J. DIFFERENTIAL GEOMETRY
34 (1991) 165-168

SOME NEW HARMONIC MAPS FROM B^3 TO S^2

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I. Introduction

It is well known that $u_0(x) = x/|x|$ is the unique minimizer of the energy functional $\int_{B^3} |\nabla u|^2\, dx$ among maps $u \in H^1(B^3, S^2)$ such that $u(x) = x$ for $x \in \partial B^3$ [2] where B^3 and S^2 are the unit 3-ball and 2-sphere respectively. Such an energy minimizer is a *weakly harmonic* map [4]. By minimizing a "relaxed energy", F. Bethuel, H. Brezis, and J.-M. Coron [1] proved that there exist infinitely many weakly harmonic maps for any nonconstant boundary data. But the regularity of such weakly harmonic maps is still unknown. Here we use a different approach to obtain the following result.

Theorem. *For any x_0 in $\overline{B}^3$, there is a harmonic map $u\colon B^3 \to S^2$ such that*

(i) $u(x) = x$ *on* ∂B^3;

(ii) u *is smooth in* $\overline{B}^3 \sim \{x_0\}$, *i.e.*, x_0 *is the only singularity of* u.

Let r, α, and z be cylindrical coordinates in $\mathbf{R}^3$, i.e., $x = r\cos\alpha$, $y = r\sin\alpha$. A map $u\colon B^3 \to S^2$ is called, as in [5], *axially symmetric* if in r, α, z

$$u(r, \alpha, z) = (\cos\alpha \sin\varphi, \sin\alpha\sin\varphi, \cos\varphi) \tag{1}$$

for some real valued function $\varphi(r, z)$. Using (1), we can simplify the formula for the energy of an axially symmetric map u,

$$\int_{B^3} |\nabla u|^2\, dx = 2\pi \int_D r\left(\frac{\partial\varphi}{\partial r}\right) + r\left(\frac{\partial\varphi}{\partial z}\right)^2 + \frac{\sin^2\varphi}{r}\, dr\, dz,$$

where $D = \{(r, z) : r^2 + z^2 < 1,\ r > 0\}$.

For any smooth $\varphi\colon D \to \mathbf{R}$, define

$$E(\varphi) = 2\pi \int_D r\left(\frac{\partial\varphi}{\partial r}\right)^2 + r\left(\frac{\partial\varphi}{\partial z}\right)^2 + \frac{\sin^2\varphi}{r}\, dr\, dz.$$

Received December 18, 1989 and, in revised form, February 22, 1990.

It is easy to see that any finite energy critical point φ of E defines by (1) an axially symmetric weakly harmonic map.

In 1987, D. Zhang [5] studied the critical points of E and obtained a smooth axially symmetric harmonic map corresponding to any given smooth axially symmetry boundary data that omits a neighborhood of the south pole. Then R. Hardt, D. Kinderlehrer, and F.-H. Lin [3] slightly improved this result to allow boundary data that can reach the south pole but not wrap around. In this paper, we will follow Zhang's method to construct harmonic maps which are axially symmetric in appropriate coordinates and have the properties stated in the theorem.

II. Proof of the theorem

First suppose that x_0 is in the interior of B^3. By rotating B^3 and S^2, we may assume that $x_0 = (0, 0, a)$, $0 < a < 1$.

Any critical point φ of E satisfies in D the partial differential equation

$$\frac{\partial}{\partial r}\left(r\frac{\partial\varphi}{\partial r}\right) + \frac{\partial}{\partial z}\left(r\frac{\partial\varphi}{\partial z}\right) - \frac{\sin 2\varphi}{2r} = 0. \tag{2}$$

Let ρ, θ be polar coordinates centered at $(0, a) \in D$, i.e., $r = \rho\sin\theta$, $z = \rho\cos\theta + a$. In coordinates ρ, θ, (2) becomes

$$\rho\sin\theta\left(\frac{\partial^2\varphi}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial\varphi}{\partial\rho} + \frac{1}{\rho}\frac{\partial^2\varphi}{\partial\theta^2}\right) + \sin\theta\frac{\partial\varphi}{\partial\rho} + \frac{\cos\theta}{\rho}\frac{\partial\varphi}{\partial\theta} - \frac{\sin 2\varphi}{2\rho\sin\theta} = 0.$$

Suppose φ is independent of ρ. Then $\partial\varphi/\partial\rho = \partial^2\varphi/\partial\rho^2 = 0$, and

$$\begin{aligned} 0 &= \sin\theta\frac{d^2\varphi}{d\theta^2} + \cos\theta\frac{d\varphi}{d\theta} - \frac{\sin 2\varphi}{2\sin\theta} \\ &= \frac{d}{d\theta}\left(\sin\theta\frac{d\varphi}{d\theta}\right) - \frac{\sin 2\varphi}{2\sin\theta}. \end{aligned}$$

Thus

$$\frac{d}{d\theta}\left(\left(\sin\theta\frac{d\varphi}{d\theta}\right)^2\right) = \sin 2\varphi\frac{d\varphi}{d\theta} = \frac{d}{d\theta}(\sin^2\varphi)$$

and

$$\left(\sin\theta\frac{d\varphi}{d\theta}\right)^2 = \sin^2\varphi + C$$

for some constant C. If we set $\varphi(0) = 0$, then $C = 0$ and

$$\left|\sin\theta\frac{d\varphi}{d\theta}\right| = |\sin\varphi|. \tag{3}$$

The general solution of (3) is:

$$\cos\varphi_c = \frac{\sinh c + \cosh c\cos\theta}{\cosh c + \sinh c\cos\theta}, \qquad -\infty < c < \infty.$$

When $c = 0$, $\varphi_0 = \theta$; when $c > 0$, $\varphi_c \le \theta$; and when $c < 0$, $\varphi_c \ge \theta$. Also $\varphi_c \to 0$ as $c \to \infty$ and $\varphi_c \to \pi$ as $c \to -\infty$.

Define $A = \{(r, z) : r^2 + z^2 = 1,\ r \ge 0\}$, $g: A \to [0, \pi]$, and $g(r, z) = \arctan(z/r)$. If we substitute φ by q in (1) and restrict r, z to $\{(r, z) : r^2 + z^2 = 1\}$, then (1) gives the identity map from S^2 to S^2. In coordinates ρ, θ, $A = \{(\rho, \theta) : \rho = \rho_0(\theta),\ 0 \le \theta \le \pi\}$ for some function ρ_0. Write $g(\theta) = g(\rho_0(\theta)\sin(\theta), \rho_0(\theta)\cos(\theta))$. Then g satisfies

$$\frac{\sin g}{\cos g - a} = \tan\theta.$$

Clearly $g(\theta) \le \theta$. Also g is monotone increasing, $g(0) = 0$, $g(\pi) = \pi$, and

$$\frac{d}{d\theta}g(\theta) = \frac{1 - 2a\cos g + a^2}{1 - a\cos g}.$$

Therefore $g'(0) = 1 - a$, and $g'(\pi) = 1 + a$. We can find some $c > 0$ such that

$$\varphi_c(\theta) \le g(\theta) \le \theta = \varphi_0(\theta).$$

Now we can proceed as in [5] and consider the following problem:

$$\textit{Minimize}\quad E(\psi) = 2\pi\int_D r\left(\frac{\partial\psi}{\partial r}\right)^2 + r\left(\frac{\partial\psi}{\partial z}\right)^2 + \frac{\sin^2\psi}{r}\,dr\,dz$$

among maps $\psi : D \to [0, \pi]$, $\psi = g$ *on* A,

$$\varphi_c(\theta) \le \psi(\rho, \theta) \le \varphi_0(\theta). \tag{4}$$

As in [5], a maximum principle implies that equality holds only for $\theta \in \{0, \pi\}$ because the constraints φ , φ are critical points of E. Thus a minimizer ψ_a is a critical point of E and is regular in D. Both φ_c and φ_0 are continuous in $\overline{D} \sim \{(0, a)\}$,

$$\begin{aligned} \varphi_c = \varphi_0 &= 0 \quad \text{for } r = 0,\ z > a, \\ \varphi_c = \varphi_0 &= \pi \quad \text{for } r = 0,\ z < a. \end{aligned}$$

By the constraint (4), we conclude that ψ_a is continuous in $\overline{D} \sim \{(0, a)\}$, $\psi_a = 0$ for $r = 0$, $z > a$, and $\psi_a = \pi$ for $r = 0$, $z < a$. Then

$$u_a(r, \alpha, z) = (\sin\psi_a\cos\alpha, \sin\psi_a\sin\alpha, \cos\psi_a)$$

is the desired axially symmetric harmonic map.

Now suppose that x_0 is on the boundary of B^3. By rotating B^3 and S^2, we may assume that $x_0 = (0, 0, 1)$.

For any $0 < a < 1$, let $v_a \colon B^3 \to S^2$ be obtained by the homogeneous extension, with respect to the point $(0, 0, a)$, of the identity map from S^2 to S^2. One can compute the energies of v_a, $0 < a < 1$, as in [2, 7.B], and see that they are uniformly bounded. If u_a is the axially symmetric harmonic map which we obtained in the above, then $E(u_a) \leq E(v_a)$ and $\{E(u_a),\ 0 < a < 1\}$ is uniformly bounded. Thus there is a subsequence $\{u_{a_i}\}$, as $a_i \to 1$, u_{a_i} converges weakly in $H^1(B^3, S^2)$ to a map u_1 which is also axially symmetric. Also, u_1 is harmonic, $u_1(x) = x$ on ∂B^3 in the sense of traces. Moreover u_1 is completely regular on $\overline{B}^3 \sim \{(0, 0, 1)\}$.

Acknowledgment

The author would like to thank Professor R. Hardt for valuable discussions on this work.

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