J. DIFFERENTIAL GEOMETRY
33 (1991) 683-715

COMPACT CONSTANT MEAN CURVATURE SURFACES IN EUCLIDEAN THREE-SPACE

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1. Introduction

The main subject of this paper is the construction of closed CMC surfaces of any genus $g \geq 3$. The abbreviation "CMC surfaces" is used throughout the paper and stands for "properly immersed complete boundaryless surfaces in E^3 of constant mean curvature $H \equiv 1$". We also talk about "compact CMC surfaces", which means the same except "with boundary" rather than without. Compact CMC surfaces of any genus $g \geq 3$ with boundary a round planar circle are also constructed. These constructions are achieved by properly strengthening the methods employed in [10] to construct CMC surfaces with ends. The main results of this paper were announced in [8].

The question of whether such surfaces exist has a long history. In 1853 J. H. Jellett proved that star-shaped closed CMC surfaces are round spheres. In 1900 Liebmann [13] proved the same for convex surfaces. S.-S. Chern [3] extended Liebman's result to a certain class of convex W-surfaces. Hopf [5] established that any CMC topological sphere is round and asked whether the same is true for all closed CMC surfaces. Alexandrov [1] gave an affirmative answer for embedded surfaces. Wu-Yi Hsiang settled in the negative the higher dimensional analogue to Hopf's question [6]. Eventually, H. C. Wente [14] settled the so-called Hopf's conjecture also in the negative by constructing infinitely many CMC tori.

This paper is self-contained in the sense that the results presented here can be understood without reference to any other papers. However, many of the proofs are extensions of proofs in [10] and it would be impossible to make them self-contained without repeating most of that paper. Familiarity with [10] would be helpful also in understanding the basic idea of the construction which we proceed to outline.

Received March 30, 1989 and, in revised form, December 1, 1989. The main results of this paper were first established while the author was visiting the University of California, San Diego. The present version was prepared at the University of California, Berkeley, where he held a Miller fellowship.

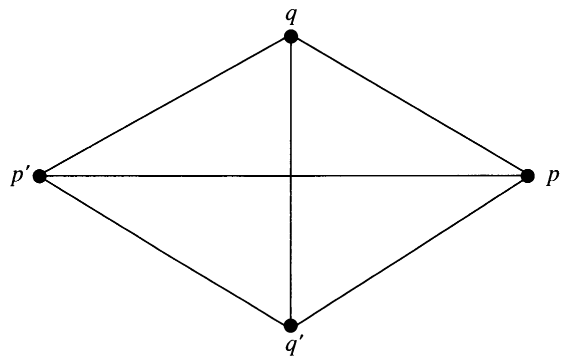

FIGURE 1.1

We start by specifying a simple graph (Figure 1.1) on which we base the construction of a CMC 3-torus. The construction is symmetric with respect to the coordinate planes of some Cartesian coordinate system $Ox_1x_2x_3$ of E^3. The vertices of the graph are p, p', q, and q'. p and p' lie on the x_1-axis and they are mirror images of each other with respect to the x_2-axis. q and q' lie on the x_2-axis and they are mirror images with respect to the x_1-axis. Each pair of vertices is connected by an edge, so we have six edges. In order to use this graph for a construction like the ones in [10], it would have to satisfy various hypotheses [10, II.1.18], most important of which are balance and flexibility. Balance amounts to specifying a nonzero real number $\hat{\tau}$ for each edge of the graph so that $\sum \hat{\tau}\vec{v} = 0$ at each vertex, where $\vec{v}$ varies over the unit vectors pointing away from the vertex in consideration in the direction of an edge with endpoint this vertex, and $\hat{\tau}$ is the number assigned to the corresponding edge. Making the graph balanced is easy because we are allowed to use negative as well as positive values for $\hat{\tau}$, corresponding to nonembedded and embedded Delaunay pieces respectively. Indeed we can arbitrarily assign a nonzero $\hat{\tau}$ to qq' which we will call τ. Symmetry and balancing at q then determines $\hat{\tau}$ at pq, and finally symmetry and balancing at p determines $\hat{\tau}$ at pp'.

Flexibility amounts to the possibility of perturbing the graph so that the lengths of the edges are slightly perturbed in any preassigned way which respects the symmetries. The graph under consideration is not flexible and this is why the construction in [10] fails: Once the lengths of qq' and pq are determined, the length of pp' is determined uniquely. To illustrate this point try to construct an initial surface based on the graph (Figure 1.2). As in [10] we have to use a round sphere for each vertex and a

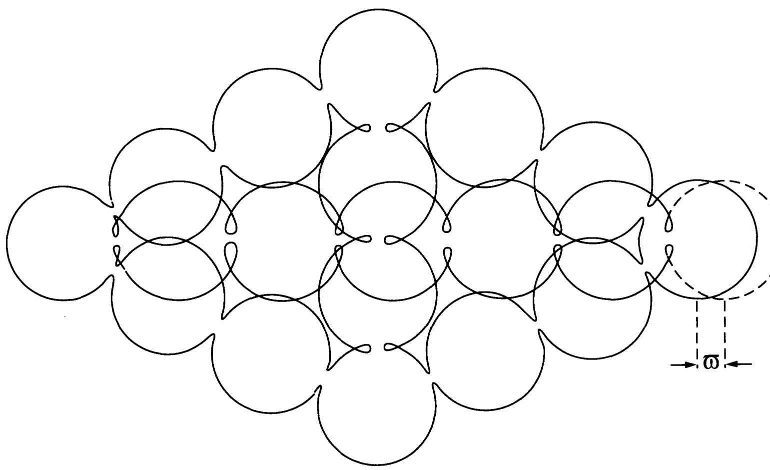

FIGURE 1.2. INTERSECTION WITH x_1x_2-PLANE.

Delaunay piece for each edge. The distance between the spheres corresponding to q and q' is determined by the Delaunay piece of parameter $\hat{\tau}(qq')$ connecting them. The position of the spheres corresponding to p and p' is then determined by the length of the Delaunay piece of parameter $\hat{\tau}(qp)$ corresponding to qp. There is no reason why the Delaunay piece corresponding to pp' should have the appropriate length to connect the spheres corresponding to p and p'.

We deal with this difficulty in the following way: Connect the sphere corresponding to p' to a new sphere near the sphere corresponding to p and repeat the whole construction around the new sphere, that is, everything is translated by carrying the sphere corresponding to p' to the new sphere. In other words we have a new group of symmetries generated by the old one and a translation by ϖ in the direction of the x_1-axis. We have replaced the initial graph with a periodic graph (Figure 1.3, next page) and this allows us to construct the corresponding initial surface (Figure 1.4). We can then perturb the initial surface in the fashion of [10] to obtain a periodic CMC surface. This periodic CMC surface is actually closed if $\varpi = 0$, hence the problem has been reduced to making ϖ vanish.

As a first step notice that we can arrange $|\varpi| < 2$ simply by choosing the correct number of "lobes" in the Delaunay piece corresponding to pp' (unlike Figure 1.4). The number of "lobes" corresponds to the number of asr's which we take the opportunity to define:

Definition 1.1. We call the closure of any connected component of $\{K_g \neq 0\}$, in a Delaunay surface or any initial surface we consider, an

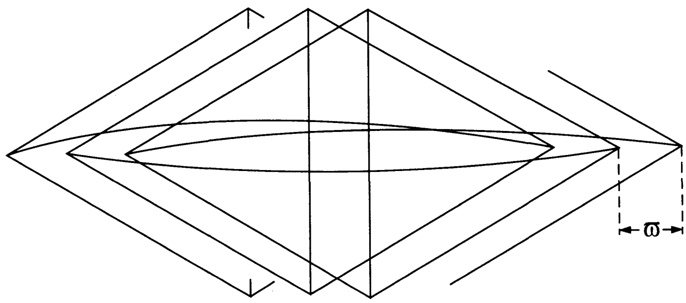

FIGURE 1.3

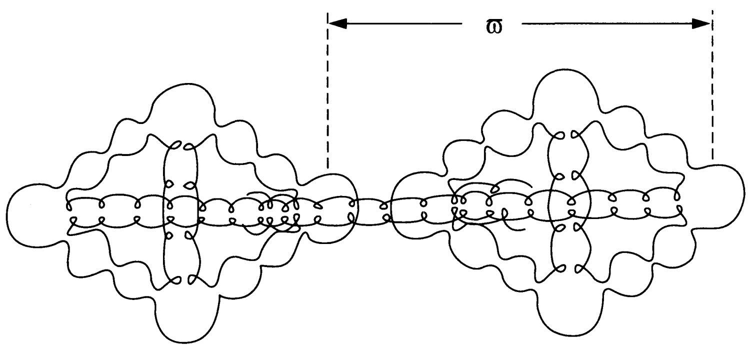

FIGURE 1.4. INTERSECTION WITH x_1x_2-PLANE DELAUNAY PIECES DRAWN SCHEMATICALLY.

almost spherical region (asr for short). If $K_g \geq 0$ on it we call it a positive asr, otherwise a negative asr.

Positive asr's are approximately round spheres minus two small discs, while negative asr's, when enlarged by a suitable large factor, approximate a large compact subset of a catenoid [10, Appendix A and Lemma III.3.8].

The perturbations of the asr's required in the construction of the CMC surface can change the period ϖ. The exponential decay of the perturbation along the edges and away from the spheres ensures that the change is at most 1 even if the number of asr's in the Delaunay pieces is arbitrarily large. This will be useful later.

Notice that the construction has τ as a free parameter. By varying τ the lengths of the Delaunay pieces change and hence they force the period ϖ to change. This change is small if the Delaunay pieces have few asr's, but if they have a lot of asr's it accumulates so that it exceeds three. Assume that ϖ depends continuously on τ. Since we can increase or decrease ϖ by varying τ in different directions, we can appeal to the intermediate value theorem to ensure that $\varpi = 0$ for some τ, obtaining thus the desired surface. Actually, we achieve the vanishing of ϖ by incorporating the above argument in the Schauder fixed theorem used in the proof of the main theorem in [10].

This paper has three sections besides the introduction. In §2 we describe a general sort of graph like the one above on which such constructions can be based. We prove the relevant result and we use it to construct various examples, in particular infinitely many closed CMC surfaces of any genus $g \geq 3$. All these surfaces have some symmetry. In §3 we generalize the construction somewhat and for all but a few genera we exhibit closed CMC surfaces which possess no symmetries. Finally in §4 we extend the construction to prove that any planar round circle of radius < 1 is the boundary of infinitely many compact CMC surfaces of any given genus $g \geq 3$.

2. Closed CMC surfaces with symmetries

In this section we prove a general theorem which enables us to construct a variety of closed CMC surfaces. Unfortunately all of them have non-trivial symmetries. To construct completely asymmetrical ones we have to overcome an extra technical difficulty. We postpone this to the next section in order to make the current construction as simple as possible. As in [10] we use graphs to codify our constructions. For constructions of closed surfaces we need new kinds of graphs which we proceed to define. We will use a tilde to denote c-graphs (Definition 2.1) and an underbar to denote central c-graphs (Definition 2.4).

Definition 2.1. A c-graph $\widetilde{\Gamma}$ is a set $\{V(\widetilde{\Gamma}), E(\widetilde{\Gamma}), \hat{\tau}, G\}$, where:

(1) $V(\widetilde{\Gamma})$ is a finite set of points in E^3 called the vertices of $\widetilde{\Gamma}$,

(2) $E(\widetilde{\Gamma})$ is a set of straight line segments whose endpoints are vertices and they are called the edges of $\widetilde{\Gamma}$. Edges with a common endpoint point in different directions,

(3) $\hat{\tau}: E(\widetilde{\Gamma}) \mapsto \mathbb{R}\backslash\{0\}$ is a function,

(4) (The group of symmetries) G is a group of Euclidean motions whose action preserves each of the above.

The above definition is similar to the definition of a graph [10, II.1.1]. A c-graph has no rays however. If $e \in E(\widetilde{\Gamma})$, we write $l_{\widetilde{\Gamma}}(e)$ for its length. This length is not expected to be close to an even integer as it is for a graph in [10]. In the construction of the surfaces e will be replaced by a Delaunay piece whose number of asr's is roughly proportional to $l_{\tilde{\Gamma}}(e)$ by a large constant N (cf. the proof of Theorem 2.7 and in particular equations (1) and (1′)). Suppose p is a vertex of $\tilde{\Gamma}$. We write E_p for the set of edges of $\tilde{\Gamma}$ which have p as an endpoint. If $e \in E_p$, we write $\vec{v}_{e,p}$ for the unit vector pointing in the direction of e from p. In analogy with [10, II.1.2], we define

$$d_{\tilde{\Gamma}}(p) \equiv \frac{3}{2} \sum_{e \in E_p} \hat{\tau}(e) \vec{v}_{e,p} . \tag{2.2}$$

We assume that E^3 is equipped with a Cartesian coordinate system $Ox_1x_2x_3$ and identify the tangent space at each point of E^3 with $\mathbb{R}^3$. $d_{\tilde{\Gamma}}$ is an element of $W(\tilde{\Gamma})$ which is defined to be the set of all functions $w: V(\tilde{\Gamma}) \mapsto \mathbb{R}^3$ invariant under the action of G. The norm of an element w of $W(\tilde{\Gamma})$ is defined by

$$|w| \equiv \max_{p \in V(\tilde{\Gamma})} |w(p)| . \tag{2.3}$$

We distinguish an edge $\mathbf{e} \in E(\tilde{\Gamma})$ and we define $L(\tilde{\Gamma}, \mathbf{e})$ to be the space of functions invariant under the action of G:

$$\tilde{\ell}: E(\tilde{\Gamma}) \backslash G\mathbf{e} \mapsto \mathbb{R} .$$

$L(\tilde{\Gamma}, \mathbf{e})$ is equipped with the maximum norm.

We call two c-graphs $\tilde{\Gamma}_1$ and $\tilde{\Gamma}_2$ isomorphic if the following (intuitively reasonable) conditions are satisfied:

(1) There are one-to-one correspondences between the vertices and the edges of the two graphs respecting the "endpoint of" relationship.
(2) The two corresponding groups of symmetries are identical and the correspondences above are equivalent with respect to the induced action.

Notice that the lengths of corresponding edges are not required to be the same. We occasionally abuse the notation by using the same symbol for corresponding vertices or edges. This applies to isomorphic graphs in the sense of [10, II.1.1] as well. To construct closed CMC surfaces we need central c-graphs which we now define.

Definition 2.4. A central c-graph $\underline{\tilde{\Gamma}}$ is a c-graph equipped with a distinguished edge $\mathbf{e}$, a collection of c-graphs $\{\tilde{\Gamma}(\tilde{d}, \tilde{\ell})\}$, each of which is called a regular perturbation of $\underline{\tilde{\Gamma}}$, and an $\epsilon > 0$. The collection of the regular perturbations is parametrized by $(\tilde{d}, \tilde{\ell})$ which takes values on the closed ϵ-ball centered at the origin of $W(\underline{\widetilde{\Gamma}}) \times L(\underline{\widetilde{\Gamma}}, \mathbf{e})$. Furthermore, the following are satisfied:

(1) $\widetilde{\Gamma}(\tilde{d}, \tilde{\ell})$ depends smoothly on $(\tilde{d}, \tilde{\ell})$ and is isomorphic to $\underline{\widetilde{\Gamma}} \equiv \widetilde{\Gamma}(0, 0)$.
(2) If $\widetilde{\Gamma} = \widetilde{\Gamma}(\tilde{d}, \tilde{\ell})$, then $d_{\widetilde{\Gamma}} = \tilde{d}$. (In particular $d_{\underline{\widetilde{\Gamma}}} = 0$; that is, $\underline{\widetilde{\Gamma}}$ is balanced).
(3) If $\widetilde{\Gamma} = \widetilde{\Gamma}(\tilde{d}, \tilde{\ell})$, then $l_{\widetilde{\Gamma}}(e) = l_{\underline{\widetilde{\Gamma}}}(e) + \tilde{\ell}(e)$ for each edge e of $\widetilde{\Gamma}$ not in the orbit of $\mathbf{e}$ under the action of G.
(4) If $\widetilde{\Gamma} = \widetilde{\Gamma}(\tilde{d}, \tilde{\ell})$, then $l_{\widetilde{\Gamma}}(\mathbf{e})$ depends only on $\tilde{\ell}$. Let $\kappa(\tilde{\ell}) \equiv l_{\widetilde{\Gamma}(\tilde{d}, \tilde{\ell})}(\mathbf{e})$. We then have

$$\kappa(0)\hat{\tau}(\mathbf{e}) \neq \sum_{e \in E(\underline{\widetilde{\Gamma}}) \setminus G\mathbf{e}} \frac{\partial \kappa}{\partial \tilde{\ell}(e)}(0) l_{\underline{\widetilde{\Gamma}}}(e)\hat{\tau}(e). \tag{2.5}$$

(5) For simplicity we also require that $\hat{\tau}(\mathbf{e})$ does not depend on $\widetilde{\Gamma}$.

Central c-graphs correspond to flexible central graphs [10, II.1.18]. Notice the following two technical differences between the two definitions: First, we require that $d_{\widetilde{\Gamma}} = \tilde{d}$ always, not just when $\tilde{\ell} = 0$; $\hat{\tau}$ depends on $\tilde{\ell}$ as well. Second, we do not require that all edges can have their lengths perturbed arbitrarily, the reason being that we did not manage to find any such closed graphs. Instead we require that all edges can have their lengths arbitrarily perturbed with one—up to the action of G—exception. This exceptional edge $\mathbf{e}$ has its length determined by the lengths of the other edges. Moreover condition (2.5) is imposed on this for reasons which will become clear later; this condition is satisfied in the generic case.

We describe now the relation between a c-graph $\widetilde{\Gamma}$ and the corresponding closed CMC surface M produced by our construction. M actually depends not only on $\widetilde{\Gamma}$ but on a large real number $N > 0$ and a small number $\tau'' > 0$ as well. (τ' is preserved to denote a different, but related to τ'', constant in the statement of Theorem 2.7.) Homothetically expand the configuration of the vertices and edges of $\widetilde{\Gamma}$ by a factor N to obtain the edges and vertices of a graph Γ (there are no rays). Define $\hat{\tau}$ at each edge of Γ to be the $\hat{\tau}$ of the corresponding edge of $\widetilde{\Gamma}$ multiplied by a factor τ''. It should be clear to the reader familiar with [10] that the collection of the vertices and edges of Γ together with the function $\hat{\tau}$ on these edges is a graph (which we call Γ) in the sense of [10, II.1.1]. Let M' be an initial surface associated to Γ. We briefly recall now what this means to help the reader in following the construction later; we avoid repeating too many details however because this would increase substantially the length of this paper.

The definition of an initial surface M' associated to Γ [10, III.2.6] assumes that Γ is a member of a family of graphs [10, II.1.9]. This is relevant only in that a small real number $\varepsilon > 0$ and a function $l\colon E(\Gamma) \mapsto \mathbb{Z}^+$ are implicitly determined in this way. l and ε determine approximately the length of the edges of Γ through the requirement that each edge e of Γ has length $2l(e) + \ell(e)$, where $|\ell(e)| \leq \varepsilon^7 l(e)$. M' is then a closed—in this case—surface invariant under the action of the symmetry group G on Γ, and which surface is the union of the following:

(1) A unit sphere centered at p for each vertex p of Γ, from which sphere small disjoint discs have been removed, one disc for each edge of Γ with p as an endpoint.
(2) A perturbed Delaunay piece for each edge e of Γ. This Delaunay piece contains $l(e)$ negative asr's, $l(e) - 1$ whole positive asr's, and two positively curved annuli of width roughly proportional to ε. The latter are neighborhoods of its two boundary circles and they are perturbed so that they are smoothly attached to the spheres minus discs above. The other (whole) asr's of the Delaunay pieces are also perturbed slightly. The perturbation of each of them is controlled by a small vector; these vectors are determined by a sequence of vectors called the configuration ξ of M'.

ε can be restricted a priori to be quite small. Γ alone determines then the number of asr's contained in the various Delaunay pieces up to a factor close to 1. The injectivity radius of M' is determined up to a factor close to 1 by τ'' and the $\hat{\tau}$ function of $\widetilde{\Gamma}$. It is close to $\pi\tau'' \min_{e\in E(\widetilde{\Gamma})} |\hat{\tau}(e)|$. (This follows from the geometry of the Delaunay surfaces.) Let $X\colon M' \mapsto E^3$ be the immersion of M' which we are considering, and $\nu\colon M' \mapsto S^2(1)$ be its Gauss map.

Definition 2.6. Suppose that M', $\widetilde{\Gamma}$, τ'', and N are as above, and there is a small (cf. (2.10)), G-invariant, smooth function φ on M' such that $X_\varphi = X + \varphi\nu$ is an immersion of constant mean curvature $H \equiv 1$. We call $M = X_\varphi(M')$ a surface based on $(\widetilde{\Gamma}, \tau'', N)$, or for simplicity, on $\widetilde{\Gamma}$.

Since we have a clear description of M' and φ is small, we feel justified to talk about the "construction of M" rather than the "proof of existence of M". The information on the injectivity radius of M' implies quite clearly that the shortest nontrivial loop on M is in $[\pi\tau''\tau, 3\pi\tau''\tau]$, where

$\tau \equiv \min_{e \in E(\widetilde{\Gamma})} |\hat{\tau}(e)|$. The reader who is not familiar with [10] might find the above definition too technical. For him an alternative (weaker) description is to say that M is built from unit spheres centered at the vertices of Γ, and Delaunay pieces corresponding to the edges, with the resulting surface being slightly perturbed. An even weaker description which avoids any mention of Delaunay surfaces is possible: Consider the support of $\widetilde{\Gamma}$—that is the union of its edges—and homothetically expand it by a factor N. Slightly perturb it to avoid any self-intersections and consider a small regular neighborhood R of it; that is, a small enough tubular neighborhood which avoids "self-intersections".

Definition 2.6′. M is based on $(\widetilde{\Gamma}, \tau'', N)$ if there is a homeomorphism f from M to the boundary of a regular neighborhood R as above, such that $\forall X \in M \ \ |f(X) - X| < 4$, M has $H \equiv 1$, it is invariant under the action of G on the homothetically expanded $\widetilde{\Gamma}$, and the length of its shortest nontrivial loop is in $[\pi\tau''\tau, 3\pi\tau''\tau]$, where $\tau \equiv \min_{e \in E(\widetilde{\Gamma})} |\hat{\tau}(e)|$.

The following theorem is valid with any of the above definitions.

Theorem 2.7. *For any central c-graph $\underline{\widetilde{\Gamma}}$ there are infinitely many closed CMC surfaces based on regular perturbations of $\underline{\widetilde{\Gamma}}$. More precisely there is $T_0(\underline{\widetilde{\Gamma}}) > 0$ such that for any $\tau' \in (0, T_0)$ there is $N_0(\tau', \underline{\widetilde{\Gamma}}) > 0$ such that for any $N > N_0$ there is a closed CMC surface M based on $(\widetilde{\Gamma}, \tau'', N)$ for some regular perturbation $\widetilde{\Gamma}$ of $\underline{\widetilde{\Gamma}}$ and a $\tau'' \in [\tau', 3\tau']$.*

Before we prove the theorem, we apply it to obtain various examples.

Examples 2.8. The examples presented here are the ones with maximum symmetry outlined in [8] and constructed in detail in [10]. Consider a g-gon of radius 1, and hence side length $2\sin\frac{\pi}{g}$. Let G be the group of Euclidean motions under which the g-gon is invariant; G is abstractly isomorphic to the dihedral group of order $2g$. We define now the central c-graph $\underline{\widetilde{\Gamma}}$ (Figure 2.1, next page). Its vertices are the center and the vertices of the g-gon. Its edges are the sides and the radii of the g-gon. The group of symmetries is G. We call e one of the radii and $\mathbf{e}$ one of the sides; all edges are G-equivalent then to either e or $\mathbf{e}$. Arbitrarily assign some nonzero value to $\hat{\tau}(\mathbf{e})$, and define $\hat{\tau}(e) = -2\sin\frac{\pi}{g}\hat{\tau}(\mathbf{e})$ so that balancing is satisfied. Because of G both $L(\underline{\widetilde{\Gamma}}, \mathbf{e})$ and $W(\underline{\widetilde{\Gamma}})$ are 1-dimensional. Clearly $\widetilde{\Gamma}(\tilde{d}, \tilde{\ell})$ is uniquely determined by the various requirements of the definitions. It remains to check (2.5). Since $(\partial\kappa/\partial\tilde{\ell}(e))(0) > 0$, and $\hat{\tau}(e)$ and $\hat{\tau}(\mathbf{e})$ have opposite signs, (2.5) is valid.

We can then apply Theorem 2.7 to establish the existence of infinitely many closed CMC surfaces of genus g and dihedral symmetry of order $2g$ (Figure 2.2). Note that the content of this paper can be described in

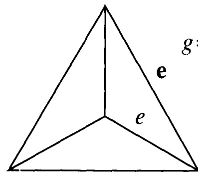

FIGURE 2.1

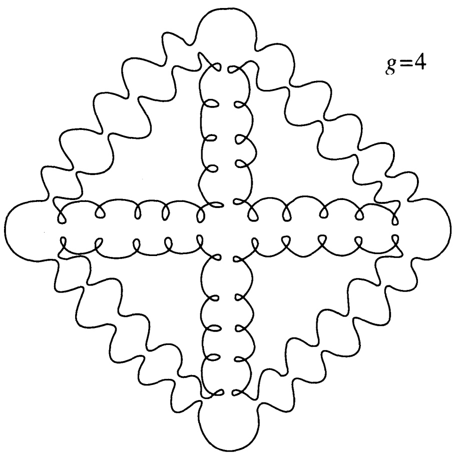

FIGURE 2.2. INTERSECTION WITH x_1x_2-PLANE (SCHEMATIC). NUMBER OF BULGES MAY NOT BE CORRECT (TOO SMALL).

a certain sense by juxtaposing this construction to (counter)example [10, II.1.20].

Examples 2.9. We now construct examples which have $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry; the group G is generated by the reflection with respect to the coordinate planes. We first construct examples of odd genus $2k+1$, $k \geq 1$. We determine the central c-graph $\tilde{\Gamma}$ (Figure 2.3): Let p_0 be a point on the positive x_2-axis, p_k a point in the positive x_1-axis, and $p_1, \cdots, p_{k-1}$ points in the first quadrant of the x_1x_2 coordinate plane. Let p_{-i} be the mirror image with respect to the x_2x_3-plane of p_i $(i = 1, \cdots, k-1)$, and p_i' the mirror image with respect to the x_1x_3-plane of p_i $(i = 1-k, \cdots, k-1)$. All these points are the vertices of $\tilde{\Gamma}$. We require that p_{i-1}, p_i, and p_{i+1} are never collinear, that p_ip_{i+1} is never parallel to any of the axes, and finally that the x_1-coordinate of p_i is strictly increasing with i. The edges are $p_{i-1}p_i$ (for $i = 1-k, \cdots, k$), their mirror images with respect to the x_1x_3-plane, p_ip_i' (for $i = 1-k, \cdots, k-1$), and $\mathbf{e} = p_{-k}p_k$. $\hat{\tau}$ is arbitrarily assigned to $\mathbf{e}$ and then uniquely determined on the other edges by considering balancing successively at $p_k, \cdots, p_0$ (and symmetry).

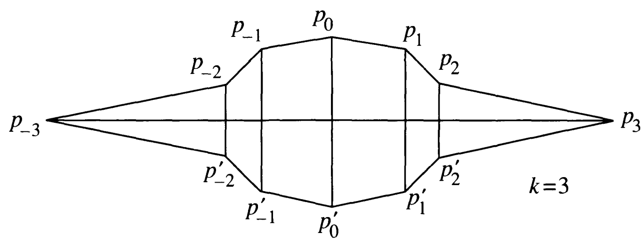

FIGURE 2.3

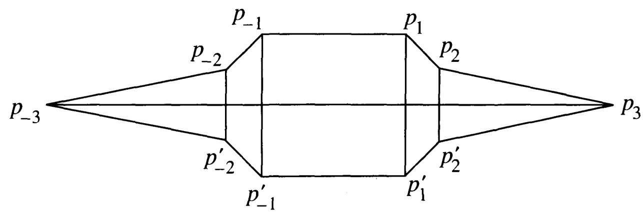

FIGURE 2.4

To define $\tilde{\Gamma}(\tilde{d}, \tilde{\ell})$, notice that the position of its vertices is uniquely determined by elementary trigonometry and it depends smoothly (only) on $\tilde{\ell}$. $\hat{\tau}$ is determined also uniquely by considering (2.2) successively at $p_k, \cdots, p_0$ for $\tilde{\Gamma}(\tilde{d}, \tilde{\ell})$. Clearly $\hat{\tau}$ depends smoothly on $\tilde{d}$ and $\tilde{\ell}$. We now restrict our attention to those cases where (2.5) is valid. This is not a strong restriction as the following argument shows. Consider a central c-graph as above and varying then the x_1-coordinate of p_k so that it tends to $+\infty$, while keeping the other vertices—except for p_{-k} because of the action of G—fixed. Also $\hat{\tau}(\mathbf{e})$ is kept fixed. We claim that (2.5) is satisfied for a large x_1 coordinate of p_k. This follows from the fact that all terms stay bounded with the exception of the ones corresponding to $\mathbf{e}$ and the edges in the orbit of p_kp_{k-1}. Since $(\partial\kappa/\partial\tilde{\ell}(p_{k-1}p_k))(0) \to \frac{1}{2}$, $\hat{\tau}(p_kp_{k-1}) \to -\frac{1}{2}\hat{\tau}(\mathbf{e})$, $\kappa(0) \to +\infty$, and $l_{\underline{\tilde{\Gamma}}}(p_kp_{k-1}) \to +\infty$, the claim is correct. We can apply the theorem then and obtain infinitely many closed CMC surfaces based on a regular perturbation of $\underline{\tilde{\Gamma}}$, and hence of genus $2k+1$ $(k \geq 1)$ and symmetry group $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

The case of even genus $2k+2$ $(k \geq 1)$ is similar and is obtained by modifying the previous construction for genus $2k+3$ as follows: Remove p_0 and p'_0 and the edges with endpoints those two vertices and put the edges p_1p_{-1} and $p'_1p'_{-1}$ instead (Figure 2.4).

Proof of Theorem 2.7. Assume we are given a central c-graph $\underline{\tilde{\Gamma}}$ as described in Definition 2.4. We fix a small $\tau' > 0$ and a large $N > 0$. We carry out a construction of a closed CMC surface under the assumption that τ' is small enough in terms of $\underline{\widetilde{\Gamma}}$ and N is large enough in terms of τ' and $\underline{\widetilde{\Gamma}}$. Given a c-graph $\widetilde{\Gamma}$, define the N-enlargement $\widetilde{\Gamma}'$ of $\widetilde{\Gamma}$ to be the c-graph with the same symmetries as $\widetilde{\Gamma}$, edges and vertices obtained from those of $\widetilde{\Gamma}$ by a homothetic expansion with factor N, and $\hat{\tau}$ the same as on the corresponding edges of $\widetilde{\Gamma}$. We specify the number of asr's the Delaunay pieces used in the construction will have: For each edge e of $\underline{\widetilde{\Gamma}}$ which is not G-equivalent to $\mathbf{e}$, we define

$$l(e) = [\tfrac{1}{2} N l_{\underline{\widetilde{\Gamma}}}(e)], \tag{1}$$

where [ ] denotes the integer part. The Delaunay pieces corresponding to e will have $2l(e) - 1$ (whole) asr's. The central graph (see below) should have edges of length $2l(e)$. Hence it should correspond to the N-enlargement of a regular perturbation of $\underline{\widetilde{\Gamma}}$ whose $\tilde{\ell}$ we call $\tilde{\ell}_0$ and is given by

$$\tilde{\ell}_0(e) = \frac{2}{N} l(e) - l_{\underline{\widetilde{\Gamma}}}(e). \tag{2}$$

We have not yet defined $l(\mathbf{e})$. Now this has to be defined carefully so that we have good control over the period of the initial surfaces. As in [10], $2 + 2p(\tau)$ is by definition the period of the Delaunay surface of parameter τ. In constructing the initial surfaces we intend to reduce all $\hat{\tau}$'s by a factor of order τ'. The construction of the initial surfaces below implies that one of them will have the Delaunay piece corresponding to e of parameter $2\tau'\hat{\tau}(e)$, where this $\hat{\tau}$ is the one of $\widetilde{\Gamma}(0, \tilde{\ell}_0)$. The $\tilde{\ell}$ of the corresponding c-graph is determined to be $\tilde{\ell}_1$ where we define

$$\tilde{\ell}_1(e) \equiv \frac{1}{N}(2 + 2p(2\tau'\hat{\tau}(e)))l(e) - l_{\underline{\widetilde{\Gamma}}}(e), \tag{3}$$

where $\hat{\tau}$ denotes the $\hat{\tau}$ of $\underline{\widetilde{\Gamma}}(0, \tilde{\ell}_0)$. Since we assume that N is as large as needed in terms of $\underline{\widetilde{\Gamma}}$, we can assert that $|\tilde{\ell}_0| < \epsilon'$, where $\epsilon' > 0$ depends only on $\underline{\widetilde{\Gamma}}$ and will be specified later. By [10, A.2.1], $p(\tau) \to 0$ as $\tau \to 0$. Choosing τ' small enough in terms of $\underline{\widetilde{\Gamma}}$ guarantees that $|\tilde{\ell}_0 - \tilde{\ell}_1| < \epsilon'$. Therefore we have

$$|\tilde{\ell}_0| < \epsilon', \qquad |\tilde{\ell}_1| < 2\epsilon'. \tag{4}$$

We will choose $\epsilon' < \frac{1}{2}\epsilon$ and then, by 2.4, $\widetilde{\Gamma}(0, \tilde{\ell}_0)$, $\widetilde{\Gamma}(0, \tilde{\ell}_1)$, and $\kappa(\tilde{\ell}_1)$ are well defined. Let

$$l(\mathbf{e}) \equiv \left[\frac{N\kappa(\tilde{\ell}_1)}{2 + 2p(2\tau'\hat{\tau}(\mathbf{e}))}\right], \tag{1'}$$

where the $\hat{\tau}$ used here is again the $\hat{\tau}$ of $\widetilde{\Gamma}(0, \tilde{\ell}_0)$.

We are ready to specify the central graphs which we need. Recall that central graphs have to be balanced and their edges should have even integer length [10, II.1.4]. The obvious candidate is the N-enlargement of $\tilde{\Gamma}(0, \tilde{\ell}_0)$ which we call $\tilde{\Gamma}'_0$. The only problem with it is that the edges corresponding to $\mathbf{e}$ do not have even integer length. Let $r_1 r_2$ be such an edge. Replace it by $r_1 r'_2$ where r'_2 is defined by the requirement that $\overrightarrow{r_1 r'_2}$ points in the same direction as $\overrightarrow{r_1 r_2}$ and has even integer length $2l(\mathbf{e})$. Let R be the parallel translation along $\overrightarrow{r_2 r'_2}$. We define the group of symmetries of the central graph $\mathbf{\Gamma}$ to be $\mathbf{G}$, the group generated by $G \cup \{R\}$. The vertices of $\mathbf{\Gamma}$ are the vertices of $\tilde{\Gamma}'_0$ and their images under the action of $\mathbf{G}$. The edges of $\mathbf{\Gamma}$ are the edges of $\tilde{\Gamma}'_0$ which are not G-equivalent to the edges corresponding to $\mathbf{e}$, $r_1 r'_2$, and the images of the above under the action of $\mathbf{G}$ (cf. Figure 1.3).

It remains to specify $\hat{\tau}$ on the edges of $\mathbf{\Gamma}$. We let this depend on a parameter $\tau'' \in [\tau', 3\tau']$ and the central graph thus defined will be denoted by $\mathbf{\Gamma}_{\tau''}$. If e is an edge of $\mathbf{\Gamma}$ corresponding to an edge e' of $\tilde{\Gamma}(0, \tilde{\ell}_0)$—$r_1 r'_2$ and its images under $\mathbf{G}$ correspond to $\mathbf{e}$ of course—define

$$\hat{\tau}(e) \equiv \tau'' \hat{\tau}(e'), \tag{5}$$

where the $\hat{\tau}$ in the right-hand side is the one corresponding to $\tilde{\Gamma}(0, \tilde{\ell}_0)$.

It is easy to see that because of the properties of $\underline{\tilde{\Gamma}}$, each of the $\mathbf{\Gamma}_{\tau''}$ is a flexible central graph and so [10, II.1.19] applies. We discuss the construction of each family of graphs $F_{\tau''}$ in more detail however in order to stress certain aspects we will need later. The central graph around which $F_{\tau''}$ is built is $\mathbf{\Gamma}_{\tau''}$. $\Gamma = \Gamma(\tau'', d, \ell)$ denotes the graph $\Gamma(d, \ell) \in F_{\tau''}$; we want Γ to depend continuously on all of its parameters, including τ''. To carry out the construction first fix some $\varepsilon > 0$ which we are free to assume as small as needed in terms of $\underline{\tilde{\Gamma}}$. Define $\tilde{\tau} \equiv \tau'$. Suppose we are given $\tau'' \in [\tau', 3\tau']$, $d: V(\mathbf{\Gamma}) \mapsto \mathbb{R}^3$, and $\ell: E(\mathbf{\Gamma}) \mapsto \mathbb{R}$, where the last two have length less than 1 in the sense of [10, II.1.6, II.1.8]. This amounts to

$$|d(p)| \leq \varepsilon^2 \tau' \quad \forall p \in V(\mathbf{\Gamma}), \qquad |\ell(e)| \leq \varepsilon^7 l(e) \quad \forall e \in E(\mathbf{\Gamma}),$$

where $l(e) = l(e')$ where e' is the edge of $\underline{\tilde{\Gamma}}$ corresponding to e. To define $\Gamma = \Gamma(\tau'', d, \ell)$ let $\tilde{d}: V(\underline{\tilde{\Gamma}}) \mapsto \mathbb{R}^3$ and $\tilde{\ell}: E(\underline{\tilde{\Gamma}}) \mapsto \mathbb{R}$ be defined by

$$\tilde{d}(p) \equiv \frac{1}{\tau''} d(p'), \qquad \tilde{\ell}(e) \equiv \tilde{\ell}_0(e) + \frac{1}{N} \ell(e'), \tag{6}$$

where p' is a vertex of $\mathbf{\Gamma}_{\tau''}$ corresponding to p and e' is an edge corresponding to e. We then have

$$(7) \qquad |\tilde{d}| \le \varepsilon^2, \quad |\tilde{\ell} - \tilde{\ell}_0| \le \varepsilon^7 \max_{e \in E(\underline{\tilde{\Gamma}})} l_{\underline{\tilde{\Gamma}}}(e), \quad |\tilde{\ell}| \le \epsilon' + \varepsilon^7 \max_{e \in E(\underline{\tilde{\Gamma}})} l_{\underline{\tilde{\Gamma}}}(e).$$

This shows that by choosing ε and ϵ' small enough we can guarantee that $\tilde{\Gamma} = \tilde{\Gamma}(\tilde{d}, \tilde{\ell})$ is well defined.

The construction of Γ once $\tilde{\Gamma}$ is determined is similar to the construction of $\mathbf{\Gamma}$: $\tilde{\Gamma}$ is enlarged by a factor N to $\tilde{\Gamma}'$, then the edges of $\tilde{\Gamma}'$ corresponding to $\mathbf{e}$ or their equivalents under the action of G are replaced with ones of the appropriate length which is $2l(\mathbf{e}) + \ell(r_1 r_2')$. Suppose the edge corresponding to $\mathbf{e}$ is $s_1 s_2$ and it is replaced by $s_1 s_2'$, then R acts by parallel translation along $\overrightarrow{s_2 s_2'}$. This defines the required action of $\mathbf{G}$, notice that its $\mathbf{O}(3)$-part is the same as that for $\mathbf{\Gamma}$ as required by [10, II.1.17(4)]. The vertices and the edges of Γ are then defined by using the action of $\mathbf{G}$ as before. It remains to define $\hat{\tau}$ on the edges of Γ. Since this has to be independent of ℓ by [10, II.1.10], we may as well assume that $\ell = 0$. In such a case, if e' is the edge of $\tilde{\Gamma}$ corresponding to an edge e of Γ, we define

$$(8) \qquad \hat{\tau}(e) = \tau'' \hat{\tau}(e').$$

We check now that $F_{\tau''}$ satisfies all the requirements of [10, II.1.9]. Conditions (1)–(4) and (1.10) of [10, II.1.9] are clear from the construction. By (1) and (6) we have

$$|\tilde{\ell} - \tilde{\ell}_0| \le \max_{e \in E(\underline{\tilde{\Gamma}})} l_{\underline{\tilde{\Gamma}}}(e) \max_{e \in \mathbf{\Gamma}} \frac{\ell(e)}{l(e)}.$$

We can then establish [10, II.1.11] because the angle in consideration is equal to the corresponding angle for $\tilde{\Gamma}$ which depends smoothly on $\tilde{\ell}$: Simply choose ε small enough so that the inverse of $\varepsilon \max_{e \in E(\underline{\tilde{\Gamma}})} l_{\underline{\tilde{\Gamma}}}(e)$ exceeds the norm of the appropriate derivatives on the compact ball $\{(\tilde{d}, \tilde{\ell})$: $|\tilde{d}| \le \epsilon, |\tilde{\ell}| \le \epsilon\}$. Similarly (1.14) of [10, II.1.9] is established. Since we have only finitely many edges and the parameters vary on compact sets, condition [10, II.1.9.(6)] can also be arranged by choosing ε small enough.

We appeal now to the lemma in [10, III.2.10] to obtain a family of initial surfaces $\mathscr{F}_{\tau''}$ based on $F_{\tau''}$. This involves the hidden hypothesis [10, III.1.1] that $\tilde{\tau}$ is small enough in terms of ε, which in our case follows through the hypothesis that τ' is small enough in terms of $\underline{\tilde{\Gamma}}$ of which ε is a function. Recall the construction of $\mathscr{F}_{\tau''}$ in [10, III.2.10]. We write

$M(\tau'', \xi)$ for the initial surface of configuration ξ in $\mathscr{F}_{\tau''}$. We recall the definition of ξ and the space in which it takes values so that we clarify what the parameter space is: $Z(F_{\tau''})$ is defined to be a set of vertices of a graph defined as follows: Consider $\Gamma_{\tau''}$. Subdivide each of its edges e to $2l(e)$ new edges by inserting $2l(e) - 1$ new vertices equally spaced on e. Notice that $Z(F_{\tau''})$ can be put in a one-to-one correspondence with the set of asr's of any initial surface associated to some graph in $F_{\tau''}$. Moreover two vertices in $Z(F_{\tau''})$ are connected by an edge in the graph just defined if and only if the corresponding asr's share a boundary circle. Notice that we can clearly identify all the $Z(F_{\tau''})$ to each other in a natural way, we write Z to denote any of them with the identification implied. $\mathscr{W}(F_{\tau''})$ is the space of functions $w: Z(F_{\tau''}) \mapsto \mathbb{R}^3$ invariant under the induced action of $\mathbf{G}$. A norm $\|\cdot\|$ for $\mathscr{W}(F_{\tau''})$ is suitably defined to impose decay away from the vertices of $\mathbf{\Gamma}$ [10, III.2.5]. The $\mathscr{W}(F_{\tau''})$'s can be clearly identified to a single space of functions $w: Z \mapsto \mathbb{R}^3$, which space we call $\mathscr{W}$.

There is a unique initial surface $M = M(\tau'', \xi) \in \mathscr{F}_{\tau''}$ for each $(\tau'', \xi) \in [\tau', 3\tau'] \times \Xi$ where Ξ is the unit ball of $\mathscr{W}$ [10, III.2.9]. Let $\Gamma = \Gamma(\tau'', d, \ell)$ be the graph to which M is associated. $d = \xi|_{V(\Gamma)}$, while the rest of ξ controls the perturbation of the Delaunay asr's of M. It is clear from the construction [10, III.2.10] and the continuous dependence of $\Gamma = \Gamma(\tau'', d, \ell)$ on its parameters that $M = M(\tau'', \xi)$ depends continuously on its parameters as well. Recall s_1, s_2, and s_2' defined in the construction of Γ above. We define the period of M to be

$$\varpi(\tau'', \xi) \equiv |\overrightarrow{s_1 s_2}| - |\overrightarrow{s_1 s_2'}|. \tag{9}$$

The period of M is therefore the signed length of the translation induced by R. The importance of ϖ is due to the fact that M is closed if and only if $\varpi(\tau'', \xi) = 0$. This motivates us to study the way ϖ varies. The following two lemmas provide a satisfactory answer for our purposes:

Lemma 1. *If ε is small enough in terms of $\underline{\widetilde{\Gamma}}$ and a given $\epsilon_0 > 0$, then for all $\tau'' \in [\tau', 3\tau']$ and $\xi \in \Xi$, we have*

$$|\varpi(\tau'', \xi) - \varpi(\tau'', 0)| < \epsilon_0(1 + N|\tau' \log \tau'|). \tag{10}$$

Proof. Let $\Gamma' = \Gamma(\tau'', 0, \ell')$ and $\Gamma'' = \Gamma(\tau'', d'', \ell'')$ be the graphs to which $M' = M(\tau'', 0)$ and $M'' = M(\tau'', \xi)$ are associated. In analogy with (6), $\tilde{d}''$, $\tilde{\ell}'$, and $\tilde{\ell}''$ are then defined. If e is an edge of $\underline{\widetilde{\Gamma}}$, we will write e' and e'' for the edges of Γ' and Γ'' which correspond to e

respectively. (9) and the various definitions involved imply that

$$\varpi(\tau'', \xi) - \varpi(\tau'', 0) = N(\kappa(\tilde{\ell}'') - \kappa(\tilde{\ell}')) - (\ell''(\mathbf{e}'') - \ell'(\mathbf{e}')).$$

Since κ is a smooth function defined on a compact set, its derivatives are bounded. (6) implies that $N(\tilde{\ell}'' - \tilde{\ell}') = \ell'' - \ell'$. Combining all these we conclude that

$$(11) \qquad |\varpi(\tau'', \xi) - \varpi(\tau'', 0)| \leq C \max_{e \in E(\tilde{\Gamma})} |\ell''(e'') - \ell'(e')|,$$

where here C denotes constants depending only on $\tilde{\Gamma}$.

Fix some e and consider $\ell''(e'') - \ell'(e')$. This is the difference of the lengths of $M''\{e''\}$ and $M'\{e'\}$, the two Delaunay pieces used to replace e'' and e' in the construction of M'' and M' respectively [10, III.2.6 and the proof of III.2.10]. Let $DP' \equiv DP(\hat{\tau}', \vec{v}', 0)$ and $DP'' \equiv DP(\hat{\tau}'', \vec{v}', \xi'')$ be the two Delaunay pieces in consideration. Clearly $\hat{\tau}' = \tau''\hat{\tau}(e)$, where $\hat{\tau}$ is the one defined for $\tilde{\Gamma}(0, \tilde{\ell}_0)$, and $\hat{\tau}'' = \tau''\hat{\tau}(e)$ where $\hat{\tau}$ is the one corresponding to $\tilde{\Gamma}(\tilde{d}'', \tilde{\ell}_0)$. $\hat{\tau}$ depends smoothly on $\tilde{d}$ which varies on a compact set, hence we conclude $|\hat{\tau}' - \hat{\tau}''| \leq C \max |d''(p)|$ (cf. (6) also). But d'' is the restriction of $\xi \in \Xi$ to the vertices, so by [10, III.2.5 and III.2.8], we have $\max |d''(p)| \leq \varepsilon^2 \tau'$. This and the way $\hat{\tau}$ was defined for Γ' allow us to conclude

$$|\hat{\tau}'| < C\tau', \qquad |\hat{\tau}'' - \hat{\tau}'| < C\varepsilon^2\tau'.$$

In order to study the length of DP'' recall [10, III.2.11 and (2) in the proof of III.2.10] and the relevant notation. If we take τ_n and $\vec{u}_n$ of those inequalities to refer to DP'', we can conclude that $|\tau_n - \hat{\tau}'| < C\varepsilon^2\tau'$. By referring to [10, A.2.2], we conclude that

$$|p(\tau_n) - p(\hat{\tau}')| < C\varepsilon^2 |\tau' \log \tau'|.$$

Using this and the estimate for $|\vec{u}_n - \vec{v}'|$, it is straightforward to analyze the lengths of the Delaunay pieces as in the proof of [10, III.2.10] to conclude that they differ by at most $C\varepsilon^2(1 + l(e)|\tau' \log \tau'|)$. By referring to (11) and since $l(e) < CN$, we conclude then

$$|\varpi(\tau'', \xi) - \varpi(\tau'', 0)| \leq C\varepsilon^2(1 + N|\tau' \log \tau'|).$$

By assuming ε small enough as in the statement of the lemma, we finish the proof.

Lemma 2. *If ε is small enough in terms of $\tilde{\Gamma}$, τ' small enough in terms of $\tilde{\Gamma}$ and ε, and N large enough in terms of $\tilde{\Gamma}$, ε, and τ', then* *there is $c > 0$ depending on $\underline{\tilde{\Gamma}}$ only, such that on $[\tau', 3\tau']$*

$$\left|\frac{d}{d\tau''}\varpi(\tau'', 0)\right| > cN|\log\tau'|.$$

Proof. Let $\Gamma_{\tau''} \equiv \Gamma(\tau'', 0, \ell_{\tau''})$ be the graph to which $M(\tau'', 0)$ is associated. In analogy with (6) we write $\tilde{\ell}_{\tau''} = \tilde{\ell}_0 + \frac{1}{N}\ell_{\tau''}$. We then have

$$\varpi(\tau'', 0) = N\kappa(\tilde{\ell}_{\tau''}) - l(\mathbf{e})(2 + 2p(\tau''\hat{\tau}(\mathbf{e}))),$$

where $\hat{\tau}$ is the $\hat{\tau}$ of $\tilde{\Gamma}(0, \tilde{\ell}_0)$. Combining the last two equalities and taking the derivative we get

$$\frac{d}{d\tau''}\varpi(\tau'', 0) = -2l(\mathbf{e})\frac{d}{d\tau''}p(\tau''\hat{\tau}(\mathbf{e})) + \sum_{e\in E(\tilde{\Gamma}_{\tau''})\backslash G\mathbf{e}} \frac{\partial\kappa}{\partial\tilde{\ell}(e)}(\tilde{\ell}_{\tau''})\frac{d}{d\tau''}\ell_{\tau''}(e),$$

where we abused the notation by writing e for the edge corresponding to e in $\underline{\tilde{\Gamma}}$. Since $\ell_{\tau''}(e) = 2l(e)p(\tau''\hat{\tau}(e))$, we conclude

$$\begin{aligned}\frac{d}{d\tau''}\varpi(\tau'', 0) = &-2l(\mathbf{e})p'(\tau''\hat{\tau}(\mathbf{e}))\hat{\tau}(\mathbf{e}) \\ &+ \sum_{e\in E(\tilde{\Gamma}_{\tau''})\backslash G\mathbf{e}} 2l(e)\frac{\partial\kappa}{\partial\tilde{\ell}(e)}(\tilde{\ell}_{\tau''})p'(\tau''\hat{\tau}(e))\hat{\tau}(e),\end{aligned}$$

where p' is the derivative of p. The following replacements in the right-hand side of the equation introduce only small errors which are acceptable as far as proving the lemma goes. $(\partial\kappa/\partial\tilde{\ell}(e))(\tilde{\ell}_{\tau''})$ can be replaced with $(\partial\kappa/\partial\tilde{\ell}(e))(0)$ by the smooth dependence of κ on $\tilde{\ell}$ and by choosing ϵ' and ε small enough so to guarantee the smallness of $\tilde{\ell}_{\tau''}$ through (7). In a similar way we can replace $\hat{\tau}$ of $\tilde{\Gamma}(0, \tilde{\ell}_0)$ with the $\hat{\tau}$ of $\underline{\tilde{\Gamma}}$. $p'(\tau)$ can be replaced by $-\log|\tau|$ by referring to [10, A.2.2] and assuming τ' small enough. The smallness of τ' allows us then to replace $\log|\tau''\hat{\tau}(e)|$ with $\log\tau'$. Finally the largeness of N and (1) imply that we can replace $l(e)$ with $\frac{N}{2}l_{\underline{\tilde{\Gamma}}}(e)$. All these replacements reduce our expression to

$$N\log\tau'\left(\kappa(0)\hat{\tau}(\mathbf{e}) - \sum_{e\in E(\underline{\tilde{\Gamma}})\backslash G\mathbf{e}} \frac{\partial\kappa}{\partial\tilde{\ell}(e)}(0)l_{\underline{\tilde{\Gamma}}}(e)\hat{\tau}(e)\right).$$

This allows us to finish the proof of the lemma by referring to (2.5).

We return now to the proof of the theorem. (1′) clearly implies that $|\varpi(2\tau', 0)| < 3$. Lemma 2 implies that $\tau'' \mapsto \varpi(\tau'', 0)$ is a monotone—

and hence invertible—function. Let $\mathscr{I}$ be its image and $\tau'': \mathscr{I} \mapsto [\tau', 3\tau']$ its (continuous) inverse. $\mathscr{I}$ is a closed interval and Lemma 2 implies that

$$\mathscr{I} \supset [3 - cN|\tau' \log \tau'|, cN|\tau' \log \tau'| - 3].$$

By choosing ϵ_0 small enough (and N large enough) and using Lemma 1, we conclude that for all $\tau'' \in [\tau', 3\tau']$ and $\xi \in \Xi$, we have $\varpi(\tau'', 0) - \varpi(\tau'', \xi) \in \mathscr{I}$. Hence we can define a continuous map

$$\hat{\varpi} : \mathscr{I} \times \Xi \mapsto \mathscr{I} \quad \text{by } (\varpi', \xi) \mapsto \varpi' - \varpi(\tau''(\varpi'), \xi). \tag{12}$$

Clearly $M(\tau''(\varpi'), \xi)$ is a closed surface if and only if $\hat{\varpi}(\varpi', \xi) = \varpi'$.

We have succeeded already in formulating the closedness question in a fixed point theorem form. It remains to modify the proof of the main theorem [10, V.2.1] so that it incorporates $\hat{\varpi}$. For completeness we discuss the argument in some detail. Recall first the lemma in [10, III.4.4]. The construction of D_ξ can be repeated verbatim to give a diffeomorphism $D_{\tau'',\xi}: M(2\tau', 0) \mapsto M(\tau'', \xi)$ which has the same properties as D_ξ and depends continuously on (τ'', ξ). $D_{\tau'',\xi}$ is clearly equivariant under the action of $\mathbf{G}$. Because we are free to assume $\tilde{\tau} = \tau'$ as small as needed in terms of ε, the lemmas in [10, IV.5.1, IV.5.5, and V.1.3] are valid on each $M = M(\tau'', \xi)$ which we are considering. Define then $\widetilde{C}$, σ, and δ as in the proof of [10, V.2.1], and assume that $\tilde{\tau} = \tau'$ is small enough to be $< T$. Let $\mathbf{M} = M(2\tau', \xi)$ and $\mathscr{K} = \{\phi \in C^{2,\overline{\alpha}}(\mathbf{M}): \phi \text{ is equivariant under } \mathbf{G}\}$. The set

$$\mathscr{N} = \{(\varpi', \xi, \phi) \in \mathbb{R} \times \mathscr{W} \times \mathscr{K} : \varpi' \in \mathscr{I}, \\ \|\xi\|_\sigma \le 1, \ \|\phi\|_{2\delta,\sigma} \le \overline{C}_1(\overline{C}_2 + \overline{C}_3)\}$$

is a convex compact subset of the Banach space $\mathbb{R} \times \mathscr{W} \times \mathscr{K}$. Fix some (ϖ', ξ, ϕ) and let $M = M(\tau''(\varpi'), \xi)$ and $\varphi = \phi \circ D^{-1}_{\tau''(\varpi'),\xi}$. We have already said that [10, III.4.4] applies to our situation and so

$$\|\varphi\|_{4\delta,\sigma} \le \widetilde{C}. \tag{2.10}$$

(Recall that $\widetilde{C}$ is a constant depending only on ε.) The definition of σ and [10, V.1.3] imply then that X_φ (defined in 2.6) is an immersion and $\|Q_\varphi\|_\sigma \le 1$, where

$$Q_\varphi \equiv 4|A|^{-2}(H_\varphi - H) - \mathscr{L}_h \varphi, \tag{2.11}$$

where H_φ is the mean curvature of X_φ. Apply [10, IV.5.1] with $f = Q_\varphi$ to obtain functions $u: M \mapsto \mathbb{R}$ and $\lambda: Z \mapsto \mathbb{R}^3$, and apply [10, IV.5.5] to

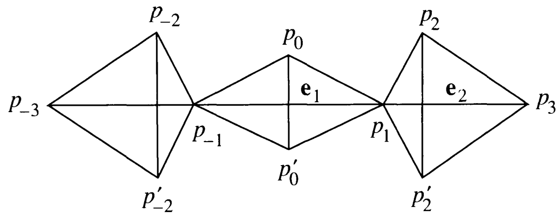

FIGURE 2.5

obtain $\overline{u}: M \mapsto \mathbb{R}$ and $\overline{\lambda}: Z \mapsto \mathbb{R}^3$, such that:

$$(2.12) \quad \mathscr{L}_h u = Q_\varphi + \Theta(\lambda), \qquad \mathscr{L}_h \overline{u} = 4|A|^{-2}(1-H) + \Theta(\xi - \overline{\lambda}),$$

$$(2.13) \quad \|u\|_{\delta,\sigma} \le \overline{C}_2, \quad \|\lambda\|_\sigma \le \tfrac{1}{2}, \quad \|\overline{u}\|_{\delta,\sigma} \le \overline{C}_3, \quad \|\overline{\lambda}\|_\sigma \le \tfrac{1}{2}.$$

The same argument as in the proof of [10, V.2.1] establishes the continuous dependence of u, λ, $\overline{u}$, and $\overline{\lambda}$ on (τ'', ξ, ϕ). (12), (2.13), and [10, III.4.4] modified as above, allow us then to define a continuous map

$$\mathscr{J}: \mathscr{N} \mapsto \mathscr{N} \quad \text{by } (\varpi', \xi, \phi) \mapsto (\hat{\varpi}(\varpi', \xi), \overline{\lambda} + \lambda, (\overline{u} - u) \circ D_{\tau''(\varpi'), \xi}).$$

The Schauder fixed point theorem provides a fixed point (ϖ', ξ, ϕ) for this map. Let $\tau'' = \tau''(\varpi')$ and M and φ be as above. Clearly the translation R acts trivially and so M is a closed surface and φ is well defined on it. (2.11) and (2.12) imply then that $H_\varphi \equiv 1$ and hence $X_\varphi(M)$ is a closed CMC surface based on $(\widetilde{\Gamma}, \tau'', N)$. (Smoothness follows from standard regularity theory.) It remains to prove that there are infinitely many such surfaces. But this is clear by giving values to N tending to ∞ or to τ' tending to 0.

Remark 2.14. Theorem 2.7 can be generalized in two different directions which we now briefly discuss. The reason we do not pursue this line in detail is that no new topological types of surfaces would be obtained and so we feel that the extra technical complications introduced would not be justified. Certain examples along these lines (3.7), are outlined later however.

Consider the first c-graph in Figure 2.5 which lies on the x_1x_3-plane. The symmetry group is generated by the reflections with respect to the coordinate planes. Let $\mathbf{e}_1 = p_{-1}p_1$ and $\mathbf{e}_2 = p_1p_3$. Both $\mathbf{e}_1$ and $\mathbf{e}_2$ have their lengths determined when the lengths of the other edges are determined. The lengths of the other edges can be arbitrarily perturbed. Although now we have two instead of one edge whose length cannot be

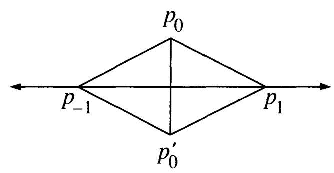

FIGURE 2.6

preassigned, on the other hand we can arbitrarily assign both $\tau_1' \equiv \hat{\tau}(\mathbf{e}_1)$ and $\tau_2' \equiv \hat{\tau}(\mathbf{e}_2)$, and then have $\hat{\tau}$ uniquely determined on the other edges. Now we can carry out a construction where we vary independently τ_1' and τ_2' to close two periods corresponding to $\mathbf{e}_1$ and $\mathbf{e}_2$.

The other generalization would be to allow rays in the graphs corresponding to ends of the CMC surface in construction. For example a construction based on the graph of Figure 2.6 (symmetry as before) would be possible with $\hat{\tau}$ predetermined on the rays and varying $\hat{\tau}(\mathbf{e} = p_{-1}p_1)$ to close the period.

3. Closed CMC surfaces with no symmetries

In this section we generalize Theorem 2.7 so that it applies in cases where there is no symmetry group acting. This allows us to construct examples of closed CMC surfaces which have no symmetries, something we did not manage to do using Theorem 2.7 as it stands. We start by explaining what the (technical) difficulty involved is, and how it can be resolved. Recall the simplest construction of CMC surface under our approach: That of a 3-ended (topological) sphere [10, II.4.2]. The relevant central graph $\mathbf{\Gamma}$ has one vertex and three rays. $W(\mathbf{\Gamma})$ is three dimensional and $\Gamma(d, 0)$ is created by keeping two of the rays—and the $\hat{\tau}$ on them—fixed, while d then uniquely determines the direction and $\hat{\tau}$ up to sign of the third ray. By applying the theorem, we obtain a CMC surface M associated to one of the graphs $\Gamma(d, 0)$ in the family under consideration. Each of the ends of M is asymptotically at infinity Delaunay with τ-parameters equal to the $\hat{\tau}$ of the corresponding ray and axis parallel to the ray. Applying then Kusner's balancing formula [10, A.3.1], we conclude that the end corresponding to the third ray has to have axis parallel to the third ray of $\mathbf{\Gamma} = \Gamma(0, 0)$ and τ-parameter the corresponding value of $\hat{\tau}$ because $\mathbf{\Gamma}$ is the only balanced graph in consideration.

It is natural to conclude from the above that the construction of the $\Gamma(d, 0)$'s is unnecessary and one should be able to construct the CMC surface by using only the $M(\xi)$'s with $\xi = 0$ at the vertex. This is indeed possible and the reason we have not discussed it in [10] is that it increases the technicalities without giving any new significant examples. In the above construction for example the desired surface can be constructed first as usual, and have the behavior of the third end specified afterwards as above by appealing to the balancing formula. The case is not so however with closed surfaces. In this case we do not have rays to perturb and the construction fails with one exception: Notice that even in the three-ended sphere construction, we can avoid the need to construct the $\Gamma(d, 0)$'s if we impose enough symmetry: Make the three rays coplanar and symmetrically arranged with the same $\hat{\tau}$. Impose the symmetry group of the central graph—dihedral of order six—as the symmetry group of the construction. $W_G(\mathbf{\Gamma})$ then becomes trivial. This is analogous to the constructions of closed CMC surfaces in the previous section: They all have enough symmetry to kill all linear functions on E^3. This reduces $W(\mathbf{\Gamma})$ enough for us to be able to construct an appropriate family of graphs around $\mathbf{\Gamma}$. To construct closed CMC surfaces without symmetries however we have to incorporate Kusner's balancing formula in the construction from the very beginning. This is the objective of this section.

Definition 3.1. A weak central c-graph $\underline{\tilde{\Gamma}}$ is a c-graph which satisfies the conditions of 2.4 with $W(\underline{\tilde{\Gamma}})$ replaced by its subspace $\widetilde{W}(\underline{\tilde{\Gamma}}) \equiv \{w: V(\underline{\tilde{\Gamma}}) \backslash \tilde{V}(\underline{\tilde{\Gamma}}) \mapsto \mathbb{R}^3\}$ where $\tilde{V}(\underline{\tilde{\Gamma}}) \subset V(\underline{\tilde{\Gamma}})$ can be any one of the following:

(1) A set containing only one vertex.
(2) A set containing two vertices provided that there is an edge in $E(\underline{\tilde{\Gamma}})$ connecting them.
(3) A set containing three vertices which are not collinear, with each pair connected by an edge in $E(\underline{\tilde{\Gamma}})$.

Condition (2) is also modified to require that $d_{\tilde{\Gamma}} = d$ only on the vertices which do not correspond to vertices in $\tilde{V}(\underline{\tilde{\Gamma}})$, with one exception: We still require that $\underline{\tilde{\Gamma}}$ is balanced, that is $d_{\tilde{\Gamma}} = 0$.

Notice that we may still have a group of symmetries G acting but the counting of the number of elements of $\tilde{V}(\underline{\tilde{\Gamma}})$ should *not* be done modulo the action of G.

Theorem 3.2. *The conclusions of Theorem* 2.7 *hold for* $\underline{\tilde{\Gamma}}$ *a weak central c-graph as well.*

Before proving the theorem we apply it to obtain various closed CMC surfaces. The next lemma simplifies the construction of weak central c-graphs. In the rest of this section we use $\tilde{V}(\tilde{\Gamma})$ to denote a set of vertices of a c-graph like the one in Definition 3.1. The set of the edges connecting vertices in $\widetilde{V}(\widetilde{\Gamma})$ is denoted by $\widetilde{E}(\widetilde{\Gamma})$. There are only three possibilities for the number of elements in the sets $\widetilde{V}(\widetilde{\Gamma})$ and $\widetilde{E}(\widetilde{\Gamma})$: $(0, 1)$, $(2, 1)$, and $(3, 3)$.

Lemma 3.3. *Assume that $\widetilde{\Gamma}$ is a c-graph and $d_{\widetilde{\Gamma}}(p) = 0$ for each vertex p except for those in a set $\widetilde{V}(\widetilde{\Gamma})$ as above. $\hat{\tau}$ can then be (re)defined in a unique way on $\widetilde{E}(\widetilde{\Gamma})$ so that $d_{\widetilde{\Gamma}} \equiv 0$.*

Proof. We proceed in a case by case basis. In the first case $\widetilde{V}(\widetilde{\Gamma})$ has only one element and $\widetilde{E}(\widetilde{\Gamma}) = \varnothing$. Recall (2.2). If e is an edge connecting p and p', then $\vec{v}_{e,p} = -\vec{v}_{e,p'}$. This implies

$$\sum_{p \in V(\widetilde{\Gamma})} d_{\widetilde{\Gamma}}(p) = 0.$$

Since all summands but one are 0 by assumption, the last one is 0 also and the proof is complete in this case.

For the other two cases we recall first some elementary facts: A system of vectors applied at points is just this: A set of vectors in $\mathbb{R}^3$ each of which has a point of E^3 assigned to it. The torque of a vector $\vec{v}$ applied at p with respect to q is defined to be $\overrightarrow{qp} \times \vec{v}$. Two systems of vectors are defined to be equivalent if they have the same sum (as vectors) and the same sum of torques with respect to each point of E^3. Such systems of vectors form a vector space which is the dual of the Lie algebra of the group of Euclidean motions of E^3. Each system of vectors is equivalent to a single vector applied to a point called the resultant of the system, or two parallel opposite vectors called the resultant couple of the system. We can always substitute the point of application of a vector with one on the line through the original point of application and parallel to the direction of the vector. If the system of two (three) vectors has trivial resultant, then the sum of the vectors is 0 and they are parallel to the line (plane) through the points of application of the vectors—assume the points of application are not collinear in the second case.

Consider now the system of the vectors $d_{\widetilde{\Gamma}}(p)$ applied at p $(p \in V(\widetilde{\Gamma}))$. This system has resultant 0 because each $\vec{v}_{e,p}$ cancels $\vec{v}_{e,p'}$ as before. Recall $d_{\widetilde{\Gamma}}(p) = 0$ for $p \notin \widetilde{V}(\widetilde{\Gamma})$. In the second case we have $\widetilde{V}(\widetilde{\Gamma}) = \{p_1, p_2\}$. By the above, $d_{\widetilde{\Gamma}}(p_1)$ is parallel to p_1p_2 and so by redefining $\hat{\tau}$ on p_1p_2, we can achieve $d_{\widetilde{\Gamma}}(p_1) = 0$. This reduces the problem to the first case. In the third case $\widetilde{V}(\widetilde{\Gamma}) = \{p_1, p_2, p_3\}$, and by the above, $d_{\widetilde{\Gamma}}(p_1)$ is parallel to the plane $p_1p_2p_3$. We can achieve $d_{\widetilde{\Gamma}}(p_1) = 0$ by redefining $\hat{\tau}$

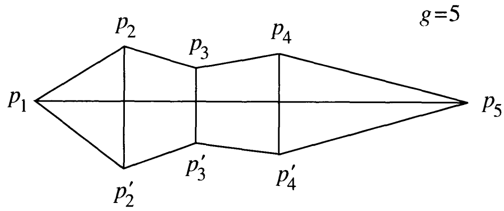

FIGURE 3.1

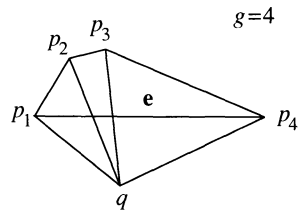

FIGURE 3.2

on p_1p_2 and p_1p_3. This reduces the problem to the second case and the proof is complete. q.e.d.

Notice that there is no guarantee in the above lemma that the redefined $\hat{\tau}$ is nonzero, although this should be generically true.

Examples 3.4. We now construct examples of closed CMC surfaces of any genus $g \geq 3$ whose symmetry group is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$. Let G be generated by reflections with respect to the x_1x_2 and x_1x_3 planes. We construct the weak central c-graph $\underset{\sim}{\Gamma}$ (Figure 3.1) first. Its vertices are p_1 and p_g on the x_1-axis, $p_2, \cdots, p_{g-1}$ on $\{x_3 = 0, x_2 > 0\}$, and their mirror reflections with respect to the x_1-axis $p_2', \cdots, p_{g-1}'$ respectively. We assume that the x_1-coordinate of p_i is increasing with i, and that p_{i-1}, p_i, p_{i+1} are never collinear. The edges are $p_1p_2, \cdots, p_{g-1}p_g$, their mirror images, $p_2p_2', \cdots, p_{g-1}p_{g-1}'$, and $\mathbf{e} = p_1p_g$. $\hat{\tau}$ is assigned an arbitrary (nonzero) value on $\mathbf{e}$. Considering balancing successively at $p_1, \cdots, p_{g-1}$, we specify $\hat{\tau}$ on all the other edges. $\widetilde{V}(\underset{\sim}{\Gamma}) \equiv \{p_g\}$ and so by Lemma 3.3 our graph is balanced. $\widetilde{\Gamma}(\tilde{d}, \tilde{\ell})$ is determined as in Examples 2.9. It remains only to check (2.5). This can be arranged as in Examples 2.9 by giving very large values to the x_1 coordinate of p_g. This also guarantees that there are no other symmetries except the intended ones.

Examples 3.5. We construct CMC closed surfaces of any genus $g \geq 3$ with symmetry group isomorphic to $\mathbb{Z}_2$. The only nontrivial symmetry is a reflection across a plane on which plane the graph lies. We first describe the weak central c-graph $\underset{\sim}{\Gamma}$ (Figure 3.2). The vertices of $\underset{\sim}{\Gamma}$ are $p_1, \cdots, p_g$ and q. The edges are $qp_1, \cdots, qp_k$, $p_1p_2, \cdots, p_{g-1}p_g$, and $\mathbf{e} = p_1p_g$. $\widetilde{V}(\underset{\sim}{\Gamma}) = \{q, p_g\}$. We assign arbitrarily $\hat{\tau}(\mathbf{e}) \neq 0$. Balancing then at $p_1, \cdots, p_{g-1}$ determines uniquely $\hat{\tau}$ on all other edges except for p_gq. Lemma 3.3 then applies and $\hat{\tau}$ is determined on this edge as well. To ensure that $\hat{\tau}$ is always nonzero we assume that no three vertices are collinear. Elementary trigonometry as usual allows us to construct the regular perturbations and check that they depend smoothly on the data.

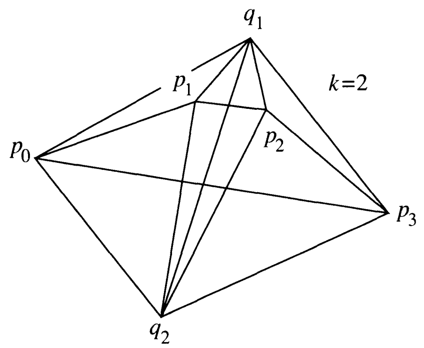

FIGURE 3.3

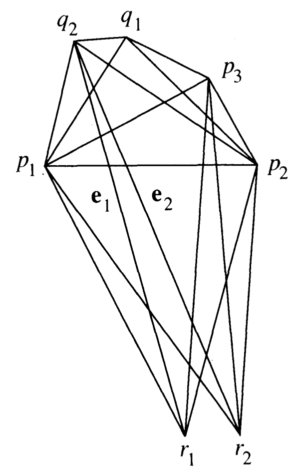

FIGURE 3.4

(2.5) can be guaranteed by sliding p_g to ∞ along p_1p_g which is kept fixed (for the details see the similar argument in 2.9).

Examples 3.6. For each $k > 1$ we construct closed CMC surfaces of genus $4 + 2k$ possessing no symmetries. We determine first the weak central c-graph $\underline{\widetilde{\Gamma}}$ (Figure 3.3). Its vertices are arranged so that no three of them are collinear and no four of them coplanar. The vertices are $q_1, q_2, p_0, p_1, \cdots, p_{k+1}$. The edges are all q_ip_j's, $p_0p_1, \cdots, p_kp_{k+1}$, q_1q_2, and $\mathbf{e} = p_0p_{k+1}$. $\widetilde{V}(\underline{\widetilde{\Gamma}}) = \{q_1, q_2, p_{k+1}\}$. A nonzero value is arbitrarily assigned for $\hat{\tau}(p_0p_{k+1})$, and then $\hat{\tau}$ is determined uniquely on the other edges except for q_1q_2, q_1p_{k+1}, q_2p_{k+1} by considering balancing successively at $p_0, p_1, \cdots, p_k$. We appeal to Lemma 3.3 to define $\hat{\tau}$ on the remaining edges. The construction of the regular perturbations is as usual and we omit the details. To guarantee (2.5) we can slide p_{k+1} to ∞ on p_0p_{k+1} as usual. Notice that the existence of no symmetries can be guaranteed by making $\underline{\widetilde{\Gamma}}$ asymmetrical enough.

Examples 3.7. We construct examples of closed CMC surfaces of any genus $2k+7$ $(k \geq 1)$ which have no symmetries. In this construction we have two edges $\mathbf{e}_1$ and $\mathbf{e}_2$ whose length is determined by the other edges (cf. Remark 2.14). Although we do not provide the details of the proof, we give enough information for the reader to fill them in.

We first determine the weak central c-graph $\underline{\widetilde{\Gamma}}$ (Figure 3.4). The vertices are p_1, p_2, p_3, $q_1, q_2, \cdots, q_k$, and r_1, r_2. $\widetilde{V}(\underline{\widetilde{\Gamma}}) = \{p_1, p_2, p_3\}$. The edges of the graph are $\mathbf{e}_1 = r_1q_k$, $\mathbf{e}_2 = r_2q_k$, and the edges of the following tetrahedra: $r_1p_1p_2p_3$, $r_2p_1p_2p_3$, $p_3q_1p_1p_2$, $\cdots$, $q_{k-1}q_kp_1q_2$. $\hat{\tau}$ is arbitrarily determined on $\mathbf{e}_1$ and $\mathbf{e}_2$. The rest of the construction is as usual except for the following: $\mathbf{G}$ is generated by two translations, we have two τ''s, τ'''s, ϖ''s, etc. one for each $\mathbf{e}_i$. At some stage of the proof condition (2.5) is replaced by the invertability of a 2×2 matrix. By assuming the edges with an r_i as an endpoint of much larger lengths than the other edges we can ignore the terms in this matrix corresponding to the other edges. The matrix then reduces to a diagonal matrix with nonzero diagonal terms (which can be arranged close to each other by assuming r_1 and r_2 close to each other if further simplification is desired). This is enough to find a fixed point and establish the existence of the desired surfaces.

Proof of Theorem 3.2. This proof is a modification of the proof of Theorem 2.7. We start by redefining once more the parameter space for the regular perturbations. The motivation for this is that $\hat{\tau}$ can be changed on the edges in $\widetilde{E}(\widetilde{\Gamma})$ of a regular perturbation and the condition $\tilde{d} = d_{\widetilde{\Gamma}}$ will still be valid on the vertices not in $\widetilde{V}(\widetilde{\Gamma})$. In this spirit we define $\overline{W}(\underset{\sim}{\widetilde{\Gamma}})$ as follows: If $\widetilde{V}(\underset{\sim}{\widetilde{\Gamma}})$ has only one element, we define $\overline{W}(\underset{\sim}{\widetilde{\Gamma}}) \equiv \widetilde{W}(\underset{\sim}{\widetilde{\Gamma}})$. If $\widetilde{V}(\underset{\sim}{\widetilde{\Gamma}}) = \{q_1, q_2\}$, we arbitrarily choose q_1 and we define $\overline{W}(\underset{\sim}{\widetilde{\Gamma}}) \equiv \widetilde{W}(\underset{\sim}{\widetilde{\Gamma}}) \times \mathbb{R}$, where the last factor is to be identified with the tangent space to the line q_1q_2 at q_1. This way we think of $\overline{W}(\underset{\sim}{\widetilde{\Gamma}})$ as being a subset of $W(\underset{\sim}{\widetilde{\Gamma}})$. If $\widetilde{\Gamma}$ is isomorphic to $\underset{\sim}{\widetilde{\Gamma}}$ (usually a regular perturbation), let q_1' and q_2' be its vertices corresponding to q_1 and q_2 respectively. We define $\overline{d}_{\widetilde{\Gamma}} \in \overline{W}(\underset{\sim}{\widetilde{\Gamma}})$ by requiring that $\overline{d}_{\widetilde{\Gamma}}$ is the same as $d_{\widetilde{\Gamma}}$ on vertices different than q_1, q_2, while at q_1 it is the x_1'-coordinate of $d_{\widetilde{\Gamma}}(q_1')$ with respect to a Cartesian coordinate system which has the origin at q_1' and q_2' on the positive first coordinate axis. Similar definitions can be given to the case when $\widetilde{V}(\underset{\sim}{\widetilde{\Gamma}}) \equiv \{q_1, q_2, q_3\}$: Let $\overline{W}(\underset{\sim}{\widetilde{\Gamma}}) \equiv \widetilde{W}(\underset{\sim}{\widetilde{\Gamma}}) \times \mathbb{R}^2 \times \mathbb{R}$, where $\mathbb{R}^2$ corresponds to the tangent plane at q_1 of $q_1q_2q_3$ and $\mathbb{R}$ to the tangent line of q_2q_3 at q_2. $\overline{d}_{\widetilde{\Gamma}}(q_1')$ specifies then the two first coordinates with respect to a coordinate Cartesian system which has the origin at q_1', q_2' on the positive first coordinate axis, and q_3' on the first coordinate plane with positive second coordinate. $\overline{d}_{\widetilde{\Gamma}}(q_2')$ is the first coordinate of $d_{\widetilde{\Gamma}}(q_2')$ with respect to a Cartesian coordinate system with the origin at q_2' and q_3' on the positive first coordinate axis.

Because of the assumptions we have it is clear that we can find some $\epsilon > 0$ such that for each $(\overline{d}, \tilde{\ell})$ in the ϵ-ball of $\overline{W}(\underset{\sim}{\widetilde{\Gamma}}) \times L(\underset{\sim}{\widetilde{\Gamma}}, \mathbf{e})$, there is $\widetilde{\Gamma}(\overline{d}, \tilde{\ell})$ so that all the conditions in Definition 2.4 are valid except (2) which is modified to read as follows:

(2′) If $\widetilde{\Gamma} = \widetilde{\Gamma}(\overline{d}, \widetilde{\ell})$, then $\overline{d}_{\widetilde{\Gamma}} = \overline{d}$. $d_{\underline{\widetilde{\Gamma}}} = 0$, that is $\underline{\widetilde{\Gamma}}$ is balanced.

As we already mentioned, this is achieved by suitably perturbing $\hat{\tau}$ on the edges in $\widetilde{E}(\underline{\Gamma})$. Arguing as in the proof of Lemma 3.3 and using the smooth dependence of $\widetilde{\Gamma}$ on its parameters, we conclude that there is a constant $\overline{c} > 0$ depending on $\underline{\widetilde{\Gamma}}$ only, such that for each $\widetilde{\Gamma}$ above we have $|d_{\widetilde{\Gamma}}| \leq \overline{c}|\overline{d}_{\widetilde{\Gamma}}|$. This allows us to repeat the construction of the initial surfaces in the proof of Theorem 2.7 with only minor modifications. The families $F_{\tau''}$ satisfy [10, II.1.9] modified by replacing $\overline{d}$ for d and $\overline{d}_{\Gamma}$ for d_{Γ}. These are defined in analogy with $\overline{d}$ and $\overline{d}_{\widetilde{\Gamma}}$. We obtain a collection of initial surfaces $M(\tau'', \zeta)$ for each $\tau'' \in [\tau', 3\tau']$, and $\zeta \in \Xi \cap \overline{\mathscr{W}}$, where $\overline{\mathscr{W}}$ is a modification of $\mathscr{W}$ in the same way $\overline{W}(\underline{\widetilde{\Gamma}})$ is a modification of $W(\underline{\widetilde{\Gamma}})$. $\overline{\mathscr{W}}$ can be identified as a subspace of $\mathscr{W}$ and this induces to it all the relevant norms. $M(\tau'', \zeta)$ depends continuously on its parameters and so does its configuration $\xi = \xi(\tau'', \zeta)$. The estimate of $d_{\widetilde{\Gamma}}$ above implies immediately that $\|\xi\|_\sigma \leq \overline{c}\|\zeta\|_\sigma$.

Repeat now the rest of the proof of Theorem 2.7 with the following modifications: First, all constants depending on ε depend on $\overline{c}$ as well. Second, $\mathscr{W}$ is replaced by $\overline{\mathscr{W}}$ and $\mathscr{J}$ is defined by

$$(\varpi', \zeta, \phi) \mapsto (\hat{\varpi}(\varpi', \zeta), \Pi(\overline{\lambda} + \lambda), (\overline{u} - u) \circ D_{\tau''(\varpi'), \zeta}),$$

where Π is defined as follows: Let $\Gamma(\tau'', d, \ell)$ be the c-graph to which $M(\tau''(\varpi'), \zeta)$ is associated. If $\widetilde{V}(\underline{\widetilde{\Gamma}})$ has only one element then $\Pi(\mu)$ is just the restriction of μ to its domain minus the vertex corresponding to the vertex in $\widetilde{V}(\underline{\widetilde{\Gamma}})$. If $\widetilde{V}(\underline{\widetilde{\Gamma}})$ has two vertices, let q_1, q_2 be the corresponding vertices of $\mathbf{\Gamma}_{\tau''}$ and q_1', q_2' the corresponding vertices of $\Gamma(\tau'', d, 0)$. We take $\Pi(\mu)(q_1)$ to be the first coordinate of $\mu(q_1)$ with respect to a coordinate system with q_1' at the origin and q_2' on the positive first axis. On the rest of the domain, $\Pi(\mu)$ agrees with μ. The definition in the remaining case is similar and in the spirit of the definition of $\overline{d}_{\widetilde{\Gamma}}$ above. Notice that these definitions imply that $\Pi(\xi) = \zeta$.

As in the proof of Theorem 2.7 we then obtain a fixed point corresponding to a closed surface $X_\varphi(M)$ such that

$$H_\varphi = 1 + \frac{|A|^2}{4}\Theta(\xi - \overline{\lambda} - \lambda).$$

Although we do not have $\xi = \overline{\lambda} + \lambda$ from the fixed point theorem anymore, we do have $\zeta = \Pi(\overline{\lambda} + \lambda)$, so $\Pi(\xi - \overline{\lambda} - \lambda) = 0$. We restrict ourselves now to the case where $\widetilde{V}(\underline{\widetilde{\Gamma}})$ has three elements, the other case being similar (and easier). Let $\Gamma(\tau'', d, \ell)$ be the graph to which M is associated.

Now let q_1, q_2, q_3 and q_1', q_2', q_3' be the vertices of Γ and $\Gamma(\tau'', d, 0)$ respectively corresponding to the vertices in $\widetilde{V}(\widetilde{\Gamma})$. Let $\mu = \xi - \overline{\lambda} - \lambda$; we know that $\mu(q_1)$ is orthogonal to the plane $q_1'q_2'q_3'$, $\mu(q_2)$ is orthogonal to the line $q_2'q_3'$, and μ vanishes on the rest of its domain except at q_3. We need to prove $\mu = 0$.

Consider now the (continuous) system of vectors $(H'(X) - 1)\nu'(X)$ applied at each $X \in M'$, where H' is the mean curvature function and ν' is the Gauss map of $M' = X_\varphi(M)$. Following R. Kusner [11], [12], let $\vec{Y}$ be a Killing vector field of E^3. M' is a closed smooth surface and it is easy to see that it is the boundary of a smooth 3-chain. An integration by parts implies then that

$$\int_{M'} (H' - 1)\nu' \cdot \vec{Y} = 0.$$

Let $\vec{a}$ be a nonzero vector and r a point in E^3. $\vec{Y} \equiv \vec{a}$ and $\vec{Y}(p) = \overrightarrow{rp} \times \vec{a}$ are Killing vector fields. Using them in the above equation we conclude that the system of vectors under consideration has trivial resultant.

$H' - 1$ is supported on the union of $X_\varphi(M[q_i])$ $(i = 1, 2, 3)$. The smallness of φ (by (2.10)) and the definition of Θ [10, IV.3.3] imply that the subsystem of vectors obtained by restricting to $X_\varphi(M[q_i])$ is equivalent to a $\vec{u}_i$ applied at a point q_i'' such that

$$|\mu(q_i) - \vec{u}_i| \leq C\varepsilon|\mu(q_i)|, \qquad |q_i''' - q_i''| < C\varepsilon,$$

where q_i''' is the vertex of $\Gamma(\tau'', d, \ell)$ corresponding to q_i. Since the resultant of the $\vec{u}_i$'s applied at q_i'' is 0, $\vec{u}_1$ is parallel to the plane $q_1''q_2''q_3''$. By [10, II.1.15] the angle between $q_1'q_2'q_3'$ and $q_1'''q_2'''q_3'''$ is small. Hence the angle between the planes $q_1''q_2''q_3''$ and $q_1'q_2'q_3'$ is small. Since $\mu(q_1)$ is orthogonal to $q_1'q_2'q_3'$, we conclude that $\mu(q_1) = 0 = \vec{u}_1$. It follows then that $\vec{u}_2$ is parallel to $q_1''q_2''$, and since $\mu(q_2)$ is orthogonal to $q_1'q_2'$, we conclude that $\mu(q_2) = 0 = \vec{u}_2$. Since we have trivial resultant, we conclude then that $\mu(q_3) = 0 = \vec{u}_3$, and this concludes the proof of Theorem 3.2.

The main theorem of [10] can be modified along the same lines as the modification of Theorem 2.7 in this section. To avoid more technicalities we do not discuss a general statement along these lines. We demonstrate the approach however and obtain some interesting examples of CMC surfaces in the following example. The details are left to the reader and they are very similar to the arguments we have given already.

Examples 3.8. Our purpose is to construct two-ended CMC surfaces of any genus $g \geq 5$ which possess no symmetries (cf. [10, II.4.1]). They

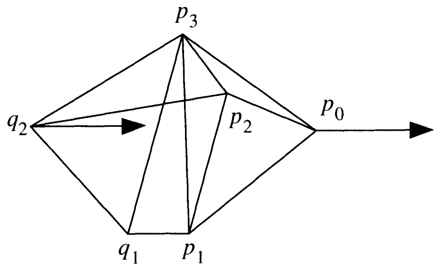

FIGURE 3.5

are also a bounded distance from a half-line; that is, they lie in a half-cylinder. We describe first the central graph $\mathbf{\Gamma}$ (Figure 3.5). The vertices are p_0, p_1, p_2, p_3 and $q_1, q_2, \cdots, q_{g-4}$. No three vertices are collinear and no four coplanar except for $p_1, p_3, q_1, \cdots, q_{g-4}$ which are required to be coplanar. There are edges connecting p_3 to each other vertex. The other edges are $p_1q_1, q_1q_2, \cdots, q_{g-5}q_{g-4}, q_{g-4}p_2, q_{g-4}p_3, p_0p_1, p_0p_2$, and p_1p_2. There are two rays r_0 and r_1, emanating from p_0 and q_{g-4} respectively. $\hat{\tau}$ is assigned an arbitrary nonzero value at p_1q_1. Considering then balancing successively at $q_1, \cdots, q_{g-5}$, and p_1, p_3, we uniquely determine $\hat{\tau}$ at the other edges with the exception of $q_{g-4}p_2$ and p_2p_0. $\hat{\tau}$ on these edges is uniquely determined by requiring that $d_\Gamma(p_2)$ is orthogonal to the plane defined by these two edges. Balancing at p_0 and q_{g-4} uniquely determines then up to the sign of $\hat{\tau}$ the direction and $\hat{\tau}$ of the rays. To check that $\mathbf{\Gamma}$ is balanced we argue as in the proof of Lemma 3.3: Consider the system of $d_\Gamma(p)$ applied at p, for each vertex p, where the summation in [10, II.1.2] is momentarily modified to exclude rays. This system has trivial resultant and only three nontrivial elements, those applied at q_{g-4}, p_2, and p_0. Therefore $d_\Gamma(p_2)$ has to be parallel to the plane $p_{g-4}p_2p_0$ and hence 0.

To determine $\Gamma(\overline{d}, 0)$ we argue as follows: We demand that p_1, the direction of p_1p_3, and the direction of $p_1p_3q_1$ are kept fixed. Considering $\overline{d}$ instead of balancing as before, we successively determine the following: First, the position of q_1 is already determined. Second, $\overline{d}(q_1)$ determines the position of q_2 and $\hat{\tau}$ on q_1q_2 and q_1p_3. Notice that in general q_2 does not lie on the plane $p_1q_1p_3$ anymore. We proceed the same way by considering $\overline{d}$ at $q_2, \cdots, q_{g-5}$. This determines the positions of q_{g-4}, p_2, and finally that of p_0. The rest of $\hat{\tau}$ and the rays are determined by considering $\overline{d}$ successively at p_1, p_3, p_2, q_{g-4}, and p_0. $\overline{d}$

differs from d_Γ only in that it does not determine the orthogonal to the plane $p_2p_0q_{g-4}$ component of $d_\Gamma(p_2)$. To construct $\Gamma(\overline{d}, \ell)$ we demand that $\hat{\tau}$, the angles between the planes intersecting at p_3q_k $(1 \le k \le g-5)$ and containing the other two edges from q_k, and the directions of the rays, are all the same as in $\Gamma(\overline{d}, \ell)$. The graph is then determined uniquely as before.

The rest of the argument goes as usual in the constructions of [10] modified in the lines of the proof of the theorem above. We miss $H_\varphi \equiv 1$ by a term corresponding to the orthogonal direction to the plane above at p_2, where we failed to prescribe d_Γ. By integrating as before we get one term corresponding to the term just mentioned and two boundary terms at ∞ corresponding to the rays. Since the resultant is trivial, arguing as in the above proof, we conclude that $H_\varphi \equiv 1$, and the two Delaunay ends have the same axis (asymptotically) at infinity. Provided that we chose the sign of $\hat{\tau}(r_0)$ correctly, the ensuing CMC surface lies in a half-cylinder. Also it has no symmetries provided we arranged the graphs to be asymmetrical enough. This construction has clearly one continuous parameter, namely $\hat{\tau}(p_1q_1)$.

4. Compact CMC surfaces with boundary a round circle

It is an interesting question whether a circle can be the boundary of a compact CMC surface of genus g. In this section we give an affirmative answer provided the circle has radius < 1 and $g \ge 3$. The proof is a relatively straightforward modification of the previous constructions in this paper. As usual the first step is to codify the data for such a construction in a graph. Hence the definitions:

Definition 4.1. A c∂-graph $\widetilde{\Gamma}$ is a c-graph equipped with a vertex $\mathbf{p}$, a unit vector $\vec{u} \in T_{\mathbf{p}}E^3$, and a real number $r \in (0, 1)$ such that:

(1) The action of G leaves both $\mathbf{p}$ and $\vec{u}$ invariant.

(2) For each edge $e \in E_{\mathbf{p}}$ we have $\measuredangle(\vec{u}, \vec{v}_{e,\mathbf{p}}) > \arcsin r$.

It will turn out in the course of the proof that there is no approximate kernel corresponding to $\mathbf{p}$. Motivated by this we redefine $W(\widetilde{\Gamma})$ for a c∂-graph to be the space of functions $w\colon V(\widetilde{\Gamma})\backslash\{\mathbf{p}\} \mapsto \mathbb{R}^3$. $d_{\widetilde{\Gamma}}$ is thought of as an element of the redefined $W(\widetilde{\Gamma})$, so $d_{\widetilde{\Gamma}}(\mathbf{p})$ is not allowed in this section. All the other definitions are the same as in the c-graph case. In particular we have:

Definition 4.2. A central c∂-graph $\underline{\widetilde{\Gamma}}$ is a c∂-graph which has the same properties as required in Definition 2.4 with the following modifications: $W(\underline{\widetilde{\Gamma}})$ is the redefined above version, and the same holds for $d_{\widetilde{\Gamma}}$.

The regular perturbations of $\widetilde{\Gamma}$ are of course $c\partial$-graphs themselves, depending smoothly on $(\tilde{d}, \tilde{\ell})$. We need to describe now the CMC surfaces obtained from such a graph. We first modify Definition 2.6 to apply in the new setting: The only difference is the following: Instead of the initial surface M', we consider the following subset M'' of M': Notice that there is an asr contained in M' which corresponds to $\mathbf{p}$. It contains the subset of a round sphere whose center we call $\mathbf{p}'$. Consider the circle of radius (in E^3) r which is the intersection of the plane through $\mathbf{p}' + \sqrt{1-r^2}\vec{u}$ orthogonal to $\vec{u}$, with the unit sphere centered at $\mathbf{p}'$. By assuming that ε is small enough and using condition (2) in 4.1 this circle is contained in the asr of M' corresponding to $\mathbf{p}$ and which is $M'[\mathbf{p}']$ by the notation of [10]. It disconnects M' into two components, one of them in a geodesic disc in a round sphere. Let M'' be the closure of the other component, and $X: M'' \mapsto E^3$ and $\nu: M'' \mapsto S^2(1)$ its immersion and Gauss map respectively. φ is required to be a smooth function on M'' which vanishes on its boundary. Definition 2.6 applies then to the current case provided we replace M' with M''.

The other versions of 2.6 apply in the current case as well, provided we add the requirement that M has a boundary round circle of radius r and lying on a plane perpendicular to the $\vec{u}$ of $\widetilde{\Gamma}$. Also in 2.6$'$ we replace the domain of f with the complement of a small open disc in ∂R, where this disc is a small distance from the vertex of the homothetically expanded $\widetilde{\Gamma}$ which corresponds to $\vec{v}$.

Theorem 4.3. *For any central $c\partial$-graph $\widetilde{\Gamma}$, there are infinitely many compact CMC surfaces with boundary a round planar circle (of predetermined radius $r < 1$), based on regular perturbations of $\widetilde{\Gamma}$. More precisely, there is $T_0(\widetilde{\Gamma}) > 0$ such that for any $\tau' \in (0, T_0)$, there is $N_0(\tau', \widetilde{\Gamma})$ such that for any $N > N_0$, there is a compact CMC surface M based on $(\widetilde{\Gamma}, \tau'', N)$ for some $\tau'' \in [\tau', 3\tau']$.*

Example 4.4. Let $r \in (0, 1)$ and a genus $g \geq 3$ be given. We construct infinitely many compact CMC surfaces of genus g whose boundary is a round planar circle of radius r. The construction is based on a central $c\partial$-graph $\widetilde{\Gamma}$ which is a modification of the central c-graph in Examples 3.4: $\vec{u}$ is a vector at $\mathbf{p} \equiv p_g$ pointing in the direction of $+\infty$ on the x_1 axis. (This is a valid for the regular perturbations as well.) r is already determined and the conditions of 4.1 are clearly satisfied. By applying Theorem 4.3 we obtain infinitely many surfaces of the desired type.

Proof of Theorem 4.3. The proof is a modification of the proof of Theorem 2.7. The differences are the following: First, $\mathscr{W}$ is modified by removing $\mathbf{p}'$ from the domain of the functions it contains, where $\mathbf{p}'$ is the vertex of $\mathbf{\Gamma}_{2\tau'}$ corresponding to $\mathbf{p}$. The construction of the initial surfaces then is similar to the construction in the proof of Theorem 2.7. Second, we remove an appropriate disc from each initial surface $M(\tau'', \xi)$ as we described above, so that $M(\tau'', \xi)$ is a surface with the appropriate circle boundary. If M is such a surface we still write $M[\mathbf{p}']$ for what is left of the asr corresponding to $\mathbf{p}'$ and which is a unit sphere minus small discs where transition regions [10, III.1.4] have been attached, and minus the smallest of two open discs into which a circle of radius r separates the unit sphere. $D_{\tau'', \xi}$ is a diffeomorphism still mapping boundary circle onto boundary circle and it is a Euclidean motion on a neighborhood of each circle. The rest of the proof goes as before with the modification that now $\mathscr{N}$ is a subset of $\mathbb{R} \times \mathscr{W} \times C_0^{2,\overline{\alpha}}(M(2\tau', 0))$; that is, ϕ is required to vanish on the boundary. We will prove shortly that the lemmas in [10, IV.5.1, IV.5.5] are still valid in the new setting provided we require that u and $\overline{u}$ vanish on the boundary, and $\overline{\lambda}$ and λ do not assign values to $\mathbf{p}'$. The fixed point then has φ vanishing on the boundary and so provides us with the desired surface.

It remains to check that the modified lemmas are valid. First we need to understand the approximate kernel of the new surfaces. Notice that $M[\mathbf{p}']$ now corresponds to $\widetilde{M}[\mathbf{p}']$ which is not a sphere anymore but a unit sphere with an open disc removed. The open disc has boundary of radius r and is the smaller of the two such discs. Let

$$0 < \mu_1(r) < \mu_2(r) \leq \mu_2(r) \leq \mu_4(r) \leq \cdots$$

be the eigenvalues of the Dirichlet problem for the Laplacian on $\widetilde{M}[\mathbf{p}']$, each appearing as many times as its multiplicity. Since the domain strictly decreases with r, standard theory [2, p. 18] implies that each $\mu_k(r)$ is strictly increasing with r increasing. For $r = 1$ the domain is a hemisphere and there is a coordinate function of fixed sign vanishing on its boundary. Hence $\mu_1(1) = 2$. On the other hand we know from [10, Appendix B], or from general theory, that $\lim_{r\to 0} \mu_1(r) = 0$ and $\lim_{r\to 0} \mu_2(r) = 2$. We conclude then that for $r \in (0, 1)$, $\mu_1(r) \in (0, 2)$ and $\mu_2(r) > 2$. In other words there are no eigenvalues close to 2. This implies that there is no approximate kernel corresponding to $\mathbf{p}'$ and this is the reason we do not need to worry about balancing at $\mathbf{p}$ etc.

Straightforward modifications which are left to the reader allow us to establish the lemmas in their new setting. We only remark that when boundary estimates are needed we resort to [4, 8.15] instead of [4, 8.17] for supremum estimates [10, III.4.3]. For $C^{2,\alpha}$ estimates we use boundary Schauder theory: Notice that the geometry of a neighborhood of the boundary depends only on r on which all our constants are allowed to depend through their dependence on $\underline{\tilde{\Gamma}}$.

Acknowledgments

We would like to thank Professors D. Christodoulou, R. Hamilton, G. Huisken, Wu-Yi Hsiang, L. Simon, H. Wente, and S.-T. Yau for their interest and encouragement. We thank also Professor Rugang Ye for mentioning to us the possibility of a construction like the one presented in §4. Most of all, we would like to thank our thesis advisor, Professor R. Schoen, for suggesting this problem and for many stimulating discussions on the subject. His advice in particular to replace the initial construction with one in the spirit of the one presented in this paper resulted in a much simpler proof and stronger results.

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