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ON THE GAUSS MAP AND TOTAL CURVATURE OF COMPLETE MINIMAL SURFACES AND AN EXTENSION OF FUJIMOTO'S THEOREM

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1. Introduction

Our object in this paper is to put into final form the connection between total curvature and the Gauss map of complete minimal surfaces in $\mathbf{R}^3$ and $\mathbf{R}^4$. The initial results were obtained by Osserman [11], [12] in 1963. We combine the methods used there with recent ideas of Fujimoto [5] and intermediate results of Hoffman and Osserman [7].

The inspiration for this paper was Fujimoto's Theorem [5]: *Let S be a complete minimal surface in* $\mathbf{R}^3$, *not a plane. Then the image of S under the Gauss map can omit at most four points on the unit sphere.*

There are a number of examples showing that the number "four" in this theorem is best possible (see [14, pp. 72–74]). An earlier paper by Xavier [15] was important in that it was the first to show that at most a finite number of points could be omitted. Using Xavier's approach, Earp and Rosenberg [3] were able to obtain results in the direction of those given here, for the case of surfaces in $\mathbf{R}^3$, under the assumption of finite topological type. However, Xavier's method does not seem able to yield the optimal results.

Fujimoto also provided an analog of his theorem for minimal surfaces in $\mathbf{R}^4$.

Among the results obtained in this paper are the following.

Theorem 1. *Let S be a complete minimal surface in* $\mathbf{R}^3$. *If the Gauss map g takes on five distinct values only a finite number of times, then S has finite total curvature.*

The behavior of the Gauss map for a complete minimal surface of finite total curvature is known in great detail [12]. In particular, the image under the Gauss map cannot omit more than three values. Thus, one consequence of the above theorem is a sharpening of Fujimoto's Theorem:

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Theorem 2. *Let S be a complete minimal surface in $\mathbf{R}^3$, not a plane. If the Gauss map g of S omits four points on the unit sphere, then every other point must be covered infinitely often.*

An examination of the various examples referred to above of surfaces whose Gauss map omits four points reveals that they exhibit precisely the behavior described in the theorem.

For a surface S in $\mathbf{R}^4$, the Gauss map is described by a pair of maps $g_k : S \to S_k$, where S_k is a sphere of radius $1/\sqrt{2}$. We have analogs of both Theorems 1 and 2 above. In particular:

Theorem 3. *Let S be a complete minimal surface in $\mathbf{R}^4$, not a plane. If each of the factors g_k of the Gauss map of S omits four distinct points, then each of the g_k must cover every other point infinitely often. If one of the g_k is constant, then the other must cover every point infinitely often with at most three exceptions.*

We shall give examples to show that both parts of the theorem are best possible.

In §2 we give a considerable simplification of Fujimoto's proof of a purely function-theoretic lemma used in the main theorems.

§§3 and 4 deal with surfaces in $\mathbf{R}^3$ and $\mathbf{R}^4$, respectively.

In §5 we summarize our results and show how they complete the description of the Gauss map of surfaces of finite and infinite total curvature in both $\mathbf{R}^3$ and $\mathbf{R}^4$.

2. A function-theoretic lemma

Lemma 2.1 (*Fujimoto* [5]). *Let $h(w)$ be analytic in $|w| < R$ and omit the points $\alpha_1, \alpha_2, \alpha_3, \alpha_4$. Let ϵ, ϵ' satisfy*

$$0 < 4\epsilon' < \epsilon < 1. \tag{2.1}$$

Then there is a positive constant B depending only on $\alpha_1, \cdots, \alpha_4, \epsilon, \epsilon'$ such that

$$\frac{(1+|h(w)|^2)^{(3-\epsilon)/2}|h'(w)|}{\prod_{j=1}^{4}|h(w)-\alpha_j|^{1-\epsilon'}} \leq B\frac{2R}{R^2-|w|^2}. \tag{2.2}$$

Proof. Let $d\sigma = \rho(z)|dz|$ be the metric of constant curvature -1 on the z-plane with the points $\alpha_1, \cdots, \alpha_4$ deleted; that is, $d\sigma$ is the pull-back of the Poincaré metric under the map F of the universal covering surface onto the unit disk. Since the map $h(w)$ can be composed with F to give a map of $|w| < R$ into the unit disk, the Schwarz-Pick Lemma, stating that noneuclidean lengths are reduced, takes the form

$$\rho(z)|dz| \leq \frac{2R}{R^2 - |w|^2}|dw|$$

or

$$\rho(h(w))|h'(w)| \leq \frac{2R}{R^2 - |w|^2}. \tag{2.3}$$

But the asymptotic behavior of $\rho(z)$ at each of the points α_j is well known [10, p. 250]:

$$\rho(z) \sim \frac{c_j}{|z - \alpha_j| \log|z - \alpha_j|}, \qquad c_j \neq 0 \ (j = 1, \cdots, 4),$$

and at $\alpha_5 = \infty$:[1]

$$\rho(z) \sim \frac{c_0}{|z| \log|z|}, \qquad c_0 \neq 0.$$

It follows that for any ϵ, ϵ' satisfying (2.1), the function

$$\frac{(1 + |z|^2)^{(3-\epsilon)/2}}{\rho(z) \prod_{j=1}^{4} |z - \alpha_j|^{1-\epsilon'}}$$

tends to zero at $\alpha_1, \cdots, \alpha_5$. Since it is positive and continuous everywhere else, it must have a positive maximum B. Thus

$$\frac{(1 + |h(w)|^2)^{(3-\epsilon)/2}}{\rho(h(w)) \prod |h(w) - \alpha_j|^{1-\epsilon'}} \leq B$$

and combining this with (2.3) proves the lemma.

3. Minimal surfaces in $\mathbf{R}^3$

We assume that all surfaces are orientable, since analogous theorems for nonorientable surfaces are easily formulated by taking the two-sheeted oriented covering surface and applying the theorems proved here.

An oriented minimal surface S in $\mathbf{R}^3$ may be described by a conformal immersion

$$X: M \to \mathbf{R}^3, \qquad X = (x_1, x_2, x_3),$$

where M is a Riemann surface. The functions

$$\varphi_k(\zeta) = \frac{\partial x_k}{\partial \xi} - i\frac{\partial x_k}{\partial_\eta}, \qquad \zeta = \xi + i\eta,$$

[1]There appears to be an incorrect sign in the reference [10, p. 250] for the asymptotic expansion at infinity.

are defined for any local conformal parameter ζ, and are holomorphic in ζ. (For the basic facts collected here, see [14, pp. 63–66].) We set

$$f(\zeta) = \varphi_1 - i\varphi_2, \qquad g(\zeta) = \frac{\varphi_3}{\varphi_1 - i\varphi_2};$$

then $g(\zeta)$ is independent of the choice of local parameter, and is in fact given by composing the Gauss map of S into the unit sphere with stereographic projection from the point $(0, 0, 1)$. If the Gauss map omits the point $(0, 0, 1)$, then f and g are both holomorphic and f is never zero.

Theorem 3.1. *Let S be a complete minimal surface in $\mathbf{R}^3$. If there are five distinct points on the unit sphere that are covered only finitely often by the Gauss map of S, then S must have finite total curvature.*

Proof. By a rotation of the surface in $\mathbf{R}^3$, we may assume that one of the five distinguished points is $(0, 0, 1)$.Then the hypothesis of the theorem implies that outside of a compact set D in S, the function g is analytic and omits certain values $\alpha_1, \alpha_2, \alpha_3, \alpha_4$. Let

$$S' = \{p \in S \backslash D\colon\ g' \neq 0 \text{ at } p\}.$$

Note that the value of $g'(\zeta)$ depends on the choice of local parameter ζ, but its vanishing does not. On S' we define a new metric

$$d\tilde{s}^2 = \left| \frac{f(\zeta)^{1/(1-p)} \prod_{j=1}^{4} (g(\zeta) - \alpha_j)^{p(1-\epsilon')/(1-p)}}{g'(\zeta)^{p/(1-p)}} \right| |d\zeta|^2, \tag{3.1}$$

where

$$0 < 4\epsilon' < \epsilon < 1 \quad \text{and} \quad p = 2/(3 - \epsilon). \tag{3.2}$$

It is easy to check that the above expression is independent of both the choice of local parameter ζ and the indeterminacy arising from fractional exponents. Since f and g are both analytic, the metric $d\tilde{s}^2$ is flat, and it can be smoothly extended over D. We thus obtain a metric on

$$S'' = S' \cup D$$

that is flat outside the compact set D. The key to our proof is showing that S'' is complete in that metric.

We proceed by contradiction. If S'' is not complete, then there is a divergent curve $\gamma(t)$ on S'' with finite length. By removing an initial segment, if necessary, we may assume that there is a positive distance d between the curve γ and the compact set D. Thus $\gamma\colon [0, 1) \to S'$, and since γ is divergent on S'', with finite length, it follows that from the point of view of S, either $\gamma(t)$ tends to a point where $g' = 0$, or else $\gamma(t)$ tends to the boundary of S as $t \to 1$. But the former case cannot occur, because if $g'(\zeta_0) = 0$, then $g(\zeta) \sim c(\zeta - \zeta_0)^m$, $m \geq 1$, and, by (3.2), $p/(1-p) = 2/(1-\epsilon)$; hence from (3.1)

$$d\tilde{s} \sim \frac{c'}{|\zeta - \zeta_0|^{2m/(1-\epsilon)}} > \frac{c'}{|\zeta - \zeta_0|^2}.$$

Thus

$$\int_0^1 d\tilde{s} = \infty,$$

contradicting the finite length of γ.

We conclude that $\gamma(t)$ must tend to the boundary of S when $t \to 1$. Choose t_0 such that

$$\int_{t_0}^1 d\tilde{s} < \frac{d}{3}; \tag{3.3}$$

that is, the length of $\gamma([t_0, 1))$ is less than $d/3$. Consider a small disk Δ with center $\gamma(t_0)$. Since $d\tilde{s}^2$ is flat, Δ is isometric to an ordinary disk in the plane. Let G be an isometry of $|w| < \eta$ onto Δ with $G(0) = \gamma(t_0)$. Extend G, as a local isometry into S', to the largest disk possible, say $|w| < R$. (Note that G may be viewed simply as the exponential map to S'' at $\gamma(t_0)$.) In view of (3.3), and the fact that γ is a divergent curve on S, we have $R \leq d/3$. Hence the image under G must be bounded away from D by a distance of at least $2d/3$. Thus, the reason that the map G cannot be extended to a larger disk must be that the image goes to the boundary of S''. Since the zeros of g' have been shown to be infinitely far away in the metric, the image must actually go the boundary of S. More specifically, there must be a point w_0 with $|w_0| = R$, such that the image under G of the line segment from 0 to w_0 is a divergent curve Γ on S. Our goal is to show that Γ has finite length in the *original* metric ds^2 on S, contradicting the completeness of the original surface.

Let $h = g \circ G$ be the Gauss map pulled back to the disk $|w| < R$. Since g omits the values $\alpha_1, \cdots, \alpha_5 = \infty$ on S' and since the image under G of $|w| < R$ lies in S', h must omit the same values. We are thus in exactly the situation covered by Lemma 2.1, and we adapt Fujimoto's argument to show that Γ has finite length.

First of all, since G is a local isometry, we have

$$d\tilde{s}^2 = |dw|^2. \tag{3.4}$$

But the expression (3.1) for $d\tilde{s}^2$ is, as we noted, independent of the choice of conformal parameter. We may therefore use w as our local parameter, and comparing (3.1) with (3.4) gives

$$\left| \frac{f(w) \prod_{j=1}^{4} (h(w) - \alpha_j)^{p(1-\epsilon')}}{h'(w)^p} \right| = 1$$

or

$$|f(w)| = \left| \frac{h'(w)}{\prod_{j=1}^{4} (h(w) - \alpha_j)^{1-\epsilon'}} \right|^p, \tag{3.5}$$

where again $h = g \circ G$ is just the representation of the Gauss map in terms of the parameter w.

We now denote by C the line segment from 0 to w_0, and by Γ the image of C on S. Then for the length L of Γ, we have

$$\begin{aligned} 2L &= \int_C |f(w)|(1 + |h(w)|^2)\,|dw| \\ &= \int_C \frac{(1 + |h(w)|^2)|h'(w)|^p}{\prod_{j=1}^{4} |h(w) - \alpha_j|^{p(1-\epsilon')}}\,|dw|, \end{aligned}$$

where the first equation is the standard expression for arc length in terms of the functions f and g (see [14, p. 65]) and the second equation follows from (3.5). We may now invoke Lemma 2.1, and recalling from (3.2) that $p = 2/(3 - \epsilon)$, we obtain

$$\begin{aligned} 2L &= \int_C \left[\frac{(1 + |h(w)|^2)^{(3-\epsilon)/2} |h'(w)|}{\prod_{j=1}^{4} |h(w) - \alpha_j|^{1-\epsilon'}} \right]^p |dw| \\ &\leq B^p \int_C \left[\frac{2R}{R^2 - |w|^2} \right]^p |dw| = \frac{(2B)^p}{R^{p-1}} \int_0^1 \frac{dt}{(1 - t^2)^p}. \end{aligned}$$

Since $p < 1$, L is finite.

To sum up, we have shown that if the surface S'' were not complete, then we could find a divergent curve on S with finite length in the original metric, so that S would not be complete. We therefore conclude that S'' is complete. Since the metric on S'' is flat outside of a compact set, we are in a familiar situation (see [12, p. 354], or [14, p. 81]. By a theorem of Huber [9], the fact that S'' has finite total curvature implies that S'' is finitely connected. We conclude first that g' can have only a finite number of zeros, and second, that the original surface S is finitely connected. Further, by [11, Theorem 2.1] (or the argument in [12, p. 354]) each annular end of S'', hence of S, is conformally equivalent to a punctured disk. Thus, the Riemann surface M on which S is based must be conformally equivalent to a compact Riemann surface $\overline{M}$ with a finite number of points removed.

In a neighborhood of each of those points, g is analytic and omits four values. By Picard's theorem, g cannot have an essential singularity, but must have at most a pole. Thus g extends to a meromorphic function on $\overline{M}$. That means that g maps $\overline{M}$ onto a compact branched covering surface of the unit sphere. If that covering surface has m sheets, then

$$\iint_S K\,dA = -4\pi m, \tag{3.6}$$

since the total curvature of S is just the negative of the area of the image of S under the Gauss map, counting multiplicities. This proves the theorem.

As we noted in the introduction, a complete minimal surface of finite total curvature whose Gauss map omits more than three points must be a plane [12, Theorem 3.3]. One consequence of Theorem 3.1 is therefore:

Theorem 3.2. *If the Gauss map to a complete minimal surface in* $\mathbf{R}^3$ *omits four points, then it must cover every other point infinitely often, unless the surface is a plane.*

A good example of the behavior described in the theorem is Scherk's surface, whose image under the Gauss map is the universal covering surface of the sphere minus four points [13, p. 572].

4. Minimal surfaces in $\mathbf{R}^4$

We shall use the following basic facts about minimal surfaces in $\mathbf{R}^4$ (see [12, §4] and [7]).

Let a minimal surface in $\mathbf{R}^4$ be given locally in isothermal parameters by

$$X(\zeta), \qquad X = (x_1, x_2, x_3, x_4),$$

and set

$$\varphi_k(\zeta) = \frac{\partial x_k}{\partial \xi} - i\frac{\partial x_k}{\partial \eta}, \qquad \zeta = \xi + i\eta, \tag{4.1}$$

$$f = \varphi_1 - i\varphi_2, \quad g_1 = \frac{\varphi_3 + i\varphi_4}{\varphi_1 - i\varphi_2}, \quad g_2 = \frac{-\varphi_3 + i\varphi_4}{\varphi_1 - i\varphi_2}. \tag{4.2}$$

Then the metric on the surface is given by

$$ds^2 = \lambda^2 |d\zeta|^2, \qquad \lambda^2 = \tfrac{1}{4}|f|^2(1 + |g_1|^2)(1 + |g_2|^2). \tag{4.3}$$

The functions g_1 and g_2 are independent of the choice of parameter ζ and may be interpreted as the two components of the Gauss map. They are both meromorphic, and each g_k may be viewed as a map onto a sphere S_k of radius $1/\sqrt{2}$ by stereographic projection followed by a similarity transformation. Then according to Blaschke's theorem [1] (see also [8, pp. 46–47]), the total curvature of the surface is the negative of the sum of the areas of the images on S_1 and S_2, counting multiplicities.

Theorem 4.1. *Let S be a complete minimal surface in $\mathbf{R}^4$. Suppose that for each of the components g_k of the Gauss map, there are four distinct points covered at most a finite number of times, and for one of the g_k there is a fifth point covered at most a finite number of times. Then S has finite total curvature.*

Proof. The argument is analogous to that used in Theorem 3.1. Our goal is to show that each of the g_k extends to a meromorphic function on a compact Riemann surface. The conclusion then follows by Blaschke's theorem.

To begin, we let S be given by a conformal immersion

$$X\colon M \to \mathbf{R}^4,$$

where M is a Riemann surface. By hypothesis, outside a compact set D on M, one of the g_k, say g_1 omits five points, while the other one, g_2, omits four points. By a rotation in $\mathbf{R}^4$, we can assume that the omitted points include $g_1 = \infty$ and $g_2 = \infty$ (see Lemma 4.6 below). Let g_1 omit in addition $\alpha_1, \cdots, \alpha_4$ and g_2 omit $\beta_1, \cdots, \beta_3$.

Assume first that neither g_1 nor g_2 is constant. Then the zeros of g_1' and g_2' are isolated. Let

$$S' = \{p \in S\backslash D\colon\, g_1' \neq 0,\ g_2' \neq 0 \text{ at } p\}.$$

On S' we define a new metric

$$d\tilde{s}^2 = \left| \frac{f(\zeta)\prod_{j=1}^{4}(g_1(\zeta)-\alpha_j)^{p(1-\epsilon')}\prod_{k=1}^{3}(g_2(\zeta)-\beta_k)^{q(1-\epsilon')}}{g_1'(\zeta)^p g_2'(\zeta)^q} \right|^{\frac{2}{1-r}} |d\zeta|^2, \tag{4.4}$$

where f, g_1, g_2 are defined in terms of the local conformal parameter ζ by (4.2), ϵ and ϵ' satisfy

$$0 < 4\epsilon' < \epsilon < 1 \tag{4.5}$$

and

$$1/(2-\epsilon) + 1/(3-\epsilon) < 1, \tag{4.6}$$

and

$$p = 1/(3-\epsilon), \quad q = 1/(2-\epsilon), \quad r = p + q. \tag{4.7}$$

As before, we may verify that this metric is independent of the parameter ζ, and is flat. The proof now runs exactly parallel to the proof of Theorem 3.1. We find a local isometry G of a disk $|w| < R$ into S' with the metric $d\tilde{s}^2$, so that

$$d\tilde{s}^2 = |dw|^2.$$

It follows from (4.4) that

$$|f(w)| = \left| \frac{h_1'(w)^p h_2'(w)^q}{\prod_{j=1}^4 (h_1(w) - \alpha_j)^{p(1-\epsilon')} \prod_{k=1}^3 (h_2(w) - \beta_k)^{q(1-\epsilon')}} \right|, \tag{4.8}$$

where $h_k(w) = g_k(G(w))$. Thus Lemma 2.1 applies to $h_1(w)$, and the same reasoning as in Lemma 2.1 shows that

$$\frac{(1 + |h_2(w)|^2)^{(2-\epsilon)/2} |h_2'(w)|}{\prod_1^3 |h_2(w) - \beta_k|^{1-\epsilon'}} \leq B_2 \left(\frac{2R}{R^2 - |w|^2} \right). \tag{4.9}$$

If C is a line segment from 0 to w_0, with $|w_0| = R$, then the length L of the image Γ of C in the original metric on the surface is given according to (4.3) by

(4.10)

$$\begin{aligned} 2L &= \int_C |f(w)| \sqrt{1 + |h_1(w)|^2} \sqrt{1 + |h_2(w)|^2} \, |dw| \\ &= \int_C \left| \frac{(1 + |h_1(w)|^2)^{\frac{3-\epsilon}{2}} h_1'(w)}{\prod_{j=1}^4 (h_1(w) - \alpha_j)^{1-\epsilon'}} \right|^p \left| \frac{(1 + |h_2(w)|^2)^{\frac{2-\epsilon}{2}} h_2'(w)}{\prod_{k=1}^3 (h_2(w) - \beta_k)^{1-\epsilon'}} \right|^q |dw| \\ &\leq B^p B_2^q \int_C \left(\frac{2R}{R^2 - |w|^2} \right)^r |dw|, \end{aligned}$$

where we have used (4.8), (4.7), (2.2), and (4.9). But by (4.6) and (4.7), $r < 1$, and therefore the last integral converges. Thus Γ has finite length in the original metric on S. But by exactly the same argument as in the proof of Theorem 3.1, if there is a divergent curve on S with finite length in the flat metric, then there is a maximal disk $|w| < R$ and a point w_0 with $|w_0| = R$, such that radius C from 0 to w_0 maps onto a divergent curve Γ. Since, by (4.10), the length on Γ is finite, this contradicts the completeness of S. We may therefore repeat the argument to show that M is conformally a compact surface $\overline{M}$, with a finite number of points removed, and by Picard's theorem, both g_1 and g_2 extend to meromorphic functions on $\overline{M}$. Hence, as indicated at the outset, S has finite total curvature.

We must also consider the case where g_1 is constant. Then, if g_2 takes on the values $\beta_1, \cdots, \beta_4 = \infty$ only a finite number of times, we may repeat the argument, using the metric

$$d\tilde{s}^2 = \left| \frac{f(\zeta) \prod_{k=1}^{3} (g_2(\zeta) - \beta_k)^{q(1-\epsilon')}}{g_2'(\zeta)^q} \right|^{2/(1-q)} |d\zeta|^2,$$

where again, $0 < 4\epsilon' < \epsilon < 1$, $q = 1/(2-\epsilon) < 1$, and we insert $h_1 =$ constant and (4.9) in the first line of (4.10). The remainder of the proof is the same.

We next recall what is known about the omitted set for complete surfaces of finite total curvature:

Proposition 4.2 [7, p. 99]. *Let S be a complete minimal surface in $\mathbf{R}^4$ with finite total curvature. If both g_1 and g_2 omit four points, or if one of them is constant and the other omits three points, then S is a plane.*

We note that despite the superficial resemblance between this result and Theorem 4.1, the proofs are unrelated, and the common role of the number "four" seems pure coincidence. Nevertheless, combining the two statements leads to the following theorem.

Theorem 4.3. *Let S be a complete minimal surface in $\mathbf{R}^4$, not a plane. If the two components of the Gauss map, g_1 and g_2, both omit four points, then they must both take on every other value infinitely often. If one of them is constant and the other omits three points, then it must take on every other value infinitely often.*

We show by examples that both parts of this theorem are best possible.

Example 4.4. Let

$$f(\zeta) = \frac{1}{\prod_1^3 (\zeta - \alpha_j)}, \qquad g_1(\zeta) = g_2(\zeta) = \zeta.$$

Setting

$$(4.11) \quad (\varphi_1, \varphi_2, \varphi_3, \varphi_4) = \tfrac{1}{2} f(1 + g_1 g_2, i(1 - g_1 g_2), g_1 - g_2, -i(g_1 + g_2))$$

and

$$(4.12) \qquad x_k = \Re \int \varphi_k(\zeta)\, d\zeta, \qquad k = 1, 2, 3, 4,$$

defines a minimal immersion

$$X \colon M \to \mathbf{R}^4,$$

where M is the universal covering surface of the plane with the points $\alpha_1, \alpha_2, \alpha_3$ removed. Furthermore, f, g_1, g_2 may be recovered from the φ_k by equations (4.2), so that g_1, g_2 are precisely the components of the Gauss map for this surface. Using the expression (4.3) for the metric on the surface, it is easy to verify that all divergent curves have infinite length. Hence it is a complete surface, and both g_1 and g_2 omit the points $\alpha_1, \cdots, \alpha_4 = \infty$.

Example 4.5. Let

$$f(\zeta) = \frac{1}{\prod_{j=1}^{2}(\zeta - \alpha_j)}, \quad g_1(\zeta) \equiv 0, \quad g_2(\zeta) = \zeta.$$

Then again using the representation (4.11), (4.12), we obtain a complete minimal surface for which g_1 is constant and g_2 omits three points.

Finally, we turn to a point in the proof of Theorem 4.1 that we feel deserves some amplification: the assumption that omitted points can be chosen to include $g_1 = \infty$ and $g_2 = \infty$. It is worth examining that assumption more closely, since in the process we obtain a clearer picture of the hypothesis of Theorem 4.1.

We first note that a condition such as "g_1 omits the value α" is equivalent to "the Gauss map omits all points of the form (α, w_2) for $w_2 \in S_2$" or that "$g(S) \cap (\{\alpha\} \times S_2) = \varnothing$". Similarly, "$g_2$ omits β" is equivalent to "$g(S) \cap (S_1 \times \{\beta\}) = \varnothing$". Thus, "$g_1$ omits α and g_2 omits β" is equivalent to "$g(S)$ does not intersect $(\{\alpha\} \times S_2) \cup (S_1 \times \{\beta\})$". But the representation of the quadric Q_2 as $S_1 \times S_2$ has the property that *the set* $(\{\alpha\} \times S_2) \cup (S_1 \times \{\beta\})$ *is precisely the intersection with* Q_2 *of the tangent plane to* Q_2 *at the point* (α, β) (see [6, p. 17–22]). Thus, the hypothesis that g_1 omits certain values α_j and g_2 omits β_k is equivalent to the statement that the image of S under the Gauss map fails to intersect a certain set of hyperplanes: the tangent hyperplanes to Q_2 at the points (α_j, β_k). In this form, Proposition 4.2 and Theorem 4.3 are reminiscent of earlier results to the effect that if the Gauss image $\hat{S}$ of a complete minimal surface S in $\mathbf{R}^4$ is nondegenerate (i.e., does not lie in a hyperplane), then $\hat{S}$ cannot fail to intersect more than N hyperplanes in general position, where $N = 9$ if S has finite total curvature (Chern-Osserman [2]) and $N = 16$ in general (Fujimoto [4]).

The hypothesis of Theorem 4.3 may be reformulated: the image of S under the Gauss map fails to intersect the 16 hyperplanes tangent to the quadric at the points (α_j, β_k), $j, k = 1, 2, 3, 4$. However, since the three tangent hyperplanes at the points (α_1, β_1), (α_1, β_2), (α_1, β_3) all contain the complex line $\alpha_1 \times S_2$, these planes are not in general position; so the older results do not apply.

Lemma 4.6. *Let* S *be a surface in* $\mathbf{R}^4$, *and* g_1, g_2 *the components of its Gauss map. If* g_1 *omits certain points* α_j, *and* g_2 *omits points* β_k, *then after a rotation in* $\mathbf{R}^4$, *we can achieve that* $g_1 \neq \infty$, $g_2 \neq \infty$.

Proof. As noted above, the fact the g_1 omits α_1 and g_2 omits β_1 is equivalent to the statement that the Gauss map omits the tangent plane to the quadric Q_2 at the point (α_1, β_1). If we make a rotation in $\mathbf{R}^4$ that carries the 2-plane corresponding to (α_1, β_1) into the 2-plane corresponding to (∞, ∞), then the induced map in $\mathbf{CP}^3$ is a linear transformation that will carry the tangent plane to Q_2 at (α_1, β_1) onto the tangent plane at (∞, ∞). If the Gauss map to the original surface omits the former tangent plane, then after the rotation the Gauss map will omit the latter. This proves the lemma.

5. Concluding remarks

As stated in the introduction, we are now in a position to give essentially definitive results for the relations between the Gauss map and total curvature of surfaces in $\mathbf{R}^3$ and $\mathbf{R}^4$.

Theorem 5.1. *Let S be a complete minimal surface in $\mathbf{R}^3$ and g its Gauss map. Then*

I. *The total curvature of S is zero $\Leftrightarrow$ g is constant $\Leftrightarrow$ S is a plane.*

II. *The total curvature S is finite but not zero $\Leftrightarrow$ $\iint_S K\,dA = -4\pi m$ for some positive integer m $\Leftrightarrow$ g is a nonconstant meromorphic function on a compact surface with a finite number of points removed $\Leftrightarrow$ g takes on every value precisely m times with a finite number of exceptions and omits at most three values altogether.*

III. *The total curvature is infinite $\Leftrightarrow$ g takes on every value infinitely often with at most four exceptions.*

The only remaining question in this theorem is whether "three" is the best possible number for the omitted points in the finite total curvature case. The example of the catenoid shows that two points can be omitted, so that the correct value is either two or three.

The story in $\mathbf{R}^4$ is a bit more complicated because the Gauss map has two components. Nevertheless, it is now equally well understood.

Theorem 5.2. *Let S be a complete minimal surface in $\mathbf{R}^4$ and let g_1, g_2 be the components of its Gauss map. Then*

I. *The total curvature of S is zero $\Leftrightarrow$ g_1 and g_2 are both constant $\Leftrightarrow$ S is a plane.*

II. *The total curvature is finite but not zero $\Leftrightarrow$ $\iint_S K\,dA = -2\pi m$, for some positive integer m $\Leftrightarrow$ each g_k is a meromorphic function of order m_k on a compact surface with a finite number of points removed, with $m = m_1 + m_2$ $\Leftrightarrow$ each g_k takes on every value m_k times with a finite number of exceptions, with $m = m_1 + m_2$, and either: one of the g_k is constant and the other omits at most two values or: neither g_k is constant and one of them omits at most three values.*

III. *The total curvature of S is infinite $\Leftrightarrow$ either g_1 or g_2 takes on every value infinitely often with at most four exceptions.*

Finally, we note that after this work was completed we received a preprint from Fujimoto [6] extending his earlier results to yield a Nevanlinna-type value distribution theory for the Gauss map. The methods and results, however, are quite different from those of the present paper.

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