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GEOMETRIC CONSTRUCTION OF HOLONOMY COVERINGS FOR ALMOST FLAT MANIFOLDS

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1. Introduction

In this paper we give a new and conceptually rather simple proof of Gromov's theorem on almost flat Riemannian manifolds ([7], [2], [12], [3]). The proof yields a generalized version of the result which can be stated as follows.

1.1. Theorem. *There is a positive constant* $\varepsilon(n)$ *depending only on* n *such that the following is true. Let* (M, g) *be a compact connected* n*-dimensional Riemannian manifold,* d *its diameter, and* ∇ *a connection on* TM *compatible with the metric* g. *If the curvature and torsion tensors* R *and* T *of* ∇ *satisfy*

$$(1.1.1) \qquad (\|R\|_\infty + \|T\|_\infty^2)\, d^2 \le \varepsilon(n),$$

then M *is diffeomorphic to an infranilmanifold* $N = \Lambda \setminus G$.

Here $\|\cdot\|_\infty$ denotes the maximum norm on tensors. Infranilmanifolds are defined in §1.2. The constant $\varepsilon(n)$ is effective, but no explicit bound will be given. The nilpotent group structure on G is determined by the fundamental group Λ of M (see [1]).

The case $T = 0$ is Gromov's theorem as sharpened by Ruh [12]. The case $R = 0$ yields a generalization of [5]. Finally, the locally homogeneous case $R = 0$ and $\nabla T = 0$ is essentially due to Kazhdan-Margulis (see [11], [4]).

Previous proofs required a detailed study of what is known as the local fundamental pseudogroup of M (see [7], [2], [3]). The present proof uses results obtained in [5], [6], and is based on a description of the structure of distance balls in the bundle P of orthonormal frames on M, making essential use of the fact that P is parallelizable with torsion bounded in terms of the Cartan curvature. The holonomy covering space $\Gamma \setminus G$ (see §1.2) associated with a flat connection with parallel torsion on M is

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constructed as a perturbation of a local soul manifold $S \subseteq P$ in the sense of [6].

1.2. We recall facts on infranilmanifolds. Let G be a Lie group. There is a flat connection D with parallel torsion tensor on G defined by $DX = 0$ for all left invariant vector fields X. The group of affine automorphisms of D is equal to a semidirect product $G \cdot \mathrm{Aut}(G)$, where an element $(a, \varphi) \in G \cdot \mathrm{Aut}(G)$ acts on $b \in G$ by $(a, \varphi)b = a\varphi(b)$. An *infranilmanifold* is then defined to be a closed connected manifold $N = \Lambda \backslash G$ whose universal covering space is a nilpotent Lie group G, and whose fundamental group Λ is a subgroup of $G \cdot \mathrm{Aut}(G)$, required to be a finite extension of a discrete cocompact subgroup $\Gamma = \Lambda \cap G$ of G. If $\Gamma = \Lambda$, then $\Lambda \backslash G$ is called a *nilmanifold*.

It follows from this description that the connection D on G descends to a flat connection on $\Lambda \backslash G$, also denoted by D, whose holonomy bundles can be identified with the nilmanifold-covering $\Gamma \backslash G$ of N. The finiteness assumption on the linear holonomy group Γ / Λ of D implies that there are left invariant Riemannian metrics g_0 on G that descend to N.

The statement of 1.1 can now be sharpened as follows.

1.2.1. *There exist a quotient g_0 of a left invariant metric on G and a diffeomorphism $f\colon M \to N$ such that $\|f^* g_0 - g\|_\infty \le \delta$ and $\|f^* D - \nabla\|_\infty d \le \delta$.*

Here δ depends only on n and on $(\|R\|_\infty + \|T\|_\infty^2)\, d^2$ and converges to zero when the latter converges to zero.

1.3. Although it will not be used in the sequel, we remark that, according to a result of Wilson, infranilmanifolds can be characterized as closed space forms of simply connected nilpotent Lie groups G equipped with left invariant Riemannian metrics g_0. More precisely, if M is a closed Riemannian manifold, whose universal covering is isometric to such a (G, g_0), then $\pi_1(M)$ acts by affine automorphisms of D (see [13]) and, according to L. Auslander's Bieberbach theorem [1], is a finite extension of a lattice in G.

1.4. We give an outline of the proof of 1.1. After multiplying the metric g by a constant, one can assume that $d^2 \le \varepsilon(n)^{1/2}$ and $\|R\|_\infty + \|T\|_\infty^2 \le \varepsilon(n)^{1/2}$. The bundle P of orthonormal frames of (M, g) carries a natural Riemannian metric depending on the connection. We study certain distance balls in P whose radius is large compared to d, but small when compared to the diameter of the fiber $O(n)$. The proof is motivated by the fact that for an almost flat metric on an infranilmanifold, neighborhoods of that size are tubular neighborhoods of holonomy bundles for the canonical flat connection D. Therefore, in the general case one can hope to find at least a perturbation of such a holonomy bundle by looking for "collapsed" directions in P.

The construction in [6] yields a nilmanifold $S \subseteq P$ imbedded with fairly large normal injectivity radius. If the metric on M was scaled suitably, then the bundle projection $\pi: P \to M$ restricted to S turns out to be a submersion and, therefore, a locally trivial fiber bundle. The fiber is shown to be zero dimensional. This essentially follows from the fact that a skew symmetric nilpotent matrix is zero. As a consequence, S is a finite regular nilmanifold covering of M almost tangent to the horizontal distribution of ∇.

In order to show that the group of deck transformations acts by affine automorphisms of suitable Maurer-Cartan coframe, S is perturbed into a principal subbundle $Q \subseteq P$. Then the deck group acts by affine isometries of the coframe obtained by pulling back the canonical one-form to Q. As a result of [5], this coframe can be deformed into a nilpotent Maurer-Cartan coframe preserving these affine isometries, and the proof is complete.

2. Coframes

We review facts on coframes that will be used in the proof.

2.1. Let M^n be a closed manifold. A *coframe* on M is an R^n-valued one-form $\eta: TM \to R^n$ such that for each $m \in M$, $\eta(m): T_mM \to R^n$ is a vector space isomorphism. The components η^i of η are defined by $\eta = \sum_{i=n}^{n} \eta^i \otimes e_i$, where e_i denotes the ith standard basis vector of R^n. η is called a *Maurer-Cartan coframe* if there are constants c_{ij}^k $(1 \le i, j, k \le n)$ such that $d\eta^k = \sum_{i<j} c_{ij}^k \eta^i \wedge \eta^j$. A Maurer-Cartan coframe is called *nilpotent* if the Lie algebra defined by the structural constants c_{ij}^k is nilpotent.

A coframe η defines a Riemannian metric $g_\eta = \sum_{i=1}^n \eta^i \otimes \eta^i$ on M as well as a flat connection D^η determined on T^*M by $D^\eta \eta^i = 0$ $(i = 1, \cdots, n)$. Using these, the space of square integrable R^n-valued forms on M can be split as

$$L^2(\Lambda^* T^* M \otimes \underline{R}^n) = \ker(D^\eta) \oplus \ker(D^\eta)^\perp,$$

and we let $\mathrm{pr}_\perp$ denote the orthogonal projection onto the second factor. A diffeomorphism $\varphi: M \to M$ will be called an *affine isometry* of η if $\varphi^*\eta = a \cdot \eta$ for some constant orthogonal matrix a; equivalently, if φ preserves D^η and g_η.

We give a reformulation of a result obtained in [5]. Let d denote the diameter $\operatorname{diam}(M, g_\eta)$ and $\kappa = \|d\eta\|_\infty$, the norm being taken with respect to g_η.

2.2. Theorem. *There is a constant $\varepsilon_1(n) > 0$ such that the following is true. If (M^n, η) satisfies $\kappa d \le \varepsilon_1(n)$, then there exists a nilpotent Maurer-Cartan coframe η' on M such that*

$$d^{-1}\|\eta - \eta'\|_\infty + \|d\eta - d\eta'\|_\infty \le c(n, \kappa d)\|\mathrm{pr}_\perp d\eta\|_\infty .$$

If $M \to M_1$ is a regular covering whose group $\mathscr{D}$ of deck transformations acts by affine isometries of η, then η' can be chosen such that $\mathscr{D}$ acts by affine isometries of η'. In particular, M_1 is diffeomorphic to an infranilmanifold.

Here the function $c = c(n, \kappa d)$ is bounded on $\{n\} \times [0, \varepsilon_1(n)]$ for each n.

Proof. We refer to [5]. The estimate follows from equations 8.2.(1), (7.4.1), (7.3.2) and Lemma 6.4 in that article. The equivariance follows from the invariance of (7.2.1) under affine isometries of ω and the fact that the solution ω_1 of that equation is unique under the constraint $\omega_1 - \omega \in \ker(D^\omega)^\perp$.

2.3. Finally, we recall from [5] the following estimates for the exponential map $\operatorname{Exp}: T_m M \to M$ of D^η. The length distortion is estimated by

$$(2.3.1) \qquad (2 - e^{\kappa\|X\|})\|Y\| \le \|d\mathrm{Exp}_X Y\| \le e^{\kappa\|X\|}\|Y\|$$

for $X \in T_m M$ and $Y \in T_X T_m M = T_m M$. The difference between Euclidean parallel translation in $T_m M$ and D^η-parallel translation in M is bounded by

$$(2.3.2) \qquad \|d\mathrm{Exp}_X W(0) - W(1)\| \le (e^{\kappa\|X\|} - 1)\|W(0)\|,$$

where W is a D^η-parallel vector field along the curve $\operatorname{Exp}(tX)$.

3. Local structure of the frame bundle

3.1. Let (M, g, ∇) satisfy the hypothesis of the theorem. We can assume that $\|R\|_\infty + \|T\|_\infty^2 \le \varepsilon(n)^{1/2}$ and $d^2 \le \varepsilon(n)^{1/2}$. Let $\pi: P \to M$ denote the bundle of orthonormal frames of (M, g). The connection form $\omega: TP \to \mathrm{so}(n)$ of ∇ together with the canonical one-form $\theta: TP \to R^n$ yields a coframe $\eta = \theta \oplus \omega$, whose exterior derivative is given by the Cartan structure equations

$$(3.1.1) \qquad d\omega = -\omega \wedge \omega + \Omega, \qquad d\theta = -\omega \wedge \theta + \Theta,$$

where Ω and Θ are the curvature and torsion forms of ∇ on P. The standard Euclidean inner product on R^n and the negative of the Cartan-Killing form on $\mathrm{so}(n)$ induce, via η, a Riemannian metric g_η on P. If R_a denotes the right action of $a \in O(n)$ on P, then the equations

$$(3.1.2) \qquad R_a^*\omega = \mathrm{Ad}(a^{-1})\omega, \qquad R_a^*\theta = a^{-1}\theta$$

show that R_a is an affine isometry of (P, η).

3.2. For $p_0 \in P$ let $\mathrm{Exp}\colon T_{p_0}P \to P$ denote the exponential map of D^η. The space of D^η-parallel vector fields on P is spanned over R by fundamental and standard horizontal vector fields. Integral curves of standard vector fields project to geodesics on M. Therefore, if $H = \ker(\omega)$ denotes the horizontal subbundle and exp the exponential map of ∇, then

$$(3.2.1) \qquad \pi \circ \mathrm{Exp}|_H = \exp \circ \pi_*|_H .$$

By definition of g_η, π_* is an isometry on horizontal vectors. Since the deviation of $d\mathrm{Exp}_X Y$ for $X \in T_{p_0}P$ and $Y \in H_{p_0} \le T_{p_0}P = T_X T_{p_0}P$ from being horizontal is bounded by (2.3.2), we obtain from (2.3.1) and (2.3.2) a rough estimate

$$(3.2.2) \qquad (2 - e^{\kappa\|v\|})\|w\| \le \|d\exp_v w\| \le e^{\kappa\|v\|}\|w\|,$$

where $\kappa = \|d\eta\|_\infty \le 4 + 2\varepsilon(n)^{1/2}$, and v, $w \in T_{m_0}M$, $m_0 = \pi(p_0)$.

3.3. In §3.5 we will rescale the metric on M suitably and then apply the local soul construction described in [6] to a distance ball $B(p_0, R)$ of suitable radius in the corresponding bundle of orthonormal frames P. Recall ([6]) that if $\|d\eta\|_\infty \le 5$, say, then there exists $\varepsilon_2(n) > 0$ such that for any $\varepsilon_3 \le \varepsilon_2(n)$ there is a radius R, $\varepsilon_3 \le R \le (100n)^n\varepsilon_3$, with the following properties.

There is a closed submanifold $p_0 \in S \subseteq B(p_0, R)$ of diameter $\le R$ such that the restriction of Exp to the subset $\nu_{10R} = \{X \in \nu\colon \|X\| < 10R\}$ of the normal bundle $\nu \le TP|S$ of S maps ν_{10R} diffeomorphically onto a neighborhood of $B(p_0, R)$. S is obtained as $S = \mathrm{Exp}(\overline{S})$ for a submanifold $\overline{S} \subseteq B(0, 50R) \subseteq T_{p_0}P$ that is C^1-close to a linear subspace V of $T_{p_0}P$ in the following sense.

3.3.1. For submanifolds N and N' of a Riemannian manifold M define the C^k-distance

$$d_{C^k}(N, N') := \inf\{d_{C^k}(f, \iota_N) + d_{C^k}(f^{-1}, \iota_{N'})\},$$

where $\iota_N\colon N \to M$ denotes the inclusion and the infimum is taken over all imbeddings $f\colon N \to M$ such that $f(N) = N'$. As usual, the infimum of the empty set is defined to be ∞.

3.3.2. For subbundles E, E' with the same fiber dimension k of a vector bundle $F \to M$ carrying a fiber metric, define the distance $\operatorname{dist}(E, E') := \sup\{\operatorname{dist}(E_m, E'_m)\colon m \in M\}$, where $\operatorname{dist}(E_m, E'_m)$ denotes the distance function on the Grassmannian manifold of k-planes in F_m canonically induced from the fiber metric.

3.3.3. $\overline{S}$ satisfies $d_{C^1}(\overline{S}, B(0, 50R) \cap V) \leq \delta_1(\varepsilon_3, n)$, where $\lim_{\varepsilon_3 \to 0} \delta_1(\varepsilon_3, n) = 0$. By §2.3 this implies that TS is δ_2-close (see §3.3.2) to the subbundle of $TP|S$ obtained by D^η-parallel translation of V, where $\lim_{\varepsilon_3 \to 0} \delta_2(\varepsilon_3, n) = o$.

3.3.4. The subspace V has the following property: The set $\Gamma = \{X \in B(0, 50R)\colon \operatorname{Exp}(X) = p_0\}$ is contained in $\{X \in T_{p_0}P\colon \operatorname{angle}(X, V) \leq \delta_3\}$, where again $\lim_{\varepsilon_3 \to 0} \delta_3 = \delta_3(\varepsilon_3, n) = o$. Γ is contained in $\overline{S}$ and $\overline{S}$ may be viewed as interpolating Γ.

3.4. Let $0 < \varepsilon_3 \leq \varepsilon_2(n)$ and $\alpha > 0$. If

$$\varepsilon(n)^{1/4} \leq 100^{-10L}\varepsilon_3, \qquad L = (100/\alpha)^{n^2},$$

then there exists a ρ_0, $100^{-L}\varepsilon_3 \leq \rho_0 \leq \varepsilon_3$, such that there is a basis $v_1, \cdots, v_n$ for $T_{m_0}M$ with the following properties: $\exp(v_i) = m_0$, $\rho_0/2 \leq |v_i| \leq 3\rho_0/4$, the angle between v_i and v_j is at least $\pi/4$ for $i \neq j$, and the linear holonomy $\operatorname{rot}(v_i)$ of the geodesic loop $\gamma_{v_i}\colon [0, 1] \to M$, $\gamma_{v_i}(t) = \exp(tv_i)$ satisfies $\operatorname{dist}(\operatorname{rot}(v_i), 1) \leq \alpha$.

Proof. Because of (3.2.2), the pigeonhole argument given in [2, p. 37] applies without change. We recall the proof for convenience.

By (3.2.2) exp is nearly a local isometry on $B(0, \varepsilon_3)$. Therefore, $\exp^{-1}(m_0)$ is at least $2d$-dense in $B(0, \varepsilon_3)$, and $d \leq \varepsilon(n)^{1/4}$.

One of the numbers $r_i = 100^{-i}\varepsilon_3$ $(i = 0, \cdots, L-1)$, call it ρ_0, has the following property: For any $v' \in B(0, r_i) \cap \exp^{-1}(m_0) \subseteq T_{m_0}M$ there exists $v'' \in B(0, r_{i+1}) \cap \exp^{-1}(m_0)$ such that $\operatorname{dist}(\operatorname{rot}(v'), \operatorname{rot}(v'')) \leq \alpha/2$. This is true because there are at most L elements in $O(n)$ with pairwise distance $> \alpha/2$. Choose $v'_1, \cdots, v'_n \in T_{m_0}M$ such that $\exp(v'_i) = m_0$, $0.6\rho_0 \leq |v'_i| \leq 0.7\rho_0$, and such that the angle between v'_i and v'_j is at least $\pi/3$ for $i \neq j$. By choice of ρ_0, there are $v''_i \in B(0, \rho_0/100)$ such that $\operatorname{dist}(\operatorname{rot}(v'_i), \operatorname{rot}(v''_i)) \leq \alpha/2$. Define v_i to be the endpoint of the curve obtained by lifting $\gamma_{v''_i}^{-1}$ to $T_{m_0}M$ with initial point v'_i. Then, up to homotopy error bounded in terms of $\varepsilon(n)^{1/2}$, $\mathrm{rot}(v_i)$ equals the linear holonomy of the loop $\gamma_{v_i'}\gamma_{v_i''}^{-1}$, which is at distance less than $\alpha/2$ from the identity. Therefore, $\mathrm{dist}(\mathrm{rot}(v_i), 1) \le \alpha$.

3.5. Rescale the metric g on M by multiplying it with the factor $(\varepsilon_3/\rho_0)^2$. The new metric will still be denoted by g, and P is the corresponding bundle of orthonormal frames. Now $\varepsilon_3/2 \le |v_i| \le 3\varepsilon_3/4$ while $\mathrm{rot}(v_i)$ is as before and the diameter has increased by a factor of at most 100^L. We will use §3.4 with $\alpha = \varepsilon_3^3$.

3.6. Apply the construction in §3.3 to P. For ε_3 small enough, S will turn out to be covering space of M, C^1-close to a reduction of the bundle P to a finite subgroup of $O(n)$.

4. S is a covering space

4.1. In the rest of the proof, δ will denote several constants $\delta(\varepsilon_3, n)$ depending only on ε_3 and n such that $\lim_{\varepsilon_3\to 0}\delta(\varepsilon_3, n) = 0$.

4.2. *The horizontal subbundle H restricted to S has distance* (*see* §3.3.2) *less than δ to a subbundle of TS. In particular, $\pi|S$ is a submersion.*

Proof. It suffices to show that H_{p_0} is δ-close to a subspace of V, where V is as described in §3.3. Let $\overline{v}_1, \cdots, \overline{v}_n$ denote the horizontal lifts of $v_1, \cdots, v_n$ to $T_{p_0}P$. Then $\varepsilon_3/2 \le |\overline{v}_i| \le 3\varepsilon_3/4$ and $\mathrm{dist}(\mathrm{Exp}(\overline{v}_i), p_0) \le \alpha = \varepsilon_3^3$. By (2.3.1) there are $Y_i \in T_{p_0}P$ such that $\mathrm{Exp}(Y_i) = p_0$ and $\mathrm{dist}(Y_i, \overline{v}_i) \le 2\varepsilon_3^3$. In particular, $Y_i \in \Gamma$ (see §3.3.2) and is nearly horizontal. The claim follows.

4.3. We describe a suitable nilmanifold structure on S. By §§3.3 and 4.2, TS is δ-close (see §4.1) to a subbundle of $TP|S$ spanned by parallel vector fields and containing $H|S$. Choose orthonormal bases $E_1, \cdots, E_n$ for R^n and $E_{n+1}, \cdots, E_n$ of $\mathrm{so}(n)$ such that the components $\eta^1 = \theta^1, \cdots, \eta^n = \theta^n, \cdots, \eta^{n+1}, \cdots, \eta^k$ of $\eta = \sum_{i=1}^N \eta_i \otimes E_i$ pull back to a coframe η_S on S, δ-close to an orthonormal coframe for the submanifold metric on S. By Theorem 2.2 and since $\mathrm{diam}(S) \le R$, there is a nilpotent Maurer-Cartan coframe $\overline{\eta}_S$ with components $\overline{\eta}^1 = \overline{\theta}^1, \cdots, \overline{\eta}^n = \theta^n, \overline{\eta}^{n+1}, \cdots, \overline{\eta}^k$ on S such that

$$\mathrm{diam}(S)^{-1}\|\overline{\eta}_S - \eta_S\|_\infty + \|d\overline{\eta}_S - d\eta_S\|_\infty \le c\|\mathrm{pr}_\perp\, d\eta_S\|_\infty, \tag{4.3.1}$$

where $c = c(n, \|d\eta_S\|_\infty \mathrm{diam}(S))$ is as in 2.2. Pulling back (3.1.1) to S we obtain $\|\mathrm{pr}_\perp\, d\eta_S\|_\infty \le \delta$.

4.4. Since $\pi|S\colon S \to M$ is a submersion and S is compact, $\pi|S$ is a locally trivial fiber bundle. We claim that *the fiber* $S_{m_0} = (\pi|S)^{-1}(m_0)$ *has dimension zero.* As a consequence, S is a covering space of M.

Proof. The idea of the proof is to compare the nilpotent structure on S to the group of Euclidean motions encoded in the structure equations (3.1.1). Suppose the vertical unit vector $X(p) \in T_pP$ was tangent to S. Extend $X(p)$ to a D^{η}-parallel (i.e., fundamental) vector field X on P. Also, extend $X(p)$ to a $D^{\overline{\eta}_s}$-parallel vector field $\overline{X}$ on S. The equation $d\theta = -\omega \wedge \theta + \Theta$ implies $\iota_X d\theta = -\omega(X)\theta$. Here $\omega(X)$ is a skew symmetric matrix of norm one. Since $\overline{\eta}_S$ is nilpotent, $(\iota_{\overline{X}} d)^k\overline{\theta} = 0$ for $k = \dim(S)$. By (4.3.1), we have

$$\|(\iota_{\overline{X}} d)^k\overline{\theta} - (\iota_X d)^k\theta\|_\infty \le \delta$$

on S. This implies $\|\omega(X)^k\| \le \delta$ and, diagonalizing $\omega(X)$ over C, $\|\omega(X)\| \le \delta^{1/k}$, in contradiction to $\|\omega(X)\| = 1$.

5. Reduction of the structure group

In §4 it was shown that M is covered by a nilmanifold $S \subseteq P$, almost tangent to the horizontal subbundle H of ∇. By §4.3, a Maurer-Cartan coframe $\overline{\eta}_S$ can be obtained as a small perturbation of the pullback of the canonical one-form θ to S.

5.1. Let $u_1, u_2 \in S$ and $a \in 0(n)$ such that $R_a u_1 = u_2$. Since R_a leaves H invariant, $d_{C^1}(S, R_a(S)) \le \delta$. In particular, the fiber S_{m_0} satisfies $d_{C^0}(S_{m_0}, R_a(S_{m_0})) \le \delta\varepsilon_3$ for all a such that $S_{m_0} \cap R_a(S_{m_0}) \ne \emptyset$. The following lemma implies that, after identifying (P_{m_0}, p_0) with $(O(n), 1)$, S_{m_0} is close to a finite subgroup Δ of $O(n)$.

5.1.1. *For any* $\varepsilon > 0$ *there exists* $\varepsilon' > 0$ *such that the following holds. Let* A *be a finite subset of* $O(n)$ *with the properties* (i) $1 \in A$, (ii) *for all* $a, a' \in A$ *either* $a = a'$ *or* $\mathrm{dist}(a, a') \ge \varepsilon$, *and* (iii) *for all* $a \in O(n)$ *such that* $Aa \cap A \ne \emptyset$ *we have* $d_{C^0}(Aa, A) \le \varepsilon'$. *Then there exists a finite subgroup* $A_0 \le O(n)$ *such that* $d_{C^0}(A, A_0) \le 10\varepsilon'$.

Proof. Define a multiplication on A by setting $a * a' = a''$, where a'' is the unique element of A closest to aa'. Then $(A, *)$ is a group, and the inclusion $\iota_A\colon A \to O(n)$ is an almost-homomorphism in the sense of [8]. Therefore, there is a homomorphism $\psi\colon (A, *) \to O(n)$ close to ι_A. Define $A_0 = \psi(A)$.

5.2. By the construction of Δ, each element $a \in \Delta$ satisfies $d_{C^0}(S, R_a(S)) \leq \delta\varepsilon_3$. Since S has normal injectivity radius $\geq 10\varepsilon_3$, this implies that Δ is isomorphic to the group of deck transformations of the covering $\pi|S$, and the covering is regular. More precise information is given by

5.3. *There is a Δ-principal subbundle $Q \subseteq P$ such that $d_{C^0}(Q, S) \leq \delta\varepsilon_3$ and* $\operatorname{dist}(TQ, H|Q) \leq \delta$.

Proof. Consider the bundle $P/\Delta \to M$ with fiber $O(n)/\Delta =: F$. Since the points of Δ are at least $9\varepsilon_3$ apart in $O(n)$, F has large injectivity radius. Let pr: $P \to P/\Delta$ denote the canonical projection. Then pr(S) is what might be called a $\delta\varepsilon_3$-reduction of the structure group: For $m \in M$ the image of the fiber pr(S_m) is a finite subset of $(P/\Delta)_m$ of $\delta\varepsilon_3$-small diameter, and pr(S_m) depends smoothly on m in the sense that it is locally the image of finitely many sections $\sigma_1, \cdots, \sigma_q: M \to P/\Delta$. Take the center of mass (see [9]) in each fiber $(P/\Delta)_m$ to obtain a global section $\sigma: M \to P/\Delta$ and define $Q = \mathrm{pr}^{-1}(\sigma(M))$.

5.4. Let $\iota_Q: Q \to P$ denote the inclusion. Then $\iota_Q^*\theta$ is an orthonormal coframe on Q with respect to the lifted metric $\overline{g} := (\pi|Q)^*g$, and $\operatorname{diam}(Q, \overline{g}) \leq 2\operatorname{diam}(S)$. The equation $R_a^*\iota_Q^*\theta = a^{-1}\iota_Q^*\theta$ for $a \in \Delta$ shows that the deck group Δ acts by affine isometries on $(Q, \iota_Q^*\theta)$. By 2.2, $\iota_Q^*\theta$ can be deformed into a nilpotent Maurer-Cartan coframe θ_0 such that Δ acts by affine isometries of θ_0. This concludes the proof of 1.1.

5.5. We note that Q, as a subset of P, is in general not yet a holonomy bundle of a flat connection with parallel torsion. However, such a bundle can be obtained by reimbedding Q into the bundle $L(M) \supseteq P$ of all linear frames as follows. Let $X_1^0, \cdots, X_n^0$ denote the frame on Q denoted by $\theta_0(X_i^0) = e_i$. Define $f: Q \to L(M)$ by

$$f(q) = (\pi_* X_1^0(q), \cdots, \pi_* X_n^0(q)).$$

Then $f(Q)$ is the required holonomy bundle.

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References

[1] L. Auslander, *Bieberbach's theorems on space groups and discrete uniform subgroups of Lie groups*, Ann. of Math. (2) **71** (1961) 579–590.

[2] P. Buser & H. Harcher, *Gromov's almost flat manifolds*, Astérisque Soc. Math. France **81** (1981).

[3] K. Fukaya, *Hausdorff convergence of Riemannian manifolds and its applications*, Advanced Studies in Pure Math., Kinokaniya, 1990.

[4] P. Ghanaat, *A note on discrete uniform subgroups of Lie groups*, Geometriae Dedicata **21** (1986) 13–17.

[5] ——, *Almost Lie groups of type* R^n , J. Reine Angew. Math. **401** (1989) 60–81.

[6] P. Ghanaat, M. Min-Oo & E. Ruh, *Local structure of Riemannian manifolds*, Indiana Univ. Math. J. **39** (1990) 1305–1312.

[7] M. Gromov, *Almost flat manifolds*, J. Differential Geometry **13** (1978) 231–241.

[8] K. Grove, H. Karcher & E. Ruh, *Groups actions and curvature*, Invent. Math. **23** (1974) 31–48.

[9] H. Karcher, *Riemannian center of mass and mollifier smoothing*, Comm. Pure Appl. Math. **30** (1977) 509–541.

[10] S. Kobayashi & K. Nomizu, *Foundations of differential geometry*, Vol. I, Wiley, New York, 1963.

[11] M. S. Ragunathan, *Discrete subgroups of Lie groups*, Ergebnisse Math. Grenzgeb., Vol. 68, Springer, Berlin, 1972.

[12] E. Ruh, *Almost flat manifolds*, J. Differential Geometry **17** (1982) 1–14.

[13] E. N. Wilson, *Isometry groups of homogeneous nilmanifolds*, Geometriae Dedicata **12** (1982) 337–346.

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