J. DIFFERENTIAL GEOMETRY
35 (1992) 723-741

RICCI FLOW OF LOCALLY HOMOGENEOUS GEOMETRIES ON CLOSED MANIFOLDS

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Abstract

Hamilton's program for using Ricci flow to study Thurston's three-dimensional geometrization conjecture requires one to understand the Ricci flow of all locally homogeneous geometries on closed three-manifolds. We study these flows and describe their characteristic behaviors

1. Introduction

Thurston's three-dimensional geometrization conjecture [9], [7], claims that any closed three-manifold M^3 may be canonically decomposed into pieces such that each of the pieces admits a locally homogeneous geometry. Hamilton has proposed a program for proving this conjecture using Ricci flow. Roughly, the idea is to choose an arbitrary metric on M^3 and then deform this metric via the (normalized) Ricci flow equation

$$\frac{\partial}{\partial t} g = -2\mathrm{Ric} + \frac{2}{3}\langle R\rangle g, \tag{1}$$

where $\langle R\rangle$ denotes the average of the scalar curvature R over M^3. One hopes to relate the local singularities of the flow to the manifold decomposition in Thurston's conjecture, and then one hopes to show that the Ricci flow of the geometry away from each of the local singularities approaches that of a locally homogeneous geometry in each disconnected piece.

While it has been shown that the Ricci flow for certain classes of three-metrics converges ([3], [2]), there are many examples known of three-metrics whose Ricci flows do not converge. It is not surprising that three-dimensional Ricci flows do not generally converge. Ricci flows can only converge to Einstein metrics (the zeros of the right-hand side of the Ricci flow equation (1)), and most three-manifolds (e.g., $S^2 \times S^1$) do not admit an Einstein metric.

Received October 19, 1990 and, in revised form, June 24, 1991. The first author was partially supported by the National Science Foundation under grant PHY-902301. The second author's current address is Department of Mathematics, University of Puget Sound, Tacoma, Washington 98416.

If one wishes to use Ricci flow to study the Thurston conjecture, one must understand completely how the Ricci flows of locally homogeneous geometries on closed manifolds behave. We have investigated this behavior, and we describe the results here. The types of behavior that occur are not very diverse. We find that, depending upon the geometry type, each of the Ricci flows either (a) converges to a constant curvature metric, (b) asymptotically approaches (as $t \to \infty$) a flat degenerate geometry of either two or one dimensions ("pancake" or "cigar" degeneracy), with the curvature decaying at the rate $1/t$, or (c) hits a curvature singularity in finite time, with this singularity being just that of the Ricci flow for the standard metric on $S^2 \times S^1$. Interestingly, in many of these cases one may avoid singularities and degeneracies by using an alternative normalization of the Ricci flow.

We discuss and prove these results in §§3 and 4 after reviewing the classification of locally homogeneous geometries in §2. A few concluding remarks appear in §5.

2. Locally homogeneous geometries on compact manifolds

A Riemannian metric g on a three-manifold M^3 is defined to be *locally homogeneous* if, for every pair of points $x, y \in M^3$, there exist neighborhoods U_x of x and V_y of y such that there is an isometry Ψ mapping $(U_x, g|_{U_x})$ to $(V_y, g|_{V_y})$, with $\Psi(x) = y$. Generally, these local isometries do not extend to isometries of the whole space (M^3, g). If the local isometries do extend, then the geometry is defined to be homogeneous. That is, (M^3, g) is *homogeneous* if, for every pair of points $x, y \in M^3$, there exists an isometry Φ of (M^3, g) with $\Phi(x) = y$. In this case, the isometry group $\mathscr{I}(M^3, g)$ of the geometry acts transitively on M^3.

We wish to study (in the next two sections) the Ricci flow of all locally homogeneous metrics on closed three-manifolds. This task is very much simplified by the following result: *For every locally homogeneous geometry* (M^3, g) *the universal cover* $(\widehat{M}^3, \hat{g})$ *is homogeneous.* This result is an immediate corollary of a theorem of Singer [8] which states that a locally homogeneous metric on a simply connected manifold must be homogeneous. One says that the locally homogeneous geometry (M^3, g) is *modelled* by the homogeneous geometry $(\widehat{M}^3, \hat{g})$. Since Ricci flow commutes with the cover map $\mu: \widehat{M}^3 \to M^3$ (for which $\hat{g} := \mu^* g$), we may study the Ricci flow of any locally homogeneous geometry by examining that of its homogeneous model.

In this work we are only concerned with locally homogeneous geometries on closed manifolds. The homogeneous geometries on simply connected manifolds which we use to model locally homogeneous geometries on closed manifolds fit into nine classes ([5], [7]). These classes are labeled by the minimal isometry group of the geometries it includes: $\mathrm{SO}(3) \times \mathbb{R}^1$, $H(3)$ [the group of isometries of hyperbolic three-space], $H(2) \times \mathbb{R}^1$ [$H(2)$ is the group of isometries of the hyperbolic plane], $\mathbb{R}^3$, $\mathrm{SU}(2)$, $\mathrm{SL}(2, \mathbb{R})$, Heisenberg, $E(1, 1)$ [the group of isometries of the plane with flat Lorentz metric], and E(2) [the group of isometries of the Euclidian plane]. Each class contains a family of geometries, labeled by the choice of g (at a point). For $H(3)$, it is a one-parameter family, with the parameter being the overall scale of the hyperbolic metric. For $\mathrm{SO}(3) \times \mathbb{R}^1$ and $H(2) \times \mathbb{R}^1$, there are two-parameter families, with the parameters being the scales of the sphere and $\mathbb{R}^1$ in the former case, and the scales of the hyperbolic space and $\mathbb{R}^1$ in the latter case. For the rest, we have three-parameter families. This information is summarized in Table 1. Also included in Table 1 is a list of the dimensions of all possible isometry groups for members of each class. For example, all of the geometries in $H(3)$ have a six-dimensional isometry group (that of hyperbolic space H^3), while those in $\mathrm{SU}(2)$ have isometry groups of either six dimensions (the round sphere), four dimensions ($G_x = \mathrm{SO}(2)$), or three dimensions.

How does all of this fit in with the Eight Geometries (X, G) which Thurston uses to model locally homogeneous Riemannian metrics on compact manifolds? The Eight Geometries are essentially the same as the nine classes we have discussed above, except Thurston considers only those metrics which have maximal symmetry in each of the nine classes. Thus both $\mathbb{R}^3$ and $E(2)$ are represented by the geometry E^3. We use the nine classes rather than the Eight Geometries to model locally homogeneous Riemannian metrics on closed manifolds because, in examining the Ricci flow of the locally homogeneous geometries, we need to consider metrics which do not have maximal symmetry. Interestingly, as we shall see, the Ricci flow of these metrics usually (but not always) tends to approach (if not converge to) the maximally symmetric members of the class.

In the next two sections we discuss the Ricci flow of the locally homogeneous Riemannian metrics on closed manifolds via that of their homogeneous models, class by class. For the purposes of this discussion, it is useful to divide the nine classes into two sets. One set, consisting of classes $\mathbb{R}^3$, $\mathrm{SU}(2)$, $\mathrm{SL}(2, \mathbb{R})$, Heisenberg $E(1, 1)$, and $E(2)$, may be studied by

TABLE 1. Classes of locally homogeneous three-geometries on compact manifolds

Class	Description[1]	Allowed Isometry Groups (dimensions)	Family of Metrics (parameters)	Eight Geometries
$\mathbb{R}^3$	Commutative	3, 4, 6	3	E^3
$SU(2)$	Simple compact	3, 4, 6	3	S^3
SL(2, $\mathbb{R}$)	Simple noncompact	3, 4	3	SL(2, $\mathbb{R}$)
Heisenberg	Nilpotent	3, 4	3	Nil
$E(1, 1)$	Solvable	3	3	Solv
$E(2)$	Solvable	3, 4, 6	3	E^3
$H(3)$	–	6	1	H^3
SO(3) $\times \mathbb{R}^1$	–	4	2	$S^2 \times \mathbb{R}^1$
$H(2) \times \mathbb{R}^1$	–	4	2	$H^2 \times \mathbb{R}^1$

treating each geometry as a left invariant metric on a Lie group (that Lie groups which labels the class). We shall call these the Bianchi classes (after the Bianchi classification of the associated Lie algebras; see Bianchi [1], or Ryan and Shepley [6]), and we examine their Ricci flow in §3. The other set, consisting of classes $H(3)$, $H(2) \times \mathbb{R}^1$, and SO(3) $\times \mathbb{R}^1$, will be studied (in §4) by using other means.

3. Ricci flow of the Bianchi classes

Let (X^3, g) be a homogeneous Riemannian metric, with the Lie group H acting simply transitively on X^3 (so $X^3 = H$), and with H being a subgroup of the isometry group of (X^3, g). It follows that one may always express g in the form

$$g = g_{ab}\theta^a\theta^b, \tag{2}$$

with $\{\theta^a\}$ being a left-invariant one-form basis on H, and with g_{ab} being a constant matrix.

The geometries in the Bianchi classes— $\mathbb{R}^3$, SU(2), SL(2, $\mathbb{R}$), Heisenberg, $E(1, 1)$ and $E(2)$—are all of this sort. Further, one can show that

[1] These descriptions refer to the Lie algebras of the minimal isometry group of each class.

for any given metric on one of these classes (all have H unimodular), one can choose the left-invariant basis $\{\theta^a\}$ so that (a) the matrix g_{ab} is diagonal, and (b) the structure constants C^a_{bd}, i.e., those constants such that

$$d\theta^a = C^a_{bd}\theta^b \wedge \theta^d, \tag{3}$$

take the form

$$C^a_{bd} = \varepsilon_{bde} m^{ae}, \tag{4}$$

with m^{ae} a diagonal matrix having entries of values only 0 or 1 or -1. (The particular expression m^{ae} takes is characteristic of the group; see, e.g., Ryan and Shepley [6].) Now, generally the basis $\{\theta^a\}$ which accomplishes this simultaneous diagonalization for one metric in a chosen class is different from that which works for a different metric in the class. However, analysis of the Ricci flow equations shows that the choice of this basis is preserved along the flow. We verify this by choosing such a basis, calculating the Ricci curvature with respect to the basis, and noting that all off-diagonal Ricci terms are zero.

As shown in [3], Ricci flow preserves isometries. Hence, the Ricci flow of a metric g contained in one of the Bianchi classes stays in that class. Based on the above discussion, we can write g_0 in the form

$$g_0 = A_0(\theta^1)^2 + B_0(\theta^2)^2 + C_0(\theta^3)^2 \tag{5}$$

for some basis $\{\theta^a\}$ and be assured that the 1-parameter family of metrics $g(t)$ satisfying the Ricci flow equations with $g(0) = g_0$ will take the form

$$g(t) = A(t)(\theta^1)^2 + B(t)(\theta^2)^2 + C(t)(\theta^3)^2. \tag{6}$$

We thus deduce that the analysis of Ricci flow for metrics in each of the Bianchi classes is an ODE analysis involving a system of three equations for the three variables $\{A(t), B(t), C(t)\}$. Indeed since volume is preserved by the normalized Ricci flow equation (1), one could set $A(t)B(t)C(t) = 1$, and reduce the analysis to two equations for two variables, but in practice it is convenient not to do so. The particular form of the equations depends upon the class. We now discuss Ricci flow in each class.

(a) $\mathbb{R}^3$: The geometries in this class are all flat. Thus the Ricci flow $g(t)$ starting at any metric g_0 in the $\mathbb{R}^3$ class is trivial:

$$g(t) = g_0 \quad \forall t \geq 0. \tag{7}$$

(b) SU(2): The SU(2) class admits Einstein metrics, namely the round metrics on the three sphere. As expected, Ricci flow does converge to these metrics in this case. We show that it does so exponentially in t.

With respect to the basis and parametrization given above, the nonzero curvature components are

(8a) $R_{11} = \frac{1}{2}A[A^2 - (B - C)^2]$,

(8b) $R_{22} = \frac{1}{2}B[B^2 - (A - C)^2]$,

(8c) $R_{33} = \frac{1}{2}C[C^2 - (A - B)^2]$,

(8d) $R = \frac{1}{2}\{[A^2 - (B - C)^2] + [B^2 - (A - C)^2] + [C^2 - (A - B)^2]\}$,

(8e) $\|\mathrm{Ric}\|^2 = \frac{1}{4}\{[A^2 - (B - C)^2]^2 + [B^2 - (A - C)^2]^2 + [C^2 - (A - B)^2]^2\}$.

The Ricci flow equations are then

(9a) $$\frac{d}{dt}A = \frac{2}{3}[-A^2(2A - B - C) + A(B - C)^2],$$

(9b) $$\frac{d}{dt}B = \frac{2}{3}[-B^2(2B - A - C) + B(A - C)^2],$$

(9c) $$\frac{d}{dt}C = \frac{2}{3}[-C^2(2C - A - B) + C(A - B)^2].$$

In terms of the metric components, the fixed point for Ricci flow occurs at $A = B = C$ (which are the metric components for the round sphere). This leads us to calculate the evolution equations for the differences between A, B, and C:

(10a) $$\frac{d}{dt}(A - B) = \frac{2}{3}[-2(A^3 - B^3) + C(A^2 - B^2) + C^2(A - B)],$$

(10b) $$\frac{d}{dt}(A - C) = \frac{2}{3}[-2(A^3 - C^3) + B(A^2 - C^2) + B^2(A - C)],$$

(10c) $$\frac{d}{dt}(B - C) = \frac{2}{3}[-2(B^3 - C^3) + A(B^2 - C^2) + A^2(B - C)].$$

We can see from the symmetry of the equations that, without loss of generality, we may assume $A_0 \geq B_0 \geq C_0$. From the evolution equations in (10) we find that it follows that $A \geq B \geq C$ for all t. With this assumption, it is easy to see from equation (9) that C is nondecreasing since $2C - A - B \leq 0$.

We now show that the difference $A - C$ decays to zero exponentially. This follows from rearranging and estimating the evolution equation for $A - C$,

$$
\begin{aligned}
\frac{d}{dt}(A - C) &= \frac{2}{3}[-2(A^2 + AC + C^2) + B(A + C) + B^2](A - C) \\
&= \frac{2}{3}[-2C^2 - AC - (A - B)(A + C) - (A^2 - B^2)](A - C) \qquad (11)\\
&\leq -2C_0^2(A - C).
\end{aligned}
$$

Integrating this inequality gives

$$A - C \leq (A_0 - C_0)e^{-2C_0^2 t}. \qquad (12)$$

Using this and the inequalities $A \geq B \geq C$ for all t, we conclude that the flow converges exponentially to the fixed point $A = B = C = 1$ (with the normalization $ABC = 1$). From (8d) and (8e), it is clear that R and $\|\text{Ric}\|$ exponentially approach fixed values, namely $\frac{3}{2}$ and $\frac{3}{4}$, respectively.

(c) $\mathrm{SL}(2, \mathbb{R})$: There is no left-invariant Einstein metric on the group $\mathrm{SL}(2, \mathbb{R})$, so we know that Ricci flow in this class cannot converge. However, we can show that the flow starting at any $\mathrm{SL}(2, \mathbb{R})$ metric has characteristic asymptotic behavior: (i) The curvature of $g(t)$ dies off as t goes to infinity; (ii) Two of the components of the metric increase without bound while the other shrinks to zero (recall that the product $ABC = 1$ for all t). We say that a Ricci flow which behaves in this way is developing a "pancake degeneracy." We now verify these results, including the $1/t$ rate of decay for the curvature.

The nonzero components of the curvature for an $\mathrm{SL}(2, \mathbb{R})$ metric of the form (5) are

(13a) $R_{11} = \frac{1}{2}A[A^2 - (B - C)^2]$,

(13b) $R_{22} = \frac{1}{2}B[B^2 - (A + C)^2]$,

(13c) $R_{33} = \frac{1}{2}C[C^2 - (A + B)^2]$,

(13d) $R = \frac{1}{2}\{[A^2 - (B - C)^2] + [B^2 - (A + C)^2] + [C^2 - (A + B)^2]\}$,

(13e) $\|\text{Ric}\|^2 = \frac{1}{4}\{[A^2 - (B - C)^2]^2 + [B^2 - (A + C)^2]^2 + [C^2 - (A + B)^2]^2\}$.

The Ricci flow equations are then

$$\frac{d}{dt}A = \frac{2}{3}[-A^2(2A+B+C)+A(B-C)^2], \tag{14a}$$

$$\frac{d}{dt}B = \frac{2}{3}[-B^2(2B+A-C)+B(A+C)^2], \tag{14b}$$

$$\frac{d}{dt}C = \frac{2}{3}[-C^2(2C+A-B)+C(A+B)^2]. \tag{14c}$$

Now, from equations (14), we see that the system is symmetric under interchange of B and C (this reflects the nature of the group $SL(2, \mathbb{R})$). So, without loss of generality, we may assume $B_0 > C_0$. Then, calculating the evolution equation for the difference $B - C$, we get

$$\frac{d}{dt}(B-C) = \frac{2}{3}[-2(B^3-C^3)-A(B^2-C^2)+A^2(B-C)]. \tag{15}$$

It follows that if $B_0 > C_0$, then $B(t) \geq C(t)$ for all $t > 0$.

Let us now use the inequality $B > C$ in doing estimates on (14c), from which we readily get

$$\frac{d}{dt}C = \frac{2}{3}[(BC^2+B^2C-2C^3)+(ABC-AC^2+ABC)] \geq \frac{2}{3}. \tag{16}$$

Integrating yields

$$C(t) \geq C_0 + \tfrac{2}{3}t \tag{17a}$$

and consequently

$$B(t) \geq C_0 + \tfrac{2}{3}t, \tag{17b}$$

so both B and C must grow at least linearly in t.

Since $ABC = 1$, we get

$$A = \frac{1}{BC} \leq \left(C_0 + \frac{2}{3}t\right)^{-2} \tag{18}$$

so that as B and C grow, A shrinks.

We next show that as B and C grow, they necessarily approach each other. Note that the above results imply the existence of a value τ such that $A \leq B$ for all $t \geq \tau$. We use this to show that after τ, the difference $B - C$ decays exponentially to zero. We estimate (15) under the assumption $A \leq B$:

$$\begin{aligned} \frac{d}{dt}(B-C) &= \frac{2}{3}[-2(B^2+BC+C^2)-A(B+C)+A^2](B-C) \\ &\leq \frac{2}{3}[-2B^2-2BC-2C^2+A^2](B-C) \\ &\leq \frac{2}{3}(-5C_\tau^2)(B-C), \end{aligned} \tag{19}$$

where $C_\tau := C(\tau)$. Integrating this gives

$$B - C \leq (B_\tau - C_\tau)e^{-kt}, \tag{20}$$

where $k := \frac{10}{3}C_\tau^2$.

In order to determine the asymptotic behavior of the curvature, we first need upper bounds on B and C. We get this by rewriting and estimating the evolution equation (14b) for B:

$$\begin{aligned} \frac{dB}{dt} &= \frac{2}{3}[-2B^3 - AB^2 + B^2C + A^2B + 2ABC + BC^2] \\ &= \frac{2}{3}[(B^2C + BC^2 - 2B^3) + AB(A-B) + 2ABC] \leq \frac{4}{3}, \end{aligned} \tag{21}$$

where the inequality holds after the value τ for which $A \leq B$. We integrate the inequality to get

$$B \leq B_\tau + \tfrac{4}{3}t, \tag{22}$$

which also gives us an upper bound on C since $C \leq B$ for all t.

We can now determine the asymptotic behavior of the scalar and Ricci curvatures. To begin, we examine the behavior of R by rearranging the right side of (13d) and taking the absolute value to get

$$|R| = \tfrac{1}{2}A^2 + \tfrac{1}{2}(B-C)^2 + AB + AC. \tag{23}$$

Each term on the right-hand side of this equation can be bounded from above by using (18), (20), and (22). Since the terms AB and AC have the slowest decay rate at t^{-1}, the overall asymptotic decay rate for $|R|$ is bounded above by t^{-1}.

To determine the asymptotic behavior of $\|\mathrm{Ric}\|$, we look separately at each of the three terms in the expression on the right-hand side of (13e). The first term can be bounded by first writing

$$[A^2 - (B-C)^2]^2 \leq A^4 + (B-C)^4. \tag{24}$$

From (18) and (20), we can see that this term is asymptotically dominated by A^4 which decays as t^{-8}. For the second term, we do a bit of algebra to get

$$[B^2-(A+C)^2]^2 = [B^2-C^2-A^2-2AC]^2 \leq (B^2-C^2)^2+A^4+2A^2C^2, \tag{25}$$

where the term $(B^2 - C^2)^2$ can be factored as

$$(B^2 - C^2)^2 = (B-C)^2(B+C)^2. \tag{26}$$

From (20) and (22) it follows that, for large t, this decays as t^2e^{-t}. The term A^4 decays as t^{-8}, while the term $2A^2C^2$ has the slowest decay rate at t^{-2}. The second term on the right-hand side of (13e) thus has an asymptotic decay rate of t^{-2}. A similar argument gives the same behavior for the remaining term in $\|\mathrm{Ric}\|^2$. Thus the overall asymptotic decay rate for $\|\mathrm{Ric}\|$ is t^{-1}.

Note that as the Ricci flows in this class evolve toward the pancake degeneracy, they exponentially approach geometries in the class with maximal symmetry: those with isometry $\mathrm{SL}(2, \mathbb{R}) \times \mathrm{SO}(2)$.

(d) *Heisenberg*: Like the $\mathrm{SL}(2, \mathbb{R})$ class, the Heisenberg (or "nil") class does not contain any Einstein metrics, and hence the Ricci flow of geometries in this class cannot converge. Again, however, we find that the Ricci flow of any metric approaches a pancake degeneracy, at the rate $\|\mathrm{Ric}\| < 1/t$. In this case, the Ricci flow equations are simple enough so that exact solutions for the flow can be found for any initial metric.

The curvature components for the metrics of the Heisenberg class are:

(27a) $$R_{11} = \tfrac{1}{2}A^3,$$

(27b) $$R_{22} = -\tfrac{1}{2}A^2 B,$$

(27c) $$R_{33} = -\tfrac{1}{2}A^2 C,$$

(27d) $$R = -\tfrac{1}{2}A^2,$$

(27e) $$\|\mathrm{Ric}\|^2 = 3R^2 = \tfrac{3}{4}A^4.$$

The Ricci flow equations are then

(28a) $$\frac{d}{dt}A = -\frac{4}{3}A^3,$$

(28b) $$\frac{d}{dt}B = \frac{2}{3}A^2 B,$$

(28c) $$\frac{d}{dt}C = \frac{2}{3}A^2 C.$$

Starting with the equation for A, these can be integrated directly to get

(29a) $$A = A_0(1 - \tfrac{16}{3}R_0 t)^{-1/2},$$

(29b) $$B = B_0(1 - \tfrac{16}{3}R_0 t)^{1/4},$$

(29c) $$C = C_0(1 - \tfrac{16}{3}R_0 t)^{1/4},$$

where $R_0 = -\tfrac{1}{2}A_0^2 < 0$. We thus see that B and C increase together at the rate $t^{1/4}$, while A decreases at the rate $t^{-1/2}$.

If we substitute these results into the expressions for the curvature (27), we obtain

(30a) $$|R(t)| = |R_0|(1 - \tfrac{16}{3}R_0 t)^{-1},$$

(30b) $$\|\mathrm{Ric}(t)\| = \|\mathrm{Ric}_0\|(1 - \tfrac{16}{3}R_0)^{-1},$$

so the curvature scalars die off at the rate t^{-1}.

While the asymptotic behaviors of the flows in the Heisenberg and $SL(2, \mathbb{R})$ classes are both pancake degenerate (i.e., two diameters expand, the other shrinks, and the curvature dies off), their behaviors differ in an important way: As shown above, the two expanding directions in an $SL(2, \mathbb{R})$ flow converge, so that a generic flow in $SL(2, \mathbb{R})$ approaches a flow of $SL(2, \mathbb{R})$ geometries with rotational isotropy. The expanding directions in Heisenberg class Ricci flows generally diverge, however: we have $B - C = (B_0 - C_0)(1 - \frac{16}{3}R_0 t)^{1/4}$. Hence generic Heisenberg flows move away from those with rotational isotropy (i.e., those with $B = C$).

(e) $E(1, 1)$: The $E(1, 1)$ (or "solv") class again fails to contain any Einstein metrics and so its Ricci flows do not converge. They are all asymptotically "cigar degeneracies": The curvature dies, and while one diameter expands without bound, the other two diameters shrink to zero. The rate of curvature decay is $1/t$.

For the $E(1, 1)$ geometries, we calculate the curvature to be

(31a) $$R_{11} = \tfrac{1}{2}A(A^2 - B^2),$$

(31b) $$R_{22} = \tfrac{1}{2}B(B^2 - A^2),$$

(31c) $$R_{33} = -\tfrac{1}{2}C(A + B)^2,$$

(31d) $$R = -\tfrac{1}{2}(A + B)^2,$$

(31e) $$\|\mathrm{Ric}\|^2 = 3R^2 = \tfrac{3}{4}A^4.$$

The Ricci flow equations are then

(32a) $$\frac{d}{dt}A = \frac{2}{3}[-2A^3 - AB(A - B)],$$

(32b) $$\frac{d}{dt}B = \frac{2}{3}[-2B^3 + AB(A - B)],$$

(32c) $$\frac{d}{dt}C = \frac{2}{3}C(A + B)^2.$$

Noting the symmetry of these equations under interchange of A and B, we may without loss of generality assume that $A_0 \geq B_0$. Then from the equation

(33) $$\frac{d}{dt}(A - B) = -\frac{4}{3}(A + B)^2(A - B),$$

it follows that $A \geq B$ for all values of t.

This assumption, substituted into (32), allows us to derive a tight set of bounds on A, B, and C. First, for A, from (32a) and $A \geq B$ we obtain the inequality

$$\frac{d}{dt}A \leq -\frac{4}{3}A^3. \tag{34}$$

Integrating this gives the upper bound

$$A \leq A_0(1 + \tfrac{8}{3}A_0^2 t)^{-1/2}, \tag{35}$$

which shows that A shrinks at least as fast as $1/t^{1/2}$. Since $B \leq A$, we have the same upper bound on B. To find a lower bound on B (and consequently on A) we substitute the assumption $A \geq B$ into (32b) to get

$$\frac{d}{dt}B \leq -\frac{4}{3}B^3. \tag{36}$$

Integrating this yields

$$B \geq B_0(1 + \tfrac{8}{3}B_0^2 t)^{-1/2} \geq B_0(1 + \tfrac{8}{3}A_0^2 t)^{-1/2}. \tag{37}$$

Then to bound C, we may use the condition $C = \frac{1}{AB}$ along with the bounds on A and B, or the evolution equation (32c) to get

$$\frac{d}{dt}C = \frac{2}{3}C(A+B)^2 = \frac{2}{3}\left(\frac{A^2 + 2AB + B^2}{AB}\right) \geq \frac{4}{3}, \tag{38}$$

which implies

$$C \geq C_0 + \tfrac{4}{3}t. \tag{39}$$

We then estimate

$$\frac{d}{dt}C = \frac{2}{3}(A+B)^2 C \leq \frac{8}{3}A^2 C \leq \frac{8}{3}\frac{A_0^2}{1 + \frac{8}{3}A_0^2 t}C, \tag{40}$$

from which we derive

$$C \leq C_0 + \frac{8}{3}\left(\frac{A_0}{B_0}\right)t. \tag{41}$$

These results show us that C expands linearly in t, while A and B shrink at the rate $1/t^{1/2}$. We now wish to show that as A and B shrink, they approach each other. Specifically, we will show that $A - B$ decreases faster than does $A + B$.

We use our bounds on A and on B to obtain the estimates

$$\frac{-16}{3}A^2(A-B) \leq \frac{d}{dt}(A-B) \leq \frac{-16}{3}B^2(A-B). \tag{42}$$

Integrating (and again using the bounds on A and B), we find

$$(43)\qquad (A_0 - B_0)(1 + \tfrac{8}{3}A_0^2 t)^{-2} \le A - B \le (A_0 - B_0)(1 + \tfrac{8}{3}A_0^2 t)^{-2}.$$

For $A + B$, we have

$$(44)\ \frac{d}{dt}(A+B) = -\frac{4}{3}[(A-B)^2 + AB](A+B) = -\frac{4}{3}[A^2 + B^2 - AB](A+B).$$

We use the first equality and the bound on B in (37) to get the estimate

$$(45)\qquad \frac{d}{dt}\ln(A+B) \le -\frac{4}{3} \le -\frac{4}{3}B^2 \le -\frac{4}{3}B_0^2(1 + \frac{8}{3}B_0^2 t)^{-1}.$$

Integrating this gives

$$(46)\qquad A + B \le (A_0 + B_0)(1 + \tfrac{8}{3}B_0^2 t)^{-1}.$$

Similarly, we use the second equality in (46) and the bound on A in (35) to get the lower estimate

$$(47)\qquad \frac{d}{dt}\ln(A+B) \ge -\frac{4}{3}(A^2 + B^2) \ge -\frac{8}{3}A^2 \ge -\frac{8}{3}A_0^2(1 + \frac{8}{3}A_0^2 t)^{-1}.$$

Integrating this yields

$$(48)\qquad A + B \ge (A_0 + B_0)(1 + \tfrac{8}{3}A_0^2 t)^{-1}.$$

From these estimates, we see that the difference $A - B$ is decreasing faster than the sum $A + B$, so that A and B approach each other as $t \to \infty$. With these bounds on $A + B$, we can also bound the scalar curvature by

$$(49)\qquad |R_0|(1 + \tfrac{8}{3}A_0^2 t)^{-2} \le |R| \le |R_0|(1 + \tfrac{8}{3}B_0^2 t)^{-1},$$

and the norm of Ricci by

$$(50)\qquad \|\mathrm{Ric}\| \le \|\mathrm{Ric}_0\|(1 + \tfrac{8}{3}B_0^2 t)^{-1}.$$

From this analysis we see that all Ricci flows in class $E(1, 1)$ approach cigar degeneracy. Further, all involve A converging to B. Note, however, that the condition $A = B$ does not characterize a subclass of $E(1, 1)$ geometries with nontrivial isotropy. We recall that there are no $E(1, 1)$ geometries which have isometry group of dimension greater than three.

(f) $E(2)$: The $E(2)$ class does admit an Einstein metric—the flat geometry. We show here that $E(2)$ Ricci flows all converge exponentially to flat metrics.

The $E(2)$ curvatures are

$$R_{11} = \tfrac{1}{2}A(A^2 - B^2), \tag{51a}$$

$$R_{22} = \tfrac{1}{2}B(B^2 - A^2), \tag{51b}$$

$$R_{33} = -\tfrac{1}{2}C(A - B)^2, \tag{51c}$$

$$R = -\tfrac{1}{2}(A - B)^2, \tag{51d}$$

$$\|\mathrm{Ric}\|^2 = \tfrac{1}{2}(A^2 - B^2)^2 + \tfrac{1}{4}(A - B)^4. \tag{51e}$$

Note that these vanish (and the geometry is flat) if and only if $A = B$. The Ricci flow equations for the $E(2)$ class are

$$\frac{d}{dt}A = -\frac{2}{3}A(2A + B)(A - B), \tag{52a}$$

$$\frac{d}{dt}B = \frac{2}{3}B(2B + A)(A - B), \tag{52b}$$

$$\frac{d}{dt}C = \frac{2}{3}C(A - B)^2. \tag{52c}$$

The usual arguments, together with the evolution equation

$$\frac{d}{dt}(A - B) = -\frac{4}{3}(A^3 - B^3) = -\frac{4}{3}(A^2 + AB + B^2)(A - B), \tag{53}$$

allow us (without loss of generality) to choose $A_0 \geq B_0$ and thence to deduce that $A \geq B$ for all t. It thus follows from (52) that A is nonincreasing while both B and C are nondecreasing, so that we have the bounds

$$B_0 \leq A \leq A_0, \tag{54a}$$

$$B_0 \leq B \leq A_0, \tag{54b}$$

$$C_0 \leq C \leq C_0 \frac{A_0}{B_0}. \tag{54c}$$

Note that the upper bound on C is obtained due to the fact that ABC is a constant along the flow.

Using the bounds (54), from (53) we derive the estimate

$$\frac{d}{dt}(A - B) \leq -4B_0^2(A - B). \tag{55}$$

Integrating gives

$$A - B \leq (A_0 - B_0)e^{-4B_0^2 t}. \tag{56}$$

This shows us that A and B approach each other exponentially, i.e., the flow converges to a flat metric. This is confirmed by examining the bounds on the scalar curvature and the norm of Ricci:

$$|R| \leq |R_0| e^{-8B_0^2 t}, \tag{57a}$$

$$\|\mathrm{Ric}\| \leq K e^{-4B_0^2 t}, \tag{57b}$$

where K depends only on A_0 and B_0.

4. Ricci flow of the non-Bianchi classes

Three classes of locally homogeneous geometries remain: $H(3)$, $\mathrm{SO}(3)\times\mathbb{R}^1$, and $H(2)\times\mathbb{R}^1$. The metrics in these classes all have isometry groups of at least four dimensions, and each class contains at most a two-parameter family of metrics. As we shall now see, the Ricci flow analysis for these classes is very simple.

(a) $H(3)$: All the geometries in this class have constant negative curvature; their metrics are all constant multiples of the standard hyperbolic metric:

$$g = \Lambda g_{H^3}. \tag{58}$$

Each metric is a fixed point of the Ricci flow, so the Ricci flow in this class is trivial:

$$g(t) = g_0 \quad \forall t \geq 0. \tag{59}$$

(b) $\mathrm{SO}(3)\times\mathbb{R}^1$: The metrics in this class all take the product form

$$g = D g_{\mathbb{R}^1} + E g_{S^2}, \tag{60}$$

where $g_{\mathbb{R}^1}$ is the metric on $\mathbb{R}^1$, g_{S^2} is the round metric on the two-sphere, and D and E are constants (the two parameters of this family of metrics). None of them are Einstein metrics, so the Ricci flows cannot converge. We find that in fact the Ricci flows of all metrics in this class go singularly in finite time.

Choosing a standard basis $\{d\psi; d\theta, \sin\theta\, d\phi\}$ we calculate the curvature and find

$$R_{11} = 0, \tag{61a}$$

$$R_{22} = 1, \tag{61b}$$

$$R_{33} = 1, \tag{61c}$$

$$R = 2/E, \tag{61d}$$

$$\|\mathrm{Ric}\|^2 = 2/E^2. \tag{61e}$$

It follows that the Ricci flow equations (in the usual normalization) take the form

(62a) $$\frac{d}{dt}E = -\frac{2}{3},$$

(62b) $$\frac{d}{dt}D = \frac{4}{3}D/E.$$

We easily integrate (62), obtaining the solutions

(63a) $$E = E_0 - \tfrac{2}{3}t,$$

(63b) $$D = D_0E_0^2/(E_0 - \tfrac{2}{3}t)^2.$$

Clearly the round two-sphere shrinks (undistorted) linearly in t, while the scale on $\mathbb{R}^1$ (which is the radius of the circle for this metric on $S^2 \times S^1$) expands at the indicated rate. At time $t = \frac{3}{2}E_0$, the radius of the sphere reaches zero, and a curvature singularity is obtained. Note that this is the only class of locally homogeneous geometries whose Ricci flows do achieve curvature singularities.

One may avoid these curvature singularities by using a different normalization for Ricci flow, namely, the one given by

(64) $$\frac{\partial}{\partial t}g = -2\mathrm{Ric} + \langle {}^2R\rangle_2 g,$$

where ${}^2R$ is the scalar curvature of the metric on S^2, and $\langle\,\rangle_2$ indicates averaging over S^2. This normalization preserves the volume of the two-sphere geometry rather than the volume of the full three-geometry. Applied to metrics of the form (60), equation (64) reads

(65a) $$\frac{d}{dt}E = 0,$$

(65b) $$\frac{d}{dt}D = 2D/E,$$

whose solution is

(66a) $$E = E_0,$$

(66b) $$D = D_0e^{2/E_0t},$$

which is nonsingular for all t.

(c) $H(2) \times \mathbb{R}^1$: These metrics take the form

(67) $$g = D\gamma_{\mathbb{R}^1} + E\gamma_{H^2},$$

where γ_{H^2} is the metric on the hyperbolic plane. Again, there are no Einstein metrics in the class, and so no convergence occurs. As we now show, Ricci flows in this class all exhibit pancake degeneracies.

In the standard basis $\{d\psi; d\theta, \sinh\theta\, d\phi\}$, the curvatures for metrics of the form (66) are as follows:

(68a) $$R_{11} = 0,$$

(68b) $$R_{22} = -1,$$

(68c) $$R_{33} = -1,$$

(68d) $$R = -2/E,$$

(68e) $$\|\mathrm{Ric}\|^2 = 2/E^2.$$

Then the Ricci flow equations take the form

(69a) $$\frac{d}{dt}E = \frac{2}{3},$$

(69b) $$\frac{d}{dt}D = -\frac{4}{3}D/E.$$

We integrate these to get solutions

(70a) $$E = E_0 + \tfrac{2}{3}t,$$

(70b) $$D = D_0E_0^2/(E_0 + \tfrac{2}{3}t)^2.$$

The signs here are crucial. The scale of the hyperbolic geometry, E, *increases* linearly in time, and hence is nonsingular. The other scale factor, D, decreases, but only at the rate $1/t^2$. Noting that the curvature decays, according to

(71) $$\|\mathrm{Ric}\| = \sqrt{2}/(E_0 + \tfrac{2}{3}t),$$

we verify pancake degeneracy behavior.

As with the $\mathrm{SO}(3) \times \mathbb{R}^1$ class of geometries, we may use an alternate Ricci flow normalization of the form (64) to prevent expansion of the hyperbolic part of the metric, $E\gamma_H$. We then have

(72a) $$E = E_0,$$

(72b) $$D = D_0e^{-2/E_0t}.$$

The flow is nonsingular for all t.

5. Conclusion

Our analysis provides a complete picture of the behavior of the Ricci flow of locally homogeneous Riemannian metrics on closed manifolds.

The results are summarized, by class, in Table 2. The key conclusions are the following:

(a) Whenever the flow *can* converge (i.e., in those classes which admit Einstein metrics: $\mathbb{R}^3$, $H(3)$, $\mathrm{SU}(2)$, $E(2)$) it *does* converge. Further, the convergence is exponential.

(b) Curvature singularities occur only in the flows in the class $\mathrm{SO}(3)\times\mathbb{R}$. These may be averted by changing the Ricci flow normalization.

(c) In all other classes— $\mathrm{SL}(2, \mathbb{R})$, Heisenberg, $E(1, 1)$, $H(2)\times\mathbb{R}$—the curvature dies along the flow as $1/t$ and the flows approach either pancake or cigar degeneracies.

(d) In all classes except Heisenberg, the generic flows approach the flows of the maximally symmetric members of the class.

TABLE 2. Summary of Ricci flow results

Class	Behavior	Rate
$\mathbb{R}^3$	Convergence to flat space	Trivial flow (all metrics flat)
$SU(2)$	Convergence to round sphere	Exponential
$\mathrm{SL}(2, \mathbb{R})$	Pancake Degeneracy	$\|\mathrm{Ric}\| \sim 1/t$
Heisenberg	Pancake Degeneracy	$\|\mathrm{Ric}\| \sim 1/t$
$E(1, 1)$	Cigar Degeneracy	$\|\mathrm{Ric}\| \sim 1/t$
$E(2)$	Convergence to flat space	Exponential
$H(3)$	Convergence to hyperbolic space	Trivial (all members hyperbolic)
$\mathrm{SO}(3) \times \mathbb{R}^1$	Curvature Singularity	$\|\mathrm{Ric}\| \sim 1/(1-t)$
$H(2) \times \mathbb{R}^1$	Pancake Degeneracy	$\|\mathrm{Ric}\| \sim 1/t$

While this work is partially motivated as a model study of the behavior of Ricci flow in three dimensions, its main motivation comes from its role in Hamilton's program for the study of Thurston's geometrization conjecture. We now see what sort of behavior to look for in generic Ricci flows, as one attempts to show that, away from local curvature singularities, they approach the Ricci flows of locally homogeneous geometries.

The main difficulty with Hamilton's program is its treatment of singularities. No way is yet known to extend solutions past the singularities which generally—and crucially—occur. To get around this difficulty, we have begun looking at families of inhomogeneous metrics which have certain symmetries and topological conditions preventing the occurrence of

S^2 and T^2 pinches. One such family consists of the collection of metrics

$$g = e^{2A} d\theta^2 + e^f [e^W dx^2 + e^{-W} dy^2] \tag{73}$$

(where A is a periodic function of θ, W is a function of θ with $W(\theta + 2\pi) = W(\theta) + \Lambda$ for the topological constant Λ, and f is a constant) defined on the manifold M^3 which is constructed as a solv-twisted T^2 bundle over the circle (with coordinate θ). For W linear in θ, these metrics are locally homogeneous, lying in the class $E(1, 1)$. Elsewhere [4], we show that the Ricci flows of metrics of the form (73) always asymptotically approach the Ricci flow of those in $E(1, 1)$. This result provides some measure of support for Hamilton's program. Further work of this nature is underway.

Acknowledgment

The first author thanks Willamette University for its hospitality during the writing of this paper. We thank Richard Hamilton and Jack Lee for very useful conversations.

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