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A REMARKABLE SYMPLECTIC STRUCTURE

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Abstract

An explicit example of an exotic symplectic structure on R^4 is given.

1. Introduction

An old theorem due to Darboux [1] asserts that about each point x in any symplectic manifold (M, ω) there exists a neighborhood of x and a local chart (q^a, p_b) such that the symplectic form ω has the local representation

$$\omega = dq^a \wedge dp_a.$$

Naturally enough, one could try to make the domain of such a chart as large as possible. It can happen that one may not be able to enlarge the domain of such a symplectic chart beyond a certain size. This obstruction is of a geometric nature and has only come to light through recent work of Gromov [2].

To explain this, let ω_0 be the standard symplectic structure on R^{2n} and let $N \subset R^{2n}$ be any closed Lagrangian submanifold.

Theorem (*Gromov*). $[\omega_0] \neq 0$ *in* $H^2(R^{2n}, N; R)$, *the second relative de Rham cohomology group of the pair* (R^{2n}, N).

Being closed, ω_0 has a potential ψ on R^{2n}, i.e., $d\psi = \omega_0$. Furthermore, $[\psi|N] \neq 0$ in $H^1(N; R)$.

In this paper we explicitly endow a manifold M diffeomorphic to R^4 with a symplectic form ω admitting a Lagrangian torus T such that $[\omega] = 0$ in $H^2(M, T; R)$. But then Gromov's theorem tells us that (M, ω) does not symplectically embed in (R^4, ω_0). Thus ω is an exotic symplectic structure on M.

We note that the existence of exotic symplectic geometries was already known to Gromov [2], although the techniques used in the course of the proof do not permit explicit construction of an example.

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2. The idea of the construction

To start with, view R^4 as $R^3 \times R$ with coordinates (x, y, z, t), with (x, y, z) coordinates on the first factor and t the coordinate on the second factor in the splitting. Let ψ and χ be one-forms on R^3, and let ζ be the one-form

$$\zeta = \psi + t\chi,$$

viewed as a one-form on $R^3 \times R \approx R^4$. We would like ζ to be a potential for an exotic symplectic form. To guarantee this, we first suppose that T^2 is an embedded two-torus in R^3 (which we view as the $t = 0$ slice in R^4). Next, construct the one-form ψ so that it satisfies the following two properties:

1. The pull-back of ψ to the torus vanishes. In other words, the tangent plane to the torus is contained in the kernel of ψ at each point of the torus.
2. The exterior derivative of ψ nowhere vanishes.

The last part of the construction is to choose the one-form χ so that $\chi \wedge d\psi$ is a volume on R^3. It is possible to do this because R^3 is orientable and $d\psi$ never vanishes. For example, we may take

$$\chi = *d\psi,$$

where $*$ is the Hodge star operator with respect to the usual Euclidean volume on R^3. Now let B be an open ball in R^3 containing the torus. The claim is now that there is an interval I about $t = 0$ so that $\omega = d\zeta$ is a symplectic for on $B \times I$. To see that this must be so, we compute

$$\omega \wedge \omega = d\zeta \wedge d\zeta = -2\chi \wedge d\psi \wedge dt + O(t),$$

which shows that $\omega \wedge \omega$ is a volume if $|t|$ is small enough. Since ω is closed because it is exact, we may then conclude that ω is a symplectic form. By construction, ζ vanishes on the torus because ψ does, and because the torus is contained in the $t = 0$ slice. Since ζ vanishes on the torus, so does $\omega = d\zeta$, and this means that the torus is not only Lagrangian, but exact Lagrangian as well. Thus the relative class

$$[\omega] \in H^2(B \times I, T^2; R)$$

vanishes, and we may conclude that the symplectic form ω is exotic.

3. Details of the construction

Let R^4 be thought of as $R^2 \times R^2$ and let (r, θ) be polar coordinates on the first factor and (s, ϕ) polar coordinates on the second factor. That is,

$$\begin{aligned} y^1 &= r\cos\theta, \\ y^2 &= r\sin\theta, \\ y^3 &= s\cos\phi, \\ y^4 &= s\sin\phi. \end{aligned}$$

Define λ to be the smooth one-form on $R^2 \times R^2$ given by

$$\lambda = r^2 \cos r^2 \, d\theta + s^2 \cos s^2 \, d\phi.$$

Let S^3 be a three sphere of radius $\sqrt{\pi}$ defined by

$$r^2 + s^2 = \pi. \tag{1}$$

We note that λ vanishes only when $r^2 = s^2 = \pi/2$. The exterior derivative of λ is

$$\begin{aligned} d\lambda = {} & 2r(\cos r^2 - r^2 \sin r^2)\, dr \wedge d\theta \\ & + 2s(\cos s^2 - s^2 \sin s^2)\, ds \wedge d\phi. \end{aligned}$$

The first term in $d\lambda$ vanishes when $r^2 = \cot r^2$, and this equation has a unique solution r^* when r is between 0 and $\sqrt{\pi}$. Similarly, the second term only vanishes when $s^2 = \cot s^2$, and we denote the solution by s^* when s is between 0 and $\sqrt{\pi}$. If we restrict ourselves to the three-sphere, we see that λ vanishes only on the two-torus T^2 defined by

$$r^2 = \pi/2, \qquad s^2 = \pi/2.$$

Let i denote the embedding of the abstract three-sphere into $R^2 \times R^2$ given by (1). We now note that $i^* d\lambda \neq 0$ anywhere on S^3. In other words, $i^* d\lambda$ is a presymplectic form on S^3. To see that this must be so, consider the first term in $d\lambda$:

$$2r(\cos r^2 - r^2 \sin r^2)\, dr \wedge d\theta.$$

Since this vanishes when $r = r^*$, and we may use r, θ, and ϕ as local coordinates on S^3 about r^*, we need to check that the second term in $d\lambda$,

$$2s(\cos s^2 - s^2 \sin s^2)\, ds \wedge d\phi,$$

does not vanish when $r = r^*$. By (1), we may write this as

$$-2r(\cos(\pi - r^2) - (\pi - r^2)\sin(\pi - r^2))\, dr \wedge d\phi.$$

This only vanishes when

$$\pi - r^2 = \cot(\pi - r^2).$$

Suppose that this term vanishes when $r = r^*$. Then we would have that

$$\pi - r^2 = \cot(\pi - r^2) = -\cot r^2 = -r^2,$$

which implies that $\pi = 0$. As this is absurd, $i^* d\lambda$ does not vanish when $r = r^*$. By the symmetry of the terms r and s, we have that $i^* d\lambda$ does not vanish when $s = s^*$ either. To see that $i^* d\lambda$ does not vanish anywhere else, we note that away from $r = r^*$ and $s = s^*$ it follows that $d\lambda \wedge d\lambda$ is a volume on $R^2 \times R^2$, and hence that $i^* d\lambda$ cannot vanish for dimensional reasons. Thus we conclude that $i^* d\lambda$ is a presymplectic form on S^3.

If we let $\eta^2 = r^2 + s^2$, and set

$$\xi = *(d\lambda \wedge d\eta^2),$$

where $*$ is the Hodge star operator with respect to the standard volume on $R^2 \times R^2$, then

$$\begin{aligned}\xi = {} & (4\cos s^2 - s^2 \sin s^2)(y^2\, dy^1 - y^1\, dy^2) \\ & + 4(\cos r^2 - r^2 \sin r^2)(y^4 dy^3 - y^3\, dy^4).\end{aligned}$$

It is clear that $i^*\xi \wedge d\lambda$ is a volume on S^3.

Define a form of stereographic projection $p\colon S^3 \to R^3$ by

$$x^1 = \frac{y^1}{\sqrt{\pi} - y^4}, \qquad x^2 = \frac{y^2}{\sqrt{\pi} - y^4}, \qquad x^3 = \frac{y^3}{\sqrt{\pi} - y^4}.$$

Let ψ be the one-form $\psi = p^{-1^*}\lambda$, and let χ be the one-form $\chi = p^{-1^*}\xi$. From our previous considerations, we know that there is a ball containing the image of the torus and an interval about $x^4 = 0$ where the form $\zeta = \psi + x^4\chi$ is a symplectic potential because we have uniform bounds on ψ and χ on the ball. Thus we may conclude that $\omega = d\zeta$ is an exotic symplectic structure on a manifold diffeomorphic to R^4.

A straightforward computation yields the explicit expression

$$\begin{aligned}
\zeta = {} & \frac{4\pi}{(1+\rho^2)^2}\cos(r^2)(-x^2\,dx^1 + x^1\,dx^2) \\
& + \frac{2\pi}{(1+\rho^2)^4}\cos(s^2)((1-\rho^4)\,dx^3 + x^3(1+3\rho^2)\,d\rho^2) \\
& + x^4\left\{\frac{16\pi}{(1+\rho^2)^2}(\cos s^2 - s^2\sin s^2)(x^2\,dx^1 - x^1\,dx^2)\right. \\
& \left. - \frac{8\pi}{(1+\rho^2)^4}(\cos r^2 - r^2\sin r^2)[(1-\rho^4)\,dx^3 + x^3(1+3\rho^2)\,d\rho^2]\right\},
\end{aligned}$$

where

$$\rho^2 = (x^1)^2 + (x^2)^2 + (x^3)^2,$$

$$\rho^2 = 4\pi\frac{(x^1)^2 + (x^2)^2}{(1+\rho^2)^2},$$

$$s^2 = \pi\frac{4(x^3)^2 + (1-\rho^2)^2}{(1+\rho^2)^2}.$$

4. Other examples

A procedure similar to the one of the previous section, but done on

$$R^6 \approx R^2 \times R^2 \times R^2,$$

will produce an exotic symplectic structure on R^6. By taking suitable products, we can then produce an explicit example of an exotic symplectic structure on any R^{2n} for $n > 2$.

It is somewhat more interesting however, to see what happens if we look at the same construction, but choose a different value for η^2. For example, if we choose $\eta^2 = 2\pi$, then we have that the same argument produces an example with 'two' (presumably nonisotopic) disjoint exact Lagrangian tori. By 'two' we count with respect to the obvious toral momentum map, as we are unable to count isotopy classes of these tori. One may continue in this manner and produce examples with n disjoint (once again, presumably nonisotopic) exact Lagrangian tori. One can show that these examples are all different from volume considerations, but we do not know how to show that there is no diffeomorphism that makes them the same up to a constant multiple.

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References

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