+1-j} \dots \tilde{e}_{n+1} \right] \tilde{e}_{n+1} \phi_n \\ &\quad \left. - \left[\sum_{j=0}^{n-2} (-1)^{n+j} \tilde{f}_1 \dots \hat{\tilde{f}}_{n-j} \dots \tilde{f}_n \right] E \left(\sum_{k=1}^{n-1} \tilde{f}_{k+1} \phi_k \right) \right\} \\ &= 2\tilde{a}_0\tilde{b}_0 \left[FE \sum_{j=0}^{n-1} \phi_{\tau(n+1-j)} + FE\phi_n + FE \sum_{j=0}^{n-2} \phi_{n-j-1} \right] \\ &= 4\tilde{a}_0\tilde{b}_0 FE \sum_{j=1}^n \phi_j. \end{aligned}$$

We have arrived at

Lemma 6.1.1. *Suppose that $G \in \mathcal{G}(n)$, set $N = 6n + 3$, and let x_1, x_2, \dots, x_N , respectively, denote the λ -length of the edge $a_0, b_0, a_1, b_1, c_1, d_1, \dots, a_n, b_n, c_n, d_n, f_1, \dots, f_n, e_1, \dots, e_{n+1}$. If we set $\tilde{x}_i = d \log x_i$ for $i = 1, \dots, N$, then the Weil-Petersson volume form on F_{n+1}^1 pulls back to*

$$\omega = -2^{4n+2} \sum_{i=1}^N (-1)^i \tilde{x}_1 \dots \hat{\tilde{x}}_i \dots \tilde{x}_N.$$

Proof. Simply substitute the expressions for V_1 and V_2 into the expression for ω and expand in terms of the λ -lengths. q.e.d.

In fact, much more is true:

Theorem 6.1.2. *Suppose that G is any trivalent fatgraph with $F(G)$ homeomorphic to F_g^1 , let $N = 6g - 3$, and let x_1, \dots, x_N denote the respective λ -lengths of the edges of G . The Weil-Petersson volume form pulls back to*

$$\omega = \pm 2^{4g-2} \sum_{i=1}^N (-1)^i d \log x_1 \wedge \dots \wedge \widehat{d \log x_i} \wedge \dots \wedge d \log x_N.$$

Proof. First notice that the expression ω above agrees with the pull-back of the Kähler two-form on \mathcal{M}_1^1 (see §4), so, by Lemma 6.1.1, the formula holds for at least one trivalent fatgraph in the complex \mathcal{G}_g^1 for each

$g \geq 1$. We show that the expression ω is invariant under Ptolemy transformations, and the result then follows from Corollary 3.3.2 and Lemma 3.3.3.

To this end, suppose that e is an edge of the trivalent fatgraph G with distinct endpoints, and adopt the notation of Figure 3(a) for the hooks of G near e , and let a, b, c, d , respectively, denote the (not necessarily distinct) edges of G containing the hooks $\hat{f}_1, \hat{g}_1, \hat{f}_2, \hat{g}_2$. As before, perform a Whitehead collapse on e to produce a trivalent fatgraph G' , identify each edge of G' with its corresponding edge of G , and let e' be the edge of G' corresponding to e . Again, we identify an edge with its λ -length, adopt the notation that $\tilde{x} = d \log x$, and omit the symbol \wedge from exterior products.

It follows from Lemma 3.3.3 that

$$\tilde{e}' = \frac{1}{ac + bd} [ac(\tilde{a} + \tilde{c}) + bd(\tilde{b} + \tilde{d})] - \tilde{e},$$

and we compute

$$\begin{aligned} & \tilde{b}\tilde{c}\tilde{d}\tilde{e}' - \tilde{a}\tilde{c}\tilde{d}\tilde{e}' + \tilde{a}\tilde{b}\tilde{d}\tilde{e}' - \tilde{a}\tilde{b}\tilde{c}\tilde{e}' + \tilde{a}\tilde{b}\tilde{c}\tilde{d} \\ &= \tilde{b}\tilde{c}\tilde{d} \left(\frac{ac\tilde{a}}{ac + bd} - \tilde{e} \right) - \tilde{a}\tilde{c}\tilde{d} \left(\frac{bd\tilde{b}}{ac + bd} - \tilde{e} \right) \\ & \quad + \tilde{a}\tilde{b}\tilde{d} \left(\frac{ac\tilde{c}}{ac + bd} - \tilde{e} \right) - \tilde{a}\tilde{b}\tilde{c} \left(\frac{bd\tilde{d}}{ac + bd} - \tilde{e} \right) + \tilde{a}\tilde{b}\tilde{c}\tilde{d} \\ &= \frac{-ac - bd - ac - bd}{ac + bd} \tilde{a}\tilde{b}\tilde{c}\tilde{d} + \tilde{a}\tilde{b}\tilde{c}\tilde{d} - \tilde{b}\tilde{c}\tilde{d}\tilde{e} + \tilde{a}\tilde{c}\tilde{d}\tilde{e} - \tilde{a}\tilde{b}\tilde{d}\tilde{e} + \tilde{a}\tilde{b}\tilde{c}\tilde{e} \\ &= -(\tilde{b}\tilde{c}\tilde{d}\tilde{e} - \tilde{a}\tilde{c}\tilde{d}\tilde{e} + \tilde{a}\tilde{b}\tilde{d}\tilde{e} - \tilde{a}\tilde{b}\tilde{c}\tilde{e} + \tilde{a}\tilde{b}\tilde{c}\tilde{d}), \end{aligned}$$

as desired.

Remark. The reader may wonder why, in light of Theorem 6.1.2, we treat each element of $\mathcal{G}(n)$ in Lemma 6.1.1 instead of just a single element; the answer is that the computation in Lemma 6.1.1 for any particular element is no easier than the general case.

6.2. A lower bound on volumes for once-punctured surfaces. We finally apply some of the previous material to derive a lower bound on the Weil-Petersson volume μ_g^1 of \mathcal{M}_g^1 . To this end, fix some fatgraph $G \in \mathcal{G}(n)$, set $N = 6n + 3$, and consider $\int_{\mathcal{D}(G)} \phi^*(\omega)$, where ω is the Weil-Petersson volume form on \mathcal{M}_g^1 and $\phi: \mathcal{F}_{n+1}^1 \rightarrow \mathcal{M}_{n+1}^1$ is the forgetful map. By Corollary 3.4.3, the region $\mathcal{D}(G)$ of integration is expressed in simplicial coordinates $\{X_i\}_{i=1}^N$ on the edges of G as the simplex

$$\mathcal{D}(G) = \{X_i > 0 \text{ for each } i = 1, \dots, N, \text{ and } \rho = \rho(X_1, \dots, X_N) = \frac{1}{2}\},$$

where $\rho = \sum_{i=1}^N X_i$ is half the hyperbolic length of the horocycle, or, in other words, ρ is the sum of all the h -lengths of sectors of G by Lemma 3.4.2.

Observe from Theorem 3.3.6 that the pullback of the Weil-Petersson Kähler two-form to $\tilde{\mathcal{F}}_{n+1}^1$ is invariant under homothety of λ -lengths, so $\phi^*(\omega)$ is similarly invariant under homothety. Since simplicial coordinates are homogeneous functions of λ -lengths, a standard application of Stokes' Theorem shows that

$$\int_{\mathcal{D}(G)} \phi^*(\omega) = 2 \int_{\Delta(G)} d\rho \wedge \phi^*(\omega),$$

where $\Delta(G)$ is the region

$$\Delta(G) = \{X_i > 0 \text{ for each } i = 1, \dots, N \text{ and } 0 < \rho < \frac{1}{2}\}.$$

On the other hand, by the first part of Proposition 3.5.3 and Theorem 6.1.2 (or Lemma 6.1.1), if $\{\lambda_i\}_{i=1}^N$ denotes the corresponding λ -lengths of edges of G , then we find that

$$\begin{aligned} d\rho \wedge \phi^*(\omega) &= \pm 2^{4n+2} \left(\sum_{i=1}^N X_i \tilde{\lambda}_i \right) \wedge \left(\sum_{i=1}^N (-1)^i \tilde{\lambda}_i \wedge \dots \wedge \hat{\tilde{\lambda}}_i \wedge \dots \wedge \tilde{\lambda}_N \right) \\ &= \pm 2^{4n+2} \rho \prod_{i=1}^N \tilde{\lambda}_i, \end{aligned}$$

where, as usual, $\tilde{\lambda}$ denotes $d \log \lambda$. It follows that

$$\int_{\mathcal{D}(G)} \phi^*(\omega) = 2^{4n+3} \int_{\Delta(G)} \rho \prod_{i=1}^N \tilde{\lambda}_i.$$

By the second part of Proposition 3.5.3, the Jacobian determinant of the transformation from $\{X_i\}_{i=1}^N$ to $\{\log \lambda_i\}_{i=1}^N$ is bounded below by $(N/3\rho)^N$, and a change of variables gives the estimate

$$\int_{\mathcal{D}(G)} \phi^*(\omega) > \frac{2^{4n+3} N^N}{3^N} \int_{\Delta(G)} \frac{\prod_{i=1}^N dX_i}{\rho^{N-1}} = \frac{2^{4n+2} N^N}{3^N \Gamma(N)}.$$

We have, therefore, derived the following general estimate.

Theorem 6.2.1. *Suppose that G is a trivalent fatgraph whose associated surface is homeomorphic to F_g^1 , let $N = 6g - 3$, and let ω be the Weil-Petersson volume form on \mathcal{M}_g^1 . Then we have the estimate*

$$\int_{\mathcal{D}(G)} \phi^*(\omega) > \frac{2^{4g-2} N^N}{3^N \Gamma(N)} \sim \left(\frac{27}{4e^3} \right)^{-2g}.$$

Now, according to Theorem B, the number n_g of distinct MC_g^1 -orbits of top-dimensional cells in $\widetilde{\mathcal{T}}_g^1$ is asymptotically

$$n_g \sim \frac{(2g)!}{6g-3} \left(\frac{e}{6}\right)^{-2g}.$$

Furthermore, since an automorphism of a fatgraph G with $s(G) = 1$ must preserve the single boundary component of the fattened graph, $\text{Aut}(G)$ is cyclic with order at most the number $(12g - 12)$ of sectors of G . Pulling together our estimates, we find

Theorem 6.2.2. *An asymptotic lower bound to the Weil-Petersson volume μ_g^1 of \mathcal{M}_g^1 is*

$$\mu_g^1 > \sim g^{-2} c^{-2g} (2g)!,$$

where $c < .15$ is a constant independent of g .

Appendix A. The Weil-Petersson Kähler two-form

The purpose of this appendix is to prove Theorem 3.3.6, which gives the pullback to $\widetilde{\mathcal{T}}_g^s$ of the Weil-Petersson Kähler two-form in terms of λ -lengths; this argument was sketched in [11, §5] and is included here for completeness.

To begin, we derive some useful identities involving λ -lengths and must first establish some notation. Fix horocycles $\{h_i\}_1^4$ in \mathbb{D} with distinct centers $\{\zeta_i\}_1^4$, and consider the corresponding “decorated ideal quadrilateral” with $\{\zeta_i\}_1^4$ as its ideal vertices, as in Figure A1.

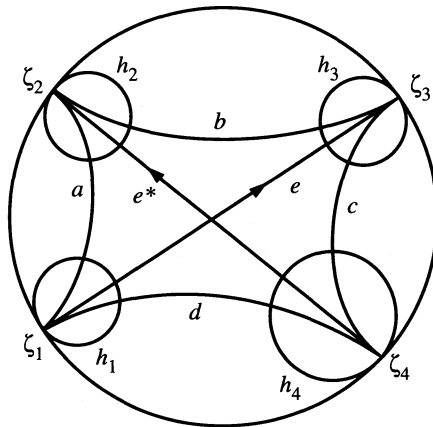


FIGURE A1

Given a geodesic x in \mathbb{D} connecting ζ_i to ζ_j , with $i, j \in \{1, 2, 3, 4\}$, define the “ λ -length” of x (as before) to be $\sqrt{2 \exp\{\delta\}}$, where δ denotes the signed hyperbolic distance between h_i and h_j ; δ is to be taken with a positive sign if and only if $h_i \cap h_j = \emptyset$.

We adopt the notation of Figure A1, identifying each geodesic with its λ -length, and give the geodesics e and e^* the indicated orientations.

Lemma A.1. *With the notation as above, we have*

- (a) $ee^* = ac + bd$,
- (b) *The cosine of the angle from e to e^* is $\gamma = (ac - bd)/(ac + bd)$.*

Proof. Let \mathbb{M} denote Minkowski three-space with the pairing

$$(x, y, z) \cdot (x', y', z') = -xx' - yy' + zz',$$

and let

$$\begin{aligned} \mathbb{H} &= \{\vec{v} = (x, y, z) \in \mathbb{M} : \vec{v} \cdot \vec{v} = 1 \text{ and } z > 0\}, \\ L^+ &= \{\vec{v} = (x, y, z) \in \mathbb{M} : \vec{v} \cdot \vec{v} = 0 \text{ and } z > 0\}, \end{aligned}$$

so that \mathbb{H} is isometric to \mathbb{D} . As shown in [10, §1], affine duality

$$\vec{v} \mapsto h(\vec{v}) = \{\vec{w} \in \mathbb{H} : \vec{v} \cdot \vec{w} = 1\}$$

establishes an isomorphism between L^+ and the collection of all horocycles in \mathbb{H} , and furthermore, if $\vec{u}, \vec{v} \in L^+$, then the λ -length of the geodesic connecting the centers of $h(\vec{u})$ and $h(\vec{v})$ is simply $\sqrt{\vec{u} \cdot \vec{v}}$.

Let $u_i \in L^+$ correspond, as above, to the horocycle h_i , respectively, for $i = 1, 2, 3, 4$. Thus, the identity in part (a) of the lemma is equivalent to

$$(a') \quad \sqrt{\vec{u}_1 \cdot \vec{u}_3} \sqrt{\vec{u}_2 \cdot \vec{u}_4} = \sqrt{\vec{u}_1 \cdot \vec{u}_2} \sqrt{\vec{u}_3 \cdot \vec{u}_4} + \sqrt{\vec{u}_2 \cdot \vec{u}_3} \sqrt{\vec{u}_1 \cdot \vec{u}_4},$$

and part (b) of the lemma asserts that

$$(b') \quad \gamma = \frac{\sqrt{(\vec{u}_1 \cdot \vec{u}_2)(\vec{u}_3 \cdot \vec{u}_4)} - \sqrt{(\vec{u}_1 \cdot \vec{u}_4)(\vec{u}_2 \cdot \vec{u}_3)}}{\sqrt{(\vec{u}_1 \cdot \vec{u}_2)(\vec{u}_3 \cdot \vec{u}_4)} + \sqrt{(\vec{u}_1 \cdot \vec{u}_4)(\vec{u}_2 \cdot \vec{u}_3)}}.$$

Observe that each of (a') and (b') is invariant under scaling each \vec{u}_i independently, so we may assume that each \vec{u}_i lies in the horizontal plane $\Pi = \{(x, y, z) \in \mathbb{M} : z = 1\}$ for $i = 1, 2, 3, 4$.

Of course, the Minkowski pairing restricts to $\sqrt{2}$ times the usual Euclidean metric on $\Pi \subset \mathbb{R}^3 \approx \mathbb{M}$, so that $\Pi \cap L^+$ is a round circle in the induced structure; furthermore, if $\vec{u}, \vec{v} \in \Pi \cap L^+$, then the λ -length $\sqrt{\vec{u} \cdot \vec{v}}$ is simply $\sqrt{2}$ times the Euclidean length of the chord with endpoints \vec{u} and \vec{v} .

Thus, (a') and (b') follow from their corresponding Euclidean analogues: (a') is Ptolemy's Theorem on quadrilaterals which inscribe in a circle, (b') is an easy consequence of the Euclidean law of cosines, and the proof is complete.

Remark. Another formula (proved in the same way) gives the cross ratio in terms of λ -lengths; namely, with the notation as above,

$$CR(\zeta_2, \zeta_3, \zeta_4; \zeta_1) = -\frac{ac}{bd},$$

where $CR(x, y, u; v)$ is the image of v under the Möbius transformation taking x, y , and u , respectively, to $0, 1$, and ∞ .

The theorem we shall prove here (which is equivalent to Theorem 3.3.6) is

Theorem A.2. *Suppose that Δ is an ideal triangulation of F_g^s , and consider λ -length coordinates on $\tilde{\mathcal{T}}_g^s$ with respect to Δ . The Weil-Petersson Kähler two-form pulls back to*

$$-2 \sum (d \log a \wedge d \log b + d \log b \wedge d \log c + d \log c \wedge d \log a),$$

where the sum is over all triangles T in $F_g^s - \Delta$ whose edges have λ -lengths a, b, c in an order compatible with that determined by the orientation of F_g^s on T .

Proof. Our strategy is as follows. Let H denote F_g^s with small horoball neighborhoods of the punctures removed. Double H along its boundary to produce a closed topological surface F with Teichmüller space \mathcal{T} , and let C denote the curves in F arising from the boundary of H . We consider limits in \mathcal{T} as the geodesic representative Γ of C is pinched to a point. By [22], the Weil-Petersson Kähler two-form on \mathcal{T} extends in the limit to the Weil-Petersson Kähler two-form on \mathcal{T}_g^s .

In order to relate λ -lengths of ideal arcs in F_g^s to hyperbolic lengths of closed geodesics in F , we observe that in pinching Γ , nearby hypercycles of Γ (i.e., curves in F equidistant to Γ) limit to horocycles in the induced structure on F_g^s . (To see this, simply consider the collection of geodesics in F which are orthogonal to Γ ; these must limit on geodesics tending to the punctures of F_g^s .)

Now, suppose that Γ is sufficiently small and consider the annular neighborhood A of Γ in F bounded by the hypercycles H_L, H_R so that each annular component of $F - (\Gamma \cup H_L \cup H_R)$ has hyperbolic area unity. If c is a closed geodesic in F , say with hyperbolic length l , meeting Γ in two points, then $c \cap (F - A)$ consists of two arcs with endpoints in H_X for $X = L, R$, and we let l_L and l_R denote the respective lengths of

these arcs. We define “generalized λ -lengths”

$$\lambda_X = \sqrt{2 \exp(l_X)}$$

for $X \in \{R, L\}$, so that

$$dl = dl_R + dl_L = 2(d \log \lambda_R + d \log \lambda_L).$$

Finally, in light of the remarks above, $d \log \lambda_X$ limits on the exterior derivative of the log of the λ -length (with respect to the horocycles about the punctures corresponding to the limiting hypercycles above) of the ideal arc in the pinched surface corresponding to c_X for $X \in \{R, L\}$.

To recognize the Weil-Petersson Kähler two-form ω on \mathcal{T} , we recall **Wolpert’s Theorem** (Theorem 3.3 and Lemma 4.5 of [21]). *If the geodesic length functions $\{l_i\}_1^n$ of geodesic curves $\{c_i\}_1^n$ give local coordinates on \mathcal{T} , then*

$$\omega_{ij} = \omega(dl_i, dl_j) = \sum_{p \in c_i \cap c_j} \cos \alpha_p,$$

where α_p is the angle between c_i and c_j at the point p . Furthermore, if $(\omega^{ij}) = (\omega_{ij})^{-1}$, then

$$\omega = \sum_{1 \leq i < j \leq n} \omega^{ij} dl_i \wedge dl_j.$$

To apply this, suppose that Δ is an ideal triangulation of $F = F_g^s$. F inherits a pants decomposition $\{c_i\}_1^N$ from $(H, H \cap \Delta)$ in the natural way, where $N = 6g - 6 + 3s$. Furthermore, if $e \in \Delta$ and e^* is the ideal arc in F_g^s derived from e (and Δ) as in Figure A1, then $(H, H \cap e^*)$ similarly gives rise to a closed curve in F . If $c_i \subset F$ arises from $e \in \Delta$, then we let $c_i^* \subset F$ denote the curve so derived from e^* , and set $\mathcal{C} = \{c_i, c_i^*\}_1^N$. Using Fenchel-Nielsen coordinates on \mathcal{T} with respect to the pants decomposition $\{c_i\}_1^N$ and convexity of hyperbolic length functions under Fenchel-Nielsen deformations [4] (together with the fact that $c_i \cap c_j^* = \emptyset$ unless $i = j$), we conclude that the hyperbolic lengths of curves in \mathcal{C} give local coordinates a.e. on \mathcal{T} , so Wolpert’s Theorem is applicable.

If x and y are oriented geodesics in F or oriented ideal arcs in F_g^s , then we let

$$\gamma(x, y) = \sum_{p \in x \cap y} \cos \alpha_p.$$

By the first part of Wolpert’s Theorem, the matrix $\{\omega_{ij}\}_{i,j=1}^{2N}$ of Kähler pairings has the form $\begin{pmatrix} 0 & D \\ -D & A \end{pmatrix}$, where $0^{N \times N}$ denotes the zero matrix,

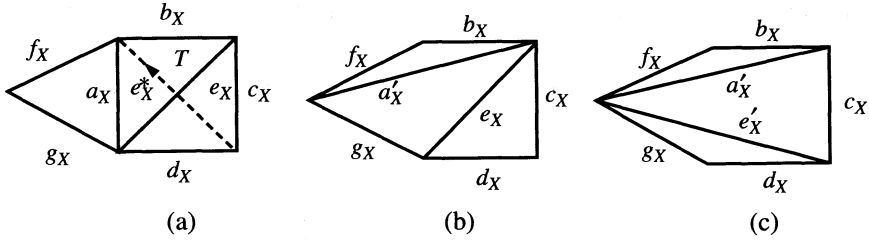


FIGURE A2

$D^{N \times N}$ is the diagonal matrix with entries $D_{ii} = \gamma(c_i, c_i^*)$, and $A^{N \times N}$ has entries $A_{ij} = \gamma(c_i^*, c_j^*)$. The inverse of (ω_{ij}) is given by

$$\begin{pmatrix} D^{-1}AD^{-1} & -D^{-1} \\ D^{-1} & 0 \end{pmatrix},$$

so by the second part of Wolpert's Theorem we find that the Kähler two-form ω on \mathcal{F} is given by

$$(†) \quad -\omega = \sum_{1 \leq i < j \leq N} \frac{\gamma(c_i^*, c_j^*)}{\gamma(c_i, c_i^*)\gamma(c_j, c_j^*)} dc_i \wedge dc_j - \sum_{i=1}^N \frac{1}{\gamma(c_i, c_i^*)} dc_i \wedge dc_i^*,$$

where we have identified a closed curve on F with its hyperbolic length for convenience.

As before, the geodesic Γ separates each $c \in \mathcal{C}$ into a left half c_L and a right half c_R , and we now pinch Γ , retaining the notation c_L and c_R for the corresponding ideal arcs. Of course, if $c, d \in \mathcal{C}$, then $\gamma(c, d)$ limits on $\gamma(c_R, d_R) + \gamma(c_L, d_L)$.

Now, suppose that $e_X = \{c_R, c_L : c \in \mathcal{C}\}$ for $X \in \{R, L\}$, adopt the notation of Figure A2(a) for the ideal arcs near e_X , and identify an ideal arc with its λ -length for convenience. By Lemma A1(a),

$$(††) \quad d \log e_X + d \log e_X^* = \left[\frac{ac}{ac + bd} (d \log a + d \log c) + \frac{bd}{ac + bd} (d \log b + d \log d) \right]_X,$$

in the obvious notation. Of course, since ω limits on the Weil-Petersson Kähler two-form ω' of F_g^s , the coefficient of $de_X \wedge df_Y$ in ω' must vanish whenever $e, d \in \mathcal{C}$ and $\{X, Y\} = \{R, L\}$; see the remark following this proof. Furthermore, notice that $c_i^* \cap c_j = \emptyset$ unless $i = j$, and $c_i^* \cap c_j^* = \emptyset$ unless c_i and c_j lie in the frontier of a common component of $F - \{c_i\}_1^N$. Inspection of (†) and (††) then shows that the coefficient of $de_X \wedge df_X$ in ω' must vanish unless e_X and f_X lie in the frontier of a common

triangle complementary to Δ in F_g^s , and indeed, there is a contribution to ω' from each such triangle.

Continuing to employ the notation of Figure A2(a) with $X = R$, we find from (†) that the contribution to $-\frac{1}{4}\omega'$ from the triangle T has the following projection into the subspace spanned by $d \log a_R \wedge d \log e_R$:

$$\begin{aligned}
 (\dagger\dagger) \quad \chi &= \frac{\gamma(a^*, e^*)}{\gamma(a, a^*)\gamma(e, e^*)} - \frac{1}{\gamma(a, a^*)} \left(\frac{ef}{ef + bg} \right)_R \\
 &\quad + \frac{1}{\gamma(e, e^*)} \left(\frac{ac}{ac + bd} \right)_R.
 \end{aligned}$$

It follows from Lemma A.1(b) that

$$\gamma(a_R, a_R^*) = \left(\frac{bg - ef}{bg + ef} \right)_R, \quad \gamma(e_R, e_R^*) = \left(\frac{ac - bd}{ac + bd} \right)_R,$$

and we claim that

$$\gamma(a_R^*, e_R^*) = \left[\frac{face - abcg - bdbg - bdef}{(ac + bd)(bg + ef)} \right]_R.$$

To see this, consider the “Whitehead moves” indicated in Figure A2(b), (c), and adopt the notation indicated there. According to Lemma A.1(a),

$$\begin{aligned}
 a'_R &= \left(\frac{bg + ef}{a} \right)_R, \\
 e'_R &= \left(\frac{a'd + cg}{e} \right)_R = \frac{1}{e_R} \left[cg + \frac{d}{a}(bg + ef) \right]_R.
 \end{aligned}$$

Thus, by Lemma A.1(b), we find

$$\begin{aligned}
 \gamma(a_R^*, e_R^*) &= \gamma(a'_R, a_R^*) = \left(\frac{cf - be'}{cf + be'} \right)_R \\
 &= \left\{ \frac{cf - \frac{b}{e} \left[cg + \frac{d}{a}(bg + ef) \right]}{cf + \frac{b}{e} \left[cg + \frac{d}{a}(bg + ef) \right]} \right\}_R,
 \end{aligned}$$

and the claim follows upon clearing denominators.

Define

$$\xi_X = (ac + bd)_X \quad \text{and} \quad \eta_X = (bg + ef)_X \quad \text{for } X \in \{R, L\},$$

and identify F_g^s with the limit of the right side of F (making, in this way,

a global sign convention). Thus, for instance,

$$\begin{aligned} \frac{1}{\gamma(e, e^*)} &= \frac{1}{\gamma(e_R, e_R^*) + \gamma(e_L, e_L^*)} \\ &= \frac{1}{[(bg - ef)/(bg + ef)]_R - [(bg - ef)/(bg + ef)]_L} \\ &= \frac{1}{2} \frac{\eta_R \eta_L}{e_L f_L b_R g_R - e_R f_R b_L g_L}. \end{aligned}$$

Similarly,

$$\begin{aligned} \frac{1}{\gamma(a, a^*)} &= \frac{1}{2} \frac{\xi_R \xi_L}{b_L d_L a_R c_R - b_R d_R a_L c_L}, \\ \frac{1}{2} \gamma(a^*, e^*) &= \left[\frac{face}{\xi \eta} \right]_R - \left[\frac{face}{\xi \eta} \right]_L. \end{aligned}$$

Finally, plugging these into (†††), we compute $2\chi = \nu/\delta$, where

$$\delta = (b_L d_L a_R c_R - b_R d_R a_L c_L)(e_L f_L b_R g_R - e_R f_R b_L g_L)$$

and

$$\begin{aligned} \nu &= \eta_L \xi_L (face)_R - \eta_R \xi_R (face)_L - (ef)_R \eta_L (b_L d_L a_R c_R - b_R d_R a_L c_L) \\ &\quad + (ac)_R \xi_L (e_L f_L b_R g_R - e_R f_R b_L g_L) \\ &= (face)_R [\xi \eta - bd(bg + ef) - bg(ac + bd)]_L \\ &\quad - (face)_L [\xi \eta - bdef - abcg]_R \\ &\quad + (bdef)_R (abcg)_L + (bdef)_L (abcg)_R \\ &= -(face)_L (bdbg)_R - (face)_R (bdbg)_L \\ &\quad + (bdef)_R (abcg)_L + (bdef)_L (abcg)_R \\ &= \delta, \end{aligned}$$

so that $\chi = \frac{1}{2}$.

Since the pair a, e was arbitrary (with this ordering induced from the orientation of F_g^s on the triangle T as in Figure A2(a)) and since χ was defined to be the coefficient in $-\frac{1}{4}\omega'$ of $d \log a_R \wedge d \log e_R$, the theorem follows.

Remark. For peace of mind, one can compute directly as above that the coefficient of $de_X \wedge df_Y$ in ω' must vanish if $\{X, Y\} = \{R, L\}$ as was asserted before.

Appendix B. An estimate on the number of cells

This appendix is dedicated to the proof of

Theorem B. *For g sufficiently large, we have the estimate*

$$n_g = \#\{[G] : G \in \mathcal{G}_g^1\} \sim \frac{(2g)!}{6g-3} \left(\frac{e}{6}\right)^{-2g}.$$

The argument closely follows the computation in [1, §6] due to J. M. Drouffe and was shown to the author by Steve Shenker.

Proof. Adopting the notation of §3.1 for fatgraphs as pairs of permutations, a fatgraph in \mathcal{G}_g^1 corresponds to a pair of type $\sigma \in \{3^k\}$ and $\tau \in \{2^{3m}\}$, where $k = 2m = 2(2g - 1)$ is the number of trivalent vertices, and $\sigma \circ \tau \in \{3k\}$.

As in [1, A.6.8], we evidently have

$$n_g = \frac{1}{3^k k!} \sum_{\tau \in \Sigma_{3k}} \delta_{\{\tau\}, \{2^{3m}\}} \delta_{\{\sigma \circ \tau\}, \{3k\}},$$

where δ denotes a delta function (and our n_g differs from n_h in [1, Appendix 6] by the factor 3^{-k}). Let $\chi^{(r)}$ denote the character of Σ_{3k} for the representation r , and let $\nu_{\{\pi\}}$ denote the number of elements in the conjugacy class of $\pi \in \Sigma_{3k}$. According to the completeness relation

$$\sum_r \chi^{(r)}(\pi) \chi^{(r)}(\pi') = \frac{(3k)!}{\nu_{\{\pi\}}} \delta_{\{\pi\}, \{\pi'\}},$$

we have

$$(†) \quad n_g = \frac{\nu_{\{2^{3m}\}} \nu_{\{3k\}}}{3^k k! (3k)!} \sum_r d_r^{-1} \chi_{\{2^{3m}\}}^{(r)} \chi_{\{3k\}}^{(r)} \chi_{\{3k\}}^{(r)},$$

where d_r denotes the dimension of the irreducible representation indexed by r as in [1, A.6.13]. Furthermore, the only representations r for which $\chi_{\{3k\}}^{(r)} \neq 0$ are of the form $(3k - p, 1^p)$ for $p = 1, \dots, 3k - 1$. Moreover, for these representations (indexed by p), one finds

$$\begin{aligned} \chi_{\{3k\}}^{(p)} &= (-1)^p, & d_p &= \binom{3k-1}{p}, \\ \chi_{\{2^{3m}\}}^{(p)} &= \delta_{p,0}^{[2]} (-1)^{p/2} \binom{3m-1}{p/2} - \delta_{p,1}^{[2]} (-1)^{(p-1)/2} \binom{3m-1}{(p-1)/2}, \\ \chi_{\{3k\}}^{(p)} &= \delta_{p,0}^{[3]} \binom{k-1}{p/3} - \delta_{p,1}^{[3]} \binom{k-1}{(p-1)/3} + \delta_{p,2}^{[3]} \binom{k-1}{(p-2)/3}, \end{aligned}$$

where $\delta_{p,q}^{[s]}$ has value zero or unity and vanishes unless $p = q$ modulo s .

Now, as $k \rightarrow \infty$, the terms dominating the sum over p (that is, over r) in (†) correspond to $p = 0, 3k - 1$, and both of these terms have positive sign since $m = 2g - 1$ is odd. Thus, as $k \rightarrow \infty$, we find

$$n_g \sim \frac{2\nu_{\{2^{3m}\}}\nu_{\{3k\}}}{3kk!(3k)!},$$

where $x \sim y$ if $x = y[1 + O(\frac{1}{k})]$. Plugging in

$$\nu_{\{2^{3m}\}} = \frac{(6m)!}{2^{3m}(3m)!} \quad \text{and} \quad \nu_{\{3k\}} = \frac{(3k)!}{3k},$$

we find that

$$n_g \sim \frac{(6m)!}{72^m(3m)(2m)!(3m)!} \sim \frac{(2g)!}{6g-3} \left(\frac{e}{6}\right)^{-2g},$$

as was asserted.

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