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GENERALIZED YANG-BAXTER EQUATIONS, KOSZUL OPERATORS AND POISSON LIE GROUPS

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0. Introduction

The notion of Poisson Lie group was first introduced by Drinfel′d in [2]. Some recent work on this subject may be found in [3], [6], [8], and [10]. General theories of Poisson manifolds may be found in [13]. It is known that every connected Poisson Lie group arises from a Lie bialgebra and an important class of Lie bialgebras, the coboundary Lie bialgebras [3] are obtained by solving the generalized Yang-Baxter equations. The solutions are called classical r-matrices.

In this paper, we first consider a class of generalized Yang-Baxter equations which are connected naturally with Manin triples of a Lie bialgebra [3] and give their solutions in terms of Koszul operators for all real semisimple Lie algebras with a compact Cartan subalgebra (§2). In these cases, the equation may be regarded as the integrability condition of an almost complex structure on a homogeneous space. Consequently, we get that every real semisimple Lie group with a maximal torus as the Cartan subgroup is a nontrivial Poisson Lie group. In the compact case, we also show that this Lie bialgebra structure is the same as that given by Lu and Weinstein [8]. It will be seen that our approach is more convenient for detailed studies because the expression for the r-matrix gives much information on the algebraic structure. In §3, we discuss the symplectic leaves and Lagrangian submanifolds of the corresponding Poisson Lie groups. In §4, we generalize an involution theorem for Poisson Lie groups to compact and noncompact symmetric spaces.

We would like to thank the referee for pointing out that our Theorem 4 has overlapped with Theorem 2 of [12].

1. Generalized Yang-Baxter equations and Manin triples

Let g be a real Lie algebra. A linear operator $R \in \mathrm{End}(g)$ is called a classical r-matrix if the bracket given by

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$$(1.1) \qquad [X, Y]_R = [RX, Y] + [X, RY] \quad \forall X, Y \in g$$

is still a Lie bracket on g. Such a pair (g, R) is called a double Lie algebra. Moreover, if there is a nondegenerate invariant bilinear form $\langle\cdot, \cdot\rangle$ on g and R is skew-symmetric, then $(g, R, \langle\cdot, \cdot\rangle)$ becomes a Lie bialgebra [3], [9]. It is known that (g, R) is a double Lie algebra iff the bilinear map B_R: $g \times g \to g$, given by

$$(1.2) \qquad B_R(X, Y) = [RX, RY] - R([X, Y]_R),$$

satisfies the equation

$$(1.3) \qquad [X, B_R(Y, Z)] + [Y, B_R(Z, X)] + [Z, B_R(X, Y)] = 0$$

for all $X, Y, Z \in g$, which means that bracket (1.1) satisfies the Jacobi identity. In particular, the equation

$$(1.4) \qquad B_R(X, Y) = \alpha[X, Y], \qquad \alpha \in \mathbb{R}, \forall X, Y \in g,$$

is called the generalized Yang-Baxter equation [6]. (1.4) is called the classical Yang-Baxter equation if $\alpha = 0$ [1] and the modified Yang-Baxter equation if $\alpha < 0$ [9]. In this paper, we consider the equation

$$(1.5) \qquad B_R(X, Y) = [X, Y].$$

A Manin triple (p, p_1, p_2) consists of a Lie bialgebra p with a nondegenerate invariant scalar product on it and two isotropic subalgebras p_1, p_2 such that $p = p_1 \oplus p_2$ as a vector space decomposition [3]. It will be seen that (1.5) is naturally connected with Manin triples of Lie bialgebras.

Let us fix the notation as follows: g is a real Lie algebra equipped with a nondegenerate invariant scalar product $\langle\cdot, \cdot\rangle$, and $\overline{g} = g + ig$ ($i = \sqrt{-1}$) is considered as a real Lie algebra.

Lemma 1. *Suppose that $R \in \mathrm{End}(g)$ is a skew-symmetric solution of* (1.5), *i.e.,* $(g, R, \langle\cdot, \cdot\rangle)$ *is a Lie algebra. Then we have the following.*

(a) $\ker R = i(g_+^* \cap g_-^*)$ *is an abelian subalgebra of g, where $g_\pm^* = \mathrm{Im}(R \pm i) \subset \overline{g}$ denotes the images of the linear operators $R \pm i$: $g \to \overline{g}$.*

(b) $\ker(R \pm i) = g \cap g_\pm^* = \{0\}$. *Consequently, $\overline{g}$ has decompositions*

$$(1.6) \qquad \overline{g} = g \oplus g_\pm^*.$$

(c) *Both $g_\pm^*$ are subalgebras of $\overline{g}$ and isomorphic to g_R, where the symbol g_R denotes the Lie algebra with bracket* (1.1) *on g.*

Proof. (a) and (b) are easily checked by the definitions. For (c), notice that the relation

$$(1.7) \qquad (R \pm i)[X, Y]_R = [(R \pm i)X, (R \pm i)Y] \quad \forall X, Y \in g$$

is only a reformulation of (1.5), and (c) follows from (1.7). q.e.d.

Now, we extend $\langle\cdot,\cdot\rangle$ from g to $\overline{g}=g+ig$ by real linearity, so both arguments will be denoted by the same symbol. Let $\mathrm{Im}\langle\cdot,\cdot\rangle$ be the imaginary part of $\langle\cdot,\cdot\rangle$

Theorem 1. *With the same notation and conditions as in Lemma* 1, *we have* $\mathrm{Im}\langle\cdot,\cdot\rangle$ *also being a nondegenerate invariant scalar product on* $\overline{g}$, *with respect to which* g, $g^*_\pm$ *are isotropic subalgebras of* $\overline{g}$. *Consequently, both* $(\overline{g}, g, g^*_\pm)$ *are Manin triples of the Lie bialgebra* $(g, R, \langle\cdot,\cdot\rangle)$.

Proof. For any $X, Y\in g$, we have

$$\mathrm{Im}\langle (R\pm i)X, (R\pm i)Y\rangle = \pm(\langle RX, Y\rangle + \langle X, RY\rangle)=0$$

since $R=-R^*$ is skew-symmetric; thus both $g^*_\pm$ are isotropic with respect to $\mathrm{Im}\langle\cdot,\cdot\rangle$. Obviously g is isotropic. Then from (b), (c) of Lemma 1 and the definition of the Manin triple, the theorem follows.

Remark. In general, it is difficult to study Manin triples of a Lie bialgebra g because the Lie algebraic structure to be defined on $g\oplus g^*$ is complicated (see [8]). But, in our case, this becomes very clear, so we think it is worthwhile to carry out a deeper study for the generalized Yang-Baxter equation (1.5).

One of the advantages of the Manin triples $(\overline{g}, g, g^*_\pm)$ is to describe dressing actions of $G^*_\pm$ on G, the Poisson Lie groups with Lie algebras $g^*_\pm$ and g respectively (see [8] and [10]). From the decomposition (1.6), one gets a local (sometimes global) factorization

$$\overline{G}=G\cdot G^*_\pm, \tag{1.8}$$

and the dressing orbit through a point $y\in G$ is then given in the following form of double cosets:

$$S_y = G\cap (G^*_\pm\cdot y\cdot G^*_\pm)=\{x\in G;\ \exists a, b\in G^*_\pm,\ \text{s.t.}\, x=ayb\}. \tag{1.9}$$

It is also known that $S_y\subset G$ is just the symplectic leaf of the Poisson Lie group G through the point $y\in G$.

Furthermore, we can get a new Lie bialgebra $(\overline{g}, \pi_R, \mathrm{Im}\langle\cdot,\cdot\rangle)$ by means of the Manin triple $(\overline{g}, g, g^*_+)$, where $\pi_R=\pi-\pi_+$, and π, π_+ are two projections from $\overline{g}$ to g, g^*_+ with respect to decomposition (1.6). According to [10, Proposition 6], we get a symplectic structure on the complex Lie group $\overline{G}$ with the nondegenerate Poisson tensor

$$\overline{\Omega_x} = l_{x^*}\pi_R + r_{x^*}\pi_R, \qquad x\in\overline{G}, \tag{1.10}$$

where l_x and r_x denote the left and right actions on $\overline{G}$ respectively. In the case when decomposition (1.8) is global, one can identify G with

$\overline{G}/G_+^*$ and G_+^* with $G\backslash\overline{G}$. Then the natural projections

$$G \cong \overline{G}/G_+^* \leftarrow \overline{G} \rightarrow G\backslash\overline{G} \cong G_+^* \tag{1.11}$$

are Poisson maps (see [10]). In the next section, it will be seen that such a symplectic structure always exists on a complex semisimple Lie group.

We end this section by noting that, if we extend R from g to $\overline{g}$ and write $\mathrm{Re}\langle\cdot\,,\cdot\rangle$ as the real part of $\langle\cdot\,,\cdot\rangle$, then both $(\overline{g}, R, \mathrm{Re}\langle\cdot\,,\cdot\rangle)$ and $(\overline{g}, iR, Re\langle\cdot\,,\cdot\rangle)$ are Lie bialgebras, where iR satisfies the modified Yang-Baxter equation (1.4) $(\alpha = -1)$. In §2, it will be seen that sometimes it is possible to get a new Lie bialgebra $g_0 \subset \overline{g}$ if g_0 is iR-invariant and $\mathrm{Re}\langle\cdot\,,\cdot\rangle|_{g_0\times g_0}$ is nondegenerate.

2. Koszul operators as r-matrices

Let us first recall the definition of Koszul operators. Let G be a real connected Lie group with the Lie algebra g and $H \subset G$, a closed subgroup of G with the Lie algebra h. Moreover, suppose that g has a decomposition such that $g = h + m$, $[h, m] \subset m$. Thus, the coset space G/H is a reductive homogeneous space. Koszul's theorem states that the coset space G/H has a G-invariant complex structure if and only if there is a linear operator J on g (the Koszul operator, see [5]) such that

(i) $J|_h = 0$, $J^2|_m = -1$,
(ii) $\mathrm{ad}(X) \circ J = J \circ \mathrm{ad}(X)$, $\forall X \in h$,
(iii) $[JX, JY] - J([JX, Y] + [X, JY]) = [X, Y] \,(\mathrm{mod}\, h)$, $\forall X, Y \in g$.

Notice that (iii) is the integrability condition of an almost complex structure on G/H, which is just (1.5) if the term $(\mathrm{mod}\, h)$ is ignored. So our idea is to take Lie algebras g, h in a suitable way such that (iii) becomes (1.5). First of all, h being abelian is a necessary condition for (iii) becoming (1.5) by part (a) of Lemma 1. On the other hand, it is also known that the coset space G/H is a homogeneous Kähler manifold if g is a compact semisimple Lie algebra and h is a Cartan subalgebra of g (see H. C. Wang's theorem in [5]). Actually, we can prove the following:

Theorem 2. *Let g be a compact semisimple Lie algebra, and h a Cartan subalgebra of g. Then the Koszul operator J making G/H a homogeneous Kähler manifold is skew-symmetric with respect to the Killing form $\langle\cdot\,,\cdot\rangle$ of g and satisfies equation* (1.5), *i.e.,*

$$[JX, JY] - J([JX, Y] + [X, JY]) = [X, Y] \quad \forall X, Y \in g.$$

Thus, J is a classical r-matrix and (g, J) is a Lie bialgebra. Consequently, every compact semisimple Lie group G is a Poisson Lie group with (g, J) as the tangent Lie bialgebra.

Proof. Let $m = h^{\perp}$ with respect to the Killing form of g. Then one has $\mathrm{ad}(h)m \subset m$. To see the skew-symmetry and explicit expression of J, consider the complexification of g, $g^{\mathbb{C}} = g + ig$, which has an orthonormal decomposition $g^{\mathbb{C}} = h^{\mathbb{C}} \oplus m^{\mathbb{C}}$ corresponding to the orthogonal decomposition $g = h \oplus m$. Let $J^{\mathbb{C}}$ represent the natural extension of J to $g^{\mathbb{C}}$. Since $m^{\mathbb{C}}$ is an invariant subspace of $J^{\mathbb{C}}$ and $(J^{\mathbb{C}})^2|_{m^{\mathbb{C}}} = -1$, $m^{\mathbb{C}}$ has the root space decomposition

$$m^{\mathbb{C}} = n^{+} \oplus n^{-} \quad \text{such that} \quad J^{\mathbb{C}}|_{n^{\pm}} = \pm i. \tag{2.1}$$

By property (ii) of J, both $n^{\pm}$ are $\mathrm{ad}(h)$-invariant subspaces and $\overline{n}^{+} = n^{-}$, where the overbar denotes the complex conjugate on $g^{\mathbb{C}}$. Let

$$g^{\mathbb{C}} = h^{\mathbb{C}} + \sum_{\alpha\in\Delta} g_{\alpha}^{\mathbb{C}} \tag{2.2}$$

be the root space decomposition of the complex semisimple Lie algebra $g^{\mathbb{C}}$ with respect to the Cartan subalgebra $h^{\mathbb{C}}$. However, one can choose a set of the simple roots in Δ such that

$$n^{\pm} = \sum_{\alpha\in\Delta^{\pm}} g_{\alpha}^{\mathbb{C}}.$$

Notice that, for every $X \in n^{+}$, both $Z = X + \overline{X}$ and $Z' = i(X - \overline{X})$ are in $m \subset g$, and $JZ = Z'$, $JZ' = -Z$. Furthermore, it is known that, for every $\alpha \in \Delta$, one can choose $0 \neq X_{\alpha} \in g_{\alpha}^{\mathbb{C}}$ such that (see [4])

$$\begin{aligned}
&\text{(a) } \overline{X}_{\alpha} = -X_{-\alpha}, \\
&\text{(b) } [X_{\alpha}, X_{\beta}] = N_{\alpha\beta}X_{\alpha+\beta}, \quad N_{\alpha\beta} \in \mathbb{R}, \quad N_{\alpha\beta} = -N_{-\alpha,-\beta}, \\
&\text{(c) the set } \{Z_{\alpha} = X_{\alpha} - X_{-\alpha},\ Z_{\alpha}' = i(X_{\alpha} + X_{-\alpha})\}_{\alpha\in\Delta^{+}} \\
&\qquad \text{forms an orthonormal basis of } m.
\end{aligned} \tag{2.3}$$

Thus we get the following explicit expression for J under the orthonormal basis of m given by (2.3):

$$J = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & I \\ 0 & -I & 0 \end{pmatrix}, \qquad I = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}. \tag{2.4}$$

Consequently, J is skew-symmetric. Next, we verify that J solves (1.5). For every $L \in h$ and $X, Y \in m$, one has

$$\langle L, [JX, JY]\rangle = \langle J[L, X], JY\rangle = \langle L, [X, Y]\rangle$$

by use of properties (i), (ii) and (2.4) of J as well as the fact that $\operatorname{ad}(h)m \subset m$. Thus

$$[JX, JY] - [X, Y] \in m \quad \forall X, Y \in m.$$

By combining this with identity (iii), it is easy to check that J satisfies (1.5) for all $X, Y \in g$. q.e.d.

In [8], Lu and Weinstein give a Lie bialgebraic structure for every compact semisimple Lie algebra by means of Iwasawa's decomposition and Manin's triple. From Theorem 1 and the following lemma, we see that it is equivalent to ours given by the Koszul operator J. Notice that g is a compact real form of $g^{\mathbb{C}}$ and can be expressed as

$$g = h \oplus \sum_{\alpha\in\Delta^+} \mathbb{R}(X_\alpha - X_{-\alpha}) \oplus \sum_{\alpha\in\Delta^+} \mathbb{R}i(X_\alpha + X_{-\alpha}). \tag{2.5}$$

Write

$$k = \sum_{\alpha\in\Delta^+} \mathbb{R}(X_\alpha - X_{-\alpha}), \qquad p = h \oplus \sum_{\alpha\in\Delta^+} \mathbb{R}i(X_\alpha + X_{-\alpha}) \tag{2.6}$$

and

$$\begin{gathered} n^\pm = \sum_{\alpha\in\Delta^\pm} (\mathbb{R}X_\alpha + \mathbb{R}iX_\alpha), \qquad n_0^\pm = \sum_{\alpha\in\Delta^\pm} \mathbb{R}X_\alpha, \\ h_0 = ih, \quad b^\pm = h_0 \oplus n^\pm, \quad b_0^\pm = h_0 \oplus n_0^\pm. \end{gathered} \tag{2.7}$$

Now, let $\overline{g} = g^{\mathbb{C}}_{\mathbb{R}}$ be considered as a real Lie algebra as in §1. Then (2.5)–(2.7) are all subalgebras of $\overline{g}$ and have the relations shown by the following diagram:

$$\begin{array}{ccccccccc} h & \subset & g & \subset & \overline{g} & \supset & b^\pm & \supset & n^\pm \\ & & \cup & & \cup & & \cup & & \cup \\ & & k & \subset & g_0 & \supset & b_0^\pm & \supset & n_0^\pm \\ & & & & & & \cup & & \\ & & & & & & h_0 & & \end{array} \tag{2.8}$$

where

$$g_0 = k \oplus ip \tag{2.9}$$

is a real split semisimple Lie algebra (a normal real form of $g^{\mathbb{C}}$).

Lemma 2. *With the notation as above, let the symbol g_J denote the Lie algebra: the vector space q equipped with the bracket*

$$[X, Y]_J = [JX, Y] + [X, JY] \quad \forall X, Y \in g.$$

Then we have

(a) *g_J is isomorphic to $b^{\pm}$ and both $(\overline{g}, g, b^{\pm})$ are Manin triples of the bialgebra (g, J).*

(b) *p_J is a subalgebra of g_J and isomorphic to $b_0^{\pm}$.*

Proof. From (2.4), (2.6), and (2.7), one can check that

$$\operatorname{Im}(J \pm i) = b^{\pm}, \quad \operatorname{Im}(J \pm i)|_p = b_0^{\pm}, \qquad [p, p]_J \subset p.$$

Thus, from Theorem 1 the lemma follows.

Remark. Notice that $\overline{g} = g \oplus b^+ = g \oplus h_0 \oplus n^+$ is just Iwasawa's decomposition of $\overline{g}$ so that the bialgebra (g, J) is the same as that given in [8].

To get the Lie bialgebraic structure for noncompact semisimple Lie algebras by the Koszul operator, let $\theta \in \operatorname{Aut}(g)$ be an involutive automorphism of g such that $\theta \circ J = J \circ \theta$. Thus one has the decomposition

$$g = k_\theta + p_\theta; \qquad \theta|_{k_\theta} = \mathrm{id}, \ \theta|_{p_\theta} = -\mathrm{id}, \tag{2.10}$$

and obtains a noncompact real semisimple Lie algebra

$$g_\theta = k_\theta + ip_\theta; \tag{2.11}$$

(2.11) gives a Cartan decomposition of g_θ. Since J is commutable with θ it can be reduced from g to g_θ. Using the same symbol to denote J reduced on g_θ, it is easy to see that J still satisfies (1.5) and is skew-symmetric with respect to the Killing form of g_θ. Thus, we get a noncompact Lie bialgebra (g_θ, J). Particularly, by property (ii) of the Koszul operator J, for any $x \in H = \exp h$ such that x^2 is in the center of G (a distinct subgroup of G), we get an involutive automorphism $\mathrm{Ad}_x \in \operatorname{Int}(g)$ satisfying $\mathrm{Ad}_x \circ J = J \circ \mathrm{Ad}_x$. In this case, one has

$$h \subset k_\theta \ (\theta = \mathrm{Ad}_x) \Rightarrow \operatorname{rank} g_\theta = \operatorname{rank} k_\theta, \tag{2.12}$$

where k_θ is a maximal compact subalgebra of g_θ, and h is a compact Cartan subalgebra of g_θ. On the other hand, it is known that every real noncompact semisimple Lie algebra with a compact Cartan subalgebra may be realized by this way (see [4, p. 424]), so we have

Theorem 3. *Every real noncompact semisimple Lie algebra with a compact Cartan subalgebra has a Lie bialgebraic structure mentioned above. Moreover, if we let g_θ denote such a Lie algebra, and g the corresponding compact semisimple Lie algebra as given in* (2.10)–(2.12), *then the dual Lie algebra of g_θ is isomorphic to the dual Lie algebra of g, i.e., b^+ given in* (2.8).

Proof. The first part is proved above. By Lemma 1, $g_{\theta J}$ is isomorphic to $(J + i)(g_\theta)$, a subalgebra of $\overline{g}_\theta = \overline{g}$. It is easy to check that

$$(J + i)(g_\theta) = (J + i)(g) = b^+$$

by means of the facts $h \subset k_\theta$, $(J+i)(h^\perp) = n^+$, and $in^+ = n^+$. Thus the theorem follows.

According to the classifications of the classical simple Lie algebras, we see that the classical noncompact simple Lie algebras in the table have the same rank as their maximal compact subalgebras (see [4, Chapter X, §2]).

g_θ	k_θ	Type	Rank
$\mathrm{su}(n, m)$	$\mathrm{su}(n) \times \mathrm{su}(m) \times \mathrm{u}(1)$	AIII	$m+n-1$
$\mathrm{so}(2n, 2m+1)$	$\mathrm{so}(2n) \times \mathrm{so}(2m+1)$	B	$n+m$
$\mathrm{sp}(n, \mathbb{R})$	$\mathrm{u}(n)$	CI	n
$\mathrm{sp}(n, m)$	$\mathrm{sp}(n) \times \mathrm{sp}(m)$	CII	$n+m$
$\mathrm{so}(2n, 2m)$	$\mathrm{so}(2n) \times \mathrm{so}(2m)$	DI	$n+m$
$\mathrm{so}^*(2n)$	$\mathrm{u}(n)$	DIII	n

Remark. Notice that $\mathrm{so}(2n, 2m)$ $(n+m>2)$ is only a part of Type DI.

We see that $\mathrm{sp}(n, \mathbb{R})$, $\mathrm{so}(2n, 2n+1)$, and $\mathrm{so}(2n, 2n)$ are the normal real forms of $\mathrm{sp}(n, \mathbb{C})$, $\mathrm{so}(4n+1, \mathbb{C})$, and $\mathrm{so}(4n, \mathbb{C})$ respectively. In general, J cannot be reduced from a compact real form to the normal form g_0 given by (2.9). But one can check that g_0 is also a bialgebra with the r-matrix $J_0 = iJ$. Write $p_0 = ip$; then $g_0 = k + p_0$ is the Cartan decomposition of g_0 with the Cartan subalgebra $h_0 \subset p_0$. It is easy to see that

$$g_0 = h_0 \oplus n_0^+ \oplus n_0^-, \quad J_0|_{n_0^\pm} = \pm 1, \quad J_0|_{h_0} = 0. \tag{2.13}$$

From (2.13), we can verify that J_0 is skew-symmetric with respect to the Killing form of g_0 and satisfies the modified Yang-Baxter equation

$$B_{J_0}(X, Y) = -[X, Y], \quad \forall X, Y \in g_0, \tag{2.14}$$

where B_{J_0} is definded by (1.2). Consequently, we have

Proposition 1. *Every real split semisimple Lie algebra g_0 is a Lie bialgebra with the r-matrix J_0 satisfying the modified Yang-Baxter equation* (2.14). *Moreover, $n_0^\pm$ and p_0 are subalgebras of $g_0^* = g_{0J_0}$.*

In [9], a number of properties of the r-matrix which satisfies (2.14) were discussed in detail. Here we only point out that the Cayley transformation of J_0 is equal to the negative of the identity map on the Cartan subalgebra h_0. Notice that, in this case, there is no simple method to realize Manin triples for the Lie bialgebra (g_0, J_0) in $\overline{g} = g_0 + ig_0$ as for (g, J), since J_0 does not satisfy equation (1.5).

3. Symplectic leaves and Lagrangian submanifolds

In this and the following section, let the corresponding capitals $\overline{G}$, G, G_θ, G_0, H, H_0, $B^\pm$, $B_0^\pm$, $N^\pm$, $N_0^\pm$, K_θ, and K denote the connected Lie groups with their Lie algebras given in §2, where $\theta = \mathrm{Ad}_x$ for $x \in H \subset K_\theta$ an involutive automorphism of both g and g_θ.

From the Lie bialgebras (g, J), (g_θ, J), and (g_0, J_0), one can get the corresponding Poisson Lie groups (G, Ω), $(G_\theta, \Omega_\theta)$, and (G_0, Ω_0) (see [3], [6], [8]), where the multiplicative Poisson tensor Ω is given by

$$\Omega_x = l_{x^*}J - r_{x^*}J\colon\ T_xG \to T_xG \quad \forall x \in G \tag{3.1}$$

(similarly for Ω_0 and Ω_θ). Here l_x and r_x denote the left and right actions on the Lie group G and we identify T_xG with T_x^*G as $g = g^*$ by use of the Killing form $\langle\cdot, \cdot\rangle$ of g, which is also considered as an invariant Riemannian metric on G.

By Theorem 3, the solvable subgroup $B^+ = H_0N^+$ of $\overline{G}$ is the dual Poisson Lie group of both G and G_θ. For the pair (G, B^+), the corresponding decomposition (1.8), $\overline{G} = GB = GH_0N^+$, is just the Iwasawa decomposition of the real semisimple Lie group $\overline{G}$, so it is global. But, for (G_θ, B^+), $\overline{G} = G_\theta B^+$ is only a local decomposition. Notice that since $\overline{G} \cong T^*G$ there is a "natural" symplectic structure on $\overline{G}$, from [7], which seems the same as those mentioned above for the pair (G, B^+).

Now we begin to discuss symplectic leaves of the Poisson Lie groups G, G_θ, and G_0. First we give some results for the compact Poisson Lie group G.

Lemma 3. *Let S_x denote the symplectic leaf through a fixed point $x \in G$. Then for any $a, b \in H$, we have*

$$S_{axb} = l_a \circ r_b S_x = aS_xb,$$

and $l_a \circ r_b\colon S_x \to S_{axb}$ is a symplectic diffeomorphism.

Proof. By property (ii) of the Koszul operator J, we have $\mathrm{Ad}_a \circ J = J \circ \mathrm{Ad}_a$ for all $a \in H$. This means that $\Omega_a = 0$ for all $a \in H$ by expression (3.1). On the other hand, because Ω is multiplicative, i.e.,

$$\Omega_{xy} = l_{x^*}\Omega_y + r_{y^*}\Omega_x, \quad \forall x, y \in G, \tag{3.2}$$

one has

$$\Omega_{axb} = l_{a^*} \circ r_{b^*}\Omega_x, \quad \forall x \in G, \ a, b \in H. \tag{3.3}$$

This means that $l_a \circ r_b \colon S_x \to aS_x b$ is a symplectic diffeomorphism. Finally, since $axb \in S_{axb} \cap aS_x b$, we have $S_{axb} = aS_x b$. q.e.d.

To distinguish the symplectic leaves of G, we consider the Weyl group $W \subset K$ for the orthogonal symmetric Lie algebra (g, k) (fix a representative of W in K) [4]. Let $w^* \in W$ be the element such that $\mathrm{Ad}_{w^*} n^+ = n^-$, i.e., $\omega^*\Delta^+ = \Delta^-$. It can be checked that the formula

$$\mathrm{Ad}_{\omega^*}^{-1} \circ J \circ \mathrm{Ad}_{\omega^*} = -J \tag{3.4}$$

holds on g since $J^{\mathbb{C}}|_{n^\pm} = \pm i$ and $J|_h = 0$.

Lemma 4. *With the notation as above, we have*

$$\dim S_{w^*} = 2\dim k, \quad \dim S_w < 2\dim k, \quad \forall w \in W\backslash\{w^*\}.$$

Proof. Notice that, for any $x \in G$, $X \in g$, one has

$$Y = l_{x^*}X \in T_x G \quad \text{and} \quad \Omega_x Y = l_{x^*}(JX - \mathrm{Ad}_x^{-1} \circ J \circ \mathrm{Ad}_x X). \tag{3.5}$$

This means that $\Omega_w|_{l_{w^*}h} = 0$ for all $w \in W$, because $J|_h = 0$ and $\mathrm{Ad}_w h \subset h$. For $w^* \in W$, let

$$Y_\alpha = l_{w_*^*}(X_\alpha - X_{-\alpha}), \qquad \overline{Y}_\alpha = l_{w_*^*} i(X_\alpha + X_{-\alpha}).$$

Then we get $\Omega_{w^*} Y_\alpha = 2\overline{Y}_\alpha$ and $\Omega_{w^*}\overline{Y}_\alpha = -2Y_\alpha$ by (3.4), i.e.,

$$\operatorname{rank}\Omega_{w^*} = \dim g - \dim h = 2\dim k.$$

Consequently, $\dim S_{w^*} = 2\dim k$. Using straightforward computations, one can check that $\dim S_w < 2\dim k$ for all $w \in W\backslash\{w^*\}$. Obviously, $S_e = \{e\}$ for the unit element e of G. q.e.d.

Notice that the orthogonal symmetric Lie algebras $(\overline{g}, g)$ and (g_0, k) have the same Weyl group W as for (g, k). Now we can use the Bruhat decomposition of a semisimple Lie group (see [4]) to prove the following theorem.

Theorem 4. *With the notation as above, we have the following*

(a) $G = \bigcup_{a\in H, w\in W} S_{aw}$.

(b) *The set* $\bigcup_{a\in H} S_{\alpha w^*}$ *is a dense open cell of* G, *and* $S_{aw^*} \neq S_{bw^*}$ $(S_{aw^*} \cap S_{bw^*} = \varnothing)$ *for all* $a \neq b \in H$.

(c) *Write* $L_w \equiv K \cap S_w$, $L_{aw} \equiv aL_w$ *for every* $w \in W$, $a \in H$. *Then we have* L_{aw} *is a Lagrangian submanifold of* S_{aw}. *Moreover,* $L_{aw} \cap L_{bw} = \varnothing$ *for all* $w \in W$, $a \neq b \in H$.

Proof. From (1.9) and Lemma 2, we see that, for any $x \in G$, S_x may be written in the form

$$S_x = G \cap (B^+ x B^+). \tag{3.6}$$

For any $w \in W$, it is easy to check that

$$HB^+ wHB^+ = HB^+ HwB^+ = HB^+ wB^+.$$

Thus the Bruhat decomposition of $\overline{G}$ corresponding to the decomposition $\overline{g} = g \oplus b^+$ has the form

$$\overline{G} = \bigcup_{w \in W} HB^+ wHB^+ = \bigcup_{w \in W} H(B^+ wB^+).$$

By (3.6) and the fact $H \subset G$, we get

$$G = G \cap \overline{G} = \bigcup_{w \in W} H(G \cap B^+ wB^+) = \bigcup_{w \in W} HS_w.$$

Thus (a) follows from Lemma 3.

For (b), the fact that $\bigcup_{a \in H} S_{aw^*}$ is dense and open in G is immediate from the Bruhat decomposition. The others may be verified by the fact that $w^* B^+ w^{*-1} = B^-$ and $H \cap B^+ B^- = \{e\}$, where $B^- \subset \overline{G}$ is the Lie group with the Lie algebra b^-.

Now we turn to the proof of (c). Here we only give a detailed proof for $w^* \in W$, since the others are similar but with more complicated computations. Notice that the real semisimple Lie group G_0 has the Bruhat decomposition in the form

$$G_0 = \bigcup_{w \in W} B_0^+ w B_0^+ \tag{3.7}$$

corresponding to the decomposition $g_0 = k \oplus b_0^+$. Thus, $K \cap (B_0^+ w^* B_0^+)$ is a dense open set in K. This means that

$$\dim K \cap (B_0^+ w^* B_0^+) = \dim k.$$

On the other hand, from Lemma 4 and the relation

$$K \cap (B_0^+ w^* B_0^+) \subset K \cap (B^+ w^* B^+) = L_{w^*} \subset K,$$

we get $\dim L_{w^*} = \dim k = \frac{1}{2} \dim S_{w^*}$. In fact, L_{w^*} is also an isotropic submanifold of the symplectic manifold S_{w^*} because it can be checked that

$$\langle \Omega_x X, Y \rangle = 0 \quad \forall x \in K, \ \forall X, Y \in T_x K \tag{3.8}$$

by means of the facts

$$Jk \subset p, \quad p = k^\perp, \quad [k, p] \subset p.$$

Here, $g = k \oplus p$ is given as in (2.5)–(2.6). Consequently, L_{w^*} is a Lagrangian submanifold of S_{w^*}. From Lemma 3, $L_{aw^*} = aL_{w^*} \subset S_{aw^*}$ is also a Lagrangian submanifold for every $a \in H$. Since $L_w \subset K$ and $H \cap K = \{e\}$, we get $L_{aw} \cap L_{bw} = \varnothing$ for all $a \neq b \in H$, $w \in W$. Thus, (c) follows.

Remarks. (1) In [8], the Bruhat decomposition is used, for the first time, to study the Poisson structure on an Ad-orbit (also see §4). The authors named it the Bruhat-Poisson structure.

(2) For any $w \in W \backslash \{w^*\}$, it is possible that $S_{aw} = S_{bw}$ for some $a \neq b \in H$, but (c) shows that L_{aw} and L_{bw} are also two nonintersecting Lagrangian submanifolds in it.

For the noncompact Poisson Lie group G_θ, Lemma 3 is obviously true, and the symplectic leaves can also be written in the form $S_x^\theta = G_\theta \cap (B^+ x B^+)$ for any $x \in G_\theta$. Moreover, G_θ has the symplectic leaf decomposition

$$G_\theta = \bigcup_{\substack{a \in H \\ w \in W}} a(G_\theta \cap (B^+ w B^+)).$$

Notice that if $w \notin G_\theta$, then $G_\theta \cap (B^+ w B^+) = S_{xwy}^\theta$ for some $x, y \in B^+$ such that $xwy \in G_\theta$.

Now we turn to the study of symplectic leaves of the noncompact Poisson Lie group (G_0, Ω_0). Notice that W is also the Weyl group for the pair (G_0, K). From the proofs of Lemmas 3 and 4, it is easy to see

Lemma 5. *All the results of Lemmas* 3 *and* 4 *are still true for the Poisson Lie group* (G_0, Ω_0) (*use* G_0, H_0, J_0, *and* Ω_0 *instead of* G, H, J, *and* Ω).

But, since J_0 does not satisfy (1.5), the symplectic leaves of G_0 cannot be realized in the form (3.6). However, we can use the infinitesimal dressing transformations, i.e., the dressing vector fields, to obtain some information on the symplectic leaves and the Lagrangian submanifolds. For any $X \in g_0$, let $X_l, X_r \in \chi(G_0)$ be the left and the right invariant vector fields on G_0. Then, the left and right dressing vector fields on G_0 are defined as ([10], [14])

$$\tilde{X}_l(x) = \Omega_0 X_l(x), \quad \tilde{X}_r(x) = \Omega_0 X_r(x) \quad \forall x \in G_0. \tag{3.9}$$

If S_x^0 denotes the symplectic leaf of G_0 through a point $x \in G_0$, it is known that S_x^0 consists of all the orbits of the flows of the left or right dressing vector fields. Now, we can get

Theorem 5. *With the notation as above, both the cosets* $N_0^+ w^*$ *and* $w^* N_0^+$ *are contained in* $S_{w^*}^0$, *and* $L_{w^*}^0 = K \cap S_{w^*}^0$ *is a Lagrangian submanifold of* $S_{w^*}^0$.

Proof. To see that $w^* N_0^+ \subset S_{w^*}^0$, notice that

$$\tilde{X}_l(x) = l_{x^*}(J_0 X - \mathrm{Ad}_x^{-1} \circ J_0 \circ \mathrm{Ad}_x X) \quad \forall x \in G_0,\ X \in g_0 \tag{3.10}$$

by (3.1) and (3.9). Now let $X \in n_0^+$, $y \in N_0^+$, i.e., $x = w^* y \in w^* N_0^+$. Since $\mathrm{Ad}_{w^*} n_0^{\pm} \subset n_0^{\mp}$ and $J_0|_{n_0^{\pm}} = \pm 1$, in this case (3.10) becomes

$$\tilde{X}_l(x) = 2 l_{x^*} X \in T_x w^* N_0^+,$$

where $l_x \colon N_0^+ \to x N_0^+ = w^* y N_0^+ = w^* N_0^+$ and $X \in n_0^+ = T_e N_0^+$. This means that $w^* N_0^+$ consists of all the orbits of the left dressing transformations of the Lie group N_0^+, which is a subgroup of the dual Lie group G_0^* because $n_0^{\pm}$ is a subalgebra of $g_0^* = g_{0J_0}$. Thus we have $w^* N_0^+ \subset S_{w^*}^0$ since $w^* \in w^* N_0^+ \cap S_{w^*}^0$. With the same reason, we have $N_0^+ w^* \subset S_{w^*}^0$.

By similar methods, we can prove that $L_{w^*}^0 = K \cap S_{w^*}^0$ consists of all the orbits of the dressing transformations of the Lie group P_0^* with the Lie algebra $p_{0J_0} \subset g_{0J_0}$, which is a dense open set in K. Thus one has $\dim L_{w^*}^0 = \dim k = \frac{1}{2} \dim S_{w^*}^0$. Finally, one can check that $L_{w^*}^0 \subset S_{w^*}^0$ is an isotropic submanifold by the same method used in the proof of Theorem 4–(c). Hence the theorem follows.

4. Reductions on the Poisson Lie groups

Let (G, Ω) and (G_0, Ω_0) be the two Poisson Lie groups given in §3. Notice that both their Lie algebras have the Cartan decompositions

$$g = k + p, \qquad g_0 = k + p_0. \tag{4.1}$$

Thus, both the coset spaces G/K and G_0/K are symmetric spaces. Let $\pi \colon G \to G/K$ and $\pi_0 \colon G_0 \to G_0/k$ denote their projections. Then we have

Lemma 6. *Both the symmetric spaces* G/K *and* G_0/K *are the Poisson manifolds with the reduced Poisson structures such that* π *and* π_0 *are Poisson maps.*

Proof. Lemma 2 and Proposition 1 tell us that both p and p_0 are subalgebras of g^* and g_0^*, the dual Lie algebras of the Lie bialgebras (g, J) and (g_0, J_0). The lemma follows from a general reduction theorem [11] and the fact that the decompositions (4.1) are orthogonal decompositions of g and g_0 with respect to the Killing forms. q.e.d.

It is known that the noncompact symmetric space G_0/K may be regarded as a submanifold of G_0, i.e.,

$$G_0/K \sim P_0 = \exp p_0,$$

by the global decomposition $G_0 = P_0 K$, where exp: $g_0 \to G_0$ denotes the exponential map. Thus, $(P_0, \Omega_0|_{P_0})$ is a Poisson manifold such that π_0: $G_0 \to P_0$ is the Poisson map. But, in general, P_0 is not a Poisson submanifold of the Poisson Lie group G_0 because the inclusion map τ: $P_0 \to G_0$ is not a Poisson map.

In most situations, involution theorems are important for studying a Poisson manifold and the integrable Hamiltonian systems defined on it. For a Poisson Lie group, say (G, Ω), there is the following well-known involution theorem ([3], [10]): If

$$\varphi, \psi \in C^{\infty}(G), \quad \varphi(xyx^{-1}) = \varphi(y), \quad \psi(xyx^{-1}) = \psi(y),$$

for all $x, y \in G$, then

$$\{\varphi, \psi\} = \Omega(d\varphi, d\psi) = 0.$$

Now we reduce this theorem to the symmetric space P_0, noticing that, for any $x \in K$ and $y \in P_0$, one has $xyx^{-1} \in P_0$ because $\mathrm{Ad}(K)p_0 = p_0$.

Theorem 6. *Let P_0 be the Poisson manifold given above, and let $\varphi, \psi \in C^{\infty}(P_0)$ both be K-invariant. Then φ and ψ are in involution.*

Proof. Extend φ to a right K-invariant function on G_0 by the decomposition $G_0 = P_0 K$, which is also left K-invariant because $\varphi(xyx^{-1}) = \varphi(y)$ for all $x \in K$, $y \in P_0$. This means that

$$\langle r_{y*}A, d\varphi(y)\rangle = \langle l_{y*}A, d\varphi(y)\rangle = 0, \quad \forall A \in k, \ y \in G_0, \tag{4.2}$$

i.e., both $r_{y*}d\varphi(y)$, $l_{y*}d\varphi(y) \in p_0 = k^{\perp}$ if we identify g_0^* with g_0 by the Killing form. The same is true for ψ. From formula (4.2), one has

$$\{\varphi, \psi\}(y) = \langle J_0 l_{y^{-1}*}X, l_{y^{-1}*}Y\rangle - \langle J_0 r_{y^{-1}*}X, r_{y^{-1}*}Y\rangle,$$

where $X = d\varphi(y)$ and $Y = d\psi(y)$. Formula (4.2) and the fact that $J_0 p_0 \subset k = p_0^{\perp}$ imply $\{\varphi, \psi\} = 0$.

Remark. Obviously, a similar involution theorem holds for all left K-invariant functions on the right coset spaces G/K and G_0/K.

Recall that a Poisson Lie subgroup U of a Poisson Lie group G is simultaneously a Lie subgroup and a Poisson submanifold of G. An equivalent condition for this is that $u^{\perp} \subset g^*$ is an ideal, where u is the Lie algebra of U (see [8], [10]). Notice that $K \subset G$ (resp. G_0) is not a

Poisson Lie subgroup because $k^{\perp} = p$ (resp. p_0) is only a subalgebra of $g^* = g_J$ (resp. g_0^*) instead of being an ideal.

Lemma 7. *Let $(g, R, \langle\cdot, \cdot\rangle)$ be a Lie bialgebra, and u an R-invariant subalgebra of g such that $u + u^{\perp} = g$. Then, $u^{\perp}$ is an ideal of g_R if it is a subalgebra of g_R.*

Proof. We only need to show $[u, u^{\perp}]_R \subset u^{\perp}$. Since R is skew-symmetric, $Ru \subset u$ means $Ru^{\perp} \subset u^{\perp}$. For any $X, Y \subset u$ and $Z \in u^{\perp}$, we see that

$$\langle X, [Y, Z]_R\rangle = \langle [X, RY], Z\rangle + \langle [X, Y], RZ\rangle = 0,$$

which means that $[u, u^{\perp}]_R \subset u^{\perp}$. Hence the lemma follows.

Now, we give a theorem for the Lie bialgebra (g, J), which is also true for (g_0, J_0). Such a theorem was proved in [8] by use of the dressing transformation.

Theorem 7. *With the same notation as used before, let $A \in h$ be fixed and let g_A be the centralizer of A in g. Then $g_A^{\perp}$ is an ideal of g_J.*

Proof. By property (ii) of the Koszul operator J, we have $\mathrm{ad}_A \circ J = J \circ \mathrm{ad}_A$, i.e., g_A is a J-invariant subalgebra of g. By Lemma 7, we only need to verify that $[g_A^{\perp}, g_A^{\perp}]_J \subset g_A^{\perp}$, i.e.,

$$(\mathrm{ad}_X \circ J - J \circ \mathrm{ad}_X)Y \in g_A \quad \forall X \in g_A, \quad Y \in g_A^{\perp}.$$

In fact, we can prove that $[\mathrm{ad}_X, J]|_{g_A^{\perp}} = 0$ for any $X \in g_A$. Let G_A be the connected Lie group with Lie algebra g_A. It is known that the coset space G/G_A is also a homogeneous Kähler manifold (see [5]). Let J_A be the corresponding Koszul operator which satisfies $[\mathrm{ad}_X, J_A] = 0$ for every $X \in g_A$. Then the relation $J|_{g_A^{\perp}} = J_A$ implies $[\mathrm{ad}_X, J] = 0$ for every $X \in g_A$. Thus, the theorem follows.

Remarks. (a) Obviously, the same holds for the Lie bialgebra (g_0, J_0).

(b) In [8], the induced Poisson structure on the homogeneous space G/G_A is called the Bruhat-Poisson structure.

References

[1] A. A. Belavin & V. G. Drinfel′d, *On the solutions of the classical Yang-Baxter equation*, Functional Anal. Appl. **16** (1982) 159.

[2] V. G. Drinfel′d, *Hamiltonian structures on Lie groups, Lie bialgebras and the geometric meaning of the Yang-Baxter equations*, Soviet Math. Dokl. **27** (1983) 68–71.

[3] ——, *Quantum groups*, Vol. 1, Proc. ICM, Berkeley, 1986, 789–820.

[4] S. Helgason, *Differential geometry, Lie groups and symmetric spaces*, Academic Press, New York, 1978.

[5] S. Kobayashi & K. Nomizu, *Foundations of differential geometry*, vol. II, Wiley, New York, 1969.

[6] Y. Kosmann-Schwarzbach, *Poisson-Drinfel′d groups*, Vol. 5, Publ. IRMA, No. 12, Lille, 1987.

[7] J. H. Lu & T. Ratiu, *On the nonlinear convexity theorem of Kostant*, preprint.

[8] J. H. Lu & A. Weinstein, *Poisson Lie groups, dressing transformations and Bruhat decomposition*, J. Differential Geometry **31** (1990) 501–526.

[9] M. A. Semenov-Tian-Shansky, *What is a classical r-matrix?* Functional Anal. Appl. **17** (1983) 259–272.

[10] ——, *Dressing transformations and Poisson Lie group actions*, Publ. Res. Inst. Math. Sci. **21** (1985) 1237–1260.

[11] ——, *Lax equations, Poisson Lie groups and dressing transformations*, Lecture Notes in Phys., Vol. 280, Springer, Berlin, 1987, 174–214.

[12] Ya. S. Soibel′man, *Irreducible representations of the function algebra on the quantum group* $\mathrm{SU}(n)$, *and Schubert cells*, Soviet Math. Dokl. **40** (1990) 34–38.

[13] A. Weinstein, *The local structure of Poisson manifolds*, J. Differential Geometry **18** (1983) 523–557.

[14] ——, *Some remarks on dressing transformations*, J. Fac. Sci. Univ. Tokyo Sect. 1A Math. **36** (1988) 163–167.

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