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RIGID REPRESENTATIONS OF KÄHLERIAN FUNDAMENTAL GROUPS

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0. Introduction

In the early 1960s, Weil proved that cocompact lattices in a semisimple Lie group had no deformations, provided the group had no local three-dimensional factors. This result has been the point of departure for many subsequent studies of local, strong, and super-rigidity of lattices in semisimple Lie groups. The goal in the current paper is to show that the local rigidity discovered by Weil in the context of locally symmetric manifolds is actually a phenomenon which holds much more generally in Kähler geometry.

Suppose M is a compact Kähler manifold, and $G_{\mathbf{R}}$ is a simple real algebraic group acting by isometries on the irreducible bounded symmetric domain $G_{\mathbf{R}}/K$. If P is a principal $G_{\mathbf{R}}$-bundle over M, we may reduce its structure group to K, and associate characteristic classes of the bundle with reduced structure group to P. In particular, there is an invariant connected with the volume form on $G_{\mathbf{R}}/K$, to be denoted by $\mathrm{vol}(P)$. It is a power of the first Chern class, up to a constant factor. We prove the following.

Theorem 0.1. *Suppose that P is a flat principal $G_{\mathbf{R}}$-bundle with* $\mathrm{vol}(P) \neq 0$, *and assume that $G_{\mathbf{R}}/K$ is not of the form*

$$U(n, 1)/U(n) \times U(1)$$

or

$$SO(2n+1, 2)/S(O(2n+1) \times O(2)).$$

Then the monodromy homomorphism of the fundamental group of M into $G_{\mathbf{R}}$ is locally rigid as a homomorphism into the complexification of $G_{\mathbf{R}}$.

An earlier result [4, Theorem 4.1] asserted the compactness of the space of equivalence classes of representations with $\mathrm{vol}(\rho) \neq 0$ under weaker assumptions on the bounded symmetric domain. This result supersedes

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the earlier one in those situations where it applies. The author has no examples which show that the exclusion of the two series of bounded symmetric domains listed is necessary, but the method of proof employed here certainly does not apply in these cases.

In a sense, Theorem 0.1 says that the stated local rigidity property is hereditary for surjective holomorphic maps. If N is a compact quotient of one of the bounded symmetric domains allowed in 0.1, then the representation of the fundamental group of M induced by a surjective holomorphic map to N (e.g. a branched covering) satisfies the hypotheses of 0.1.

It should be mentioned that there is an interesting result which drives in the converse direction, due to Carlos Simpson [14]. His result is the following.

Theorem 0.2. *Suppose $\rho: \pi_1(M) \to Gl(n, \mathbf{C})$ is a locally rigid representation of the fundamental group of a compact Kähler manifold. Then the associated flat vector bundle is the underlying vector bundle of a variation of Hodge structure.*

The homomorphisms which enter into Theorem 0.1 are the monodromy representations of variations of Hodge structure of principal type, as follows almost immediately from [3].

The method of the present paper was inspired by a reading of [1], as well as discussions of it and related matters with Domingo Toledo. John Millson told me of the relevance of Kostant's theorem [7] to the goals of §3 and, together with Bill Goldman, introduced the author to the philosophy of looking for a differential graded Lie algebra which guides a deformation theory. This philosophy plays a role in the proof of 3.3. The author would like to thank all of them for their advice and encouragement, and also the Mathematical Sciences Research Institute and the University of Maryland, each of which provided shelter while part of this work was being done.

1. Families of representations

We shall require some basic facts about the algebraic geometry of the representation and character varieties of finitely generated groups. Much of what we need is taken from Johnson and Millson [6] and Morgan [10].

Let Γ be a finitely generated group, and G a simple algebraic group defined over the real numbers. $G_{\mathbf{R}}$ (resp. $G_{\mathbf{C}}$) will represent the real (resp. complex) points of G. Choose a finite generating set $\{\gamma_1, \cdots, \gamma_n\}$ for Γ and a set of defining relations: $R_i(\gamma_1, \cdots, \gamma_n) = 1$. The set $\mathrm{Hom}(\Gamma, G_{\mathbf{C}})$ of homomorphisms from Γ to $G_{\mathbf{C}}$ is then the set of complex points of an affine algebraic variety defined over $\mathbf{R}$, since it coincides with the set of solutions of the polynomial equations $R_i(\gamma_1, \cdots, \gamma_n) = 1$ in $G^n_{\mathbf{C}}$, and these equations have integer coefficients.

Definition 1.1. A homomorphism $\rho \in \mathrm{Hom}(\Gamma, G_{\mathbf{C}})$ is called *stable* if the image of ρ is not contained in any proper parabolic subgroup of $G_{\mathbf{C}}$. In particular, homomorphisms with Zariski dense image in $G_{\mathbf{C}}$ are stable.

Proposition 1.2. *The set of all stable* $\rho \in \mathrm{Hom}(\Gamma, G_{\mathbf{C}})$ *is a Zariski open subset of* $\mathrm{Hom}(\Gamma, G_{\mathbf{C}})$.

For a proof, see Johnson and Millson [6, §1].

Let the subset of stable homomorphisms be denoted by $\mathrm{Hom}^s(\Gamma, G_{\mathbf{C}})$. The character variety $X(\Gamma, G_{\mathbf{C}})$ is, naively, the quotient

$$\mathrm{Hom}(\Gamma, G_{\mathbf{C}})/G_{\mathbf{C}},$$

where $G_{\mathbf{C}}$ acts by conjugation on homomorphisms. However, this quotient presents the usual array of pathologies, so one must use the more sophisticated notions of geometric invariant theory to give the proper definition of the character variety. We will only need to work in the Zariski open subset of equivalence classes of stable homomorphisms, however, and this obviates the need for subtleties in the definition of the character variety. Thus, we define

$$X(\Gamma, G_{\mathbf{C}}) = \mathrm{Hom}^s(\Gamma, G_{\mathbf{C}})/G_{\mathbf{C}}.$$

Consult the paper of Johnson and Millson for verification of the fact that $X(\Gamma, G_{\mathbf{C}})$ is the set of complex points of a quasiprojective variety defined over the real numbers.

Our main concern in this paper is with homomorphisms into the set of real points of G, so define

$$X(\Gamma, G_{\mathbf{R}}) = \mathrm{Hom}^s(\Gamma, G_{\mathbf{C}}) \cap \mathrm{Hom}(\Gamma, G_{\mathbf{R}})/G_{\mathbf{R}}.$$

The map $\pi: X(\Gamma, G_{\mathbf{R}}) \to X(\Gamma, G_{\mathbf{C}})$ induced by the inclusion of $G_{\mathbf{R}}$ in $G_{\mathbf{C}}$ is a map into the set of real points of $X(\Gamma, G_{\mathbf{C}})$ [6]. The basic observation which will be used in this paper is:

Lemma 1.3. *Let* x *be any point in* $X(\Gamma, G_{\mathbf{R}})$. *If, for any real analytic curve* $f: [-1, 1] \to X(\Gamma, G_{\mathbf{C}})$ *with* $f(0) = \pi(x)$, *there is an open neighborhood of zero with image contained in* $X(\Gamma, G_{\mathbf{R}})$, *then* $\pi(x)$ *is an isolated point in* $X(\Gamma, G_{\mathbf{C}})$.

Proof. This is a direct consequence of the Curve Selection Lemma [9]. The set of nonreal points of $X(\Gamma, G_{\mathbf{C}})$ is an open real semialgebraic subset which contains x in its closure if x is not isolated. The Curve Selection

Lemma then guarantees that there is a real analytic map $f\colon [-1, 1] \to X(\Gamma, G_{\mathbf{C}})$ with $f(0) = x$ and $f([-1, 0) \cup (0, 1])$ contained in the set of nonreal points.

2. Harmonic maps, liftings, and horizontality

Let M be a compact Kähler manifold with fundamental group Γ. $G_{\mathbf{R}}$ will be as in the previous section, with the additional assumption that $X = G_{\mathbf{R}}/K$ is an irreducible bounded symmetric domain, where K is a maximal compact subgroup of $G_{\mathbf{R}}$. We assign n to be $\dim_{\mathbf{C}} G_{\mathbf{R}}/K$, and assume $n \geq 2$.

Let $\rho\colon \Gamma \to G_{\mathbf{R}}$ be a homomorphism. It determines a principal $G_{\mathbf{R}}$ bundle P with flat connection over M, and an associated fiber bundle $Y \to M$ with fiber X. As described in [3], we can manufacture topological invariants of P from powers of the Kähler form ω on X, in the following manner. Because ω is invariant under the action of $G_{\mathbf{R}}$, there is a closed two-form on Y (again denoted by ω) which coincides with the Kähler form along any fiber and vanishes when restricted to the leaves of the horizontal foliation defined by the flat connection, obtained by pulling back ω to the product of the universal cover of M with X and dividing by the natural action of the fundamental group. Since X is contractible, we can choose a section f of Y, and $f^*\omega^m$ is a closed $2m$-form on M for any $m \geq 1$. It therefore determines a class $[f^*\omega^{2m}] \in H^{2m}(M, \mathbf{R})$. On the other hand, f determines a reduction of the structure group of P to K. Our assumptions on $G_{\mathbf{R}}$ imply that K has a one-dimensional center, so there is a complex line bundle $L_\lambda \to M$ determined by any nontrivial character λ of the center. One can easily check that, up to normalization, $f^*\omega$ is the Chern–Weil representative of the first Chern class of L_λ, hence a topological invariant of P. In particular, $[f^*\omega^m]$ does not vary if ρ is allowed to roam over a component of $\mathrm{Hom}(\Gamma, G_{\mathbf{R}})$.

Of particular interest is the invariant associated to the volume form $\omega^n/n!$; we obtain from it an invariant which will be denoted by $\mathrm{vol}(\rho) \in H^{2n}(M, \mathbf{R})$.

Proposition 2.1. *If $\rho\colon \Gamma \to G_{\mathbf{R}}$ is a homomorphism with* $\mathrm{vol}(\rho) \neq 0$, *then ρ has Zariski dense image.*

Proof. Suppose not. Then the Zariski closure of $\mathrm{im}(\rho)$ has a reductive Levi factor G' which is a proper subgroup of $G_{\mathbf{R}}$. ρ may be deformed to a representation ρ' with image in G'. $G'/G' \cap K$ is a submanifold of positive codimension in X, and there is a section f' of the flat $G'/G' \cap K$

bundle associated to ρ'. However, the pullback of $\omega^n/n!$ by f' vanishes, so ρ must have had Zariski dense image. q.e.d.

Let D be the natural flat connection on P and let $\mathrm{ad}(P)$ be the natural flat vector bundle associated to P by the adjoint representation of $G_{\mathbf{R}}$. Once we have fixed a section $f\colon M \to Y$, a decomposition

$$\mathrm{ad}(P) = V \oplus W$$

is determined, according to the decomposition $\mathbf{g}_{\mathbf{R}} = \mathbf{k} \oplus \mathbf{p}$ of the Lie algebra under the action of K. There is a corresponding decomposition

$$D = D^+ + \theta,$$

where D^+ is a connection which preserves the reduction of structure group determined by f and θ is a one-form with values in W.

There are two invariant complex structures on X, and we may use either one to give Y a complex structure. The complex structures on X are reflected on the level of Lie algebras in the following manner. Let $\mathbf{g}$ be the Lie algebra of $G_{\mathbf{R}}$. The choice of a maximal compact subgroup K determines a direct sum decomposition

$$\mathbf{g} = \mathbf{k} \oplus \mathbf{p},$$

where $\mathbf{k}$ is the Lie algebra of K. The vector space $\mathbf{p}_{\mathbf{C}} = \mathbf{p} \otimes \mathbf{C}$ decomposes under the action of the center of $\mathbf{k}$ into two maximal abelian subalgebras of $\mathbf{g} \otimes \mathbf{C}$. Either one of these can be taken to define the $(1, 0)$ directions for an invariant complex structure on X, and one which is so chosen will be denoted by $\mathbf{p}^{1,0}$, while the other will be denoted by $\mathbf{p}^{0,1}$.

Proposition 2.2. *Suppose $\rho\colon \Gamma \to G_{\mathbf{R}}$ is a homomorphism with* $\mathrm{vol}(\rho) \neq 0$. *If* $\dim_{\mathbf{C}} X > 1$, *then there is a choice of complex structure on X such that Y has a holomorphic section.*

Proof. The existence of a harmonic section f follows from Proposition 2.1 and [3] (cf. [5]). Let $\theta = \theta^{1,0} + \theta^{0,1}$ be the decomposition of θ into $(1, 0)$ and $(0, 1)$ forms with values in $W \otimes \mathbf{C}$. By Theorem 5.1 of [3] (or Proposition 2.4 below), the image of $\theta^{1,0}$ is pointwise an abelian subalgebra of $\mathbf{p}_{\mathbf{C}}$. Since $\mathrm{vol}(\rho) \neq 0$, the image of $\theta^{1,0}$ has half the dimension of $\mathbf{p}_{\mathbf{C}}$ at a generic point. Thus, by Theorem 3.1 of [1], $\mathrm{im}(\theta^{1,0})$ coincides with $\mathbf{p}^{1,0}$ for some choice of invariant complex structure on X. This is true on an open set in M, so f is holomorphic on an open subset of M relative to the specified complex structure on Y. The argument that f is holomorphic everywhere is given in [15]. q.e.d.

Now suppose we are presented with a real analytic curve $\chi_t\colon [-1, 1] \to X(\Gamma, G_{\mathbf{C}})$ with χ_0 the character of ρ. Without loss of generality, we may assume that each χ_t represents an equivalence class of representations with Zariski dense image in $G_{\mathbf{C}}$, since the latter is an open condition on characters. This family of characters determines a family of equivalence classes of flat connections on $P_{\mathbf{C}}$, the principal $G_{\mathbf{C}}$-bundle obtained from P by extending the structure group. Each χ_t corresponds to an equivalence class of representations with Zariski dense image in $G_{\mathbf{C}}$, so any flat connection in the corresponding equivalence class admits a harmonic metric, in the sense of [3]. The next result asserts that we can choose representatives which vary smoothly with t. Let $\mathbf{P}_{\mathbf{C}}$ be the pullback of $P_{\mathbf{C}}$ along the projection from $\mathbf{M} = M \times [-1, 1]$ to M. Let M_t and $P_{\mathbf{C},t}$ be the inverse images of $t \in [-1, 1]$ under the natural projections from $\mathbf{M}$ and $\mathbf{P}_{\mathbf{C}}$. Let H be a maximal compact subgroup of $G_{\mathbf{C}}$, and define $X_{\mathbf{C}}$ to be $G_{\mathbf{C}}/H$. $\mathbf{Y}_{\mathbf{C}}$ and $Y_{\mathbf{C},t}$ are the fiber bundles associated to $\mathbf{P}_{\mathbf{C}}$ and $P_{\mathbf{C},t}$ by the action of $G_{\mathbf{C}}$ on $X_{\mathbf{C}}$. Let T_1 be the bundle of tangent vectors to the fibers of the projection $\mathbf{P}_{\mathbf{C}} \to [-1, 1]$, and T_2 the tangent bundle to the fibers of the projection $\mathbf{P}_{\mathbf{C}} \to \mathbf{M}$. Define a vertical connection on $\mathbf{P}_{\mathbf{C}}$ to be a projection $T_1 \to T_2$ which is a connection in the usual sense upon restriction to any $P_{\mathbf{C},t}$.

Proposition 2.3. *There is a vertical connection* $\mathbf{D}$ *on* $\mathbf{P}_{\mathbf{C}}$ *and a smooth section* $\mathbf{f}\colon \mathbf{M} \to \mathbf{Y}_{\mathbf{C}}$ *such that the restriction of* $\mathbf{D}$ *to any* $P_{\mathbf{C},t}$ *is a flat connection with holonomy representation in the equivalence class determined by* χ_t, *and* $\mathbf{f}$ *restricted to any* M_t *is a harmonic section of* $Y_{\mathbf{C},t}$.

Proof. Let us recall the framework of [3], [4]. $\mathscr{C}$ will be the space of smooth connection on $P_{\mathbf{C}}$, and $\mathscr{G}$ will be the group of smooth automorphisms of $P_{\mathbf{C}}$ as a principal $G_{\mathbf{C}}$-bundle covering the identity on M. Let $\mathscr{C}'$ be the subset of $\mathscr{C}$ consisting of flat connections with Zariski dense holonomy. Fix a section f of $Y_{\mathbf{C}}$, and let $\mathscr{U}$ be the subgroup of $\mathscr{G}$ which fixes f. As described in [3], [4], this data allows us to think of $\mathscr{C}$ as an infinite-dimensional symplectic manifold where $\mathscr{U}$ acts in a Hamiltonian fashion. A zero for the associated moment map Φ along a $\mathscr{G}$-orbit representing an equivalence class of flat connections is simply a flat connection for which f is harmonic. $\Phi^{-1}(0) \cap \mathscr{C}'$ is a principal $\mathscr{U}$ bundle over $\mathscr{C}'/\mathscr{G}$ (we may assume without loss of generality that $G_{\mathbf{C}}$ has trivial center, so $\mathscr{G}$ acts freely on $\mathscr{C}'$). The family of representations ρ_t determines a family of equivalence classes of connections, i.e., a path in $\mathscr{C}'/\mathscr{G}$. This can be lifted to a smooth path in $\Phi^{-1}(0) \cap \mathscr{C}'$. Pulling back the universal vertical connection on $P_{\mathbf{C}} \times \mathscr{C}$ and the section f of $Y_{\mathbf{C}}$ gives the required smooth family. q.e.d.

We now recall the various consequences of the Bochner-Siu formula for a harmonic section of $Y_{\mathbf{C},t}$ from [1] (cf. [3], [12], [15]).

Proposition 2.4. *For any harmonic section* f_t *of* $Y_{\mathbf{C},t}$, *we have*

(i) $\overline{\partial}^+ \circ \overline{\partial}^+ = 0$;

(ii) $\overline{\partial}^+ \theta^{1,0} = 0$; *and*

(iii) $[\theta^{1,0}, \theta^{1,0}] = 0$.

Proof. Calculating as in [3] gives

$$\Delta|\theta^{1,0}|^2 = 2|\overline{\partial}^+ \theta^{1,0}|^2 + 2|[\theta^{1,0}, \theta^{1,0}]|^2.$$

This must vanish, since the left side integrates to zero while the right side is nonnegative. (i) follows from the fact that D_t is flat, since the curvature of D_t^+ must then be $-\frac{1}{2}[\theta_t, \theta_t]$. Taking the $(2, 0)$-components on both sides yields $\overline{\partial}^+ \circ \overline{\partial}^+ = -\frac{1}{2}[\theta_t^{1,0}, \theta_t^{1,0}] = 0$. q.e.d.

Let $\mathrm{ad}(P_{\mathbf{C}})$ be the Lie algebra bundle associated to $P_{\mathbf{C}}$ by the adjoint representation of $G_{\mathbf{C}}$. Each fiber is isomorphic to $\mathbf{g}_{\mathbf{C}}$, as a Lie algebra. As noted in the proof of Proposition 2.4, (iii) implies that the image of $\theta_t^{1,0}$ is, pointwise, an abelian subalgebra of the corresponding fiber of $\mathrm{ad}(P_{\mathbf{C}})$. For $t = 0$ and a generic point of M, the image of $\theta_t^{1,0}$ is $\mathbf{p}^{1,0}$. For the remainder of this section, we will operate under the following assumption:

Assumption 2.5. $\mathbf{p}^{1,0}$ *is locally rigid as an abelian subalgebra of* $\mathbf{g}_{\mathbf{C}}$.

In other words, if $\mathbf{a}_t$ is any smooth family of abelian subalgebras of $\mathbf{g}_{\mathbf{C}}$ with $\mathbf{a}_0 = \mathbf{p}^{1,0}$, then each $\mathbf{a}_t$ is conjugate to $\mathbf{p}^{1,0}$, for t in some open neighborhood of zero. This assumption will be defended in certain cases in the next section.

Granted this assumption, for t sufficiently small (e.g. less than ε in absolute value), there is a nonempty open subset U of M such that the pointwise image of $\theta^{1,0}$ on U is an abelian subalgebra of the fiber of $\mathrm{ad}(P_{\mathbf{C}})$ which is in the conjugacy class represented by $\mathbf{p}^{1,0}$. One can interpret this as saying that the 1-jets of f_t on U are those of holomorphic maps from U into totally geodesically embedded copies of X in $X_{\mathbf{C}}$, for $|t| < \varepsilon$. Our task is then to show that these totally geodesic copies of X given by the various 1-jets of f_t are all identical.

Furthermore, we can assume that $M - U$ has real codimension at least two, for the following reason. By Proposition 2.4(i), $\overline{\partial}^+$ induces a holomorphic structure on $\mathrm{ad}(P_{\mathbf{C}})$. Relative to this structure, 2.4(ii) asserts that $\theta^{1,0}$ is a holomorphic $(1, 0)$-form with values in $\mathrm{ad}(P_{\mathbf{C}})$. Hence, the set on which $\theta^{1,0}$ has rank smaller than $\dim_{\mathbf{C}} \mathbf{p}^{1,0}$ is a complex analytic subset of M. In particular, it does not disconnect M. Thus, any point in the set on which $\theta^{1,0}$ attains maximal rank can be joined to a point in

U by a path in this set. Consequently, the image of $\theta^{1,0}$ at any point of this set is a conjugate of $\mathbf{p}^{1,0}$.

Define $Z = G_{\mathbf{C}}/K$. Z fibers over $X_{\mathbf{C}}$ with fiber H/K. Points of Z are in bijective correspondence with points of $X_{\mathbf{C}}$ together with a totally geodesically embedded copy of X passing through the given point (this observation seems to have been made first by Cartan [2]). Lift f_t to a map $\tilde{f}_t$ from the universal cover $\widetilde{M}$ of M to $X_{\mathbf{C}}$, and let $\widetilde{U}$ be the inverse image of U under the covering map. The claim of the previous paragraph then implies that there is a natural lift of $\tilde{f}_t$ to a smooth map $F_t\colon \widetilde{U} \to Z$, determined by the image of $\theta^{1,0}$ at each point.

Z carries a natural foliation $\mathscr{F}$ whose leaves are horizontal relative to the submersion onto $X_{\mathbf{C}}$; each leaf is the lift of a copy of X embedded in $X_{\mathbf{C}}$. We wish to show that the image of F_t on $\widetilde{U}$ lies in a single leaf of $\mathscr{F}$. Under the action of H, $\mathbf{g}_{\mathbf{C}}$ decomposes (as a real vector space) into a direct sum of the Lie algebra of H and a complement:

$$\mathbf{g}_{\mathbf{C}} = \mathbf{h} \oplus \mathbf{q}.$$

When restricted to K, the representation reduces further:

$$\mathbf{g}_{\mathbf{C}} = \mathbf{k} \oplus \mathbf{h}/\mathbf{k} \oplus \mathbf{p} \oplus \mathbf{q}/\mathbf{p}.$$

There is a corresponding decomposition of D_t, upon restriction to $\widetilde{U}$, determined by F_t:

$$D_t = D^{\natural} + \varrho + \theta_1 + \theta_2.$$

The last term vanishes because the image of θ coincides with $\mathbf{p}$, so $\theta = \theta_1$. On the other hand, the condition that the image of F_t be contained in a leaf of $\mathscr{F}$ is equivalent to the vanishing of $\varrho + \theta_2$, since the foliation of Z is defined by the one-form corresponding to the projection from $\mathbf{g}_{\mathbf{C}}$ to $\mathbf{h}/\mathbf{k} \oplus \mathbf{q}/\mathbf{p}$. By Proposition 2.4, we get

$$\bar{\partial}^{+}\theta^{1,0} = \bar{\partial}^{\natural}\theta^{1,0} + [\varrho^{0,1}, \theta^{1,0}] = 0.$$

The two terms in the middle take values in different subspaces of $\mathbf{g}_{\mathbf{C}}$, so each vanishes. In particular, since the image of $\theta^{1,0}$ is a maximal abelian subalgebra of $\mathbf{g}_{\mathbf{C}}$, the vanishing of $[\varrho^{0,1}, \theta^{1,0}]$ implies that the image of $\varrho^{0,1}$ is contained in $\mathbf{p}^{1,0}$, so $\varrho^{1,0}$ takes values in $\mathbf{p}^{0,1}$. From the fact that D_t is flat, it follows that

$$D^{+}\theta = D^{\natural}\theta + [\varrho, \theta] = 0.$$

Again, both terms vanish, and taking $(2, 0)$-components gives

$$[\varrho^{1,0}, \theta^{1,0}] = 0.$$

Let V be the image at a point in U of $\varrho^{1,0}+\theta^{1,0}$. The projection of $\mathbf{g}_{\mathbf{C}}$ onto $\mathbf{p}^{1,0}$ restricts to a surjection on V. Choose a basis $\{v_\alpha\}\cup\{v'_\beta\}$ of V^* such that the dual of $\{v_\alpha\}$ projects to a basis $\{p_\alpha\}$ of $\mathbf{p}^{1,0}$, and the dual of $\{v'_\beta\}$ is in the kernel of the projection onto $\mathbf{p}^{1,0}$. For some choice of Cartan subalgebra $\mathbf{c}$, assume that each p_α is a nonzero element of a root space, and let $\{p_{\bar\alpha}\}$ be the conjugate basis of $\mathbf{p}^{0,1}$. We will take $\{\alpha\}$ as a set of labels for roots. Let $\pi_\pm\in\operatorname{Hom}(V,\mathbf{p}_\pm)$ be the elements induced by the projections from $\mathbf{g}_{\mathbf{C}}$. We need to show that the vanishing of $[\pi_+,\pi_-]\in\operatorname{Hom}(\bigwedge^2 V,\mathbf{g}_{\mathbf{C}})$ implies the vanishing of π_-. A basis for $\operatorname{Hom}(V,\mathbf{p}^{1,0})$ is given by $\{v_\alpha\otimes p_\beta\}\cup\{v'_\alpha\otimes p_\beta\}$, while $\{v_\alpha\otimes p_{\bar\beta}\}\cup\{v'_\alpha\otimes p_{\bar\beta}\}$ is a basis for $\operatorname{Hom}(V,\mathbf{p}^{0,1})$. In terms of these bases, we have

$$\pi_+=\sum v_\alpha\otimes p_\alpha,$$
$$\pi_-=\sum \pi_{\alpha\bar\beta}v_\alpha\otimes p_{\bar\beta}+\sum \pi'_{\alpha\bar\beta}v'_\alpha\otimes p_{\bar\beta}.$$

If $\alpha\neq\beta$, then we can show that $\pi_{\alpha\bar\beta}=0$ by considering the coefficient in $\mathbf{g}_{\mathbf{C}}$ of $v_\alpha\wedge v_\beta$ in the expansion of $[\pi_+,\pi_-]$:

$$[\pi_+,\pi_-]_{\alpha\beta}=\sum \pi_{\beta\bar\gamma}[p_\alpha,p_{\bar\gamma}]+\pi_{\alpha\bar\gamma}[p_\beta,p_{\bar\gamma}].$$

Each of these terms belongs to some nontrivial root space, with the exception of

$$\pi_{\alpha\bar\beta}[p_\beta,p_{\bar\beta}]+\pi_{\beta\bar\alpha}[p_\alpha,p_{\bar\alpha}],$$

which lies in $\mathbf{c}$. If both terms are nonzero, then they are linearly independent. Thus, the vanishing of $[\pi_+,\pi_-]$ implies that $\pi_{\alpha\bar\beta}=0$ when $\alpha\neq\beta$. The vanishing of $\pi'_{\alpha\bar\beta}$ for all α,β follows by an analogous argument.

For any α, let p_γ be an element such that $[p_\gamma,p_{\bar\alpha}]$ is a nonzero element of a nontrivial root space. Consider the coefficient of $v_\gamma\wedge v_\alpha$:

$$[\pi_+,\pi_-]_{\gamma\alpha}=\sum \pi_{\gamma\bar\delta}[p_\alpha,p_{\bar\delta}]+\pi_{\alpha\bar\delta}[p_\gamma,p_{\bar\delta}].$$

All of these terms lie in $\mathbf{c}$ or a root space other than that corresponding to $[p_\gamma,p_{\bar\alpha}]$, except

$$\pi_{\alpha\bar\alpha}[p_\gamma,p_{\bar\alpha}]+\pi_{\gamma\bar\delta}[p_\alpha,p_{\bar\delta}],$$

where $\delta=2\alpha-\gamma$. However, since $\alpha\neq\gamma$, we have $\gamma\neq\delta$, so the second term vanishes. Thus, $\pi_{\alpha\bar\alpha}=0$.

As a consequence, π_- vanishes, so $\varrho^{1,0}$ vanishes on U, and F_t maps $\widetilde U$ to a single leaf of $\mathscr{F}$. Hence, $\tilde f_t$ maps $\widetilde U$ to a totally geodesically embedded copy of X in $X_{\mathbf{C}}$. By continuity, it must map all of $\widetilde M$ to that copy of X. The stabilizer in $G_{\mathbf{C}}$ of any leaf of $\mathscr{F}$ is a conjugate on $G_{\mathbf{R}}$, and since $\tilde{f}_t$ is equivariant relative to ρ_t, the character of ρ_t must be a real point of $X(\Gamma, G_{\mathbf{C}})$ for all $|t| < \varepsilon$.

3. Rigidity of abelian subalgebras

The completion of the proof of the main result for several of the Hermitian symmetric domains is provided by a theorem of Mal′cev.

Theorem 3.1 (Mal′cev [8]). *Suppose G is $Sp(2n, \mathbf{R})$, $n \geq 2$, $SO^*(2n)$, $n \geq 2$, or the automorphism group of either of the exceptional bounded symmetric domains. Then $\mathbf{p}^{1,0}$ is locally rigid as an abelian subalgebra of $\mathbf{g}_{\mathbf{C}}$.*

Schur [13] proved the analogous result for $SU(n, n)$ and $SU(n, n+1)$, but we will not rely on his result, since the argument we give for $SU(m, n)$ will apply to these cases. Mal′cev actually characterizes the abelian subalgebras of maximal dimension in all the semisimple complex Lie algebras, showing that there is a unique conjugacy class of such subalgebras in each case. For many of the cases we must consider, $\mathbf{p}^{1,0}$ does not have maximal dimension, so Mal′cev's result does not apply.

To treat the remaining cases, we recall the statement of a theorem of Kostant [7], specialized to the case at hand. Let $\mathbf{c}$ be a Cartan subalgebra of $\mathbf{g}_{\mathbf{C}} = \mathbf{g} \otimes \mathbf{C}$, and let $\Phi \subset \mathbf{c}^*$ be a system of positive roots for $\mathbf{g}_{\mathbf{C}}$. Φ is a union of a set of compact roots, corresponding to the roots of $\mathbf{k} \otimes \mathbf{C}$ and denoted by Φ^c, and a set of noncompact roots, corresponding to a choice of $\mathbf{p}^{1,0}$ and denoted by Φ^n. We may assume that each positive noncompact root is larger than any compact root. We denote by λ the largest root in Φ, and by η we will mean

$$\frac{1}{2} \sum_{\alpha \in \Phi} \alpha.$$

Let W be the Weyl group of $\mathbf{g}_{\mathbf{C}}$ relative to this set of roots. W^1 is the subset of W containing any σ such that $\sigma(-\Phi) \cap \Phi$ contains only noncompact roots, while $W^1(i)$ will be the subset of elements in W^1 such that $\sigma(-\Phi) \cap \Phi$ has cardinality i. For $\sigma \in W^1(1)$, α_σ will be the unique element of $\sigma(-\Phi) \cap \Phi$, and ν_σ will be any nonzero element of the dual in $\mathbf{g}_{\mathbf{C}}^*$ of the root space corresponding to α_σ, under the identification of $\mathbf{g}_{\mathbf{C}}^*$ and $\mathbf{g}_{\mathbf{C}}$ given by the Killing form. For any $\sigma \in W^1(1)$, let $\xi_\sigma = \sigma(\eta + \lambda) - \eta$. It turns out that ξ_σ is in the weight lattice for $\mathbf{k}$.

Theorem 3.2 (Kostant [7]). *Let $H^1(\mathbf{p}^{1,0}, \mathbf{g}_{\mathbf{C}})$ be the first Lie algebra cohomology group of $\mathbf{p}^{1,0}$ with coefficients in the complexified adjoint representation of $\mathbf{g}$, and, for any ξ in the weight lattice of $\mathbf{k}$, let $H^1(\mathbf{p}^{1,0}, \mathbf{g}_{\mathbf{C}})^\xi$*

be the subspace which transforms under the natural action of $\mathbf{k}$ *as a direct sum of representations of* $\mathbf{k}$ *with highest weight* ξ.

Then $H^1(\mathbf{p}^{1,0}, \mathbf{g}_{\mathbf{C}})^{\xi}$ *is irreducible as a* $\mathbf{k}$*-module, and it is nontrivial if and only if* $\xi = \xi_\sigma$ *for some* $\sigma \in W^1(1)$. *Furthermore, the highest weight for the action of* $\mathbf{k}$ *on* $H^1(\mathbf{p}^{1,0}, \mathbf{g}_{\mathbf{C}})^{\xi_\sigma}$ *is represented by*

$$\nu_\sigma \otimes v_{\sigma\lambda},$$

where $v_{\sigma\lambda}$ *is a nonzero element of the root space for* $\sigma\lambda$.

We will interpret $H^1(\mathbf{p}^{1,0}, \mathbf{g}_{\mathbf{C}})$ as infinitesimal deformations of the homomorphism $\mathbf{p}^{1,0} \to \mathbf{g}_{\mathbf{C}}$ given by the inclusion, and apply the information from Kostant's theorem to show that, in the cases at hand, all of these infinitesimal deformations leave the image fixed.

Proposition 3.3. *Suppose the inclusion* $\imath: \mathbf{p}^{1,0} \to \mathbf{g}_{\mathbf{C}}$ *induces a surjection*

$$\imath_*: H^1(\mathbf{p}^{1,0}, \mathbf{p}^{1,0}) \to H^1(\mathbf{p}^{1,0}, \mathbf{g}_{\mathbf{C}}).$$

Then $\mathbf{p}^{1,0}$ *is locally rigid as an abelian subalgebra of* $\mathbf{g}_{\mathbf{C}}$.

Proof. We will construct a differential graded Lie algebra which governs the deformation theory of Lie subalgebras, in the sense of Nijenhuis and Richardson [11]. Let $C^\cdot$ be the graded Lie algebra obtained by tensoring $\mathbf{g}_{\mathbf{C}}$ with the graded algebra $\bigwedge^\cdot \mathbf{p}^{1,0,*}$, and $d: C^\cdot \to C^\cdot$ the differential of degree one which computes the Lie algebra cohomology of $\mathbf{p}^{1,0}$ with coefficients in $\mathbf{g}_{\mathbf{C}}$. Let $L^\cdot$ be the graded Lie algebra obtained by taking the semidirect product of $gl(\mathbf{p}^{1,0})$ with $C^\cdot$, where $gl(\mathbf{p}^{1,0})$ acts on $C^\cdot$ via the natural representation on $\bigwedge^\cdot \mathbf{p}^{1,0,*}$. Thus, $L^k = C^k$ for $k > 0$, while $L^0 = C^0 \oplus gl(\mathbf{p}^{1,0})$. Extend d to $L^\cdot$ by defining $d\xi \in \mathbf{p}^{1,0,*} \otimes \mathbf{p}^{1,0} \subset \mathbf{p}^{1,0,*} \otimes \mathbf{g}_{\mathbf{C}}$ to be the element representing $\xi \in gl(\mathbf{p}^{1,0})$. The verification that $(L^\cdot, d)$ is a differential graded Lie algebra is routine.

Elements ϕ of L^1 satisfying the deformation equation

$$R(\phi) := d\phi + \tfrac{1}{2}[\phi, \phi] = 0$$

are in bijective correspondence with Lie algebra homomorphisms from $\mathbf{p}^{1,0}$ to $\mathbf{g}_{\mathbf{C}}$; ϕ simply represents the difference between $\imath$ and another homomorphism. One has a natural action of $Gl(\mathbf{p}^{1,0}) \times G_{\mathbf{C}}$ on $L^\cdot$, and this action preserves $R^{-1}(0)$. Subalgebras of $\mathbf{g}_{\mathbf{C}}$ which can be represented as images of homomorphisms from $\mathbf{p}^{1,0}$ are in bijective correspondence with $R^{-1}(0)/Gl(\mathbf{p}^{1,0} \times G_{\mathbf{C}})$. Thus, by Theorem 23.4 of [11], there is a locally complete family of deformations of $\mathbf{p}^{1,0}$ as a subalgebra of $\mathbf{g}_{\mathbf{C}}$, and $\mathbf{p}^{1,0}$ is locally rigid if $H^1(L^{\cdot}, d) = 0$, by Theorem 22.1 of the same paper. But the one-cocycles are given by $Z^1(L^{\cdot}, d) = Z^1(C^{\cdot}, d)$, since $L^{\cdot}$ and $C^{\cdot}$ coincide as complexes in positive degrees, while the coboundaries are given by

$$B^1(L^{\cdot}, d) = B^1(C^{\cdot}, d) + \mathbf{p}^{1,0,*} \otimes \mathbf{p}^{1,0} = B^1(C^{\cdot}, d) + H^1(\mathbf{p}^{1,0}, \mathbf{p}^{1,0}).$$

Hence,

$$H^1(L^{\cdot}, d) = H^1(C^{\cdot}, d)/\iota_* H^1(\mathbf{p}^{1,0}, \mathbf{p}^{1,0}) = H^1(\mathbf{p}^{1,0}, \mathbf{g}_{\mathbf{C}})/\iota_* H^1(\mathbf{p}^{1,0}, \mathbf{p}^{1,0}).$$

q.e.d.

We now verify the assumption of the lemma for the remaining bounded symmetric domains.

Case 1. $G_{\mathbf{R}} = SU(m, n)$, $m, n \geq 2$.

Identify $\mathbf{c}^*$ with $\{(a_1, \cdots, a_{m+n}) | \sum a_i = 0, a_i \in \mathbf{C}\}$. For $i \neq j$, let $\alpha_{ij} \in \mathbf{c}^*$ be the element with $a_i = 1$, $a_j = -1$, and $a_k = 0$ for $k \neq i, j$. Then

$$\Phi = \{\alpha_{ij} | i < j\},$$
$$\Phi^c = \{\alpha_{ij} | 1 \leq i < j \leq m \text{ or } m+1 \leq i < j \leq m+n\},$$
$$\Phi^n = \{\alpha_{ij} | 1 \leq i \leq m, m+1 \leq j \leq m+n\}.$$

The Weyl group W is the symmetric group on $m+n$ letters, acting by permutation of the a_i. For any $\sigma \in W$, one has $\sigma(-\Phi) \cap \Phi \subset \Phi^n$ if and only if $\sigma^{-1}\alpha \notin -\Phi$ for all $\alpha \in \Phi^c$. Hence, if $1 \leq i < j \leq m$ or $m+1 \leq i < j \leq m+n$, we must have $\sigma^{-1}(i) < \sigma^{-1}(j)$. Thus, elements of W^1 are in bijective correspondence with increasing functions $f: \{1, \cdots, m\} \to \{1, \cdots, m+n\}$; f extends canonically to give σ^{-1} by assigning $\sigma^{-1}(m+k)$ to be the kth value not in the image of f. Furthermore, the cardinality of $\sigma(-\Phi) \cap \Phi$ is the number of pairs $i < j$ such that $\sigma^{-1}(i) > \sigma^{-1}(j)$. Thus, $W^1(1)$ consists of a single element σ, namely, the permutation which exchanges m and $m+1$ and leaves all other values fixed. Then $\alpha_\sigma = \alpha_{m,m+1}$, while $\lambda = \alpha_{1,m+n}$. Since $m, n \geq 2$, we have $\sigma\lambda = \lambda$, so $\nu_\sigma \otimes v_{\sigma\lambda}$ is in the image of the map from $H^1(\mathbf{p}^{1,0}, \mathbf{p}^{1,0})$. Since this map is a morphism of $\mathbf{k}$-modules, it must be surjective.

Note that, in case $G_{\mathbf{R}} = SU(1, n)$, we get that $\sigma\lambda \notin \mathbf{p}^{1,0}$, so that the map from $H^1(\mathbf{p}^{1,0}, \mathbf{p}^{1,0})$ is not surjective. In this case, there are nontrivial deformations of $\mathbf{p}^{1,0}$ as an abelian subalgebra of $\mathbf{g}_{\mathbf{C}}$.

Case 2. $G_{\mathbf{R}} = SO(2, 2n)$, $n > 1$. Identify $\mathbf{c}^*$ with $\{(a_1, \cdots, a_{n+1}) | a_i \in \mathbf{C}\}$. For $i \neq j$, let α_{ij} be as above, while β_{ij} is the element with $a_i = a_j = 1$ and $a_k = 0$ for $k \neq i, j$. Then

$$\Phi = \{\alpha_{ij}, \beta_{ij} | i < j\},$$
$$\Phi^c = \{\alpha_{ij}, \beta_{ij} | 1 < i < j\},$$
$$\Phi^n = \{\alpha_{1j}, \beta_{1j} | 1 < j < n+1\}.$$

The Weyl group W is generated by permutations and changes in the signs of any pair of entries. Thus, any $\sigma \in W$ factors as $\sigma = \sigma_s \sigma_p$, where σ_p is a permutation, and σ_s is a change in the sign of an even number of entries. Hence, $\sigma \in W^1$ implies that, for $1 < i < j$, we always have $\sigma_p^{-1}(i) < \sigma_p^{-1}(j)$, since otherwise σ^{-1} will convert either α_{ij} or β_{ij} into a negative root. Furthermore, for $1 < i < n+1$, σ_s^{-1} cannot change the sign of a_i, since otherwise σ^{-1} will convert any α_{ij} to a negative root. Thus, σ_p^{-1} is order-preserving when restricted to $\{2, \cdots, n+1\}$, and σ_s^{-1} is either the identity or changes the signs of the first and last entries since it must change an even number. If σ_s is the identity, then the cardinality of $\sigma(-\Phi) \cap \Phi$ is the number of pairs $i < j$ such that $\sigma^{-1}(i) > \sigma^{-1}(j)$. If $\sigma_p^{-1}(1) = k$, then this is $k - 1$, so the only contribution of this sort to $W^1(1)$ is the element which exchanges a_1 and a_2. In the second case, the cardinality of $\sigma(-\Phi) \cap \Phi$ is $2n + 1 - k$, where $\sigma_p^{-1}(1) = k$. We find that $2n + 1 - k = 1$ only when $k = 2n$, which is possible only when $n = 1$. Take σ to be the unique element of $W^1(1)$. Then $\alpha_\sigma = \alpha_{12}$, while $\lambda = \beta_{12}$. Again, $\sigma\lambda = \lambda \in \mathbf{p}^{1,0}$, so $\nu_\sigma \otimes v_{\sigma\lambda}$ is in the image of the map from $H^1(\mathbf{p}^{1,0}, \mathbf{p}^{1,0})$, and this map is surjective.

When $n = 1$, $G_{\mathbf{R}}$ is locally isomorphic to a product of two three-dimensional factors, so representations in $G_{\mathbf{R}}$ do not in general enjoy the local rigidity property under discussion.

For $G_{\mathbf{R}} = SO(2, 2n+1)$, the methods above can be used to show that $\mathbf{p}^{1,0}$ does admit nontrivial deformations as an abelian subalgebra of $\mathbf{g}_{\mathbf{C}}$, but we refrain from giving the details.

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