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GEOMETRIC QUANTIZATION OF CHERN-SIMONS GAUGE THEORY

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Abstract

We present a new construction of the quantum Hilbert space of Chern-Simons gauge theory using methods which are natural from the three-dimensional point of view. To show that the quantum Hilbert space associated to a Riemann surface Σ is independent of the choice of complex structure on Σ, we construct a natural projectively flat connection on the quantum Hilbert bundle over Teichmüller space. This connection has been previously constructed in the context of two-dimensional conformal field theory where it is interpreted as the stress energy tensor. Our construction thus gives a $(2+1)$-dimensional derivation of the basic properties of $(1+1)$-dimensional current algebra. To construct the connection we show generally that for affine symplectic quotients the natural projectively flat connection on the quantum Hilbert bundle may be expressed purely in terms of the intrinsic Kähler geometry of the quotient and the Quillen connection on a certain determinant line bundle. The proof of most of the properties of the connection we construct follows surprisingly simply from the index theorem identities for the curvature of the Quillen connection. As an example, we treat the case when Σ has genus one explicitly. We also make some preliminary comments concerning the Hilbert space structure.

Introduction

Several years ago, in examining the proof of a rather surprising result about von Neumann algebras, V. F. R. Jones [20] was led to the discovery of some unusual representations of the braid group from which invariants of links in S^3 can be constructed. The resulting "Jones polynomial" of links has proved in subsequent work to have quite a few generalizations, and to be related to two-dimensional lattice statistical mechanics and to quantum groups, among other things.

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Tsuchiya and Kanie [38] recognized that the Jones braid representations and their generalizations coincide with certain representations of braid groups and mapping class groups that have quite independent origins in conformal field theory [4] and that have been intensively studied by physicists [11], [40], [23], [31]. (The representations in question are actually projective representations, for reasons that will be clear later.) The conformal field theory viewpoint leads to a rigorous construction of these representations [34], [39].

Conformal field theory alone, however, does not explain why these particular representations of braid groups and mapping class groups are related to *three-dimensional* invariants. It was conjectured [2] that some form of three- or four-dimensional gauge theory would be the key to understanding the three-dimensional invariances of the particular braid traces that lead to the Jones polynomial. Recently it has been shown [41] that three-dimensional Chern-Simons gauge theory for a compact gauge group G indeed leads to a natural framework for understanding these phenomena. This involves a nonabelian generalization of old work by A. Schwarz relating analytic torsion to the partition functions of certain quantum field theories with quadratic actions [32], and indeed Schwarz had conjectured [33] that the Jones polynomial was related to Chern-Simons gauge theory.

Most of the striking insights that come from Chern-Simons gauge theory depend on use of the Feynman path integral. To make the path integral rigorous would appear out of reach at present. Of course, results predicted by the path integral can be checked by, e.g., showing that the claimed three-manifold invariants transform correctly under surgery, a program that has been initiated in [30]. Such combinatorial methods—similar to methods used in the original proofs of topological invariance of the Jones polynomial—give a verification but not a natural explanation of the three-dimensional symmetry of the constructions.

In this paper, we pursue the more modest goal of putting the Hamiltonian quantization of Chern-Simons gauge theory—which has been discussed heuristically in [42] and in [7]—on a rigorous basis. In this way we will obtain new insights about the representations of braid and mapping class groups that arise in this theory. These representations have been constructed, as we have noted, from other points of view, and most notably from the point of view of conformal field theory. However, three-dimensional quantum field theory offers a different perspective, in which the starting point is the fact that affine spaces and their symplectic quotients can be quantized in a natural way. Our goal in this paper is to give a rigorous construction of the representations of mapping class groups that are associated with the Jones polynomial, from the point of view of the three-dimensional quantum field theory.

Canonical quantization. The goal is to associate a Hilbert space to every closed oriented 2-manifold Σ by canonical quantization of the Chern-Simons theory on $\Sigma \times R$. As a first step, we construct the physical phase space, $\mathscr{M}$. It is the symplectic quotient of the space, $\mathscr{A}$, of G connections on Σ by the group $\mathscr{G}$ of bundle automorphisms. It has a symplectic form ω which is k times the most fundamental quantizable symplectic form ω_0; here k is any positive integer. $\mathscr{M}$ is the finite-dimensional moduli space of flat G connections on Σ. We then proceed to quantize $\mathscr{M}$ as canonically as possible. We pick a complex structure J on Σ. This naturally induces a complex structure on $\mathscr{M}$, making it into a Kähler manifold. We may then construct the Hilbert space $\mathscr{H}_J(\Sigma)$ by Kähler quantization. If $\mathscr{T}$ denotes the space of all complex structures on $\mathscr{M}$, we thus have a bundle of Hilbert spaces $\mathscr{H}(\Sigma) \to \mathscr{T}$. This "quantum bundle" will be denoted $\tilde{\mathscr{H}}_Q$. For our quantization to be "canonical" it should be independent of J, at least up to a projective factor. This is shown by finding a natural projectively flat connection on the quantum bundle.

The essential relation between Chern-Simons gauge theory and conformal field theory is that this projectively flat bundle is the same as the bundle of "conformal blocks" which arises in the conformal field theory of current algebra for the group G at level k. This bundle together with its projectively flat connection is relatively well understood from the point of view of conformal field theory. (In particular, the property of "duality" which describes the behavior of the $\mathscr{H}_J(\Sigma)$ when Σ degenerates to the boundary of moduli space has a clear physical origin in conformal field theory [4], [40]. The property is essential to the computability of the Jones polynomial.) The conformal field theory point of view on the subject has been developed rigorously from the point of view of loop groups by Segal [34], and from an algebra-geometric point of view by Tsuchiya et al. [39]. Also, there is another rigorous approach to the quantization of $\mathscr{M}$ due to Hitchin [18]. Finally, in his work on non-abelian theta functions, Fay [8] (using methods more or less close to arguments used in the conformal field theory literature) has described a heat equation obeyed by the determinant of the Dirac operator which is closely related to the construction of the connection and may in fact lead to an independent construction of it.

We will be presenting an alternative description of the connection on $\mathscr{H}(\Sigma)$ which arises quite naturally from the theory of geometric quantization. In fact, this entire paper is the result of combining three simple facts.

(1) The desired connection and all of its properties are easily understood for Kähler quantization of a finite-dimensional affine symplectic manifold $\mathscr{A}$. In that case the connection 1-form is a simple second order differential operator on $\mathscr{A}$ acting on vectors in the quantum Hilbert space (which are sections of a line bundle over $\mathscr{A}$).

(2) By geometric invariant theory we can present a simple abstract argument to "push down" this connection "upstairs" for quantization of $\mathscr{A}$ to a connection "downstairs" for the quantization of $\mathscr{M}$. Here, $\mathscr{M}$ is the symplectic quotient of $\mathscr{A}$ by a suitable group of affine symplectic transformations that preserves the complex structure which is used in quantizing $\mathscr{A}$.[1] The one-form $\mathscr{O}$ for the connection downstairs is a second-order differential operator on $\mathscr{M}$.

(3) Even in the gauge theory case where the constructions upstairs are not well-defined since $\mathscr{A}$ is infinite dimensional, we may present the connection downstairs in a well-defined way. We first work in the finite dimensional case and write out an explicit description of $\mathscr{O}$. We then interpret the "downstairs" formulas in the gauge theory case in which the underlying affine space is infinite dimensional though its symplectic quotient is finite dimensional. As is familiar from quantum field theory, interpreting the "downstairs" formulas in the gauge theory context requires regularization of some infinite sums. This can, however, be done satisfactorily.

What has just been sketched is a very general strategy. It turns out that we have some "luck"—the definition of $\mathscr{O}$ and the proof of most of its properties can all be written in terms of the Kähler structure of $\mathscr{M}$ and a certain regularized determinant which is independent of the quantization machinery. Since these objects refer only to $\mathscr{M}$, our final results are independent of geometric invariant theory. One consequence of this independence is that our results apply for an arbitrary prequantization line bundle on $\mathscr{M}$, and not just for line bundles which arise as pushdowns of prequantum line bundles on $\mathscr{A}$.

The infinite dimensionality of the affine space that we are studying shows up at one key point. Because of what physicists would call an "anomaly", one requires a rescaling of the connection 1-form from the normalization it would have in finite dimensions. This has its counterpart in conformal field theory as the normalization of the Sugawara construction [14], which is the basic construction giving rise to the connection from that point of view. This rescaling does not affect the rest of the calculation

[1] For physicists, geometric invariant theory is just the statement that, in this situation, one gets the same result by imposing the constraints corresponding to $\mathscr{G}$ invariance before or after quantization.

except to rescale the final answer for the central curvature of the connection. This reproduces a result in conformal field theory. From our point of view, though we can describe what aspects of the geometry of the moduli space lead to the need to rescale the connection, the deeper meaning of this step is somewhat mysterious.

Outline. This paper is quite long. Essentially this is because we rederive the connection several times from somewhat different viewpoints and because we describe the special case of genus one in considerable detail. Most readers, depending on their interests, will be able to omit some sections of the paper.

For physicists, the main results of interest are mostly in §§2 and 5, and amount to a $(2+1)$-dimensional derivation of the basic properties of $(1+1)$-dimensional current algebra, including the values of the central charge and the conformal dimensions. This reverses the logic of previous treatments in which the understanding of the $(2+1)$-dimensional theory ultimately rested, at crucial points, on borrowing known results in $(1+1)$-dimensions. This self-contained $(2+1)$-dimensional approach should make it possible, in future, to understand theories whose $(1+1)$-dimensional counterparts are not already understood. On the other hand, a mathematically precise statement of the majority of results of this paper is given at the beginning of §4. This discussion is essentially self-contained.

In §1, we present a detailed, although elementary, exposition of the basic concepts of Kähler quantization of affine spaces and their symplectic quotients. We define the desired connection abstractly. As an example, in the last subsection we show explicitly how for quantization of the quotient of a vector space by a lattice, the connection is the operator appearing in the heat equation for classical theta functions.

The remainder of the paper is devoted to making the results of §1 explicit in such a way that they essentially carry over to the gauge theory case. In §2 we discuss this case in detail and construct the desired connection in a notation that is probably most familiar to physicists.

In §3 we present a more precise and geometric formulation of the results of §2 in notation suitable for arbitrary affine symplectic quotients. We derive a formula for the connection that may be written intrinsically on $\mathscr{M}$. This derivation is, of course, only formal for the gauge theory problem.

In §4, we state and prove most of the main results. Using an ansatz suggested by the results of §§2 and 3 and properties of the intrinsic geometry of $\mathscr{M}$, we find a well-defined connection. The properties of the intrinsic geometry of $\mathscr{M}$ which we need follow from the local version of the families index theorem and geometric invariant theory. Using these properties, and one further fact, we show that the connection is projectively flat. (Actually, there are several candidates for the "further fact" in question. One argument uses a global result—the absence of holomorphic vector fields on $\mathscr{M}$—while a second argument is based on a local differential geometric identity proved in §7. It should also be possible to make a third proof on lines sketched at the end of §6.) This section is the core section analyzing the properties of the connection and is rigorous since all the required analysis has already been done in the proof of the index theorem.

In §5 we shall concentrate on the gauge theory case when Σ is a torus. We give explicit formulas for our connection and a basis of parallel sections of the quantum Hilbert bundle $\mathscr{H}(\Sigma)$. We also show directly that our connection is unitary and has the curvature claimed. The parallel sections are identified with the Weyl-Kac characters for the representations of the loop group of G. This result is natural from the conformal field theory point of view, and was originally discussed from the point of view of quantization of Chern-Simons gauge theory in [7].

In §6 we make some preliminary comments about the unitarity of our connection.

In §7 we develop an extensive machinery allowing us to prove in a systematic way the one identity left unproved in §4. Our discussion, however, is incomplete in that we have not checked some details of the analysis of regularization.

The appendix contains further formulas relevant to §5.

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1. Geometric setup and pushed down connection

In this section we consider the quantization of a finite-dimensional symplectic manifold $\mathscr{M}$ which is the symplectic quotient of an affine symplectic manifold $\mathscr{A}$ by a suitable subgroup of the affine symplectic group. Quantization of $\mathscr{M}$ is carried out by choosing a suitable complex structure J on $\mathscr{A}$ which induces one on $\mathscr{M}$. We describe the projectively flat connection whose existence shows that quantization of $\mathscr{M}$ is independent of the choice of J. This is an interesting, though fairly trivial, result about geometric quantization. Its real interest comes in the generalization to gauge theory, which will occupy the rest of the paper.

Most of this section is a review of concepts which are well known, although possibly not in precisely this packaging [22], [36], [37], [43]. We review this material in some detail in the hope of making the paper accessible.

1a. Symplectic geometry and Kähler quantization. To begin with, we consider a symplectic manifold $\mathscr{A}$, that is, a manifold with a closed and nondegenerate two-form ω. Nondegeneracy means that if we regard ω as a map from $\omega : T\mathscr{A} \to T^*\mathscr{A}$, then there is an inverse map ω^{-1} : $T^*\mathscr{A} \to T\mathscr{A}$. In local coordinates, a^i, if

$$\omega = \omega_{ij}\, da^i \wedge da^j, \tag{1.1}$$

and

$$\omega^{-1} = \omega^{ij} \frac{\partial}{\partial a^i} \otimes \frac{\partial}{\partial a^j}, \tag{1.2}$$

then the matrices ω_{ij} and ω^{ij} are inverses,

$$\omega_{ij}\omega^{jk} = \delta_i^{\ k}. \tag{1.3}$$

Let $C^\infty(\mathscr{A})$ denote the smooth functions on $\mathscr{A}$. Given $h \in C^\infty(\mathscr{A})$, we form the vector field $V_h = \omega^{-1}(dh)$ called the flow of h. It is a symplectic vector field—that is, the symplectic form ω is annihilated by the Lie derivative $\mathscr{L}_{V_h}$—since $\mathscr{L}_{V_h}(\omega) = (i_{V_h} d + d i_{V_h})\omega = d(i_{V_h}\omega)$ (since ω is closed) and by the definition of V_h one has $i_{V_h}(\omega) = -dh$. Conversely, given a symplectic vector field V, that is a vector field V such that $\mathscr{L}_V(\omega) = 0$, one has a closed one-form $\alpha = i_V(\omega)$. A function h such that $\alpha = dh$ is called a Hamiltonian function or moment map for V. If the first Betti number of $\mathscr{M}$ is zero, then every symplectic vector field on $\mathscr{A}$ can be derived from some Hamiltonian function.

The symplectic vector fields on $\mathscr{A}$ form a Lie algebra. If two symplectic vector fields V_f and V_g can be derived from Hamiltonian functions f and g, then their commutator $[V_f, V_g]$ can likewise be derived from a Hamiltonian function; in fact

$$[V_f, V_g] = V_{[f,g]_{\mathrm{PB}}}, \tag{1.4}$$

where $[f, g]_{\mathrm{PB}}$ denotes the so-called Poisson bracket

$$[f, g]_{\mathrm{PB}} = \omega^{-1}(df, dg) = \omega^{ij}\partial_i f \cdot \partial_j g. \tag{1.5}$$

Therefore, the symplectic vector fields that can be derived from Hamiltonians form a Lie subalgebra of the totality of symplectic vector fields.

Essentially by virtue of (1.4) the Poisson bracket obeys the Jacobi identity

$$(1.6)\qquad [[f\,,\,g]_{\mathrm{PB}}\,,\,h]_{\mathrm{PB}} + [[g\,,\,h]_{\mathrm{PB}}\,,\,f]_{\mathrm{PB}} + [[h\,,\,f]_{\mathrm{PB}}\,,\,g]_{\mathrm{PB}} = 0\,,$$

so that under the $[\ ,\ ]_{\mathrm{PB}}$ operation, the smooth functions on $\mathscr{A}$ have a Lie algebra structure. It is evident that the center of this Lie algebra consists of functions f such that $df = 0$; in other words, if $\mathscr{A}$ is connected, it consists of the constant functions. The smooth functions on $\mathscr{A}$ are also a commutative, associative algebra under ordinary pointwise multiplication, and the two structures are compatible in the sense that

$$(1.7)\qquad [f\,,\,gh]_{\mathrm{PB}} = [f\,,\,g]_{\mathrm{PB}}\cdot h + [f\,,\,h]_{\mathrm{PB}}\cdot g.$$

These compatible structures $[\ ,\ ]_{\mathrm{PB}}$ and pointwise multiplication give $C^\infty(\mathscr{A})$ a structure of "Poisson-Lie algebra".

According to quantum mechanics textbooks, "quantization" of a symplectic manifold $\mathscr{A}$ means constructing "as nearly as possible" a unitary Hilbert space representation of the Poisson-Lie algebra $C^\infty(\mathscr{A})$. This would mean finding a Hilbert space H and a linear map $f \to \widehat{f}$ from smooth real-valued functions on $\mathscr{A}$ to selfadjoint operators on H such that $\widehat{(fg)} = \widehat{f}\cdot\widehat{g}$ and $\widehat{[f\,,\,g]}_{\mathrm{PB}} = i[\widehat{f}\,,\,\widehat{g}]$. One also requires (or proves from an assumption of faithfulness and irreducibility), that if 1 denotes the constant function on A, then $\widehat{1}$ is the identity operator on H.

This notion of what quantization should mean is however far too idealized; it is easy to see that such a Hilbert space representation of the Poisson-Lie algebra $C^\infty(\mathscr{A})$ does not exist. Quantum mechanics textbooks therefore instruct one to construct something that is "as close as possible" to a representation of $C^\infty(\mathscr{A})$. This is of course a vague statement. In general, a really satisfactory notion of what "quantization" should mean exists only in certain special classes of examples. The proper study of these examples, on the other hand, leads to much information. The examples we will be considering in this paper are affine spaces and their symplectic quotients by subgroups of the affine symplectic group obeying certain restrictions.

Prequantization. If one considers $C^\infty(\mathscr{A})$ purely as a Lie algebra, a Hilbert space representation can be constructed via the process of "prequantization".

Actually, for prequantization one requires that $\frac{1}{2\pi}\omega$ represents an integral cohomology class. This condition ensures the existence of a Hermitian line bundle $\mathscr{L}$ over $\mathscr{A}$ with a connection ∇ that is compatible with the Hermitian metric $\langle\ ,\ \rangle_{\mathscr{L}}$ and has curvature $-i\omega$. The isomorphism class of $\mathscr{L}$ (as a unitary line bundle with connection) may not be unique; given one choice of $\mathscr{L}$, any other choice is of the form $\mathscr{L}' = \mathscr{L} \otimes S$, where S is a flat unitary line bundle, determined by an element of $H^1(\mathscr{A}, U(1))$. The problem of prequantization has a solution for every choice of $\mathscr{L}$.

Let $\mathscr{C}$ be the group of diffeomorphisms of the total space of the line bundle $\mathscr{L}$ which preserves all the structure we have introduced (the fibration over $\mathscr{A}$, the connection, and the Hermitian structure). Let $H_{L^2}(\mathscr{A}, \mathscr{L})$ be the "prequantum" Hilbert space of all square integrable sections of $\mathscr{L}$. Since $\mathscr{C}$ acts on $\mathscr{L}$, it also acts on $H_{L^2}(\mathscr{A}, \mathscr{L})$. An element, D, of the Lie algebra of $\mathscr{C}$ is just a vector field on $\mathscr{M}$ lifted to act on $\mathscr{L}$. Acting on $H_{L^2}(\mathscr{A}, \mathscr{L})$, this corresponds to a first-order differential operator,

$$D = \nabla_T + ih. \tag{1.8}$$

Here T is the vector field representing the action of D on the base space $\mathscr{A}$ and h is a function on $\mathscr{A}$. The conditions that D preserves the connection is that for any vector field v we have

$$[\nabla_T + ih, \nabla_v] = \nabla_{\mathscr{L}_T v}. \tag{1.9}$$

Since the curvature of ∇ is $-i\omega$, this is true if and only if $T = V_h$. One may easily check that the map ρ_{pr} from $C^\infty(\mathscr{A})$ to the Lie algebra of $\mathscr{C}$ defined by

$$\rho_{\mathrm{pr}}(h) = \tfrac{1}{i}\nabla_{V_h} + h \tag{1.10}$$

is an isomorphism of Lie algebras. In addition, the function 1 maps to the unit operator.

Prequantization, as just described, is a universal recipe which respects the Lie algebra structure of $C^\infty(\mathscr{A})$ at the cost of disregarding the other part of the Poisson-Lie structure, coming from the fact that $C^\infty(\mathscr{A})$ is a commutative associative algebra under multiplication of functions. Quantization, as opposed to prequantization, is a compromise between the two structures, and in contrast to prequantization, there is no universal recipe for what quantization should mean. We now turn to the case of affine spaces, the most important case in which there is a good recipe.

Quantization of affine spaces. Let $\mathscr{A}$ be a $2n$-dimensional affine space, with linear coordinates a^i, $i = 1 \dots 2n$ and an affine symplectic structure

$$\omega = \omega_{ij}\, da^i\, da^j, \tag{1.11}$$

with ω_{ij} being an invertible (constant) skew matrix.

The Poisson brackets of the linear functions a^i are

$$[a^i, a^j]_{\mathrm{PB}} = \omega^{ij}. \tag{1.12}$$

In contrast to prequantization, in which one finds a Lie algebra representation of all of $C^\infty(\mathscr{A})$ in a Hilbert space $\mathscr{H}$, in quantization we content ourselves with finding a Hilbert space representation of the Poisson brackets of the linear functions, that is, a Hilbert space representation of the Lie algebra

$$[\widehat{a^i}, \widehat{a^j}] = -i\omega^{ij}. \tag{1.13}$$

Actually, we want Hilbert space representations of the "Heisenberg Lie algebra" (1.13) that integrate to representations of the corresponding group. This group, the Heisenberg group, is simply the subgroup of $\mathscr{C}$ that lifts the affine translations. According to a classic theorem by Stone and von Neumann, the irreducible unitary representation of the Heisenberg group is unique up to isomorphism, the isomorphism being unique up to multiplication by an element of $U(1)$ (the group of complex numbers of modulus one). Representing only the Heisenberg group—and not all of $C^\infty(\mathscr{A})$—means that quantization can be carried out in a small subspace of the prequantum Hilbert space.

Action of the affine symplectic group. Before actually constructing a representation of the Heisenberg group, let us discuss some properties that any such representation must have.

The affine symplectic group $\mathscr{W}$—the group of affine transformations of $\mathscr{A}$ that preserve the symplectic structure—acts by outer automorphisms on the Lie algebra (1.13). The pullback of a representation ρ of (1.13) by an element w of $\mathscr{W}$ is another representation ρ' of (1.13) in the same Hilbert space H. The uniqueness theorem therefore gives a unitary operator $U(w) : H \to H$ such that $\rho' = U(w) \circ \rho$. The $U(w)$ are unique up to multiplication by an element of $U(1)$, and therefore it is automatically true that for $w, w' \in H$, $U(ww') = U(w)U(w')\alpha(w, w')$ where $\alpha(w, w')$ is a $U(1)$-valued two-cocycle of $\mathscr{W}$. Thus, the $U(w)$ give a representation of a central extension by $U(1)$ of the group $\mathscr{W}$. It can be shown that if one restricts to the *linear* symplectic group—the subgroup of $\mathscr{W}$ that fixes a point in $\mathscr{A}$—then (for finite-dimensional affine spaces) the kernel of this central extension can be reduced to $\mathbb{Z}/2\mathbb{Z}$.

Now, let us investigate the extent to which a representation ρ of the Lie algebra (1.13) can be extended to a representation of the Poisson-Lie algebra $C^\infty(\mathscr{A})$. One immediately sees that this is impossible, since one would require both $\rho(a^i a^j) = \rho(a^i)\rho(a^j)$ and $\rho(a^i a^j) = \rho(a^j a^i) = \rho(a^j)\rho(a^i)$; but

$$\rho(a^i)\rho(a^j) - \rho(a^j)\rho(a^i) = -i\omega^{ij}. \tag{1.14}$$

Thus, ρ cannot be extended to a representation of $C^\infty(\mathscr{A})$. However, the right-hand side of (1.14), though not zero, is in the center of $C^\infty(\mathscr{A})$, and this enables us to take one more important step. Defining

$$\rho(a^i a^j) = \tfrac{1}{2}(\rho(a^i)\rho(a^j) + \rho(a^j)\rho(a^i)), \tag{1.15}$$

one verifies that

$$\begin{aligned} [\rho(a^i a^j), \rho(a^k a^l)] &= -i\rho([a^i a^j, a^k a^l]_{\mathrm{PB}}) \\ [\rho(a^i), \rho(a^k a^l)] &= -i\rho([a^i, a^k a^l]_{\mathrm{PB}}). \end{aligned} \tag{1.16}$$

To interpret (1.16) observe that the linear and quadratic functions on $\mathscr{A}$ form, under Poisson bracket, a Lie algebra which is a central extension of the Lie algebra of the affine symplectic group. In other words, the Hamiltonian functions from which the generators of the affine symplectic group can be derived are simply the linear and quadratic functions on $\mathscr{A}$. Equation (1.16), together with (1.13), means that *any* representation of the Lie algebra (1.13) automatically extends to a projective representation of the Lie algebra of the affine symplectic group $\mathscr{W}$. This is an infinitesimal counterpart of a fact that we have already noted: by virtue of the uniqueness theorems for irreducible representations of (1.13), the group $\mathscr{W}$ automatically acts projectively in any such representation.

The verification of (1.16) depends on the fact that the ambiguity in the definition of $\rho(a^i a^j)$—the difference between $\rho(a^i)\rho(a^j)$ and $\rho(a^j)\rho(a^i)$—is central. For polynomials in the a^i of higher than second order, different orderings differ by terms that are no longer central, and it is impossible to extend ρ to a representation of $C^\infty(\mathscr{A})$ even as a Lie algebra, let alone a Poisson-Lie algebra. It is natural to adopt a symmetric definition

$$\rho(a^{i_1}a^{i_2}\cdots a^{i_n}) = \frac{1}{n!}\left(a^{i_1}a^{i_2}\cdot a^{i_n} + \text{permutations}\right), \tag{1.17}$$

but for $n > 2$ this does not give a homomorphism of Lie algebras.

Quantization. There remains now the problem of actually constructing Hilbert space representations of (1.13). There are two standard constructions (which are equivalent, of course, in view of the uniqueness theorem). Each construction involves a choice of a "polarization", that is, a maximal linearly independent commuting subset of the linear functions on $\mathscr{A}$. In the first approach, one takes these functions to be real valued. In the second approach, they are complex valued and linearly independent over $\mathbb{C}$.

We will describe the second approach; it is the approach that will actually be useful in what follows.

Pick a complex structure J on $\mathscr{A}$, invariant under affine translations, such that ω is positive and of type $(1,1)$. Then one can find n linear functions z^i that are holomorphic in the complex structure J such that

$$\omega = +i dz^i \wedge d\overline{z}^i. \tag{1.18}$$

Let $\mathscr{L}$ be the prequantum line bundle introduced in our discussion of prequantization. We recall that $\mathscr{L}$ is to be a Hermitian line bundle with a connection ∇ whose curvature form is $-i\omega$. Since the $(0,2)$ part of ω vanishes, the connection ∇ gives $\mathscr{L}$ a structure as a holomorphic line bundle. In fact, $\mathscr{L}$ may be identified as the trivial holomorphic line bundle whose holomorphic sections are holomorphic functions ψ and with the Hermitian structure $|\psi|^2 = \exp(-h)\cdot\overline{\psi}\psi$, with $h = \sum_i \overline{z}^i z^i$. Indeed, the connection ∇ compatible with the holomorphic structure and with this Hermitian structure has curvature $\overline{\partial}\partial(-h) = \sum dz^i d\overline{z}^i = -i\omega$. Since $H^1(\mathscr{A}, U(1)) = 0$, the prequantum line bundle just constructed is unique up to isomorphism.

We now define the quantum Hilbert space $\mathscr{H}_Q|_J$, in which the Heisenberg group is to be represented, to be the Hilbert space $H^0_{L^2}(\mathscr{A}, \mathscr{L})$ of *holomorphic* L^2 sections of $\mathscr{L}$. We recall that, by contrast, the prequantum Hilbert space consists of all L^2 sections of $\mathscr{L}$ without the holomorphicity requirement.

The required representation ρ of the Heisenberg group is the restriction of the prequantum action to the quantum Hilbert space. At the Lie algebra level, the z^i act as multiplication operators,

$$\rho(z^i)\psi = z^i\psi, \tag{1.19}$$

and the $\overline{z}^i$ act as derivatives with respect to the z^i,

$$\rho(\overline{z}^i)\psi = \frac{\partial}{\partial z^i}\psi. \tag{1.20}$$

That this representation is unitary follows from the identity

$$\langle z^i\chi, \psi\rangle = \left\langle \chi, \frac{\partial}{\partial z^i}\psi \right\rangle, \tag{1.21}$$

which asserts that $\rho(\overline{z}^i)$ is the Hermitian adjoint of $\rho(z^i)$. (Of course, with the chosen Hermitian structure on $\mathscr{L}$, $\langle\chi, \psi\rangle = \int \exp(-\sum_i \overline{z}_i z_i)\cdot \overline{\chi}\psi$.)

Irreducibility of this representation of the Heisenberg group can be proved in an elementary fashion. This irreducibility is a hallmark of quantization as opposed to prequantization.

Because of the uniqueness theorem for irreducible unitary representations of the Heisenberg group, the Hilbert space $\mathscr{H}_Q|_J$ that we constructed above is, up to the usual projective ambiguity, independent of the choice of J (as long as J obeys the restrictions we imposed: it is invariant under the affine translations, and ω is positive and of type $(1, 1)$). It automatically admits a projective action of the group of all affine symplectic transformations, including those that do not preserve J.

Infinitesimally, the independence of J is equivalent to the existence of a projective action of the Lie algebra of the affine symplectic group. Its existence follows from what we have said before; we have noted in (1.16) that given *any* representation $x^i \to \rho(x^i)$ of the Heisenberg Lie algebra, no matter how constructed, one can represent the Lie algebra of the affine symplectic group by expressions quadratic in the $\rho(x^i)$. In the representation that we have constructed of the Heisenberg Lie algebra, since the $\rho(z^i)$ and $\rho(\overline{z}^i)$ are differential operators on $\mathscr{A}$ of order 0 and 1, respectively, and the Lie algebra of the affine symplectic group is represented by expressions quadratic in these, this Lie algebra is represented by differential operators of (at most) second order. By the action of this Lie algebra, one sees the underlying symplectic geometry of the affine space $\mathscr{A}$, though an arbitrary choice of a complex structure J of the allowed type has been used in the quantization.

Quantization of Kähler manifolds. In this form, one can propose to "quantize" symplectic manifolds more general than affine spaces. Let $(\mathscr{A}, \omega)$ be a symplectic manifold with a chosen complex structure J such that ω is positive and of type $(1, 1)$ (and so defines a Kähler structure on the complex manifold $\mathscr{M}$). Any prequantum line bundle $\mathscr{L}$ automatically has a holomorphic structure, since its curvature is of type $(1, 1)$, and the Hilbert space $H^0_{L^2}(\mathscr{A}, \mathscr{L})$ can be regarded as a quantization of $(\mathscr{A}, \omega)$. In this generality, however, Kähler quantization depends on the choice of J and does not exhibit the underlying symplectic geometry. What is special about the Kähler quantization of affine spaces is that in that case, through the action of the affine symplectic group, one can see the underlying symplectic geometry even though a complex structure is used in quantization.

Most of this paper will in fact be concerned with quantization of special Kähler manifolds that are closely related to affine spaces. So we will now discuss Kähler quantization in detail, considering first some general features and then properties that are special to affine spaces. To begin, we review some basic definitions to make our notation clear.

An almost complex structure J on a manifold $\mathscr{A}$ is a linear operator from $T\mathscr{A}$ to itself with $J^2 = -1$, i.e. a complex structure on $T\mathscr{A}$. On $T\mathscr{A} \otimes \mathbb{C}$ we can form the projection operators $\pi_z \equiv \frac{1}{2}(1 - iJ)$ and $\pi_{\bar{z}} \equiv \frac{1}{2}(1 + iJ)$. The image of π_z is the subspace of $T\mathscr{A} \otimes \mathbb{C}$ on which J acts by multiplication by i. It is called $T^{(1,0)}\mathscr{A}$ or the holomorphic tangent space. Similarly, $T^{(0,1)}\mathscr{A}$ is the space on which J acts by multiplication by $-i$. The transpose maps π_z^{T} and $\pi_{\bar{z}}^{\mathrm{T}}$ act on $T^*\mathscr{A} \otimes \mathbb{C}$. We define $T^{*(1,0)}$ and $T^{*(0,1)}$ as their images. Given local coordinates a^i, we may define

$$da^{\underline{i}} = \pi_z(da^i), \qquad da^{\bar{i}} = \pi_{\bar{z}} da^i. \tag{1.22}$$

The statement that J is a complex structure means that we may pick our coordinates a^i so that $da^{\underline{i}}$ actually is the differential of a complex valued function $a^{\underline{i}}$ and $da^{\bar{i}}$ is the differential of the complex conjugate $a^{\bar{i}}$.

We should make contact with more usual notation. The usual complex and real coordinates are

$$\begin{gathered} z^i = x^i + iy^i \quad \text{for } i = 1, \dots, n, \\ a^i = \begin{cases} x^i & \text{for } i = 1, \dots, n, \\ y^{i-n} & \text{for } i = n+1, \dots, 2n. \end{cases} \end{gathered} \tag{1.23}$$

So we have

$$a^{\underline{i}} = \begin{cases} \frac{1}{2} z^i & \text{for } i = 1, \dots, n, \\ -\frac{i}{2} z^{i-n} & \text{for } i = n+1, \dots, 2n. \end{cases} \tag{1.24}$$

We may decompose a 2-form σ as the sum of its $(2, 0)$, $(1, 1)$, and $(0, 2)$ components:

$$\begin{gathered} \sigma^{(2,0)} = \sigma_{\underline{i}\underline{j}} da^{\underline{i}} da^{\underline{j}}, \qquad \sigma^{(0,2)} = \sigma_{\bar{i}\bar{j}} da^{\bar{i}} da^{\bar{j}}, \\ \sigma^{(1,1)} = \sigma_{\underline{i}\bar{j}} da^{\underline{i}} da^{\bar{j}} + \sigma_{\bar{i}\underline{j}} da^{\bar{i}} da^{\underline{j}}. \end{gathered} \tag{1.25}$$

In general, any real tensor can be thought of as a complex tensor with the indices running over $\underline{i}$ and $\bar{i}$ which correspond to a basis for $T\mathscr{A} \otimes \mathbb{C}$.

We also assume that J is compatible with ω in the sense that $\omega(Jv, Jw) = \omega(v, w)$ for any $v, w \in T\mathscr{A}$. This amounts to the assumption that

$$\begin{gathered} J^T \omega = -\omega J \\ J^j{}_i \omega_{jk} = -\omega_{ij} J^j{}_k. \end{gathered} \tag{1.26}$$

This is so exactly when ω is purely of type $(1, 1)$.

We may form the map $g = \omega \circ J$ from $T\mathscr{A}$ to $T^*\mathscr{A}$. Equivalently g is the J compatible nondegenerate symmetric bilinear form:

$$g(v, w) = \omega(v, Jw) \quad \text{for } v, w \in \mathscr{A}. \tag{1.27}$$

Finally, we assume that J is chosen so that g is a positive definite metric. In summary, $T\mathscr{A}$ is a complex manifold with a Riemannian metric, g, which is compatible with the complex structure and so that $\omega = -g \circ J$ is a symplectic form. This is just the definition of a Kähler manifold; ω is also called the Kähler form.

A connection ∇ on a vector bundle $\mathscr{V}$ over a Kähler manifold which obeys the integrability condition

$$0 = [\nabla_{\bar{i}}, \nabla_{\bar{j}}] \tag{1.28}$$

induces a holomorphic structure on $\mathscr{V}$, the local holomorphic sections being the sections annihilated by $\nabla_{\bar{i}}$. In particular, since ω is of type $(1, 1)$, the prequantum line bundle $\mathscr{L}$, which is endowed with a unitary connection obeying

$$[\nabla_i, \nabla_j] = -i\omega_{ij}, \tag{1.29}$$

is always endowed with a holomorphic structure. It is this property that enables one to define the quantum Hilbert space $\mathscr{H}_Q|_J$ as $H^0_{L^2}(\mathscr{A}, \mathscr{L})$.

Variation of complex structure. In general, given a symplectic manifold $\mathscr{A}$ with symplectic structure ω, it may be impossible to find a Kähler polarization—that is, a complex structure J for which ω has the properties of a Kähler form. If however a Kähler polarization exists, it is certainly not unique, since it can be conjugated by any symplectic diffeomorphism. To properly justify the name "quantization", which implies a process in which one is seeing the underlying symplectic geometry and not properties that depend on the choice of a Kähler structure, one would ideally like to have a canonical identification of the $\mathscr{H}_Q|_J$ as J varies. This, however, is certainly too much to hope for.

In many important problems, there is a natural choice of Kähler polarization—for instance, a unique choice compatible with the symmetries of the problem. We will be dealing with situations in which there is not a single natural choice of Kähler polarization, but a preferred family $\mathscr{T}$. For instance, for $\mathscr{A}$ affine we take $\mathscr{T}$ to consist of translationally invariant complex structures such that ω is a Kähler form. In such a case, the spaces $\mathscr{H}_Q|_J = H^0_{L^2}(\mathscr{A}, \mathscr{L})$ are the fibers of a Hilbert bundle $\mathscr{H}_Q$ over $\mathscr{T}$. $\mathscr{H}_Q$ is a subbundle of the trivial Hilbert bundle with total space

$\mathscr{H}_{\mathrm{pr}} = H_{L^2}(\mathscr{A}, \mathscr{L}) \times \mathscr{T}$. We will aim to find a canonical (projective) identification of the fibers $\mathscr{H}_Q|_J$, as J varies, by finding a natural projectively flat Hermitian connection $\delta^{\mathscr{H}_Q}$ on $\mathscr{H}_Q$. The parameter spaces $\mathscr{T}$ will be simply connected, so such a connection leads by parallel transport to an identification of the fibers of $\mathscr{H}_Q$.

Let $\mathscr{C}'$ be the subgroup of $\mathscr{C}$ consisting of those elements whose action on the space of complex structures on $\mathscr{A}$ leaves $\mathscr{T}$ invariant setwise. An element $\phi \in \mathscr{C}'$ maps $\mathscr{H}_Q|_J$ to $\mathscr{H}_Q|_{\phi J}$ in an obvious way. Using the projectively flat connection $\delta^{\mathscr{H}^Q}$ to identify $\mathscr{H}_Q|_{\phi J}$ with $\mathscr{H}_Q|_J$, we get a unitary operator $\phi|_J : \mathscr{H}_Q|_J \to \mathscr{H}_Q|_J$. We consider the association $\phi \to \phi|_J$ to represent a quantization of the symplectic transformation ϕ if the $\phi|_J$ are invariant (at least projectively) under parallel transport by $\delta^{\mathscr{H}^Q}$. In this case we say that ϕ is quantizable. It is evident that the symplectic transformations that are quantizable in this sense form a group $\mathscr{C}''$; for any J, $\phi \to \phi|_J$ is a projective representation of this group (the representations obtained for different J's are of course conjugate under parallel transport by $\delta^{\mathscr{H}^Q}$).

Repeating this discussion at the Lie algebra level, we obtain the following definition of quantization of functions h whose flow leaves $\mathscr{T}$ invariant. (For $\mathscr{A}$ affine, h is any quadratic function.) Let $\underset{h}{\delta} J = \mathscr{L}_{V_h}(J) \in T_J\mathscr{T}$ be the infinitesimal change in J induced by h. Let δ be the trivial connection on the trivial bundle $\mathscr{H}_{\mathrm{pr}}$. The quantization of h can be written as a sum of first-order differential operators on $\Gamma(\mathscr{T}, \mathscr{H}_{\mathrm{pr}})$,

$$i\hat{h} = i\rho_{\mathrm{pr}}(h) + \delta_{\underset{h}{\delta} J} - \delta^{\mathscr{H}_Q}{}_{\underset{h}{\delta} J} + \text{constant}. \tag{1.30}$$

The first term is the naive prequantum contribution. The second term represents the fact that the prequantum operator should also be thought of as moving the complex structure. The third term is our use of $\delta^{\mathscr{H}_Q}$ to return to the original complex structure so that $\hat{h}$ is just a linear transformation on the fibers of $\mathscr{H}_{\mathrm{pr}}$. To check that (1.30) leaves the subbundle $\mathscr{H}_Q$ invariant we observe that acting on sections of $\mathscr{H}_Q$

$$\begin{aligned} \nabla_{\pi_{\bar{z}} v} \circ \hat{h} &= [\nabla_{\pi_{\bar{z}} v}, \rho_{\mathrm{pr}}(h) - i\delta_{\underset{h}{\delta} J}] \\ &= -\nabla_{-i\mathscr{L}_{V_h}(\pi_{\bar{z}} v) + i(\delta_{\underset{h}{\delta} J})(\pi_{\bar{z}} v)} = 0. \end{aligned} \tag{1.31}$$

In the first line of (1.31), $\delta_{\underset{h}{\delta} J}$ is the trivial connection acting in the $\underset{h}{\delta} J$ direction on sections of the trivial bundle $\mathscr{H}_{\mathrm{pr}} \to \mathscr{T}$. In the second line it is the trivial connection on $T\mathscr{A} \otimes \mathbb{C} \times \mathscr{T} \to \mathscr{T}$. The first equality in (1.31) follows from the facts that $\pi_{\bar{z}} v$ annihilates holomorphic sections and that $\delta^{\mathscr{H}_Q}$ takes holomorphic sections to holomorphic sections. The second equality follows from (1.9). Equation (1.31) shows that $\hat{h}$ preserves holomorphicity as desired.

The requirement that $\hat{h}$ is independent of complex structure is the statement that

$$\delta^{\mathscr{H}_Q} \hat{h} = 0. \tag{1.32}$$

If $\delta^{\mathscr{H}_Q}$ is projectively flat this implies that quantization is a projective representation:

$$\widehat{[h_1, h_2]}_{\mathrm{PB}} = i[\hat{h}_1, \hat{h}_2] + \text{constant}. \tag{1.33}$$

Connection for quantization of affine space. We now turn to the case in which $\mathscr{A}$ is an affine symplectic manifold and $\mathscr{T}$ consists of translationally invariant complex structures. We take a^i to be global affine coordinates. By the uniqueness theorem for irreducible unitary representations of the Heisenberg algebra we know that the projectively flat connection $\delta^{\mathscr{H}_Q}$ must exist. It may be defined in several equivalent ways which we shall discuss in turn.

1. We first present a simple explicit formula for $\delta^{\mathscr{H}_Q}$ and then show that it corresponds to any of the definitions below. The connection $\delta^{\mathscr{H}_Q}$ is given by

$$\delta^{\mathscr{H}_Q} = \delta - \mathscr{O}^{up}, \tag{1.34.1}$$

$$\mathscr{O}^{up} = M^{\underline{ij}} \nabla_{\underline{i}} \nabla_{\underline{j}} \quad \text{with} \quad M^{\underline{ij}} = -\tfrac{1}{4}(\delta J \omega^{-1})^{\underline{ij}}. \tag{1.34.2}$$

Here δJ is a one form on $\mathscr{T}$ with values in $\mathrm{Hom}(T\mathscr{A}, T\mathscr{A})$. We call $\mathscr{O}^{up}$ the connection one-form for $\delta^{\mathscr{H}_Q}$. It is a second-order differential operator on $\mathscr{A}$ acting on sections of $\mathscr{L}$. We use the superscript '*up*' to distinguish it from the connection one-form which we will construct for quantization of the symplectic quotient $\mathscr{M}$.

To demonstrate that $\delta^{\mathscr{H}_Q}$ preserves holomorphicity and is projectively flat, we need the variation with respect to J of the statements that $J^2 = -1$ and that J is ω-compatible (1.26), that is,

$$0 = J\delta J + \delta J J = J^i{}_j \delta J^j{}_k + \delta J^i{}_j J^j{}_k = 2i\delta J^{\underline{i}}{}_{\underline{k}} - 2i\delta J^{\bar{i}}{}_{\bar{k}} \tag{1.35}$$

$$(\omega\delta J)_{ij} = (\omega\delta J)_{\underline{ij}} + (\omega\delta J)_{\bar{i}\bar{j}} \quad \text{is symmetric.} \tag{1.36}$$

Using these identities as well as the fact that ∇ has curvature $-i\omega$, it is easy to check that $\delta^{\mathscr{H}_Q}$ preserves holomorphicity (so that it does in fact give a connection on the bundle $\mathscr{H}_Q$ over $\mathscr{T}$). To calculate the curvature $R^{\delta^{\mathscr{H}_Q}} = (\delta^{\mathscr{H}_Q})^2$ we first observe that $\mathscr{O}^{up} \wedge \mathscr{O}^{up} = 0$ because holomorphic derivatives commute. To calculate $\delta\mathscr{O}^{up}$ we must remember that the meaning of the indices $\underline{i}$ and $\bar{i}$ change as we change J. One way to account for this is to use only indices of type i and explicitly write π_z wherever needed. Using the formulas

$$\delta\pi_z = -\frac{i}{2}\delta J \tag{1.37.1}$$

$$\delta\pi_{\bar{z}} = +\frac{i}{2}\delta J, \tag{1.37.2}$$

we find

$$R^{\delta^{\mathscr{H}_Q}} = -\delta\mathscr{O}^{up} = -\tfrac{1}{8}\delta J^{\underline{i}}{}_{\bar{j}}\delta J^{\bar{j}}{}_{\underline{i}} = -\tfrac{1}{8}\mathrm{Tr}(\pi_z \delta J \wedge \delta J). \tag{1.38}$$

This is a two-form on $\mathscr{T}$ whose coefficients are multiplication operators by constant functions, i.e., $\delta^{\mathscr{H}_Q}$ is projectively flat as desired.

2. The essential feature of the connection that we have just defined is that $\mathscr{O}^{up} = \delta - \delta^{\mathscr{H}^Q}$ is a second-order differential operator. The reason for this key property is that in quantization of an affine space, the Lie algebra of the affine symplectic group is represented by second-order differential operators. Indeed, a change δJ of complex structure is induced by the flow of the Hamiltonian function

$$h = -\tfrac{1}{4}(\omega J \delta J)_{ij} a^i a^j. \tag{1.39}$$

In the discussion leading to (1.15), we have already defined the quantization of a quadratic function

$$h = h_{ij} a^i a^j + h_i a^i + h_0 \tag{1.40}$$

by symmetric ordering,

$$\rho(h) = \hat{h} = h_{ij}\tfrac{1}{2}\{\hat{a}^i \hat{a}^j + \hat{a}^j \hat{a}^i\} + h_i \hat{a}^i + h_0. \tag{1.41}$$

This preserves holomorphicity for any complex structure J and gives a representation of the quadratic Hamiltonian functions on $\mathscr{H}_Q|_J$. According to (1.30) (dropping the constant), $\mathscr{O}^{up} = \delta - \delta^{\mathscr{H}^Q}$ is to be simply $i\hat{h} - i\rho_{\mathrm{pr}}(h)$, with h in (1.41). This leads to the definition (1.34) of the connection $\delta^{\mathscr{H}_Q}$.[2]

[2]Note that (1.30), with a constant included, holds true for arbitrary h, and not just those of the form (1.39). By properly including the "metaplectic correction" we can actually find a flat connection and arrange for all unwanted constant factors to vanish.

3. One natural candidate, which exists quite generally, for the connection on $\mathscr{H}_Q$ is the "orthogonally projected" connection for $\mathscr{H}_Q$ considered as a subbundle of the trivial Hilbert bundle $\mathscr{H}_{\text{pr}}$. It may be defined by

$$< \psi|\delta^{\mathscr{H}_Q}\psi' >=< \psi|\delta\psi' > \quad \text{for } \psi, \psi' \in \Gamma(\mathscr{T}, \mathscr{H}_Q). \tag{1.42}$$

Although we may write this formula down quite generally, there is no general reason that it should yield a projectively flat connection. However, in the case at hand, we may check that this definition agrees with that of points 1 and 2 above. This is so because $\mathscr{O}^{up}$ does not change form after integrating by parts, and $\mathscr{O}^{up}\overline{\psi} = 0$. This implies that our connection is in fact unitary.

4. Closely related to point 3 is the fact that $\delta^{\mathscr{H}_Q}$ may also be described as the unique unitary connection compatible with the holomorphic structure on $\mathscr{H}_Q$. We discussed the holomorphic structure on $\mathscr{H}_Q$ previously. To describe it explicitly we first note that a complex structure on the space $\mathscr{T}$ is defined by stating that the forms $\delta J^{j}{}_{\bar{i}}$ are of type $(1, 0)$ and that the forms $\delta J^{\bar{j}}{}_{i}$ are of type $(0, 1)$. In other words, the $(1, 0)$ and $(0, 1)$ pieces of δJ are

$$\delta J^{(1,0)} = \pi_z \delta J \pi_{\bar{z}} \quad \text{and} \quad \delta J^{(0,1)} = \pi_{\bar{z}} \delta J \pi_z. \tag{1.43}$$

Let $\delta^{(1,0)}$ and $\delta^{(0,1)}$ be the holomorphic and antiholomorphic pieces of the trivial connection on $\mathscr{H}_{\text{pr}}$. Holomorphic sections of $\mathscr{H}_{\text{pr}}$ are those sections which are annihilated by the $\overline{\partial}$ operator $\delta^{(0,1)}$. We define sections of $\mathscr{H}_Q$ to be holomorphic if they are holomorphic as sections of $\mathscr{H}_{\text{pr}}$. The integrability condition that we can find a local holomorphic trivialization of $\mathscr{H}_Q$ is satisfied if we can show that $\delta^{(0,1)}$ leaves $\mathscr{H}_Q$ invariant. But this is true since for ψ a section of $\mathscr{H}_Q$, we have

$$\nabla_{\bar{k}}\delta^{(0,1)}\psi = [\pi^{j}_{\bar{z}k}\nabla_j, \delta^{(0,1)}]\psi = -\tfrac{i}{2}(\delta J^{(0,1)})^{\bar{j}}{}_{k}\nabla_{\bar{j}}\psi = 0. \tag{1.44}$$

The statement that $\delta^{\mathscr{H}_Q}$ as defined in point 1 above is compatible with the holomorphic structure is just the observation that $\delta^{\mathscr{H}_Q (0,1)} = \delta^{(0,1)}$ since $\mathscr{O}$ only depends on $\delta J^{(1,0)}$.

1b. Symplectic quotients and pushing down geometric objects. Affine spaces by themselves are comparatively dull. The facts just described get considerably more interest because they have counterparts for symplectic quotients of affine spaces. Our applications will ultimately come by considering finite-dimensional symplectic quotients of infinite-dimensional affine spaces.

Symplectic quotients. To begin, we discuss symplectic quotients of a general symplectic manifold $\mathscr{A}$, ω. Suppose that a group $\mathscr{G}$ acts on $\mathscr{A}$ by symplectic diffeomorphisms. We would like to define the natural notion of the "quotient" of a symplectic manifold by a symplectic group action. This requires defining the "moment map".

Let $\mathbf{g}$ be the Lie algebra of $\mathscr{G}$ and $T : \mathbf{g} \to \mathrm{Vect}(\mathscr{A})$ be the infinitesimal group action. Since $\mathscr{G}$ preserves ω, the image of T consists of symplectic vector fields. A comoment map for the $\mathscr{G}$ action is a $\mathscr{G}$ invariant map, F, from $\mathbf{g}$ to the Hamiltonian functions on $\mathscr{A}$ (where $\mathscr{G}$ acts on $\mathbf{g}$ by the adjoint action). To express this in component notation, let L_a be a basis for $\mathbf{g}$ and $T_a = T(L_a)$. Since T is a representation, we have

$$[T_a, T_b] = f_{ab}{}^c T_c, \tag{1.45}$$

where $f_{ab}{}^c$ are the structure constants of $\mathscr{G}$. The comoment map is given by functions F_a whose flow is T_a. Invariance of F under the connected component of $\mathscr{G}$ is equivalent to the statement that F is a Lie algebra homomorphism,

$$\{F_a, F_b\}_{\mathrm{PB}} = f_{ab}{}^c F_c. \tag{1.46}$$

For each $A \in \mathscr{A}$, $F_a(A)$ are the components of a vector in the dual space $\mathbf{g}^\vee$. We may view F as a map from $\mathscr{A}$ to $\mathbf{g}^\vee$. Viewed this way it is called a moment map.

Since the moment map and $\{0\} \subset \mathscr{G}$ are $\mathscr{G}$ invariant, so is $F^{-1}(0)$. The quotient space $\mathscr{M} = F^{-1}(0)/\mathscr{G}$ is called the symplectic or Marsden-Weinstein quotient of $\mathscr{A}$ by $\mathscr{G}$.[3] With mild assumptions, $\mathscr{M}$ is a nonsingular manifold near the points corresponding to generic orbits of $\mathscr{G}$ in $F^{-1}(0)$. We will always restrict ourselves to nonsingular regions of $\mathscr{M}$, although we do not introduce any special notation to indicate this. We have the quotient map:

$$\begin{aligned} \pi : F^{-1}(0) &\to F^{-1}(0)/\mathscr{G} = \mathscr{M} \\ A &\mapsto \tilde{A}. \end{aligned} \tag{1.47}$$

We may define a symplectic structure, $\tilde{\omega}$, on $\mathscr{M}$ by

$$\tilde{\omega}_{\tilde{A}}(\tilde{v}, \tilde{w}) = \omega_A(v, w) \quad \text{for } \tilde{v}, \tilde{w} \in T_{\tilde{A}}\mathscr{M}. \tag{1.48}$$

[3]This quotient plays a role in elementary physics. If $\mathscr{A}$ is the phase space of a physical system, and $\mathscr{G}$ as a group of symmetries, then $\mathscr{M}$ is simply the phase space for the effective dynamics after one restricts to the level sets of the conserved momenta and solves the equations that can be integrated trivially due to group invariance. Alternatively, if the F_a are constraints generating gauge transformations of an unphysical phase space, then $\mathscr{M}$ is the physical phase space left after solving the constraints and identifying gauge equivalent configurations.

By $\mathscr{G}$-invariance and the fact that

$$\omega(T_a, u) = 0 \quad \text{for } u \in TF^{-1}(0)\,, \tag{1.49}$$

the definition is independent of our choice of A, v, and w. This is the first example of our theme of "pushing down" geometric objects from $\mathscr{A}$ to $\mathscr{M}$. The basic principle is that the objects (symplectic form, complex structure, bundles, connections, etc.) that we consider on $\mathscr{A}$ are $\mathscr{G}$-invariant so that when restricted to $F^{-1}(0)$ they push down to the corresponding objects on $\mathscr{M}$.

Pushing down the prequantum line bundle. In order to push down the prequantum line bundle we must assume we are given a lift of the $\mathscr{G}$-action on $\mathscr{A}$ to a $\mathscr{G}$-action on $\mathscr{L}$ which preserves the connection and Hermitian structure, i.e., an action by elements of $\mathscr{C}$. The Lie algebra version of such a lift is just a moment map. We may define the pushdown bundle $\tilde{\mathscr{L}}$ by stating its sections,

$$\Gamma(\mathscr{M}, \tilde{\mathscr{L}}) = \Gamma(F^{-1}(0), \mathscr{L})^{\mathscr{G}}. \tag{1.50}$$

The $\mathscr{G}$ superscript tells us to take the $\mathscr{G}$-invariant subspace. A line bundle on $\mathscr{M}$ with (1.50) as its sheaf of sections will exist if $\mathscr{G}$ acts freely on $F^{-1}(0)$ (or more generally if for all $x \in F^{-1}(0)$, the isotropy subgroup of x in $\mathscr{G}$ acts trivially on the fiber of $\mathscr{L}$ at x). A section $\psi \in \Gamma(F^{-1}(0), \mathscr{L})$ is invariant under the connected component of $\mathscr{G}$ precisely if

$$0 = i\rho(F_a)_{\mathrm{pr}} = \nabla_{T_a}\psi. \tag{1.51}$$

The pushdown connection may be defined by

$$\tilde{\nabla}_{\tilde{v}}\psi = \nabla_v\psi. \tag{1.52}$$

Here v is any vector field on $F^{-1}(0)$ which pushes forward to $\tilde{v}$ on $\mathscr{M}$. By (1.51) the right-hand side of (1.52) is independent of our choice of v. To show that (1.52) is a good definition we must show that the right-hand side is annihilated by ∇_{T_a}. This can be done using (1.29) and (1.49). Similarly, one can check that $\tilde{\nabla}$ has curvature $-i\tilde{\omega}$.

Pushing down the complex structure. To proceed further, we assume that $\mathscr{A}$ is an affine space and that $\mathscr{G}$ is a Lie subgroup of the affine symplectic group such that (i) there is an invariant metric on the Lie algebra $\mathbf{g}$, and (ii) the action of $\mathscr{G}$ on $\mathscr{A}$ leaves fixed an affine Kähler polarization. We continue to assume that the $\mathscr{G}$-action on $\mathscr{A}$ has been lifted to an action on $\mathscr{L}$ with a choice of moment map. We let $\mathscr{T}$ be the space of Kähler polarizations of $\mathscr{A}$ that are invariant under the affine translations and also $\mathscr{G}$-invariant. $\mathscr{T}$ is nonempty and contractible.

Since $\mathscr{G}$ acts linearly, there is a unique extension of the $\mathscr{G}$-action to an action of the complexification $\mathscr{G}_c$ which is holomorphic as a function from $\mathscr{G}_c \times \mathscr{A}$ to $\mathscr{A}$. In a closely related context of compact group actions on projective spaces, a basic theorem of Mumford, Sternberg, and Guillemin [16; 24, p. 158] asserts that the symplectic quotient $\mathscr{M}$ of $\mathscr{A}$ by $\mathscr{G}$ is naturally diffeomorphic to the quotient, in the sense of algebraic geometry, of $\mathscr{A}$ by $\mathscr{G}_c$. Since the latter receives a complex structure as a holomorphic quotient, $\mathscr{M}$ receives one also.

Those results about group actions on projective spaces carry over almost directly to our problem of certain types of group actions on affine spaces, using the fact that subgroups of the affine symplectic group obeying our hypotheses are actually extensions of compact groups by abelian ones. We will not develop this explicitly as actually the properties of the geometry of $\mathscr{M}$ that we need can be seen directly by local considerations near $F^{-1}(0)$, without appeal to the "global" results of geometric invariant theory. For instance, let us give a direct description of an almost complex structure $\tilde{J}$ obtained on $\mathscr{M}$ which coincides with the one given by its identification with $\mathscr{A}/\mathscr{G}_c$ when geometric invariant theory holds. (By the methods of §3a below, this almost complex structure can be shown to be integrable without reference to geometric invariant theory.) Let $\mathbf{g}_c$ be the complexification of the Lie algebra $\mathbf{g}$. The action of $\mathscr{G}_c$ is determined by the action of $\mathscr{G}$ and $\mathbf{g}_c$. Since we want it to be holomorphic in $\mathscr{G}_c$, the complex Lie algebra action $T_c : \mathbf{g}_c \to \mathrm{Vect}(\mathscr{A})$ must be

$$T_c(L_a) = T_a, \qquad T_c(iL_a) = JT_a. \tag{1.53}$$

At every $A \in F^{-1}(0)$, we have the following inclusion of spaces:

$$\begin{array}{ccc} TF^{-1}(0) & \subset & T\mathscr{A} \\ \cup & & \cup \\ T(\mathbf{g}) & \subset & T_c(\mathbf{g}_c) \end{array} \tag{1.54}$$

So we have the map

$$T_{\tilde{A}}\mathscr{M} \cong [T_{F^{-1}(0)}/T(\mathbf{g})]_A \to T\mathscr{A}/T_c(\mathbf{g}_c). \tag{1.55}$$

One can show that this is an isomorphism by simple dimension counting. As a quotient of complex vector spaces, the right-hand side of (1.55) receives a complex structure. Therefore, under the identification (1.55), $T_{\tilde{A}}\mathscr{M}$ receives a complex structure. (By $\mathscr{G}$-invariance, the choice of a point A in the orbit above $\tilde{A}$ is immaterial.) The fact that (1.55) is an isomorphism is just the infinitesimal version of the statement that $\mathcal{M} \cong \mathcal{A}/\mathcal{G}_c$. This isomorphism may also be proved using the Hodge theory description that we will present in §3. For instance, surjectivity of (1.55) follows since the representative of shortest length of any vector in $T\mathcal{A}/T_c(\mathbf{g}_c)$ actually lies in $TF^{-1}(0)$. This argument is just the infinitesimal version of the proof that $\mathcal{M} \cong \mathcal{A}/\mathcal{G}_c$, which uses a distance function to choose preferred elements on the $\mathcal{G}_c$ orbits.

Geometric invariant theory also constructs a holomorphic line bundle $\tilde{\mathcal{L}}$ over $\mathcal{A}/\mathcal{G}_c$, such that, if $\pi : \mathcal{A} \to \mathcal{A}/\mathcal{G}_c$ is the natural projection, then $\mathcal{L} = \pi^*(\tilde{\mathcal{L}})$. Moreover, $\tilde{\mathcal{L}}$ has the property that

$$H^0(\mathcal{A}, \mathcal{L})^G = H^0(\mathcal{A}/\mathcal{G}_c, \tilde{\mathcal{L}}). \tag{1.56}$$

This equation is a holomorphic analog of (1.50). Under the identification of $\mathcal{M}$ with $\mathcal{A}/\mathcal{G}_c$, the two definitions of $\tilde{\mathcal{L}}$ agree.

Connection for quantization of $\mathcal{M}$. We let $\tilde{\mathcal{H}}_{\mathrm{pr}}$ be the trivial prequantum bundle over $\mathcal{T}$ whose fibers are the sections of $\tilde{\mathcal{L}}$. We let $\tilde{\mathcal{H}}_Q$ be the bundle whose fibers consist of holomorphic sections of $\tilde{\mathcal{L}}$. The quantum bundle $\tilde{\mathcal{H}}_Q$ that arises in quantizing $\mathcal{M}$ may be identified with the $\mathcal{G}$-invariant subbundle of the quantum bundle $\mathcal{H}_Q$ that entered in our discussion of the quantization of the affine space $\mathcal{A}$:

$$\tilde{\mathcal{H}}_Q|_J = H^0_{L^2}\begin{pmatrix}\tilde{\mathcal{L}}\\ \downarrow \\ \mathcal{M}\end{pmatrix} \cong H^0_{L^2}\begin{pmatrix}\mathcal{L}\\ \downarrow \\ \mathcal{A}\end{pmatrix}^{\mathcal{G}_c} = H^0_{L^2}\begin{pmatrix}\mathcal{L}\\ \downarrow \\ \mathcal{A}\end{pmatrix}^{\mathcal{G}} = (\mathcal{H}_Q|_J)^{\mathcal{G}} \tag{1.57}$$

The second to last equality in (1.57) is due to the fact that, for holomorphic sections, $\mathcal{G}_c$-invariance is equivalent to $\mathcal{G}$-invariance. The $\mathcal{G}$-action on $\mathcal{H}_Q$ is just the prequantum action. It is invariant under parallel transport by $\delta^{\mathcal{H}_Q}$.

Thus we may identify $\tilde{\mathcal{H}}_Q$ with a subbundle of $\mathcal{H}_Q$ which is preserved under parallel transport by $\delta^{\mathcal{H}_Q}$. Therefore $\delta^{\mathcal{H}_Q}$ restricts to the desired projectively flat connection $\delta^{\tilde{\mathcal{H}}_Q}$ on the subbundle $\tilde{\mathcal{H}}_Q$. Of course, the Hermitian structure of $\tilde{\mathcal{H}}_Q$ is the one it inherits as a subbundle of $\mathcal{H}_Q$.

In finite dimensions, this is a complete description of the projectively flat connection on $\tilde{\mathcal{H}}_Q$; there is no need to say more. However, even in finite dimensions, one obtains a better understanding of the projectively flat connection on $\tilde{\mathcal{H}}_Q$ by describing it as much as possible in terms of the intrinsic geometry of $\mathcal{M}$. Moreover, the main application that we envision is to a gauge theory problem in which $\mathcal{A}$ and $\mathcal{G}$ are infinite dimensional, though the symplectic quotient $\mathcal{M}$ is finite dimensional. In this situation, the "upstairs" quantum bundle $\mathscr{H}_Q$ is difficult to define rigorously, though the "downstairs" bundle $\tilde{\mathscr{H}}_Q$ is certainly well defined. Under these conditions, we cannot simply define a connection on $\tilde{\mathscr{H}}_Q$ by restricting the connection on $\mathscr{H}_Q$ to the $\mathscr{G}$-invariant subspace. At best, we can construct a "downstairs" connection on $\tilde{\mathscr{H}}_Q$ by imitating the formulas that one would obtain if $\mathscr{H}_Q$, with a projectively flat connection of the standard form, did exist. Lacking a suitable construction of $\mathscr{H}_Q$, one must then check *ex post facto* that the connection that is constructed on $\tilde{\mathscr{H}}_Q$ has the correct properties. This is the program that we will pursue in this paper.

The desired connection on $\tilde{\mathscr{H}}_Q$ should have the following key properties. The connection form should be a second order differential operator on $\mathscr{M}$, since the connection upstairs has this property and since a second-order differential operator, restricted to act on $\mathscr{G}_c$-invariant functions, will push down to a second-order differential operator. The connection should be projectively flat. And it should be unitary.

In the gauge theory problem, we will be able to understand the first two of these properties. In fact, we will see that there is a natural connection on $\tilde{\mathscr{H}}_Q$ that is given by a second-order differential operator, and that this connection is projectively flat. Most of these properties (except for vanishing of the $(2, 0)$ part of the curvature) can be understood in terms of the local differential geometry of $\mathscr{M}$. Unitarity appears more difficult and perhaps can only be understood by referring back to the underlying infinite-dimensional affine space $\mathscr{A}$. It is even conceivable that a construction of the "upstairs" bundle $\mathscr{H}_Q$ is possible and would be the best approach along the lines of gauge theory to understanding the unitarity of the induced connection on the $\mathscr{G}_c$-invariant subbundle.

1c. Theta functions. As an example of the above ideas, we consider the case in which $\mathscr{A}$ is a $2n$-dimensional real vector space, with affine symplectic form ω and prequantum line bundle $\mathscr{L}$, and $\mathscr{G}$ is the discrete group of translations by a lattice Λ in $\mathscr{A}$ whose action on $\mathscr{A}$ lifts to an action on $\mathscr{L}$. Picking such a lift, and taking the Λ-invariant sections of $\mathscr{L}$, we get a prequantum bundle $\tilde{\mathscr{L}}$ over the torus $\mathscr{M} = \mathscr{A}/\Lambda$.

To quantize $\mathscr{M}$, we pick an affine complex structure J on $\mathscr{A}$ which defines a Kähler polarization; it descends to a complex structure $\tilde{J}$ on $\mathscr{M}$. The existence of the prequantum line bundle $\tilde{\mathscr{L}}$, with curvature of type $(1, 1)$, means that the complex torus $\mathscr{M}$ is actually a polarized abelian variety. The Hilbert space $\tilde{\mathscr{H}}_Q|_J = H^0(\mathscr{M}, \tilde{\mathscr{L}})$ serves as a quantization of $\mathscr{M}$.

This Hilbert space $\tilde{\mathscr{H}}_Q|_J$ is, in classical terminology, the space of theta functions for the polarized abelian variety $(\mathscr{M}, \tilde{\mathscr{L}})$.

Classically, one then goes on to consider the behavior as J varies in the Siegel upper half plane Ω, which parametrizes the affine Kähler polarizations of $\mathscr{A}$. Thus, the quantum Hilbert spaces $\tilde{\mathscr{H}}_Q|_J$, as J varies, fit together into a holomorphic bundle $\tilde{\mathscr{H}}_Q$ over Ω.

Since the work of Jacobi, it has been known that it is convenient to fix the theta functions, in their dependence on J, to obey a certain "heat equation". While it is well known that the theta functions, for fixed J, have a conceptual description as holomorphic sections of the line bundle $\tilde{\mathscr{L}}$, the conceptual origin of the heat equation which fixes the dependence on J is much less well known. In fact, this heat equation is most naturally understood in terms of the concepts that we have introduced above. The heat equation is simply the projectively flat connection $\delta^{\tilde{\mathscr{H}}_Q}$ on the quantum bundle $\tilde{\mathscr{H}}_Q$ over Ω which expresses the fact that, up to the usual projective ambiguity, the quantization of $\mathscr{M}$ is canonically independent of the choice of Kähler polarization $\tilde{J}$. The usual theta functions, which obey the heat equations, are projectively parallel sections of $\tilde{\mathscr{H}}_Q$. The fact that they obey the heat equation means that the quantum state that they represent is independent of J.

Actually, the connection $\delta^{\tilde{\mathscr{H}}_Q}$ as we have defined it in equation (1.34) is only projectively flat. The central curvature of this connection can be removed by twisting the bundle $\tilde{\mathscr{H}}$ by a suitable line bundle over Ω (with a connection whose curvature is minus that of $\delta^{\tilde{\mathscr{H}}_Q}$). The heat equation as usually formulated differs from $\delta^{\tilde{\mathscr{H}}_Q}$ by such a twisting. In the literature on geometric quantization, the twisting to remove the central curvature is called the "metaplectic correction". We have not incorporated this twisting in this paper because it cannot be naturally carried out in the gauge theory problem of interest.

In the rest of this subsection, we shall work out the details of the relation of the heat equation to the connection $\delta^{\tilde{\mathscr{H}}_Q}$. These details are not needed in the rest of the paper and can be omitted without loss.

Prequantization of $\mathscr{M}$. A prequantum Hermitian line bundle $\tilde{\mathscr{L}}$ on $\mathscr{M}$ with connection with curvature $-i\omega$ is, up to isomorphism, the quotient of $\mathscr{A} \times C$ under the identifications

$$(A, v) \sim (A + \lambda, e_\lambda(A)v),$$

where the "multipliers" e_λ can be taken to be of the form

$$e_\lambda(A) = \epsilon(\lambda)\exp(-\tfrac{i}{2}\omega(A - A_0, \lambda)). \tag{1.58}$$

Here A_0 is a point in $\mathscr{A}/\Lambda$ and may be considered as parametrizing the possible flat line bundles over $\mathscr{M}$, and $\epsilon(\lambda) \in \{\pm 1\}$ satisfies

$$\epsilon(\lambda_1 + \lambda_2) = \epsilon(\lambda_1)\epsilon(\lambda_2)(-1)^{\omega(\lambda_1, \lambda_2)/2\pi}. \tag{1.59}$$

A section of $\tilde{\mathscr{L}}$ is a function $s : \mathscr{A} \to C$ satisfying

$$s(A + \lambda) = e_\lambda(A)s(A). \tag{1.60}$$

The connection and metric can be taken to be

$$\nabla_i s(A) = \left(\frac{\partial}{\partial A^i} + \frac{i}{2}\omega_{ij}A^j\right) s(A), \qquad ||s||(A) = |s(A)|^2. \tag{1.61}$$

Family of complex structures on $\mathscr{M}$. The Siegel upper half-space Ω of complex, symmetric $n \times n$ matrices Z with positive imaginary part parametrizes affine Kähler polarizations of $\mathscr{A}$ and $\mathscr{M}$. For Z in Ω we may define the complex structure J_Z on $\mathscr{A}$ as follows. Fix an integral basis λ_i for Λ so that in terms of the dual coordinates $\{x_i\}$ on $\mathscr{A}$,

$$\frac{\omega}{2\pi} = \sum_i \delta_i dx^i \wedge dx^{i+n}. \tag{1.62}$$

Such a basis always exists (see [15]). (The δ_i are nonzero integers called elementary divisors; they depend on the choice of Λ.) The complex structure J_Z is defined by saying that the functions

$$A^{\underline{i}} = \frac{1}{2\pi}\left(\delta_i x^i + \sum_j Z_{ij}x^{j+n}\right), \qquad i = 1 \dots n, \tag{1.63}$$

are holomorphic. In terms of these,

$$\omega = \pi i \sum_{ij} dA^{\underline{i}}\,(\operatorname{Im} Z)^{-1}_{ij}\, dA^{\bar{j}}. \tag{1.64}$$

The map $Z \mapsto J_Z$ is a holomorphic map, which may be shown to map onto $\mathscr{T}$.

We easily compute

$$(\delta^{(1,0)} J_Z)^{\underline{i}}{}_{\bar{j}} = -((\delta^{(1,0)} Z)(\operatorname{Im} Z)^{-1})_{ij}. \tag{1.65}$$

The bundle $\tilde{\mathscr{H}}_Q$ *and the connection* $\delta^{\tilde{\mathscr{H}}_Q}$. The quantum Hilbert space $\tilde{\mathscr{H}}_Q|_{J_Z}$ is the space of holomorphic sections of $\tilde{\mathscr{L}}$ and is thus identified with functions s satisfying (1.60) and

$$0 = \nabla_{\bar{i}} s(Z, A) = \left(\frac{\partial}{\partial A^{\bar{i}}} + \frac{\pi}{2}(\operatorname{Im} Z)^{-1}_{ij} A^{\underline{j}}\right) s(Z, A). \tag{1.66}$$

As Z varies, such functions correspond to sections of the Hilbert bundle $\tilde{\mathscr{H}}_Q \to \mathscr{T}$ (pulled back to Ω).

It is not hard to write down explicitly the action of the connection $\delta^{\tilde{\mathscr{H}}_Q}$ on sections of $\tilde{\mathscr{H}}_Q$ realized in this way. Recall that

$$(1.67)\qquad \delta^{\tilde{\mathscr{H}}_Q} = \delta - M^{\underline{i}\underline{j}}\nabla_{\underline{i}}\nabla_{\underline{j}}, \qquad M^{\underline{i}\underline{j}} = -\tfrac{1}{4}(\delta J\omega^{-1})^{\underline{i}\underline{j}}.$$

(1.61) gives the actions of the covariant derivatives $\nabla_{\underline{i}}$ on s; and (1.64) and (1.65) give expressions for ω and $\delta^{(1,0)}J$. We need an expression for the action of δ on s. For any function $s(Z, A)$, let $\delta^Z s(Z, A)$ denote the exterior derivative in the Z directions when A is considered independent. It is given by the formula

$$(1.68)\quad \delta = \left[\delta^Z + \frac{1}{2i}(A^{\underline{i}} - A^{\bar{i}})\left(((\delta^{(1,0)}Z)(\operatorname{Im} Z)^{-1})_{ij}\frac{\partial}{\partial A^{\underline{j}}} + ((\delta^{(0,1)}Z)(\operatorname{Im} Z)^{-1})_{ij}\frac{\partial}{\partial A^{\bar{j}}}\right)\right],$$

where the second line takes into account the dependence of the coordinates $A^{\underline{i}}$ and $A^{\bar{i}}$ on Z. Substituting this expression and the formulas for $\nabla_{\underline{i}}$, ω, and δJ into (1.67) gives a formula for $\delta^{\tilde{\mathscr{H}}_Q}$ acting on sections of $\tilde{\mathscr{H}}_Q$ realized as functions s satisfying (1.60).

A more convenient formula, from the point of view of making contact with the traditional expressions for theta functions, however, is obtained by changing the trivialization of $\mathscr{L} = \mathscr{A} \times C$ so that holomorphic sections are represented by holomorphic functions of Z and A. Such a change in trivialization corresponds to a function $g : \Omega \times \mathscr{A} \to \mathbb{C}$ satisfying

$$(1.69)\qquad g\nabla_{\bar{i}}g^{-1} = \frac{\partial}{\partial A^{\bar{i}}} \quad \text{and} \quad g\delta^{(0,1)}g^{-1} = \delta^{Z(0,1)}.$$

To obtain the usual theta functions, we take

$$(1.70)\qquad g(Z, A) = \exp\left(\frac{\pi}{2}A^{\underline{i}}(\operatorname{Im} Z)^{-1}_{ij}(A^{\bar{j}} - A^{\underline{j}})\right).$$

If we now write $\theta(Z, A) = s(Z, A)g(Z, A)$, then the conditions that $s(Z, A)$ represent a holomorphic section of $\tilde{\mathscr{H}}_Q$ are that θ is holomorphic as a function of Z and A and has the periodicities

$$(1.71)\qquad \begin{aligned} \theta(Z, A + \lambda_i) &= \epsilon(\lambda_i)\theta(Z, A) \\ \theta(Z, A + \lambda_{i+n}) &= \epsilon(\lambda_{i+n})\exp(-2\pi i A^{\underline{i}} - \pi i Z_{ii})\theta(Z, A). \end{aligned}$$

The classical theta functions satisfy these conditions. The choice of ϵ is called the theta characteristic.

Acting on $\theta(Z, A)$, the holomorphic derivatives become

$$\begin{aligned}
\nabla_{\underline{i}}\theta(Z, A) &= \left(\frac{\partial}{\partial A^{\underline{i}}} - \pi(\operatorname{Im} Z)^{-1}_{ij}(A^{\bar{j}} - A^{\underline{j}})\right)\theta(Z, A) \\
\delta^{(1,0)}\theta(Z, A) &= \Big[\delta^{Z(1,0)} + ((\delta^{(1,0)}Z)(\operatorname{Im} Z)^{-1})_{ij}(A^{\underline{j}} - A^{\bar{j}})\frac{\partial}{\partial A^{\underline{i}}} \\
&\quad + \frac{\pi}{4i}((\operatorname{Im} Z)^{-1}(\delta^{(1,0)}Z)(\operatorname{Im} Z)^{-1})_{ij} \\
&\quad \times (A^{\underline{i}} - A^{\bar{i}})(A^{\underline{j}} - A^{\bar{j}})\Big]\,\theta(Z, A).
\end{aligned} \tag{1.72}$$

Combining the equations (1.64), (1.65), and (1.72), we find after a short calculation

$$\delta^{\tilde{\mathscr{H}}_Q} = \delta^{\tilde{\mathscr{H}}_Q{}'} - \frac{i}{4}\operatorname{Tr}(\operatorname{Im} Z^{-1}\delta^{(1,0)}Z), \tag{1.73}$$

where

$$\delta^{\tilde{\mathscr{H}}_Q{}'(1,0)}\theta(Z, A) = \left[\delta^{Z(1,0)} - \frac{1}{4\pi i}(\delta^{(1,0)}Z)_{ij}\frac{\partial}{\partial A^i}\frac{\partial}{\partial A^j}\right]\theta(Z, A). \tag{1.74}$$

The modified connection $\delta^{\tilde{\mathscr{H}}_Q{}'}$ has zero curvature. The equation $\delta^{\tilde{\mathscr{H}}_Q{}'(1,0)}\theta(Z, A) = 0$ is the heat equation satisfied by the classical theta functions.

It may be shown that the modified connection $\delta^{\tilde{\mathscr{H}}_Q{}'}$ is that obtained when account is taken of the "metaplectic correction". Thus the dependence of the classical theta functions on Z is naturally interpreted from this point of view as the statement that as Z varies, the theta functions $\theta(Z, \cdot) \in \tilde{\mathscr{H}}_Q|_Z$ represent the same quantum state.

2. The gauge theory problem

In this section we will describe the concrete problem that actually motivated the investigation in this paper. It is the problem of quantizing the moduli space $\mathscr{M}$ of flat connections on a two dimensional surface Σ (of genus g, oriented, connected, compact, and without boundary). This moduli space can be regarded (as shown in [3]) as the symplectic quotient of an underlying infinite dimensional affine space by the action of the gauge group. Our goal is to explain concretely how this viewpoint leads to a projectively flat connection that makes possible quantization of

$\mathscr{M}$. In this section, we will aim for simplicity rather than precision and rigor. The precision and rigor will be achieved in later sections (which are independent of this one). Our goal in this section is to explain as directly as possible and without any unnecessary machinery the precise definition of the projectively flat connection that is used for quantization of $\mathscr{M}$, for the benefit of readers who may have use for the formulas. Also, we will explain in a language that should be familiar to physicists how one sees from this $(2+1)$-dimensional point of view a subtlety that is well known in $(1+1)$-dimensions, namely the replacement in many formulas of the "level" k by $k+h$, with h being the dual Coxeter number of the gauge group.

Preliminaries. Let G be a compact Lie group, which for convenience we take to be simple. The simple Lie algebra $\mathrm{Lie}(G)$ admits an invariant positive definite Killing form $(\, , \,)$, unique up to multiplication by a positive number. If F is the curvature of a universal G-bundle over the classifying space BG, a choice of $(\, , \,)$ enables us to define an element $\lambda = (F, \bigwedge F)$ of $H^4(BG, \mathbb{R})$. We normalize $(\, , \,)$ so that $\lambda/2\pi$ is a de Rham representative for a generator of $H^4(BG, \mathbb{Z}) \cong \mathbb{Z}$. This basic inner product $(\, , \,)$ is defined in down to earth terms in the appendix.

Let E be a principal G-bundle on the surface Σ. Let A be a connection on E. Locally, after picking a trivialization of E, A can be expanded

$$A = \sum A^a \cdot T_a, \tag{2.1}$$

where T_a is a basis of the Lie algebra $\mathrm{Lie}(G)$ of G. We can take T_a to be an orthonormal basis in the sense that $(T_a, T_b) = \delta_{ab}$. In this basis the Lie algebra $\mathrm{Lie}(G)$ may be described explicitly in terms of the structure constants $f_{ab}{}^c$:

$$[T_a, T_b] = f_{ab}{}^c T_c. \tag{2.2}$$

Defining $f_{abc} = f_{ab}{}^d \delta_{cd}$, invariance of $(\, , \,)$ implies that f_{abc} is completely antisymmetric.

One has

$$f_{ab}{}^c f_{cd}{}^b = 2\delta_{ad} \cdot h, \tag{2.3}$$

where h is the dual Coxeter number of G (as defined in the appendix).

Let $\mathscr{A}$ be the space of smooth connections on E. $\mathscr{A}$ is an affine space; its tangent space $T\mathscr{A}$ consists of one forms on Σ with values in $\mathrm{ad}(E)$. $\mathscr{A}$ has a natural symplectic structure, determined by the symplectic form

$$\omega_0 = \frac{1}{4\pi} \int_\Sigma \delta A^a \wedge \delta A^b \, \delta_{ab}. \tag{2.4}$$

The symplectic form that we will actually use in quantization is

$$\omega = k \cdot \omega_0 , \tag{2.5}$$

with k a positive integer called the "level".

After picking local coordinates on Σ, the connection d_A can be described by the explicit formula for covariant derivatives

$$D_\mu \phi = (\partial_\mu + A_\mu{}^a T_a)\phi , \tag{2.6}$$

with ϕ denoting a section of any associated bundle to E. And the curvature form is

$$F_{\mu\nu}{}^a = \partial_\mu A_\nu{}^a - \partial_\nu A_\mu{}^a + f_{cd}{}^a A_\mu{}^c A_\nu{}^d . \tag{2.7}$$

The "gauge group" $\mathcal{G}$ is the group of automorphisms of E as a principal bundle. An element g of the gauge group transforms the connection d_A by

$$d_A \to g \cdot d_A \cdot g^{-1} . \tag{2.8}$$

$\mathcal{G}$ is an infinite-dimensional Lie group whose Lie algebra $\mathbf{g}$ consists of the smooth sections of $\mathrm{ad}(E)$. The action of $\mathbf{g}$ on $\mathcal{A}$ is described by the map

$$\epsilon \to -d_A \epsilon \tag{2.9}$$

from $\Gamma(\mathrm{ad}(E))$ to $T_A(\mathcal{A})$.

The action of the group $\mathcal{G}$ on the space $\mathcal{A}$ of connections preserves the symplectic structure ω. The moment map $F : \mathcal{A} \to \mathbf{g}^\vee$ is the map which takes a connection A to its curvature. The curvature F is a two-form with values in $\mathrm{ad}(E)$ and is identified with an element of $\mathbf{g}^\vee$ by the pairing

$$< F , \epsilon >= \frac{k}{4\pi} \int_\Sigma (\epsilon , F) \tag{2.10}$$

for $\epsilon \in \Gamma(\mathrm{ad}(E))$.

The zeros of the moment map $F^{-1}(0)$ thus consist of flat connections, and the quotient $\mathcal{M} = F^{-1}(0)/\mathcal{G}$ is therefore the moduli space of flat connections on E, up to gauge transformation. If G is connected and simply connected, $\mathcal{M}$ is simply the moduli space of flat G bundles over Σ. If, however, G is not connected and simply connected, there may be several topological types of flat G bundles on Σ, and $\mathcal{M}$ is the moduli space of flat bundles with the topological type of E.

The general arguments about symplectic quotients apply in this situation, so that the symplectic structure ω_0 on $\mathcal{A}$ descends to a symplectic structure on $\mathscr{M}$, which we will call $\tilde{\omega}_0$. Our normalization convention on $(\, , \,)$ ensures that $\tilde{\omega}_0/2\pi$ represents an integral element of $H^2(\mathscr{M}, \mathbb{R})$, so that $\tilde{\omega}_0$ is at least prequantizable.

Holomorphic interpretation. Our goal is to quantize $\mathscr{M}$ with the symplectic structure ω_0 and more generally with the symplectic structure $\tilde{\omega} = k\tilde{\omega}_0$.

One of the important ingredients will be a construction of a suitable complex structure on $\mathscr{A}$. To do so, we pick a complex structure J on Σ (such that the orientation on Σ determined by J is the given one). This choice induces a complex structure $J_{\mathscr{A}}$ on $\mathscr{A}$, as follows. The tangent space $T\mathscr{A}$ consists of one forms on Σ with values in $\mathrm{ad}(E)$. Given a complex structure J on Σ, we define

$$J_{\mathscr{A}}\delta A = -J\,\delta A, \qquad \delta A \in T\mathscr{A}. \tag{2.11}$$

Relative to this complex structure

$$T\mathscr{A} = T^{(1,0)}\mathscr{A} \oplus T^{(0,1)}\mathscr{A}, \tag{2.12}$$

where $T^{(1,0)}\mathscr{A}$ and $T^{(0,1)}\mathscr{A}$ consist respectively of $(0, 1)$-forms and $(1, 0)$-forms on Σ with values in $\mathrm{ad}(E)$. (This is opposite to the choice of complex structure on $\mathscr{A}$ which appears frequently in the physics literature in which the holomorphic directions are represented by holomorphic one-forms on Σ. The above choice, however, is more natural since it is the antiholomorphic one-forms which couple to the $\overline{\partial}$ operator on Σ, and it is this operator, which defines a complex structure on the bundle E, which we want to vary holomorphically as a function of the complex structures on Σ and $\mathscr{A}$.) It is evident that with the choice (2.12) the symplectic form ω on $\mathscr{A}$ is positive and of type $(1, 1)$.

By analogy with the discussion in §1 of group actions on finite-dimensional affine spaces, one might expect that once the complex structure $J_{\mathscr{A}}$ is picked, the action of the gauge group $\mathscr{G}$ can be analytically continued to an action of the complexified gauge group $\mathscr{G}_c$ (which, in local coordinates, consists of smooth maps of Σ to G_c, the complexification of G). It is easy to see that this is so. Once the complex structure J is picked on Σ, the connection d_A can be decomposed as

$$d_A = \partial_A + \overline{\partial}_A, \tag{2.13}$$

where ∂_A and $\overline{\partial}_A$ are the $(1, 0)$ and $(0, 1)$ pieces of the connection, respectively. The $\mathscr{G}_c$ action on connections is then determined by the formula

$$\overline{\partial}_A \to g \cdot \overline{\partial}_A \cdot g^{-1}. \tag{2.14}$$

It is evident that this action is holomorphic. (2.14) implies the complex conjugate formula

$$\partial_A \to \overline{g} \cdot \partial_A \cdot \overline{g}^{-1}, \tag{2.15}$$

with $\overline{g}$ the complex conjugate of g.

Equation (2.14) has the following interpretation. For dimensional reasons, the $(0, 2)$ part of the curvature of any connection on a Riemann surface Σ vanishes. Therefore, for any connection A, the $\overline{\partial}_A$ operator gives a holomorphic structure to the principal G_c-bundle E_c (E_c is the complexification of E). The holomorphic structures determined by two such operators $\overline{\partial}_A$ and $\overline{\partial}_{A'}$ are equivalent if and only $\overline{\partial}_A$ and $\overline{\partial}_{A'}$ are conjugate by a transformation of the kind (2.14), that is, if and only if they are on the same orbit of the action of $\mathscr{G}_c$ on $\mathscr{A}$. Therefore, the set $\mathscr{A}/\mathscr{G}_c$ can be identified with the set $\mathscr{M}_J{}^{(0)}$ of equivalence classes of holomorphic structures on E_c.

Under a suitable topological restriction, for instance, if $G = SO(3)$ and E is an $SO(3)$ bundle over Σ with nonzero second Stieffel-Whitney class, $\mathscr{G}_c$ acts freely on $\mathscr{A}$. In this case, we will simply refer to $\mathscr{M}_J{}^{(0)}$ as $\mathscr{M}_J$; it is the moduli space of holomorphic G_c bundles over Σ, of specified topological type. The subscript in $\mathscr{M}_J$ is meant to emphasize that these bundles are holomorphic in the complex structure J. In general, reducible connections correspond to singularities in the quotient $\mathscr{A}/\mathscr{G}_c$; in this case, instead of the naive set theoretic quotient $\mathscr{A}/\mathscr{G}_c$, one should take the quotient in the sense of geometric invariant theory. Doing so, one gets the moduli space $\mathscr{M}_J$ of semistable G_c bundles on Σ, of a fixed topological type.

For finite-dimensional affine spaces, we know that the symplectic quotient by a compact group can be identified with the ordinary quotient by the complexified group. Does such a result hold for the action of the gauge group $\mathscr{G}$ on the infinite-dimensional affine space $\mathscr{A}$? The symplectic quotient of $\mathscr{A}$ by $\mathscr{G}$ is the moduli space $\mathscr{M}$ of flat connections on E; the ordinary quotient of $\mathscr{M}$ by $\mathscr{G}_c$ gives the moduli space $\mathscr{M}_J$ of holomorphic structures on E_c. In fact, there is an obvious map $i : \mathscr{M} \to \mathscr{M}_J$ coming from the fact that any flat structure on E determines a holomorphic structure on E_c. Using Hodge theory, it is easy to see that the map i induces an isomorphism of the tangent spaces of $\mathscr{M}$ and $\mathscr{M}_J$. Indeed, $T\mathscr{M} = H^1(\Sigma, \mathrm{ad}(E_A))$. (Here $H^*(\Sigma, \mathrm{ad}(E_A))$ denotes de Rham cohomology of Σ with values in the flat bundle $\mathrm{ad}(E_A)$.) According to the

Hodge decomposition, the complexification of $H^1(\Sigma, \mathrm{ad}(E_A))$ is

$$H^1_{\mathbb{C}}(\Sigma, \mathrm{ad}(E_A)) = H^{(0,1)}(\Sigma, \mathrm{ad}(E_A)) \oplus H^{(1,0)}(\Sigma, \mathrm{ad}(E_A)), \tag{2.16}$$

and on the right-hand side we recognize the $(1, 0)$ and $(0, 1)$ parts of the complexified tangent bundle of $\mathscr{M}_J$.

Actually, it is a fundamental theorem of Narasimhan and Seshadri that the map i is an isomorphism between $\mathscr{M}$ and $\mathscr{M}_J$. Just as in our discussion of the comparison between symplectic quotients and holomorphic quotients of finite-dimensional affine spaces, the symplectic form ω becomes a Kähler form on $\mathscr{M}_J$ under this isomorphism. In particular, ω is of type $(1, 1)$, so any prequantum line bundle $\tilde{\mathscr{L}}$ over $\mathscr{M}$ becomes a holomorphic line bundle on $\mathscr{M}_J$ (and in fact taking the holomorphic sections of powers of $\tilde{\mathscr{L}}$ leads to an embedding of $\mathscr{M}_J$ in projective space).

Thus, the Narasimhan-Seshadri theorem gives us a situation similar to the situation for symplectic quotients of affine spaces in finite dimensions. As the complex structure J on Σ varies, the $\mathscr{M}_J$ vary as Kähler manifolds, but as symplectic manifolds they are canonically isomorphic to a fixed symplectic variety $\mathscr{M}$.

Since diffeomorphisms that are isotopic to the identity act trivially on $\mathscr{M}$, isotopic complex structures J and J' give the *same* Hodge decompositions (2.16), not just "equivalent" ones. Therefore, the complex structure of $\mathscr{M}_J$ depends on the complex structure J on Σ only up to isotopy.

The moduli space of complex structures on Σ up to isotopy is usually called the Teichmuller space of Σ; we will denote it as $\mathscr{T}$. A point $t \in \mathscr{T}$ does not determine a canonical complex structure on Σ (it determines one only up to isotopy). But in view of the above, the choice of t does determine a canonical complex structure J_t on the moduli space $\mathscr{M}$ of flat connections. The complex structure J_t varies holomorphically in t. Therefore, the product $\mathscr{M} \times \mathscr{T}$, regarded as a bundle over $\mathscr{T}$, gets a natural complex structure, with the fibers being isomorphic as symplectic manifolds but the complex structure of the fibers varying with t.

Prequantization and the action of the mapping class group. There are several rigorous approaches to constructing a prequantum line bundle $\tilde{\mathscr{L}}$ over $\mathscr{M}$—that is, a unitary line bundle with a connection of curvature $-i\tilde{\omega}$. Since for G a compact, semisimple Lie group, $b_1(\mathscr{M})$ vanishes and $H^1(\mathscr{M}, U(1))$ is a finite set, there are finitely many isomorphism classes of such prequantum line bundles.

Holomorphically, one can pick a complex structure J on Σ, and take $\tilde{\mathscr{L}}$ to be the determinant line bundle Det_J of the $\overline{\partial}_J$ operator coupled to the associated bundle $E(R)$ determined by some representation R of the gauge group. If this line bundle is endowed with the Ray-Singer-Quillen metric, then the main result of [27] shows that its curvature is $-il(R)\cdot\tilde{\omega}_0$, where $l(R)$ is defined by $-\mathrm{Tr}_R(T_aT_b) = l(R)\delta_{ab}$. Though this construction depends on a choice of complex structure on Σ, the line bundle Det_J, as a unitary line bundle with connection, is independent of the complex structure chosen since the space of complex structures is connected and (according to the remark at the end of the last paragraph) the set of isomorphism classes of prequantum bundles is a finite set. (The isomorphism among the Det_J as J varies can also be seen more explicitly by using the Quillen connection to define a parallel transport on the Det_J bundle as J varies.) Thus, if k is of the form $2c_2(R)$ for some not necessarily irreducible representation of G, the prequantum bundle can be defined as a determinant line bundle.

From the point of view of the present paper, it is more natural to construct the prequantum line bundle by pushing down a trivial prequantum bundle $\mathscr{L}$ from the underlying infinite-dimensional affine space $\mathscr{A}$. This can be done rigorously [28].[4]

Once E is fixed, there is some subgroup $\Gamma_{\Sigma,E}$ of the mapping class group of Σ consisting of diffeomorphisms ϕ that fix the topological type of E. $\Gamma_{\Sigma,E}$ has an evident action on $\mathscr{M}$ coming from the interpretation of the latter as a moduli space of representations of $\pi_1(\Sigma)$. The goal of the present paper is to construct an action of $\Gamma_{\Sigma,E}$ on the Hilbert spaces obtained by quantizing $\mathscr{M}$. For this aim, we lift the action of the mapping class group of Σ on $\mathscr{M}$ to an action (or at least a projective action) on the prequantum line bundle $\tilde{\mathscr{L}}$.

Actually, if the prequantum line bundle is unique up to isomorphism, which occurs if G is connected and simply connected in which case $H^1(\mathscr{M}, U(1)) = 0$, then at least a projective action of the mapping class group is automatic. Even if the prequantum line bundle is not unique up to isomorphism, on a prequantum line bundle constructed as a determinant line bundle one automatically gets a projective action of the mapping class group. (If ϕ is a diffeomorphism of Σ, and $\tilde{\mathscr{L}}$ has been constructed as Det_J for some J, then ϕ naturally maps $\det_J$ to $\det_{\phi J}$, which has a projective identification with $\det_J$ noted in the last paragraph.)

The construction of prequantum line bundles via pushdown is also a natural framework for constructing actions (not just projective actions) of the

[4]The authors of that paper consider explicitly the case of $G = SU(2)$, but it should be straightforward to generalize their constructions.

mapping class group. We will now sketch how this construction arises from the three-dimensional point of view. This will be discussed more precisely elsewhere. To start with, we choose an element of $H^4(BG, \mathbb{Z})$. This allows us to define an $\mathbb{R}/2\pi\mathbb{Z}$ valued "Chern-Simons" functional of connections on G bundles over three-manifolds with boundary, as discussed in [6]. The functional S obeys the factorization property that $e^{iS(M_0, E_0, A_0)} = e^{iS(M_1, E_1, A_1)} \cdot e^{iS(M_2, E_2, A_2)}$ if the three-manifold M_0 with bundle E_0 and connection A_0 is obtained by gluing M_1, E_1, A_1 to M_2, E_2, A_2. The gluing is accomplished by an identification, $\Phi : E_1|_{\Sigma_1} \to E_2|_{\Sigma_2}$, of the restriction of E_2 to some boundary component Σ_1 of M_1 with the restriction of E_2 to some boundary component Σ_2 of M_2. Now fix a bundle E over a surface Σ. Let A be an element of the space $\mathscr{A}$ of connections on E. For $i = 1, 2$, let (M_i, E_i, A_i) be obtained by crossing (Σ, E, A) with an interval. So for each automorphism Φ of E (not necessarily base preserving) we may form M_0, E_0, A_0 by gluing. We thus obtain a function $\rho(\Phi, A) = e^{iS(M_0, E_0, A_0)}$ from $\mathrm{Aut}(E) \times \mathscr{A}$ to $U(1)$. By factorization, ρ is a lift of the action of $\mathrm{Aut}(E)$ on $\mathscr{A}$ to the trivial line bundle over $\mathscr{A}$. By restricting to flat connections and factoring out by the normal subgroup $\mathrm{Aut}'(E)$ of $\mathrm{Aut}(E)$ consisting of automorphism which lift diffeomorphisms of Σ which are connected to the identity, we obtain the line bundle $\tilde{\mathscr{L}}$ over $\mathscr{M}$ with an action of the mapping class group $\Gamma_{\Sigma, E} = \mathrm{Aut}(E)/\mathrm{Aut}'(E)$.

Finally, we can introduce the action of the mapping class group $\Gamma_{\Sigma, E}$ on the quantum bundle $\tilde{\mathscr{H}}_Q$ over $\mathscr{T}$. The fiber $\tilde{\mathscr{H}}_Q|_t$ of $\tilde{\mathscr{H}}_Q$, over a point $t \in \mathscr{T}$, is simply $H^0(\mathscr{M}_{J_t}, \tilde{\mathscr{L}})$. Obviously, since the mapping class group has been seen to act on $\tilde{\mathscr{L}}$, its action on $\mathscr{T}$ lifts naturally to an action on $\tilde{\mathscr{H}}_Q$. Our goal is to construct a natural, projectively flat connection ∇ on the bundle $\tilde{\mathscr{H}}_Q \to \mathscr{T}$. Naturalness will mean in particular that ∇ is invariant under the action of $\Gamma_{\Sigma, E}$. A projectively flat connection on $\mathscr{T}$ that is $\Gamma_{\Sigma, E}$-invariant determines a projective representation of $\Gamma_{\Sigma, E}$. Thus, in this way we will obtain representations of the genus g mapping class groups. These representations are genus g counterparts of the Jones representations of the braid group.

The precise connection with Jones's work depends on the following. At least formally, one can generalize the constructions to give representations of the mapping class groups $\Gamma_{g, n}$ for a surface of genus g with n marked points $P_1, \cdots, P_n$. This is done by considering flat connections on $\Sigma - \bigcup_i P_i$ with prescribed monodromies around the P_i. Jones's representations would then correspond to some of the representations so obtained for $\Gamma_{0,n}$. For simplicity, we will only consider the case without marked points.

Construction of the connection. We will now describe how, formally, one can obtain the desired connection on the bundle $\tilde{\mathscr{H}}_Q \to \mathscr{T}$, by formally supposing that one has a quantization of the infinite-dimensional affine space $\mathscr{A}$, and "pushing down" the resulting formulas from $\mathscr{A}$ to $\mathscr{M}$.

To begin with, we must (formally) quantize $\mathscr{A}$. This is done by using the complex structure $J_{\mathscr{A}}$ on $\mathscr{A}$ that comes (as described above) from a choice of a complex structure J on Σ. The prequantum line bundle $\mathscr{L}$ over $\mathscr{A}$ is a unitary line bundle with a connection ∇ of curvature $-i\omega$. To describe this more explicitly, let z be a local complex coordinate on σ, and define $\delta/\delta A_z{}^a(z)$ and $\delta/\delta A_{\overline{z}}{}^a(z)$ by

$$\begin{aligned}\nabla_u\psi(A) &= \int_\Sigma d^2z\, u_z^a \frac{\delta}{\delta A_z^a(z)}\psi(A),\\ \nabla_{\overline{u}}\psi(A) &= \int_\Sigma d^2z\, \overline{u}_{\overline{z}}^a \frac{\delta}{\delta A_{\overline{z}}^a(z)}\psi(A)\end{aligned} \tag{2.17}$$

for u and $\overline{u}$ adjoint valued $(1,0)$ and $(0,1)$ forms on Σ and $d^2z = idzd\overline{z}$. (We will sometimes abbreviate $\int_\Sigma d^2z$ as $\int_\Sigma$.) Then the connection ∇ is characterized by

$$\left[\frac{\delta}{\delta A_w{}^a(w)}, \frac{\delta}{\delta A_{\overline{z}}{}^b(z)}\right] = -i\frac{k}{4\pi}\delta_{ab}\delta_{z\overline{w}}(z, w), \tag{2.18}$$

along with

$$\left[\frac{\delta}{\delta A_z{}^a(z)}, \frac{\delta}{\delta A_w{}^b(w)}\right] = \left[\frac{\delta}{\delta A_{\overline{z}}{}^a(z)}, \frac{\delta}{\delta A_{\overline{w}}{}^b(w)}\right] = 0. \tag{2.19}$$

(In (2.18), $\delta_{z\overline{w}}(z, w)dzd\overline{w}$ represents the identity operator on $\Gamma(K\otimes \mathrm{ad}(E))$; that is $\int_{\Sigma_w} d\overline{w}dw\delta_{z\overline{w}}u_w = u_z$. We also have

$$\int_{\Sigma_z} d\overline{z}dz\,\overline{u}_{\overline{z}}\delta_{z\overline{w}} = \overline{u}_{\overline{w}}.)$$

The "upstairs" quantum Hilbert space $\mathscr{H}_Q|_J$ consists of holomorphic sections Ψ of the prequantum bundle, that is, sections obeying

$$\frac{\delta}{\delta A_z{}^a(z)}\Psi = 0. \tag{2.20}$$

What we actually wish to study is the object $\tilde{\mathscr{H}}_Q|_J = H^0(\mathscr{M}_J, \tilde{\mathscr{L}})$ introduced in the last subsection. The latter is perfectly well defined. Formally, this well-defined object should be the $\mathcal{G}_c$-invariant subspace of the larger space $\mathcal{H}_Q|_J$. At present, the latter is ill defined.

But proceeding formally, we will attempt to interpret $\tilde{\mathcal{H}}_Q|_J$ as the $\mathcal{G}_c$-invariant subspace of $\mathcal{H}_Q|_J$. Supposing for simplicity that $\mathcal{G}$ is connected, the $\mathcal{G}_c$-invariant subspace is the same as the subspace invariant under the Lie algebra $\mathbf{g}_c$ of $\mathcal{G}_c$. The condition for $\mathbf{g}_c$-invariance is

$$\left(-D_{\bar{z}}\frac{\delta}{\delta A_{\bar{z}}{}^{a}(z)}+\frac{k}{4\pi}\delta_{ab}F^{b}_{\bar{z}z}(z)\right)\Psi=0. \tag{2.21}$$

Here $F^a_{\bar{z}z}(z)$ is the curvature of the connection A, which enters because it is the moment map in the action of the gauge group on the space of connections. Also, $D_{\bar{z}}=\frac{\partial}{\partial\bar{z}}+A_{\bar{z}}$ is the $(0,1)$ component of the exterior derivative coupled to A.

Green's functions. To proceed further, it is convenient to introduce certain useful Green's functions that arise in differential geometry on the smooth surface Σ. In what follows, we will be working with flat connections on Σ—corresponding to zeros of the moment map for the $\mathcal{G}$-action on $\mathcal{A}$. For flat connections, the relevant Green's functions can all be expressed in terms of the Green's function of the Laplacian, or equivalently the operator $\bar{\partial}\partial$. For simplicity, we shall assume that this operator has no kernel. Let $\pi_i:\Sigma\times\Sigma\to\Sigma$, for $i=1,2$, be the projections on the first and second factors respectively. Let $E_i=\pi_i^*(E)$, for $i=1,2$. Similarly, let K be the canonical line bundle of Σ, regarded as a complex Riemann surface with complex structure J, and let $K_i=\pi_i^*(K)$.

The Green's function for the operator $\bar{\partial}\partial$ is a section ϕ of $\mathrm{ad}(E_1)\otimes\mathrm{ad}(E_2)^\vee$ over $\Sigma\times\Sigma-\Delta$ (Δ being the diagonal), such that

$$D_{\bar{z}}D_z\phi^a{}_b(z,w)=\delta^a{}_b\delta_{\bar{z}z}(z,w). \tag{2.22}$$

(Here $\delta_{\bar{z}z}(z,w)$ satisfies $\int_{\Sigma_w}d\bar{w}dw\,\delta_{\bar{z}z}v_{\bar{w}w}=v_{\bar{z}z}$.) It is convenient to also introduce

$$L_z{}^a{}_b(z,w)=D_z\phi^a{}_b(z,w) \tag{2.23}$$

and

$$\bar{L}_{\bar{z}}{}^a{}_b(z,w)=D_{\bar{z}}\phi^a{}_b(z,w). \tag{2.24}$$

They are sections, respectively, of $K_1\otimes\mathrm{ad}(E_1)\otimes\mathrm{ad}(E_2)^\vee$ and $\bar{K}_1\otimes\mathrm{ad}(E_1)\otimes\mathrm{ad}(E_2)^\vee$, over $\Sigma\times\Sigma-\Delta$, and obviously obey (for A a flat connection)

$$D_{\bar{z}}L_z{}^a{}_b(z,w)=-D_z\bar{L}_z{}^a{}_b(z,w)=\delta^a{}_b\delta_{\bar{z}z}(z,w). \tag{2.25}$$

On $H^0(K \otimes \mathrm{ad}(E))$, there is a natural Hermitian structure given by

$$|\lambda|^2 = \frac{1}{i}\int_\Sigma \overline{\lambda}^a \wedge \lambda^b \delta_{ab}. \tag{2.26}$$

Let $\lambda_{(i)}{}^a(z)$, $i = 1 \ldots (g-1)\cdot \mathrm{Dim}(G)$, be an orthonormal basis for $H^0(K \otimes \mathrm{ad}(E))$, that is, an orthonormal basis of $\mathrm{ad}(E)$-valued $(1, 0)$ forms obeying

$$D_{\overline{z}} \lambda_{(i)z}{}^a(z) = 0. \tag{2.27}$$

Obviously, their complex conjugates $\overline{\lambda}_{(i)}{}^a$ are $\mathrm{ad}(E)$-valued $(0, 1)$-forms that furnish an orthonormal basis of solutions of the complex conjugate equation

$$D_z \overline{\lambda}_{(i)\overline{z}}{}^a(z) = 0. \tag{2.28}$$

Finally, we have

$$D_{\overline{w}} L_z{}^a{}_b(z, w) = -\delta^a{}_b \delta_{z\overline{w}}(z, w) + \frac{1}{i}\sum_i \lambda_{(i)z}{}^a(z)\overline{\lambda}^b_{(i)\overline{w}}(w), \tag{2.29}$$

together with the complex conjugate equation.

We can now re-express (2.21) in a form that is convenient for constructing the well-defined expressions on $\mathscr{M}_J$ that are formally associated with ill-defined expressions on $\mathscr{A}$. Let

$$\mathscr{D}_{(i)} = \nabla_{\overline{\lambda}_{(i)}} = \int_\Sigma \overline{\lambda}_{(i)\overline{z}}{}^a(z) \frac{\delta}{\delta A_{\overline{z}}{}^a(z)}. \tag{2.30}$$

And for future use, let

$$\overline{\mathscr{D}}_{(i)} = \nabla_{\lambda_{(i)}} = \int_\Sigma \lambda_{(i)z}{}^a(z) \frac{\delta}{\delta A_z{}^a(z)}. \tag{2.31}$$

Then

$$\frac{\delta}{\delta A_{\overline{z}}{}^a(z)} = i\int_{\Sigma_w} L_z{}^a{}_b(z, w) D_{\overline{w}} \frac{\delta}{\delta A_{\overline{w}}{}^b(w)} + \sum_i \lambda_{(i)z}{}^a(z)\mathscr{D}_{(i)}. \tag{2.32}$$

The symbol $\int_{\Sigma_w}$ is just an instruction to integrate over the w variable. (2.32) is proved by integrating D_w by parts and using (2.29).

At last, we learn that on a section Ψ of $\mathscr{H}_Q|_J$ that obeys (2.21), we can write

$$\frac{\delta}{\delta A_{\overline{z}}{}^a(z)} = \frac{k}{4\pi}\int_{\Sigma_w} d\overline{w} dw L_z{}^a{}_b(z, w) F^b_{\overline{w}w}(w) + \sum_i \lambda_{(i)z}{}^a(z)\mathscr{D}_{(i)}. \tag{2.33}$$

The point of this formula is that arbitrary derivatives with respect to $A_{\bar{z}}$, appearing on the left, are expressed in terms of derivatives $\mathscr{D}_{(i)}$ in finitely many directions that correspond exactly to the tangent directions to $\mathscr{M}$.

We also will require formulas for the change in the $\lambda_{(i)}{}^a(z)$ and their complex conjugates $\overline{\lambda}_{(i)}{}^a(z)$ under a change in the flat connection A. There is some arbitrariness here, since although the sum

$$\frac{1}{i}\sum_i \lambda_{(i)}{}^a(z)\overline{\lambda}_{(i)}{}^b(w) \tag{2.34}$$

is canonical—it represents the projection operator onto the kernel of $\overline{\partial}_A$ acting on one-forms—the individual $\lambda_{(i)}{}^a(z)$ are certainly not canonically defined. However, expanding around a particular flat connection A, it is convenient to choose the $\overline{\lambda}_{(i)}$ so that, to first order in δA,

$$\frac{\delta}{\delta A_{\bar{z}}{}^a(z)}\overline{\lambda}_{(i)\overline{w}}{}^b(w) = 0. \tag{2.35}$$

This choice is natural since $A_{\bar{z}}$ does not appear in the equation obeyed by the $\overline{\lambda}_{(i)}$ (in other words this equation depends antiholomorphically on the connection). Varying (2.28) with respect to the connection and requiring that the orthonormality of the $\lambda_{(i)}$ should be preserved then leads to

$$\frac{\delta}{\delta A_{\bar{z}}{}^a(z)}\lambda_{(i)w}{}^b(w) = -iL_w{}^b{}_d(w\,,\,z)\,f_{ac}{}^d\lambda_{(i)z}{}^c(z). \tag{2.36}$$

Construction of the connection. We are now in a position to construct the desired connection on the quantum bundle $\tilde{\mathscr{H}}_Q \to \mathscr{T}$. First, we work "upstairs" on $\mathscr{A}$. To define a complex structure on $\mathscr{A}$, we need an actual complex structure on Σ, not just one defined up to isotopy. Accordingly, we shall also work over the space of complex structures on Σ. We discuss below why, for the final answer, these complex structures need actually only be defined up to isotopy. The projectively flat connection on $\mathscr{H}_Q$ that governs quantization of $\mathscr{A}$ is formally

$$\delta^{\mathscr{H}_Q} = \delta - \frac{it}{4}\cdot\frac{4\pi}{k}\int_\Sigma \delta J_{\bar{z}}{}^z\frac{\delta}{\delta A_{\bar{z}}{}^a(z)}\frac{\delta}{\delta A_{\bar{z}}{}^a(z)}. \tag{2.37}$$

Given the formulas for the symplectic structure and complex structure of $\mathscr{A}$, (2.37) is an almost precise formal transcription of the basic formula—equation (1.34)—for quantization of an affine space. However, for finite-dimensional affine spaces, one would have $t = 1$, as we see in (1.34); in the present infinite-dimensional situation, it is essential, as we will see, to permit ourselves the freedom of taking $t \neq 1$.

We now wish to restrict $\delta^{\mathscr{H}_Q}$ to act on $\mathscr{G}_c$-invariant sections Ψ of $\tilde{\mathscr{H}}_Q$, that is, sections that obey (2.33). On such sections, we can use (2.33) to write

(2.38)
$$\begin{aligned}\delta^{\mathscr{H}_Q} = \delta - \frac{it\pi}{k}\int_\Sigma \delta J_{\bar z}{}^z \frac{\delta}{\delta A_{\bar z}{}^a(z)} \\ \times\left(\frac{k}{4\pi}\int_{\Sigma_w} d\bar w dw L_z{}^a{}_b(z, w)F^b_{\bar w w}(w) + \sum_i \lambda_{(i)z}{}^a(z)\mathscr{D}_{(i)}\right).\end{aligned}$$

We now want to move $\delta/\delta A_{\bar z}{}^a(z)$ to the right on (2.41), so that—acting on $\mathscr{G}_c$ invariant sections—we can use (2.33) again. At this point, however, it is convenient to make the following simplification. Our goal is to obtain a well-defined connection on sections of $\tilde{\mathscr{L}}$ over $\mathscr{M}\times\{J\}$ which are holomorphic in $\mathscr{M}$. Since we are working at flat connections, after moving $\delta/\delta A_{\bar z}{}^a(z)$ to the right in (2.38), we are entitled to set $F = 0$. This causes certain terms to vanish.

In moving $\delta/\delta A_{\bar z}{}^a$ to the right, we encounter a term

$$\frac{\delta}{\delta A_{\bar z}{}^a(z)}F^b_{\bar w w}(w) = -iD_w\left(\delta_a{}^b\delta_{z\bar w}(z, w)\right). \tag{2.39}$$

We also pick up a term

$$\frac{\delta}{\delta A_{\bar z}{}^a(z)}\lambda_{(i)z}{}^a(z) = -iL_z{}^a{}_d(z, z)f_{ac}{}^d\lambda_{(i)z}{}^c(z), \tag{2.40}$$

where we have blindly used (2.36); the meaning of this formal expression that involves the value of a Green's function on the diagonal will have to be discussed later. With the help of (2.39) and (2.40), one finds that upon moving $\delta/\delta A_{\bar z}{}^a(z)$ to the right and setting $F = 0$ one gets

$$\begin{aligned}\delta^{\mathscr{H}_Q} = \delta + \frac{t\pi}{k}\int_\Sigma\Bigg(-\frac{k}{4\pi}\int_{\Sigma_w} d\bar w dw L_z{}^a{}_b(z, w)(D_w(\delta^b{}_a\delta(z, w))) \\ - L_z{}^a{}_d(z, z)f_{ac}{}^d\lambda_{(i)z}{}^c(z)\mathscr{D}_{(i)} \\ + \frac{1}{i}\sum_i \lambda_{z(i)}{}^a(z)\mathscr{D}_{(i)}\frac{\delta}{\delta A_{\bar z}{}^a(z)}\Bigg).\end{aligned} \tag{2.4.1}$$

Now we use (2.33) again. The resulting expression can be simplified by setting F to zero, and using the fact that $\mathscr{D}_{(i)}F = 0$ at $F = 0$ (by virtue of (2.39) and (2.27)). Also it is convenient to integrate by parts in the w variable in the first line of (2.41), using the delta function to eliminate the

w integration. After these steps, the connection on $\tilde{\mathscr{H}}_Q$ turns out to be

$$\delta^{\tilde{\mathscr{H}}_Q} = \delta - t\mathscr{O}, \tag{2.42.1}$$

$$\begin{aligned}\mathscr{O} = -\frac{\pi}{k}\int_\Sigma \delta J_{\bar{z}}{}^{z} \Bigg(& \frac{k}{4\pi}(\delta^b{}_a D_w L_z{}^a{}_b(z, w))|_{w=z} \\ & - L_z{}^a{}_d(z, z) f_{ac}{}^d \lambda_{(i)z}{}^c(z)\mathscr{D}_{(i)} \\ & + \frac{1}{i}\sum_i \lambda_{(i)z}{}^a(z)\mathscr{D}_{(i)} \sum_j \lambda_{(j)z}{}^a(z)\mathscr{D}_{(j)}\Bigg).\end{aligned} \tag{2.42.2}$$

Notice that the combination

$$\sum_i \lambda_{(i)}{}^a(z)\mathscr{D}_{(i)}, \tag{2.43}$$

which is the only expression through which the $\lambda_{(i)}$ appear in (2.42), is independent of the choice of an orthonormal basis of the $\lambda_{(i)}{}^a$.

Regularization. The first problem in understanding (2.42) is to make sense of the Green's functions on the diagonal

$$f_{ac}{}^d L_z{}^a{}_d(z, z) \quad \text{and} \quad \left(\delta^b{}_a D_w L_z{}^a{}_b(z, w)\right)|_{w=z}. \tag{2.44}$$

These particular Green's functions on the diagonal have an interpretation familiar to physicists. Consider a free field theory with an anticommuting spin zero field c and a $(1, 0)$-form b, both in the adjoint representation, and with the Lagrangian

$$\mathscr{L} = \int_\Sigma d\bar{z}\, dz D_{\bar{z}} c^a(z) b^a_z(z). \tag{2.45}$$

Let us introduce the current $J_{zc}(z) = f_{ca}{}^d b^a_z(z) c^d(z)$ and the stress tensor $T_{zz}(z) = b^a_z(z) D_z c^a(z)$. Then the Green's functions appearing in (2.44) are formally

$$f_{ca}{}^d L_z{}^a{}_d(z, z) = \langle J_{zc}\rangle', \tag{2.46}$$

and

$$\left(\delta^b{}_a D_w L_z{}^a{}_b(z, w)\right)|_{w=z} = \langle T_{zz}(z)\rangle', \tag{2.47}$$

where the symbol $\langle\ \rangle'$ means to take an expectation value with the kernel of the kinetic operator D_z projected out. Interpreted in this way, the desired Green's functions on the diagonal have been extensively studied in the physics literature on "anomalies". Thus, the crucial properties of these particular Green's functions are well known to physicists.

The interpretation of the Green's functions on the diagonal that appear in (2.44) as the expectation values of the current J^a and the stress tensor T can also be recast in a language that will be recognizable to mathematicians. Introduce a metric $g_{\mu\nu}$ on Σ which is a Kähler metric for the complex structure J. Then one has the Laplacian $\triangle : \Gamma(\Sigma, \mathrm{ad}(E)) \to \Gamma(\Sigma, \mathrm{ad}(E))$, defined by $\triangle = -g^{\mu\nu} D_\mu D_\nu$. Let

$$H = \det{}'(\Delta) \tag{2.48}$$

be the regularized determinant of the Laplacian. H is a functional of the connection A (which appears in the covariant derivative D_μ) and the metric g. We can interpret (2.46) and (2.47) as the statements that

$$f_{ca}{}^{d} L_z{}^{a}{}_{d}(z, z) = -i \frac{\delta}{\delta A_{\bar{z}}{}^{c}(z)} \ln H, \tag{2.49}$$

and

$$(\delta^b{}_a D_w L_z{}^a{}_b(z, w))|_{w=z} = \frac{\delta}{\delta g^{zz}} \ln H. \tag{2.50}$$

Given a regularization of the determinant of the Laplacian—for instance, Pauli-Villars regularization, often used by physicists, or zeta function regularization, usually preferred in the mathematical theory—the right-hand sides of (2.49) and (2.50) are perfectly well defined, and can serve as definitions of the left-hand sides. Since these were the problematic terms in the formula (2.42) for the connection on the quantum bundle, we have now made this formula well defined. It remains to determine whether this connection has the desired properties.

Conformal and diffeomorphism invariance. In defining the Green's functions on the diagonal, we have had to introduce a metric, not just a complex structure. To ensure diffeomorphism invariance we will choose the metric to depend on the complex structure in a natural way. Before making such a choice, however, we shall explain the simple way in which the connection (2.42) transforms under conformal rescalings of the metric.

One knows from the theory of regularized determinants (or the theory of the conformal anomaly in $(1+1)$-dimensions) that under a conformal rescaling $g \to e^{\phi} g$ of the metric, with ϕ being a real-valued function on Σ, the regularized determinant H transforms as

$$H \to \exp(S(\phi, g)) \cdot H, \tag{2.51}$$

where $S(\phi, g)$ is the Liouville action with an appropriate normalization and may be defined by

$$\frac{\partial}{\partial \phi} S(0, g) = \frac{\mathrm{Dim}(G)}{24\pi} \sqrt{g} R, \tag{2.52}$$

together with the group laws $S(0, g) = 1$ and $S(\phi_1 + \phi_2, g) = S(\phi_1, e^{\phi_2} g) \cdot S(\phi_2, g)$. Here, $\sqrt{g}R$ is the scalar curvature density of the metric g. The crucial property of $S(\phi, g)$ is that it is independent of the connection A. We conclude that under a conformal rescaling of the metric, the current expectation value defined in (2.49) is invariant. On the other hand, the expectation value (2.50) of the stress tensor transforms as

$$\begin{aligned} \left(\delta^b{}_a D_w L_z{}^a{}_b(z, w)\right)|_{w=z} \to \left(\delta^b{}_a D_w L_z{}^a{}_b(z, w)\right)|_{w=z} \\ + \frac{\delta}{\delta g^{zz}} (S(\phi, g)). \end{aligned} \tag{2.53}$$

This means that under a conformal transformation of the metric of Σ, the connection $\delta^{\mathscr{H}_Q}$ defined in (2.42) transforms as

$$\delta^{\mathscr{H}_Q} \to \delta^{\mathscr{H}_Q} + t\pi \int_\Sigma \delta J_{\bar{z}}{}^z \frac{\delta}{\delta g^{zz}} (S(\phi, g)). \tag{2.54}$$

The second term on the right-hand side of (2.54), being independent of the connection A, is a function on the base in the fibration $\mathscr{M} \times \mathscr{T} \to \mathscr{T}$. On each fiber, this function is a constant, and this means that the second term on the right in (2.54) is a central term. Up to a projective factor, $\delta^{\mathscr{H}_Q}$ is conformally invariant. The central term in the change of $\delta^{\mathscr{H}_Q}$ under a conformal transformation has for its $(1+1)$-dimensional counterpart the conformal anomaly in current algebra.

To show that our connection lives over Teichmüller space and is invariant under the mapping class group one must check that $\delta^{\mathscr{H}_Q}$ is invariant under a diffeomorphism of Σ, and that the connection form $\mathscr{O}$ vanishes for the variation $\delta J_{\bar{z}}{}^z = \partial_{\bar{z}} v^z$ of the complex structure induced by a vector field v on Σ. The first assertion is automatic if we always equip Σ with a metric that is determined in a natural way by the complex structure (for instance, the constant curvature metric of unit area or the Arakelov metric), since except for the choice of metric the rest of our construction is natural and so diffeomorphism invariant. The second point may be verified directly by substituting $\delta J_{\bar{z}}{}^z = \partial_{\bar{z}} v^z$ into (2.42), integrating by parts, and using (2.28). It follows more conceptually from the fact that the connection may be written in a form intrinsic on $\mathscr{M}$ (see §4) and from the fact that H and the Kähler structure on $\mathscr{M}$ depend on the complex structure J on Σ only up to isotopy.

Properties to be verified. Let us state precisely what has been achieved so far. Over the moduli space $\mathscr{M}$ of flat connections, we fix a Hermitian line bundle $\tilde{\mathscr{L}}$ of curvature $-i\tilde{\omega}$ with an action of the mapping class group. The prequantum Hilbert space is $\Gamma(\mathscr{M}, \tilde{\mathscr{L}})$. The prequantum bundle over Teichmuller space $\mathscr{T}$ is the trivial bundle $\tilde{\mathscr{H}}_{\mathrm{pr}} = \Gamma(\mathscr{M}, \mathscr{L}) \times \mathscr{T}$. The connection $\delta - t\mathscr{O}$ (with the regularization defined in (2.49), (2.50)) is rigorously well defined as a connection on the prequantum bundle. What remains is to show that this connection has the following desired properties.

(i) The quantum bundle $\tilde{\mathscr{H}}_Q$ over $\mathscr{T}$ is the subbundle of $\tilde{\mathscr{H}}_{\mathrm{pr}}$ consisting of *holomorphic* sections of the prequantum line bundle; that is, $\tilde{\mathscr{H}}_Q|_{J_t} = H^0(\mathscr{M}_J, \tilde{\mathscr{L}})$. We would like to show that (with the correct choice of the parameter t) the connection $\delta - t\mathscr{O}$ on $\tilde{\mathscr{H}}_{\mathrm{pr}}$ preserves holomorphicity and thus restricts to a connection on $\tilde{\mathscr{H}}_Q$.

(ii) We would like to show that this restricted connection, $\delta^{\tilde{\mathscr{H}}_Q}$, is projectively flat, and that it is unitary for the correct choice of a unitary structure on $\tilde{\mathscr{H}}_Q$.

Of course, for symplectic quotients of finite-dimensional affine spaces, these properties would be automatic consequences of simple "upstairs" facts that are easy to verify. The reason that there is something to be done is that here the underlying affine space is infinite dimensional. Since we do not have a rigorous quantization of the "upstairs" space, we need to verify *ex post facto* that the connection $\delta^{\tilde{\mathscr{H}}_Q}$ has the desired properties. Except for unitarity, which we will not be able to understand except in genus one (see §5), this will be done in §§3, 4, and 7. The computations will be done in a framework that is expressed directly, to the extent possible, in terms of the intrinsic geometry of the moduli space $\mathscr{M}$. These computations could be carried out directly in the framework and notation of the present section, but they are simpler if expressed in terms of the intrinsic geometry of $\mathscr{M}$. However, we will here describe (nonrigorously, but in a language that may be quite familiar to some readers) a small piece of the direct, explicit verification of property (i). This piece of the verification of (i) is illuminating because it explains a phenomenon that is well known in conformal field theory, namely the replacement of the "level" k by $k+h$ in many formulas.

The anomaly. In (2.30) and (2.31), we have introduced bases $\mathscr{D}_{(i)}$ and $\overline{\mathscr{D}}_{(m)}$ of $T^{(1,0)}\mathscr{M}$ and $T^{(0,1)}\mathscr{M}$, respectively. The symplectic structure of $\mathscr{M}$ can be described by the statement that, acting on sections of $\tilde{\mathscr{L}}$,

$$[\overline{\mathscr{D}}_{(m)}, \mathscr{D}_{(i)}] = -\frac{k}{4\pi}\delta_{im}. \tag{2.55}$$

This can be verified directly using (2.18) and the orthonormality of the λ's (and the fact that a section of $\tilde{\mathscr{L}}$ over $\mathscr{M}$ is the same as a $\mathscr{G}_c$-invariant section on $\mathscr{A}$).

Now, property (i) above—that the connection $\delta - t\mathscr{O}$ preserves holomorphicity—amounts to the statement that, at least when acting on holomorphic sections of $\tilde{\mathscr{L}}$,

$$0 = [\delta - t\mathscr{O}, \overline{\mathscr{D}}_{(m)}] = [\delta, \overline{\mathscr{D}}_{(m)}] + t[\overline{\mathscr{D}}_{(m)}, \mathscr{O}]. \tag{2.56}$$

The analogue of (2.56) would of course be true for symplectic quotients of finite-dimensional affine spaces. For the present problem, (2.56) can be verified directly although tediously. In doing so, one meets many terms that would be present in the finite-dimensional case. There is really only one point at which one meets an "anomaly" that would not be present in the finite-dimensional case. This comes from the term in $t[\overline{\mathscr{D}}_{(m)}, \mathscr{O}]$ with the structure

$$-\frac{t\pi}{k}\int_\Sigma \delta J_{\bar z}{}^{z}\left(\overline{\mathscr{D}}_{(m)}\langle J_{za}(z)\rangle'\right)\sum_{(i)}\lambda_{(i)z}{}^{a}(z)\mathscr{D}_{(i)}. \tag{2.57}$$

Now, formally the current $J_z{}^a(z)$ is defined in a Lagrangian (2.45) that depends holomorphically on the connection A, that is, $A_{\bar z}$ and not A_z appears in this Lagrangian. Naively, therefore, one might expect that $< J_{za}(z) >'$ or any other quantity computed from this Lagrangian would be independent of A_z and therefore annihilated by $\overline{\mathscr{D}}_{(m)}$.

However, the quantum field theory defined by the Lagrangian (2.45) is anomalous. As a result of the anomaly in this theory, there is a clash between gauge invariance and the claim that the current is independent of A_z. At least for Σ of genus zero, where there are no zero modes to worry about, one can indeed define the quantum current $< J_z{}^a(z) >'$ so as to be annihilated by $\delta/\delta A_z{}^a(z)$, but in this case $< J_{za} >'$ is not gauge invariant. In the case at hand, we must insist on gauge invariance since otherwise the basic formulas such as the definition of the connection (2.42) do not make sense on the moduli space $\mathscr{M}$. Indeed, in (2.49) we have regulated the current in a way that preserves gauge invariance. The anomaly is the assertion that the gauge invariant current defined in (2.49) cannot be independent of A_z; rather one has

$$\frac{\delta}{\delta A_w{}^b(w)} < J_{za}(z) >' = \frac{1}{2\pi}\delta_{ab}\delta_{z\overline{w}}(z, w)h + \cdots . \tag{2.58}$$

Here h is the dual Coxeter number, defined in (2.3). The $\cdots$ terms in (2.58), which would be absent for Σ of genus zero, arise because in addition to the anomalous term that comes from the short distance anomaly in quantizing the chiral Lagrangian (2.45), there is an additional dependence of $\langle J_z\rangle'$ on A_z that comes because of projecting away the zero modes present in (2.45) in defining $\langle\ \rangle'$. These $\cdots$ terms have analogs for symplectic quotients of finite-dimensional manifolds, and cancel in a somewhat elaborate way against other terms that arise in evaluating (2.56). We want to focus on the implications of the anomalous term.

The contribution of the anomalous term to (2.57) and (2.56) is

$$-\left(\frac{ith}{2k}\right)\int_\Sigma \delta J_{\overline{z}}{}^z \lambda_{(m)z}{}^a(z)\sum_{(i)}\lambda_{(i)z}{}^a(z)\mathscr{D}_{(i)}. \tag{2.59}$$

Terms of the same structure come from two other sources. As we see in (2.42), the last term in the connection form $\mathscr{O}$ is a second-order differential operator $\mathscr{O}_2$. In computing $t[\overline{\mathscr{D}}_{(m)}, \mathscr{O}_2]$, one finds with the use of 2.55 a term

$$-\left(\frac{it}{2}\right)\int_\Sigma \delta J_{\overline{z}}{}^z \lambda_{(m)z}{}^a(z)\sum_{(i)}\lambda_{z(i)}{}^a(z)\mathscr{D}_{(i)}. \tag{2.60}$$

The last contribution of a similar nature comes from

$$\begin{aligned}[\delta, \overline{\mathscr{D}}_{(m)}] &= \left[\delta, \int_\Sigma \lambda_{(m)z}{}^a(z)\frac{\delta}{\delta A_z{}^a(z)}\right] \\ &= \int_\Sigma \lambda_{(m)z}{}^a(z)\left[\delta, \frac{\delta}{\delta A_z{}^a(z)}\right] + \cdots .\end{aligned} \tag{2.61}$$

The $\cdots$ terms are proportional to $\delta/\delta A_z$ and annihilate holomorphic sections of $\mathscr{L}$. On the other hand,

$$\left[\delta, \frac{\delta}{\delta A_z{}^a(z)}\right] = \frac{i}{2}\delta J_{\overline{z}}{}^z \frac{\delta}{\delta A_{\overline{z}}{}^a(z)}. \tag{2.62}$$

Therefore, on holomorphic sections, after using (2.33), we get

$$[\delta, \overline{\mathscr{D}}_{(m)}] = \frac{i}{2}\int_\Sigma \delta J_{\overline{z}}{}^z \lambda_{(m)z}{}^a(z)\sum_i \lambda_{(i)z}{}^a(z)\mathscr{D}_{(i)}. \tag{2.63}$$

In the absence of the anomalous term (2.59), the two terms (2.63) and (2.60) would cancel precisely if $t = 1$. This is why the correct value in the quantization of a finite-dimensional affine space is $t = 1$. However, including the anomalous term, (2.59), the three expressions (2.59), (2.60), and (2.63) sum to zero if and only if $t = k/(k+h)$. The connection on the quantum bundle $\mathscr{H}_Q \to \mathscr{T}$ is thus finally pinned down to be

$$\begin{aligned}\nabla = \delta + \frac{\pi}{k+h}\int_\Sigma \Bigg(&\frac{k}{4\pi}(\delta^b{}_a D_w L_z{}^a{}_b(z, w))|_{w=z} \\ &- L_z{}^a{}_d(z, z)\, f_{ac}{}^d \lambda_{(i)z}{}^c(z)\mathscr{D}_{(i)} \\ &+ \frac{1}{i}\sum_i \lambda_{(i)z}{}^a(z)\mathscr{D}_{(i)} \sum_j \lambda_{(j)z}{}^a(z)\mathscr{D}_{(j)}\Bigg).\end{aligned} \tag{2.64}$$

That the connection form is proportional to $1/(k+h)$ rather than $1/k$, which one would obtain in quantizing a finite dimensional affine space, is analogous to (and can be considered to explain) similar phenomena in two-dimensional conformal field theory.

The rest of the verification of (2.56) is tedious but straightforward. No further anomalies arise; the computation proceeds just as it would in the quantization of a finite-dimensional affine space. We forego the details here, since we will give a succinct and rigorous proof of (2.56) in §4.

3. Hodge theory derivation of the pushdown connection

In §1 we saw, assuming results from geometric invariant theory, that the Hilbert space for quantization of $\mathscr{M}$ can be identified with the $\mathscr{G}$-invariant subspace of the Hilbert space for quantization of $\mathscr{A}$. A priori, we concluded that the connection for quantization of $\mathscr{A}$ pushes down to a connection for quantization of $\mathscr{M}$. The connection one-form $\mathscr{O}^{up}$, which is a second-order differential operator on $\mathscr{A}$, pushes down to a second-order differential operator, $\mathscr{O}$, on $\mathscr{M}$. In the gauge theory case $\mathscr{O}^{up}$ is ill defined due to an infinite sum over the partial derivatives in all directions on the space of connections. Proceeding formally, however, we were able in §2 to outline a "derivation" of $\mathscr{O}$. The final answer we obtained has a sensible regularization. In this section we will describe this construction in a way that highlights the relevant geometry for general affine symplectic quotients.

In §3a we generalize the Hodge decomposition on a Riemann surface, which applies for the gauge theory example, to a decomposition of the tangent space of $\mathscr{A}$ into '$\mathscr{G}$' and '$\mathscr{M}$' directions. The $\mathscr{G}$ directions are those along the $\mathscr{G}_c$ orbits and the $\mathscr{M}$ directions are orthogonal to the $\mathscr{G}_c$ orbits. Vectors in the $\mathscr{M}$ directions are identified with tangent vectors to $\mathscr{M}$. We define the "Green's function" T_z^{-1} as the inverse of the $(1, 0)$ part of the $\mathbf{g}_c$-action (with a suitable definition of 'inverse' in the presence of a kernel and cokernel). The reason for the name Green's function is that, in the gauge theory case, T_z^{-1} is an integral operator whose kernel is standardly called a Green's function. When the second-order differential operator $\mathscr{O}^{up}$ acts on $\mathscr{G}$-invariant holomorphic sections of $\mathscr{L}$, we may, with the aid of the Green's function, solve for the derivatives in the $\mathscr{G}_c$ directions. We obtain a differential operator, $\mathscr{O}$, which only involves partial derivatives in the $\mathscr{M}$ directions. This is naturally identified with the desired operator $\mathscr{O}$ on $\mathscr{M}$. In making this identification we obtain explicit formulas for the coefficients of $\mathscr{O}$ as pushdowns of tensors on $\mathscr{A}$. In the gauge theory case these formulas for the coefficients may be written as the multiple integrals over Σ which we found in our discussion of §2.

In §3b, we derive identities which tell us how the Green's function and the projection operator onto the $\mathscr{M}$ directions vary as we move in $F^{-1}(0) \times \mathscr{T}$. Ultimately, all of the properties of $\mathscr{M}$ follow from its definition as a symplectic quotient. All identities about the local geometry of $\mathscr{M}$ can be derived in terms of the given tensors on $\mathscr{A}$ and multiple derivatives of the Green's function and projection operators. Thus, the formulas of this subsection are a powerful general tool.

In §3c we will apply some of the results of §3b in order to express $\mathscr{O}$ purely in terms of intrinsic objects on $\mathscr{M}$ and a certain determinant. In the gauge theory case, this is just the determinant of a Laplacian. $\mathscr{O}$ is well defined in that case once we define a regularized gauge invariant determinant.

3a. Explicit form of the connection for quantization of $\mathscr{M}$.

Generalization of Hodge theory. Consider the tangent space to $T\mathscr{A}$ at points $A \in F^{-1}(0)$. It has a metric g and a complex structure J which is an orthogonal transformation of the tangent bundle. The subspace $TF^{-1}(0) \subset T\mathscr{A}$ consists of vectors u which satisfy

$$0 = \langle dF_a, u\rangle = \omega(u, T_a) = -g(u, JT_a) \quad \text{for all } a. \tag{3.1}$$

Equation (3.1) means that $TF^{-1}(0)$ is the orthogonal complement in $T\mathscr{A}$ of $T(i\mathbf{g}) = JT(\mathbf{g})$. Let $\widehat{T\mathscr{M}}$ be the orthocomplement in $TF^{-1}(0)$ of the tangent space, $T(\mathbf{g})$, to the $\mathscr{G}$ orbits. We have a $\mathscr{G}$-invariant orthogonal decomposition:

$$T\mathscr{A} = T(\mathbf{g}) \oplus JT(\mathbf{g}) \oplus \widehat{T\mathscr{M}} = T(\mathbf{g}_c) \oplus \widehat{T\mathscr{M}}. \tag{3.2}$$

The derivative of the projection map from $F^{-1}(0)$ to $\mathscr{M}$ maps $\widehat{T\mathscr{M}}|_A$ isomorphically onto $T\mathscr{M}|_{\tilde{A}}$. Hence we may identify $\widehat{T\mathscr{M}}$ with the pullback of $T\mathscr{M}$. Since J is orthogonal and leaves $T(\mathbf{g}_c)$ invariant, it restricts to a $\mathscr{G}$-invariant complex structure on the bundle $\widehat{T\mathscr{M}}$. The complex structure J on $\widehat{T\mathscr{M}}$ is the pullback of a complex structure $\tilde{J}$ on $T\mathscr{M}$.

We would also like to decompose the holomorphic tangent space of $\mathscr{A}$ into holomorphic '$\mathscr{G}$' and '$\mathscr{M}$' directions. Recall that $T_c\mathscr{A} = T\mathscr{A} \otimes \mathbb{C}$ has a notion of complex conjugation and a Hermitian metric:

$$\overline{v + iw} = v - iw \quad \text{for } v, w \in T\mathscr{A}, \tag{3.3}$$

$$< v, w >_{T_c\mathscr{A}} = \overline{v}^i g_{ij} w^j = \overline{v}^i (\omega J)_{ij} w^j \quad \text{for } v, w \in T_c\mathscr{A}. \tag{3.4}$$

The projection operators $\pi_z, \pi_{\overline{z}} : T_c\mathscr{A} \to T_c\mathscr{A}$ onto the holomorphic and antiholomorphic tangent spaces are orthogonal projection operators. The $(1, 0)$ and $(0, 1)$ components of the $\mathbf{g}_c$ action are

$$T_z = \pi_z \circ T_c : \mathbf{g}_c \to T^{(1,0)}\mathscr{A} \tag{3.5}$$

and

$$T_{\overline{z}} = \pi_{\overline{z}} \circ T_c : \mathbf{g}_c \to T^{(0,1)}\mathscr{A}. \tag{3.6}$$

It is not hard to verify that the holomorphic and antiholomorphic tangent space decomposes as

$$T^{(1,0)}\mathscr{A} = T_z(\mathbf{g}_c) \oplus \widehat{T\mathscr{M}}^{(1,0)} \tag{3.7}$$

and

$$T^{(0,1)}\mathscr{A} = T_{\overline{z}}(\mathbf{g}_c) \oplus \widehat{T\mathscr{M}}^{(0,1)}. \tag{3.8}$$

The orthogonal decompositions (3.7) and (3.8) are a refinement of the complexification of (3.2). We identify $\widehat{T\mathscr{M}}^{(1,0)}$ and $\widehat{T\mathscr{M}}^{(0,1)}$ with the pullbacks of $T^{(1,0)}\mathscr{M}$ and $T^{(0,1)}\mathscr{M}$.

In the gauge theory case, the orthogonal decompositions which we have been discussing are standard statements of Hodge theory on the Riemann surface Σ. First observe that

$$T = d_A : \mathbf{g} = \Omega^0(\Sigma, \mathrm{Lie}(G)) \to T\mathscr{A} = \Omega^1(\Sigma, \mathrm{Lie}(G)) \tag{3.9}$$

and

$$\begin{aligned} T_z = \overline{\partial}_A : \mathbf{g}_c = \Omega^{(0,0)}(\Sigma, \mathrm{Lie}(G)_c) &\to \Omega^{(0,1)}(\Sigma, \mathrm{Lie}(G)_c) \\ &= T^{(1,0)}\mathscr{A} \end{aligned} \tag{3.10}$$

are the d and $\overline{\partial}$ operators coupled to the flat G connection A acting in the adjoint representation. The decomposition (3.2) becomes:

$$\begin{aligned}\Omega^1(\Sigma, \mathrm{Lie}(G)) &= \mathrm{Im}(d) \oplus \mathrm{Im}(d^\dagger) \oplus H^1_{harm}(\Sigma, d_A) \\ &= \mathrm{Im}(d + d^\dagger) \oplus H^1_{harm}(\Sigma, d_A).\end{aligned} \tag{3.11}$$

Similarly, (3.7) now reads:

$$\Omega^{(0,1)}(\Sigma, \mathrm{Lie}(G)_c) = \mathrm{Im}(\overline{\partial}) \oplus H^{(0,1)}_{harm}(\Sigma, \partial_A). \tag{3.12}$$

Definition of Green's function. In order to define the Green's function for T_z we need to assume that we are given a $\mathscr{G}$-invariant metric on $\mathbf{g}$. We give $\mathbf{g}_c$ the corresponding Hermitian metric. In the gauge theory case we take the metric on $\mathbf{g}$ to be of the form

$$< \epsilon_1 , \epsilon_2 >_{\mathbf{g}} = \int_\Sigma \mu < \epsilon_1 , \epsilon_2 >_{\mathrm{Lie}(G)} , \qquad \epsilon_1 , \epsilon_2 \in \Omega^0(\Sigma, \mathrm{Lie}(G)), \tag{3.13}$$

where μ is a volume form on Σ and $\langle\ ,\ \rangle_{\mathrm{Lie}(G)}$ is an invariant metric on the Lie algebra of the gauge group G. When Σ is given a complex structure, the choice of a volume form is the same as a choice of a metric.

The operator T_z is a map from $\mathbf{g}_c$ to $T^{(1,0)}\mathscr{A}$. By restricting the domain and range of T_z, we obtain an invertible map

$$(T_z)_r : [\mathrm{Ker}(T_z)]^\dagger \to T_z(\mathbf{g}_c). \tag{3.14}$$

Here, $[\mathrm{Ker}(T_z)]^\dagger$ is the orthogonal complement in $\mathbf{g}_c$ of $[\mathrm{Ker}(T_z)]$. Let $\mathscr{H}$ be the projection operator onto the subspace $\widehat{T\mathscr{M}}^{(1,0)}$ of $T^{(1,0)}\mathscr{A}$. So, $1 - \mathscr{H}$ is the projection operator onto $T_z(\mathbf{g}_c)$. We define

$$T_z^{-1} = (T_z)_r^{-1} \circ (1 - \mathscr{H}) : T^{(1,0)}\mathscr{A} \to \mathbf{g}_c. \tag{3.15}$$

Equivalently, we could define T_z^{-1} by requiring that its kernel is $\widehat{T\mathscr{M}}^{(1,0)}$, its image is orthogonal to $\mathrm{Ker}(T_z)$, and it satisfies

$$\mathscr{H} + T_z T_z^{-1} = 1. \tag{3.16}$$

Our final result for the pushdown connection in finite dimensions must a priori be independent of the choice of metric on $\mathbf{g}$. If T_z has no kernel we may actually define the Green's function without choosing a metric. Even when there is no kernel, however, we find it convenient to choose a metric so that we may define a useful operator which we call the Laplacian and which specializes in the gauge theory case to the usual Laplacian for the operator d_A acting on $\mathrm{Lie}(G)$-valued zero forms on Σ. The Laplacian is

$$\triangle = T_z^\dagger T_z = T_{\overline{z}}^\dagger T_{\overline{z}} = \tfrac{1}{2} T_c^\dagger T_c. \tag{3.17}$$

This is a linear operator on $\mathbf{g}_c$, which is the complexification of the operator $\frac{1}{2}T^\dagger T$ on $\mathbf{g}$. In the gauge theory case, the fact that all the terms on the right are the same is just the usual identity that the d, ∂, and $\overline{\partial}$ Laplacians are proportional. Equation (3.17) follows quite generally in this framework from the fact that we are on the zeros of the moment map. From (3.17), we obtain

$$\mathrm{Ker}(T_z) = \mathrm{Ker}(T_{\bar{z}}) = \mathrm{Ker}(T_c) = \mathrm{Ker}(T) \otimes \mathbb{C} \tag{3.18}$$

which is familiar in the gauge theory case.

Derivation of pushed down connection. We are now ready to essentially independently repeat the derivation of the pushed connection one-form given for the gauge theory case in §2. In §1, we found that the connection one-form for quantization of $\mathscr{A}$ is

$$\mathscr{O}^{up} = M^{\underline{i}\underline{j}}\nabla_{\underline{i}}\nabla_{\underline{j}} \quad \text{with } M^{\underline{i}\underline{j}} = -\tfrac{1}{4}(\delta J\omega^{-1})^{\underline{i}\underline{j}}. \tag{3.19}$$

We will show that, when acting on $\mathscr{G}$ invariant holomorphic sections (GIHS), $\mathscr{O}^{up}$ is equal to an operator $\mathscr{O}$ which only involves derivatives in the $\mathscr{M}$ directions.

First recall that as operators acting on GIHS, we have

$$\nabla_{\bar{i}} = 0, \qquad T^i{}_a\nabla_i + iF_a = 0. \tag{3.20}$$

Using (3.16) and (3.20), we see that acting on GIHS

$$\begin{aligned}\nabla_{\underline{i}} &= \mathscr{K}^{\underline{j}}{}_{\underline{i}}\nabla_{\underline{j}} + (T_z^{-1})^a{}_{\underline{i}}T^{\underline{k}}_{z\,a}\nabla_{\underline{k}}\\ &= \mathscr{K}^{\underline{j}}{}_{\underline{i}}\nabla_{\underline{j}} - i(T_z^{-1})^a{}_{\underline{i}}F_a.\end{aligned} \tag{3.21}$$

We work at a point $A \in F^{-1}(0)$ and massage $\mathscr{O}^{up}$ when acting on GIHS:

$$\begin{aligned}\mathscr{O}^{up} &= \nabla_{\underline{i}}M^{\underline{i}\underline{j}}\nabla_{\underline{j}}\\ &= \nabla_{\underline{i}}M^{\underline{i}\underline{j}}[\mathscr{K}^{\underline{k}}{}_{\underline{j}}\nabla_{\underline{k}} - i(T_z^{-1})^a{}_{\underline{j}}F_a]\\ &= M^{\underline{i}\underline{j}}\mathscr{K}^{\underline{k}}{}_{\underline{j}}\nabla_{\underline{k}}[\mathscr{K}^{\underline{l}}{}_{\underline{i}}\nabla_{\underline{l}} - i(T_z^{-1})^a{}_{\underline{i}}F_a]\\ &\quad + M^{\underline{i}\underline{j}}(\nabla_{\underline{i}}\mathscr{K}^{\underline{k}}{}_{\underline{j}})\mathscr{K}^{\underline{l}}{}_{\underline{k}}\nabla_{\underline{l}} - iM^{\underline{i}\underline{j}}[(T_z^{-1})^a{}_{\underline{j}}(\nabla_{\underline{i}}F_a)].\end{aligned} \tag{3.22}$$

The last expression only involves derivatives in the $\mathscr{M}$ directions and so is equal to the desired operator $\mathscr{O}$. To simplify the first term, we observe that $\mathscr{K}^{\underline{k}}{}_{\underline{j}}\nabla_{\underline{k}}F_a = 0$ on $F^{-1}(0)$. To simplify the last term, recall that $\nabla_{\underline{i}}F_a = \omega_{\underline{i}\bar{\underline{k}}}T^{\bar{\underline{k}}}_{\bar{z}}a$. Using these facts, we obtain

$$\mathscr{O} = \mathscr{K}^{\underline{k}}{}_{\underline{j}}\nabla_{\underline{k}}M^{\underline{j}\underline{i}}\mathscr{K}^{\underline{l}}{}_{\underline{i}}\nabla_{\underline{l}} + M^{\underline{i}\underline{j}}(\nabla_{\underline{i}}\mathscr{K}^{\underline{k}}{}_{\underline{j}})\mathscr{K}^{\underline{l}}{}_{\underline{k}}\nabla_{\underline{l}} + \frac{i}{4}\mathrm{Tr}(T_z^{-1}\delta J T_{\bar{z}}). \tag{3.23}$$

This can be translated into the formula (2.42.2) for $\mathscr{O}$ given in §2. (For example, in the notation of §2, $\mathscr{K}$ corresponds to (2.34). The translation is fairly direct after using (2.35) and (2.36) to evaluate the derivative of the projection operator $\mathscr{K}$, given by (2.34), appearing in the second term of (3.23).) Before simplifying this further and writing it as a differential operator on $\mathscr{M}$, we derive some useful identities.

3b. Variation of Green's function and projection operators. In this subsection we will derive formulas for the derivatives of the Green's function and projection operators onto the kernel and image of T_z as functions on $\mathscr{A} \times \mathscr{T}$.

It turns out that we will not actually need to use most of these formulas since we manage to mostly supersede them with the index theory argument of §4. We include them since they are a powerful tool that would allow us to investigate any question about the local geometry of $\mathscr{M}$, not merely results which follow from the index theorem. To physicists, what we are saying is that the Green's function identities below allow us to calculate all correlation functions we would need, not merely the ones that calculate anomalies.

We start by deriving a general formula for the variation of "Hodge theory" inverses and projection operators. Let $N : \mathscr{H}_- \to \mathscr{H}_+$ be a linear map between two Hilbert spaces. Let I be the image of N, K the kernel, and $I^\dagger$ and $K^\dagger$ their orthocomplements. Also let π_I, π_K, $\pi_{I^\dagger}$, $\pi_{K^\dagger}$ be the orthogonal projection operators onto the indicated spaces. By restricting the domain and range of N, we obtain an invertible operator N_r from $K^\dagger$ to I. We define the "Hodge theory" inverse of N to be

$$N^{-1} = i_{K^\dagger} \circ (N_r)^{-1} \circ \pi_I : \mathscr{H}_+ \to \mathscr{H}_-, \tag{3.24}$$

where $i_{K^\dagger}$ is just the inclusion map of $K^\dagger$ into $\mathscr{H}_-$. Equivalently, we may have defined the projection operators and Green's function by

$$\begin{array}{llll} \pi_K = \pi_K^2, & \pi_K = (\pi_K)^\dagger, & \pi_{K^\dagger} = 1 - \pi_K, & 0 = N\pi_K, \\ \pi_I = \pi_I^2, & \pi_I = (\pi_I)^\dagger, & \pi_{I^\dagger} = 1 - \pi_I, & 0 = \pi_{I^\dagger} N, \\ \pi_{K^\dagger} = N^{-1}N, & \pi_I = NN^{-1}, & 0 = \pi_K N^{-1}, & 0 = N^{-1}\pi_{I^\dagger}. \end{array} \tag{3.25}$$

For our application we want to evaluate holomorphic and antiholomorphic derivatives. So, for $\delta_1 N$ and $\delta_2 N$ any two variations of N, let $\delta V = \delta_1 V + i\delta_2 V$ and $\overline{\delta} V = \delta_1 V - i\delta_2 V$ for V equal to N, N^{-1}, or any of the projection operators. By requiring that the defining equations (3.25) remain true when one subjects N to a variation we find that the variations of π_K, π_I, and N^{-1} are given by

$$\delta\pi_I = [\pi_{I\dagger}\delta N N^{-1}] + [\pi_{I\dagger}\overline{\delta} N N^{-1}]^\dagger, \tag{3.26.1}$$

$$\delta\pi_K = -[N^{-1}\delta N\pi_K] - [N^{-1}\overline{\delta} N\pi_K]^\dagger, \tag{3.26.2}$$

$$\begin{aligned}\delta N^{-1} &= -N^{-1}\delta N N^{-1} + N^{-1}\delta\pi_I + \delta\pi_{K\dagger}N^{-1} \\ &= -N^{-1}\delta N N^{-1} + N^{-1}N^{-1\dagger}(\overline{\delta} N)^\dagger\pi_{I\dagger} \\ &\quad + \pi_K(\overline{\delta} N)^\dagger N^{-1\dagger}N^{-1}.\end{aligned} \tag{3.26.3}$$

Note that to derive (3.26) we had to assume that the operators vary smoothly, which is the condition that the dimensions of I and K do not jump discontinuously as we vary N. This is equivalent to $\pi_{I\dagger}\delta N\pi_K = 0$.

We would like to apply the above results to the problem at hand. For every point $(A, J) \in \mathscr{A} \times \mathscr{T}$, we obtain an operator $T_z = N : \mathbf{g}_c \to T^{(1,0)}\mathscr{A}$. Since the range of T_z depends on the complex structure J, we think of $\mathscr{H}_-$ and $\mathscr{H}_+$ as bundles over $\mathscr{A} \times \mathscr{T}$. The variation of T_z must be defined using a covariant derivative. $\mathscr{H}_-$ is the trivial bundle with fiber $\mathbf{g}_c$ and we give it the trivial connection. The fibers of $\mathscr{H}_+$ are $\mathscr{H}_+|_{(A,J)} = T_A^{(1,0)}\mathscr{A}$. Since $\mathscr{H}_+$ is a subbundle of the trivial bundle $T\mathscr{A} \otimes \mathbb{C} \times \mathscr{T} \to \mathscr{A} \times \mathscr{T}$, we may give it the induced Hermitian metric and the projected connection. These are compatible. In the J directions the projected connection is defined by

$$\delta^{\mathscr{H}_+} \equiv \pi_z\delta^{T\mathscr{A}}. \tag{3.27}$$

Since there is no twisting in the $\mathscr{A}$ directions, we have

$$\nabla^{\mathscr{H}_+} \equiv \pi_z\nabla = \nabla. \tag{3.28}$$

From now on we drop the superscript $\mathscr{H}_+$ in the notation for the connection on $\mathscr{H}_+$.

The variational formulas that we wish to obtain are greatly simplified due to the fact that T_z depends holomorphically on A and J, where the holomorphic structure on $\mathscr{H}_+$ is defined by requiring that holomorphic sections are annihilated by the $(0, 1)$ piece of the connection on $\mathscr{H}_+$. Holomorphicity of T_z in A follows from the assumption that $\mathscr{G}$ acts holomorphically on $\mathscr{A}$. To show that T_z is also holomorphic as a function of J we must show that $\delta^{(0,1)}T^{\underline{i}}_{za} = 0$. This follows from

$$\delta T^{\underline{i}}_{za} = -\frac{i}{2}\delta J^{\underline{i}}{}_j T^j{}_a = -\frac{i}{2}\delta J^{\underline{i}}{}_{\overline{j}} T^{\overline{j}}_{\overline{z}a} = \delta^{(1,0)}T^{\underline{i}}_{za}. \tag{3.29}$$

A further simplification of the variational formulas arises by observing that $\mathrm{Ker}(T_z) = \mathrm{Ker}(T) \otimes \mathbb{C}$ is independent of J. Therefore $\delta^{(1,0)}\pi_K = \delta^{(0,1)}\pi_K = 0$.

Recall that $\mathscr{K} = \pi_{I^\dagger}$. By a simple plug into (3.26), we obtain

$$\nabla_{\underline{i}}\mathscr{K} = -\mathscr{K}[(\nabla_{\underline{i}}T_z)T_z^{-1}], \tag{3.30.1}$$

$$\nabla_{\bar{i}}\mathscr{K} = [\nabla_{\underline{i}}\mathscr{K}]^\dagger = -[(\nabla_{\underline{i}}T_z)T_z^{-1}]^\dagger\mathscr{K}, \tag{3.30.2}$$

$$\nabla_{\underline{i}}\pi_K = -[T_z^{-1}(\nabla_{\underline{i}}T_z)]\pi_K, \tag{3.30.3}$$

$$\nabla_{\bar{i}}\pi_K = [\nabla_{\underline{i}}\pi_K]^\dagger = -\pi_K[T_z^{-1}(\nabla_{\underline{i}}T_z)]^\dagger\mathscr{K}, \tag{3.30.4}$$

$$\nabla_{\underline{i}}T_z^{-1} = -T_z^{-1}[(\nabla_{\underline{i}}T_z)T_z^{-1}], \tag{3.30.5}$$

$$\nabla_{\bar{i}}T_z^{-1} = -T_z^{-1}(\nabla_{\bar{i}}\mathscr{K})\mathscr{K} - \pi_K(\nabla_{\bar{i}}\pi_K)T_z^{-1}, \tag{3.30.6}$$

$$\delta^{(1,0)}\mathscr{K} = \frac{i}{2}\mathscr{K}\delta J T_{\bar{z}}T_z^{-1}, \tag{3.30.7}$$

$$\delta^{(0,1)}\mathscr{K} = -\frac{i}{2}[\delta J T_{\bar{z}}T_z^{-1}]^\dagger\mathscr{K} = -\frac{i}{2}T_zT_{\bar{z}}^{-1}\delta J\mathscr{K}, \tag{3.30.8}$$

$$\delta^{(1,0)}T_z^{-1} = \frac{i}{2}T_z^{-1}\delta J T_{\bar{z}}T_z^{-1}, \tag{3.30.9}$$

$$\delta^{(0,1)}T_z^{-1} = -T_z^{-1}\delta^{(0,1)}\mathscr{K} = \frac{i}{2}T_{\bar{z}}^{-1}\delta J\mathscr{K}. \tag{3.30.10}$$

To obtain the second equality in (3.30.8) we observe that $\delta J^\dagger = \delta J$ and $[T_{\bar{z}}T_z^{-1}]^\dagger = T_zT_{\bar{z}}^{-1}$, which follows from $T_z^\dagger T_z = T_{\bar{z}}^\dagger T_{\bar{z}}$.

3c. Intrinsic expression for $\mathscr{O}$. In this section, we will be establishing correspondences between $\mathscr{G}$-invariant objects on $F^{-1}(0)$ and their push forwards to $\mathscr{M}$. We have been using indices i, j, etc. for $T\mathscr{A}$, $\bar{i}$, $\bar{j}$, etc. for $T^{(0,1)}\mathscr{A}$, and $\underline{i}$, $\underline{j}$, etc. for $T^{(1,0)}A$. It is convenient to similarly use I, J, etc. for $T\mathscr{M}$, $\bar{I}$, $\bar{J}$, etc. for $T^{(0,1)}\mathscr{M}$, and $\underline{I}$, $\underline{J}$ for $T^{(1,0)}\mathscr{M}$.

By the identification of $\widehat{T\mathscr{M}}$ as the pullback of $T\mathscr{M}$, a vector field $\tilde{v}^I$ on $\mathscr{M}$ corresponds to a vector field v^i on $F^{-1}(0)$ which is $\mathscr{G}$-invariant and lies in $\widehat{T\mathscr{M}}$. Similarly, a $(1,0)$ vector field $\tilde{w}^{\underline{I}}$ corresponds to a $\mathscr{G}$-invariant vector field $w^{\underline{i}}$ which lies in $\widehat{T\mathscr{M}}^{(1,0)}$. As operators on $\Gamma(\mathscr{M}, \tilde{\mathscr{L}})$, which is identified with $\Gamma(F^{-1}(0), \mathscr{L})^{\mathscr{G}}$, the operator $\tilde{v}^I\tilde{\nabla}_I$ corresponds to the operator $v^i\nabla_i$.

Observe that the orthogonally projected connection for $\widehat{T\mathscr{M}}^{(1,0)}$ as a subbundle of $T\mathscr{A}$ is unitary, holomorphic, and $\mathscr{G}$-invariant. Therefore it pushes down to a unitary holomorphic connection on $T^{(1,0)}\mathscr{M}$. The unique such connection is the Riemannian connection on $\mathscr{M}$, which we will also call $\tilde{\nabla}$. In component notation, this says that the vector field $\tilde{v}^I(\tilde{\nabla}_I \tilde{w}^{\underline{J}})$ lying in $T^{(1,0)}\mathscr{M}$ corresponds to the vector field $v^i(\mathscr{K}^{\underline{j}}{}_{\underline{k}}\nabla_i w^{\underline{k}})$ lying in $\widehat{T\mathscr{M}}^{(1,0)}$.[5]

In order to understand what differential operator on $\mathscr{M}$ the leading order term in the expression (3.23) for $\mathscr{O}$ corresponds to, we define $B^{\underline{I}\underline{J}} = -\frac{1}{4}(\delta\tilde{J}\tilde{\omega}^{-1})^{\underline{I}\underline{J}}$. Observe that $B^{\underline{I}\underline{J}}$ corresponds to $\mathscr{K}^{\underline{i}}{}_{\underline{k}}M^{\underline{k}\underline{l}}\mathscr{K}^{\underline{j}}{}_{\underline{l}}$ under the identification of $\Gamma(\mathscr{M}, \mathrm{Sym}^2(T^{(1,0)}\mathscr{M}))$ with $\Gamma(F^{-1}(0), \mathrm{Sym}^2(\widehat{T\mathscr{M}}^{(1,0)}))^{\mathscr{G}}$. The operator $\tilde{\nabla}_{\underline{J}}B^{\underline{J}\underline{I}}\tilde{\nabla}_{\underline{I}}$ on $\Gamma(\mathscr{M}, \tilde{\mathscr{L}})$ corresponds to the following operator on $\Gamma(F^{-1}(0), \mathscr{L})^{\mathscr{G}}$:

$$\begin{aligned}\mathscr{K}^{\underline{k}}{}_{\underline{j}}\nabla_{\underline{k}}(\mathscr{K}M\mathscr{K})^{\underline{j}\underline{i}}\mathscr{K}^{\underline{l}}{}_{\underline{i}}\nabla_{\underline{l}} = {}&\mathscr{K}^{\underline{k}}{}_{\underline{j}}\nabla_{\underline{k}}M^{\underline{j}\underline{i}}\mathscr{K}^{\underline{l}}{}_{\underline{i}}\nabla_{\underline{l}}\\ &+\mathscr{K}^{\underline{k}}{}_{\underline{j}}(\nabla_{\underline{k}}\mathscr{K}^{\underline{j}}{}_{\underline{m}})M^{\underline{m}\underline{i}}\mathscr{K}^{\underline{l}}{}_{\underline{i}}\nabla_{\underline{l}}.\end{aligned} \tag{3.31}$$

Comparing with the expression (3.23) for $\mathscr{O}$ as an operator on $F^{-1}(0)$, we see that $\mathscr{O}$ written as an operator on $\mathscr{M}$ is

$$\mathscr{O} = \tilde{\nabla}_{\underline{J}}B^{\underline{J}\underline{I}}\tilde{\nabla}_{\underline{I}} + \tilde{W}^{\underline{L}}\tilde{\nabla}_{\underline{L}} + \lambda, \tag{3.32}$$

where

$$\begin{aligned}&\lambda \leftrightarrow \frac{i}{4}\mathrm{Tr}(T_z^{-1}\delta J T_{\bar{z}})\\ &\tilde{W}^{\underline{L}} \leftrightarrow W^{\underline{l}} \equiv M^{\underline{i}\underline{j}}(\nabla_{\underline{i}}\mathscr{K}^{\underline{k}}{}_{\underline{j}})\mathscr{K}^{\underline{l}}{}_{\underline{k}} - \mathscr{K}^{\underline{k}}{}_{\underline{j}}(\nabla_{\underline{k}}\mathscr{K}^{\underline{j}}{}_{\underline{m}})M^{\underline{m}\underline{i}}\mathscr{K}^{\underline{l}}{}_{\underline{i}}\end{aligned} \tag{3.33}$$

is the correspondence between the function λ on $\mathscr{M}$ and a $\mathscr{G}$-invariant function on $F^{-1}(0)$ and between the $(1,0)$ vector field $\tilde{W}$ on $\mathscr{M}$ and a $\mathscr{G}$-invariant section of $\widehat{T\mathscr{M}}^{(1,0)}$.

To simplify the expression for λ, let $H \equiv \det{}'(\triangle) = \det(\triangle + \pi_K)$, where the prime on the first determinant tells us to take the determinant of $\triangle$ as a linear transformation on the orthocomplement of its zero modes. Observe that

$$\delta^{(1,0)}(\ln H) = \mathrm{Tr}(\triangle^{-1}\delta^{(1,0)}\triangle) = -\frac{i}{2}\mathrm{Tr}(T_z^{-1}\delta J T_{\bar{z}}). \tag{3.34}$$

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[5]We have actually been slightly imprecise in the description above. In order for a connection on $F^{-1}(0)$ to push down it must not only be $\mathscr{G}$-invariant, but also satisfy the condition that the components of the connection in the $\mathscr{G}$ directions annihilate $\mathscr{G}$-invariant sections. This latter condition is not necessarily satisfied by the orthogonally projected connection. It is satisfied, however, by a connection which only differs from the orthogonally projected connection in the $\mathscr{G}$ directions. Since the pullback of v^I has no component in the $\mathscr{G}$ directions, we see that $\tilde{v}^I(\tilde{\nabla}_I\tilde{w}^{\underline{J}})$ corresponds to $v^i(\mathscr{K}^{\underline{j}}{}_{\underline{k}}\nabla_i w^{\underline{k}})$.

Here we have used the identity

$$T_z{}^{-1} = \Delta^{-1} T_z{}^{\dagger} \tag{3.35}$$

and the fact that $\delta^{(1,0)} T_z{}^{\dagger} = 0$, and $\delta^{(1,0)} T_z = -\frac{i}{2}\delta J T_{\bar{z}}$.

To summarize, we have shown so far that, as a differential operator on $\mathscr{M}$,

$$\mathscr{O} = \tilde{\nabla}_{\underline{I}} B^{\underline{I}\underline{J}} \tilde{\nabla}_{\underline{J}} + \tilde{W}^{\underline{J}} \nabla_{\underline{J}} - \tfrac{1}{2}\delta^{(1,0)}(\ln H), \tag{3.36}$$

where $B^{\underline{I}\underline{J}} = -\frac{1}{4}(\delta \tilde{J} \tilde{\omega}^{-1})^{\underline{I}\underline{J}}$. In the next section we will take this as an ansatz for the connection one-form for the connection $\delta^{\tilde{\mathscr{H}}_Q}$. We will then be able to solve uniquely for $\tilde{W}$ so that $\delta^{\tilde{\mathscr{H}}_Q}$ is holomorphicity preserving. The solution is

$$\tilde{W}^{\underline{J}} = \frac{i}{2}\tilde{\omega}^{\underline{J}\overline{K}} \tilde{\nabla}_{\overline{K}} \delta^{(1,0)} \ln(H) - \tilde{\nabla}_{\underline{I}} B^{\underline{I}\underline{J}} = B^{\underline{I}\underline{J}} \tilde{\nabla}_{\underline{I}} \ln(H). \tag{3.37}$$

(The equality of the last two expressions is shown in §3.)

We shall complete this section by proving the first equality in (3.37). Although, for $\mathscr{A}$ finite dimensional, this result follows from the calculations of §4, we prove it here as an illustration of the use of the identities of §3b for the derivatives of the Green's function and projection operators. Also, an appropriately regularized version of the calculations below is needed as part of the demonstration in §7 that the zeroth order piece of the $(2, 0)$ curvature of $\delta^{\tilde{\mathscr{H}}_Q}$ vanishes.

We will need the following three identities which follow from $\mathscr{G}$-invariance of M and ω and from selfadjointness of the projection operator $\mathscr{K}$:

$$0 = -\mathscr{L}_{T_a} M^{\underline{k}\underline{l}} = (\nabla_{\underline{i}} T^{\underline{k}}_{z\,a}) M^{\underline{i}\underline{l}} + (\nabla_{\underline{i}} T^{\underline{l}}_{z a}) M^{\underline{k}\underline{i}}, \tag{3.38}$$

$$0 = \mathscr{L}_{T_a} \omega_{\bar{k}\underline{i}} = (\nabla_{\underline{i}} T^{\underline{m}}_{z\;a}) \omega_{\bar{k}\underline{m}} - (\nabla_{\bar{k}} T^{\overline{m}}_{\bar{z}\;a}) \omega_{\underline{i}\overline{m}}, \tag{3.39}$$

$$\overline{\mathscr{K}} \omega^{-1} = \omega^{-1} \mathscr{K}^T. \tag{3.40}$$

Let

$$\tilde{W}_0^{\underline{J}} = \tilde{W}^{\underline{J}} + \tilde{\nabla}_{\underline{I}} B^{\underline{I}\underline{J}}. \tag{3.41}$$

The corresponding tensor upstairs is

$$W_0^{\underline{j}} = W^{\underline{j}} + \nabla_{\underline{i}} (\mathscr{K} M \mathscr{K}^T)^{\underline{i}\underline{j}}. \tag{3.42}$$

So the identity we wish to show is

$$W_0^{\underline{j}} = \frac{i}{2} \mathscr{K}^{\underline{j}}{}_{\underline{l}} \omega^{\underline{l}\bar{k}} \nabla_{\bar{k}} \delta^{(1,0)} \ln H. \tag{3.43}$$

Commuting $\delta^{(1,0)}$ to the left of $\nabla_{\bar{k}}$, remembering that it acts on the $\bar{k}$ index, we see that we must show

$$W_0^{\underline{j}} = \frac{i}{2}\mathscr{K}^{\underline{j}}{}_{\underline{l}}\omega^{\underline{l}\bar{k}}\left[-\frac{i}{2}\delta J^{\underline{i}}{}_{\bar{k}}\nabla_{\underline{i}}\ln H + \delta^{(1,0)}\nabla_{\bar{k}}\ln H\right]. \tag{3.44}$$

Plugging (3.33) for W into (3.42) and evaluating the derivative of $\mathscr{K}$, we obtain

$$W_0^{\underline{j}} = X^{\underline{j}} + Y^{\underline{j}}, \tag{3.45}$$

where

$$X^{\underline{j}} = -[\mathscr{K}(\nabla_{\underline{i}}T_z)T_z^{-1}M]^{\underline{j}\underline{i}} = -\mathscr{K}^{\underline{j}}{}_{\underline{k}}[(\nabla_{\underline{i}}T^{\underline{k}}_{z\,a})M^{\underline{i}\underline{l}}](T_z^{-1})^a{}_{\underline{l}}, \tag{3.46}$$

$$\begin{aligned} Y^{\underline{j}} &= -[\mathscr{K}(\nabla_{\underline{i}}T_z)T_z^{-1}M\mathscr{K}^T]^{\underline{j}\underline{i}} \\ &= \frac{i}{2}\mathscr{K}^{\underline{j}}{}_{\underline{l}}\omega^{\underline{l}\bar{k}}[\omega_{\bar{k}\underline{m}}\nabla_{\underline{i}}T^{\underline{m}}_{z\,a}]\left(-\frac{i}{2}T_z^{-1}\delta J\omega^{-1}\mathscr{K}^T\right)^{a\underline{i}}. \end{aligned} \tag{3.47}$$

Using (3.38) to rewrite X, we find

$$X^{\underline{j}} = \mathscr{K}^{\underline{j}}{}_{\underline{k}}M^{\underline{k}\underline{i}}\operatorname{Tr}[(\nabla_{\underline{i}}T_z)T_z^{-1}] = \mathscr{K}^{\underline{j}}{}_{\underline{k}}M^{\underline{k}\underline{i}}\nabla_{\underline{i}}\ln H. \tag{3.48}$$

Using (3.39) on the term in square brackets in the expression (3.48) for Y and (3.40) on the term in round brackets, we have

$$\begin{aligned} Y^{\underline{j}} &= \frac{i}{2}\mathscr{K}^{\underline{j}}{}_{\underline{l}}\omega^{\underline{l}\bar{k}}[\omega_{\underline{i}\bar{m}}\nabla_{\bar{k}}T^{\bar{m}}_{\bar{z}\,a}](-\frac{i}{2}T_z^{-1}\delta J\overline{\mathscr{K}}\omega^{-1})^{a\underline{i}} \\ &= \frac{i}{2}\mathscr{K}^{\underline{j}}{}_{\underline{l}}\omega^{\underline{l}\bar{k}}\operatorname{Tr}\left[(\nabla_{\bar{k}}T_{\bar{z}})\left(-\frac{i}{2}T_z^{-1}\delta J\overline{\mathscr{K}}\right)\right]. \end{aligned} \tag{3.49}$$

Making use of the complex conjugate of the variational formula (3.30.10) for T_z^{-1}, yields

$$\begin{aligned} Y &= \frac{i}{2}\mathscr{K}^{\underline{j}}{}_{\underline{l}}\omega^{\underline{l}\bar{k}}\delta^{(1,0)}\operatorname{Tr}[(\nabla_{\bar{k}}T_{\bar{z}})T_{\bar{z}}^{-1}] \\ &= \frac{i}{2}\mathscr{K}^{\underline{j}}{}_{\underline{l}}\omega^{\underline{l}\bar{k}}\delta^{(1,0)}\nabla_{\bar{k}}\ln H. \end{aligned} \tag{3.50}$$

Plugging in (3.48) and (3.50) for X and Y into $W_0 = X + Y$, we see that we have proved (3.44).

4. Proof of the basic properties of $\delta^{\tilde{\mathscr{H}}_Q}$

In the previous section we saw that the desired connection can be expressed purely in terms of intrinsic objects on $\mathscr{M}$ and a regularized determinant of the Laplacian. This motivates us to study the role that this determinant plays in relation to the intrinsic geometry of $\mathscr{M}$. This can be understood in the framework of determinant line bundles as worked out by Quillen, Bismut, and Freed [27], [5]. In this framework, they give a proof of the local family index theorem for the first Chern class of the index bundle of a holomorphic family of operators. Of course these constructions have been known to physicists for a long time from the study of anomalies. This particular index theorem would be phrased by a physicist as the calculation of the anomaly in holomorphic factorization [1]. We present the result in the Quillen formalism in order to highlight the geometrical concepts.

For our problem, we apply the index theorem to the family T_z of operators over $B = \mathscr{M} \times \mathscr{T}$. The cokernel of this family is the tangent bundle to the fibers of the projection $\mathscr{M} \times \mathscr{T} \to \mathscr{T}$. If the $\mathscr{G}$-action is generically free, the kernel vanishes, and in that case the determinant line bundle of the family T_z is the canonical bundle $K_{\mathscr{M}}$ of $\mathscr{M}$, as a holomorphic line bundle over B. The Ray-Singer-Quillen metric on $K_{\mathscr{M}}$ is defined as $H = \det'(T_z{}^{\dagger} T_z)$ times the metric on $K_{\mathscr{M}}$ given by the Kähler structure on $\mathscr{M}$. We denote by ∇^Q the unique connection compatible with the Quillen metric and the holomorphic structure on $K_{\mathscr{M}}$. The local index theorem yields an expression for the curvature of ∇^Q. For our problem, this curvature is of the form used in the proof of Theorem 1 below.

It turns out that we will be able to demonstrate most of the properties of $\delta^{\tilde{\mathscr{H}}_Q}$ only using the intrinsic identities following from the index theorem as well as the additional fact that the $(2, 0)$ piece of $\delta \tilde{J} \tilde{\omega}^{-1}$ is holomorphic. (This latter fact follows easily from geometric invariant theory or by the Hodge theory methods of §3.) As a consequence, these considerations apply in a more general context than that of symplectic quotients.

To state the result, let $(\mathscr{M}, \tilde{\omega})$ be a symplectic manifold. We assume $\mathscr{M}$ has a prequantum line bundle $\tilde{\mathscr{L}}$, that is a Hermitian line bundle with compatible connection $\tilde{\nabla}$ which has curvature $-i\tilde{\omega}$. Let $\mathscr{T}$ be a holomorphic family of Kähler structures with Kähler form $\tilde{\omega}$. We write $\pi_{\mathscr{M}}$ and $\pi_{\mathscr{T}}$ for the projection operators from $B \equiv \mathscr{M} \times \mathscr{T}$ to $\mathscr{M}$ and $\mathscr{T}$, respectively.

We assume $\mathscr{M}$ is compact, although this assumption is not essential. We assume it here only so that we may state the second assumption of Theorem 1 in a simple topological manner and so that we may forego questions about domains of operators.

Let $\tilde{\mathscr{H}}_{\mathrm{pr}}$ be the prequantum bundle; that is the trivial bundle over $\mathscr{T}$ with fiber the sections of $\tilde{\mathscr{L}} \to \mathscr{M}$. Let $\tilde{\mathscr{H}}_Q$ be the quantum bundle, that is the subbundle of $\tilde{\mathscr{H}}_{\mathrm{pr}}$ whose fiber $\tilde{\mathscr{H}}_Q|_J$ consists of holomorphic sections of $\tilde{\mathscr{L}}$.

Finally, let $\underline{I}$, $\underline{J}$, etc. be indices for $T^{(1,0)}\mathscr{M}$ and $T^{*(1,0)}\mathscr{M}$, and let $\bar{I}$, $\bar{J}$, etc. be indices for $T^{(0,1)}\mathscr{M}$ and $T^{*(0,1)}\mathscr{M}$.

Theorem 1. *Assume the following*:

1. *For* $\delta\tilde{J}$ *any variation of the complex structure in the family* $\mathscr{T}$, $(\delta\tilde{J}^{(1,0)})\tilde{\omega}^{-1}$ *is holomorphic as a section of* $\mathrm{Sym}^2(T^{(1,0)}\mathscr{M})$.

2. *As a holomorphic line bundle over* B, *some power of the canonical line bundle of* $\mathscr{M}$ *is isomorphic to the tensor product of some power of* $\tilde{\mathscr{L}}$ *with a line bundle pulled back from* $\mathscr{T}$ *and a line bundle admitting a flat metric. For fixed* $J\in\mathscr{T}$ *the first Chern class of* $K_{\mathscr{M}}\oplus\tilde{\mathscr{L}}^2$ *does not vanish* (*so that one can divide by* $1+r$ *in what follows*).

Then there exists a connection $\delta^{\tilde{\mathscr{H}}_Q}$ *which is compatible with the natural holomorphic structure on* $\tilde{\mathscr{H}}_Q$ *and whose connection one-form is a second-order differential operator on* $\mathscr{M}$. *The* $(0,2)$ *curvature of* $\delta^{\tilde{\mathscr{H}}_Q}$ *vanishes. The* $(1,1)$ *curvature is a purely projective factor, i.e., the components of* $R^{(1,1)}$ *are multiples of the identity operator on* $\tilde{\mathscr{H}}_Q$. *The* $(2,0)$ *curvature is a first-order differential operator which preserves holomorphicity.*

Sketch of the proof. The first sentence of assumption (2) is equivalent to the condition that the canonical bundle $K_{\mathscr{M}}$ of $\mathscr{M}$, considered as a holomorphic vector bundle over B, admits a connection ∇^Q which is compatible with the holomorphic structure and which has curvature of the form

$$2r\pi^*_{\mathscr{M}}(-i\tilde{\omega})+\pi^*_{\mathscr{T}}(\rho), \tag{4.1}$$

where r is a real number and ρ is a $(1,1)$-form on $\mathscr{T}$.

For a given choice of ∇^Q, let $(\mathscr{J},\lambda)$ be the $(1,0)$-form on $\mathscr{M}\times\mathscr{T}$ which is the difference between ∇^Q and the connection induced on $K_{\mathscr{M}}$ by the Kähler structure on $\mathscr{M}$. The connection $\delta^{\tilde{\mathscr{H}}_Q}$ may then be written in the form:

$$\delta^{\tilde{\mathscr{H}}_Q}=\delta-\frac{1}{1+r}\mathscr{O},$$
$$\mathscr{O}=\tilde{\nabla}_{\underline{I}}B^{\underline{I}\underline{J}}\tilde{\nabla}_{\underline{J}}+B^{\underline{I}\underline{J}}\mathscr{J}_{\underline{I}}\tilde{\nabla}_{\underline{J}}-\tfrac{1}{2}\lambda,$$

where δ is the trivial connection on $\tilde{\mathscr{H}}_{\mathrm{pr}}$ and $B^{\underline{I}\underline{J}}=-\frac{1}{4}(\delta\tilde{J}\tilde{\omega}^{-1})^{\underline{I}\underline{J}}$.

As written, $\delta^{\tilde{\mathscr{H}}_Q}$ is manifestly a connection on $\tilde{\mathscr{H}}_{\mathrm{pr}}$. The calculation that this restricts to a connection on the subbundle $\tilde{\mathscr{H}}_Q$ is given in §4b (under the heading "holomorphicity preservation"). Since the connection one-form $\mathscr{O}$ is purely $(1,0)$, $\delta^{\tilde{\mathscr{H}}_Q}$ is compatible with the natural holomorphic structure on $\tilde{\mathscr{H}}_Q$ (defined by the $\overline{\partial}$-operator $\delta^{(0,1)}$) and $R^{(0,2)}$ vanishes.

The $(1,1)$ curvature $R^{(1,1)}$ is $\frac{1}{1+r}\frac{1}{2}\rho$ as calculated in §4b (under the heading "$(1,1)$ curvature"). This is clearly purely projective. That $R^{(2,0)}$ is a first order operator is also calculated in §4b (under the heading "$(2,0)$ curvature"). The fact that $R^{(2,0)}$ preserves holomorphicity follows trivially from the fact that $\delta^{\tilde{\mathscr{H}}_Q}$ does. q.e.d.

We should note that the calculations of §4b referred to above are actually in a notation specialized to the case where $\mathscr{M}$ is an affine symplectic quotient. Assumption (2) follows in this case (with a slight additional assumption on the $\mathscr{G}$ action) since, as described in §4a, we may take the connection ∇^Q used in the proof of Theorem 1 to be the Quillen connection. In the notation of the proof, it may be defined by

$$(\mathscr{J}_{\underline{I}}, \lambda) = (\tilde{\nabla}_{\underline{I}} \ln H, \delta^{(1,0)} \ln H).$$

The general proofs just amount to a slight change of notation where we allow for arbitrary $(\mathscr{J}, \lambda)$ rather than specializing to the values appropriate for the Quillen connection.

In the case when $\mathscr{A}$ is finite dimensional, the calculations of §3 show that the connection constructed in the proof of Theorem 1 is the connection on $\tilde{\mathscr{H}}_Q$ obtained by pushing down the natural connection for quantization of $\mathscr{A}$. In this case, we have

$$\rho = -\tfrac{1}{4}\mathrm{Tr}(\delta J \wedge \delta J \pi_z), \quad r = 0, \quad \text{and} \quad R^{(1,1)} = \tfrac{1}{2}\rho.$$

The real interest of this paper comes from the gauge theory problem (described in detail in §2). Our principal results are summarized in the following theorem. (We must point out that, although we give two arguments for the vanishing of the $(2,0)$ curvature, there are technical details in both proofs which we do not provide here.)

Theorem 2. *Let $\mathscr{M}$ be the moduli space of flat connections on a principal bundle E, with compact structure group G, over a closed oriented 2-manifold Σ of genus g. Equip $\mathscr{M}$ with the symplectic form $\tilde{\omega} = k\tilde{\omega}_0$, where $\tilde{\omega}_0$ is the basic symplectic form on $\mathscr{M}$ (described in §2). Define $\Gamma_{\Sigma,E}$ to be the subgroup of the mapping class group of Σ which leaves $\mathscr{M}$ invariant, i.e., which fixes the topological type of E. Let $\tilde{\mathscr{L}}$ be a prequantum line bundle on $\mathscr{M}$ which has an action of $\Gamma_{\Sigma,E}$ (lifting the action on $\mathscr{M}$ and preserving the connection and Hermitian metric). Let $\mathscr{T}$*

denote the Teichmüller space of Σ, *considered as a family of positive complex structures on* $\mathscr{M}$ *compatible with* $\tilde{\omega}$.

In this case, there exists a projectively flat connection $\delta^{\tilde{\mathscr{H}}_Q}$ *on* $\tilde{\mathscr{H}}_Q$ *which is compatible with the natural holomorphic structure on* $\tilde{\mathscr{H}}_Q$ *and whose connection one-form is a second-order differential operator on* $\mathscr{M}$. *Furthermore,* $\delta^{\tilde{\mathscr{H}}_Q}$ *has curvature of type* $(1, 1)$.

The connection $\delta^{\tilde{\mathscr{H}}_Q}$ *is invariant under* $\Gamma_{\Sigma, E}$ *and therefore determines a projective representation of* $\Gamma_{\Sigma, E}$.

Sketch of the proof. The case when Σ has genus one is special because the $\mathscr{G}$-action at a generic flat connection has an isotropy subgroup. We describe that case explicitly in the next section. We restrict ourselves here to the case when $g > 1$, although many of the constructions below apply to the genus one case.

Let $\mathscr{M}^s$ be the submanifold of $\mathscr{M}$ consisting of the points where $\mathscr{M}$ is smooth. Since $\mathscr{M}$ is a normal projective algebraic variety, any holomorphic function (or section of $\tilde{\mathscr{L}}$) which is defined on $\mathscr{M}^s$ has a unique extension to a continuous function (section) on all of $\mathscr{M}$. Thus, the quantum Hilbert bundle for quantization of $\mathscr{M}$ may be identified with that for quantization of $\mathscr{M}^s$. Therefore to define a connection for quantization of $\mathscr{M}$, we may restrict ourselves to $\mathscr{M}^s$.

To construct a connection which has vanishing $(0, 2)$ curvature, purely projective $(1, 1)$ curvature, and $(2, 0)$ curvature which is a first-order operator, it is sufficient to define the Quillen connection ∇^Q and show it has curvature of the form required for the proof of Theorem 1. Invariance under $\Gamma_{\Sigma, E}$ will follow from the naturality of the construction.

In order to define the Quillen connection, we need not only a complex structure on Σ but in fact a metric. A complex structure, however, can be considered to canonically determine a metric, for instance, the constant curvature metric or the Arakelov metric. Given a metric on Σ and flat connection A on E, one has the Laplacian $\triangle \equiv \partial_A{}^\dagger \partial_A$ acting on $\Gamma(\Sigma, \mathrm{ad}(E))$. We define H as the zeta function regularized determinant of $\triangle$. (For $g = 1$ we take the determinant of $\triangle$ restricted to the orthocomplement of its zero modes.) From this definition, it is clear that H only depends on the isotopy class of the complex structure on Σ, and is invariant under $\Gamma_{\Sigma, E}$.

We define the Ray-Singer-Quillen metric to be H times the metric on $\mathscr{K}_{\mathscr{M}}$ given by the Kähler structure on $\mathscr{M}$. The Quillen connection is defined as the unique connection compatible with the Ray-Singer-Quillen metric and holomorphic structure on $K_{\mathscr{M}}$. As discussed in §4a, ∇^Q

has curvature of the form stated in the proof of Theorem 1 with $\rho = c_1(\mathrm{Ind}(\overline{\partial}))$ and $r = h/k$. The rigorous analysis that ∇^Q has the curvature claimed is contained in [27, 5].

The vanishing of the $(2, 0)$ curvature follows on purely global grounds if $H^0(\mathscr{M}^s, T^{(1,0)}) = 0$. For in such a case $R^{(2,0)}$ is a two-form on $\mathscr{T}$ with values in the holomorphic functions on $\mathscr{M}^s$, and these must be constant (on each connected component of $\mathscr{M}$) because of normality of $\mathscr{M}$. Thus, if $H^0(\mathscr{M}^s, T^{(1,0)})$ vanishes, then $R^{(2,0)}$ is purely projective, permitting us to conclude that $\delta^{\mathscr{H}_Q}$ is projectively flat. Actually, the Bianchi identity for $\delta^{\mathscr{H}_Q}$ shows in this situation that $R^{(2,0)}$ is a closed $(2, 0)$-form on $\mathscr{T}$, which descends to a closed $(2, 0)$-form on the quotient $\mathscr{T}/\Gamma_{\Sigma,E}$. If $\Gamma_{\Sigma,E} = \Gamma_\Sigma$, which is true, for instance, if G is connected, then $\mathscr{T}/\Gamma_{\Sigma,E}$ is the moduli space of curves. In this case, by a result of Harer [17], $R^{(2,0)}$ must vanish.

The key step in the above argument, the vanishing of $H^0(\mathscr{M}^s, T^{(1,0)})$, was proved by Narasimhan and Ramanan [25] for certain nonsingular components of moduli spaces (in fact, for $G = SU(N)/\mathbb{Z}_n$, $g \geq 2$, and E of a nontrivial topological type obeying certain restrictions). Hitchin [19] gave an alternative proof of this result for $G = SO(3)$ and E a bundle of nonzero second Stieffel-Whitney class. The main requirement in Hitchin's proof is the construction of a moduli space $\mathscr{M}_H$ of "Higgs bundles", and the construction of a proper map from this moduli space to a vector space V of generalized quadratic differentials. As these steps have been carried out by Simpson for arbitrary compact G [35], the vanishing of $H^0(\mathscr{M}^s, T^{(1,0)})$ holds as long as $\mathscr{M} = \mathscr{M}^s$. Even for the singular components of the moduli space, essentially the same argument should give $H^0(\mathscr{M}^s, T^{(1,0)}) = 0$, since the singularities of $\mathscr{M}_H$ lie above a subspace of V of rather high codimension. However, we will not try to make this argument rigorous here.

The argument of the previous two paragraphs uses global facts about $\mathscr{M}$. In the spirit of this paper, a proof using local differential geometry is more natural. Under the hypotheses of the theorem, we will show at the end of this section that $R^{(2,0)}$ vanishes if its zeroth order piece vanishes as a function on $\mathscr{M}$. This is equivalent to a certain differential equation (7.1) for the determinant of the Laplacian. In §7 we present the proof of this equation in the finite-dimensional case in a way that should carry over to the gauge theory case. To complete the details of this rather technical proof, we would need to define a regularization of the expressions involving Green's functions which appear there such that the manipulations carried out are valid with the regularization in place. Although we do not carry out the analysis here, it is clear (based on the vast experience of physicists with these type of manipulations) that no essential complications would arise. q.e.d.

The connection $\delta^{\tilde{\mathscr{H}}_Q}$ that we have constructed actually coincides with the genus g analogs of the differential equations originally introduced by Knizhnik and Zamolodchikov [21] to determine the correlation functions in $(1+1)$-dimensional current algebra. Obtaining those same differential equations from the point of view of quantization of $(2+1)$-dimensional Chern-Simons gauge theory comes close to a demonstration that the full content of the $(1+1)$-dimensional theory can be extracted from the $(2+1)$-dimensional theory. What is required to complete the deduction of the $(1+1)$-dimensional theory from $(2+1)$-dimensions is actually an understanding of the unitarity of $\delta^{\tilde{\mathscr{H}}_Q}$, a thorny question that we will address in §6.

For $g \geq 2$, the fact that the connection $\delta^{\tilde{\mathscr{H}}_Q}$ coincides with that constructed in conformal field theory would follow from the fact that $\delta^{\tilde{\mathscr{H}}_Q}$ is the unique connection on $\tilde{\mathscr{H}}_Q$ whose connection one-form is a second-order operator. This latter fact follows from global considerations, essentially from the results about $H^0(\mathscr{M}, T^{(1,0)}\mathscr{M})$ and $H^0(\mathscr{M}, \operatorname{Sym}^2(T^{(1,0)}\mathscr{M}))$ in [19].

4a. The index theorem. In this subsection we review the geometry of determinant line bundles in the holomorphic setting. This leads to a finite dimensional version of the family index theorem which generalizes to infinite dimensions. We then apply the theorem to our symplectic quotient problem to find the identities (4.17)–(4.20). These formulas may also be proved without reference to determinant line bundles by using the tools of §3.

Let $\mathscr{H}_-$ and $\mathscr{H}_+$ be two holomorphic vector bundles with Hermitian metrics over the base complex manifold B, and let $\nabla^{\mathscr{H}_\pm}$ be the unique compatible connections. Let T_z be a linear map from $\mathscr{H}_-$ to $\mathscr{H}_+$ which depends holomorphically on B. The kernel and cokernel of T_z form holomorphic bundles $\operatorname{Ker}(T_z)$ and $\operatorname{Cok}(T_z) = \mathscr{H}_+/\operatorname{Im}(T_z)$ over any region of B where their dimensions are constant. We will restrict our attention to such regions. For our applications this corresponds to working away from the singularities of $\mathscr{M}$.

We identify $\operatorname{Cok}(T_z)$ and $\operatorname{Im}(T_z)^\dagger$ by the natural isomorphism. $\operatorname{Ker}(T_z)$ and $\operatorname{Im}(T_z)^\dagger$ inherit metrics and compatible connections as subbundles of $\mathscr{H}_-$ and $\mathscr{H}_+$, respectively. One may easily check that these induced connections are compatible with the holomorphic structures on $\mathrm{Ker}(T_z)$ and $\mathrm{Cok}(T_z)$.

We now specialize to the case where the fibers of $\mathscr{H}_\pm$ are finite-dimensional so that we may compare the (local) geometry of the determinant bundle of $\mathscr{H}_+ - \mathscr{H}_-$ with that of the determinant of the index bundle $\mathrm{Ind}(T_z) \equiv \mathrm{Cok}(T_z) - \mathrm{Ker}(T_z)$. The bundle $\mathrm{Det}(\mathscr{H}_+ - \mathscr{H}_-) \equiv \mathrm{Det}(\mathscr{H}_+) \otimes \mathrm{Det}(\mathscr{H}_-)^{-1}$ inherits a holomorphic structure, metric g^Q, and compatible connection ∇^Q from the corresponding objects on $\mathscr{H}_\pm$. (These are the Ray-Singer-Quillen metric and Quillen connection.) Similarly, $\mathrm{Det}(\mathrm{Ind}(T_z)) \equiv \mathrm{Det}(\mathrm{Cok}(T_z)) \otimes \mathrm{Det}(\mathrm{Ker}(T_z))^{-1}$ inherits a holomorphic structure, metric g, and compatible connection ∇ from $\mathrm{Cok}(T_z)$ and $\mathrm{Ker}(T_z)$. Since we are assuming $\mathscr{H}_\pm$ are finite dimensional we may form $D \equiv \mathrm{Det}(\mathrm{Im}(T_z)) \otimes \mathrm{Det}(\mathrm{Ker}(T_z)^\dagger)^{-1}$ which is also a holomorphic unitary bundle with connection. We have a natural isomorphism which is holomorphic and unitary:

$$\begin{aligned}\mathrm{Det}(\mathscr{H}_+ - \mathscr{H}_-) &= \mathrm{Det}([\mathrm{Cok}(T_z) \oplus \mathrm{Im}(T_z)] - [\mathrm{Ker}(T_z) \oplus \mathrm{Ker}(T_z)^\dagger]) \\ &= \mathrm{Det}(\mathrm{Ind}(T_z)) \otimes D.\end{aligned} \tag{4.2}$$

The isomorphism

$$T_z : \mathrm{Ker}(T_z)^\dagger \to \mathrm{Im}(T_z), \tag{4.3}$$

obtained by restricting the domain and range of T_z, induces an isomorphism of D with the trivial line bundle. This isomorphism is holomorphic. It is also unitary if we give the trivial line bundle the metric which is H times the trivial metric, where

$$H \equiv \det{}'(\Delta), \qquad \Delta = T_z^\dagger T_z. \tag{4.4}$$

We call Δ the Laplacian. The prime on the determinant reminds us that we are supposed to restrict the operator to the subspace orthogonal to its zero modes.

Putting this together, we may identify $\mathrm{Det}(\mathrm{Ind}(T_z))$ with $\mathrm{Det}(\mathscr{H}_+ - \mathscr{H}_-)$ as a holomorphic vector bundle. Under this identification, $\mathrm{Det}(\mathrm{Ind}(T_z))$ receives a unitary structure $g^Q = Hg$ and the compatible connection $\nabla^Q = \nabla + H^{-1}\nabla^{(1,0)}H$. The curvature R^Q of ∇^Q is related to the curvature R of ∇ by

$$R^Q = [\bar\partial\partial \ln(H)] + R. \tag{4.5}$$

Since the curvature of $\mathrm{Det}(\mathscr{V})$ is the trace of the curvature of $\mathscr{V}$ for any vector bundle $\mathscr{V}$, we have

$$R^Q = \operatorname{Tr}(R_{\mathscr{H}_+}) - \operatorname{Tr}(R_{\mathscr{H}_-}), \tag{4.6}$$

$$R = \operatorname{Tr}(R_{\operatorname{Cok}(T_z)}) - \operatorname{Tr}(R_{\operatorname{Ker}(T_z)}), \tag{4.7}$$

and therefore,

$$R^Q = [\bar{\partial}\partial \ln(H)] + \operatorname{Tr}(R_{\operatorname{Cok}(T_z)}) - \operatorname{Tr}(R_{\operatorname{Ker}(T_z)}). \tag{4.8}$$

Equation (4.6) may be construed as a statement of the local family index theorem in the finite-dimensional case: There exists a connection ∇^Q on $\operatorname{Ind}(T_z)$ whose curvature R^Q is a natural representative for the first Chern class of $\operatorname{Ind}(T_z)$ which is independent of T_z. Furthermore, R^Q is also given by (4.8).

For the usual index theorem we allow $\mathscr{H}_\pm$ to be infinite dimensional and take T_z to be a holomorphic family of elliptic differential operators. In that case, $\operatorname{Ker}(T_z)$ and $\operatorname{Cok}(T_z)$ are finite dimensional and $\operatorname{Det}(\operatorname{Ind}(T_z))$ may still be defined as a holomorphic vector bundle with metric g as before. Given a $\mathscr{G}$ invariant regularized determinant of the Laplacian, we may still define $g^Q = Hg$. We still have (4.5), (4.7), and (4.8), but now the right-hand side of (4.6) is ill defined. It is replaced by the appropriate local index form.

Application of the index theorem to our problem. We will now apply the index theorem to the $(1, 0)$ component, T_z, of the $\mathscr{G}_c$-action. As in §3, for $(A, J) \in \mathscr{A} \times \mathscr{T}$ we let

$$\mathscr{H}_-|_{(A,J)} = \operatorname{Lie}(\mathscr{G}_c) = \Omega^{(0,0)}(\Sigma, \operatorname{Lie}(G_c)), \tag{4.9}$$

$$\mathscr{H}_+|_{(A,J)} = T^{(1,0)}\mathscr{A} = \Omega^{(0,1)}(\Sigma, \operatorname{Lie}(G_c)), \tag{4.10}$$

$$T_z|_{(A,J)} = (1,0) \text{ piece of } \mathscr{G}_c \text{ action} = \bar{\partial}_{(A,J)}. \tag{4.11}$$

The far right expressions are for the gauge theory case. $\mathscr{H}_\pm$ are bundles over $\mathscr{A} \times \mathscr{T}$ with a $\mathscr{G}$ action and T_z is a $\mathscr{G}$-invariant linear transformation between them.

If $\mathscr{G}$ acts freely on $F^{-1}(0)$ then $\mathscr{H}_\pm$ and T_z push down to Hermitian bundles over B and a linear map between them. Since T_z upstairs depends holomorphically on $\mathscr{A} \times \mathscr{T}$, it depends holomorphically on B downstairs. So we may apply the index theorem to T_z as a family over B. If, on the other hand, $\mathscr{G}$ does not act freely then we cannot necessarily push $\mathscr{H}_\pm$ down by defining the sections of the bundles downstairs to be the $\mathscr{G}$-invariant sections of the bundles upstairs. The problem is that the fibers of the bundles on B would be isomorphic to the subspaces of the fibers of $\mathscr{H}_\pm$ upstairs which are invariant under the isotropy subgroups of the $\mathscr{G}$-action on $F^{-1}(0)$. Whether $\mathscr{G}$ acts freely or not, we may apply the index theorem upstairs and push the resulting expressions for the Quillen curvature down to B.

As described in §3, the bundle $\mathrm{Cok}(T_z) \to F^{-1}(0) \times \mathscr{T}$ can be naturally identified as the pullback of $T^{(1,0)}\mathscr{M} \to B$ by the projection map $\pi : F^{-1}(0) \times \mathscr{T} \to B$. So the bundle $\mathrm{Det}(\mathrm{Cok}(T_z))$ pushes down to the canonical bundle of $\mathscr{M}$. By Hodge theory, also explained in §3, we have that $\mathrm{Ker}(T_z)$ is the complexification of the real bundle $\mathrm{Ker}(T)$. A real bundle, with orthogonal structure group, has traceless curvature; and this remains true after complexification. Therefore $\mathrm{Tr}(R_{\mathrm{Ker}(T_z)}) = 0$ and we may ignore the contribution of the kernel in the local index theorem. The index theorem gives us an expression for the curvature of the holomorphic connection on $\mathrm{Det}(\mathrm{Cok}(T_z))$ which is compatible with a certain metric, namely H times the metric induced from the Hermitian structure of $\mathrm{Cok}(T_z)$. Let $\nabla^{\mathrm{Det}(\mathrm{Cok}(T_z))}$ be the holomorphic connection on $\mathrm{Det}(\mathrm{Cok}(T_z))$ which is compatible with the metric induced from $\mathrm{Cok}(T_z)$. Also let $\tilde{\nabla}^{K_\mathscr{M}}$ be the natural holomorphic connection on $K_\mathscr{M}$ compatible with the metric induced from $T^{(1,0)}\mathscr{M}$. It is not difficult to see that $\nabla^{\mathrm{Det}(\mathrm{Cok}(T_z))}$ and $\pi^*(\tilde{\nabla}^{K_\mathscr{M}})$ agree when acting in the $\widehat{T\mathscr{M}}$ directions (i.e., the horizontal directions orthogonal to the $\mathscr{G}$ orbits) and that their difference is $\mathscr{G}$ invariant. This implies that the horizontal part of the two form $\mathrm{Tr}(R_{\mathrm{Cok}(T_z)})$ pushes forward to $\mathrm{Tr}(R_{T^{(1,0)}\mathscr{M}})$.

In the formulas below involving forms on $\mathscr{M} \times \mathscr{T}$ we will use a component notation for forms on $\mathscr{M}$ but we will include the differentials $\delta\tilde{J}$ and δJ for forms on $\mathscr{T}$. Also the notation $(1,0)$, $(0,1)$, and $(1,1)$ will refer to the type of form in the $\mathscr{T}$ directions. For forms involving differentials in the $\mathscr{M}$ and $\mathscr{T}$ directions we choose the sign of the components appropriate to placing the $\mathscr{M}$ differentials to the right of the $\mathscr{T}$ differentials.

In order to write down $\mathrm{Tr}(R_{T^{(1,0)}\mathscr{M}})$, we will need the Ricci tensor of $\mathscr{M}$. Let $\mathscr{R}_{IJ} = [\tilde{\nabla}_I, \tilde{\nabla}_J]$ be the Riemann tensor. For each I and J it is a linear transformation with components $(\mathscr{R}_{IJ})^K_L$. The Ricci tensor is

$$R_{IJ} = (\mathscr{R}_{KI})^K{}_J. \tag{4.12}$$

For a Kähler manifold, this vanishes unless I and J are of opposite type.

Also, the Bianchi identity implies:

$$R_{\underline{M}\overline{K}} = R_{\overline{K}\underline{M}} = (\mathscr{R}_{\underline{I}\overline{K}})^{\underline{I}}{}_{\underline{M}} = -(\mathscr{R}_{\overline{K}\underline{M}})^{\underline{I}}{}_{\underline{I}}. \tag{4.13}$$

So we have

$$[\mathrm{Tr}(R_{\mathrm{Cok}(T_z)})]_{\overline{K}\underline{M}} = \mathrm{Tr}([\tilde{\nabla}_{\overline{K}}, \tilde{\nabla}_{\underline{M}}]) = -R_{\underline{M}\overline{K}}, \tag{4.14.1}$$

$$\begin{aligned}[\mathrm{Tr}(R_{\mathrm{Cok}(T_z)})]^{(1,0)}_{\overline{M}} &= \mathrm{Tr}([\pi_z\delta^{(1,0)}, \tilde{\nabla}_{\overline{M}}]) \\ &= (\delta^{(1,0)}\Gamma)^{\underline{I}}_{\overline{M}\underline{I}} = -\frac{i}{2}\tilde{\nabla}_{\underline{I}}\delta\tilde{J}^{\underline{I}}{}_{\overline{M}}\end{aligned} \tag{4.14.2}$$

$$\begin{aligned}[\mathrm{Tr}(R_{\mathrm{Cok}(T_z)})]^{(0,1)}_{\underline{M}} &= \mathrm{Tr}([\pi_z\delta^{(0,1)}, \tilde{\nabla}_{\underline{M}}]) \\ &= (\delta^{(0,1)}\Gamma)^{\overline{I}}_{\underline{M}\overline{I}} = -\frac{i}{2}\tilde{\nabla}_{\overline{I}}\delta\tilde{J}^{\overline{I}}{}_{\underline{M}},\end{aligned} \tag{4.14.3}$$

$$\begin{aligned}[\mathrm{Tr}(R_{\mathrm{Cok}(T_z)})]^{(1,1)} &= \mathrm{Tr}([\pi_z\delta^{(1,0)}, \pi_z\delta^{(0,1)}]_+) \\ &= -\frac{1}{4}\mathrm{Tr}(\pi_z\delta\tilde{J}\wedge\delta\tilde{J}),\end{aligned} \tag{4.14.4}$$

where $(\tilde{\nabla}, \pi_z\delta)$ is the connection on $T^{(1,0)}\mathscr{M}$ as a subbundle of $T\mathscr{M}_c$ over $\mathscr{M}\times\mathscr{T}$. $\delta^{(1,0)}\Gamma$ is the variation of the Christoffel symbol for the metric connection on $\mathscr{M}$; its components are easily calculated to give the results indicated. All the traces are over $T^{(1,0)}\mathscr{M}$. In (4.14.4), the anticommutator $[\ ,\ ]_+$ appears because the connection δ is valued in one-forms; in (4.14.1)–(4.14.3), ordinary commutators appear because the expressions are written in terms of the components of $\tilde{\nabla}$.

To write down the right-hand side of (4.6) in the finite-dimensional case, we first observe that $\mathrm{Tr}(R_{\mathscr{H}_-})$ vanishes because $\mathscr{H}_-$ is the trivial bundle with trivial connection. The curvature of $\mathrm{Det}(\mathscr{H}_+) = \mathrm{Det}(T^{(1,0)}\mathscr{A})$ is just the upstairs version of (4.14). Since δJ is covariantly constant upstairs and since the Ricci tensor upstairs vanishes, we have

$$[\mathrm{Tr}(R_{\mathscr{H}_+})]_{\underline{M}\overline{K}} = 0, \qquad [\mathrm{Tr}(R_{\mathscr{H}_+})]^{(1,0)}_{\overline{M}} = 0. \tag{4.15}$$

The analog of (4.14.4) is

$$\begin{aligned}[\mathrm{Tr}(R_{\mathscr{H}_+})]^{(1,1)} &= \mathrm{Tr}_{T\mathscr{A}}([\pi_z\delta^{(1,0)}, \pi_z\delta^{(0,1)}]_+) \\ &= -\frac{1}{4}\mathrm{Tr}_{T\mathscr{A}}(\pi_z\delta J\wedge\delta J).\end{aligned} \tag{4.16}$$

We now write down the index theorem in a way which is true for both the gauge theory and finite-dimensional cases. In the formulas below, the left-hand side is the curvature of the Quillen connection on $K_{\mathcal{M}}$ as given by (4.8). The right-hand side is the local index density in the gauge theory case and $\mathrm{Tr}(R_{\mathcal{H}_+})$ in the finite-dimensional case. For the $\mathcal{M}-\mathcal{M}$ directions, the index theorem states that

$$\tilde{\nabla}_{\overline{K}}\tilde{\nabla}_{\underline{M}} \ln H - R_{\underline{M}\overline{K}} = 2\frac{h}{k}(-i\tilde{\omega}_{\overline{K}\underline{M}}). \tag{4.17}$$

In the finite-dimensional case h equals 0; and, in the gauge theory case $\frac{1}{k}\tilde{\omega}$ is the basic symplectic form and h is the quadratic Casimir of the adjoint representation of G, as defined in the appendix. The other components of the index theorem are

(4.18)
$$\delta^{(1,0)}(\tilde{\nabla}_{\overline{M}} \ln H) - (\delta^{(1,0)}\Gamma)^{\underline{I}}_{\underline{I}\overline{M}} = 0, \qquad (\delta^{(1,0)}\Gamma)^{\underline{I}}_{\underline{I}\overline{M}} = -\frac{i}{2}\tilde{\nabla}_{\underline{I}}\delta\tilde{J}^{\underline{I}}{}_{\overline{M}}$$

(4.19)
$$\delta^{(0,1)}(\tilde{\nabla}_{\underline{M}} \ln H) + (\delta^{(0,1)}\Gamma)^{\underline{I}}_{\underline{I}\underline{M}} = 0, \qquad (\delta^{(0,1)}\Gamma)^{\underline{I}}_{\underline{I}\underline{M}} = -\frac{i}{2}\tilde{\nabla}_{\overline{I}}\delta\tilde{J}^{\overline{I}}{}_{\underline{M}},$$

$$\delta^{(0,1)}\delta^{(1,0)} \ln H - \frac{1}{4}\mathrm{Tr}[\delta\tilde{J} \wedge \delta\tilde{J}\pi_z] = c_1(\mathrm{Ind}(T_z)). \tag{4.20}$$

Note that a minus sign appears in the first equation of (4.18) because the holomorphic derivative of $\ln H$ appears to the left of the antiholomorphic derivative; as a check of the signs observe that (4.19) is just the complex conjugate of (4.18). Here $c_1(\mathrm{Ind}(T_z))$ is the $\mathcal{T}-\mathcal{T}$ piece of the local index form for the first Chern class of the index bundle. In finite dimensions, it is just equal to $-\frac{1}{4}\mathrm{Tr}(\pi_z\delta J \wedge \delta J)$. In the gauge theory case, it is also a constant in its dependence on $\mathcal{M}$. In the expression $\delta^{(1,0)}(\tilde{\nabla}_{\overline{M}} \ln H)$, the operator $\delta^{(1,0)}$ acts on the index $\overline{M}$. Formally, $\tilde{\nabla}_{\overline{M}} \ln H$ are the components of an antiholomorphic form, and $\delta^{(1,0)}$ acts as the projected connection on $T^{*(0,1)}\mathcal{M}$; or more explicitly

$$\delta^{(1,0)}(\tilde{\nabla}_{\overline{M}} \ln H) = \tilde{\nabla}_{\overline{M}}(\delta^{(1,0)} \ln H) + \frac{i}{2}\delta\tilde{J}^{\underline{K}}{}_{\overline{M}}\tilde{\nabla}_{\underline{K}} \ln H. \tag{4.21}$$

Equations (4.17)–(4.20) mean that the Quillen connection on $K_{\mathcal{M}}$ satisfies assumption (2) of Theorem 1.

4b. Basic properties of $\delta^{\mathcal{H}_Q}$ following from the index theorem.

Holomorphicity preservation. With the results of §§2 and 3 as motivation, we take the following ansatz for the connection on the quantum Hilbert bundle for $\mathcal{M}$:

$$\delta^{\mathcal{H}_Q} = \delta - \frac{k}{k+h}\mathcal{O}, \tag{4.22}$$

$$\mathscr{O} = \tilde{\nabla}_{\underline{I}} B^{\underline{I}\underline{J}} \tilde{\nabla}_{\underline{J}} + \tilde{W}^{\underline{J}} \tilde{\nabla}_{\underline{J}} - \tfrac{1}{2}\delta^{(1,0)} \ln(H), \tag{4.23}$$

$$B^{\underline{I}\underline{J}} = -\tfrac{1}{4}(\delta \tilde{J} \tilde{\omega}^{-1})^{\underline{I}\underline{J}} = B^{\underline{J}\underline{I}}. \tag{4.24}$$

Here we treat $\tilde{W}$ as an unknown $(1, 0)$ vector field on $\mathscr{M}$ to be solved for so that the $\delta^{\tilde{\mathscr{H}}_Q}$ preserves holomorphicity. We will see below that the solution is

$$\tilde{W}^{\underline{J}} = B^{\underline{I}\underline{J}} \tilde{\nabla}_{\underline{I}} \ln(H). \tag{4.25}$$

The holomorphicity preservation equation is

$$\begin{aligned} 0 = [\delta^{\tilde{\mathscr{H}}_Q}, \tilde{\nabla}_{\overline{K}}] &= \frac{k}{k+h}[\tilde{\nabla}_{\overline{K}}, \mathscr{O}] + \frac{i}{2}\delta \tilde{J}^{\underline{J}}{}_{\overline{K}} \tilde{\nabla}_{\underline{J}} \\ &= \frac{k}{k+h}\Big\{ [\tilde{\nabla}_{\overline{K}}, \tilde{\nabla}_{\underline{I}}] B^{\underline{I}\underline{J}} \tilde{\nabla}_{\underline{J}} + \tilde{\nabla}_{\underline{I}} (\tilde{\nabla}_{\overline{K}} B^{\underline{I}\underline{J}}) \tilde{\nabla}_{\underline{J}} \\ &\qquad + \tilde{\nabla}_{\underline{I}} B^{\underline{I}\underline{J}} [\tilde{\nabla}_{\overline{K}}, \tilde{\nabla}_{\underline{J}}] \\ &\qquad + (\tilde{\nabla}_{\overline{K}} \tilde{W}^{\underline{J}}) \tilde{\nabla}_{\underline{J}} + \tilde{W}^{\underline{J}} [\tilde{\nabla}_{\overline{K}}, \tilde{\nabla}_{\underline{J}}] \\ &\qquad - \frac{1}{2}(\tilde{\nabla}_{\overline{K}} \delta^{(1,0)} \ln(H)) \Big\} + \frac{i}{2}\delta \tilde{J}^{\underline{J}}{}_{\overline{K}} \tilde{\nabla}_{\underline{J}}. \end{aligned} \tag{4.26}$$

We examine this equation order by order. (We define the mth order piece to be the sum of the terms with m covariant derivative operators when all of the covariant operators are commuted to the right.) To show that the second order piece of (4.26) vanishes we need to use the fact that $\tilde{\nabla}_{\overline{K}} (\delta \tilde{J} \tilde{\omega}^{-1})^{\underline{I}\underline{J}} = 0$. This is assumption (1) of Theorem 1. In this case, it may be shown by observing that $\delta J \omega^{-1}$ is holomorphic on $\mathscr{A}$ and then applying either geometric invariant theory or the Hodge theory methods of §3.

The zeroth order piece of (4.26) is

$$0 = \frac{k}{k+h}\left[-i\tilde{\omega}_{\overline{K}\underline{J}} (\tilde{\nabla}_{\underline{I}} B^{\underline{I}\underline{J}}) - i\tilde{\omega}_{\overline{K}\underline{J}} \tilde{W}^{\underline{J}} - \frac{1}{2}\tilde{\nabla}_{\overline{K}} \delta^{(1,0)} \ln(H) \right]. \tag{4.27}$$

This equation uniquely determines $\tilde{W}$ to be

$$\begin{aligned} \tilde{W}^{\underline{J}} &= \frac{i}{2}\tilde{\omega}^{\underline{J}\overline{K}} \tilde{\nabla}_{\overline{K}} \delta^{(1,0)} \ln(H) - \tilde{\nabla}_{\underline{I}} B^{\underline{I}\underline{J}} \\ &= \frac{i}{2}\tilde{\omega}^{\underline{J}\overline{K}} \big(-\frac{i}{2}\delta \tilde{J}^{\underline{I}}{}_{\overline{K}} \tilde{\nabla}_{\underline{I}} \ln(H) + \delta^{(1,0)} \tilde{\nabla}_{\overline{K}} \ln(H) + \frac{i}{2}\tilde{\nabla}_{\underline{I}} \delta \tilde{J}^{\underline{I}}{}_{\overline{K}} \big), \end{aligned} \tag{4.28}$$

where for the second equality we have used (4.21). By the index theorem identity (4.18), the second and third terms cancel and we find that $\tilde{W}$ is indeed given by (4.25).

We must show that, with this choice of $\tilde{W}$, the first order piece of (4.26) is satisfied. That is, we need to show that

$$
\begin{aligned}
0 &= \frac{k}{k+h}([\tilde{\nabla}_{\overline{K}}, \tilde{\nabla}_{\underline{I}}]B^{\underline{I}\underline{J}}\tilde{\nabla}_{\underline{J}} + B^{\underline{I}\underline{J}}(-i\tilde{\omega}_{\overline{K}\underline{J}})\tilde{\nabla}_{\underline{I}} \\
&\qquad + (\tilde{\nabla}_{\overline{K}}\tilde{W}^{\underline{J}})\tilde{\nabla}_{\underline{J}}) + \frac{i}{2}\delta\tilde{J}^{\underline{J}}{}_{\overline{K}}\tilde{\nabla}_{\underline{J}} \\
&= \frac{k}{k+h}\left(\left(1+\frac{h}{k}\right)\frac{i}{2}\delta\tilde{J}^{\underline{J}}{}_{\overline{K}} - 2i\tilde{\omega}_{\overline{K}\underline{I}}B^{\underline{I}\underline{J}}\right. \\
&\qquad \left. + (\mathscr{R}_{\overline{K}\underline{I}})^{\underline{I}}_{\underline{L}}B^{\underline{L}\underline{J}} + (\tilde{\nabla}_{\overline{K}}\tilde{W}^{\underline{J}})\right)\tilde{\nabla}_{\underline{J}}.
\end{aligned}
\tag{4.29}
$$

Using the solution (4.25) for $\tilde{W}$ and the definition of the Ricci tensor and of B, the right-hand side becomes

$$
\frac{k}{k+h}B^{\underline{M}\underline{J}}\left(2i\frac{h}{k}\tilde{\omega}_{\overline{K}\underline{M}} - R_{\underline{M}\overline{K}} + (\tilde{\nabla}_{\overline{K}}\tilde{\nabla}_{\underline{M}}\ln H)\right)\tilde{\nabla}_{\underline{J}}.
\tag{4.30}
$$

This vanishes by the index theorem identity (4.17). This concludes the proof of holomorphicity preservation. Along the way we have obtained the simple formula (4.25) which expresses $\tilde{W}$ purely in terms of intrinsic objects on $\mathscr{M}$ and a regularized determinant of the Laplacian.

Projective flatness. We will now show that $\delta^{\tilde{\mathscr{H}}_Q}$ has the expected curvature as a pushdown in the finite-dimensional case (where we know the result a priori anyway) and has curvature agreeing with the results of conformal field theory in the gauge theory case. In both cases, $\delta^{\tilde{\mathscr{H}}_Q}$ is projectively flat.

The curvature of $\delta^{\tilde{\mathscr{H}}_Q}$ is

$$
R = \frac{-k}{k+h}\delta\mathscr{O} + \left(\frac{k}{k+h}\right)^2\mathscr{O}\wedge\mathscr{O}.
\tag{4.31}
$$

Since $\mathscr{O}$ is purely $(1, 0)$ on $\mathscr{T}$, $\delta^{\tilde{\mathscr{H}}_Q}$ is compatible with the complex structure on $\tilde{\mathscr{H}}_Q$ and $R^{(0,2)} = 0$.

$(1, 1)$ *curvature.* Since $\mathscr{O}$ is $(1, 0)$, the $(1, 1)$ curvature is also quite simple:

$$
R^{(1,1)} = \frac{-k}{k+h}\delta^{(0,1)}\mathscr{O}
\tag{4.32}
$$

$$
\begin{aligned}
\delta^{(0,1)}\mathscr{O} &= -\left[\frac{i}{2}\delta\tilde{J}^{\overline{K}}{}_{\underline{I}}\tilde{\nabla}_{\overline{K}}B^{\underline{I}\underline{J}}\tilde{\nabla}_{\underline{J}}\right] + [(\delta^{(0,1)}\Gamma)^{\underline{I}}_{\underline{I}\underline{K}}B^{\underline{K}\underline{J}}\tilde{\nabla}_{\underline{J}}], \\
&\quad + [(\delta^{(0,1)}(\tilde{\nabla}_{\underline{I}}\ln H)B^{\underline{I}\underline{J}})\tilde{\nabla}_{\underline{J}}] - \left[\frac{1}{2}\delta^{(0,1)}\delta^{(1,0)}\ln H\right].
\end{aligned}
\tag{4.33}
$$

The second and third terms of (4.33) cancel by the index theorem identity (4.19). Since $\tilde{\nabla}_{\overline{K}}$ annihilates holomorphic wave functions, and since

$$[\tilde{\nabla}_{\overline{K}}, B^{\underline{I}\underline{J}}\tilde{\nabla}_{\underline{J}}] = -i\tilde{\omega}_{\overline{K}\underline{J}}B^{\underline{I}\underline{J}} = -\frac{i}{4}\delta\tilde{J}^{\underline{I}}{}_{\overline{K}}, \tag{4.34}$$

we see that, when acting on holomorphic wave functions, the first term in (4.33) is just

$$-\tfrac{1}{8}\mathrm{Tr}[\delta\tilde{J}\wedge\delta\tilde{J}\pi_{\bar{z}}] = \tfrac{1}{8}\mathrm{Tr}[\delta\tilde{J}\wedge\delta\tilde{J}\pi_{z}]. \tag{4.35}$$

By the index theorem identity (4.20), the first and fourth terms of (4.33) sum to $-\frac{1}{2}c_1(\mathrm{Ind}\,T_z)$. Our final result is

$$R^{(1,1)} = \frac{k}{k+h}\left[\frac{1}{2}c_1(\mathrm{Ind}T_z)\right]. \tag{4.36}$$

This agrees with the expected result in both the finite-dimensional and gauge theory cases.

(2, 0) *curvature.* We now show that the vanishing of the $R^{(2,0)}$ is equivalent to the vanishing of its zeroth order piece. It is not hard to verify that $\delta^{(1,0)}\mathscr{O}$ is a first order operator,

$$\delta^{(1,0)}\mathscr{O} = (\delta^{(1,0)}\tilde{\nabla}_{\underline{I}}\ln H)B^{\underline{I}\underline{J}}\tilde{\nabla}_{\underline{J}}. \tag{4.37}$$

A priori, $\mathscr{O}\wedge\mathscr{O}$ is a third order operator. To examine it order by order, we write

$$\begin{aligned}
\mathscr{O} &= \mathscr{O}_2 + \mathscr{O}_1 + \mathscr{O}_0, \\
\mathscr{O}_2 &= B^{\underline{I}\underline{J}}\tilde{\nabla}_{\underline{I}}\tilde{\nabla}_{\underline{J}}, \\
\mathscr{O}_1 &= [(\tilde{\nabla}_{\underline{I}}B^{\underline{I}\underline{J}}) + B^{\underline{I}\underline{J}}(\tilde{\nabla}_{\underline{I}}\ln H)]\tilde{\nabla}_{\underline{J}}, \\
\mathscr{O}_0 &= -\tfrac{1}{2}(\delta^{(1,0)}\ln H), \\
\mathscr{O}_{2+1} &= \mathscr{O}_2 + \mathscr{O}_1.
\end{aligned} \tag{4.38}$$

Notice that all terms of $\mathscr{O}_i\wedge\mathscr{O}_j$ are of order at least j. Also,

$$\mathscr{O}_i\wedge\mathscr{O}_j + \mathscr{O}_j\wedge\mathscr{O}_i$$

has order strictly less than $i+j$. This implies, for example, that the zeroth order piece of $\mathscr{O}\wedge\mathscr{O}$ is

$$\begin{aligned}
[\mathscr{O}\wedge\mathscr{O}]_0 &= (\mathscr{O}_{2+1}(\mathscr{O}_0)) \\
&= \{[\tilde{\nabla}_{\underline{I}}B^{\underline{I}\underline{J}}\tilde{\nabla}_{\underline{J}} + B^{\underline{I}\underline{J}}(\tilde{\nabla}_{\underline{I}}\ln H)\tilde{\nabla}_{\underline{J}}](-\tfrac{1}{2}\delta^{(1,0)}\ln H)\}.
\end{aligned} \tag{4.39}$$

To show that the second and third order pieces of $\mathscr{O}\wedge\mathscr{O}$ vanish, we first observe that

$$\delta^{(1,0)}\tilde{\nabla}_{\overline{K}}\delta\tilde{J}^{\underline{I}}{}_{\overline{J}} = 0. \tag{4.40}$$

This implies that

$$B^{I\underline{L}}(\tilde{\nabla}_{\underline{L}} B^{\underline{J}\underline{K}}) = 0. \tag{4.41}$$

Using (4.41) and the fact that holomorphic derivatives $\tilde{\nabla}_{\underline{I}}$ commute, one quickly sees that the second and third order pieces vanish.

So far we have shown that the $(2, 0)$ curvature has the form

$$R^{(2,0)} = X^{\underline{I}}\tilde{\nabla}_{\underline{I}} + [R^{(2,0)}]_0, \tag{4.42}$$

where

$$[R^{(2,0)}]_0 = \left(\frac{k}{k+h}\right)^2 (\mathscr{O}_{2+1}(\mathscr{O}_0)) \tag{4.43}$$

and $X^{\underline{I}}$ is a $(1, 0)$ vector field. Since $\delta^{\tilde{\mathscr{H}}_Q}$ preserves holomorphicity for local reasons, its curvature does also, that is:

$$0 = [\tilde{\nabla}_{\overline{K}}, R] = [-i\tilde{\omega}_{\overline{K}\underline{I}} X^{\underline{I}} + (\tilde{\nabla}_{\overline{K}}[R^{(2,0)}]_0)] + (\tilde{\nabla}_{\overline{K}} X^{\underline{I}})\tilde{\nabla}_{\underline{I}}. \tag{4.44}$$

The first order piece of (4.44) says that $X^{\underline{I}}$ is a holomorphic vector field on $\mathscr{M}$.

If $\mathscr{M}$ has no nonzero holomorphic vector fields then X must vanish, and so, by the zeroth order piece of (4.44), $[R^{(2,0)}]_0$ is holomorphic. If, in addition to having no holomorphic vector fields, $\mathscr{M}$ has no nonconstant holomorphic functions then we may conclude that $R^{(2,0)}$ is purely projective. This fact was used in our first demonstration in the proof of Theorem 2 that $R^{(2,0)}$ vanishes.

Though this argument applies simply to the gauge theory problem that has motivated our investigation, the use of global holomorphic considerations is not in the spirit of our investigation. We would like to give a proof of the vanishing of $R^{(2,0)}$ using the local differential geometry of $\mathscr{M}$. So in §7 we will return to this matter and use the local differential geometry of symplectic quotients to prove the vanishing of $[R^{(2,0)}]_0$. This suffices for proving that $R^{(2,0)} = 0$, since the zeroth order piece of (4.44) tells us that $[R^{(2,0)}]_0 = 0$ implies that $X^{\underline{I}} = 0$.

5. Explicit description of the gauge theory problem on the torus

As an illustration of the previous ideas, we will consider the gauge theory problem, discussed in §2, for Σ a torus and G a compact, simple, and simply connected Lie group with maximal torus T and Weyl group W. In §5a we explain conceptually why the connection $\delta^{\tilde{\mathscr{H}}_Q}$ can essentially be reduced to the connection for quantization of an abelian variety. In §5b we make the computation more explicit, remove some ambiguous central factors in §5a, and exhibit a basis of a parallel section of $\tilde{\mathscr{H}}_Q$ in terms of Weyl-Kac characters for the level k representations of the loop group of G. Most of the results of this section were first obtained by Elitzur et al. [7] by physical arguments starting from Feynman path integrals. That paper is useful background for this section.

5a. Reduction to quantization of abelian varieties. Since the fundamental group of Σ is abelian, $\pi_1(\Sigma) \cong \mathbb{Z} \oplus \mathbb{Z}$, every representation of $\pi_1(\Sigma)$ in G is conjugate to a representation in T, uniquely up to the action of the Weyl group. The space of representations of $\pi_1(\Sigma)$ in G is thus $\mathscr{M} = T \times T/W$. The symplectic structure that $\mathscr{M}$ obtains as a moduli space of representations is $\tilde{\omega} = k\tilde{\omega}_0$, where $\tilde{\omega}_0$ is the fundamental quantizable symplectic form, and k is a positive integer, the "level", introduced in §2. The symplectic structure $\tilde{\omega}$ lifts to a standard translation invariant and W-invariant symplectic structure on the torus $T \times T$, which we call ω. The basic prequantum line bundle $\tilde{\mathscr{L}}_0$ on $\mathscr{M}$ lifts to a prequantum line bundle $\mathscr{L}_0$ on $T \times T$ with an action of W so that sections of $\tilde{\mathscr{L}}_0$ may be identified with W-invariant sections of $\mathscr{L}_0$. To quantize the level k symplectic form, we consider the prequantum line bundle $\tilde{\mathscr{L}}_0^{\otimes k}$.

According to the general theory, $\mathscr{M}$ is to be quantized by picking a complex structure J on Σ, whereupon $T \times T$ becomes an abelian variety $\mathscr{M}_J$ with a W-invariant complex structure. The quantum bundle $\tilde{\mathscr{H}}_Q$ is to be defined over the Teichmüller space $\mathscr{T}$ of isotopy classes of complex structures on Σ. Its fiber $\tilde{\mathscr{H}}_Q|_J$ at $J \in \mathscr{T}$ is to be $H^0(\mathscr{M}, \tilde{\mathscr{L}}_0^{\otimes k})$. Because of the singularities in $\mathscr{M}$ (coming from fixed points in the action of the Weyl group on the space of representations of $\pi_1(\Sigma)$ in T), it is actually necessary to explain what one means by a holomorphic line bundle or a holomorphic section of such a bundle over $\mathscr{M}$. Though such questions can be addressed by constructing $\mathscr{M}$ as an algebraic variety, for our purposes it is sufficient to define

$$H^0(\mathscr{M}, \tilde{\mathscr{L}}^{\otimes k}) = H^0(T \times T, \mathscr{L}^{\otimes k})^W. \tag{5.1}$$

In general, rather than dealing with objects on the singular manifold $\mathscr{M}$, we will work with the corresponding W-invariant objects on $T \times T$.

At first sight, the relation of $\mathscr{M}$ to the abelian variety $T \times T$ makes it appear that it would be trivial to quantize $\mathscr{M}$. Indeed, in §1 we have considered the quantization of abelian varieties. A family of abelian varieties, with a fixed symplectic structure but a variable complex structure parametrized by a parameter space $\mathscr{T}$, can be quantized using the connection

$$\delta_0^{\tilde{\mathscr{H}}_Q} = \delta + \frac{1}{4k}(\delta J\omega_0{}^{-1})^{\underline{i}\underline{j}}\nabla_{\underline{i}}\nabla_{\underline{j}} \tag{5.2}$$

on the quantum bundle over $\mathscr{T}$. In the case at hand, where one considers a family of abelian varieties with an action of a finite group W, the connection (5.2) is compatible with the W-action and thus descends to a connection on the W-invariant subspace.

Though this is thus a projectively flat connection on the quantum bundle $\tilde{\mathscr{H}}_Q$, it is not the one that comes by specializing the general, genus g formulas of this paper to the case of genus one. The formula (5.2) is "wrong" in the sense that it is *ad hoc* and does not have a genus g generalization. Our goal in this section is to investigate the projectively flat connection on $\tilde{\mathscr{H}}_Q$ obtained by specializing the general formulas to $g = 1$. We will see that this latter connection can be conjugated to a form very similar to (5.2), but with k replaced by $k + h$.

The projectively flat connection that comes from the general formula is

$$\begin{aligned}\delta^{\tilde{\mathscr{H}}_Q} = \delta &+ \frac{1}{4(k+h)}(\delta J\omega_0{}^{-1})^{\underline{i}\underline{j}}\left(\nabla_{\underline{i}}\nabla_{\underline{j}} + (\nabla_{\underline{i}}\ln H)\nabla_{\underline{j}}\right)\\ &+ \frac{k}{2(k+h)}\delta^{(1,0)}\ln H.\end{aligned} \tag{5.3}$$

This acts on W invariant sections of $\mathscr{L}_0^{\otimes k} \to (T \times T) \times \mathscr{T}$ which are holomorphic in their dependence on $T \times T$. This formula contains, in addition to the Kähler geometry of $T \times T$, the regularized determinant of the Laplacian $H = \det' \triangle$. We must now discuss the basic properties of this function.

Preliminary discussion of H. Because the fundamental group of the genus one surface Σ is abelian, a flat connection A on a G-bundle E can up to a gauge transformation be assumed to take values in a maximal commutative subalgebra t of $\mathrm{Lie}(G)$. The choice of t gives a decomposition $\mathrm{Lie}(G) = \mathrm{Lie}(G)_\pm \oplus t$, where $\mathrm{Lie}(G)_\pm$ is a sum of nontrivial representations of t (its complexification is the sum of the nonzero root spaces in $\mathrm{Lie}(G)_c$). The adjoint bundle $\mathrm{ad}(E)$ has a corresponding decomposition $\mathrm{ad}(E) = \mathrm{ad}(E)_\pm \oplus \mathrm{ad}(E)_0$, and the Laplacian decomposes as $\triangle = \triangle_\pm \oplus \triangle_0$. Letting $H_\pm$ and H_0 denote the determinants of $\triangle_\pm$ and $\triangle_0$, we have

$$H = H_0 \cdot H_\pm. \tag{5.4}$$

Now, to begin with, H is a W-invariant function on $(T \times T) \times \mathscr{T}$. However, in fact, H_0 is the pullback of a function on $\mathscr{T}$ because t is abelian and acts trivially on its own adjoint bundle. Looking back to (5.3), we see therefore that H_0 contributes only a central term to the connection $\delta^{\mathscr{H}_Q}$ (since $\nabla_{\underline{i}} \ln H_0 = 0$, and $\delta^{(1,0)} \ln H_0$ is the pullback of a one form on $\mathscr{T}$ and so is central). Thus, it is $H_\pm$ that we must study.

The operator $\triangle_\pm$ is $\triangle_\pm = \overline{\partial}_\pm^\dagger \overline{\partial}_\pm$, where $\overline{\partial}_\pm$ is the $\overline{\partial}$ operator on the bundle $E_\pm$ with the connection A. By the index theorem, the determinant line bundle $\operatorname{Det}(\overline{\partial}_\pm)$ has curvature $-i(2h\omega_0)$, where h is the quadratic Casimir of the Lie algebra of G. Thus, $\operatorname{Det}(\overline{\partial}_\pm)$ can be identified with $\mathscr{L}_0^{\otimes 2h}$.

By its construction, the line bundle $\operatorname{Det}(\overline{\partial}_\pm)$ has a natural action of the Weyl group W. Actually, since W has a nontrivial one-dimensional character ϵ of order two (in which the elementary reflections act as multiplication by -1), it abstractly can act on $\operatorname{Det}(\overline{\partial}_\pm)$ in two possible ways. The action τ that arises naturally in thinking of $\operatorname{Det}(\overline{\partial}_\pm)$ as a determinant bundle is the action in which W acts trivially on the fiber above the trivial connection. An object that transforms as the character ϵ of the Weyl group will be called Weyl anti-invariant in what follows.

The operator $\overline{\partial}_\pm$ is generically invertible, so according to [26], its determinant is a holomorphic section s of the determinant line bundle $\tilde{\mathscr{L}}_0^{\otimes 2h}$. Moreover

$$H_\pm = ||s||^2, \tag{5.5}$$

where $||\ ||$ is the Hermitian structure on the bundle $\tilde{\mathscr{L}}_0^{\otimes 2h}$ that has a compatible connection with curvature $-2ih\tilde{\omega}_0$.

The section s is essentially the square of the denominator in the Weyl-Kac character formula for the affine Lie algebra $\widehat{\mathfrak{g}}$, and is extensively studied in that context. It has the following key properties:

(i) From its construction, it is invariant under the Weyl group W.

(ii) It has a natural square root $s^{1/2}$, which is a holomorphic section of $\mathscr{L}_0^{\otimes h}$. This square root is simply the partition function for a system of Majorana fermions coupled to the bundle $E_\pm$. It can be rigorously constructed using the theory of *Pfaffian* line bundles [10]. (The theory of Pfaffian bundles applies to appropriate Dirac operators twisted by vector bundles with quadratic forms. In genus one the $\overline{\partial}$-operator is equivalent to the Dirac operator D associated with the trivial spin bundle. The bundle $E_\pm$ has a quadratic form coming from the pairing of positive and negative roots. The square root of s is just the Pfaffian of the Dirac operator twisted by $E_\pm$.)

(iii) Since s is Weyl invariant, its square root must be either Weyl invariant or anti-invariant. In fact $s^{1/2}$ is Weyl anti-invariant, something that can be seen by inspecting its zeros, which are manifest from the description of s as a determinant. (s has a double zero and $s^{1/2}$ has a single zero at generic fixed points of an elementary reflection. Consideration of the zeros of s provides another elementary approach to understanding the existence of $s^{1/2}$.) Moreover, $s^{1/2}$ has only the zeros required by anti-invariance. Therefore, every anti-invariant section of $\mathscr{L}^{\otimes h}$ is divisible by $s^{1/2}$, so the anti-invariant subspace of $H^0(T \times T, \mathscr{L}^{\otimes h})$ is one dimensional, generated by $s^{1/2}$.

(iv) $s^{1/2}$ obeys a heat equation

$$\left(\delta^{(1,0)} + \frac{1}{4h}(\delta^{(1,0)} J \omega_0{}^{-1})^{\underline{ij}} \nabla_{\underline{i}} \nabla_{\underline{j}} + \text{central}\right) s^{1/2} = 0, \tag{5.6}$$

where "central" denotes the pullback of a one-form from $\mathscr{T}$. The fact that $s^{1/2}$ obeys such a heat equation, with the correct choice of the central term, is an inevitable consequence of the fact that theta functions obey heat equations if their dependence on $\mathscr{T}$ is fixed correctly, and the fact that the space of anti-invariant theta functions at level h is one dimensional. (Actually, the heat equation (5.6) is well known in the theory of affine Lie algebras and is essentially equivalent to the MacDonald identities.)

Reduction of the connection. We are now in a position to simplify the connection $\delta^{\tilde{\mathscr{H}}_Q}$ on the quantum bundle $\tilde{\mathscr{H}}_Q$ over $\mathscr{T}$. The quantum bundle $\tilde{\mathscr{H}}_Q$ is isomorphic to what we might call the mock quantum bundle, namely the bundle $\widehat{\mathscr{H}_Q}$ whose fiber at J is the Weyl anti-invariant subspace of $H^0(T \times T, \mathscr{L}^{\otimes k+h})$. The isomorphism between $\tilde{\mathscr{H}}_Q$ and $\widehat{\mathscr{H}_Q}$ is multiplication by $s^{1/2}$. (Multiplication by $s^{1/2}$ clearly defines a map from $\tilde{\mathscr{H}}_Q$ to $\widehat{\mathscr{H}_Q}$, and this map is an isomorphism since, as $s^{1/2}$ has only the zeros required by anti-invariance, every anti-invariant section of $\mathscr{L}^{\otimes k+h}$ is divisible by $s^{1/2}$.) Conjugating by $s^{1/2}$, we get a connection $\delta^{\widehat{\mathscr{H}_Q}}$ on the mock quantum bundle, namely

$$\delta^{\widehat{\mathscr{H}_Q}} = s^{1/2} \cdot \delta^{\tilde{\mathscr{H}}_Q} \cdot s^{-1/2}. \tag{5.7}$$

In evaluation of (5.7) using the heat equation (5.6) one finds some cancellations; after a short computation, one obtains

$$\delta^{\widehat{\mathscr{H}_Q}} = \delta + \frac{1}{4(k+h)} \left((\delta J \omega_0{}^{-1})^{\underline{ij}} \nabla_{\underline{i}} \nabla_{\underline{j}}\right) + \text{central}. \tag{5.8}$$

(5.8) coincides with the projectively flat connection (5.2) which might have been guessed naively from the beginning, but (i) k is replaced by $k + h$, and (ii) we are to restrict ourselves to Weyl anti-invariant sections of $\mathcal{L}^{\otimes k+h}$.

The question now arises of what would be required to sharpen the above computation and determine the central term in (5.8). There were various points in the above derivation in which central factors were omitted. Even the starting point, (5.3), had an ambiguous central factor, because the determinant H, which is a function on $\mathcal{M} \times \mathcal{T}$, is defined only up to multiplication by a function on $\mathcal{T}$, until one picks a particular family of metrics on Σ, which we have not done in the above. We will see in §5b that if one uses flat metrics on Σ of unit area in defining H, then the central term in (5.8) equals the conventional central term in the usual heat equation for theta functions.

For future reference, it is useful to note that up to multiplication by a function on Teichmüller space,

$$H \sim \|s\|^2, \tag{5.9}$$

since in fact $H = H_0 \cdot \|s\|^2$, and H_0 is a function on $\mathcal{T}$ alone.

Unitarity. It remains to discuss unitary. According to the standard discussion of quantization of abelian varieties, the connection (5.8) is compatible with a simple unitary structure. If θ is a section of $\widehat{\mathcal{H}_Q}$ which is parallel for (5.8), the unitary structure is

$$\langle \theta , \theta \rangle \sim \int_{\mathcal{M}} \|\theta\|^2, \tag{5.10}$$

where $\int_{\mathcal{M}}$ is integration on $\mathcal{M}$ with the measure determined by its symplectic structure, and $\|\theta\|^2$ is the norm of θ as a section of the Hermitian line bundle $\mathcal{L}^{\otimes k+h}$. The symbol " $\sim$ " in (5.10) means that, since we have not pinned down the central term in (5.8), the unitary structure (5.10) is invariant under parallel transport by the connection (5.8) only up to multiplication by a function on $\mathcal{T}$.

It is now easy to derive a similar expression for a projectively unitary structure respected by the connection $\delta^{\mathcal{H}_Q}$ on the original quantum bundle $\mathcal{H}_Q$. Since the relation between a parallel section ψ of $\mathcal{H}_Q$ and a parallel section θ of $\widehat{\mathcal{H}_Q}$ is $\theta = s^{1/2} \cdot \psi$, the unitary structure on $\mathcal{H}_Q$ is given by

$$\langle \psi , \psi \rangle \sim \int_{\mathcal{M}} \|s^{1/2}\|^2 \cdot \|\psi\|^2. \tag{5.11}$$

In view of (5.9), this amounts to

$$\langle\psi, \psi\rangle \sim \int_{\mathscr{M}} H^{1/2} \cdot ||\psi||^2. \tag{5.12}$$

The occurrence of a $\sim$ sign in (5.12) is inevitable at this point since we have not yet chosen metrics on Σ and defined H precisely. We will see below that if H is defined using flat metrics of unit area, then the unitary structure defined on the right hand side of (5.12) is invariant under parallel transport by (5.3), not just projectively so.

5b. Explicit computations. We will now present explicit formulas fleshing out the conceptual treatment just sketched. In addition to making the treatment more explicit, this will give us the chance to demonstrate that the Weyl-Kac characters of affine Lie algebras are parallel sections of the quantum bundle, and to fix the projective ambiguity of the above discussion.

Basic objects upstairs. The space $\mathscr{A}$ of connections on the trivial G-bundle over Σ may be identified, after choosing a base connection and a trivialization of the tangent bundle of Σ, with the direct sum $\Omega^0(\Sigma, \mathrm{Lie}(G)) \oplus \Omega^0(\Sigma, \mathrm{Lie}(G))$ of two copies of the space of Lie algebra valued functions on Σ. Explicitly, we will represent Σ by the quotient

$$\Sigma = R^2/\mathbb{Z}\times\mathbb{Z}, \quad (x^1, x^2) \sim (x^1+m^1, x^2+m^2) \quad \text{for } m^1, m^2 \in \mathbb{Z}, \tag{5.13}$$

and write any connection A as

$$A = A_1 dx^1 + A_2 dx^2 \quad \text{for } A_i \in \Omega^0(\Sigma, \mathrm{Lie}(G)). \tag{5.14}$$

The symplectic structure on $\mathscr{A}$ was given in §2, where it was also shown how the relevant complex structures on $\mathscr{A}$ are induced from complex structures on Σ. We explicitly parametrize the (Teichmuller space of) complex structures on Σ by the upper half plane $\mathrm{Im}\,\tau > 0$. In the complex structure determined by τ, the coordinate

$$z = x^1 + \tau x^2 \tag{5.15}$$

is holomorphic.

The Kähler structure on $\mathscr{M}$. Let t be the Cartan subalgebra of G, and let S denote the set of flat connections $\theta = \theta_1(2\pi i dx^1) + \theta_2(2\pi i dx^2)$ whose components, θ_i, are constant on Σ and valued in t. Every point in $\mathscr{M}$ is represented by some point in S. Two points θ and θ' in $\mathscr{S}$ are gauge equivalent if and only if their holonomies are conjugate; that is,

$$\theta' \sim \theta \leftrightarrow e^{2\pi i\theta_i} = g e^{2\pi i\theta'_i} g^{-1} \tag{5.16}$$

for some g in G. Equation (5.16) is equivalent to

$$\theta' \sim w \cdot \theta + \lambda, \tag{5.17}$$

where $\lambda = (\lambda_1, \lambda_2)$ with λ_i in the coroot lattice $\Lambda^R = \{\lambda \in t : e^{2\pi i\lambda} = 1\}$, $w \in W$, and with $w \cdot \theta \equiv (w \cdot \theta_1, w \cdot \theta_2)$. To summarize, we have

$$\mathscr{M} = F^{-1}(0)/\mathscr{G} = T \times T/W = t \times t/(\Lambda^R \times \Lambda^R)\tilde{\times} W, \tag{5.18}$$

where $\tilde{\times}$ denotes semidirect product.

We may calculate the Kähler structure on $\mathscr{M}$ by regarding it as the quotient of the finite-dimensional affine Kähler manifold $S \subset F^{-1}(0)$ by the discrete subgroup $(\Lambda^R \times \Lambda^R)\tilde{\times} W$ of $\mathscr{G}$. Let (θ_1, θ_2) be a point in $t \times t \cong \mathscr{S}$. We fix a basis $\{e_i\}$, $i = 1, \cdots, \mathrm{Rank}(G)$, for t, and denote the components of a point $\phi \in t$ by ϕ^i. We let $C_{ij} \equiv e_i \cdot e_j$ be the matrix for the basic inner product on $\mathrm{Lie}(G)$ restricted to t (see the Appendix for normalization), and C^{ij} be the inverse matrix. We will sometimes write u^2 for $C_{ij}u^i u^j$, and $u \cdot v$ for $C_{ij}u^i v^j$. We will denote the imaginary part of τ by τ_2.

The symplectic structure on moduli space restricts to a symplectic structure on $t \times t$ given by

$$\omega = 2\pi k\, d\theta_1 \cdot d\theta_2 \equiv 2\pi k C_{ij} d\theta_1^i d\theta_2^j. \tag{5.19}$$

The complex structure on $\mathscr{A}$ determined by τ restricts to a complex structure on $t \times t$ in which

$$u^i = \theta_2^i - \tau\theta_1^i, \qquad i = 1, \cdots, \mathrm{Rank}(G), \tag{5.20}$$

are the holomorphic coordinates. We will denote by $t_{\mathbb{C}}$ the space $t \times t$ endowed with this complex structure.

In complex coordinates

$$\omega = \frac{ik\pi}{\tau_2} C_{ij} d_0 u^i \wedge d_0 \overline{u}^j, \tag{5.21}$$

where $d_0 u^i = d\theta_2^i - \tau d\theta_1^i$ is the differential of u^i at fixed τ, and similarly for $d_0\overline{u}^i$. Observe that ω is positive and (1,1) as expected. Also we have

$$\delta J_\tau^{(1,0)} = -\frac{d\tau}{\tau_2}\frac{\partial}{\partial u^i} \otimes d_0\overline{u}^i. \tag{5.22}$$

The determinant $\det{}'(\overline{\partial}_A^\dagger \overline{\partial}_A)$. To make the arguments of §5a explicit, we need to evaluate $\det{}'(\overline{\partial}_A^\dagger \overline{\partial}_A)$, $s = \det(\overline{\partial}_\pm)$, and $s^{1/2}$. We actually find it convenient to work not with the section $s^{1/2}$ of the Pfaffian bundle of $\bar\partial_\pm$, but with the function $\tilde\Pi$ which appears below and is a section of the Pfaffian bundle of the operator $\bar\partial$ coupled to all of $\mathrm{Lie}(G)$. The two Pfaffian bundles differ by the pullback of a bundle from $\mathscr{T}$, and $\tilde\Pi$ differs from $s^{1/2}$ only by a central factor, which was ambiguous in §5a to begin with. To keep the calculations concrete, we will calculate the results without reference to the theory of determinant line bundles, but rather in the language of holomorphic factorization which may be more familiar to some readers. We will see, after the next subheading, how the choice of Quillen counterterms in these calculations implicitly trivialize certain line bundles that would appear in a more conceptual description.

We choose the J independent metric on the Lie algebra $\Omega^0(\Sigma, \mathrm{Lie}(G))$ given by the J independent measure $dx_1\, dx_2$ on Σ. The determinant $\det{}'\bar\partial_A^\dagger \bar\partial_A$ in this metric has been evaluated in [12]. For completeness, and to fix notation, we now give a derivation of the required formula. The result is stated in equation (5.28) below.

We first observe that the determinant factorizes as the product of determinants of Laplacians on the root spaces. To see this, recall that the complexified Lie algebra $\mathrm{Lie}(G)_c$ of G decomposes under the action of the Cartan subalgebra t as

$$\mathrm{Lie}(G)_c = E_0 \oplus \left(\bigoplus_{\alpha>0} E_\alpha\right) \oplus \left(\bigoplus_{\alpha<0} E_\alpha\right). \tag{5.23}$$

Here E_α are the root spaces on which t acts by the root α. The space E_0 is the complexified Cartan subalgebra $t_{\mathbb{C}}$ and is $r = \mathrm{Rank}(G)$ dimensional. The root spaces E_α, for $\alpha \neq 0$, are one dimensional. The nonzero roots are divided into positive and negative roots.

The operator $\bar\partial_A$ has a simple form relative to the decomposition (5.23) when $A = \theta = 2\pi i(\theta_1 dx^1 + \theta_2 dx^2)$ with $\theta_i \in t$. Any $\psi \in \Omega^0(\Sigma, \mathrm{Lie}(G)_c)$ may be decomposed as $\psi = \sum_\alpha \psi_\alpha$, where for $\alpha \neq 0$, $\psi_\alpha \in \Omega^0(\Sigma, E_\alpha)$, and for $\alpha = 0$ the sum is understood to run over r copies of $\mathbb{C}$ corresponding to writing $E_0 = \bigoplus_1^r \mathbb{C}$. Then

$$\bar\partial_\theta \psi = \sum_\alpha \bar\partial_{\langle\alpha, u\rangle} \psi_\alpha, \qquad \langle\alpha, u\rangle = \langle\alpha, \theta_1\rangle - \tau\langle\alpha, \theta_2\rangle, \tag{5.24}$$

where $\bar\partial_v$ denotes the $\bar\partial$ operator on the trivial complex line bundle over Σ coupled to the flat connection $2\pi i(v_1 dx^1 + v_2 dx^2)$.

For v not gauge equivalent to zero[6], the operator $\bar\partial_v$ is invertible, while $\bar\partial_0$ has a one-dimensional kernel and a one-dimensional cokernel

[6]That is $(v_1, v_2) \not\sim (0, 0) \mod (1, 1)$.

spanned by the constant functions and constant one-forms respectively. Correspondingly, the operator $\overline{\partial}_A$ has (generically) an r-dimensional kernel and an r-dimensional cokernel spanned by constant functions and constant one-forms valued in $t_{\mathbb{C}}$.

As a consequence of (5.24), we have the factorization

$$\det{}'\overline{\partial}_\theta^\dagger\overline{\partial}_\theta = (\det{}'\overline{\partial}_0^\dagger\overline{\partial}_0)^r \prod_{\alpha\neq 0} \det\overline{\partial}^\dagger_{\langle\alpha, u\rangle}\overline{\partial}_{\langle\alpha, u\rangle}. \tag{5.25}$$

The ζ function regulated determinant of the Laplacian $\overline{\partial}_v^\dagger\overline{\partial}_v$ on Σ was computed originally by Ray-Singer in connection with analytic torsion [29]. They found, for v not equivalent to 0,

$$\begin{aligned} \det(\overline{\partial}_v^\dagger\overline{\partial}_v) &= \left(\exp\frac{\pi}{2\tau_2}(v-\overline{v})^2\right)|\det\overline{\partial}_v|^2, \\ \det\overline{\partial}_v &= e^{\pi i\tau/6}(e^{\pi i v}-e^{-\pi i v})\prod_{n=1}^{\infty}(1-e^{2\pi i v}q^n)(1-e^{-2\pi i v}q^n), \\ & \qquad q = e^{2\pi i \tau}. \end{aligned} \tag{5.26}$$

Some explanation of the definition of $\det\overline{\partial}_v$ is required. Since $\overline{\partial}_v$ varies holomorphically in both τ and v and has no zero modes, if there were no "anomalies" introduced by regularization, $\det\overline{\partial}_v^\dagger\overline{\partial}_v$ would be the absolute value squared of a holomorphic function. Because of anomalies, this is only true up to a "counterterm": While the function $\det\overline{\partial}_v$ appearing in (5.26) is holomorphic in both τ and v, the "counterterm" correction factor $\exp((\pi/2\tau_2)(v-\overline{v})^2)$ cannot be written as the absolute value squared of a function holomorphic in both τ and v.[7]

For $v = 0$, the determinant is given by

$$\det{}'\overline{\partial}_0^\dagger\overline{\partial}_0 = \tau_2|\det{}'\overline{\partial}_0|^2, \qquad \det{}'\overline{\partial}_0 \equiv e^{\pi i\tau/6}\prod_{n=1}^{\infty}(1-q^n)^2, \tag{5.27}$$

where $\det{}'\overline{\partial}_0$ is holomorphic in τ. The correction factor τ_2 by which (5.27) fails to be an absolute value squared arises due to zero modes.

[7] In $\exp((\pi/2\tau_2)(v-\overline{v})^2)$ we recognize the contribution of the original "Quillen" counterterm

$$\exp\frac{1}{2\pi i}\int_\Sigma A_z A_{\overline{z}} = \exp -\frac{\pi}{\tau_2}\overline{v}v,$$

which is the usual counterterm extracted from (5.26) to obtain holomorphic factorization with τ held fixed.

Substituting (5.26) and (5.27) into (5.25) and doing a little algebra (using (A.8)) give

$$\begin{aligned}
&H = \det{}'\overline{\partial}_\theta^\dagger\overline{\partial}_\theta = e^Q|\det{}'\overline{\partial}_\theta|^2,\\
&Q \equiv r\ln\tau_2 + \frac{h\pi}{\tau_2}(u-\overline{u})^2,\\
&\det{}'\overline{\partial}_\theta \equiv \tilde{\Pi}(\tau, u)^2,\\
&\tilde{\Pi}(\tau, u) \equiv \Pi(\tau, u)e^{-2\pi i(<u,\rho>-\tau|G|/24)}(-1)^{(|G|-r)/2},\\
&\rho \equiv \frac{1}{2}\sum_{\alpha>0}\alpha = \text{the Weyl vector}, \qquad |G| = \mathrm{Dim}(G),\\
&\Pi(\tau, u) \equiv \left[\prod_{n>0}(1-q^n)\right]^r \prod_{\alpha\neq 0, n>0}(1-q^n e^{2\pi i<\alpha, u>})\\
&\qquad\times\prod_{\alpha>0}(1-e^{2\pi i\langle\alpha, u\rangle}).
\end{aligned} \tag{5.28}$$

The definitions of Π and $\tilde{\Pi}$ agree with that given in the Appendix. From the Macdonald identity, $\tilde{\Pi} = \theta^-_{\rho,h}$, we conclude that $\tilde{\Pi} = \det{}'(\overline{\partial}_\theta)^{1/2}$ is a Weyl anti-invariant level h theta function.

We see from the above formulas that H is e^Q times the absolute value squared of the function $\det{}'(\overline{\partial}_\theta)$ which is holomorphic in τ and u. That H has this form could be deduced from the index theorem. The main point is that

$$\overline{\partial}_{\mathcal{T}\times t\times t}\partial_{\mathcal{T}\times t\times t}\ln H = \overline{\partial}_{\mathcal{T}\times t\times t}\partial_{\mathcal{T}\times t\times t}Q, \tag{5.29}$$

where $\partial_{\mathcal{T}\times t\times t}$ and $\overline{\partial}_{\mathcal{T}\times t\times t}$ are the ∂- and $\overline{\partial}$-operators on $\mathcal{T}\times t\times t$. We first observe that $t\times t$ has vanishing Ricci tensor because it is flat. Also the $\mathcal{T}-\mathcal{T}$ piece of the local index density for $c_1(\mathrm{Ind}(\overline{\partial}))$ vanishes since Σ is flat. As a result, the index theorem identities (4.17)–(4.20) read

$$\begin{aligned}
\overline{\partial}_{\mathcal{T}\times t\times t}\partial_{\mathcal{T}\times t\times t}\ln H &= -2i\frac{h}{k}\omega + \frac{1}{4}\mathrm{Tr}[\delta J^{(1,0)}\wedge\delta J^{(0,1)}]\\
&= \frac{2h\pi}{\tau_2}C_{ij}d_o u^i\wedge d_0\overline{u}^j + \frac{r}{4\tau_2^2}d\tau\wedge d\overline{\tau}.
\end{aligned} \tag{5.30}$$

Writing

$$Q = 2hQ_0 + r\ln\tau_2, \qquad Q_0 = \frac{\pi}{2\tau_2}(u-\overline{u})^2, \tag{5.31}$$

and noting that $du^i = d_0u + ((u^i-\overline{u}^i)/2i\tau_2)d\tau$, we see that the two terms of $\overline{\partial}_{\mathscr{T}\times t\times t}\partial_{\mathscr{T}\times t\times t}Q$ equal the two terms of (5.30). We will find the identity

$$\overline{\partial}_{\mathscr{T}\times t\times t}\partial_{\mathscr{T}\times t\times t}Q_0 = -i\frac{1}{k}\omega \equiv -i\omega_0 \tag{5.32}$$

particularly useful.

The prequantum line bundle $\tilde{\mathscr{L}}$ *and its connection.* Let $\mathscr{L}_l$ be the trivial holomorphic line bundle over $\mathscr{T} \times t \times t$. We give it the Hermitian structure

$$\langle \psi_l , \psi_l' \rangle_{\mathscr{L}_l} = \psi_l^* \, \psi_l' \, e^{lQ_0} \tag{5.33}$$

and the compatible connection with component ∇ in the $t \times t$ directions and δ in the $\mathscr{T}$ directions. By (5.32), we see that the connection has curvature $-il\omega_0$. The action of the Weyl group on the trivial bundle $\mathscr{L}_l$ is the trivial lift of the action on $t \times t$. Define the action of $\Lambda^R \times \Lambda^R$ on $\mathscr{L}_l$ so that invariant sections of $\mathscr{L}_l$ have periodicities

$$\begin{aligned} \psi_l(\tau , u + \lambda_1) &= \psi_l(\tau , u) \\ \psi_l(\tau , u + \tau\lambda_2) &= \exp(-i\pi l\tau\lambda_2^2 - 2\pi i l\lambda_1 \cdot u)\psi_l(\tau , u) \end{aligned} \tag{5.34}$$

for $\lambda_i \in \Lambda^R$. The holomorphic and Hermitian structures on $\mathscr{L}_l$ are invariant under this action. For fixed τ, sections of $\mathscr{L}_l$ with the above periodicities are identified with sections of the pushed down line bundle over $T \times T$ and are called level l theta functions. The pushdown of the bundle $\mathscr{L}_{2h}$ is isomorphic with the determinant line bundle $\mathrm{Det}(\overline{\partial})$. Thus the bundle $\mathscr{L}_k \to t \times t$ pushes down to the bundle $\mathscr{L}_0^{\otimes k} \to T \times T$ appearing in §5a. $\mathscr{L}_0^{\otimes k}$ further pushes down by the action of W to the prequantum line bundle $\tilde{\mathscr{L}}_0^{\otimes k}$ over $\mathscr{M}$. The quantum Hilbert space consists of the holomorphic sections of $\tilde{\mathscr{L}}_0^{\otimes k}$, that is the Weyl invariant theta functions at level k.

An important fact that we need for our calculation is that the multiplication map from $\mathscr{L}_{l_1} \otimes \mathscr{L}_{l_2}$ to $\mathscr{L}_{l_2}$ respects the holomorphic and Hermitian structures, as well as the action of the group $(\Lambda^R \times \Lambda^R)\tilde{\times} W$. Therefore we may use Leibniz's rule to evaluate covariant derivatives of products of theta functions. Also the holomorphicity of $\tilde{\Pi}$ together with

$$H^{1/2} = \tau_2^{r/2} < \tilde{\Pi} , \tilde{\Pi} >_{\mathscr{L}_h} \tag{5.35}$$

implies that

$$\tfrac{1}{2}(\tilde{\nabla}_{\underline{i}} \ln H) = \tilde{\Pi}^{-1} \tilde{\nabla}_{\underline{i}} \tilde{\Pi} \tag{5.36}$$

and

$$\tfrac{1}{2}(\delta^{(1,0)}\ln H) = \tilde{\Pi}^{-1}\delta^{(1,0)}\tilde{\Pi} + d\tau\frac{r}{4i\tau_2}. \tag{5.37}$$

Heat equations for theta functions. Allowing τ to vary, we require that level l theta functions are holomorphic in τ and satisfy the heat equation

$$\Box_l\psi_l = 0, \qquad \Box_l \equiv \delta^{(1,0)} - \frac{k}{l}B^{\underline{ij}}\tilde{\nabla}_{\underline{i}}\tilde{\nabla}_{\underline{j}} + d\tau\frac{r}{4i\tau_2}. \tag{5.38}$$

To write this out explicitly, we let $d/d\tau$ denote the τ derivative in coordinates $(\tau, \bar{\tau}, \theta_1, \theta_2)$, and let $\partial/\partial\tau$ denote the τ derivative in coordinates $(\tau, \bar{\tau}, u, \bar{u})$. We have

$$\frac{d}{d\tau} = \frac{\partial}{\partial\tau} + \frac{u^i - \bar{u}^i}{2i\tau_2}\frac{\partial}{\partial u^i}. \tag{5.39}$$

The connection on $\mathscr{L}_l$ is given by

$$\begin{aligned}
\nabla_{\bar{i}}\psi_l &= \frac{\partial}{\partial\bar{u}^i}\psi_l, \\
\delta^{(0,1)}\psi_l &= d\bar{\tau}\frac{d}{d\bar{\tau}}\psi_l, \\
\nabla_{\underline{i}}\psi_l &= \left[\frac{\partial}{\partial u^i} + \left(\frac{\partial}{\partial u^i}lQ_0\right)\right]\psi_l \\
&= \left[\frac{\partial}{\partial u^i} + \frac{l\pi}{\tau_2}C_{ij}(u-\bar{u})^j\right]\psi_l, \\
\delta^{(1,0)} &= d\tau\left[\frac{d}{d\tau} + \left(\frac{d}{d\tau}lQ_0\right)\right] \\
&= d\tau\left[\frac{\partial}{\partial\tau} + \frac{(u-\bar{u})^i}{2i\tau_2}\frac{\partial}{\partial u^i} + \frac{l\pi}{4i\tau_2^2}(u-\bar{u})^2\right].
\end{aligned} \tag{5.40}$$

Finally, note that

$$B^{\underline{ij}} = -\frac{1}{4}(\delta J\omega^{-1})^{\underline{ij}} = \frac{d\tau}{4ik\pi}C^{ij}. \tag{5.41}$$

Putting this altogether, we find

$$\Box_l = d\tau\left[\frac{\partial}{\partial\tau} - \frac{1}{4\pi il}C^{ij}\frac{\partial}{\partial u^i}\frac{\partial}{\partial u^j}\right], \tag{5.42}$$

which is the heat operator at level l found in the appendix. So the level l theta functions are just the theta functions $\theta_{\gamma,l}$ given in the Appendix.

Proof that Weyl-Kac characters are parallel. Let

$$\psi_{\gamma,k}(\tau, u) \equiv \chi^{H.W.}_{\gamma,k}(e^{2\pi i\tau}, e^{2\pi iu})e^{\pi i\tau(|\rho|^2/h - |\gamma+\rho|^2/(k+h))} \tag{5.43}$$

be (up to the overall constant included) the Weyl-Kac character at level k for the highest weight $\gamma \in \Lambda^R$. According to the character formula (see the Appendix),

$$\psi_{\gamma,k}(\tau, u) = \frac{\theta^-_{\gamma+\rho,k+h}(\tau, u)}{\tilde{\Pi}(\tau, u)}, \tag{5.44}$$

where $\theta^-_{\gamma+\rho,k+h}$ is the Weyl odd combinations of level k theta functions defined in the Appendix. For each τ, $\psi_{\gamma,k}$ is the ratio of a level $k+h$ to a level h Weyl odd theta function, and is thus a Weyl invariant level k theta function. So $\psi_{\gamma,k}$ is a section of $\tilde{\mathscr{H}}_Q$.

The vanishing of $\delta^{\tilde{\mathscr{H}}_Q (0,1)}\psi_k$ follows trivially from holomorpicity in τ. Since $\theta^-_{\gamma+\rho,k+h}$ satisfies the level $k+h$ heat equation, $\psi_{\gamma,k}$ is annihilated by

$$\begin{aligned}\tilde{\Pi}^{-1} \circ \Box_{k+h} \circ \tilde{\Pi} = \delta^{(1,0)} &+ (\tilde{\Pi}^{-1}\delta^{(1,0)}\tilde{\Pi}) + d\tau\frac{r}{4i\tau_2} \\ &- \frac{k}{k+h}[B^{\underline{ij}}\tilde{\nabla}_{\underline{i}}\tilde{\nabla}_{\underline{j}} + 2B^{\underline{ij}}(\tilde{\Pi}^{-1}\tilde{\nabla}_{\underline{i}}\tilde{\Pi})\tilde{\nabla}_{\underline{j}} \\ &+ B^{\underline{ij}}(\tilde{\Pi}^{-1}\tilde{\nabla}_{\underline{i}}\tilde{\nabla}_{\underline{j}}\tilde{\Pi})].\end{aligned} \tag{5.45}$$

Now using (5.36), the heat equation at level h for $\tilde{\Pi}$, and (5.37) twice, this operator equals

$$\begin{aligned}&\delta^{(1,0)} + \frac{1}{2}(\delta^{(1,0)}\ln H) \\ &- \frac{k}{k+h}\left[B^{\underline{ij}} + 2B^{\underline{ij}}\left(\frac{1}{2}\tilde{\nabla}_{\underline{i}}\ln H\right)\tilde{\nabla}_{\underline{j}} + \frac{h}{k}\frac{1}{2}(\delta^{(1,0)}\ln H)\right].\end{aligned} \tag{5.46}$$

Combining the two constant terms, we see that this equals $\delta^{\tilde{\mathscr{H}}_Q (1,0)}$. Thus $\psi_{\gamma,k}$ is annihilated by $\delta^{\tilde{\mathscr{H}}_Q}$.

Orthonormality of the Weyl-Kac characters. We define the inner product on $\tilde{\mathscr{H}}_Q$ by

$$\langle\psi_{\gamma,k}, \psi_{\gamma',k}\rangle = \int_{T\times T}\omega^r(\det{}'\overline{\partial}_\theta^\dagger\overline{\partial}_\theta)^{1/2}\langle\psi_{\gamma,k}, \psi_{\gamma',k}\rangle_{\mathscr{L}}. \tag{5.47}$$

Up to a τ independent factor, the integrand is

$$\left(\frac{1}{\tau_2^r}d^r u\, d^r\overline{u}\right)(e^{hQ_0}\tau_2^{r/2}|\tilde{\Pi}|^2)(e^{kQ_0}\psi^*_{\gamma,k}\psi_{\gamma',k}). \tag{5.48}$$

So, up to a constant,

$$\langle \psi_{\gamma,k}, \psi_{\gamma',k}\rangle = \int_{T\times T} d^r u d^r \overline{u}\, \tau_2^{-r/2}\, e^{(k+h)\pi(u-\overline{u})^2/2\tau_2} \times \theta^-_{\gamma+\rho,k+h}(\tau,u)^* \theta^-_{\gamma'+\rho,k+h}(\tau,u). \tag{5.49}$$

From the orthonormality properties given in the Appendix for the theta functions $\theta_{\gamma+\rho,k+h}$, it follows that the $\psi_{\gamma,k}$ are orthogonal and have τ independent norms. Since the $\psi_{\gamma,k}$ form a basis of parallel sections, this shows that the connection $\delta^{\tilde{\mathscr{H}}_Q}$ is unitary relative to the inner product (5.47).

6. Hilbert space structure on $\tilde{\mathscr{H}}_Q$

In this section we will briefly discuss the question of the existence of a Hilbert space structure on $\tilde{\mathscr{H}}_Q$ relative to which the connection $\delta^{\tilde{\mathscr{H}}_Q}$ is unitary. We make no attempt at complete results, but merely point out possibly useful observations for later consideration.

In the finite-dimensional case, a Hilbert space structure on $\tilde{\mathscr{H}}_Q$ relative to which $\delta^{\tilde{\mathscr{H}}_Q}$ is unitary may be obtained by pushing down the trivial Hilbert space structure on $\mathscr{H}_Q$. In this case, the inner product on $\tilde{\mathscr{H}}_Q|_J$ is given by

$$\langle \tilde{\psi}_1, \tilde{\psi}_2\rangle = \int_{\mathscr{A}} \omega^{n/2} \langle \psi_1, \psi_2\rangle_{\mathscr{L}}, \tag{6.1}$$

where the $\tilde{\psi}_i$ are elements of $\tilde{\mathscr{H}}_Q|_J$ and the ψ_i are the corresponding $\mathscr{G}_c$-invariant sections of $\tilde{\mathscr{L}}$; $\langle\cdot\rangle_{\mathscr{L}}$ is the inner product on $\mathscr{L}$; and $\omega^{n/2}$ is the symplectic volume form. The sections ψ are determined by their values along $F^{-1}(0)$. Integrating along the $\mathscr{G}_c$ orbits then gives

$$\langle \tilde{\psi}_1, \tilde{\psi}_2\rangle = \int_{\mathscr{M}} \tilde{\omega}^{m/2} \langle \tilde{\psi}_1, \tilde{\psi}_2\rangle_{\tilde{\mathscr{L}}} U, \tag{6.2}$$

where U is a function on $\mathscr{M}\times\mathscr{T}$ expressible as an integral of a top form σ along the $\mathscr{G}_c$ orbit. We will give below an explicit expression for σ along $F^{-1}(0)$ together with the differential equation determining σ along the orbit.

It was shown in [7] that in the gauge theory case, σ may be expressed in terms of the exponent of a gauged WZW action. The $\mathscr{G}_c$ integral determining U is then a (formal) functional integral, which, in the genus one case, has been studied in [12] in connection with coset models. Using the results of [12], the authors of [7] showed that on the torus, $U = H^{1/2}$, where H is the determinant $\det'(\triangle)$. We have already rigorously checked in §5 that in this case and with this choice of U the connection $\delta^{\mathscr{H}_Q}$ is unitary.

In the gauge theory case for genus bigger than one, and in the general finite-dimensional case, U depends on k; there does not seem to be a simple explicit expression for U. We may, however, attempt to evaluate U perturbatively in powers of $1/(k+h)$. The solution to leading order is still $H^{1/2}$, but now there are subleading terms.

It would be very interesting to investigate the higher order corrections. For example, in the next few paragraphs we will explain why an explicit expression for the subleading term could lead to a rigorous proof (perhaps simpler than that given in §7) that $[R^{(2,0)}]_0$ vanishes.

Differential equation for U. Since in the gauge theory case it will be difficult to make rigorous the pushdown construction of U, we begin by simply formulating in terms of the geometry of $\mathscr{M}$ the conditions that U must obey in order to give a unitary structure.

Unitarity of $\delta^{\mathscr{H}_Q}$ is the statement that for holomorphic sections $\tilde{\psi}_1$, $\tilde{\psi}_2$, one should have

$$\delta\langle\tilde{\psi}_1, \tilde{\psi}_2\rangle = \langle\delta^{\mathscr{H}_Q}\tilde{\psi}_1, \tilde{\psi}_2\rangle + \langle\tilde{\psi}_1, \delta^{\mathscr{H}_Q}\tilde{\psi}_2\rangle. \tag{6.3}$$

Writing $\delta^{\mathscr{H}_Q} = \delta^{(1,0)} + \delta^{(0,1)} - k\mathscr{O}/(k+h)$, and recalling that $\mathscr{O}$ is of type $(1, 0)$, (6.3) is equivalent to

$$\int_{\mathscr{M}} \langle\tilde{\psi}_1, \tilde{\psi}_2\rangle_{\mathscr{L}}\delta^{(1,0)}U = \int_{\mathscr{M}} \langle\tilde{\psi}_1, -\frac{k}{k+h}\mathscr{O}\tilde{\psi}_2\rangle_{\mathscr{L}}U, \tag{6.4}$$

together with the complex conjugate equation. Integrating by parts (neglecting the effects of singularities) and then requiring that the coefficient of $\langle\tilde{\psi}_1, \tilde{\psi}_2\rangle$ in the integrand vanishes, we find

$$\left(\delta^{(1,0)} + \frac{k}{k+h}\left(\nabla_{\underline{i}}B^{\underline{ij}}\nabla_{\underline{j}} - \nabla_{\underline{i}}W^{\underline{i}} - \frac{1}{2}\delta^{(1,0)}\log H\right)\right)U = 0. \tag{6.5}$$

This is the local form of the unitarity condition.

To analyze (6.5), it is convenient to write $U = H^{1/2}F$ for F a function on $\mathscr{M} \times \mathscr{T}$. With a little algebra, the local unitarity condition (6.5) may be written as

$$0 = -\frac{1}{k+h}(H^{-1/2}(\Box H^{1/2})F - k\nabla_{\underline{i}}B^{\underline{ij}}\nabla_{\underline{j}}F) + \delta^{(1,0)}F. \tag{6.6}$$

In (6.6), $\Box$ denotes the partial differential operator

$$\Box = -h\delta^{(1,0)} + k\nabla_{\underline{i}}B^{\underline{ij}}\nabla_{\underline{j}}. \tag{6.7}$$

(Note: The expression $k\nabla_{\underline{i}} B^{\underline{ij}} \nabla_{\underline{j}}$ is independent of k since B is proportional to $1/k$.)

To solve this equation order by order in $1/(k+h)$ write

$$F = 1 + \frac{1}{k+h} F_1 + \cdots + \left(\frac{1}{k+h}\right)^N F_N + \cdots . \tag{6.8}$$

Then,

$$\begin{aligned} \delta^{(1,0)} F_1 &= H^{-1/2} \Box H^{1/2}, \\ \delta^{(1,0)} F_N &= (-k\nabla_{\underline{i}} B^{\underline{ij}} \nabla_{\underline{j}} + H^{-1/2} \Box H^{1/2}) F_{N-1} \quad \text{for } N > 1. \end{aligned} \tag{6.9}$$

We conclude:

(i) The choice $U = H^{1/2}$ obeys the local unitary condition to leading order in $1/k$.

(ii) This leading order solution is exact if and only if $H^{-1/2} \Box H^{1/2}$ vanishes. (In the gauge theory problem, this is true in genus one but not otherwise.)

(iii) Since $(\delta^{(1,0)})^2 = 0$, the first order correction F_1 exists only if

$$\delta^{(1,0)} (H^{-1/2} \Box H^{1/2}) = 0. \tag{6.10}$$

The fact that, in the gauge theory case, $H^{-1/2} \Box H^{1/2}$ vanishes in genus one is essentially equivalent to the heat equation for $s^{1/2}$ (or $\tilde{\Pi}$) discussed in §5. It is because $H^{-1/2} \Box H^{1/2}$ vanishes in genus one that we were able to give an explicit proof of unitarity in genus one, with, of course, $U = H^{1/2}$.

Point (iii) on the above list is of particular importance, since (6.10) is equivalent to the identity required at the very end of §4 to complete the proof of the vanishing of the $(2, 0)$ part of the curvature of the connection $\delta^{\tilde{\mathscr{H}}_Q}$ without appealing to global holomorphic considerations. While it is no surprise that unitarity would imply vanishing of the $(2, 0)$ curvature, the point to be noted here is that unitarity has to be understood only approximately, up to terms of order $1/k$, to get an exact result for vanishing of the $(2, 0)$ curvature.

§7 will be devoted to a proof of (6.10); evidently the identity could also be proved, perhaps more directly, by constructing the object F_1. F_1 could be constructed in principle, and probably also in practice, by carrying out the pushdown construction of U perturbatively in $1/k$. For completeness we now present the simple calculation that (6.10) is equivalent to the criterion in §4 for vanishing of the $(2, 0)$ curvature. In §4 we showed

$$[R^{(2,0)}]_0 = [\nabla_{\underline{i}} B^{\underline{ij}} \nabla_{\underline{j}} + B^{\underline{ij}} (\nabla_{\underline{i}} \log H) \nabla_{\underline{j}}](-\tfrac{1}{2} \delta^{(1,0)} \log H). \tag{6.11}$$

We compute

$$(6.12.1)\quad -\nabla_{\underline{i}}B^{\underline{ij}}\nabla_{\underline{j}}\delta^{(1,0)}(\log H^{\frac{1}{2}}) = -\nabla_{\underline{i}}B^{\underline{ij}}(\delta^{(1,0)}\nabla_{\underline{j}}\log H^{\frac{1}{2}})$$

$$(6.12.2)\quad = +\nabla_{\underline{i}}(\delta^{(1,0)}B^{\underline{ij}}\nabla_{\underline{j}}\log H^{\frac{1}{2}}) = +\delta^{(1,0)}(\nabla_{\underline{i}}B^{\underline{ij}}\nabla_{\underline{j}}\log H^{\frac{1}{2}})$$

$$(6.12.3)\quad = \delta^{(1,0)}(H^{-\frac{1}{2}}(\nabla_{\underline{i}}B^{\underline{ij}}\nabla_{\underline{j}}H^{\frac{1}{2}}) - B^{\underline{ij}}(\nabla_{\underline{i}}\log H^{\frac{1}{2}})(\nabla_{\underline{j}}\log H^{\frac{1}{2}}))$$

$$(6.12.4)\quad = \delta^{(1,0)}(H^{-\frac{1}{2}}(\nabla_{\underline{i}}B^{\underline{ij}}\nabla_{\underline{j}}H^{\frac{1}{2}}) - \tfrac{1}{4}B^{\underline{ij}}(\nabla_{\underline{i}}\log H)(\nabla_{\underline{j}}\log H)).$$

In (6.12.1) and (6.12.2) the derivatives $\delta^{(1,0)}$ and $\nabla_{\underline{i}}$ commute because there is no curvature in the holomorphic-holomorphic directions. The sign change in (6.12.3) comes from commuting forms. Similarly,

$$(6.13)\quad \begin{aligned} &-B^{\underline{ij}}(\nabla_{\underline{i}}\log H)\nabla_{\underline{j}}\delta^{(1,0)}\log H^{1/2} \\ &= \tfrac{1}{4}\delta^{(1,0)}(B^{\underline{ij}}(\nabla_{\underline{i}}\log H)(\nabla_{\underline{j}}\log H)). \end{aligned}$$

Combining (6.12) and (6.13) proves the required result.

The pushdown inner product. Finally, we will now describe a general theoretical formula for the pushdown construction of the function U that is needed for unitarity.

Let $\pi_c : \mathscr{A} \mapsto \mathscr{A}/\mathscr{G}_c \cong \mathscr{M}$ be the projection map, and for a point $A \in F^{-1}(0)$, let $i_A : \mathscr{G}_c \mapsto \mathscr{A}$ be the map $g \mapsto g \cdot A$. Let ψ_1 and ψ_2 be $\mathscr{G}_c$ invariant sections of $\mathscr{L}$ over $\operatorname{Im} i_A$ such that $<\psi_1, \psi_2>= 1$ at the point A. On the $\mathscr{G}_c$ orbit $\operatorname{Im} i_A$,

$$(6.14)\quad \omega^{n/2}\langle\psi_1, \psi_2\rangle_{\mathscr{L}} = \pi_c^*(\tilde{\omega})^{m/2}\sigma$$

for some form top form σ along $\operatorname{Im} i_A$. The pullback $i_A^*\sigma$ is a top form on $\mathscr{G}_c$ (we assume that $\mathscr{G}_c$ has discrete isotropy subgroups).

According to (6.2), U is determined by σ; in fact, the value of U at the point $\pi_c(A)$ is

$$(6.15)\quad U(\pi_c(A)) = \int_{\mathscr{G}_c} i_A^*\sigma.$$

The form σ can be determined as follows. On $F^{-1}(0)$, we have $T\mathscr{A} \cong \widehat{T\mathscr{M}} \oplus T(\mathbf{g}_c)$ as symplectic vector spaces. Of course, $T(\mathbf{g}_c) = i_{A*}(\mathbf{g}_c)$. Since $\omega^{n/2}$ and $\pi_c^*(\tilde{\omega})^{m/2}$ are the natural symplectic volumes on $T\mathscr{A}$ and $\widehat{T\mathscr{M}}$, and since $<\psi_1, \psi_2>= 1$ on $F^{-1}(0)$, σ must be the natural symplectic volume form on $T(\mathbf{g}_c)$. By the definition of the determinant H, this natural symplectic volume form pulls back to

$$(6.16)\quad i_A^*\sigma = H \cdot \sigma_0,$$

where σ_0 is a fixed volume form on $T(\mathbf{g}_c)$.

The variation of σ along the G_c orbit can then be determined by computing its Lie derivatives. In fact

$$\mathscr{L}_{T_a}\sigma = 0, \qquad \mathscr{L}_{JT_a}\sigma = 2F_a\sigma. \tag{6.17}$$

Equations (6.16) and (6.17) determine σ. It has Gaussian decay as one moves slightly away from $F^{-1}(0)$, and so the integral (6.15) which gives U may be evaluated for large k by steepest descent. The leading order solution for U is $H^{1/2}$, as claimed above.

7. Explicit proof that $R^{(2,0)}$ vanishes

This is the most technical section of the paper. Its main purpose is to complete the proof, begun in §4, that the $(2, 0)$ curvature of the connection $\delta^{\mathscr{H}_Q}$ vanishes for the gauge theory problem where there is no a priori argument available. In §4 it was shown that the vanishing of $R^{(2,0)}$ followed from the vanishing of its zeroth order piece $[R^{(2,0)}]_0$. In principle, however, had we not been able to make such a simplification by the intrinsic considerations of §4, we could have proved the vanishing of all of $R^{(2,0)}$ by the methods used below.

The identity which we wish to prove the determinant $H = \det{}'(T_z^\dagger T_z)$ satisfies is

$$\begin{aligned} 0 &= \left(\frac{k+h}{k}\right)^2 [R^{(2,0)}]_0 = [\mathscr{O}_{2+1}(\mathscr{O}_0)]_0 \\ &= (\tilde\nabla_{\underline{i}} B^{\underline{i}\underline{j}} \tilde\nabla_{\underline{j}} + W^{\underline{j}}\tilde\nabla_{\underline{j}})\left(-\frac{1}{2}\delta^{(1,0)}\ln H\right). \end{aligned} \tag{7.1}$$

This is an identity for intrinsic objects on $\mathscr{M}$. As discussed in §3, $\mathscr{O}_{2+1}$ and $\mathscr{O}_0$ may be written in terms of Green's functions and their derivatives on $F^{-1}(0)$:

$$\begin{aligned} \mathscr{O}_{2+1} &= \mathscr{K}^{\underline{k}}{}_i \nabla_{\underline{k}} M^{\underline{i}\underline{j}} \mathscr{K}^{\underline{l}}{}_j \nabla_{\underline{l}} + \mathscr{K}^{\underline{k}}{}_j (\nabla_{\underline{k}} \mathscr{K}^{\underline{j}}{}_{\underline{l}}) M^{\underline{l}\underline{i}} \mathscr{K}^{\underline{m}}{}_{\underline{i}} \nabla_{\underline{m}}, \\ \mathscr{O}_0 &= \tfrac{i}{4}\mathrm{Tr}(T_z^{-1}(\delta^{(1,0)}J)T_{\bar z}), \end{aligned} \tag{7.2}$$

where the derivatives in $\mathscr{O}_{2+1}$ act on the function $\mathscr{O}_0$.

The proof of (7.1) will be motivated by the following philosophy. A priori, in finite dimensions, the vanishing of $[R^{(2,0)}]_0$ follows immediately from two facts. First, the $(2, 0)$ curvature of the connection $\delta^{\mathscr{H}_Q}$ on $\mathscr{H}_Q$ vanishes trivially; and, second, the pushdown connection $\delta^{\mathscr{H}_Q}$ has the same curvature as $\delta^{\mathscr{H}_Q}$. In the gauge theory case, this is only a formal argument since it involves ill-defined operators. It can however be used to provide an outline for a rigorous argument by rephrasing the two facts above not in terms of the operators themselves, but in terms of the *coefficients* of the pushdown operators. More precisely, as we saw in §3, if we look at operators such as those appearing in the above argument, the coefficients of terms which do not vanish when acting on GIHS (gauge invariant holomorphic sections) over $F^{-1}(0)$ are given by explicit expressions written in terms of Green's functions. Looking at the degree zero piece of the operators appearing in $R^{(2,0)}$ provides the desired outline for the proof: On the one hand, the first fact, that the $(2, 0)$ curvature of $\delta^{\mathscr{H}_Q}$ upstairs vanishes, suggests that a certain nontrivial combination of explicit functions on $F^{-1}(0)$ sum to zero. The second fact, that the curvature of the pushdown connection is the same as the curvature of $\delta^{\mathscr{H}_Q}$, suggests that this same combination of functions must sum to desired expression $[\mathscr{O}_{2+1}(\mathscr{O}_0)]_0$ appearing in (7.1).

The proof that $[R^{(2,0)}]_0$ vanishes then amounts to showing that what is suggested formally is in fact true. In this section we shall write down in complete detail the manipulations of Green's functions which show the desired result. (The proof also relies on the manipulation of §3 which shows that the pushdown connection is the same as the connection discussed in §4.) For the gauge theory problem, one must then show that, with a suitable regularization, these manipulations are rigorously valid. We do not carry through the analysis of regularization here, although we strongly believe that no essential complications would arise in doing so. Our basis for this belief is that the vast experience of physicists shows that the relevant physical system, the $b-c$ system, has only the anomalies discussed in §2; but these anomalies should not spoil the manipulations below. We should point out, however, that the manipulations of this section do not appear to simplify when expressed in a more physical notation. Since the anomalies are not relevant for our consideration here, we will not see the shift $k \to k+h$ in our discussion.

This section is divided into several parts. In §§7a and 7b we develop some machinery which allows us to systematize the procedure of pushing down operators and identities which was discussed in §3. In §7a we define some useful geometric objects. In §7b we define a means to extract from an operator on $\mathscr{L}$ the coefficient of the zero-grading piece of the pushdown operator. In §7c we rederive the connection $\delta^{\mathscr{H}_Q}$ using this machinery; along the way we find some useful identities. In §7d we review the a priori argument that $[R^{(2,0)}]_0$ vanishes in a way which will motivate the calculations in §7e. In §7e we present the heart of the calculation: By looking at the zero-grading pieces of the pushdown operators, we interpret the a priori argument that $[R^{(2,0)}]_0$ vanishes in terms of relationships between explicit functions on $F^{-1}(0)$. Neglecting issues of regularization, we then prove these relationships using manipulations with Green's functions.

7a. Some useful geometric objects and identities. In this subsection we will develop some machinery which makes the pushdown procedure more transparent. To begin, we define the holomorphic derivatives $\nabla^{\mathscr{M}}$ and $\nabla^{\mathscr{G}}$ in the $\mathscr{M}$ and $\mathscr{G}$ directions by

$$\nabla^{\mathscr{M}}_{\underline{i}} = \mathscr{K}^{\underline{j}}{}_{\underline{i}}\nabla_{\underline{j}}, \qquad \nabla^{\mathscr{G}}_{\underline{i}} = (\pi_I)^{\underline{j}}{}_{\underline{i}}\nabla_{\underline{j}}, \tag{7.3}$$

so that $\nabla_{\underline{i}} = \nabla^{\mathscr{M}}_{\underline{i}} + \nabla^{\mathscr{G}}_{\underline{i}}$. We may then compute the following commutators:

$$\begin{aligned} &[\nabla^{\mathscr{M}}_{\underline{i}}, \nabla^{\mathscr{M}}_{\underline{j}}] = (\mathscr{M}\mathscr{M})^{\underline{k}}_{\underline{ij}}\nabla^{\mathscr{M}}_{\underline{k}}, \\ &(\mathscr{M}\mathscr{M})^{\underline{k}}_{\underline{ij}} = \mathscr{K}^{\underline{l}}{}_{\underline{i}}\nabla_{\underline{l}}\mathscr{K}^{\underline{k}}{}_{\underline{j}} - \mathscr{K}^{\underline{l}}{}_{\underline{j}}\nabla_{\underline{l}}\mathscr{K}^{\underline{k}}{}_{\underline{i}}, \end{aligned} \tag{7.4.1}$$

$$\begin{aligned} &[\nabla^{\mathscr{M}}_{\underline{i}}, \nabla^{\mathscr{G}}_{\underline{j}}] = (\mathscr{M}G)^{\underline{k}}_{\underline{ij}}\nabla^{\mathscr{M}}_{\underline{k}}, \\ &(\mathscr{M}\mathscr{G})^{\underline{k}}_{\underline{ij}} = \mathscr{K}^{\underline{l}}{}_{\underline{i}}\nabla_{\underline{l}}(\pi_I)^{\underline{k}}{}_{\underline{j}} - (\pi_I)^{\underline{l}}{}_{\underline{j}}\nabla_{\underline{l}}\mathscr{K}^{\underline{k}}{}_{\underline{i}}, \end{aligned} \tag{7.4.2}$$

$$[\nabla^{\mathscr{G}}_{\underline{i}}, \nabla^{\mathscr{G}}_{\underline{j}}] = 0. \tag{7.4.3}$$

The important point here is that the commutator of any two derivatives is necessarily in the $\mathscr{M}$ direction. This is a consequence of the fact that the holomorphic variation $\nabla_{\underline{i}}\mathscr{K}$ maps into the image of $\mathscr{K}$. The commutators (7.4) are easily derived using the explicit formulas (3.30) for $\nabla\mathscr{K}$. For example, for (7.4.3), we compute

$$\begin{aligned} (7.5.1)\quad (GG)^{\underline{k}}_{\underline{ij}} &= (\pi_I)^{\underline{l}}{}_{\underline{i}}\nabla_{\underline{l}}(\pi_I)^{\underline{k}}{}_{\underline{j}} - (\pi_I)^{\underline{l}}{}_{\underline{j}}\nabla_{\underline{l}}(\pi_I)^{\underline{k}}{}_{\underline{i}} \\ (7.5.2)\quad &= (T_z^{-1})^a{}_{\underline{i}}(T_z^{-1})^b{}_{\underline{j}}\mathscr{K}^{\underline{k}}{}_{\underline{m}}(-T^{\underline{l}}_{za}\nabla_{\underline{l}}T^{\underline{m}}_{z\ b} + T^{\underline{l}}_{zb}\nabla_{\underline{l}}T^{\underline{m}}_{z\ a}) \\ (7.5.3)\quad &= (T_z^{-1})^a{}_{\underline{i}}(T_z^{-1})^b{}_{\underline{j}}\mathscr{K}^{\underline{k}}{}_{\underline{m}}f_{ab}{}^c T^{\underline{m}}_{z\ c} \\ (7.5.4)\quad &= 0. \end{aligned}$$

(7.5.3) follows since the T_z form a group representation, and (7.5.4) follows since $\mathscr{K}T_z = 0$.

Next, to make the group action more transparent, we introduced some slightly modified covariant derivatives. Recall that the quantum mechanical generators are

$$\hat{F}_a = \frac{1}{i} T^{\underline{i}}_{za} \nabla_{\underline{i}} + F_a. \tag{7.6}$$

We define modified covariant derivatives $\nabla'^{\mathcal{G}}_{\underline{i}}$ and $\nabla'_{\underline{i}}$ by

$$\nabla'^{\mathcal{G}}_{\underline{i}} = i(T_z^{-1})^a{}_{\underline{i}} \hat{F}_a = \nabla^{\mathcal{G}}_{\underline{i}} + iF_{\underline{i}}, \qquad F_{\underline{i}} \equiv (T_z^{-1})^a{}_{\underline{i}} F_a, \tag{7.7.1}$$

$$\nabla'_{\underline{i}} \equiv \nabla^{\mathcal{M}}_{\underline{i}} + \nabla'^{\mathcal{G}}_{\underline{i}}. \tag{7.7.2}$$

That is, $\nabla'_{\underline{i}}$ equals $\nabla^{\mathcal{M}}_{\underline{i}}$ in the $\mathcal{M}$ directions and the quantum mechanical generators in the G directions. Consistent with these definitions we write

$$\nabla'^{\mathcal{G}}_a = i\hat{F}_a = T^{\underline{i}}_{za} \nabla'^{\mathcal{G}}_{\underline{i}}, \qquad a = \text{a group index.} \tag{7.8}$$

The requirement that the group generators form a representation leads us to expect

$$[\nabla'^{\mathcal{G}}_{\underline{i}}, \nabla'^{\mathcal{G}}_{\underline{j}}] = 0. \tag{7.9}$$

Using the fact that $[\nabla^{\mathcal{G}}_{\underline{i}}, \nabla^{\mathcal{G}}_{\underline{j}}] = 0$ this is seen to be equivalent to

$$\nabla^{\mathcal{G}}_{\underline{i}} F_{\underline{j}} = \nabla^{\mathcal{G}}_{\underline{j}} F_{\underline{i}}. \tag{7.10}$$

This identity is fundamental in what follows.

We will prove (7.10) as a consequence of two other identities. The first is a "structure equation" along the $\mathcal{G}_c$ orbits: Think of $(T_z^{-1})^c{}_{\underline{i}}$ as a basis of $\mathcal{G}_c$ invariant $(1, 0)$-forms along the fibers of the $\mathcal{G}_c$-action. We have

$$\begin{aligned}(dT_z^{-1})^c_{\underline{ij}} &\equiv \nabla_{\underline{i}}(T_z^{-1})^c{}_{\underline{j}} - \nabla_{\underline{j}}(T_z^{-1})^c{}_{\underline{i}} \\ &= -f_{ab}{}^c (T_z^{-1})^a{}_{\underline{i}} (T_z^{-1})^b{}_{\underline{j}} \\ &\quad - [T_z^{-1}(\nabla_{\underline{k}} T_z) T_z^{-1}]^c{}_{\underline{l}} [\mathcal{K}^{\underline{k}}{}_{\underline{i}} (\pi_I)^{\underline{l}}{}_{\underline{j}} - \mathcal{K}^{\underline{l}}{}_{\underline{j}} (\pi_I)^{\underline{k}}{}_{\underline{i}}].\end{aligned} \tag{7.11}$$

To prove (7.11), first note that

$$\begin{aligned}(\nabla_{\underline{i}} T_z^{-1})^c{}_{\underline{j}} &= -(T_z^{-1}((\nabla^{\mathcal{G}}_{\underline{i}} + \nabla^{\mathcal{M}}_{\underline{i}}) T_z) T_z^{-1})^c{}_{\underline{j}} \\ &= -(T_z^{-1})^a{}_{\underline{i}} (T_z^{-1})^c{}_{\underline{m}} (T^{\underline{l}}_{za} \nabla_{\underline{l}} T_z^{\underline{m}}{}_b)(T_z^{-1})^b{}_{\underline{j}} \\ &\quad - (T_z^{-1}(\nabla^{\mathcal{M}}_{\underline{i}} T_z) T_z^{-1})^c{}_{\underline{j}},\end{aligned} \tag{7.12}$$

so

$$(dT_z^{-1})^c_{\underline{i}\underline{j}} = -(T_z^{-1})^a{}_{\underline{i}}(T_z^{-1})^b{}_{\underline{j}}[T^{\underline{l}}_{za}(\nabla_{\underline{l}}(T_z^{-1})^b{}_{\underline{m}}) - T^{\underline{l}}_{zb}(\nabla_{\underline{l}}T^{\underline{m}}_{z\ a})](T_z^{-1})^c{}_{\underline{m}} - [T_z^{-1}(\nabla_{\underline{k}}T_z)T_z^{-1}]^c_{\underline{l}}(\mathscr{K}^{\underline{k}}{}_{\underline{i}}(\pi_I)^{\underline{l}}{}_{\underline{j}} - \mathscr{K}^{\underline{k}}{}_{\underline{j}}(\pi_I)^{\underline{l}}{}_{\underline{i}}). \tag{7.13}$$

But, by the group law

$$\begin{aligned}[T^{\underline{l}}_{za}(\nabla_{\underline{l}}(T_z^{-1})^{\underline{m}}_{b}) - T^{\underline{l}}_{zb}(\nabla_{\underline{l}}T^{\underline{m}}_{z\ a})](T_z^{-1})^c{}_{\underline{m}} &= f_{ab}{}^d T^{\underline{m}}_{z\ d}(T_z^{-1})^c{}_{\underline{m}} \\ &= f_{ab}{}^c,\end{aligned} \tag{7.14}$$

completing the proof of (7.11).

The second identity needed to prove (7.10) is

$$[(\omega T_{\bar{z}}T_z^{-1}) - (\omega T_{\bar{z}}T_z^{-1})^T]_{\underline{i}\underline{j}} = (T_z^{-1})^a{}_{\underline{i}}(T_z^{-1})^b{}_{\underline{j}}f_{ab}{}^c F_c. \tag{7.15}$$

To prove this, note that, since F is a moment map,

$$f_{ab}{}^c F_c = \omega(T_a, T_b) = (T_z{}^T\omega T_{\bar{z}})_{ab} + (T_{\bar{z}}{}^T\omega T_z)_{ab}. \tag{7.16}$$

Multiplying by $(T_z^{-1})^a{}_{\underline{i}}(T_z^{-1})^b{}_{\underline{j}}$ then yields (7.15).

Finally, to prove (7.10), note that

$$\begin{aligned}\nabla_{\underline{i}}F_{\underline{j}} = \nabla_{\underline{i}}(F_a(T_z^{-1})^a{}_{\underline{j}}) &= (\nabla_{\underline{i}}F_a)(T_z^{-1})^a{}_{\underline{j}} + F_a(\nabla_{\underline{i}}(T_z^{-1})^a{}_{\underline{j}}) \\ &= (\omega T_{\bar{z}}T_z^{-1})_{\underline{i}\underline{j}} - F_a(T_z^{-1}(\nabla_{\underline{i}}T_z)T_z^{-1})^a_{\underline{j}}.\end{aligned} \tag{7.17}$$

Applying (7.11) and (7.15) then gives

$$\nabla_{\underline{i}}F_{\underline{j}} - \nabla_{\underline{j}}F_{\underline{i}} = -F_a(T_z^{-1}\nabla_{\underline{k}}T_z^{-1})^a{}_{\underline{l}}(\mathscr{K}^{\underline{k}}{}_{\underline{i}}(\pi_I)^{\underline{l}}{}_{\underline{j}} - \mathscr{K}^{\underline{k}}{}_{\underline{j}}(\pi_I)^{\underline{l}}{}_{\underline{i}}). \tag{7.18}$$

This vanishes if $\underline{i}$ and $\underline{j}$ are both in the $\mathscr{G}$ direction, thus proving (7.10).

7b. Operator grading. In §7e we will express the a priori argument that $[R^{(2,0)}]_0$ vanishes, in terms of the coefficients of the zeroth order pieces of pushdown operators. To do this we will need to assign to a differential operator D on $\mathscr{L}$ with only holomorphic derivatives a function $[D]_0$ on $F^{-1}(0)$ which is the zeroth order piece of the pushdown differential operator.

A suitable definition is the following: Let D be a differential operator on $\mathscr{L}$ with only holomorphic derivatives. Choose any decomposition of D of the form

$$D = [D]_0 + \sum D^{\underline{i}_1\cdots\underline{i}_n\underline{j}_1\cdots\underline{j}_m}\nabla^{\mathscr{M}}_{\underline{i}_1}\cdots\nabla^{\mathscr{M}}_{\underline{i}_n}\nabla'^{\mathscr{G}}_{\underline{j}_1}\cdots\nabla'^{\mathscr{G}}_{\underline{j}_m}, \tag{7.19}$$

where all the $\nabla'^{\mathscr{G}}$ derivatives are to the right, and all the $\nabla^{\mathscr{M}}$ derivatives are to the left. Define $[D]_0$ as indicated to be the piece of this decomposition with no derivative operators.

Such a decomposition of D always exists. It is not unique because $\nabla^{\mathscr{M}}_{\underline{i}}$ does not commute with $\nabla^{\mathscr{M}}_{\underline{j}}$. However, using the commutation relations derived above, it may be checked that the zero grading piece $[D]_0$ is unique. This grading pushes down to the natural notion of grading of operators with only holomorphic derivatives on $\mathscr{M}$ since the operators $\nabla'^{\mathscr{G}}$ annihilate GIHS.

7c. The pushdown connection $\delta^{\mathscr{H}_Q}$.

Verification that $\delta^{\mathscr{H}_Q}$ respects the $\mathscr{G}$-action. We have already seen that the connection $\delta^{\mathscr{H}_Q}$ preserves holomorphic sections. Before rederiving the connection $\delta^{\mathscr{H}_Q}$ in the notation of the previous subsection, we check here that $\delta^{\mathscr{H}_Q}$ also respects the group action. Equivalently, we must show that the group generators $\nabla'^{\mathscr{G}}_a$ are parallel: We have

$$[\delta^{\mathscr{H}_Q}, -i\nabla'^{\mathscr{G}}_a] = [\delta^{(1,0)} - M^{\underline{ij}}\nabla_{\underline{i}}\nabla_{\underline{j}}, -iT^{\underline{k}}_{z\,a}\nabla_{\underline{k}} + F_a]$$

$$= -\tfrac{1}{2}\delta J^{\underline{i}}{}_{\bar{j}}T^{\bar{j}}_{\bar{z}\,a}\nabla_{\underline{i}} \tag{T7.20.1}$$

$$- 2(\nabla_{\underline{i}}T^{\underline{k}}_{z\,a})M^{\underline{ij}}\nabla_{\underline{j}}\nabla_{\underline{k}} \tag{T7.20.2}$$

$$- 2(\nabla_{\underline{i}}F_a)M^{\underline{ij}}\nabla_{\underline{j}} \tag{T7.20.3}$$

$$- M^{\underline{ij}}(\nabla_{\underline{i}}\nabla_{\underline{j}}F_a). \tag{T7.20.4}$$

(The above is a definition of the terms (T7.20.1)–(T7.20.4). We will refer to the above equation by (T7.20). We use similar notation throughout the section.) Now

$$\nabla_{\underline{i}}\nabla_{\underline{j}}F_a = 0 \tag{7.21}$$

since $\mathscr{G}$ acts holomorphically and by the definition of F_a. Thus (T7.20.4) is zero. Also using the definition of F_a, we find

$$(\text{T7.20.3}) = -2\omega_{\underline{i}\bar{k}}T^{\bar{k}}_{\bar{z}\,a}(\tfrac{-1}{4}\delta J\omega^{-1})^{\underline{ij}}\nabla_{\underline{j}} = -(\text{T7.20.1}). \tag{7.22}$$

Finally, to see that (T.20.2) vanishes, recall first that ω and δJ and so also M are group invariant. Therefore,

$$0 = (\nabla_{\underline{i}}T^{\underline{k}}_{z\,a})M^{\underline{ij}} + (\underline{j} \leftrightarrow \underline{k}). \tag{7.23}$$

But then

$$(7.24)\qquad (\mathrm{T}7.20.2) = -2\,\underbrace{(\nabla_{\underline{i}} T^{\underline{k}}_{z\,a}) M^{\underline{ij}}}_{\text{odd in } (\underline{j}\leftrightarrow\underline{k})}\;\underbrace{\nabla_{\underline{j}}\nabla_{\underline{k}}}_{\text{even in } (\underline{j}\leftrightarrow\underline{k})} = 0.$$

(The holomorphic derivatives in (7.24) commute since ω is of type $(1,1)$.)

Combining (7.21)–(7.24) we see that $[\delta^{\mathscr{H}_Q}, \nabla'^{\mathscr{G}}_a]$ vanishes. The connection $\delta^{\tilde{\mathscr{H}}_Q}$ thus respects the group action.

Deriving the pushdown connection. For $\mathscr{A}$ finite dimensional, $\delta^{\tilde{\mathscr{H}}_Q} = \delta - \mathscr{O}^{up}$ is holomorphicity preserving and respects the group action and so pushes down to the desired projectively flat connection $\delta^{\tilde{\mathscr{H}}_Q} = \delta - \mathscr{O}$ on $\tilde{\mathscr{H}}_Q$ (see §1). We found an operator $\mathscr{O}$ in §3 which was equivalent to $\mathscr{O}^{up}$ when acting on $\mathscr{G}$-invariant holomorphic sections over $F^{-1}(0)$, but which involves only derivatives in the $\mathscr{M}$ directions—the $\mathscr{G}$ derivatives were solved by using

$$(7.25)\qquad \nabla'^{\mathscr{G}}_a = \nabla^{\mathscr{G}}_a + iF_a = 0 \quad \text{on GIHS.}$$

The operator $\mathscr{O}$ has a well-defined regularization in the gauge theory case.

We now want to rederive the expression for $\mathscr{O}$ in a more systematic fashion. We decompose $\mathscr{O}^{up}$ in the form (7.19), and further separate out those terms whose coefficients vanish on $F^{-1}(0)$. First, we compute the following identities

$$(7.26)\qquad \begin{aligned} \nabla_{\underline{i}}\nabla_{\underline{j}} &= [\nabla'_{\underline{i}} - iF_{\underline{i}}][\nabla'_{\underline{j}} - iF_{\underline{j}}] \\ &= -i[F_{\underline{i}}\nabla'_{\underline{j}} + F_{\underline{j}}\nabla'_{\underline{i}}] - F_{\underline{i}}F_{\underline{j}} - i\nabla^{\mathscr{M}}_{\underline{i}} F_{\underline{j}} - i(\nabla^{\mathscr{G}}_{\underline{i}} F_{\underline{j}}) + \nabla^{\mathscr{G}}_{\underline{i}}\nabla^{\mathscr{G}}_{\underline{j}}, \end{aligned}$$

$$(7.27)\qquad \nabla'_{\underline{i}}\nabla'_{\underline{j}} = \nabla^{\mathscr{M}}_{\underline{i}}\nabla^{\mathscr{M}}_{\underline{j}} + [\nabla'^{\mathscr{G}}_{\underline{i}}, \nabla^{\mathscr{M}}_{\underline{j}}] + \nabla^{\mathscr{M}}_{\underline{i}}\nabla'^{\mathscr{G}}_{\underline{j}} + \nabla^{\mathscr{M}}_{\underline{j}}\nabla'^{\mathscr{G}}_{\underline{i}} + \nabla'^{\mathscr{G}}_{\underline{i}}\nabla'^{\mathscr{G}}_{\underline{j}},$$

$$(7.28)\qquad \begin{aligned} [\nabla'^{\mathscr{G}}_{\underline{i}}, \nabla^{\mathscr{M}}_{\underline{j}}] &= [\nabla^{\mathscr{G}}_{\underline{i}}, \nabla^{\mathscr{M}}_{\underline{j}}] - i(\nabla^{\mathscr{M}}_{\underline{j}} F_{\underline{i}}) \\ &= -(\mathscr{MG})^{\underline{k}}_{\underline{ij}}\nabla^{\mathscr{M}}_{\underline{k}} - i(\nabla^{\mathscr{M}}_{\underline{j}} F_{\underline{i}}). \end{aligned}$$

For (7.28) we have used the fact (see (7.4)) that the $\underline{k}$ index of $(\mathscr{MG})^{\underline{k}}_{\underline{ij}}$ is necessarily in the $\mathscr{M}$ direction.

Using (7.26)–(7.28) we obtain the decomposition

$$(7.29)\qquad \mathscr{O}^{up} = M^{\underline{ij}}\nabla_{\underline{i}}\nabla_{\underline{j}} = \mathscr{O} + \mathscr{O}^G + \mathscr{O}^F,$$

where

$$\mathscr{O} = \mathscr{O}_{2+1} + \mathscr{O}_0, \tag{7.30.1}$$

$$\mathscr{O}_{2+1} = M^{\underline{ij}}(\nabla^{\mathscr{M}}_{\underline{i}} \nabla^{\mathscr{M}}_{\underline{j}} - (\mathscr{M}\mathscr{G})^{\underline{k}}_{\underline{ij}} \nabla^{\mathscr{M}}_{\underline{k}}), \tag{7.30.2}$$

$$\mathscr{O}_0 = -iM^{\underline{ij}}(\nabla^{\mathscr{G}}_{\underline{i}} F_{\underline{j}}), \tag{7.30.3}$$

$$\mathscr{O}^G = M^{\underline{ij}}(\nabla'_{\underline{i}} \nabla'^{\mathscr{G}}_{\underline{j}} + \nabla^{\mathscr{M}}_{\underline{j}} \nabla'^{\mathscr{G}}_{\underline{i}}) = M^{\underline{ij}}(2\nabla^{\mathscr{M}}_{\underline{i}} + \nabla'^{\mathscr{G}}_{\underline{i}})\nabla'^{\mathscr{G}}_{\underline{j}}, \tag{7.30.4}$$

$$\mathscr{O}^F = M^{\underline{ij}}(-2iF_{\underline{i}}\nabla'_{\underline{j}} - F_{\underline{i}}F_{\underline{j}} - 2i\nabla^{\mathscr{M}}_{\underline{j}} F_{\underline{i}}). \tag{7.30.5}$$

The various operators (7.30) appearing in the decomposition (7.29) of $\mathscr{O}^{up}$ have the following properties:

1. The operator $\mathscr{O}^F$ vanishes on $F^{-1}(0)$ since $F_{\underline{i}}$ and $\nabla^{\mathscr{M}}_{\underline{j}} F_{\underline{i}}$ vanish there.

2. The operator $\mathscr{O}^G$ vanishes when acting on GIHS since it involves a derivative $\nabla'^{\mathscr{G}}_{\underline{i}}$ on the right.

3. As desired, the operator $\mathscr{O}$ only involves derivatives in the $\mathscr{M}$ directions and agrees with the operator $\mathscr{O}^{up}$ when acting on GIHS over $F^{-1}(0)$.

4. $\mathscr{O}$ is further decomposed into two pieces $\mathscr{O}_{2+1}$ and $\mathscr{O}_0$. $\mathscr{O}_0$ is the zeroth order piece of $\mathscr{O}$. $\mathscr{O}_{2+1}$ is the sum of the purely first and second order pieces of $\mathscr{O}$.

Using the explicit expression for $(\mathscr{M}\mathscr{G})^{\underline{k}}_{\underline{ij}}$ given in (7.4), we obtain explicit expressions for $\mathscr{O}_{2+1}$ and $\mathscr{O}_0$ in terms of Green's functions. It may be checked that these expressions agree with those derived in §3.

7d. The formal argument that $R^{(2,0)}$ vanishes. We briefly present, in a way that will be useful for the arguments to follow, the reason why $[R^{(2,0)}]_0$ vanishes in the finite dimensional case. We first observe that

$$\mathscr{O}^{up} \wedge \mathscr{O}^{up} = 0, \tag{7.31.1}$$

$$\mathscr{O}^F \wedge \mathscr{O} = 0 \quad \text{on } F^{-1}(0), \tag{7.31.2}$$

$$[\delta^{(1,0)} - \mathscr{O}^{up}, \mathscr{O}^{\mathscr{G}}] = 0 \quad \text{on GIHS}, \tag{7.31.3}$$

$$[\delta^{(1,0)} \mathscr{O}^{\mathscr{G}}]_0 = 0. \tag{7.31.4}$$

These facts are simple to verify. For example, (7.31.3) follows since $\mathscr{O}^{\mathscr{G}}$ annihilates GIHS and $\delta^{(1,0)} - \mathscr{O}^{up}$ preserves GIHS.

Using (7.31.1)–(7.31.4) we obtain two different expressions for $[\mathscr{O}^{\mathscr{G}} \wedge \mathscr{O}^{up}]_0$. On the one hand, from the decomposition $\mathscr{O}^{up} = \mathscr{O} + \mathscr{O}^{\mathscr{G}} + \mathscr{O}^F = \mathscr{O}_{2+1} + \mathscr{O}_0 + \mathscr{O}^{\mathscr{G}} + \mathscr{O}^F$, we obtain

$$\begin{aligned}[\mathscr{O}^{\mathscr{G}} \wedge \mathscr{O}^{up}]_0 &= [\mathscr{O}^{up} \wedge \mathscr{O}^{up}]_0 - [(\mathscr{O} + \mathscr{O}^F) \wedge \mathscr{O}]_0 \\ &= -[\mathscr{O} \wedge \mathscr{O}]_0 = -[\mathscr{O}_{2+1}(\mathscr{O}_0)]_0\end{aligned} \tag{7.32}$$

on $F^{-1}(0)$. (In the second equality above we have used (7.31.1) and (7.31.2).) On the other hand, from (7.31.3) and (7.31.4) we get

$$[\mathscr{O}^{\mathscr{G}} \wedge \mathscr{O}^{up}]_0 = [\delta^{(1,0)} - \mathscr{O}^{up}, \mathscr{O}^{\mathscr{G}}]_0 = 0. \tag{7.33}$$

Thus $[\mathscr{O}_{2+1}(\mathscr{O}_0)]_0$ vanishes and hence $[R^{(2,0)}]_0$ vanishes.

7e. The rigorous argument that $[R^{(2,0)}]_0$ vanishes. In gauge theory, the argument of the previous subsection is formal. The difficulty is that the operator $\mathscr{O}^{up}$ on $\mathscr{L} = \mathscr{A} \times C$ appearing in equations (7.32) and (7.33) involves ill-defined sums. We can, however, use the argument as an outline for a rigorous proof. We do this by writing (formally, in the gauge theory case) $[\mathscr{O}^G \wedge \mathscr{O}^{up}]_0$ as the sum of nine explicit functions on $F^{-1}(0)$ given in terms of Green's functions. Equation (7.33) suggests that these nine terms sum to zero. Equation (7.32) suggests that these terms sum to the expression $[-\mathscr{O}_{2+1}(\mathscr{O}_0)]_0$. To prove that $[\mathscr{O}_{2+1}(\mathscr{O}_0)]_0$ vanishes we must show that both these conclusions are true. Since everything is now expressed in terms of explicit functions on $F^{-1}(0)$, we can give a proof completely in terms of Green's functions.

In this subsection, foregoing questions of regularization, we will carry out the details of this proof. The more difficult analytical question of regulating the arguments given here involves making the nine terms discussed above well defined using a regularization scheme which is consistent with the scheme used in defining the determinant of the Laplacian. After this, one must check that the manipulations, carried out in this subsection, of these nine terms are valid within the given regularization scheme. To simplify the exposition, we will present the discussion below as it would be presented if we had such a regularization scheme in place.

The basic ideas involved are simple, and the procedure is more or less algorithmic. Since the details are tedious and involve a large number of equations, it is probably best to outline the steps involved.

Step 1: We formally derive a decomposition of $[\mathscr{O}^{\mathscr{G}} \wedge \mathscr{O}^{up}]_0$ into the sum of nine "primitive" terms. These terms are expressed explicitly on $F^{-1}(0)$ in terms of Green's functions.

Step 2: In this step, we massage the terms found in Step 1 to show rigorously that they sum to zero (7.33). The abstract proof of this result was based on the fact that $[\delta^{\mathscr{H}_Q}, \nabla'^{\mathscr{G}}_a] = 0$. Accordingly, we group the terms of Step 1 so as to make the role of the $\nabla'^{\mathscr{G}}_a$ explicit. Keeping in mind the explicit proof in §7c that $\delta^{\mathscr{H}_Q}$ respects the group generators, we shall use the crucial identity (7.23) at the appropriate point.

Step 3: In this step, we massage the terms found in Step 1 to show rigorously that they sum to $-[\mathscr{O}_{2+1}(\mathscr{O}_0)]_0$. This corresponds to the rigorous version of equation (7.32). The abstract proof of this result was based on the technique of breaking operators up into derivatives acting in the $\mathscr{M}$ and $\mathscr{G}$ directions, which is done systematically by the machinery introduced above. Accordingly, we group the terms of Step 1 to make manifest the derivatives in the $\mathscr{M}$ directions and the $\mathscr{G}$ directions. We now give the details.

Step 1. *Deriving the primitive terms.* Our goal is to decompose $[\mathscr{O}^{\mathscr{G}} \wedge \mathscr{O}^{up}]_0$ into primitive terms which can be grouped in two ways corresponding to Steps 2 and 3 above. We find it simplest to first make a preliminary decomposition in a way suggested by Step 2. So, to make the group generators manifest, we write:

$$(7.34) \qquad [\mathscr{O}^{\mathscr{G}} \wedge \mathscr{O}^{up}]_0 = [M^{\underline{i}\underline{j}}(2\nabla^{\mathscr{M}}_{\underline{i}} + \nabla'^{\mathscr{G}}_{\underline{i}})[\nabla'^{\mathscr{G}}_{\underline{j}}, \mathscr{O}^{up}]]_0$$

$$(7.35) \qquad [\nabla'^{\mathscr{G}}_{\underline{j}}, \mathscr{O}^{up}] = (T_z^{-1})^a{}_{\underline{j}}[\nabla'^{\mathscr{G}}_a, \mathscr{O}^{up}] + [(T_z^{-1})^a{}_{\underline{j}}, \mathscr{O}^{up}]\nabla'^{\mathscr{G}}_a .$$

Now we will decompose this into more primitive terms, designed so that they may be grouped into the natural preliminary decomposition (T7.56) of Step 3. To do this, we must first compute the commutator

$$\begin{aligned}
&(\mathrm{T}7.36.1) \qquad [\nabla'^{\mathscr{G}}_{\underline{j}}, \mathscr{O}^{up}] = 2M^{\underline{k}\underline{l}}(T_z^{-1})^a{}_{\underline{j}}[\nabla'^{\mathscr{G}}_a, \nabla_{\underline{k}}]\nabla_{\underline{l}} \\
&(\mathrm{T}7.36.2) \qquad\qquad + M^{\underline{k}\underline{l}}(T_z^{-1})^a{}_{\underline{j}}[\nabla_{\underline{l}}, [\nabla'^{\mathscr{G}}_a, \nabla_{\underline{k}}]] \\
&(\mathrm{T}7.36.3) \qquad\qquad - M^{\underline{k}\underline{l}}(\nabla_{\underline{k}}\nabla_{\underline{l}}(T_z^{-1})^a{}_{\underline{j}})\nabla'^{\mathscr{G}}_a \\
&(\mathrm{T}7.36.4) \qquad\qquad - 2M^{\underline{k}\underline{l}}(\nabla_{\underline{k}}(T_z^{-1})^a{}_{\underline{j}})\nabla_{\underline{l}}\nabla'^{\mathscr{G}}_a .
\end{aligned}$$

Now recall

$$(7.37) \qquad \nabla'^{\mathscr{G}}_a = T_z{}^{\underline{m}}{}_a\nabla_{\underline{m}} + iF_a ,$$

$$(7.38) \qquad \nabla_{\underline{k}}\nabla_{\underline{l}}F_a = 0 , \qquad \nabla_{\underline{k}}\nabla_{\underline{l}}T_z{}^{\underline{m}}{}_a = 0 ,$$

so that

$$[\nabla'^{\mathscr{G}}_a, \nabla_{\underline{k}}] = -(\nabla_{\underline{k}} T^{\underline{m}}_{z\ a})\nabla_{\underline{m}} - i\nabla_{\underline{k}} F_a \tag{7.39.1}$$

$$[[\nabla'^{\mathscr{G}}_a, \nabla_{\underline{k}}], \nabla_{\underline{l}}] = 0. \tag{7.39.2}$$

The term (T7.36.2) is therefore zero. Substituting (T7.39.1) into (T7.36.1) and (7.37) into (T7.36.3) and (T7.36.4), and expanding we deduce

$$\begin{aligned}
&[\nabla'^{\mathscr{G}}_{\underline{j}} \wedge \mathscr{O}^{up}]_0 \\
&\quad \left.\begin{aligned}
&= -2M^{\underline{k}\,\underline{l}}((\nabla_{\underline{k}} T_z)T_z^{-1})^{\underline{m}}{}_{\underline{j}}\nabla_{\underline{m}}\nabla_{\underline{l}} && \text{(T7.40.1)} \\
&\quad - 2iM^{\underline{k}\,\underline{l}}(\nabla_{\underline{k}} F_a)(T_z^{-1})^a{}_{\underline{j}}\nabla_{\underline{l}} && \text{(T7.40.2)}
\end{aligned}\right\} \text{from (T7.36.1)} \\
&\quad \left.\begin{aligned}
&- 2M^{\underline{k}\,\underline{l}}(\nabla_{\underline{k}}(T_z^{-1})^a{}_{\underline{j}})(\nabla_{\underline{l}} T^{\underline{m}}_{z\ a})\nabla_{\underline{m}} && \text{(T7.40.3)} \\
&- 2M^{\underline{k}\,\underline{l}}(\nabla_{\underline{k}}(T_z^{-1})^a{}_{\underline{j}})T^{\underline{l}}_{za}\nabla_{\underline{m}}\nabla_{\underline{l}} && \text{(T.7.40.4)} \\
&- 2iM^{\underline{k}\,\underline{l}}(\nabla_{\underline{k}}(T_z^{-1})^a{}_{\underline{j}})(\nabla_{\underline{l}} F_a) && \text{(T7.40.5)}
\end{aligned}\right\} \text{from (T7.36.3)} \\
&\quad \left.\begin{aligned}
&- 2iM^{\underline{k}\,\underline{l}}(\nabla_{\underline{k}}(T_z^{-1})^a{}_{\underline{j}})F_a\nabla_{\underline{l}} && \text{(T7.40.6)} \\
&- M^{\underline{k}\,\underline{l}}(\nabla_{\underline{k}}\nabla_{\underline{l}}(T_z^{-1})^a{}_{\underline{j}})T^{\underline{m}}_{z\ a}\nabla_{\underline{m}} && \text{(T7.40.7)} \\
&- iM^{\underline{k}\,\underline{l}}(\nabla_{\underline{k}}\nabla_{\underline{l}}(T_z^{-1})^a{}_{\underline{j}})F_a. && \text{(T7.40.8)}
\end{aligned}\right\} \text{from (T7.36.4)}
\end{aligned}$$

(In writing down the terms (T7.40) we have used the convention that a derivative operator acts only on those terms to its left within its parenthesis level.)

The desired expansion. We now obtain the desired expansion of $[\mathscr{O}^{up} \wedge \mathscr{O}^G]_0$. First, we apply $M^{\underline{i}\underline{j}}(2\nabla^{\mathscr{M}}_{\underline{i}} + \nabla'^{\mathscr{G}}_{\underline{i}})$ to both sides of (T7.40). Each term on the left-hand side gives rise to two terms. In the term labelled (T7.42. n B) below, the operator $M^{\underline{i}\underline{j}}(2\nabla^{\mathscr{M}}_{\underline{i}} + \nabla'^{\mathscr{G}}_{\underline{i}})$ acts on the coefficient of the term (T7.40. n); the term labelled (T7.42. n A) is the remainder of the contribution from (T7.40. n). Next we evaluate the zeroth order pieces of these terms using the the following identities, valid on $F^1(0)$:

$$\begin{gathered}
[\nabla_{\underline{l}}]_0 = 0, \qquad [\nabla_{\underline{m}}\nabla_{\underline{l}}]_0 = -i(\nabla^{\mathscr{G}}_{\underline{m}} F_{\underline{l}}), \\
[(2\nabla^{\mathscr{M}}_{\underline{i}} + \nabla'^{\mathscr{G}}_{\underline{i}})\nabla_{\underline{l}}]_0 = -i(\nabla^{\mathscr{G}}_{\underline{l}} F_{\underline{i}}), \\
[(2\nabla^{\mathscr{M}}_{\underline{i}} + \nabla'^{\mathscr{G}}_{\underline{i}})\nabla_{\underline{m}}\nabla_{\underline{l}}]_0 = -2i(\nabla^{\mathscr{G}}_{\underline{m}}\nabla^{\mathscr{M}}_{\underline{i}} F_{\underline{l}}) - i(\nabla_{\underline{m}}\nabla_{\underline{l}} F_{\underline{i}}).
\end{gathered} \tag{7.41}$$

We obtain the following expansion:

$$
\begin{aligned}
&[\mathscr{O}^{up}\wedge\mathscr{O}^{G}]_0\\
&= \left.\begin{aligned}
&+2iM^{\underline{i}j}M^{\underline{k}l}((\nabla_{\underline{i}}+\nabla^{\mathscr{M}}_{\underline{i}})\\
&\qquad\times[(\nabla_{\underline{k}}T_z)T_z^{-1}]^{\underline{m}}{}_{\underline{j}})(\nabla^{\mathscr{G}}_{\underline{m}}F_{\underline{l}}) && \text{(T7.42.1B)}\\
&-2M^{\underline{i}j}M^{\underline{k}l}[(\nabla_{\underline{k}}T_z)T_z^{-1}]^{\underline{m}}{}_{\underline{j}}\\
&\qquad\times[-2i(\nabla^{\mathscr{G}}_{\underline{m}}\nabla^{\mathscr{M}}_{\underline{i}}F_{\underline{l}})-i(\nabla_{\underline{m}}\nabla_{\underline{l}}F_{\underline{i}})] && \text{(T7.42.1A)}
\end{aligned}\right\}\text{ from (T7.40.1)}\\
&\quad -2M^{\underline{i}j}M^{\underline{k}l}(\nabla_{\underline{k}}F_a)(T_z^{-1})^{a}{}_{\underline{j}}(\nabla'^{\mathscr{G}}_{\underline{l}}F_{\underline{i}}) \qquad \text{(T7.42.2A)}\}\text{ from (T7.40.2)}\\
&\quad +2iM^{\underline{i}j}M^{\underline{k}l}((\nabla_{\underline{l}}T_z)(\nabla_{\underline{k}}T_z^{-1}))^{\underline{m}}{}_{\underline{j}}(\nabla^{\mathscr{G}}_{\underline{m}}F_{\underline{i}}) \qquad \text{(T7.42.3A)}\}\text{ from (T7.40.3)}\\
&\quad \left.\begin{aligned}
&+2iM^{\underline{i}j}M^{\underline{k}l}((\nabla_{\underline{i}}+\nabla^{\mathscr{M}}_{\underline{i}})\\
&\qquad\times(T_z\nabla_{\underline{k}}T_z^{-1})^{\underline{m}}{}_{\underline{j}})(\nabla^{\mathscr{G}}_{\underline{m}}F_{\underline{l}}) && \text{(T7.42.4B)}\\
&-2M^{\underline{i}j}M^{\underline{k}l}[T_z\nabla_{\underline{k}}T_z^{-1}]^{\underline{m}}{}_{\underline{j}}\\
&\qquad\times[-2i(\nabla^{\mathscr{G}}_{\underline{m}}\nabla^{\mathscr{M}}_{\underline{i}}F_{\underline{l}})-i(\nabla_{\underline{m}}\nabla_{\underline{l}}F_{\underline{i}})] && \text{(T7.42.4A)}
\end{aligned}\right\}\text{ from (T7.40.4)}\\
&\quad \left.\begin{aligned}
&-2iM^{\underline{i}j}M^{\underline{k}l}((\nabla_{\underline{i}}+\nabla^{\mathscr{M}}_{\underline{i}})\\
&\qquad\times(\nabla_{\underline{l}}F_i)(\nabla_{\underline{k}}(T_z^{-1})^{a}{}_{\underline{j}})) && \text{(T7.42.5B)}
\end{aligned}\right\}\text{ from (T7.40.5)}\\
&\quad +iM^{\underline{i}j}M^{\underline{k}l}(T_z\nabla_{\underline{k}}\nabla_{\underline{l}}T_z^{-1})^{\underline{m}}{}_{\underline{j}}(\nabla^{\mathscr{G}}_{\underline{m}}F_{\underline{i}}) \qquad \text{(T7.42.7A)}\}\text{ from (T7.40.7)}\\
&\quad -iM^{\underline{i}j}M^{\underline{k}l}(\nabla^{\mathscr{G}}_{\underline{i}}F_a)(\nabla_{\underline{k}}\nabla_{\underline{l}}T_z^{-1})^{a}{}_{\underline{j}}. \qquad \text{(T7.42.8B)}\}\text{ from (T7.40.8)}
\end{aligned}
$$

In writing down (T7.42) we have omitted some terms which are trivially zero. For example, the term (T7.42.2B) which arises when $M^{\underline{i}j}(2\nabla^{\mathscr{M}}_{\underline{i}} + \nabla'^{\mathscr{G}}_{a})$ acts on the coefficient of (T7.40.2) is zero since it involves $[\nabla_{\underline{l}}]_0$ which is zero. (In (T7.42) we have again used the convention that a derivative operator acts only on those objects within its parenthesis level.)

While we have used formal manipulations to arrive at the decomposition (T7.42), the terms appearing on the right-hand side are all functions on $F^{-1}(0)$ which have well-defined regularizations in the gauge theory case.

Step 2. *Proof that the primitive terms sum to zero.* We will now show that the terms (T7.42) sum to zero. (This is the rigorous version of (7.33).)

Derivation of the equations to be proved. Our derivation of the primitive terms in Step 1 was tailor made to allow us to see why they group to zero. We need only trace the decomposition given there backwards.

To begin, notice that the terms (T7.42.7A) and (T7.42.8B) arise from the terms (T7.40.7) and (T7.40.8). These in turn arise from (T7.36.4), which, because it contains a $\nabla_a'^{\mathscr{G}}$ to the right annihilates GIHS. We thus expect:

$$(7.43)\qquad (\mathrm{T7.42.7A}) + (\mathrm{T7.42.8B}) = 0.$$

This is indeed the case, as we see below. Similarly, a glance at equations (T7.42) and (T7.40) show that the terms (T7.42.3A), (T7.42.4A), (T7.42.4B), and (T7.42.5B) have their origins in the term (T7.36.3), which also vanishes on GIHS. We thus expect (and prove below) that

$$(7.44)\qquad (\mathrm{T7.42.3A}) + (\mathrm{T7.42.4A}) + (\mathrm{T7.42.4B}) + (\mathrm{T7.42.5B}) = 0.$$

The terms (T7.42.1A), (T7.42.1B), and (T7.42.2A) arise from (T7.40.1) and (T7.40.2). These in turn arise from the first term on the right-hand side of (7.35). Since $[\delta, \nabla_a'^{\mathscr{G}}] = 0$ and $\delta^{\mathscr{H}_Q}$ respects the group action, the first term on the right side of (7.35) vanishes. Thus we expect to find that (T7.42.1A), (T7.42.1B), and (T7.42.2A) sum to zero. In fact we shall show that they vanish separately:

$$(7.45)\qquad (\mathrm{T7.42.1A}) = 0, \quad (\mathrm{T7.42.1B}) = 0, \quad (\mathrm{T7.42.2A}) = 0.$$

We will see that $(\mathrm{T7.42.2A}) = 0$ follows from a simple symmetry argument. To prove explicitly that (T7.42.1A) and (T7.42.1B) vanish, we will need to use (7.23) which was the crucial equation in proving that $\delta^{\mathscr{H}_Q}$ respects the group action.

Proving the equations. We now proceed with the proofs of the above claims in terms of manipulations of Green's functions.

Proof of (7.43). First recall that $F_a = T_{z\ a}^{\underline{m}} F_{\underline{m}}$, so

$$(7.46)\qquad \nabla_{\underline{i}}^{\mathscr{G}} F_a = (\nabla_{\underline{i}}^{\mathscr{G}} T_{z\ a}^{\underline{m}}) F_{\underline{m}} + (\nabla_{\underline{i}}^{\mathscr{G}} F_{\underline{m}}) T_{z\ a}^{\underline{m}}.$$

On $F^{-1}(0)$ the first term on the right in (7.46) vanishes. Thus (T7.42.7A)

+ (T7.42.8B) equals

$$M^{\underline{ij}}M^{\underline{kl}}\left[\underbrace{(-i(\nabla^{\mathscr{G}}_{\underline{i}}F_{\underline{m}})T_z{}^{\underline{m}}{}_a)(\nabla_{\underline{k}}\nabla_{\underline{l}}(T_z^{-1})^a{}_{\underline{j}})}_{\text{from (T7.42.8B) using (7.46)}} \right. \tag{7.47}$$

$$\left. + \underbrace{iT_z{}^{\underline{m}}{}_a(\nabla^{\mathscr{G}}_{\underline{m}}F_{\underline{i}})(\nabla_{\underline{k}}\nabla_{\underline{l}}(T_z^{-1})^a{}_{\underline{j}})}_{\text{from (T7.42.7A)}}\right].$$

Since $\nabla^{\mathscr{G}}_{\underline{i}}F_{\underline{m}} = \nabla^{\mathscr{G}}_{\underline{m}}F_{\underline{i}}$, the two terms on the right-hand side of (7.47) cancel and (7.43) is proved.

Proof of (7.44). This is slightly more involved. Writing $F_a = T_z{}^{\underline{m}}{}_a F_{\underline{m}}$ and substituting into (T7.42.5B) gives

$$-(\text{T7.42.5B}) = 2iM^{\underline{ij}}M^{\underline{kl}}\Big((\nabla_{\underline{i}} + \nabla^{\mathscr{M}}_{\underline{i}})((\nabla_{\underline{l}}F_{\underline{m}})(T_z\nabla_{\underline{k}}T_z^{-1})^{\underline{m}}{}_{\underline{j}}$$
$$- F_a(\nabla_{\underline{l}}(T_z^{-1})^a{}_{\underline{m}})(T_z(\nabla_{\underline{k}}T_z^{-1}))^{\underline{m}}{}_{\underline{j}})\Big)$$

$$= 2iM^{\underline{ij}}M^{\underline{kl}}\Big((\nabla_{\underline{i}} + \nabla^{\mathscr{M}}_{\underline{i}})(T_z\nabla_{\underline{k}}T_z^{-1})^{\underline{m}}{}_{\underline{j}}\Big)(\nabla_{\underline{l}}F_{\underline{m}}) \tag{T7.48.1}$$

$$+ 2iM^{\underline{ij}}M^{\underline{kl}}((\nabla_{\underline{i}} + \nabla^{\mathscr{M}}_{\underline{i}})(\nabla_{\underline{l}}F_{\underline{m}})\Big)(T_z\nabla_{\underline{k}}T_z^{-1})^{\underline{m}}{}_{\underline{j}} \tag{T7.48.2}$$

$$- 2iM^{\underline{ij}}M^{\underline{kl}}(\nabla^{\mathscr{G}}_{\underline{i}}F_a)(\nabla_{\underline{l}}(T_z^{-1})^a{}_{\underline{m}})(T_z\nabla_{\underline{k}}T_z^{-1})^{\underline{m}}{}_{\underline{j}}. \tag{T7.48.3}$$

We now claim that

$$(\text{T7.48.1}) = (\text{T7.42.4B}), \quad (\text{T7.48.2}) = (\text{T7.42.4A}), \quad (\text{T7.48.3}) = (\text{T7.42.3A}), \tag{7.49}$$

which immediately imply (7.44). The first equality of (7.49) is immediate. For the second, we first note that on $F^{-1}(0)$,

$$(\nabla_{\underline{i}} + \nabla^{\mathscr{M}}_{\underline{i}})(\nabla_{\underline{l}}F_{\underline{m}}) = 2(\nabla^{\mathscr{G}}_{\underline{m}}\nabla^{\mathscr{M}}_{\underline{i}}F_{\underline{l}}) + (\nabla^{\mathscr{G}}_{\underline{m}}\nabla_{\underline{l}}F_{\underline{i}}). \tag{7.50}$$

Substituting (7.50) into (T7.48.2) gives

$$(\text{T7.48.2}) = 2iM^{\underline{ij}}M^{\underline{kl}}(T_z\nabla_{\underline{k}}T_z^{-1})^{\underline{m}}{}_{\underline{j}} \times [2\nabla^{\mathscr{G}}_{\underline{m}}\nabla^{\mathscr{M}}_{\underline{i}}F_{\underline{l}} + \nabla^{\mathscr{G}}_{\underline{m}}\nabla_{\underline{l}}F_{\underline{i}}] = (\text{T7.42.4A}). \tag{7.51}$$

(For the last equality of (7.49), we observed that the $\underline{m}$ index in (T7.42.4A) is in the G direction "for free".)

The proof of the last equality of (7.49) is similar and (7.44) is proved.

Proof of (7.45). To prove that (T7.42.1A) is zero we use a symmetry argument:

$$(7.52)\qquad \begin{aligned}(\text{T7.42.1A}) = &-2M^{\underline{ij}}\underbrace{[M^{\underline{kl}}(\nabla_{\underline{k}}T_z)T_z^{-1}]^{\underline{m}}_{\underline{j}}}_{\text{odd under}(\underline{l}\leftrightarrow\underline{m})}\\ &\times\underbrace{[-2i(\nabla_{\underline{i}}^{\mathscr{M}}\nabla_{\underline{m}}^{\mathscr{G}}F_{\underline{l}})-i\nabla_{\underline{m}}\nabla_{\underline{l}}F_{\underline{i}}]}_{\text{even under}(\underline{l}\leftrightarrow\underline{m})}.\end{aligned}$$

The odd symmetry of the first term in (7.52) is the identity (7.23) used in proving that $\delta^{\mathscr{H}_Q}$ respects the group action.

A similar argument involving $(\underline{l}\leftrightarrow\underline{m})$ proves that (T7.42.1B) $=0$.

To complete the proof of (7.45) and hence Step 2, we prove that (T7.42.2A) vanishes, also using a symmetry argument. On $F^{-1}(0)$, we have

$$(7.53)\qquad (\text{T7.42.2A}) = -2M^{\underline{ij}}M^{\underline{kl}}(\nabla_{\underline{k}}^{\mathscr{G}}F_{\underline{j}})(\nabla_{\underline{l}}^{\mathscr{G}}F_{\underline{i}}).$$

Now, $\nabla_{\underline{k}}^{\mathscr{G}}F_{\underline{j}}$ is even under $(\underline{k}\leftrightarrow\underline{j})$, and $\nabla_{\underline{l}}^{\mathscr{G}}F_{\underline{i}}$ is even under $(\underline{i}\leftrightarrow\underline{l})$. On the other hand because $M^{\underline{ij}}$ is a form, the combination $M^{\underline{ij}}M^{\underline{kl}}$ is antisymmetric in the simultaneous interchanges $(\underline{k}\leftrightarrow\underline{j})$, $(\underline{i}\leftrightarrow\underline{l})$. Thus (T7.42.2A) vanishes as claimed.

This completes the proof that the terms of (T7.42) sum to zero.

Step 3. *Proof that the primitive terms sum to* $-[\mathscr{O}_{2+1}(\mathscr{O}_0)]_0$. Roughly speaking, our demonstration that the primitive terms sum to $-[\mathscr{O}_{2+1}(\mathscr{O}_0)]_0$ is organized so that the splitting between the $\mathscr{M}$ directions and the G directions becomes increasingly apparent as the calculation progresses.

A preliminary decomposition of $[\mathscr{O}^{up}\wedge\mathscr{O}^{\mathscr{G}}]_0$. We first regroup the primitive terms so as to make the split up of derivatives into $\mathscr{M}$ and $\mathscr{G}$ directions more apparent. The grouping we want is motivated by the following consideration. In Step 1, as a first step to obtaining the decomposition of $\mathscr{O}^G\wedge\mathscr{O}^{up}$ into primitive terms we expanded $[\nabla_{\underline{j}}^{\prime\mathscr{G}},\mathscr{O}^{up}]$ by first writing $\nabla_{\underline{j}}^{\prime\mathscr{G}} = (T_z^{-1})^a{}_{\underline{j}}\nabla_a^{\prime\mathscr{G}}$ and arrived at (T7.40). As a consequence, the group generators $\nabla_a^{\prime\mathscr{G}}$ (with a group index a) appeared manifestly in the final result. To make the $\mathscr{M}$ and $\mathscr{G}$ directions more evident, we leave the $\underline{j}$ index on the covariant derivative and instead write

$$(7.54)\qquad [\nabla_{\underline{j}}^{\prime\mathscr{G}},\mathscr{O}^{up}] = M^{\underline{kl}}(2[\nabla_{\underline{j}}^{\prime\mathscr{G}},\nabla_{\underline{k}}]\nabla_{\underline{l}}+[\nabla_{\underline{l}},[\nabla_{\underline{j}}^{\prime\mathscr{G}},\nabla_{\underline{k}}]]).$$

Using the commutation relations (7.4), we then obtain

$$[\nabla'^{\mathscr{G}}_{\underline{j}}, \mathscr{O}^{up}] = 2M^{\underline{k}\,\underline{l}}(\nabla_{\underline{k}}\mathscr{K}^{\underline{m}}{}_{\underline{j}})\nabla_{\underline{m}}\nabla_{\underline{l}} \tag{T7.55.1}$$

$$- 2iM^{\underline{k}\,\underline{l}}(\nabla_{\underline{k}}F_{\underline{j}})\nabla_{\underline{l}} \tag{T7.55.2}$$

$$+ M^{\underline{k}\,\underline{l}}(\nabla_{\underline{l}}\nabla_{\underline{k}}\mathscr{K}^{\underline{m}}{}_{\underline{j}})\nabla_{\underline{m}} \tag{T7.55.3}$$

$$- iM^{\underline{k}\,\underline{l}}(\nabla_{\underline{l}}\nabla_{\underline{k}}F_{\underline{j}}). \tag{T7.55.4}$$

This is the analog of (T7.40). We now apply $M^{\underline{i}\underline{j}}(2\nabla^{\mathscr{M}}_{\underline{i}} + \nabla'^{\mathscr{G}}_{\underline{i}})$ to (T7.55) and extract the zeroth order piece of the resulting expression using (7.41) in a manner similar to the derivation of equation (T7.40). We obtain (with a similar labelling convention for the terms):

$$[\mathscr{O}^{up} \wedge \mathscr{O}^{G}]_0$$

$$= -2iM^{\underline{i}\underline{j}}M^{\underline{k}\,\underline{l}}((\nabla_{\underline{i}} + \nabla^{\mathscr{M}}_{\underline{i}})(\nabla_{\underline{k}}\mathscr{K}^{\underline{m}}{}_{\underline{j}}))(\nabla^{\mathscr{G}}_{\underline{m}}F_{\underline{l}}) \tag{T7.56.1B}$$

$$+ 2M^{\underline{i}\underline{j}}M^{\underline{k}\,\underline{l}}(\nabla_{\underline{k}}\mathscr{K}^{\underline{m}}{}_{\underline{j}})[-2i(\nabla^{\mathscr{G}}_{\underline{m}}\nabla^{\mathscr{M}}_{\underline{i}}F_{\underline{l}}) - i(\nabla_{\underline{m}}\nabla_{\underline{l}}F_{\underline{i}})] \tag{T7.56.1A}$$

$$- 2M^{\underline{i}\underline{j}}M^{\underline{k}\,\underline{l}}(\nabla_{\underline{k}}F_{\underline{j}})(\nabla_{\underline{l}}F_{\underline{i}}) \tag{T7.56.2A}$$

$$- iM^{\underline{i}\underline{j}}M^{\underline{k}\,\underline{l}}(\nabla_{\underline{k}}\nabla_{\underline{l}}\mathscr{K}^{\underline{m}}{}_{\underline{j}})(\nabla^{\mathscr{G}}_{\underline{m}}F_{\underline{i}}) \tag{T7.56.3A}$$

$$- iM^{\underline{i}\underline{j}}M^{\underline{k}\,\underline{l}}((\nabla_{\underline{i}} + \nabla^{\mathscr{M}}_{\underline{i}})(\nabla_{\underline{l}}\nabla_{\underline{k}}F_{\underline{j}})). \tag{T7.56.4B}$$

These five terms must sum to the nine terms of (T7.40). In fact the definition of the primitive terms in (T7.40) was arranged so that this fact may be seen by a simple regrouping.

Proof that the primitive terms sum to the preliminary decomposition. We have completed all the formal manipulations to motivate the proofs of Step 3. All subsequent arguments will involve only manipulations with explicit functions on $F^{-1}(0)$.

To begin, we show that the five terms (T7.56) do sum to the nine terms (T7.40). We need the following identities:

$$-\nabla_{\underline{k}}\mathscr{K}^{\underline{m}}{}_{\underline{j}} = \nabla_{\underline{k}}(T_z T_z^{-1})^{\underline{m}}{}_{\underline{j}}$$

$$= (\nabla_{\underline{k}}T^{\underline{m}}_{z\ a})(T_z^{-1})^{a}{}_{\underline{j}} + T^{\underline{m}}_{z\ a}(\nabla_{\underline{k}}T_z^{-1})^{a}{}_{\underline{j}}, \tag{7.57.1}$$

$$\nabla_{\underline{k}}F_{\underline{j}} = (\nabla_{\underline{k}}F_a)(T_z^{-1})^{a}{}_{\underline{j}} + (\nabla_{\underline{k}}T_z^{-1})^{a}{}_{\underline{j}}F_a, \tag{7.57.2}$$

$$\begin{aligned}\nabla_{\underline{k}}\nabla_{\underline{l}}\mathscr{K}^{\underline{m}}{}_{\underline{j}} = &- (\nabla_{\underline{k}}\nabla_{\underline{l}}T_z^{-1})^a{}_{\underline{j}}T^{\underline{m}}_{z\ a} \\ &- [(\nabla_{\underline{k}}T_z^{-1})^a{}_{\underline{j}}(\nabla_{\underline{l}}T^{\underline{m}}_{z\ a}) + (\underline{k} \leftrightarrow \underline{l})],\end{aligned} \tag{7.57.3}$$

$$\nabla_{\underline{k}}\nabla_{\underline{l}}F_{\underline{j}} = (\nabla_{\underline{k}}\nabla_{\underline{l}}T_z^{-1})^a{}_{\underline{j}}F_a + [(\nabla_{\underline{k}}T_z^{-1})^a{}_{\underline{j}}(\nabla_{\underline{l}}F_a) + (\underline{k} \leftrightarrow \underline{l})]. \tag{7.57.4}$$

(To obtain (7.57.3) we differentiate (7.57.1) and use $\nabla_{\underline{k}}\nabla_{\underline{l}}T_z = 0$.)

Using these identities we easily obtain:

$$\begin{aligned}
(\text{T}7.42.1\text{B}) + (\text{T}7.42.4\text{B}) &= (\text{T}7.56.1\text{B}) \quad (\text{using } (7.57.1))\\
(\text{T}7.42.1\text{A}) + (\text{T}7.42.4\text{A}) &= (\text{T}7.56.1\text{A}) \quad (\text{using } (7.57.1))\\
(\text{T}7.42.2\text{A}) &= (\text{T}7.56.2\text{A})\\
(\text{T}7.42.3\text{A}) + (\text{T}7.42.7\text{A}) &= (\text{T}7.56.3\text{A}) \quad (\text{using } (7.57.2))\\
(\text{T}7.42.5\text{B}) + (\text{T}7.42.8\text{B}) &= (\text{T}7.56.4\text{B}) \quad (\text{using } (7.57.4)).
\end{aligned}$$

(For (T7.56.4) we have also used $\nabla^{\mathscr{G}}_{\underline{i}}F_a = (\nabla_{\underline{i}} + \nabla^{\mathscr{M}}_{\underline{i}})F_a$ on $F^{-1}(0)$.) These equalities prove that the terms of (T7.56) sum to those of (T7.42).

Proof that the preliminary decomposition sums to $-[\mathscr{O}_{2+1}(\mathscr{O}_0)]_0$. To show that the five terms of (T7.56) sum to $-[\mathscr{O}_{2+1}(\mathscr{O}_0)]_0$ we note first an immediate simplification. The $\underline{m}$ index arising from the holomorphic variations of $\mathscr{K}^{\underline{m}}{}_{\underline{j}}$ in (T7.56.1A), (T7.56.1B), and (T7.56.3A) range only over the $\mathscr{M}$ directions. Since the $\underline{m}$ index of $\nabla^{\mathscr{G}}_{\underline{m}}$ lies in the G directions, we conclude that

$$(\text{T}7.56.1\text{B}) = 0, \quad (\text{T}7.56.3\text{A}) = 0, \quad \text{the first term of } (\text{T}7.56.1\text{A}) = 0.$$

Recalling that $(\text{T}7.42.2\text{A}) = 0$, so that $(\text{T}7.56.2\text{A}) = 0$, we thus see that (T7.56) takes the simplified form:

$$\begin{aligned}
[\mathscr{O}^G \wedge \mathscr{O}^{up}]_0 = &- 2iM^{\underline{i}\underline{j}}M^{\underline{k}\underline{l}}(\nabla_{\underline{k}}\mathscr{K}^{\underline{m}}{}_{\underline{j}})(\nabla_{\underline{m}}\nabla_{\underline{l}}F_{\underline{i}}) && (\text{T}7.58.1)\\
&- iM^{\underline{i}\underline{j}}M^{\underline{k}\underline{l}}(\nabla_{\underline{i}}\nabla_{\underline{l}}\nabla_{\underline{k}}F_{\underline{j}}) && (\text{T}7.58.2)\\
&- iM^{\underline{i}\underline{j}}M^{\underline{k}\underline{l}}(\nabla^{\mathscr{M}}_{\underline{i}}\nabla_{\underline{l}}\nabla_{\underline{k}}F_{\underline{j}}). && (\text{T}7.58.3)
\end{aligned}$$

The terms of (T7.58) allegedly sum to $-[\mathscr{O}_{2+1}(\mathscr{O}_0)]_0$. To show this, we could at this point proceed completely algorithmically and expand all derivatives in (T7.58) in terms of $\nabla^{\mathscr{M}}$ and $\nabla^{\mathscr{G}}$ and attempt to massage what resulted. It is convenient, however, to first simplify the problem slightly. Using the commutation relations (7.4), we compute

$$\begin{aligned} -iM^{\underline{k}\,\underline{l}}(\nabla^{\mathscr{M}}_{\underline{i}}\nabla_{\underline{l}}\nabla_{\underline{k}}F_{\underline{j}}) = &- iM^{\underline{k}\,\underline{l}}\nabla_{\underline{k}}\nabla_{\underline{l}}(\nabla^{\mathscr{M}}_{\underline{i}}F_{\underline{j}}) \\ &+ 2iM^{\underline{k}\,\underline{l}}((\nabla_{\underline{k}}\mathscr{K}^{\underline{m}}{}_{\underline{i}})(\nabla_{\underline{l}}\nabla_{\underline{m}}F_{\underline{j}})) \\ &+ iM^{\underline{k}\,\underline{l}}(\nabla_{\underline{k}}\nabla_{\underline{l}}\mathscr{K}^{\underline{m}}{}_{\underline{i}})(\nabla^{\mathscr{M}}_{\underline{m}}F_{\underline{j}}). \end{aligned} \tag{7.59}$$

The third term above vanishes on $F^{-1}(0)$. Substituting (7.59) into (T7.58), we see that

$$[\mathscr{O}^G \wedge \mathscr{O}^{up}]_0 = -iM^{\underline{i}\underline{j}}M^{\underline{k}\,\underline{l}}((\nabla_{\underline{k}}\nabla_{\underline{l}})(\nabla_{\underline{i}} + \nabla^{\mathscr{M}}_{\underline{i}})F_{\underline{j}}). \tag{7.60}$$

We are thus left with the task of massaging (7.60) into the function $-(\mathscr{O}_{2+1}(\mathscr{O}_0))$. Consistent with the philosophy of Step 3, this is achieved by making manifest the splitting into $\mathscr{M}$ and G directions. The key identity is:

$$\begin{aligned} M^{\underline{k}\,\underline{l}}\nabla_{\underline{k}}\nabla_{\underline{l}} &= M^{\underline{k}\,\underline{l}}(\nabla^{\mathscr{M}}_{\underline{k}}\nabla^{\mathscr{M}}_{\underline{l}} + [\nabla^{\mathscr{G}}_{\underline{k}}, \nabla^{\mathscr{M}}_{\underline{l}}]) \\ &\quad + M^{\underline{k}\,\underline{l}}(\nabla^{\mathscr{M}}_{\underline{k}}\nabla^{\mathscr{G}}_{\underline{l}} + \nabla^{\mathscr{M}}_{\underline{l}}\nabla^{\mathscr{M}}_{\underline{k}} + \nabla^{\mathscr{G}}_{\underline{k}}\nabla^{\mathscr{G}}_{\underline{l}}) \\ &= \mathscr{O}_{2+1} + M^{\underline{k}\underline{l}}[(\nabla_{\underline{k}} + \nabla^{\mathscr{M}}_{\underline{k}})\nabla^{\mathscr{G}}_{\underline{l}}]. \end{aligned} \tag{7.61}$$

This is just the decomposition of $\mathscr{O}^{up}$ when acting on functions rather than a section of $\mathscr{L}$.

Substituting (7.61) into (7.60) gives

$$\begin{aligned} &[\mathscr{O}^G \wedge \mathscr{O}^{up}]_0 = i\mathscr{O}_{2+1}(M^{\underline{i}\underline{j}}(\nabla_{\underline{i}} + \nabla^{\mathscr{M}}_{\underline{i}})F_{\underline{j}}) && \text{(T7.62.1)} \\ &\qquad - iM^{\underline{i}\underline{j}}M^{\underline{k}\,\underline{l}}((\nabla_{\underline{k}} + \nabla^{\mathscr{M}}_{\underline{k}})\nabla^{\mathscr{G}}_{\underline{l}}(\nabla_{\underline{i}} + \nabla^{\mathscr{M}}_{\underline{i}})F_{\underline{j}}). && \text{(T7.62.2)} \end{aligned}$$

But,

$$\text{(T7.62.1)} = \mathscr{O}_{2+1}(iM^{\underline{i}\underline{j}}(\nabla^{\mathscr{G}}_{\underline{i}}F_{\underline{j}})) = -\mathscr{O}_{2+1}(\mathscr{O}_0) \tag{7.63}$$

Hence, to complete Step 3 and therefore the proof that $[R^{(2,0)}]_0 = 0$, we need only show that (T7.62.2) $= 0$. We have that (T7.62.2) equals

$$-iM^{\underline{i}\underline{j}}M^{\underline{k}\,\underline{l}}\left[\underbrace{(\nabla_{\underline{k}} + \nabla^{\mathscr{M}}_{\underline{k}})(\nabla_{\underline{i}} + \nabla^{\mathscr{M}}_{\underline{i}})}_{\text{(I)}}(\nabla^{\mathscr{G}}_{\underline{l}}F_{\underline{j}}) + \underbrace{(\nabla_{\underline{k}} + \nabla^{\mathscr{M}}_{\underline{k}})}_{\text{(II)}}[\nabla^{\mathscr{G}}_{\underline{l}}, \nabla_{\underline{i}} + \nabla^{\mathscr{M}}_{\underline{i}}]F_{\underline{j}}\right]. \tag{7.64}$$

By symmetry, we may replace the operator labelled (I) above by a commutator. Also, since derivatives of $F_{\underline{j}}$ along $\mathscr{M}$ vanish on $F^{-1}(0)$, the operator labelled (II) may be commuted all the way to the right and then replaced by $\nabla^{\mathscr{G}}_{\underline{k}}$. A little algebra (using the fact that $\mathscr{G}$ derivatives commute) yields:

$$(\mathrm{T7.62.2}) = 0,$$

thus concluding the proof.

Appendix

We collect here some relevant formulas about Weyl-Kac characters. The material here is only used in §5 and to specify some normalization conventions used throughout the paper. We closely follow the exposition and conventions of [26] (see also [13]).

Let G be a compact simple and simply connected group with maximal torus T and Lie algebra g. The complexified Lie algebra of G decomposes under the action of T by conjugation as

$$E_0 \oplus_{\alpha\neq 0} \left(\bigoplus E_\alpha\right). \tag{A.1}$$

Here, E_α is the vector subspace on which T acts by the homomorphism $\alpha : T \mapsto S^1$, and E_0 is the complexified Cartan subalgebra t_C. The "roots" α occurring in this decomposition form a finite subset of the character group $\hat{T} = \mathrm{Hom}(T, S^1)$. The nonzero roots may be divided into positive and negative roots. An element α of $\hat{T}$ is determined by its derivative $\dot{\alpha}$ at the identity:

$$\alpha = e^{i\dot{\alpha}}, \qquad \dot{\alpha} \in t^* \equiv \mathrm{Hom}(t, R). \tag{A.2}$$

Identifying α and $\dot{\alpha}$, the space $\hat{T}$ is a lattice in t^* (the weight lattice).

The root spaces E_α, for $\alpha \neq 0$, are one-dimensional and $E_\alpha = \overline{E}_{-\alpha}$. The Cartan subalgebra E_0 is $r = \mathrm{Rank}(G)$ dimensional. It is standard to choose $e_\alpha \in E_\alpha$ such that the three vectors

$$e_\alpha \in E_\alpha, \quad e_{-\alpha} \equiv \overline{e}_\alpha \in E_{-\alpha}, \quad h_\alpha = -i[e_\alpha, e_{-\alpha}] \in E_0 \tag{A.3}$$

satisfy the $SU(2)$ commutation relations

$$[h_\alpha, e_\alpha] = 2ie_\alpha, \quad [h_\alpha, e_{-\alpha}] = -2ie_\alpha, \quad [e_\alpha, e_{-\alpha}] = ih_\alpha. \tag{A.4}$$

The element h_α satisfies $\alpha(h_\alpha) = 2$, and is called the coroot corresponding to α. It lies in the group $\check{T} \equiv \Lambda^R \equiv \mathrm{Hom}(S^1, T)$, which may be thought of as a lattice in t (the coroot lattice). Since G is simply connected the coroots h_α generate the lattice $\check{T}$.

The Weyl group W is the group of outer automorphisms of T which are obtained by conjugating by elements of G. The group W preserves the lattice $\check{T}$ and permutes the roots in $\hat{T}$.

Since G is simple, all invariant inner products on g are proportional. The basic inner product is defined by the condition that $\langle h_{\alpha_0}, h_{\alpha_0}\rangle = 2$ for α_0 the highest root. For the representation ϕ of lowest weight λ, we may define the inner product

$$\langle t_1, t_2\rangle_\phi = -\mathrm{Tr}(\phi(t_1)\,\phi(t_2)). \tag{A.5}$$

This is a positive integer, $l(\phi)$, times the basic inner product. The quadratic Casimir of the representation ϕ is the operator $-1/2\sum_a \phi(T_a)\phi(T_a)$, where $\{T_a\}$ is any orthonormal basis of g relative to the basic inner product. The quadratic Casimir acts as multiplication by a constant $c(\phi)$. By taking traces we see that $\dim(G)l(\phi) = 2c(\phi)d(\phi)$, where $d(\phi)$ is the dimension of the representation ϕ. The quadratic Casimir is also given by

$$c_\lambda = \tfrac{1}{2}(||\lambda - \rho|| - ||\rho||). \tag{A.6}$$

Here the norm is taken in the basic inner product and ρ is the Weyl vector

$$\rho = \frac{1}{2}\sum_{\alpha>0}\alpha. \tag{A.7}$$

The dual Coxeter number h is the quadratic Casimir of the adjoint representation. In §5 we need the formula

$$h||u||^2 = -\frac{1}{2}\mathrm{Tr}_{\mathrm{adj}}(u\,u) = \sum_{\alpha>0}\alpha(u)\alpha(u) \quad \text{for } u \in t. \tag{A.8}$$

This follows by evaluating the trace in the e_α basis and observing that $[u, [u, e_\alpha]] = -\alpha(u)\alpha(u)e_\alpha$.

Loop groups. Let LG denote the loop group $\mathrm{Map}(S^1, G)$ with pointwise multiplication. It has Lie algebra $Lg = \mathrm{Map}(S^1, g)$. Let $\tilde{L}g = Lg \oplus R$ be a central extension by $\mathbb{R}$ of Lg, so that $\tilde{L}g = Lg \oplus R$ with the Lie bracket

$$\begin{aligned}[(\epsilon_1, \zeta_1), (\epsilon_2, \zeta_2)] &= ([\epsilon_1, \epsilon_2], \omega(\epsilon_1, \epsilon_2)),\\ \omega(\epsilon_1, \epsilon_2) &= \frac{1}{2\pi}\int_0^{2\pi}\left\langle \epsilon_1(\theta), \frac{d}{d\theta}\epsilon_2(\theta)\right\rangle d\theta.\end{aligned} \tag{A.9}$$

(Here $\langle\cdot\,,\,\cdot\rangle$ is an invariant inner product on G.) This is the most general form of a central extension of Lg. The Lie algebra extension determined by ω corresponds to a group extension

$$S^1 \to \tilde{L}G \to LG \tag{A.10}$$

if and only if $\omega/2\pi$ represents an integral cohomology class in LG. This is true if and only if $\langle h_\alpha\,,\,h_\alpha\rangle$ is an even integer for each coroot h_α. For G simple, the basic inner product is the smallest one satisfying this condition.

Let $\mathbb{T}$ denote the group of rigid rotations of S^1. If we identify the Lie algebra of $\mathbb{T}$ with $\mathbb{R}$ by $a \leftrightarrow a\frac{d}{d\theta}$, then the Lie algebra of the semidirect product $\mathbb{T}\tilde{\times}LG$ is $R \oplus Lg$ with the bracket

$$[(x_1\,,\,\epsilon_1)\,,\,(x_2\,,\,\epsilon_2)] = \left(0\,,\,[\epsilon_1\,,\,\epsilon_2] + x_1\frac{d}{d\theta}\epsilon_2 - x_2\frac{d}{d\theta}\epsilon_1\right). \tag{A.11}$$

$\mathbb{T}\times T$ is a maximal abelian subgroup of $\mathbb{T}\tilde{\times}LG$. If we let $z \in S^1$ then the complexified Lie algebra of $\mathbb{T}\tilde{\times}LG$ has the Fourier decomposition

$$(\mathbb{C}\oplus t_{\mathbb{C}}) \oplus \left(\oplus_{n\neq 0} t_{\mathbb{C}} z^n\right) \oplus \left(\bigoplus_{(n\,,\,\alpha)} E_\alpha z^n\right). \tag{A.12}$$

The pieces of this decomposition are indexed by homomorphisms $\overline{\alpha} = (n\,,\,\alpha)$ from $\mathbb{T}\times T \mapsto S^1$ (called affine roots). Here α is a root of g and $n \in Z$. Acting on $(z\,,\,e^{2\pi i u}) \in \mathbb{T}\times T$ we have

$$e^{i\overline{\alpha}}(z\,,\,e^{2\pi i u}) = z^n e^{2\pi i\langle u\,,\,\alpha\rangle}, \tag{A.13}$$

where, as in (A.2), we have identified $\overline{\alpha}$ and its derivative. The affine root $\overline{\alpha}$ is positive when

$$n > 0 \quad \text{or} \quad n = 0 \text{ and } \alpha > 0. \tag{A.14}$$

The affine Weyl group W_{aff} is the semidirect product of $\check{T}$ and W. Here, the elements of $\check{T}$ are thought of as lying in LG and act by conjugation.

The Lie algebra of $\mathbb{T}\tilde{\times}\tilde{L}G$ is $\mathbb{R}\oplus Lg \oplus \mathbb{R}$, with the bracket given by combining (A.9) and (A.11).

The Weyl-Kac character formula. The weights $\overline{\lambda} = (n\,,\,\lambda\,,\,k)$ of $\mathbb{T}\tilde{\times}\tilde{L}g$ lie in $\mathbb{Z}\times\hat{T}\times\mathbb{Z} \subset \mathbb{R}\times t^*\times\mathbb{R}$. An element $\overline{w} = w\eta$ for $w \in W$ and $\eta \in \check{T}$ acts on the weights by

$$\begin{aligned} w(n\,,\,\lambda\,,\,k) &= (n\,,\,w(\lambda)\,,\,k) \\ \eta(n\,,\,\lambda\,,\,k) &= (n + \langle\lambda\,,\,\eta\rangle + \tfrac{1}{2}k\|\eta\|^2\,,\,\lambda + k\eta\,,\,k). \end{aligned} \tag{A.15}$$

Here t and t^* are identified using the inner product.

Every irreducible representation has a unique lowest weight $\bar{\lambda}$. This weight is antidominant, that is,

$$-\frac{k}{2}||h_\alpha||^2 \leq \lambda(h_\alpha) \leq 0 \tag{A.16}$$

for α any positive root. The integer k is called the level of the representation. Equation (A.16) implies that there are only a finite number of irreducible representations at each level. The isomorphism classes of irreducible representations of $\mathbb{T}\tilde{\times}\tilde{L}G$ are in one-to-one correspondence with the antidominant weights.

The Weyl-Kac character formula gives an expression for the character

$$\chi_{\bar{\lambda}} = \sum_{\bar{\gamma}} e^{i\bar{\gamma}}$$

of an irreducible representation. Here the sum is over all weights $\bar{\gamma}$ in the representation, counted according to multiplicity. The character formula is

$$\chi_{\bar{\lambda}} = \left(\prod_{\bar{\alpha}>0}(1-e^{i\bar{\alpha}})\right)^{-1} \sum_{\overline{w}\in W_{\mathrm{aff}}} (-1)^{l(\overline{w})} e^{i(\bar{\rho}+w(\bar{\lambda}-\bar{\rho}))}, \tag{A.17}$$

where $l(\overline{w})$ is the number of positive roots $\bar{\alpha}$ for which $\overline{w\alpha}$ is negative and $\bar{\rho} = (0, \rho, -h)$. The product is over all positive roots counting multiplicities. In particular, the root $\bar{\alpha} = (n, 0)$ is repeated $r = \mathrm{Rank}(G)$ times.

Theta functions. In order to make contact with theta functions, we need to make (A.17) more explicit. We write $z \in \mathbb{T}$ as $z = e^{2\pi i\tau}$, for $\tau \in R$. The formulas below make sense for τ in the complex upper half plane; we use them in this form in §5. Evaluated at $(z, e^{2\pi i u}) \in \mathbb{T} \times T$, the denominator in (A.17) is

$$\begin{aligned}\Pi(\tau, u) &\equiv \prod_{\bar{\alpha}>0}(1-e^{i\bar{\alpha}})(z, e^{2\pi i u}) \\ &= \prod_{\alpha>0}(1-e^{2\pi i\langle u,\alpha\rangle}) \\ &\quad \times \prod_{n>0}\left((1-z^n)^r \prod_{\alpha>0}(1-e^{2\pi i<u,\alpha>}z^n)(1-e^{-2\pi i<u,\alpha>}z^n)\right).\end{aligned} \tag{A.18}$$

Note that, for convenience in §5, the definition of $\Pi(\tau, u)$ is chosen to agree with that of [12], *not* [26].

For the numerator of (A.17), we write the sum over $W_{\mathrm{aff}} = W \tilde{\times} \check{T}$ as a sum over $\check{T}$ followed by a sum over W. For $\overline{w} = w\eta$, $w \in W$, $\eta \in \check{T}$, and $\overline{\lambda} = (0, \lambda, k)$, we have

$$\begin{aligned}\overline{w}(\overline{\lambda} - \overline{\rho}) + \overline{\rho} &= \overline{w}(0, \lambda - \rho, k+h) + (0, \rho, -h) \\ &= \left(\langle \lambda - \rho, \eta \rangle + \frac{1}{2}(k+h)\|\eta\|^2, \right. \\ &\qquad \left. \rho + w(\lambda - \rho) + (k+h)w(\eta), k\right).\end{aligned} \tag{A.19}$$

Thus,

$$\begin{aligned} e^{i\overline{w}(\overline{\lambda}-\overline{\rho})}&(z, e^{2\pi i u}) \\ &= \exp\left(2\pi i \tau \left(\frac{1}{2(k+h)}\|\eta\|^2 + \langle \lambda - \rho, \eta\rangle\right)\right) \\ &\quad \times \exp(2\pi i(\langle \rho + w(\lambda - \rho) + (k+h)w(\eta), u\rangle)) \\ &= \exp\left(2\pi i\left(\langle \rho, u\rangle - \frac{\tau}{2(k+h)}\|\lambda - \rho\|^2\right)\right) \\ &\quad \times \exp\left(2\pi i(k+h)\left(\frac{\tau}{2}\left\|w(\eta) + \frac{w(\lambda-\rho)}{k+h}\right\|^2\right.\right. \\ &\qquad\qquad \left.\left. + \left\langle w(\eta) + \frac{w(\lambda-\rho)}{k+h}, u\right\rangle\right)\right),\end{aligned} \tag{A.20}$$

where we have used the invariance of $\|\cdot\|$ under W.

And so,

$$\begin{aligned}\sum_{\eta \in \check{T}} e^{i(\overline{\rho} + \overline{w}(\overline{\lambda}-\overline{\rho}))}&(z, e^{2\pi i u}) \\ &= e^{2\pi i(<\rho, u> - (\tau/2)\|\lambda-\rho\|^2/2(k+h))}\theta_{w(\lambda-\rho), k+h}(\tau, u),\end{aligned} \tag{(A.21)}$$

where

$$\theta_{\gamma, k}(\tau, u) \equiv \sum_{\alpha \in \check{T}} \exp\left(i\pi k\tau \left\|\alpha + \frac{\gamma}{k}\right\|^2 + 2\pi i k\left\langle u, \left(\alpha + \frac{\gamma}{k}\right)\right\rangle\right). \tag{A.22}$$

To obtain (A.21) we have shifted the sum over $\check{T}$ from η to $w(\eta)$.

Finally, substituting into (A.17), we obtain

$$\begin{aligned}\chi_{\lambda,k}(e^{2\pi i\tau}, e^{2\pi i u}) &= \Pi(\tau, u)^{-1}\left[\sum_{w\in W}(-1)^{l(w)}\theta_{w(\lambda-\rho),k+h}(\tau, u)\right]\\ &\quad\times\exp\left(2\pi i\left(\langle\rho, u\rangle - \frac{\tau}{2(k+h)}\|\lambda-\rho\|^2\right)\right)\\ &= \frac{\theta^-_{\lambda-\rho,k+h}(\tau, u)}{\Pi(\tau, u)\exp(-2\pi i(\langle\rho, u\rangle - \tau|G|/24))}\\ &\quad\times\exp\left(2\pi i\tau\left(\frac{\|\rho\|^2}{2h} - \frac{\|\rho-\lambda\|^2}{2(k+h)}\right)\right),\end{aligned} \tag{A.23}$$

where

$$\theta^-_{\gamma,k} = \sum_{w\in W}(-1)^{l(w)}\theta_{w(\gamma),k}. \tag{A.24}$$

For the second equality in (A.23), we have used the Freudenthal strange formula:

$$\|\rho\|^2/2h = |G|/24, \tag{A.25}$$

where $|G|$ is the dimension of G.

The functions $\theta_{\gamma,k}$ defined in (A.22) are theta functions for the torus $t\times t/\Lambda_R\times\Lambda_R$ with the symplectic structure and complex structure

$$\begin{aligned}\omega &= 2\pi k C_{ij}\,d\theta_1^i d\theta_2^j\\ u^i &= \theta_2^i - \tau\theta_1^i \qquad i = 1, \dots, r.\end{aligned} \tag{A.26}$$

Here (θ_1^i, θ_2^i) are real coordinates for $t\times t$, C_{ij} is the matrix of the basic inner product, and u^i are holomorphic coordinates.

For consistency with [7], in §5 we shall actually use the characters of *highest* weight representations at level k. If γ is a highest weight, then

$$\chi^{\text{H.W.}}_{\gamma,k}(e^{2\pi i\tau}, e^{2\pi i u}) = \chi^{\text{L.W.}}_{-\gamma,k}(e^{2\pi i\tau}, e^{2\pi i(-u)}), \tag{A.27}$$

where $\chi^{\text{L.W.}}$ is the lowest weight character given above. It is easy to see that $\theta_{-\gamma-\rho}(\tau, -u) = \theta_{\gamma+\rho}(\tau, u)$ and that

$$\pi(\tau, -u)e^{-2\pi i<\rho, -u>} = \pi(\tau, u)e^{-2\pi i<\rho, u>}(-1)^{(|G|-r)/2}. \tag{A.28}$$

So the highest weight character is

$$\begin{aligned}\chi^{\text{H.W}}_{\gamma,k}(e^{2\pi i\tau}, e^{2\pi i u}) &= \frac{\theta^-_{\gamma+\rho,k+h}(\tau, u)}{\tilde\Pi(\tau, u)e^{\pi i\tau(\|\rho\|^2/h-\|\rho+\gamma\|^2/(k+h))}},\\ \tilde\Pi(\tau, u) &= \Pi(\tau, u)e^{-2\pi i(\langle\rho, u\rangle-\tau|G|/24)}(-1)^{(|G|-\tau)/2}.\end{aligned} \tag{A.29}$$

Since $\chi_{0,0}^{\text{H.W.}} = 1$, we get the Macdonald identity:

(A.30)
$$\tilde{\Pi}(\tau, u) = \theta^{-}_{\rho,h}(\tau, u).$$

In §5, we also use the following facts.

(a) Differentiating (A.22) term by term we see that $\theta_{\gamma,k}$ satisfies the heat equation

$$\left[\frac{\partial}{\partial\tau} - \frac{1}{4\pi i k} C^{ij} \frac{\partial}{\partial u^i} \frac{\partial}{\partial u^j}\right] \theta_{\gamma,k} = 0.$$

(b) The functions $\theta_{\gamma,k}$ are (up to a normalization independent of τ and u) orthonormal in the inner product

$$\langle\theta_{\gamma,k}, \theta_{\gamma',k}\rangle(\tau, u) = e^{\frac{\pi k}{2\tau_2}\|u-\overline{u}\|^2} \theta_{\gamma,k}(\tau, u)^* \theta_{\gamma',k}(\tau, u),$$
$$\langle\theta_1, \theta_2\rangle = \int_{T\times T} d^r u\, d^r \overline{u}\, \tau_2^{-r/2} \langle\theta_1, \theta_2\rangle(\tau, u).$$

References

[1] L. Alvarez-Gaumé, G. Moore & C. Vafa, *Theta functions, modular invariance, and strings*, Comm. Math. Phys. **106** (1986) 1–40.

[2] M. F. Atiyah, *New invariants of three and four dimensional manifolds*, The Mathematical Heritage of Hermann Weyl (R. Wells, ed.), Proc. Sympos. Pure Math., Vol. 48, Amer. Math. Soc., Providence, RI, 1988.

[3] M. F. Atiyah & R. Bott, *The Yang-Mills equations over Riemann surfaces*, Philos. Trans. Roy. Soc. London Ser. A **308** (1982) 523.

[4] A. Belavin, A. M. Polyakov & A. Zamolodchikov, *Infinite conformal symmetry in two-dimensional quantum field theory*, Nuclear Phys. B **241** (1984) 333–380.

[5] J. Bismut & D. Freed, *The analysis of elliptic families*. I, Comm. Math. Phys. **106** (1986) 159–176.

[6] R. Dijkgraaf & E. Witten, *Topological gauge theories and group cohomology*, Institute for Advanced Study, preprint HEP-89/33.

[7] S. Elitzur, G. Moore, A. Schwimmer & N. Seiberg, *Remarks on the canonical quantization of the Chern-Simons-Witten theory*, Institute for Advanced Study, preprint HEP-89/20.

[8] J. Fay, *Perturbations of analytic torsion on Riemann surfaces*, preprint, 1980.

[9] D. Freed, *Determinants, torsion, and strings*, Comm. Math. Phys. **107** (1986) 485–513.

[10] ——, *On determinant line bundles*, Math. Aspects of String Theory (S. T. Yau, ed.), World Scientific, Singapore, 1987.

[11] D. Friedan & S. Shenker, *The analytic geometry of two-dimensional conformal field theory*, Nuclear Phys. B **281** (1987) 509–545.

[12] K. Gawkedzki & A. Kupiainen, *Coset construction from functional integrals*, Inst. Hautes Études Sci. preprint P/88/45.

[13] D. Gepner & E. Witten, *String theory on group manifolds*, Nuclear Phys. B **278** (1986) 493–549.

[14] P. Goddard & D. Olive, *Kac-Moody and Virasoro algebras in relation to quantum physics*, Internat. J. Modern Phys. A **1** (1986) 303–314. (And references therein.)

[15] P. Griffiths & J. Harris, *Principles of algebraic geometry*, Wiley, New York, 1978.

[16] V. Guillemin & S. Sternberg, *Geometric quantization and multiplicities of group representations*, Invent. Math. **67** (1982) 515–538.

[17] J. Harer, *The second homology group of the mapping class group of an orientable surface*, Invent. Math. **72** (1982) 221–239.

[18] N. Hitchin, *Flat connections and geometric quantization*, Oxford preprint, 1989.

[19] ——, *Stable bundles and integrable systems*, Duke Math. J. **54** (1987) 91–114.

[20] V. F. R. Jones, *Hecke algebra representations of braid groups and link polynomials*, Ann. of Math. (2) **126** (1987) 335–388.

[21] V. G. Knizhnik & A. B. Zamolodchikov, *Current algebra and Wess-Zumino model in two dimensions*, Nuclear Phys. B **247** (1984) 83.

[22] B. Kostant, *Orbits, symplectic structures, and representation theory*, Proc. U.S.-Japan Seminar in Differential Geometry (Kyoto, 1965), Quantization And Unitary Representations, Lecture Notes in Math., Vol. 170, Springer, Berlin, 1970, 87; Line Bundles And the Prequantized Schrodinger Equation, Coll. Group Theoretical Methods in Physics, Marseille, 1972, 81.

[23] G. Moore and N. Seiberg, *Polynomial equations for rational conformal field theories*, Phys. Lett. B **212** (1988) 360; Naturality in conformal field theory, Institute for Advanced Study, preprint HEP/88/31 (to appear in Nuclear Phys. B); Classical and quantum conformal field theory, Institute for Advanced Study, preprint HEP/88/39 (to appear in Nuclear Phys. B).

[24] D. Mumford & J. Fogarty, *Geometric invariant theory*, Springer, Berlin, 1982.

[25] M. S. Narasimhan & S. Ramanan, *Deformations of the moduli space of vector bundles over an algebraic curve*, Ann. of Math. (2) **101** (1975) 391–417.

[26] A. Pressley & G. Segal, *Loop groups*, Oxford Univ. Press, Oxford, 1988.

[27] D. Quillen, *Determinants of Cauchy-Riemann operators over a Riemann surface*, Functional Anal. Appl. **19** (1985) 31–34.

[28] T. R. Ramadas, I. M. Singer & J. Weitsman, *Some comments on Chern-Simons gauge theory*, MIT, preprint.

[29] D. Ray and I. Singer, *R-Torsion and the Laplacian on Riemannian manifolds*, Advances in Math. **7** (1971) 145–210; *Analytic torsion of complex manifolds*, Ann. of Math. (2) **98** (1973) 154–177.

[30] N. Yu. Reshetikhin & V. G. Turaev, *Invariants of three-manifolds via link polynomials and quantum groups*, Math. Sci. Res. Inst. preprint, 1989.

[31] B. Schroer, *Operator approach to conformal invariant quantum field theories and related problems*, Nuclear Phys. **295** (1988) 586–616; K.-H. Rehren & B. Schroer, *Einstein causality and Artin braids*, FU preprint, 1987.

[32] A. Schwarz, *The partition function of degenerate quadratic functional and Ray-Singer invariants*, Lett. Math. Phys. **2** (1978) 247–252.

[33] A. S. Schwarz, *New topological invariants arising in the theory of quantized fields*, Abstract, Baku International Topological Conference, 1987.

[34] G. Segal, *Two-dimensional conformal field theories and modular functors*, IXth Internat. Conf. on Math. Physics (Swansea, July, 1988) (B. Simon, A. Truman, and I. M. Davies, eds.), Adam Hilger, 1989, 22–37.

[35] C. Simpson, *Higgs bundles and local systems, moduli of representations of the fundamental group of a smooth projective variety*, preprints.

[36] J. Sniatycki, *Geometric quantization and quantum mechanics*, Springer, New York, 1980.

[37] J.-M. Souriau, *Quantification geometrique*, Comm. Math. Phys. **1** (1966) 374; *Structures des systemes dynamiques*, Dunod, Paris, 1970.

[38] A. Tsuchiya & Y. Kanie, *Conformal field theory and solvable lattice models*, Advanced Studies Pure Math. **16** (1988) 297; Lett. Math. Phys. **13** (1987) 303–312.

[39] A. Tsuchiya, K. Ueno & Y. Yamada, *Conformal field theory on universal family of stable curves with gauge symmetries*, Max Planck Institute, preprint, 1989.
[40] E. Verlinde, *Fusion rules and modular transformations in 2d conformal field theory*, Nuclear Phys. B **300** (1988) 360–376.
[41] E. Witten, *Topological quantum field theory*, Comm. Math. Phys. **117** (1988) 353–386.
[42] ——, *Quantum field theory and the Jones polynomial*, Comm. Math. Phys. **121** (1989) 351–399.
[43] N. Woodhouse, *Geometric quantization*, Oxford Univ. Press, Oxford, 1980.

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