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# FLOW OF NONCONVEX HYPERSURFACES INTO SPHERES

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## 0. Introduction

The flow of surfaces by functions of their principal curvatures has been intensively studied. It started with the work of Brakke [1], who used the formalism of geometric measure theory; a more classical approach had been chosen by Huisken, who looked at the so-called *inward flow* in [5]. The outward flow of surfaces (this term will be explained in the sequel) is scaling invariant and therefore more natural than the inward flow, producing more general results so far. Huisken [6] and Urbas [8] proved that the outward flow of *convex* surfaces by suitable functions of their principal curvatures converges to spheres. The convexity assumption is essential in their work.

In this paper we would like to present a different method for proving the convergence of star-shaped surfaces into spheres via the outward flow.

Let $f$ be a symmetric, positive function homogeneous of degree one being defined on an open cone $\Gamma$ of $\mathbf{R}^n$ with vertex in the origin, which contains the positive diagonal, i.e., all $n$-tuples of the form

(0.1) $$(\lambda, \cdots, \lambda), \qquad \lambda \in \mathbf{R}_+.$$

Assume that

(0.2) $$f \in C^0(\overline{\Gamma}) \cap C^2(\Gamma)$$

is *monotone*, i.e.,

(0.3) $$\frac{\partial f}{\partial \lambda^i} > 0, \qquad i = 1, \cdots, n, \ \text{in}\, \Gamma,$$

*concave*

(0.4) $$\frac{\partial^2 f}{\partial \lambda^i \partial \lambda^j} \leq 0,$$

and that

(0.5) $$f = 0 \quad \text{on}\ \partial\Gamma.$$

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We also use the normalization convention

$$f(1, \cdots, 1) = n. \tag{0.6}$$

Then we shall prove the following theorem.

**Theorem 0.1.** *Let $M_0$ be a compact, star-shaped $C^{2,\alpha}$-hypersurface of $\mathbf{R}^{n+1}$, which is given as an embedding*

$$x_0 : S^n \to \mathbf{R}^{n+1} \tag{0.7}$$

*whose principal curvatures are contained in $\Gamma$. Then the evolution equation*

$$\dot{x} = f^{-1}\nu, \qquad x(0) = x_0 \tag{0.8}$$

*on $S^n \times \mathbf{R}_+$, where $\nu$ is the outward unit normal of the surface $x(t)$ and $f$ is evaluated at the principal curvatures of $x(t)$, has a unique solution of class $C^{2,\alpha}$. The rescaled surfaces*

$$\tilde{x} = e^{-t/n}x \tag{0.9}$$

*converge exponentially fast to a uniquely determined sphere of radius $r_f$, which is estimated by*

$$\left(\frac{|M_0|}{|S^n|}\right)^{1/n} \le r_f, \tag{0.10}$$

*where $|M_0|$ and $|S^n|$ denote the n-dimensional measures of the corresponding surfaces. The lower bound is obtained iff $f$ is equal to the mean curvature function during the evolution process.*

## 1. A reformulation of the problem

Let the surface $M(t)$ be represented as a graph over $S^n$, i.e., the embedding vector $x = (x^\alpha)$ now has the components

$$x^{n+1} = u(x, t), \qquad x^i = x^i(t), \tag{1.1}$$

where the $(x^i)$ are local coordinates of $S^n$, and the notation $u = u(x, t)$ is slightly ambiguous. In other words, we have introduced polar coordinates.

Furthermore, let $\xi = (\xi^i)$ be a local coordinate system of $M(t)$. Then

$$u = u(x(\xi), t), \tag{1.2}$$

the outward unit normal in $(x, u)$ has the form

$$\nu = (\nu_\alpha) = v^{-1}(-D_i u, 1), \tag{1.3}$$

where

$$D_i u = \frac{\partial u}{\partial x^i}, \tag{1.4}$$

$$v = (1 + u^{-2}|Du|^2)^{1.2} = (1 + u^{-2}\sigma^{ij} D_i u D_j u)^{1/2}, \tag{1.5}$$

$(\sigma_{ij})$ being the metric of $S^n$ in the coordinates $(x^i)$. The euclidean metric now has the form

$$ds^2 = dr^2 + r^2 \sigma_{ij}\, dx^i\, dx^j. \tag{1.6}$$

As usual we denote by $(\sigma^{ij})$ the inverse of $(\sigma_{ij})$, and we agree to raise and lower indices only with respect to this metric.

The evolution equation (0.8) now yields

$$\frac{d}{dt} u = f^{-1} v^{-1}, \qquad \dot{x}^i = -f^{-1} v^{-1} D^i u \cdot u^{-2}, \tag{1.7}$$

from which we deduce

$$\frac{\partial}{\partial t} u = \frac{v}{f}. \tag{1.8}$$

We prefer to interpret this as an equation on $S^n \times \mathbf{R}_+$, i.e., to overlook the time-dependence of $(x^i)$, and rewrite it in more convenient notation as

$$\dot{u} - \frac{v}{f} = 0, \qquad u(0) = u_0. \tag{1.9}$$

Furthermore, let $M(\Gamma)$ be the class of all real $(n \times n)$-matrices the eigenvalues of which belong to $\Gamma$. Then, there is a natural way to define a function $F$ on $M(\Gamma)$:

$$F(a^{ij}) = f(\lambda^i), \tag{1.10}$$

where the $(\lambda^i)$ are the eigenvalues of $(a^{ij})$. It has been shown in [2] that the conditions (0.3) and (0.4) on $f$ now take the form

$$F_{ij} = \frac{\partial F}{\partial a^{ij}} \quad \text{is positive definite}, \tag{1.11}$$

and

$$F_{ij,rs} = \frac{\partial^2 F}{\partial a^{ij} \partial a^{rs}} \quad \text{is negative semidefinite}. \tag{1.12}$$

If we evaluate $f$ at the principal curvatures of a surface $M$, then we can define $F = F(h_{ij})$ as a function of the second fundamental form in any fixed coordinate system. But, if we use the covariant tensor notation of the second fundamental form, then we also have to take the metric $(g_{ij})$ of the surface into account.

It is better to consider the mixed tensor

$$h^i_j = g^{ik} h_{kj}, \tag{1.13}$$

the eigenvalues of which are precisely the principal curvatures.

For a graph $M$ over $S^n$ the metric has the components

$$g_{ij} = u_i u_j + u^2 \sigma_{ij} = u^2(\sigma_{ij} + \varphi_i \varphi_j), \tag{1.14}$$

where

$$\varphi = \log u; \tag{1.15}$$

the inverse is

$$g^{ij} = u^{-2}(\sigma^{ij} - \varphi^i \varphi^j / v^2), \tag{1.16}$$

where $v$ can be expressed as

$$v = (1 + |D\varphi|^2)^{2-1/2}. \tag{1.17}$$

The second fundamental form is given as

$$h_{ij} = \frac{u}{v}(\sigma_{ij} + \varphi_i \varphi_j - \varphi_{ij}), \tag{1.18}$$

where all derivatives are covariant derivatives with respect to the metric $(\sigma_{ij})$ of the sphere.

For the mixed tensor $h^i_j$ we obtain

$$h^i_j = \frac{1}{u \cdot v}\{\delta^i_j + [-\sigma^{ik} + \varphi^i \varphi^k / v^2]\varphi_{kj}\}. \tag{1.19}$$

It is not difficult to see that the symmetric tensor

$$\hat{h}_{ij} = \tfrac{1}{2}\{\tilde{\sigma}_{ik} h^k_j + \tilde{\sigma}_{jk} h^k_i\} \tag{1.20}$$

has the same eigenvalues (with respect to $\tilde{\sigma}_{ij}$) as the mixed tensor (1.19), where

$$\tilde{\sigma}_{ij} = \sigma_{ij} + \varphi_i \varphi_j. \tag{1.21}$$

Let us now define

$$\tilde{h}_{ij} = \frac{u}{v} \cdot \hat{h}_{ij} \tag{1.22}$$

and use the homogeneity of $F$. Then we conclude from (1.9)

$$\dot{\varphi} - \frac{1}{F(\tilde{h}_{ij})} = 0, \qquad \varphi(0) = \varphi_0. \tag{1.23}$$

This is a nonlinear parabolic equation on $S^n \times \mathbf{R}_+$, where the nonlinearity $F$ only depends on $D\varphi$ and $D^2\varphi$. Therefore, we deduce the existence of a solution on a maximal time interval $(0, T)$ with $T > 0$. Uniform estimates for $\varphi$, $\dot{\varphi}$, and $D\varphi$ on this time interval are now simple consequences of the maximum principal, although the equation is not yet known to be uniformly elliptic.

## 2. First order estimates

Let us first proof the following simple lemma.

**Lemma 2.1.** *If we regard $F(\tilde{h}_{ij})$ as a function depending on $D\varphi$ and $D^2\varphi$, then*

$$a^{ij} = -\partial F/\partial\varphi_{ij} \tag{2.1}$$

*is positive definite.*

*Proof.* First we remark that $\partial F/\partial\varphi_{ij}$ is a contravariant tensor, so that the notation $a^{ij}$ is justified. The positivity follows from (1.11), since

$$\tilde{h}_{ij} = v^{-2}(\sigma_{ij} + \varphi_i\varphi_j - \varphi_{ij}), \tag{2.2}$$

$$\frac{\partial \tilde{h}_{rs}}{\partial\varphi_{ij}} = -v^{-2}\delta^{ij}_{rs}, \tag{2.3}$$

and consequently

$$-\frac{\partial F}{\partial\varphi_{ij}} = -\frac{\partial F}{\partial\tilde{h}_{rs}} \cdot \frac{\partial\tilde{h}_{rs}}{\partial\varphi_{ij}} = v^{-2}F^{ij} \tag{2.4}$$

where

$$F^{ij} = \frac{\partial F}{\partial\tilde{h}_{ij}}. \tag{2.5}$$

**Lemma 2.2.** *Let $Q_T = S^n \times (0, T)$. Then the following estimate is valid in $Q_T$:*

$$\inf_{S^n}\varphi_0 \le \varphi - t/n \le \sup_{S^n}\varphi_0. \tag{2.6}$$

*Proof.* We only prove the upper estimate. Let $\lambda$ be a small parameter whose sign will be chosen later, and consider the function

$$\tilde{\varphi} = (\varphi - t/n)e^{\lambda t}. \tag{2.7}$$

If the supremum of $\tilde{\varphi}$ in the cylinder $Q_{\tilde{T}}$ for $0 < \tilde{T} < T$ is attained at a point $(x_0, t_0)$ with $0 < t_0 \leq \tilde{T}$, then we have at this point

$$D\tilde{\varphi} = 0, \qquad D^2\tilde{\varphi} \leq 0, \tag{2.8}$$

$$\dot{\tilde{\varphi}} \geq 0. \tag{2.9}$$

Hence, from (1.23) and the positivity of $(F^{ij})$ we deduce

$$0 \geq -\frac{e^{\lambda t}}{F} + \frac{e^{\lambda t}}{n} - \lambda\tilde{\varphi} \geq -\lambda\tilde{\varphi}, \tag{2.10}$$

in view of (2.8),

$$F(\tilde{h}_{ij}) \geq F(\sigma_{ij}) = n. \tag{2.11}$$

Now, we have to consider two cases separately. First let us assume, that

$$\sup_{S^n} \varphi_0 \geq 0. \tag{2.12}$$

Then we choose $\lambda < 0$, and deduce

$$\sup_{Q_{\tilde{T}}} \tilde{\varphi} = \sup_{S^n} \varphi_0. \tag{2.13}$$

Thus by letting $\lambda$ tend to zero, the estimate is proved because of the arbitrariness of $\tilde{T}$.

If $\sup_{S^n} \varphi_0$ is negative, then we choose $\lambda$ positive and deduce that on any cylinder $Q_{\tilde{T}}$, where

$$\sup_{Q_{\tilde{T}}} \tilde{\varphi} < 0, \tag{2.14}$$

we have

$$\sup_{Q_{\tilde{T}}} \tilde{\varphi} \leq \sup_{S^n} \varphi_0 < 0. \tag{2.15}$$

A simple continuity argument then leads to the final result.

To derive estimates for $\dot{\varphi}$, we differentiate (1.23) with respect to $t$ and obtain

$$\ddot{\varphi} + \frac{1}{F^2}\{-a^{ij}D_jD_j\dot{\varphi} + a^iD_i\dot{\varphi}\} = 0, \tag{2.16}$$

where

$$a^i = \frac{\partial F}{\partial \varphi_i}. \tag{2.17}$$

The maximum principle, modified as in the proof of Lemma 2.2, then yields

**Lemma 2.3.** *Let $\varphi$ be a solution of* (1.23) *on* $Q_T$. *Then*

$$\inf_{S^n} \dot{\varphi}(0) \le \dot{\varphi} \le \sup_{S^n} \dot{\varphi}(0). \tag{2.18}$$

Finally, let us estimate

$$w = \tfrac{1}{2}|D\varphi|^2. \tag{2.19}$$

By differentiating (1.23) with respect to the operator

$$D^k \varphi D_k \tag{2.20}$$

we obtain

$$\dot{w} + \frac{1}{F^2}\{-a^{ij} D_k(D_i D_j \varphi)\dot{D}^k \varphi + a^i D_i w\} = 0. \tag{2.21}$$

If we apply the rule for interchanging derivatives

$$\varphi_{ijk} = \varphi_{ikj} + R^m{}_{ijk}\varphi_m \tag{2.22}$$

and use the fact that on $S^n$

$$R_{mijk} = \sigma_{mj}\sigma_{ik} - \sigma_{mk}\sigma_{ij}, \tag{2.23}$$

we deduce

$$\dot{w} + \frac{1}{F^2}\{-a^{ij} D_i D_j w + |D\varphi|^2 a^i_i - a^{ij} D_i \varphi D_j \varphi + a^{ij} D_i D_k \varphi D_j D^k \varphi\} = 0.$$

Hence, we have proved

**Lemma 2.4.** *Let $\varphi$ be a solution of* (1.23) *on* $Q_T$. *Then*

$$|D\varphi|^2 \le \sup_{S^n} |D\varphi_0|^2. \tag{2.24}$$

As an immediate corollary we obtain

**Lemma 2.5.** *Let $F^{ij}$ be uniformly elliptic. Then the $a^{ij}$ are uniformly elliptic, and $|D\varphi|^2$ decays exponentially, or more precisely, there exists a positive constant $\lambda$, independent of $T$, such that the estimate*

$$|D\varphi|^2 e^{\lambda t} \le \sup_{S^n} |D\varphi_0|^2 \tag{2.25}$$

*is valid on* $Q_T$.

*Proof.* The uniform ellipticity of the $a^{ij}$ follows immediately from (2.4). Let $\mu$ be the smallest eigenvalue of the $a^{ij}$, and consider the function

$$\tilde{w} = w \cdot e^{\lambda t} \tag{2.26}$$

with some positive constant $\lambda$. Then $\tilde{w}$ satisfies the inequality

$$\dot{\tilde{w}} + \frac{1}{F^2}\{-a^{ij}D_iD_j\tilde{w} + (\mu - \lambda F^2)\tilde{w}\} \leq 0, \tag{2.27}$$

and hence the result provided

$$\lambda \leq \mu \cdot F^{-2}. \tag{2.28}$$

Let us finish this section with a comparison lemma:

**Lemma 2.6.** *Let $\varphi$ and $\tilde{\varphi}$ be two solutions of the initial value problem* (1.22) *with initial values $\varphi_0$ and $\tilde{\varphi}_0$ respectively. Then they satisfy the estimate*

$$\inf_{S^n}(\varphi_0 - \tilde{\varphi}_0) \leq \varphi - \tilde{\varphi} \leq \sup_{S^n}(\varphi_0 - \tilde{\varphi}_0) \tag{2.29}$$

*in their common domain of definition.*

The proof of the lemma is a modification of the proof of Lemma 2.1 and is omitted.

## 3. Second order estimates

So far we have not yet used the concavity of $F$. We shall need it to derive a priori estimates for the second fundamental form of the surfaces.

For this purpose it is also convenient to consider the original equation (0.8), which we shall write now in the form

$$\dot{x} = F^{-1} \cdot \nu, \tag{3.1}$$

and deal directly with the geometric quantities of the surfaces.

Let us first derive the evolution equations for the normal, the metric, and the second fundamental form.

**Lemma 3.1.** *The normal vector $\nu$ satisfies the evolution equation*

$$\dot{\nu} = F^{-2}\delta F = F^{-2}D^k F x_k. \tag{3.2}$$

In this section we use the notation $g_{ij}$ for the induced metric on the surfaces $M(t)$, covariant differentiation is always understood with respect to it, and the same observation applies for raising or lowering indices.

*Proof of Lemma* 3.1. The proof is identical to that of the corresponding result in [5, Lemma 3.3].

**Lemma 3.2.** *The metric $g_{ij}$ satisfies the equation*

$$\dot{g}_{ij} = \frac{2}{F}h_{ij}. \tag{3.3}$$

*Proof.* Compare the proof of [5, Lemma 3.2].

From this we deduce immediately

**Lemma 3.3.** *The volume element* $\sqrt{g}$ *satisfies*

$$\frac{d}{dt}\sqrt{g} = \frac{H}{F}\sqrt{g}, \tag{3.4}$$

*where* $H$ *is the mean curvature.*

Before we establish the evolution equation for $h_{ij}$, some preliminary remarks are in order. We recall the Gauss formula

$$x_{ij} = h_{ij}\nu, \tag{3.5}$$

the Weingarten equations

$$\nu_i = h_i^k x_k, \tag{3.6}$$

the Codazzi equations

$$h_{ij,k} = h_{ik,j}, \tag{3.7}$$

and the Gauss equations

$$R_{ijkl} = h_{jk}h_{il} - h_{il}h_{jk} \tag{3.8}$$

connecting the Riemann curvature tensor of hypersurfaces with its second fundamental form. We also indicate with a comma the start of covariant differentiation if the notation would become ambiguous otherwise.

We finally observe that by using the Codazzi and the Gauss equations, and the rule for interchanging the orders of derivatives, the following relation holds:

$$\begin{aligned} h_{rs,ij} &= h_{ir,sj} = h_{ir,js} + R^k{}_{rsj}h_{ik} + R^k{}_{isj}h_{kr} \\ &= h_{ij,rs} + (h_{ks}h_{rj} - h_{kj}h_{rs})h_i^k + (h_{ks}h_{ij} - h_{kj}h_{is})h_r^k. \end{aligned} \tag{3.9}$$

From the Gauss formula we deduce

$$h_{ij} = \langle x_{ij}, \nu\rangle, \tag{3.10}$$

where we can use ordinary derivatives instead of covariant derivatives, and hence

$$\dot{h}_{ij} = \langle (F^{-1}\nu)_{ij}, \nu\rangle - \langle x_{ij}, \dot{\nu}\rangle. \tag{3.11}$$

The last term is zero because of (3.2), and therefore

$$\begin{aligned} \dot{h}_{ij} &= F^{-2}F^{rs}h_{rs,ij} - 2F^{-3}F^{rs}h_{rs,i}F^{rs}h_{rs,j} \\ &\quad + F^{-2}F^{rs,lm}h_{rs,i}h_{lm,j} - F^{-1}\langle \nu_{ij}, \nu\rangle. \end{aligned} \tag{3.12}$$

We obtain, from the Weingarten equations,

$$\nu_{ij} = h^k{}_{ij}, x_k - h_i^k h_{kj} \nu, \tag{3.13}$$

and, from (3.9),

$$\begin{aligned} F^{-2}F^{rs}h_{rs,ij} &= F^{-2}F^{rs}h_{ij,rs} + F^{-2}F^{rs}h_{ks}h_i^k h_{rj} - F^{-1}h_{kj}h_i^k \\ &\quad + F^{-2}F^{rs}h_{ks}h_r^k h_{ij} - F^{-2}F^{rs}h_{is}h_{kr}h_j^k \\ &= F^{-2}F^{rs}h_{ij,rs} + F^{-2}F^{rs}h_{ks}h_r^k h_{ij} - F^{-1}h_{kj}h_i^k \\ &\quad + F^{-2}F^{rs}(h_{ks}h_{rj}h_i^k - h_{si}h_{kr}h_j^k), \end{aligned} \tag{3.14}$$

where we have used the homogeneity of $F$:

$$F = F^{ij}h_{ij}. \tag{3.15}$$

Combining these relations we deduce

**Lemma 3.4.** *The second fundamental form satisfies the evolution equation*

$$\begin{aligned} \dot{h}_{ij} &= F^{-2}F^{rs}h_{ij,rs} + F^{-2}F^{rs}h_{ks}h_r^k h_{ij} + F^{-2}F^{rs,lm}h_{rs,i}h_{lm,j} \\ &\quad - 2F^{-3}F^{rs}h_{rs,i}F^{rs}h_{rs,j} + F^{-2}F^{rs}(h_{ks}h_{rj}h_i^k - h_{si}h_{kr}h_j^k). \end{aligned} \tag{3.16}$$

For later purposes we need the evolution equation for the mixed tensor $h_i^i$ (no summation).

At first we notice that

$$\begin{aligned} \frac{d}{dt}(h_i^i) &= \frac{d}{dt}(g^{ki}h_{ki}) = \dot{g}^{ki}h_{ki} + g^{ki}\dot{h}_{ki} \\ &= -\frac{2}{F}h^{ki}h_{ki} + g^{ki}\dot{h}_{ki}. \end{aligned} \tag{3.17}$$

If we choose coordinates such that at a fixed point $g_{ik} = \delta_{ik}$, we deduce

$$\begin{aligned} \dot{h}_i^i = &-\frac{2}{F}h^{ki}h_{ki} + F^{-2}F^{rs}h_{ii,rs} + F^{-2}F^{rs}h_{ks}h_r^k h_{ii} \\ &+ F^{-2}F^{rs,lm}h_{rs,i}h_{lm,i} - 2F^{-3}F^{rs}h_{rs,i}F^{rs}h_{rs,i} \end{aligned} \tag{3.18}$$

since the last term in (3.16) vanishes. Taking the concavity of $F$ into account we conclude

$$\begin{aligned} \dot{h}_i^i \le &-\frac{2}{F}h_i^k h_k^i + F^{-2}F^{rs}h^i_{i,rs} + F^{-2}F^{rs}h_{ks}h_r^k h_i^i \\ &- 2F^{-3}F^{rs}h_{rs,i}F^{rs}h_{rs,i} \end{aligned} \tag{3.19}$$

at a point where $g_{ik} = \delta_{ik}$.

We shall now derive the evolution equation for the crucial term that controls the star-shapedness of a surface.

**Lemma 3.5.** *Let* $\varphi = \langle x, \nu \rangle^{-1}$. *Then we have*

$$\dot{\varphi} = F^{-2}F^{rs}\varphi_{rs} - 2\varphi^{-1} \cdot F^{-2}F^{rs}\varphi_r\varphi_s - F^{-2}F^{rs}h_{kr}h_s^k\varphi. \tag{3.20}$$

Before proving the lemma let us remark that $\varphi$ is always well defined since

$$\varphi = |x|^{-1} \cdot v, \tag{3.21}$$

where $v$ is the quantity defined in (1.17).

*Proof of Lemma* 3.5. We deduce from (3.2) that

$$\dot{\varphi} = -\varphi^2 F^{-1} - \varphi^2 F^{-2}F^{rs}h^k{}_{rs}, \ \langle x_k, x \rangle, \tag{3.22}$$

and from (3.13) that

$$\varphi_{rs} = -\varphi^2 h_{rs} + 2\varphi^{-1}\varphi_r\varphi_s + \varphi h_r^k h_{ks} - \varphi^2 h_{rs}^k, \ \langle x_k, x \rangle, \tag{3.23}$$

hence the result in view of (3.15).

We want to derive a priori estimates for the second fundamental form of the rescaled surfaces

$$\tilde{x} = x \cdot e^{-t/n}, \tag{3.24}$$

so let us remark that the right-hand side of the evolution equation (3.18) or (3.20) is a scaling invariant, i.e., the rescaled quantity $\tilde{h}_i^i$ or $\tilde{\varphi}$ satisfies the same equation with the additional term

$$\tfrac{1}{n}\tilde{h}_i^i \tag{3.25}$$

or

$$\tfrac{1}{n}\tilde{\varphi} \tag{3.26}$$

on the right-hand side.

Let $\tilde{\psi}$ be defined by

$$\tilde{\psi} = \sup\{\tilde{h}_{ij}\xi^i\xi^j : \tilde{g}_{ij}\xi^i\xi^j = 1\}. \tag{3.27}$$

Then we are able to prove

**Lemma 3.6.** *Let the evolution equation for the surfaces* $\tilde{M}$ *be defined in the maximal time interval* $(0, T)$. *Then the a priori estimate*

$$\tilde{\psi}\tilde{\varphi} \le \sup_{M_0} \tilde{\psi}(0)\tilde{\varphi}(0) \tag{3.28}$$

*is valid in* $(0, T)$.

*Proof.* As before, let $0 < \widetilde{T} < T$ be arbitrary and look at the point $\tilde{x}_0 = \tilde{x}(t_0)$ where

$$\sup_{0\le t\le \widetilde{T}} \sup_{\tilde{M}(t)} \tilde{\psi}\tilde{\varphi} \tag{3.29}$$

is attained. At this point $\tilde{\psi}$ can be expressed as

$$\tilde{\psi} = \tilde{h}_{ij}\eta^i\eta^j \tag{3.30}$$

with a certain unit vector $\eta$. We may now choose a Riemannian normal coordinate system $(\xi^i)$ such that at $\tilde{x}_0$ we have

$$(\eta^i) = (0, 0, \cdots, 1), \tag{3.31}$$

$$\tilde{g}_{ij} = \delta_{ij}. \tag{3.32}$$

In this coordinate system the surfaces $\widetilde{M}(t)$ are locally described as $\tilde{x} = \tilde{x}(\xi, t)$, where we may assume that $\tilde{x}_0 = \tilde{x}(0, t_0)$.

If we now define

$$w = \tilde{h}_{ij}\eta^i\eta^j / \tilde{g}_{ij}\eta^i\eta^j \tag{3.33}$$

for all $(\xi, t)$ in a neighbourhood of $(0, t_0)$, then we have for $t = t_0$

$$\dot{w} = \dot{\tilde{h}}{}^n_n, \tag{3.34}$$

$$w = \tilde{h}^n_n. \tag{3.35}$$

Furthermore, $w \cdot \tilde{\varphi}$ attains its maximum for $t$ less than $\widetilde{T}$ in $(0, t_0)$.

If we assume that $t_0$ is positive, then in view of the maximum principle from (3.19), (3.20), (3.25), and (3.26) we conclude that at $\tilde{x}_0$ the inequality

$$0 \le -\frac{2}{F}(\tilde{h}^n_n)^2\tilde{\varphi} + \frac{2}{n}\tilde{h}^n_n\tilde{\varphi} \tag{3.36}$$

is valid, where we also have used the fact that $\tilde{h}^n_n$ and $\tilde{\varphi}$ are nonnegative, and hence

$$n \cdot \tilde{h}^n_n \le F. \tag{3.37}$$

Obviously, $\tilde{h}^n_n$ is the largest principal curvature of $\widetilde{M}(t_0)$ at $\tilde{x}_0$, i.e., we have

$$\widetilde{H} \le F. \tag{3.38}$$

On the other hand, the opposite inequality

$$F \le \tilde{H} \tag{3.39}$$

is always valid due to the concavity of $F$ and our normalization (compare [8, Lemma 3.3]). Thus, we have a contradiction unless $\tilde{x}_0$ is an umbilic. But, by using the same trick as in the proof of Lemma 2.2, we can also overcome this obstacle, and prove the assertion.

Let us summarize what we have proved so far. From the results in §2 we know that $|\tilde{x}|$ and $\langle \tilde{x}, \nu \rangle$ and bounded from below and above by positive constants independent of $t$. Lemma 3.6 then yields an upper bound for largest principal curvature of $\widetilde{M}(t)$. Moreover, the estimate (3.39) shows that all principal curvatures have to be bounded independently of $t$.

The principal curvatures of $\widetilde{M}(t)$ therefore stay in a compact subset of $\overline{\Gamma}$. But because of the assumption (0.5) and the results of Lemmas 2.3 and 2.4 we conclude that they even stay in a compact subset of $\Gamma$ from which we deduce the important information.

**Lemma 3.7.** *During the evolution of $\widetilde{M}(t)$ the nonlinear operator $F(\tilde{h}_{ij})$ is uniformly elliptic.*

## 4. Convergence to a sphere

Let us now return to the setting and notation described in §§1 and 2. We shall assume that $F$ is concave, so that during the evolution process $\tilde{h}_{ij}$ or, equivalently, $\varphi_{ij}$ are uniformly bounded and $F(\tilde{h}_{ij})$ is uniformly elliptic.

Applying the known a priori estimates for a uniformly parabolic equation of the kind

$$\dot{\varphi} - 1/F = 0, \tag{4.1}$$

we first obtain, in addition to the estimates derived before,

**Lemma 4.1.** *$\dot{\varphi}$ and $D\varphi$ are Hölder continuous in $S^n \times (0, T)$. The Hölder norm is bounded independently of $T$.*

*Proof.* Differentiating (4.1) with respect to $t$ and $x^k$ gives a system of $(n+1)$ linear uniformly parabolic equations to which we apply the results in [7, Theorem 4.3.4].

Using now the concavity of $F$ and the trick of increasing the number of independent variables we obtain uniform a priori estimates for $D^2\varphi$ in $C^{0,\alpha}(Q_T)$.

**Theorem 4.2.** *The second derivatives of $\varphi$ are uniformly bounded in $C^{0,\alpha}(Q_T)$, where the estimate only depends on $|\varphi_0|_{2,\alpha}$, $n$, and the ellipticity constants of $F$, but not on $T$ and the second derivatives of $F$.*

For a proof we refer to [7, §5.5].

We therefore know that the $\widetilde{M}(t)$ viewed as graphs over $S^n$ are $C^{2,\alpha}$ surfaces with uniform bounds. If we would know that the original embedding $\tilde{x}(t)$ remains an embedding in the limit $t \to T$, we could deduce that $T = \infty$.

The only thing that could go wrong is that

$$\tilde{g}_{ij} = \langle \tilde{x}_i, \tilde{x}_j \rangle \tag{4.2}$$

could degenerate, i.e., its eigenvalues could approach zero or tend to infinity. But this cannot happen, since in view of Lemma 3.3

$$\tfrac{d}{dt}\sqrt{\tilde{g}} = (H/F - 1)\sqrt{\tilde{g}}, \tag{4.3}$$

where

$$0 \le H/F - 1 \le \text{const}. \tag{4.4}$$

Moreover, from (3.3) and the boundedness of the principal curvatures it follows that

$$\dot{\tilde{g}}_{ij} = \dot{g}_{ij} e^{-2t/n} - \frac{2}{n}\tilde{g}_{ij} = \frac{2}{F} h_{ij} e^{-2t/n} - \frac{2}{n}\tilde{g}_{ij} \le c \cdot \tilde{g}_{ij}, \tag{4.5}$$

and hence that

$$\tilde{g}_{ij}(t) \le \tilde{g}_{ij}(0) \cdot e^{c\cdot t}, \tag{4.6}$$

while (4.3) yields

$$\sqrt{\tilde{g}(t)} = \sqrt{g(0)} \exp\left\{ \int_0^t (H/F - 1) \right\}. \tag{4.7}$$

Thus, no degeneracy can develop in finite time.

Let us define

$$\tilde{u} = u e^{-t/n}. \tag{4.8}$$

The family $\tilde{u}(t)$ is uniformly bounded in $C^{2,\alpha}(S^n)$, and $D\tilde{u}$ decays exponentially fast. Using the well-known interpolation theorems we deduce that the second derivatives of $\tilde{u}$ decay exponentially fast. The rescaled second fundamental form $uh^i_j$ therefore satisfies the estimate

$$|uh^i_j - \delta^i_j| \le c \cdot e^{-\beta\cdot t} \tag{4.9}$$

with some $\beta > 0$, which furthermore yields

$$0 \le H/F - 1 \le c \cdot e^{-\beta t} \tag{4.10}$$

with some different constant $c$.

We can now prove the final assertions of Theorem 0.1. First, we note that $|\widetilde{M}(t)|$ is a Cauchy sequence if $t$ tends to infinity, since in view of (4.7) we have

$$(4.11)\quad |\widetilde{M}(t)| - |\widetilde{M}(t')| = \int_{S^n} \left[\exp\left\{\int_0^t (H/F - 1)\right\} - \exp\left\{\int_0^{t'} (H/F - 1)\right\}\right] \sqrt{g(0)}.$$

From this we conclude that $\widetilde{M}(t)$ converges exponentially fast to a sphere with a uniquely determined radius $r_F$, for subsequences always converge to spheres $S_r^n$ and their radius is given by

$$(4.12)\quad r^n \cdot |S^n| = \lim |\widetilde{M}(t)|.$$

Suppose now that $F = H$. Then from (4.3) we derive

$$(4.13)\quad r_H^n = |M_0|/|S^n|.$$

To prove the sharp estimate

$$(4.14)\quad r_H \leq r_F$$

with strict inequality unless $H = F$ for the surfaces, we integrate (4.7) to obtain

$$(4.15)\quad |\widetilde{M}(\infty)| - |M_0| = \int_{S^n} \left[\exp\left\{\int_0^\infty (H/F - 1)\right\} - 1\right] \sqrt{\tilde{g}(0)},$$

hence the result.

Finally, let us prove that even in the limit $\tilde{x}$ remains an embedding.

We use the evolution equation for the metric, and write it in the form

$$(4.16)\quad \dot{\tilde{g}}_{ij} = \frac{2}{F(\tilde{h}_{ij})} \cdot \tilde{h}_{ij} - \frac{2}{n}\tilde{g}_{ij}.$$

Let $\tilde{\kappa}$ be the largest principal curvature of $\widetilde{M}$ and set $\mu = |\tilde{x}| \cdot \tilde{\kappa}$. Then

$$(4.17)\quad \dot{\tilde{g}}_{ij} \leq \left\{\frac{2\mu}{F(|\tilde{x}| \cdot \tilde{h}_{ij})} - \frac{2}{n}\right\} \tilde{g}_{ij}.$$

The terms in the braces converge exponentially fast to zero, thus we obtain an upper estimate for the eigenvalues of $\tilde{g}_{ij}$. Hence the relation (4.7) shows that the smallest eigenvalue cannot tend to zero.


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