J. DIFFERENTIAL GEOMETRY
31 (1990) 547-599

CASSON'S INVARIANT AND GAUGE THEORY

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Recently, Andrew Casson [7] (see [2] or [16]) defined an integer valued invariant for closed 3-manifolds with the homology of S^3. Casson gave a topological definition of his invariant. An analytic definition of Casson's invariant is the subject of this article. Roughly speaking, Casson's invariant can be defined using gauge theory as an infinite dimensional generalization of the classical Euler characteristic.

The article begins below with an introduction to the relevant geometry of the space of connections on a homology 3-sphere. In §2, the definition of an integer valued invariant of an oriented homology 3-sphere is given. The construction of Casson's integer valued invariant is reviewed in §3 where the main theorem is stated: These two invariants are equal. §§4–8 are occupied with the proof of the main theorem. There is also a technical appendix.

1. Gauge theory in 3 dimensions

The new definition of Casson's invariant requires some basic facts from gauge theory (connections, curvature and covariant derivatives); the reader is referred to [11] and [15] for these definitions. Related material is present in the recent work of Andreas Floer [10].

Fix an oriented, closed 3-manifold, M, with the homology of S^3. Every principal SU(2) bundle over M is isomorphic to the trivial bundle, $P \equiv M \times \mathrm{SU}(2)$. It is convenient (though not necessary) to fix a trivialization of P and the associated product connection, Γ.

The space of smooth connections on P, $\mathscr{A} \equiv \mathscr{A}(P)$, is an affine space; the choice of Γ gives an affine isomorphism of $\mathscr{A}$ with $\Omega^1 \times \mathrm{su}(2)$. Here, su(2) is the Lie algebra of SU(2), and Ω^p ($p = 0, 1, 2, 3$) is the space of smooth p-forms on M. Use the L_1^2-inner product on $\Omega^1 \times \mathrm{su}(2)$ to define

Received April 19, 1988 and, in revised form, December 21, 1988. The author was supported in part by the National Science Foundation.

$\mathscr{A}$ as a smooth manifold, modelled on a (pre-) Hilbert space (see, e.g. [17).]

With the product structure fixed, the group $\mathscr{G}$ of smooth automorphisms of P is identical to $C^\infty(M; \mathrm{SU}(2))$. It acts on $\mathscr{A}$ in the usual way (as $(g, A) \to gA \equiv g \cdot A \cdot g^{-1} + g \cdot dg^{-1}$); let $\mathscr{B} \equiv \mathscr{A}/\mathscr{G}$ with the quotient topology. Let $\mathscr{R} \subset \mathscr{A}$ denote the space of reducible connections. The space $\mathscr{R}$ has infinite codimension in $\mathscr{A}$, and the group $\mathscr{G}$ acts on $\mathscr{A}^\# \equiv \mathscr{A}\backslash\mathscr{R}$ with stabilizer ± 1. The stabilizer of $\mathscr{G}$ on $\mathscr{R}$ is 1-dimensional except for connections in the $\mathscr{G}$-orbit of Γ, where the stabilizer is SU(2). Set $\mathscr{B}^\# \equiv (\mathscr{A}^\#)/\mathscr{G}$. Think of $\mathscr{B}^\#$ as an infinite dimensional manifold which is modeled on a pre-Hilbert space by using the L_1^2-theory in [17]. This manifold structure makes the projection from $\mathscr{A}^\# \to \mathscr{B}^\#$ a principal $\mathscr{G}$-bundle. The reader is referred to [17] for the details.

The curvature of a connection A is the su(2)-valued 2-form $F_A = dA + A \wedge A$. Because M is 3-dimensional, the assignment of a connection to its curvature can be interpreted as defining a 1-form on $\mathscr{A}$ which has been pulled up from $\mathscr{B}$ via the projection. Indeed, a 1-form on $\mathscr{A}$ assigns to a connection, A, a homomorphism from $T\mathscr{A}|_A$ ($\equiv \Omega^1 \times \mathrm{su}(2)$) into the real numbers. Define such a homomorphism by sending $a \in \Omega^1 \times \mathrm{su}(2)$ to

$$\mathscr{f}_A(a) \equiv \int_M \mathrm{tr}(a \wedge F_A). \tag{1.1}$$

Here, $\mathrm{tr}(\cdot)$ is a fixed, Ad-invariant trace on su(2).

The $\mathscr{G}$-invariance of the 1-form $\mathscr{f}$ is guaranteed by the $\mathscr{G}$-equivariance of the curvature ($F_{gA} = g \cdot F_A \cdot g^{-1}$). The Bianchi identity insures that $\mathscr{f}_A(\cdot)$ annihilates the tangent space to the $\mathscr{G}$ orbit through A. (This vector subspace of $T\mathscr{A}|_A$ is $\{d_A\varphi = d\varphi + [A, \varphi] : \varphi \in \Omega^0 \times \mathrm{su}(2)\}$, while the Bianchi identity says that $d_A F_A \equiv dF_A + A \wedge F_A + F_A \wedge A = 0$.) Thus, one can think of $\mathscr{f}$ as being the pull-back of a 1-form (also denoted by $\mathscr{f}$) on $\mathscr{B}$.

As an aside, note that $\mathscr{f}$ is the exterior derivative of the Chern-Simons functional which assigns to A the real number

$$\mathscr{c}(A) \equiv \int_M \mathrm{tr}(A \wedge F_A - 2/3 \cdot A \wedge A \wedge A). \tag{1.2}$$

(Atiyah-Patodi-Singer [5] make this observation.)

The Chern-Simons functional does not descend to $\mathscr{B}$ as a function. This is because $\mathscr{c}(gA) = \mathscr{c}(A) + \alpha \cdot \mathrm{degree}(g)$. Here, $\alpha > 0$ is a real constant, and degree(g) is the degree of the automorphism g as a map from M to $\mathrm{SU}(2) = S^3$. Indeed $\pi_0(C^\infty(M; \mathrm{SU}(2))) \approx \mathbb{Z}$, and $\mathscr{c}(gA) = \mathscr{c}(A)$ only for g in the connected component of the constant map.

One can show that the projection from $\mathscr{A}^\#$ to $\mathscr{B}^\#$ is a principal fibration [17]. Thus, $\pi_1(\mathscr{B}^\#) \approx H^1(\mathscr{B}^\#) \approx \mathbb{Z}$, and $\mathscr{c}(\cdot)/\alpha$ descends to $\mathscr{B}^\#$ as a map into S^1 which generates H^1. This identifies the 1-form $\mathscr{f}$ as a generator on $H^1(\mathscr{B}^\#; \mathbb{Q})$.

Note that the zeros of $\mathscr{f}$ on $\mathscr{B}$ are precisely the $\mathscr{G}$-orbits of the flat connections on $M \times \mathrm{SU}(2)$. That is, $\mathscr{f}^{-1}(0) \equiv \{[A] \in \mathscr{B}: F_A \equiv 0\}$. A classical theorem in differential geometry asserts that this set is identical to the set $\mathrm{Hom}(\pi_1(M); \mathrm{SU}(2))/\mathrm{SU}(2)$ where SU(2) acts on $\mathrm{Hom}(\cdot)$ by conjugation.

For a 1-form defined on a compact, oriented manifold, there is a standard way to obtain an invariant from the set of zeros; this invariant is the Euler characteristic of the manifold. Recall that to compute the Euler characteristic in this situation, use a Riemannian metric to make the 1-form into a vector field, v. If necessary, perturb the vector field to a vector field v' with nondegenerate zeros. Then, the Euler characteristic is given by the formula

$$\chi \equiv \sum_{p:v'(p)=0} \mathrm{sign}(\det(\nabla v'|_p)). \tag{1.3}$$

Here, $\nabla v'$ is the covariant derivative of v' (its value at a zero of v' is apriori independent of the choice of the connection on the tangent bundle).

The preceding formula makes sense on a finite dimensional noncompact manifold provided that the zero set of v is compact. On such a manifold, $\chi \equiv \chi(v)$ depends on the vector field v; but $\chi(v_1) = \chi(v_2)$ when $|v_1| > |v_1 - v_2|$ on the complement of a compact set.

In the gauge theory situation, consider using (1.3) to define an Euler characteristic $\chi(\mathscr{f})$. The first requirement is a derivative of $\mathscr{f}$, and a natural derivative can be defined by exploiting the affine structure of $\mathscr{A}$. Indeed, for any connection A and su(2)-valued 1-forms (a, b), define

$$\partial_b \mathscr{f}|_A(a) \equiv \frac{d}{dt}(\mathscr{f}_{A+tb}(a))|\partial t = 0 = \int_M \mathrm{tr}(a \wedge d_A b). \tag{1.4}$$

Here, d_A is the covariant exterior derivative.

Notice that the assignment of (a, b) to $\partial_b \mathscr{f}|_A(a)$ defines a bilinear and symmetric form on $\Omega^1 \times \mathrm{su}(2)$. The bilinear form is symmetric because $\mathscr{f}$ defines a closed 1-form on $\mathscr{A}$; $\mathscr{f} = d\mathscr{c}$ so $d\mathscr{f} = 0$.

Give M a Riemannian metric, and let $*: \bigwedge^p T^* \to \bigwedge^{3-p} T^*$ denote the associated Hodge star. The metric defines an L^2 inner product on $\Omega^p \times \mathrm{su}(2)$; the inner product of p-form a with p-form b is

$$\langle a, b\rangle_{L^2} \equiv -\int_M \mathrm{tr}(a \wedge *b). \tag{1.5}$$

This inner-product makes $\partial\not{f}_A$ a symmetric map from $T\mathscr{A}|_A$ to itself:

$$\partial\not{f}_A \cdot a \equiv *d_A a, \tag{1.6}$$

which identifies $\partial\not{f}_A$ as a symmetric, 1st-order differential operator on $\Omega^1 \times \mathrm{su}(2)$. This operator is $\mathscr{G}$-equivariant in the sense that $*d_{gA} = *g \cdot d_A \cdot g^{-1}$.

When A is a flat connection, $\partial\not{f}_A$ has a huge null space; it annihilates, in particular, the tangents to the $\mathscr{G}$-orbit through A. This is because $\not{f}$ on $\mathscr{A}$ is the pull-back of a 1-form on $\mathscr{B}$. But, consider $\partial\not{f}$ rightfully downstairs as a section of the bundle of endomorphisms of the tangent bundle to $\mathscr{B}^\#$, and the null space becomes finite dimensional.

The tangent space to $\mathscr{B}^\#$ at some orbit [A] (note necessarily flat) is isomorphic to the orthogonal complement (with respect to the L^2-metric in (1.5)) of $T(\mathscr{G} \cdot A)$ in $T\mathscr{A}|_A$. This is the vector space

$$\mathscr{T}_A \equiv \{a \in \Omega^1 \times \mathrm{su}(2) : d_A^* a \equiv - * d_A * a \equiv 0\}. \tag{1.7}$$

By composing with orthogonal projection onto $\mathscr{T}_A$, $\partial\not{f}_A$ defines an endomorphism of $\mathscr{T}_A$ which will be denoted $\nabla\not{f}_A$ since it is a covariant derivative of $\not{f}$ at the orbit [A]. Explicitly, $\nabla\not{f}_A$ sends $a \in \mathscr{T}_A$ to

$$\nabla\not{f}_A \cdot a \equiv *d_A a - d_A u(a), \tag{1.8}$$

where $u(a)$ is the unique solution in $\Omega^0 \times \mathrm{su}(2)$ of the equation

$$*d_A * d_A u(a) = *(F_A \wedge a - a \wedge F_A). \tag{1.9}$$

Lemma 1.1. *Let A be a smooth connection on $M \times \mathrm{SU}(2)$. As a linear operator from $\mathscr{T}_A$ to itself, $\nabla\not{f}_A$ has at most a finite dimensional kernel. The operator $\nabla\not{f}_A$ defines a closed, essentially selfadjoint, Fredholm operator on the L_2 completion of $\mathscr{T}_A$, and its eigenvectors form an L^2-complete orthonormal basis for $\mathscr{T}_A$. The domain of $\nabla\not{f}_A$ is the L_1^2-Sobolev space completion of $\mathscr{T}_A$. The eigenvalues form a discrete subset of the real line which has no accumulation points, and which is unbounded in both directions. Each eigenvalue has finite multiplicity.*

The proof of this lemma involves standard elliptic theory on compact manifolds; see Part 3 of the Appendix.

A nondegenerate zero of $\not{f}$ on $\mathscr{B}^\#$ is, by definition, the orbit of a flat connection for which 0 is not in the spectrum of $\nabla\not{f}_A$.

Lemma 1.2. *A nondegenerate zero of $\not{f}$ on $\mathscr{B}^\#$ is isolated.*

This lemma is a straightforward consequence of the Sobolev inequalities with Lemma 1.1; the proof is outlined in §8b.

As M has the homology of S^3, the only zero of $\not{f}$ on $\mathscr{R}$ is the orbit of the product connection (there is only the trivial representations of $\pi_1(M)$

in S^1). This representation is also isolated (see also [2]) in the following sense:

Lemma 1.3. *Let M have the homology of S^3. There exists $\varepsilon > 0$ with the following significance: Let $[\Gamma]$ denote the $\mathscr{G}$-orbit of the product connection on $M \times \mathrm{SU}(2)$. Let $[A] \neq [\Gamma]$ be the $\mathscr{G}$-orbit of another flat connection. Then*

$$\sup_{g\in\mathscr{G}} \int_M \{|\nabla_\Gamma(A - g d_\Gamma g^{-1})|^2 + |(A - g d_\Gamma g^{-1})|^2\} > \varepsilon.$$

Lemma 1.3 has introduced the following notation: For a connection A, ∇_A is the covariant derivative on $\Omega^p \times \mathrm{su}(2)$ ($p = 0, 1, 2, 3$). The proof of Lemma 1.3 is discussed at the end of §8b.

The zeros of $\not{f}$ in $\mathscr{B}^\#$ need not be isolated. However, since $\nabla \not{f}$ has index zero, a suitably generic perturbation of $\not{f}$ will have isolated zeros in $\mathscr{B}^\#$.

The class of perturbations must be constrained by the following considerations: The class of perturbations must form an affine space. Then, the perturbations must be of the form $\not{f} + d\mathscr{u}$ for some function $\mathscr{u}$ so as not to change the fact that $\not{f}$ is a closed form on $\mathscr{B}^\#$. Also, the linearization of the perturbation must differ from $\nabla \not{f}$ at each orbit by a compact operator. Finally, the zero set of a perturbation must be compact.

The construction of a reasonable class of perturbations follows an analogous construction in [8]. To begin, consider a smoothly embedded loop γ in M. Use the Riemannian metric to construct a tubular neighborhood, $\nu(\gamma)$, for γ. Let D^2 denote the unit disc in $\mathbb{R}^2$ and choose a diffeomorphism from $\varphi_\gamma : S^1 \times D^2 \to \nu(\gamma)$ which identifies $S^1 \times \{0\}$ with γ. Fix a smooth, rotationally symmetric bump function $\eta : D^2 \to [0, 1]$, which is identically 1 on the ball of radius $1/2$ and is identically zero on the complement of the ball of radius $1/4$.

Let A be a connection on $M \times \mathrm{SU}(2)$. For each $y \in D^2$, the trace of parallel transport around the loop $\varphi_\gamma(\cdot, y)$ using A yields a number $p_\gamma[y; A]$ which depends only on the $\mathscr{G}$-orbit of A. Thus a function $p_\gamma[\cdot] : \mathscr{B} \to [-2\pi, 2\pi]$ is defined by the formula

$$p_\gamma[A] \equiv \int_{D^2} p_\gamma[y; A] \cdot \eta(y) \cdot d^2y. \tag{1.10}$$

Definition 1.4. Let M be a compact, oriented, 3-dimensional Riemannian manifold. An *admissible* function $\mathscr{u}$ on $\mathscr{B}$ is obtained by first fixing a finite set of smoothly embedded loops in M, $\Lambda \equiv \{\gamma_i\}_{i=1}N$. Then, fix a smooth function $f : \mathbb{R}^N \to \mathbb{R}$. Finally, set $\mathscr{u} \equiv f(\{p_\gamma\}_{\gamma\in\Lambda})$, where p_γ is given in (1.10). A *perturbation* of $\not{f}$ is a section $\not{f}' \equiv \not{f} + d\mathscr{u}$ of $T^*\mathscr{B}^\#$ where $\mathscr{u}$ is an admissible function on $\mathscr{B}$.

When $\mathscr{u}$ is an admissible function, define the covariant derivative of $d\mathscr{u}$ at an orbit $[A]$ to be the symmetric, bilinear form on $\mathscr{T}_A$ which assigns

$$\nabla d\mathscr{u}_A(a,b) \equiv \frac{d^2}{dtds}(u([A + t\cdot a + s\cdot b)) |_{t=s=0}$$

to a pair $(a,b) \in \mathscr{T}_A$. A *nondegenerate* perturbation of $\mathscr{f}$ is a perturbation $\mathscr{f}' \equiv \mathscr{f} + d\mathscr{u}$ with the following additional property: Require that the zeros in $\mathscr{B}^\#$ of $\mathscr{f} + d\mathscr{u}$ form a finite set and that the kernel of $\nabla(\mathscr{f} + d\mathscr{u})$ at each zero is empty.

Proposition 1.5. *Let M be a compact, oriented, 3-dimensional Riemannian manifold. There exist admissible functions $\mathscr{u}$ as described in Definition 1.4 such that the perturbation $\mathscr{f} + d\mathscr{u}$ of $\mathscr{f}$ is nondegenerate. In fact, there exists $\varepsilon_0 > 0$ which is such that for any given $0 < \varepsilon < \varepsilon_0$, there is a nondegenerate perturbation of $\mathscr{f}$, $\mathscr{f} + d\mathscr{u}$, which satisfies:*

(1) *The image of $\mathscr{u}$ lies in $[-\varepsilon, \varepsilon]$.*

(2) *The closure of the support of $\mathscr{u}$, $\operatorname{supp}\mathscr{u}$, is disjoint in $\mathscr{B}$ from the orbits of the reducible connections, and from the set $\mathscr{M}^* \subset \mathscr{B}^\#$ of orbits of flat connections for which $\nabla\mathscr{f}_A$ has empty kernel. In fact, let $\Delta = \operatorname{supp}\mathscr{u} \times \mathscr{G} \subset \mathscr{B} \times \mathscr{G}$. For each $[A] \in \mathscr{M}^* \cup (\mathscr{B}\backslash\mathscr{B}^\#)$,*

$$\sup_{([A'],g)\in\Delta} \int_M \{|\nabla_A(A - A' - g d_{A'} g^{-1})|^2 + (A - A' - g D_{A'} g^{-1})|^2\} > \varepsilon.$$

(3) *The differential $d\mathscr{u}$ obeys $|d\mathscr{u}| < \varepsilon$ when the metric in (1.5) is used to measure norms.*

(4) *With the metric in (1.5), $\nabla d\mathscr{u}$ is bounded everywhere by ε.*

(5) *For each zero, $[A']$, of $\mathscr{f}'$ there is a zero, $[A]$, of $\mathscr{f}$ such that*

$$\sup_{g\in\mathscr{G}} \int_M \{|\nabla_A(A - A' - g d_{A'} g^{-1})|^2 + |(A - A' - g d_{A'} g^{-1})|^2|\} < \varepsilon.$$

This proposition is proved in §8, where other properties of the perturbations from Definition 1.4 are discussed.

2. Spectral flow and the Euler class

Take a nondegenerate perturbation $\mathscr{f}' \equiv \mathscr{f} + d\mathscr{u}$ of $\mathscr{f}$, where $\mathscr{u}$ is described in Definition 1.4. Since (1.3) is the model for the Euler class of $\mathscr{f}$, a sign must be defined for the determinant of $\nabla\mathscr{f}'$ where $\nabla\mathscr{f}'_A$ is to be considered as an endomorphism of $T^*\mathscr{B}^\#|_{[A]} \equiv \mathscr{T}_A$. A problem instantly arises: For any orbit $[A]$ in $\mathscr{B}^\#$, the spectrum of $\nabla\mathscr{f}_A$ on the L^2-completion of $\mathscr{T}_A$ is unbounded in both directions on the line, and likewise, is the spectrum of $\nabla\mathscr{f}_A + \nabla d\mathscr{u}_A$. As a consequence, the existence of a determinant is problematical. (One can define various "regularized" determinants, a complicated procedure.)

However, the present interests require only the sign of the determinant. This sign is subtle, but the sign difference between the determinants at any two zeros of $\not{f}'$ is easy to define. Formally, the mod-2 spectral flow of the family of Fredholm operators $\nabla \not{f}'$ along a path in $\mathscr{B}^\#$ between the two zeros gives the relevant sign difference. The *relative* signs in the $\not{f}'$ version of (1.3) will be defined here directly via the mod-2 spectral flow.

The spectral flow for a continuous family of selfadjoint Fredholm operators was the subject of an intensive study by Atiyah-Patodi-Singer in [5]. Recall the basic idea: To say that an operator is selfadjoint and Fredholm is to say that its spectrum near 0 is that of a finite dimensional, selfadjoint matrix. Move on a continuously differentiable path in the space of such operators, and the eigenvalues near 0 move in a continuously differentiable manner. Suppose that the operators at the path's endpoints have empty kernel. Then, the number of eigenvalues which cross zero with positive slope minus the number which cross zero with negative slope is well defined and finite along a suitably generic path. This number is the spectral flow along the path.

When two such generic paths are homotopic (rel endpoints), the spectral flows agree. Therefore, the spectral flow defines a locally constant function on the space of continuous paths between the two endpoints.

Since the spectral flow is only a locally constant function, there can be nonzero spectral flow around a noncontractible, closed curve in the space of selfadjoint, Fredholm operators. Indeed, as remarked in [5], the spectral flow around closed loops gives an isomorphism between $\mathbb{Z}$ and the fundamental group of the Banach space of selfajdoint, Fredholm operators on a real, infinite dimensional, separable Hilbert space.

The relevance of spectral flow to (1.3) stems from the following observation: When considering a path of selfadjoint operators (matrices) on a finite dimensional vector space, the mod(2) spectral flow gives the relative sign between the determinants of the matrices at the two endpoints of the path. (Assuming both endpoints parametrize nondegenerate matrices.)

In the present infinite dimensional context, spectral flow is a well-defined concept (the precise statement is Proposition 2.1, below); it is used to define the relative sign between two operator determinants.

Proposition 2.1. *Let M be a compact, oriented 3-dimensional manifold with a Riemannian metric. Take a perturbation of $\not{f}' \equiv \not{f} + d\mathscr{a}$ of $T^*\mathscr{B}^\#$, where $\mathscr{a}$ is described in Definition 1.4 and Proposition 1.5. The assignment to an orbit $[A]$ of the operator $\nabla \not{f}'_A$ on the L^2-completion of $\mathscr{T}_A$ defines a* *smooth map from* $\mathscr{B}^{\#}$ *into the Banach space of real, selfadjoint operators on a separable Hilbert space. Let* $[A_{1,2}]$ *be a pair of orbits in* $\mathscr{B}^{\#}$ *for which the kernel of* $\nabla f'_{A_{1,2}}$ *is empty. Then the spectral flow defines a locally constant function on the space of continuous paths between* $[A_1]$ *and* $[A_2]$. *This function depends on the homotopy class of the path between* $[A_1]$ *and* $[A_2]$, *but its* mod*(8) reduction does not.*

This proposition is proved in Part 3 of the Appendix.

Together, Propositions 1.5 and 2.1 define the absolute value of the Euler number $\chi(f)$.

Definition 2.2: $|\chi(f)|$. Choose a nondegenerate perturbation f' of f as described in Definition 1.4, and let Z denote the finite set of zeros of f' in $\mathscr{B}^{\#}$. Pick one such zero, $[A]$. To each zero, $[A'] \in Z$, of f', let $\Delta[A, A'] = \pm 1$ denote the mod(2) spectral flow for $\nabla f'$ along paths in $\mathscr{B}^{\#}$ between $[A]$ and $[A']$. Finally, define

$$|\chi(f)| = \left| \sum_{[A'] \in Z} \Delta[A, A'] \right| .$$

The justification for this definition is provided by

Proposition 2.3. *Let* M *be a homology* 3*-sphere. Then the number* $|\chi(f)|$ *depends only on the differential structure on* M. *In particular, it is independent of the choice of the Riemannian metric on* M, *and of the choice of nondegenerate perturbation,* f', *of* f.

Proof of Proposition 2.3. The proof that the invariant is independent of the choice of nondegenerate perturbation is an application of the theory of nonlinear Fredholm maps in [9]. Details are given in §8b.

As for the metric dependence, note first that the set $f'^{-1}(0)$ is metric independent as the definition of f' requires no metric. Furthermore, the condition of nondegeneracy for f' is also metric independent: Indeed, $\nabla f'_A$ is defined as a quadratic form on $\Omega^1(M) \times \mathrm{su}(2)$ without the need of a metric (see (8.3)). And, if $f'(A) = 0$, then this quadratic form annihilates the image under d_A of $\Omega^0(M) \times \mathrm{su}(2)$; so it descends to define a quadratic form on the quotient. Thus, $\nabla f'_A$ is nondegenerate if and only if the associated quadratic form on said quotient is nondegenerate.

Therefore, the metric can only affect $|\chi(f)|$ through the spectral flow between two zeros of f'. But, since the space of metrics on M is path connected (indeed, convex), the said spectral flow is also metric independent. q.e.d.

To define the overall sign of $\chi(f)$, it is sufficient to associate a sign to one particular flat connection orbit. And, Mother Nature blesses a homology 3-sphere with one fiducial flat connection orbit, the orbit $[\Gamma]$ of the product connection on $M \times SU(2)$. To exploit Nature's bounty, it is necessary to consider the spectral flow along paths in $\mathscr{B}$ which are based at $[\Gamma]$.

The singularity of $\mathscr{B}$ at $[\Gamma]$ can be avoided by working $\mathscr{G}$ equivariantly on $\mathscr{A}$. Up on $\mathscr{A}$, one exploits the canonical sequence

$$0 \to T\mathscr{G}|_1 \xrightarrow{d} T\mathscr{A} \to \pi^* T\mathscr{B} \to 0, \tag{2.1}$$

where $d|_A \equiv d_A$, and $1 \in \mathscr{G}$ is the constant map to the identity in SU(2). This sequence is not exact at a reducible connection, but it is exact at an irreducible connection where one has

$$0 \to T\mathscr{G}|_1 \xrightarrow{d} T\mathscr{A}^{\#} \to \pi^* T\mathscr{B}^{\#} \to 0. \tag{2.2}$$

Here, $\pi: \mathscr{A} \to \mathscr{B}$ is the projection.

The L^2-metric on $T\mathscr{A}$ in (1.5) defines $d^*: T\mathscr{A}|_A \to T\mathscr{G}|_1$ which gives a $\mathscr{G}$-equivariant splitting of (1.9). Here, $d^*|_A \equiv d_A^*$ ($\equiv -{*d_A*}$). Over $\mathscr{B}^{\#}$ this splitting defines a connection on the principal $\mathscr{G}/\{\pm 1\}$-bundle $\mathscr{A}^{\#} \to \mathscr{B}^{\#}$.

This splitting can be exploited to extend an endomorphism of $T\mathscr{B}^{\#}$ to a $\mathscr{G}$-equivariant one of $T\mathscr{A} \oplus T\mathscr{G}|_1$. An endomorphism, L, of $T\mathscr{B}^{\#}$ is extended to:

$$\begin{matrix} L & d \\ d^* & 0 \end{matrix} \tag{2.3}$$

Here, L is thought of as a $\mathscr{G}$-equivariant endomorphism of $T\mathscr{A}^{\#}$ which annihilates $\mathrm{Im}(d)$ in (2.2).

(The physicists exploit this extension from $T\mathscr{B}^{\#}$ to $T\mathscr{A} \oplus T\mathscr{G}|_1$. They have introduced the term "ghost" to refer to the $T\mathscr{G}|_1 \oplus \mathrm{Im}(d)$ summand of $T\mathscr{G}|_1 \oplus T\mathscr{A} \approx T\mathscr{G}|_1 \oplus \mathrm{Im}(d) \oplus \mathrm{Ker}(d^*)$.)

Let $\not{f}'$ be a perturbation of $\not{f}$ as described in Definition 1.4. The covariant derivative $\nabla \not{f}'$ defines an endomorphism of $T\mathscr{B}^{\#}$ at each orbit, and so extends according to (2.3) to define an endomorphism, $K(\not{f}')$, of $(\Omega^1 \oplus \Omega^0) \times su(2)$. For example, take $\not{f}' = \not{f}$. At a connection $A \in \mathscr{A}^{\#}$, the extension $K(\not{f})_A$ sends $\omega \equiv (a, \varphi)$ to

$$K(\not{f})_A \omega \equiv ({*d_A}(a - d_A v(a)) - d_A u(a) + d_A \varphi, d_A^* a), \tag{2.4}$$

where $v(a) \subset \Omega^0 \times su(2)$ obeys

$$d_A^* d_A v(a) = d_A^* a, \tag{2.5}$$

and $u(a) \in \Omega^0 \times su(2)$ obeys

$$d_A^* d_A u(a) = -*(F_A \wedge (a - d_A v(a)) - (a - d_A v(a)) \wedge F_A). \tag{2.6}$$

Both $v(a)$ and $u(a)$ are uniquely defined when A is irreducible; when A is irreducible, the covariant Laplacian $d_A^* d_A\colon \Omega^0 \times \mathrm{su}(2) \to \Omega^0 \times \mathrm{su}(2)$ has no kernel.

Lemma 2.4. *Let M be a compact, oriented 3-manifold with a Riemannian metric. Let $\not{f}'$ be a perturbation of $\not{f}$ as described in Definition 1.4. For any connection A, the operator $K(\not{f}')_A$ extends to $L^2((T^*M \oplus \mathbb{R}) \times \mathrm{su}(2))$ as a closed, essentially selfadjoint, Fredholm operator. It has discrete spectrum with no accumulation points, and each eigenvalue has finite multiplicity. The spectrum is unbounded from above and below. The assignment of A to $K(\not{f}')_A$ gives a smooth map from $\mathscr{A}^\#$ to the Banach space of selfadjoint Fredholm operators on an infinite dimensional, separable Hilbert space.*

Proof of Lemma 2.4. This lemma is a corollary to Lemma A.5. q.e.d.

The operator $K(\not{f}')_A$ is relevant for the spectral flow computations because it has, by construction, the same spectral flow as $\nabla \not{f}'_A$. The next lemma summarizes:

Lemma 2.5. *Let M be a compact, oriented 3-manifold with a Riemannian metric. Fix an irreducible connection A. Let S_1 denote the spectrum of the operator $K(\not{f}')_A$ on $L^2((T^*M \oplus \mathbb{R}) \times \mathrm{su}(2))$. Let S_2 denote the spectrum of $\nabla \not{f}'_A$ on the L^2 completion of $\mathscr{T}_A$. Then S_1 and S_2 agree on a neighborhood of* 0.

The preceding two lemmas have the following consequence: In computing the spectral flow of $\nabla \not{f}'$ along a $\gamma \subset \mathscr{B}^\#$, one can lift the path to a path $\gamma_0 \subset \mathscr{A}$ and compute the spectral flow of $K(\not{f}')_{(\cdot)}$ along γ_0.

For a flat connection A, the endomorphism $*d_A$ of $\Omega^1 \times \mathrm{su}(2)$ extends via (2.3) to define the endomorphism, K_A, of $(\Omega^1 \oplus \Omega^0) \times \mathrm{su}(2)$ which is given by:

$$\begin{matrix} *d_A & d_A \\ d_A^* & 0 \end{matrix} \tag{2.7}$$

(This is the (twisted) signature complex in [5].)

Note that K_A makes sense for any connection A, flat or not, irreducible or not. And, K_A is $\mathscr{G}$-equivariant as $K_{gA} \equiv g \cdot K_A \cdot g^{-1}$. The next lemma describes K_A.

Lemma 2.6. *Let M be a compact, oriented 3-manifold with a Riemannian metric. For any connection A, the operator K_A extends to $L^2((T^*M \oplus \mathbb{R}) \times \mathrm{su}(2))$ as a closed, essentially selfadjoint, Fredholm operator. It has pure point spectrum, all real. The multiplicity of any eigenvalue is finite, and there are no accumulation points. Furthermore, the spectrum is unbounded in both directions. The assignment of A to K_A gives a smooth map from $\mathscr{A}$ to the Banach space of selfadjoint Fredholm operators on an infinite dimensional, separable Hilbert space. If A is flat, then*

$K_A = K(\mathscr{f})_A$. In general, for any irreducible connection A and any perturbation $\mathscr{f}'$ of $\mathscr{f}$ as described in Definition 1.5, the difference $K_A - K(\mathscr{f}')_A$ defines a bounded operator from $L^2((T^*M \oplus \mathbb{R}) \times \mathrm{su}(2))$ which is a relatively compact perturbation of K_A.

Proof of Lemma 2.6. This is a corollary to Lemma A.5. q.e.d.

As $K_{(\cdot)} - K(\mathscr{f}')_{(\cdot)}$ is compact, a sign for the determinant of $K_{(\cdot)}$ defines one for the determinant of $K(\mathscr{f}')_{(\cdot)}$. Just compute the mod(2) spectral flow for the path of operators $t \in [0,1] \to K_{(\cdot)} - t \cdot (K_{(\cdot)} - K(\mathscr{f}')_{(\cdot)})$. The advantage $K_{(\cdot)}$ has over $K(\mathscr{f}')_{(\cdot)}$ is that the former has continuous variation over the whole of $\mathscr{A}$. This makes $K_{(\cdot)}$ easier to use near the product connection.

At the product connection, Γ, the operator K_Γ has a 3-dimensional kernel given by the ∇_Γ-covariantly constant elements in $\Omega^0 \times \mathrm{su}(2)$. Thus, the sign of the determinant of K_Γ is not well defined. However, a sign can be given to the determinant of $K_{\Gamma+a}$ for a generic, small tangent vector to $\mathscr{A}$ at Γ.

For the purpose of defining such a sign, introduce the Hodge decomposition $\Omega^1 \times \mathrm{su}(2) = \mathrm{Im}(d_\Gamma) \oplus \ker(d_\Gamma^*)$. Let $\Pi_\Gamma: \Omega^1 \times \mathrm{su}(2) \to \Omega^1 \times \mathrm{su}(2)$ denote the L^2-orthogonal projection onto $\ker(d_\Gamma^*)$. Note that $*d_\Gamma: \ker(d_\Gamma^*) \to \ker(d_\Gamma^*)$ is invertible when M is a (rational) homology sphere.

For $a \in \Omega^1 \times \mathrm{su}(2)$ ($\equiv T\mathscr{A}|_\Gamma$), introduce the following symmetric bilinear form on su(2):

$$\tau_a(\sigma_0, \sigma_1) \equiv \int_M \mathrm{tr}([\sigma_0, \Pi_\Gamma \cdot a] \wedge [\sigma_1, (*d_\Gamma)^{-1} \Pi_\Gamma \cdot a]). \tag{2.8}$$

The lemma below motivates the introduction of τ_a.

Lemma 2.7. *Let M be an oriented homology 3-sphere with a Riemannian metric. There exists $\varepsilon_0 > 0$ with the following significance: Let $a \in \Omega^1 \times \mathrm{su}(2)$. For all $0 \le s < \varepsilon_0$, the operator $K_{\Gamma+s\cdot a}$ has exactly three eigenvectors with eigenvalue in $(-\varepsilon_0, \varepsilon_0)$. Let $\{\lambda_1, \lambda_2, \lambda_3\}$ be the three eigenvalues of τ_a. Then, to order s^3, $\{s^2 \cdot \lambda_1, s^2 \cdot \lambda_2, s^2 \cdot \lambda_3\}$ are the eigenvalues of $K_{\Gamma+s\cdot a}$ in $(-\varepsilon_0, \varepsilon_0)$.*

Proof of Lemma 2.7. The assertion is obtained using standard perturbation theory techniques (see e.g. [13]). The application is straightforward, so the details are left to the reader. q.e.d.

Think of $\tau_{(\cdot)}$ as a map from $\Omega^1 \times \mathrm{su}(2)$ into the vector space of endomorphisms of su(2).

Definition 2.8. Let M be an oriented homology 3-sphere with Riemannian metric. For $a \in \Omega^1 \times \mathrm{su}(2)$, suppose that τ_a is nondegenerate. Define $\lim_{s\to 0} \mathrm{sign}(\det(K_{\Gamma+s\cdot a})) \equiv \mathrm{sign}(\det(\tau_a))$.

The definition above is justified by the following remark.

Lemma 2.9. *Let M be an oriented homology 3-sphere with a Riemannian metric. There is a dense, open set in $\Omega^1 \times \mathrm{su}(2)$ for which $\tau_{(\cdot)}$ is nondegenerate. Let $a_1, a_2 \in \Omega^1 \times \mathrm{su}(2)$ be such that both $\tau_{a_{1,2}}$ are nondegenerate. Choose $s_0 > 0$ such that $K_{\Gamma+s\cdot a_{1,2}}$ has empty kernel for all $0 < s < s_0$. For such s, let δ_K denote the $\mathrm{mod}(2)$ spectral flow for $K_{(\cdot)}$ between $\Gamma + s\cdot a_1$ and $\Gamma + s\cdot a_2$. Then $\mathrm{sign}(\det(\tau_{a_1})) = \delta_K \cdot \mathrm{sign}(\det(\tau_{a_2}))$.*

Proof of Lemma 2.9. The first assertion is an observation which is based simply on linear algebra. The last assertion follows directly from Lemma 2.7. q.e.d.

As an aside, there is a geometric interpretation of Definition 2.8: On a homology sphere, the tangent space to $\mathscr{A}$ at Γ has the Hodge decomposition $\Omega^1 \times \mathrm{su}(2) \approx \mathrm{Im}(d_\Gamma) \oplus \ker(d_\Gamma^*)$. The summand $\ker(d_\Gamma^*)$ can be identified with the tangent space at $[\Gamma]$ to the quotient of $\mathscr{A}$ by the pointed gauge group $\mathscr{G}_0 \equiv \{g \in \mathscr{G}: g(x_0) = 1\}$, where $x_0 \in M$ is a fixed point. This quotient is a smooth manifold which defines a principal $\mathscr{G}_0$-bundle.

The quotient $\mathscr{G}/\mathscr{G}_0 \approx \mathrm{SU}(2)$. This $\mathrm{SU}(2)$ acts on $\mathscr{A}/\mathscr{G}_0$; the action has stabilizer $\{\pm 1\}$ on $(\mathscr{A}\backslash\mathscr{R})/\mathscr{G}_0$, and the quotient $(\mathscr{A}\backslash\mathscr{R})/\mathscr{G}_0 \to \mathscr{B}^\#$ defines a principal $\mathrm{SO}(3)$ bundle.

Definition 2.8 is associating an $\mathrm{SO}(3)$ equivariant sign to an open, dense set of points in the tangent space to $[\Gamma] \in \mathscr{A}/\mathscr{G}_0$; that is, an equivariant sign is given to points in the "exceptional fiber" in the "blow up" of $\mathscr{A}/\mathscr{G}_0$ at the point $[\Gamma]$.

Definition 2.10: $\chi(\not{f})$. Choose a nondegenerate perturbation $\not{f}'$ of $\not{f}$ as described in Definition 1.5. Choose $a \in \Omega^1 \times \mathrm{su}(2)$ for which τ_a is nondegenerate. Use Definition 2.8 to define $\mathrm{sign}(\det(K_{\Gamma+s\cdot a}))$ for all s sufficiently small. Fix an irreducible connection $A \in \not{f}'^{-1}(0)$. For small s, use the spectral flow for $t \in [0,1] \to K_{\Gamma+s\cdot a} - t\cdot(K_{\Gamma+s\cdot a} - K(\not{f}')_A)$ to define $\mathrm{sign}(\det(K'_A))$. Use Definition 2.2 to define $\Delta(A, A')$ for each orbit $[A']$ in $Z \equiv \not{f}'^{-1}(0)$. Finally, set

$$\chi(\not{f}) \equiv \mathrm{sign}(\det(K(f')_A)) \cdot \sum_{[A']\in Z} \Delta[A, A'].$$

The next proposition summarizes the discussion up to this point.

Proposition 2.11. *Let M be an oriented homology 3-sphere. The number $\chi(\not{f})$ $(\equiv \chi_M(\not{f}))$ is an invariant of the differential structure of M.*

Proof of Proposition 2.11. This is a direct corollary of Proposition 2.3 and Lemma 2.9 using the fact that the space of metrics on M is path connected. q.e.d.

As a final remark, note that the invariant $\chi_M(\not\ell)$ does depend on the choice of orientation. Changing the orientation will not change the relative spectral flow, but for $a \in \Omega^1 \times \mathrm{su}(2)$, the change in orientation reverses the sign of τ_a. Thus,

Lemma 2.12. *Let M and $-M$ denote the same underlying homology 3-sphere, but with reversed orientations. Then $\chi_M(\not\ell) = -\chi_{-M}(\not\ell)$.*

Proof of Lemma 2.12. Change the orienation of M, and the positive spectrum of $\nabla\not\ell$ becomes negative, and vice-versa since the changed orientation puts a factor of -1 in front of the Hodge $*$.

3. The Casson invariant

In [7], Casson introduces his new invariant, $\lambda(M)$, for an oriented homology 3-sphere. The reader is referred to [2] for a detailed exposition. The purpose of this section is provide a brief description of Casson's construction and to introduce some of the associated, gauge theory structure.

To define his invariant, Casson introduces a Heegard splitting of M. This is an embedded Riemann surface, $\Sigma \subset M$, which divides M into two handlebodies, $M_{+,-}$. To obtain such a surface, take a self-indexing Morse function, $f: M \to [0,3]$, and take $\Sigma \equiv f^{-1}(3/2)$, $M_+ \equiv f^{-1}([3/2,3])$ and $M_- \equiv f^{-1}([0,3/2])$. Note that M_+ is diffeomorphic to M_-; both are obtained from the 3-ball by adding $g \equiv \mathrm{genus}(\Sigma)$ standard 1-handles.

It is convenient for the analysis to enlarge both $M_\pm$ to overlap. Define $M_1 \equiv f^{-1}([0,7/4])$ and $M_2 \equiv f^{-1}([5/4,3])$. Since f has no critical values in the interval $(1,2)$, the gradient flow of f produces a diffeomorphism between M_1 and M_-; and likewise, between M_2 and M_+. The overlap, $M_1 \cap M_2 \equiv M_0$, is diffeomorphic to $\Sigma \times [5/4, 7/4]$.

For $\alpha = 0,1,2$, let $\mathcal{M}_\alpha = \mathrm{Hom}(\pi_1(M_\alpha); \mathrm{SU}(2))/\mathrm{Ad}(\mathrm{SU}(2))$ denote the set of conjugacy classes of representations of $\pi_1(M_\alpha)$ in SU(2). Let $\mathcal{M}^0_\alpha$ denote the subset of conjugacy clases of irreducible flat connections.

The $\mathcal{M}_\alpha$ as point sets are well understood. Both $M_{1,2}$ are handlebodies, so their fundamental groups are free on $g \equiv \mathrm{genus}(\Sigma)$ generators. Hence, $\mathcal{M}_{1,2} = \times_g \mathrm{SU}(2)/\mathrm{Ad}(\mathrm{SU}(2))$. The fundamental group of M_0 is the same as that of Σ. The fundamental group of $\Sigma_0 = \Sigma\backslash\{\text{point}\}$ is free on $2g$ generators, so $\mathrm{Hom}(\pi_1(\Sigma_0); \mathrm{SU}(2)) = \times_{2g}\mathrm{SU}(2)$. The surface relation defines an Ad(SU(2))-equivariant map, $\theta: \times_{2g}\mathrm{SU}(2) \to \mathrm{SU}(2)$, and $\mathcal{M}_0 = \theta^{-1}(1)/\mathrm{Ad}(\mathrm{SU}(2))$.

Each $\mathcal{M}^0_\alpha$ is an open subset of the corresponding $\mathcal{M}_\alpha$, and each is naturally a manifold. The manifold structure of $\mathcal{M}^0_{1,2}$ is obtained via its definition as a quotient of the open set in $\times_g \mathrm{SU}(2)$ on which conjugation defines an action SU(2) with stabilizer $\{\pm 1\}$. The manifold structure of $\mathcal{M}_0^0$ is obtained in a similar way since the critical set of the map θ is the set of reducible representations. The dimension of $\mathcal{M}_{1,2}^0$ is $3g-3$ and that of $\mathcal{M}_0^0$ is $6g-6$.

The inclusions of Σ into $M_{1,2}$ as the boundary induce embeddings $j_{1,2}^*\colon \mathcal{M}_{1,2}^0 \to \mathcal{M}_0^0$. The points of intersection of $j_1^*\mathcal{M}_1^0$ with $j_2^*\mathcal{M}_2^0$ are in 1-1 correspondence with $\mathrm{Hom}(\pi_1(M); \mathrm{SU}(2))/\mathrm{Ad}(\mathrm{SU}(2))$.

Casson assigns an integer to $\mathrm{Hom}(\pi_1(M); \mathrm{SU}(2))/\mathrm{Ad}(\mathrm{SU}(2))$ by making sense of an intersection number for the intersection of $j_1^*\mathcal{M}_1^0$ with $j_2^*\mathcal{M}_2^0$ in $\mathcal{M}_0^0$. (See [2] for details.)

Definition 3.1 (*Casson's invariant*). Let M be an oriented homology 3-sphere. Choose a Heegard splitting, $\Sigma \subset M$, which splits M into handlebodies M_1 and M_2. When the intersection of $j_1^*\mathcal{M}_1^0$ with $j_2^*\mathcal{M}_2^0$ is transverse in $\mathcal{M}_0^0$, then Casson's invariant is

$$\lambda(M) \equiv \frac{1}{2} \sum_{p \in j_1^*\mathcal{M}_1^0 \cap j_2^*\mathcal{M}_2^0} (\pm 1)_p.$$

Here, $(\pm 1)_p$ is a sign that is obtained by first orienting each $\mathcal{M}_\alpha^0$ and then comparing the orientations on $Tj_1^*\mathcal{M}_1^0 \oplus Tj_2^*\mathcal{M}_2^0$ and $T\mathcal{M}_0^0$ at the point p (see below).

When the intersection of $j_1^*\mathcal{M}_1^0$ with $j_2^*\mathcal{M}_2^0$ is not transverse in $\mathcal{M}_0^0$, take generic, compactly supported, transversely intersection perturbations of $j_1^*\mathcal{M}_1^0$ and $j_2^*\mathcal{M}_2^0$ to define $\lambda(M)$. The formula is given above with such perturbations, $\mathcal{M}_1'^0$ and $\mathcal{M}_2'^0$ replacing $j_1^*\mathcal{M}_1^0$ and $j_2^*\mathcal{M}_2^0$ respectively.

Casson's orientation of $\mathcal{M}_\alpha^0$ can be defined in the following way: An orientation of Σ defines a symplectic form on $H^1(\Sigma; \mathbb{R})$; send (a, b) to $(a \cup b)[\Sigma] \in \mathbb{R}$ with $[\Sigma] \in H_2(\Sigma)$ the generator which is defined by the orientation.

As M_1 is a handlebody, the inclusion $\Sigma \to M_1$ induces $0 \to H^1(M_1; \mathbb{R}) \to H^1(\Sigma; \mathbb{R})$. Likewise, one has $0 \to H^1(M_2; \mathbb{R}) \to H^1(\Sigma; \mathbb{R})$, and Stoke's theorem implies that the symplective form on $H^1(\Sigma; \mathbb{R})$ induces a perfect pairing between $H^1(M_1; \mathbb{R})$ and $H^1(M_2; \mathbb{R})$.

Via the symplectic pairing, an orientation of $H^1(M_1; \mathbb{R})$ induces an orientation of $H^1(M_2; \mathbb{R})$ and also an orientation of $H^1(\Sigma; \mathbb{R})$. (As a symplectic vector space, $H^1(\Sigma; \mathbb{R})$ has a canonical orientation—this is it.) The induced orientation on $H^1(\Sigma; \mathbb{R})$ is independent of the choice of orientation for $H^1(M_1; \mathbb{R})$.

Introduce $\Sigma_0 \equiv \Sigma \backslash \{\text{pt.}\}$. Fix a small disk about the missing point. The orientation on Σ orients the boundary circle, γ.

A choice of basis (consistent with the orientation) for $H^1(\Sigma;\mathbb{R})$ identifies $\mathrm{Hom}(\pi_1(\Sigma_0);\mathrm{SU}(2))$ with $\times_{2g}\mathrm{SU}(2)$. Introduce the map $\theta\colon \times_{2g}\mathrm{SU}(2) \to \mathrm{SU}(2)$ which is defined by parallel transport around γ. Choose an orientation for SU(2). This orients the nonsingular part of $\theta^{-1}(1)$, and also $\mathcal{m}_0^0 \subset \theta^{-1}(1)/\mathrm{Ad}(\mathrm{SU}(2))$. This orientation of $\mathcal{m}_0^0$ is independent of the initial choice of orientation of SU(2), but it does depend on the orientation of Σ through the sign of the symplectic form and the orientation of γ. Change the orientation of Σ, and the orientation of $\mathcal{m}_0^0$ changes by a factor of $(-1)^{g+1}$.

A choice of basis (consistent with the orientation) for $H^1(M_1;\mathbb{R})$ identifies $\mathrm{Hom}(\pi_1(M_1);\mathrm{SU}(2))$ with $\times_g\mathrm{SU}(2)$. The orientation of SU(2) orients this space, and thus $\mathcal{m}_1^0 \subset (\times_g\mathrm{SU}(2))/\mathrm{Ad}(\mathrm{SU}(2))$. If g is even, the choice of SU(2) orientation affects the orientation of $\mathcal{m}_1^0$. Obviously, the orientation of $H^1(M_1;\mathbb{R})$ affects the orientation of $\mathcal{m}_1^0$.

Then via the symplectic pairing in $H^1(\Sigma;\mathbb{R})$, these orientations of $H^1(M_1;\mathbb{R})$ and of SU(2) also orient $\mathcal{m}_2^0$.

The orientations of both $\mathcal{m}_{1,2}^0$ change when the orientation of $H^1(M_1;\mathbb{R})$ is changed; and, in the g even case, when the orientation of SU(2) is changed. In both cases, the orientation of $T\mathcal{m}_1^0 \oplus T\mathcal{m}_2^0|_p$ at a point, p, of intersection is insensitive to these choices. However, changing the orientation of Σ changes the orientation of $T\mathcal{m}_2^0$ by a factor of $(-1)^g$. Therefore, change the orientation of Σ and the sign of $\lambda(M)$ changes.

The handlebody called M_1 played a special role above. Switching the roles of M_1 and M_2 changes the orientation of $H^1(\Sigma;\mathbb{R})$ when g is odd but not when g is even. Hence, this switch changes the orientation of $T\mathcal{m}_0^0$ by a factor of $(-1)^g$. When g is even (but not odd), the switch of M_1 and M_2 changes the orientation of $T\mathcal{m}_1^0 \oplus T\mathcal{m}_2^0|_p$ at a point, p, or intersection. Therefore, the switch of role, $M_1 \leftrightarrow M_2$, changes the sign of $\lambda(M)$.

Thus, the sign $(\pm 1)_p$ in (3.12) is well defined given the orientation of Σ and the specification of the handlebody to call M_1. Changing both choices leaves $\lambda(M)$ invariant so it is only the induced orientation of M which must be specified to avoid an ambiguity. (The specification of the handlebody to call M_1 specifies a normal vector to Σ—this, with an orientation of Σ, orients M.)

Main Theorem. *Let M be an oriented homology 3-sphere. Introduce the invariant $\chi(\mathcal{f})$ from Definition 2.10 and introduce Casson's invariant, $\lambda(M)$ (see Definition 3.1). Then $\chi(\mathcal{f}) = 2\cdot\lambda(M)$.*

The remainder of this paper is occupied with proving this theorem.

4. Gauge theory and Heegard splittings

To prove the main theorem, it is necessary to consider the extra structure that the Heegard splitting gives to the gauge theory. The introduction of this extra structure occupies this section.

For $\alpha \equiv 0, 1, 2$, introduce the space $\mathscr{A}_\alpha$ of smooth connections on the principal SU(2)-bundle $M_\alpha \times \mathrm{SU}(2)$.

The inclusions $M_0 \xrightarrow[j_{1,2}]{} M_1$, $M_2 \xrightarrow[i_{i,2}]{} M$ induce by pull-back the maps

$$\mathscr{A} \xrightarrow[I]{} \mathscr{A}_1 \times \mathscr{A}_2 \xrightarrow[J]{} \mathscr{A}_0 \times \mathscr{A}_0, \tag{4.1}$$

where $I \equiv i_1^* \times i_2^*$ and $J \equiv j_1^* \times j_2^*$. (4.1) is "exact" in the following sense: The map I is an embedding, and J is a submersion. Furthermore, the image of I is equal to the inverse image under J of the diagonal, $\Delta \subset \mathscr{A}_0 \times \mathscr{A}_0$. Such is the case if the spaces in question are considered as Fréchet spaces using the C^∞-topology, or if they are considered as pre-Hilbert manifolds with the L_1^2-Sobolev structure (or with any L_k^p-Sobolev structure with $p \geq 2$ and $k \geq 1$).

Each of the arrows in (4.1) is naturally equivariant with respect to the respective gauge group actions. (The gauge group $\mathscr{G}_\alpha \equiv \mathrm{Aut}(M_\alpha \times \mathrm{SU}(2)) \equiv C^\infty(M_\alpha; \mathrm{SU}(2))$ acts on $\mathscr{A}_\alpha$ in the usual way.)

To describe the invariant theory analytically, introduce the quotients $\mathscr{B}, \mathscr{B}_\alpha$ of $\mathscr{A}, \mathscr{A}_\alpha$ by the respective gauge groups. It proves useful to introduce $\mathscr{A}^0 \subset \mathscr{A}$ and $\mathscr{A}_\alpha^0 \subset \mathscr{A}_\alpha$, as the subsets of connections which restrict to irreducible connections on $([11/8, 13/8] \times \Sigma) \times \mathrm{SU}(2)$. In each case, $\mathscr{A}^0 \subset \mathscr{A}$, $\mathscr{A}_\alpha^0 \subset \mathscr{A}_\alpha$ is open and dense and the complement of a set of infinite codimension. Introduce the quotients $\mathscr{B}^0, \mathscr{B}_\alpha^0$ of $\mathscr{A}^0, \mathscr{A}_\alpha^0$ by the respective gauge groups.

Proposition 4.1. *With the L_1^2-Hilbert space structure, each of $\mathscr{B}^0, \mathscr{B}_\alpha^0$ is naturally a manifold modelled on a pre-Hilbert space, and the defining quotient is a principal bundle with structure group $\mathscr{G}/\{\pm 1\}$, respectively $\mathscr{G}_\alpha/\{\pm 1\}$. Indeed, $\mathscr{B}^0$ is an open dense set in $\mathscr{B}^\#$ and the complement of a set of infinite codimension. The tangent space to $\mathscr{B}_\alpha^0$ at an orbit $[A]$ is the vector space*

$$\mathscr{T}_{\alpha A} \equiv \{a \in \Omega^1(M_\alpha) \times \mathrm{su}(2) \colon d_A^* a \equiv 0 \text{ and } i^*(*a) \equiv 0\},$$

where $i \colon \partial M_\alpha \to M_\alpha$ is the inclusion. Furthermore, for each $[A] \in \mathscr{B}_\alpha^0$, there exists $\varepsilon > 0$ such that the assignment of $[A + a]$ to

$$a \in \left\{ b \in \mathscr{T}_{\alpha A} \colon \int_{M_\alpha} (|\nabla_\Gamma b|^2 + |b|^2) < \varepsilon \right\}$$

is a homeomorphism onto an open set and defines a smooth coordinate chart for a neighborhood of $[A]$.

This proposition is a straightforward modification of the results in [17]. The details are omitted.

The equivariant version of (4.1) is

$$\mathscr{B} \underset{I}{\longrightarrow} \mathscr{B}_1 \times \mathscr{B}_2 \underset{J}{\longrightarrow} \mathscr{B}_0 \times \mathscr{B}_0, \tag{4.2a}$$

and

$$\mathscr{B}^0 \underset{I}{\longrightarrow} \mathscr{B}_1^0 \times \mathscr{B}_2^0 \underset{J}{\longrightarrow} \mathscr{B}_0^0 \times \mathscr{B}_0^0. \tag{4.2b}$$

On the level of tangent spaces, (4.2b) induces $I_*: \mathscr{T}_A \to \mathscr{T}_{1A_1} \times \mathscr{T}_{2A_2}$ and $J_*: \mathscr{T}_{1A_1} \times \mathscr{T}_{2A_2} \to \mathscr{T}_{0A_0} \times \mathscr{T}_{0A_0}$. These maps have the following explicit descriptions: At $[A] \in \mathscr{B}^0$,

$$I_* \cdot a \equiv (a - d_A\varphi_1(a), a - d_A\varphi(a)), \tag{4.3}$$

where $\varphi_{1,2} \in \Omega^0(M_{1,2}) \times \mathrm{su}(2)$ obeys

$$d_A^* d_A \varphi_{1,2} = 0 \quad \text{and} \quad i_{\partial M_{1,2}}^*(*(a - d_A\varphi_{1,2}) = 0. \tag{4.4}$$

Meanwhile, at $([A_1], [A_2]) \in \mathscr{B}_1^0 \times \mathscr{B}_2^0$,

$$J_* \cdot (a_1, a_2) \equiv (a_1 - d_{A_1}\psi_1(a_1), a_2 - d_{A_2}\psi_2(a_2)), \tag{4.5}$$

where $\psi_{1,2} \in \Omega^0(M_0) \times \mathrm{su}(2)$ obeys

$$d_{A_{1,2}}^* d_{A_{1,2}} \psi_{1,2} = 0 \quad \text{and} \quad i_{\partial M_0}^*(*(a_{1,2} - d_{A_{1,2}}\psi_{1,2}) = 0. \tag{4.6}$$

Lemma 4.2. *The sequence in* (4.2b) *is exact in that* I *is an embedding,* J *is a submersion, and* $\mathrm{image}(I) = J^{-1}(\Delta)$.

On the tangent level, the assertion of Lemma 4.2 can be expressed in terms of the sequence of linear maps

$$0 \to \mathscr{T}_A \xrightarrow{\Phi} \mathscr{T}_{1A_1} \times \mathscr{T}_{2A_2} \xrightarrow{\Psi} \mathscr{T}_{0A_0} \to 0, \tag{4.7}$$

which is exact. Here $(A_1, A_2) \equiv I(A)$, $A_0 \equiv j_1^* i_i^*(A) = j_2^* i_2^*(A)$; in addition, $\Phi \equiv I_*$ while $\Psi \equiv (j_1^*)_* - (j_2^*)_*$.

Lemma 4.3. *The sequence in* (4.7) *defines an exact, acyclic, Fredholm complex.*

Proof of Lemmas 4.2 *and* 4.3. If A restricts irreducibly to M_α, then the covariant Laplacian $d_A^* d_A: \Omega^0(M_\alpha) \times \mathrm{su}(2) \to \Omega^0(M_\alpha) \times \mathrm{su}(2)$ is invertible with specified Neumann data on the boundary, ∂M_α. In fact, for any $k \geq 0$, the inverse of $d_A^* d_A$ defines a bounded linear map from $(L_k^2(M_\alpha) \times \mathrm{su}(2)) \times (L_k^2(\partial M_\alpha) \times \mathrm{su}(2)) \to L_{k+2}^2(M_\alpha) \times \mathrm{su}(2)$. That is, for each (ω, ν) in the former space, there exists a unique $\sigma \in L_{k+2}^2(M_\alpha) \times \mathrm{su}(2)$ which obeys $d_A^* d_A \sigma = \omega$ and $i^*(*d_A\sigma) = \mu$. Furthermore, the L^2_{k+2}-norm of σ is estimated by the L^2_k-norm of (ω, ν).

The assertions of Lemma 4.2 and the asserted properties of the differentials I_* and J_* are simple algebraic consequences of the facts from the preceding paragraph. The assertions in Lemma 4.2 that I is 1-1 and that J is surjective use the fact that the space of maps from a Riemann surface to SU(2) is path connected. The details are left to the reader. q.e.d.

The orbits of irreducible flat connections on $M_\alpha \times \mathrm{SU}(2)$ are the zeros of a section of a vector bundle, $\mathscr{L}_\alpha$, over $\mathscr{B}^0_\alpha$, but the bundle in question is not the tangent bundle. The fiber of $\mathscr{L}_\alpha$ at an orbit $[A] = \mathscr{B}^0_\alpha$ is defined to be the vector space

$$\mathscr{L}_{\alpha A} \equiv \{f \in \Omega^1(M_\alpha) \times \mathrm{su}(2) : d_A^* f \equiv 0\}. \tag{4.8}$$

This subspace is equivariantly defined because $\mathscr{L}_{\alpha g A} \equiv g \cdot \mathscr{L}_{\alpha A}$ as subspaces of $\Omega^1(M_\alpha) \times \mathrm{su}(2)$. Give $\mathscr{L}_{\alpha A}$ the structure of a (pre-)Hilbert space by using the L^2 inner product on $\Omega^1(M_\alpha) \times \mathrm{su}(2)$. Note that $\mathscr{L}_{\alpha A}$ is well defined even when A is reducible.

Lemma 4.4. *The assignment of A to the vector space $\mathscr{L}_{\alpha A}$ defines a smooth vector bundle $\mathscr{L}_\alpha \to \mathscr{B}^0_\alpha$; in fact, $\mathscr{L}_\alpha$ is a sub-bundle of $(A^0_\alpha \times (\Omega^1(M_\alpha) \times \mathrm{su}(2)))/\mathscr{G}_\alpha)$ when $\Omega^1(M_\alpha) \times \mathrm{su}(2)$ is given the structure of a pre-Hilbert space with the L^2-inner product. For any connection A on $M_\alpha \times \mathrm{su}(2)$, the inclusion maps $i_{1,2}$ and $j_{1,2}$ induce the following exact sequence of Fredholm maps*:

$$0 \to \mathscr{L}_A \xrightarrow{\underline{\Phi}} \mathscr{L}_{1A_1} \times \mathscr{L}_{\alpha A_2} \xrightarrow{\underline{\Psi}} \mathscr{L}_{-A_0} \to 0,$$

where $\underline{\Phi} \cdot f \equiv (i_1^* f, i_2^* f)$ and $\underline{\Psi} \cdot (f_1, f_2) \equiv j_1^* f_1 - j_2^* f_2$.

PROOF OF LEMMA 4.4. The first assertion of the lemma is a straightforward modification of results in [17] (see Part 1 of the Appendix). The second assertion should be obvious. q.e.d.

The assignment of curvature to a connection on $M_\alpha \times \mathrm{SU}(2)$ defines a section $\mathscr{f}_\alpha$ of $\mathscr{L}_\alpha$, which sends $[A]$ to $[A, *F_A] \in \mathscr{L}_\alpha$.

It is crucial to observe that $\mathscr{m}_\alpha \subset \mathscr{B}_\alpha$ is the set of orbits of flat connections, and $\mathscr{m}^0_\alpha$ is the subset of orbits of irreducible flat connections. That is, $\mathscr{m}_\alpha = \mathscr{f}_\alpha^{-1}(0)$ and $\mathscr{m}^0_\alpha = f_\alpha^{-1}(0) \cap \mathscr{B}^0_\alpha$; this last equivalence is a consequence of the fact that the inclusion of Σ into M induces an epimorphism of fundamental groups.

The identifications $\mathscr{m}_\alpha = \mathscr{f}_\alpha^{-1}(0)$ and $\mathscr{m}^0_\alpha = \mathscr{f}_\alpha^{-1}(0) \cap \mathscr{B}^0_\alpha$ give these sets added structure.

Proposition 4.5. *Let M be a rational homology sphere, and assume that the Riemannian metric on M is the product metric on M_0. Then the set $\mathscr{m}^0_{1,2}$*

as an open submanifold of $(\times_g \mathrm{SU}(2))/\mathrm{Ad}(\mathrm{SU}(2))$ *is an embedded submanifold of* $\mathscr{B}^0_{1,2}$, *the set* m^0_0 *as an open submanifold of* $\theta^{-1}(1)/\mathrm{Ad}(\mathrm{SU}(2)) \subset (\times_{2g}\mathrm{SU}(2))/\mathrm{Ad}(\mathrm{SU}(2))$ *is an embedded submanifold of* $\mathscr{B}^0_0$, *and the map* J *of* (4.2) *restricts to* $m^0_1 \times m^0_2$ *as an embedding into* $m^0_0 \times m^0_0$.

The proof of Proposition 4.5 is given at the end of this section.

The Fredholm properties of the section f_α and its perturbations are of critical importance; a digression to study these properties follows.

Since perturbations of f as described in §1 are required to define the invariant χ, and since a Heegard splitting of M is required to define Casson's invariant, it is necessary to study the behavior of the perturbations given the choice of a Heegard splitting.

Definition 4.6. Let M be a homology 3-sphere with a choice of Heegard splitting. An admissible function u in the sense of Definition 1.4 will be said to be *compatible* when the loops of the set Λ and their associated tubular neighborhoods lie in the interior of M_0.

Choose a set of loops, Λ, of size $N < \infty$, and the loops to lie in the interior of M_0. In defining the functions $\{p_\gamma\}_{\gamma\in\Lambda}$, make sure the tubular neighborhood about each loop γ lies inside M_0. Set $u \equiv f((p_\gamma))$, where $f\colon \mathbb{R}^N \to \mathbb{R}$ is a smooth function. Set $f' \equiv f + du$, which is a perturbation of f as described in Definition 1.4.

Note that $du|_A$ defines a section of $\Omega^1 \times \mathrm{su}(2)$, which is denoted $v[A]$ and compactly supported on the union of the tubular neighborhoods about the loops which comprise the set Λ.

To be explicit, for each such loop γ, introduce the diffeomorphism φ_γ between $S^1 \times D^2$ and the tubular neighborhood about γ. Let ∂_t denote the unit, oriented tangent vector to S^1. Let η_γ denote the 1-form on M which is metrically dual at each point to $\varphi_{\gamma *}(\eta \cdot \partial_t)$; this is a 1-form on M with compact support in the tubular neighbohrood about γ. The section $v[A]$ is given by the formula

$$v[A](x) \equiv \sum_{\gamma\in\Lambda} f_\gamma \cdot \mathrm{Im}(P_\gamma[\varphi_\gamma^{-1}(x); A]) \cdot \eta_\gamma, \tag{4.9}$$

where $\mathrm{Im}\colon \mathrm{SU}(2) \to \mathrm{su}(2)$ is defined by identifyiing SU(2) with the unit quaternions, and su(2) with the imaginary quaternions. In (4.9), each f_γ is a smooth function on $\mathbb{R}^N$ which is evaluated on the point $\{p_\gamma\} \in \mathbb{R}^N$.

The section $v[A]$ respects the decomposition of M by the Heegard splitting in that it depends only on the restriction of A to M_0. This means that $v[A]$ is well defined when $[A]$ is only in $\mathscr{B}_0$ or $\mathscr{B}_{1,2}$. The assignment of $v[A]$ in (4.9) to $[A] \in \mathscr{B}_{0,1,2}$ defines a section, $du_{0,1,2}$, of the vector bundle $\mathscr{L}_{0,1,2}$ of (4.8).

For $\alpha = 0, 1, 2$, define $f'_\alpha \equiv f_\alpha + d u_\alpha$ as a section of $\mathscr{L}_\alpha \to \mathscr{B}_\alpha$. Apropos the discussion of the preceding paragraph, one has

Lemma 4.7. *Introduce the maps I and J of* (4.2) *and the maps* $\underline{\Phi}$ *and* $\underline{\Psi}$ *of Lemma* 4.4. *Let* $f'_\alpha \equiv f_\alpha + d u_\alpha$ *as described above. Then the following identities hold*:

(1) $\underline{\Phi} \cdot f' \equiv (f'_1, f'_2) \circ I$.

(2) $\underline{\Psi} \cdot (f'_1, 0) \times \underline{\Psi} \cdot (0, f'_2) \equiv (f'_0, f'_0) \circ J$.

(3) $f'^{-1}(0) = I^{-1}((f_1'^{-1}(0) \times f_2'^{-1}(0)) \cap J^{-1}(\Delta))$.

Lemma 4.8. *For* $\alpha = 0, 1, 2$, *the section* f'_α *as defined above is a smooth section of* $\mathscr{L}_\alpha$ *over* $\mathscr{B}^0_\alpha$.

Proof of Lemma 4.8. This is a consequence of (8.2), (8.3) and (8.7); the details are straightforward applications of Sobolev inequalities and are omitted. q.e.d.

A covariant derivative of f_α or of a perturbation f'_α is defined with the help of the affine structure of $\mathscr{A}_\alpha$. In each case $\alpha = 0, 1, 2$, or $\alpha = \varnothing$, the covariant derivative of $\nabla f'_\alpha$ of f'_α is defined over $\mathscr{B}^0_\alpha$; it is the linear map $\nabla f'_{\alpha A}: \mathscr{T}_{\alpha A} \to \mathscr{L}_{\alpha A}$ which sends a to

$$\nabla f'_{\alpha A} \cdot a \equiv *d_A a + \partial v[A] \cdot a - d_A u'_\alpha(a), \tag{4.10}$$

where $\partial v[A] \cdot a \equiv \frac{d}{ds}(v[A + s \cdot a])|_{s=0}$ (see (8.3)), and $u_\alpha(a) \in \Omega^0(M_\alpha) \times \mathrm{su}(2)$ obeys

$$d_A^* d_A u'_\alpha(a) - (F_A \wedge a - a \wedge F_A) - d_A^* \partial v[A] \cdot a = 0, \quad i^*(u'_\alpha(a)) = 0. \tag{4.11}$$

Here $i: \partial M_\alpha \to M_\alpha$ is the inclusion.

Note that $\nabla f'_{\alpha A}$ is a $\mathscr{G}_\alpha$-equivariant, linear map from $\mathscr{T}_{\alpha A}$ to $\mathscr{L}_{\alpha A}$.

As A varies through $\mathscr{B}^0_\alpha$ the vector spaces $\mathscr{T}_{\alpha(\cdot)}$ and $\mathscr{L}_{\alpha(\cdot)}$ fit together to define smooth vector bundles $\mathscr{T}_\alpha, \mathscr{L}_\alpha \to \mathscr{B}^0_\alpha$. Let $d \equiv 3g - 3$ if $\alpha = 1, 2$, and $d \equiv 6g - 6$ if $\alpha = 0$; if $\alpha = \varnothing$, let $d = 0$. A smooth fiber bundle $\mathrm{Fred}_d(\mathscr{T}_\alpha, \mathscr{L}_\alpha) \to \mathscr{B}^0_\alpha$ is defined by taking the fiber over $[A]$ to be the Banach manifold of bounded, Fredholm operators from $\mathscr{T}_\alpha$ to $\mathscr{L}_\alpha$ of index d. (These assertions are considered in Part 1 of the Appendix.)

Proposition 4.9. *Let M be a homology* 3-*sphere with a Riemannian metric. Suppose that M has a Heegard splitting, and define* $M_{0,1,2}$ *as above. Let* u *be an admissible function on* $\mathscr{B}$ *which is compatible in the sense of Definition* 4.6. *For* $\alpha = 0, 1, 2, \varnothing$, *let* $f'_\alpha \equiv f_\alpha + d u_\alpha$, *where* u_α *is the restriction of* u *to* $\mathscr{B}_\alpha$. *Let A be an irreducible connection on* $M_\alpha \times \mathrm{SU}(2)$. *Then the following hold*:

(1) $\nabla f'_{\alpha A}$ *defines a bounded operator from* $\mathscr{T}_{\alpha A}$ *to* $\mathscr{L}_{\alpha A}$ *when these spaces are completed in the Sobolev* L^2_1 *and* L^2 *norms, respectively.*

(2) *This operator is Fredholm with index* 0 *for* $\alpha = \emptyset$, *with index* $3g - 3$ *for* $\alpha = 1, 2$ *and with index* $6g - g$ *for* $\alpha = 0$.

(3) *The difference* $\nabla \not{f}'_{\alpha A} - \nabla \not{f}_{\alpha A}$ *is a compact operator, in fact, bounded when the* L^2*-metric is used for* $\mathscr{T}_{\alpha A}$.

(4) *If* $\not{f}'_{\alpha A} \equiv 0$, *then* $\nabla \not{f}'_{\alpha A} \cdot a = \frac{d}{ds}(\not{f}'_\alpha([A + s \cdot a]))|_{s=0}$.

(5) *The assignment of* $\nabla \not{f}'_{\alpha A}$ *to* $[A] \in \mathscr{B}^0_\alpha$ *defines a smooth section over* $\mathscr{B}^0_\alpha$ *of* $\mathrm{Fred}_d(\mathscr{T}_\alpha, \mathscr{L}_\alpha)$.

Proof of Proposition 4.9. Assertions (1)–(3), (5) are Lemma A.4 in the Appendix. Assertion (4) can be verified by a direct calculation; it is the infinite dimensional analog of the following finite dimensional phenomenon: The covariant derivative of a section of a vector bundle is independent of the connection when such a derivative is taken at a point where the section vanishes. q.e.d.

This section ends with the

Proof of Proposition 4.5. To show that m^0_α is an embedded submanifold of $\mathscr{B}^0_\alpha$, it is sufficient to show that $\dim(\ker(\nabla \not{f}_{\alpha A})) = \mathrm{index}(\nabla \not{f}_{\alpha A})$ for $[A] \in m^0_\alpha$. Suppose that $\omega \in \mathrm{coker}(\nabla \not{f}_{\alpha A})$. Then $d_A \omega = 0$, $d^*_A \omega = 0$, and $i^*(\omega) = 0$ where $i: \partial M_\alpha \to M_\alpha$ is the inclusion. Thus, ω defines a class, $[\omega]$, in the relative first cohomology, $H^1(M_\alpha, \partial M_\alpha; d_A)$, of the (twisted) DeRham complex, $\Omega^* \times \mathrm{su}(2)$, with differential d_A. (Since A is flat, $d^2_A = 0$.)

Lemma 4.10. *Let* M_1 *be a smooth,* 3*-dimensional handlebody with boundary* Σ. *That is,* M_1 *is obtained from the standard* 3*-ball by attaching some* $g \geq 0$ *one-handles. Let* $M_0 = [0, 1] \times \Sigma$ *be a (closed) product neighborhood of* ∂M_1. *Let* A *be a flat connection on* $M_{1,0} \times \mathrm{SU}(2)$. *Then the inclusion* $\Sigma = \partial M_1 \to M_{0,1}$ *induces monomorphisms* $0 \to H^1(M_{0,1}; d_A) \to H^1(\Sigma; d_A)$.

Proof of Lemma 4.10. For M_0, use the fact that $\Sigma \to M_0$ is a strong deformation retract. For M_1, use the fact that M_1 is a handlebody, so that the relative cell complex (M_1, Σ) has 1-dimensional cells. q.e.d.

As A is irreducible, $H^0(\partial M_1; d_A) = 0$. Plug this fact plus Lemma 4.10 into the long exact homology sequence to find $0 = H^1(M_1, \partial M_1; d_A)$. Likewise, $H^1(M_2, \partial M_2; d_A) = 0$ and $H^1(M_0, \partial M_0; d_A) = 0$. Thus, in all cases, $\omega \equiv d_A \mu$ with $d^*_A d_A \mu \equiv 0$ and (as A is irreducible) $\mu = 0$ on ∂M_α. This means that μ and hence ω vanish identically and $\mathrm{coker}(\nabla \not{f}_{\alpha A}) = 0$ as claimed.

The vanishing of $\mathrm{coker}(\nabla \not{f}_{\alpha A})$ implies that $m^0_\alpha \subset \mathscr{B}^0_\alpha$ is an embedded submanifold. Its tangent space at $[A]$ is identified with $\ker(\nabla \not{f}_\alpha|_A) = H^1(M_\alpha; d_A)$. Since M_0 is a product, $H^1(M_0; d_A) \approx H^1(\Sigma; d_A)$. Hence, Lemma 4.8 implies that J is an embedding. (The epimorphism $\pi_1(\Sigma) \to \pi_1(M_{1,2}) \to 0$ implies that J is 1-1.)

The completed proof of Proposition 4.5 requires an identification of the smooth structure on $\mathcal{m}_\alpha^0$. For this purpose, identify $\ker(\nabla \mathcal{f}_{\alpha A})$ with $H^1(M_\alpha; d_A)$. The cohomology group $H^1(M_\alpha; d_A)$ has a (twisted) Čeck-cocycle interpretation. The identification of $\mathcal{m}_\alpha$ with $\mathrm{Hom}(\pi(M_\alpha); \mathrm{SU}(2))/\mathrm{Ad}(\mathrm{SU}(2))$ has a Čeck-cocycle proof. The reader can compare these Čeck descriptions to see that the smooth structure on $\mathcal{m}_\alpha^0$ is the standard one.

5. $\chi(\mathcal{f})$ and Casson's invariant

The comparison of the two invariants is based, ultimately, upon the observation that $\mathcal{f}^{-1}(0)$ is the set of conjugacy classes of irreducible representations of $\pi_1(M)$ in SU(2), and that this set is in 1-1 correspondence with the points of intersection of $\mathcal{m}_1^0$ and $\mathcal{m}_2^0$ in $\mathcal{m}_0^0$. This observation is summarized by the identity

$$\mathcal{m}^0 \equiv f^{-1}(0) = I^{-1}((\mathcal{f}_1^{-1}(0) \times \mathcal{f}_2^{-1}(0)) \cap J^{-1}(\Delta)). \tag{5.1}$$

It is the Fredholm properties of the maps which appear in (5.1) which allow for the comparison of spectral flow data on $\mathcal{B}^0$ with intersection data on $\mathcal{m}_0^0$.

Both invariants are defined, in general, only after perturbations are made. Choosing compatible perturbations for the two cases simplifies the comparison. The first proposition provides a useful set of perturbations.

Proposition 5.1. *Let M be a homology 3-sphere with Heegard splitting. Given $\varepsilon > 0$, there exist admissible (in the sense of Definition 1.4) and compatible (in the sense of Definition 4.6) functions $\mathcal{u}$ on $\mathcal{B}$ which have the following properties: For $\alpha = 0, 1, 2$, introduce the section $\mathcal{f}'_\alpha \equiv \mathcal{f}_\alpha + d\mathcal{u}_\alpha$ of $\mathcal{L}_\alpha$, and introduce as notation, $\mathcal{m}'^0_\alpha \equiv f'^{-1}_\alpha(0) \cap \mathcal{B}_\alpha^0$.*

(1) *The sets $\mathcal{f}'^{-1}_\alpha(0)$ and $\mathcal{f}_\alpha^{-1}(0)$ are identical on an open neighborhood of their intersections with $\mathcal{B}_\alpha \backslash \mathcal{B}_\alpha^0$, that is, on the complement of compact sets in $\mathcal{m}'^0_\alpha$ and $\mathcal{m}_\alpha^0$.*

(2) *The set $\mathcal{m}'^0_\alpha$ is a smoothly embedded submanifold of $\mathcal{B}_\alpha^0$, which is smoothly isotopic to $\mathcal{m}_\alpha^0$ by an ambient isotopy which is the identity on an open neighborhood of the intersection of $\mathcal{m}'^0_\alpha$ with $\mathcal{B}_\alpha \backslash \mathcal{B}_\alpha^0$.*

(3) *The L_1^2-distance moved by a point in $\mathcal{m}'^0_\alpha$ by the isotopy in (2) is less than ε.*

(4) *The function $\mathcal{u}$ obeys all of the assertions of Proposition 1.5; in particular, the perturbation $\mathcal{f}' \equiv \mathcal{f} + d\mathcal{u}$ is nondegenerate in the sense of Definition 1.4.*

(5) *The intersection of $\mathcal{m}'^0_1$ with $\mathcal{m}'^0_2$ in $\mathcal{m}'^0_0$ is transverse.*

The first four assertions are direct corollaries of Proposition 8.7 to which the reader is referred. The last assertion is proved below as Lemma 5.5 after some technical machinery is introduced. But, note that assertion (4) insures that $\not{f}'^{-1}(0)$ is a finite set, and Lemma 4.7 identifies this set with $m_1'^0 \cap m_2'^0$; thus, the latter set is also finite. Transversality is the issue which Lemma 5.5 settles.

To define Casson's invariant and $\chi(\not{f})$, a sign is associated to each point in the finite set $\not{f}'^{-1}(0)$. Comparing these signs is a two-step procedure. One first proves

Proposition 5.2. *Let M be a homology 3-sphere with Heegard splitting, and let u be as given in Proposition 5.1. Use the same notation as in Proposition 5.1. Let $[A]$, $[A'] \subset \mathscr{B}^0$ be two points in $\not{f}'^{-1}(0)$, and let $[A_0] \equiv j_1^* i_1^*[A]$ and $[A_0'] \equiv j_1^* i_1^*[A']$ denote the corresponding two points in $m_1'^0 \cap m_2'^0 \subset m_0'^0$. Then the* mod(2) *spectral flow for $\nabla \not{f}'$ between $[A]$ and $[A']$ vanishes if and only if the local intersection number at $[A_0]$ for $m_1'^0 \cap m_2'^0 \subset m_0'^0$ agrees with that for $[A_0']$.*

The proof of this proposition occupies the remainder of this section and also §6.

Propositions 5.1 and 5.2 have the following immediate corollary:

Proposition 5.3. *Let M be an oriented, homology 3-sphere. Then twice the absolute value of Casson's invariant is equal to $|\chi(\not{f})|$.*

In §7, the signs of the two invariants are compared.

The proofs of Proposition 5.2 and assertion (5) of Proposition 5.1 require the introduction of a new Fredholm operator which comes with the Heegard splitting. This operator allows the finite dimensional intersection theory on m^0 to be reinterpreted as a spectral invariant of an elliptic operator. This reinterpretation occupies most of the present section. In what follows, assume always that a function u on $\mathscr{B}^0$ has been given by Proposition 5.1 and that this function is used to define the perturbationas $\not{f}_\alpha' \equiv f_\alpha + du_\alpha$ for $\alpha = 0, 1, 2$ and $\varnothing$.

Let $[A] \in \mathscr{B}^0$ and set $[A_{1,2}] \equiv i_{1,2}^*[A]$ and $[A_0] \equiv j_1^* i_1^*[A]$. Introduce the (pre-)Hilbert spaces

$$(5.2) \qquad \mathscr{E}_A^0 \equiv \mathscr{T}_{1A} \oplus \mathscr{T}_{2A} \oplus \mathscr{L}_{0A}, \qquad \mathscr{E}_A^1 \equiv \mathscr{L}_{1A} \oplus \mathscr{L}_{2A} \oplus \mathscr{T}_{0A}$$

and introduce the operator $H_A: \mathscr{E}_A^0 \to \mathscr{E}_A^1$ which sends $\omega \equiv (a_1, a_2, u_0)$ to

$$(5.3) \qquad H_A\omega \equiv (\nabla \not{f}_1'|_{A_1} \cdot a_1, \nabla \not{f}_2'|_{A_2} \cdot a_2, Y(a_1, a_2) - (\nabla \not{f}_0'|_{A_0})^* u^0),$$

where, $\nabla \not{f}_\alpha'|_{(\cdot)}$ is defined in (4.10). The adjoint, $(\nabla \not{f}_\alpha'|_A)^*: \mathscr{L}_{\alpha A} \to \mathscr{T}_{\alpha A}$, is discussed in Part 2 of the Appendix. It is defined via integration by parts and the identification $(\mathcal{T}_{\alpha A})^* \approx \mathcal{T}_{\alpha A}$, which is provided by the metric $\langle\cdot,\cdot\rangle_A$ in (A.1): For $a \in \mathcal{T}_{\alpha A}$ and $u \in \mathcal{L}_{\alpha A}$,

$$(5.4)\ \langle(\nabla \not{f}_\alpha'|_A)^* u, a\rangle_{L^2} \equiv \int_{\partial M_\alpha} \operatorname{tr}(u \wedge a) - \int_{M_\alpha} \operatorname{tr}(d_A u \wedge a + *\partial v[A] \cdot u \wedge a)$$

with $\partial v[A]\cdot$ defined in (4.10). The map Y in (5.3) is the composition of three maps; the first is $\Psi\colon T_{1A} \oplus T_{2A} \to T_{0A}$ of (4.7). Follow Ψ by the map from $\mathcal{T}_{0A}$ into $(\mathcal{T}_{0A})^*$ which is given by the L^2-pairing. Finally, use the metric $\langle\cdot,\cdot\rangle_A$ in (A.1) to identify $(\mathcal{T}_{\alpha A})^*$ with $\mathcal{T}_{\alpha A}$.

As the connection A varies in $\mathcal{A}^\#$, the pre-Hilbert spaces $\mathcal{E}_A^{1,0}$ fit together to define smooth vector bundles, $\mathcal{E}^{1,0} \to \mathcal{B}^0$. Let $\operatorname{Fred}_0(\mathcal{E}^1, \mathcal{E}^0) \to \mathcal{B}^0$ denote the fiber bundle whose fiber over $[A]$ is the Banach manifold of bounded, index 0, Fredholm operators from $\mathcal{E}_A^1$ to $\mathcal{E}_A^0$. (These bundles are discussed in Part 1 of the Appendix.)

Proposition 5.4. *Let M be a homology 3-sphere with Heegard splitting. Let $[A] \in \mathcal{B}^0$. Then the operator H_A defines a bounded, Fredholm operator from $\mathcal{E}_A^0$ to $\mathcal{E}_A^1$, and the index of H_A is zero. Furthermore, the assignment of $H_A\colon \mathcal{E}_A^0 \to \mathcal{E}_A^1$ to an orbit $[A] \in \mathcal{B}^0$ defines a smooth section, H, over $\mathcal{B}^0$ of* $\operatorname{Fred}_0(\mathcal{E}^1, \mathcal{E}^0)$.

Proof of Proposition 5.4. The assignment of (a_1, a_2, u_0) in $\mathcal{E}_A^0$ to $(\nabla \not{f}_1'|_{A_a} \cdot a_1, \nabla \not{f}_2'|_{A_2} \cdot a_2, -(\nabla \not{f}_0'|_{A_0})^* u_0) \in \mathcal{E}_A^1$ defines a bounded, index 0, Fredholm operator. This is a direct corollary to Lemma A.4. The assignment of $(a_1, a_2) \in \mathcal{T}_{1A} \oplus \mathcal{T}_{2A}$ to $Y(a_1, a_2) \in \mathcal{T}_{0A}$ defines a compact operator, courtesy of the Rellich lemma which insures that the forgetful map from L_1^2 to L^2 defines a compact operator. This proves the first two assertions of Proposition 5.4. The last assertion follows from Lemma A.4 and the differentiability over $\mathcal{A}$ of the metric $\langle\cdot,\cdot\rangle_A$.

Lemma 5.5. *Let M be a homology 3-sphere with Heegard splitting. Let $[A] \in \mathcal{B}^0$. Then the kernel of H_A is isomorphic to the set of triples* $\{(a_1, a_2, u) \in \ker(\nabla \not{f}_a'|_{A_1}) \times \ker(\nabla \not{f}_2'|_{A_2}) \times \operatorname{coker}(\nabla \not{f}_0'|_{A_0})$: *the L^2-projection of $\Psi(a_1, a_2)$ onto* $\ker(\nabla \not{f}_0'|_{A_0})$ *vanishes*$\}$. *In particular, if $[A] \in \not{f}'^{-1}(0)$, then* $\ker(H_A)$ *is in* 1-1 *correspondence with* $\ker(\nabla \not{f}'|_A)$. *Thus, the intersection of $j_1^* m_1'^0$ with $j_2^* m_0'^0$ is transverse if and only if $[A]$ is a nondegenerate zero of $\not{f}'$.*

Proof of Lemma 5.5. To say that there exists $u \in \underline{\mathcal{L}}_{\alpha A}$ (the completion of $\mathcal{L}_{\alpha A}$) such that $Y(a_1, a_2) - (\nabla \not{f}_0'|_{A_0})^* u = 0$ is to say that the L^2-projection of $\Psi(a_1, a_2)$ onto $\ker(\nabla \not{f}_0'|_{A_0})$ is zero. This proves the first assertion of the lemma. If $\not{f}'(A) = 0$, and $(a_1, a_2) \in \ker(\nabla \not{f}_1'|_{A_1}) \times \ker(\nabla \not{f}_2'|_{A_2})$, then $\Psi(a_1, a_2)$ already lies in $\ker(\nabla \not{f}_0'|_{A_0})$; this is a consequence of assertion (5) of Proposition 4.9 and assertion (3) of Lemma 4.7. A direct calculation can also verify the assertion. Therefore, if $[A] \in \not{f}'^{-1}(0)$, then $\ker(H_A) = \{(a_1, a_2) \in \ker(\nabla \not{f}_1'|_{A_1}) \times \ker(\nabla \not{f}_2'|_{A_2}) : \Psi(a_1, a_2) = 0\}$. (Remember that u has been chosen so that $\not{f}'$ has only nondegenerate zeros.) Lemma 4.3 provides a unique $a \in \mathscr{T}_A$ such that $(a_1, a_2) \equiv \Phi(a)$. Furthermore, $a \in \ker(\nabla \not{f}'|_A)$ when $[A] \in \not{f}'^{-1}(0)$ (an exercise for the reader). This proves the second assertion of the lemma. To establish the third assertion, use assertion (2) of Proposition 5.1. q.e.d.

Let $\mathscr{F}_0$ denote the Banach manifold of real, bounded, index 0, Fredholm operators on a separable Hilbert space. It is a model for the classifying space for real K-theory. There is a class, $\omega_1 \in H^1(\mathscr{F}_0; \mathbb{Z}/2)$, the first Stieffel-Whitney class. The determinant line bundle over $\mathscr{F}_0$ is a real line bundle which is classified by ω_1.

Let $\mathscr{X}$ be a smooth manifold, and let $h: \mathscr{X} \to \mathscr{F}_0$ be a smooth map for which $h^*\omega_1 = 0$. Let $x_{1,2}$ be two points of $\mathscr{X}$ for which the operator $h(x_{1,2})$ is invertible. By transgression, ω_1 defines a relative class, $\delta_h(x_1, x_2) \in H^0(\{x_2, x_2\}; \mathbb{Z}/2)$.

This relative class is defined as follows: Pick a path $\varphi: [0,1] \to \mathscr{X}$ which starts at x_1 and ends at x_2. The composition, $h(\varphi)$, defines a path of Fredholm operators. Let $\mathscr{F}_0^1 \subset \mathscr{F}_0$ denote the subset of operators which have nonempty kernel. Then $\mathscr{F}_0^1$ is a codimension 1 subvariety of $\mathscr{F}_0$ which represents ω_1 [14].

For a generic smooth homotopy (rel$\{0,1\}$) of the map $h[\varphi]$, the image is a smooth path in $\mathscr{F}_0$ which intersects $\mathscr{F}_0^1$ transversly in a finite set. The mod(2) cardinality of this set is, by definition, the number $\delta_h(x_1, x_2)$.

Standard transversality theory insures that this number is independent of the chosen homotopy. For a closed loop in $\mathscr{X}$, the number $\delta_h(x_1, x_2)$ computes the restriction to the loop of $h^*\omega_1$. Since this is assumed to be trivial, $\delta_h(x_1, x_2)$ is independent of the chosen path between x_1 and x_2 and depends only on the endpoints and the map h.

More generally, let $\mathscr{K} \to \mathscr{X}$ be a smooth Banach space bundle with fiber $\mathscr{F}_0$, and let H be a smooth section. Since the general linear group of a Hilbert space is contractible, there are trivializations of $\mathscr{K}$. With respect to a chosen trivialization, h defines a smooth map (also denoted by h) from $\mathscr{X}$ into $\mathscr{F}_0$. The characteristic class $h^*\omega_1$ is independent of the trivialization of $\mathscr{K}$. If $h^*\omega_1 = 0$, the secondary characteristic class $\delta_h(\cdot,\cdot)$ is well defined, and it is also independent of the chosen trivialization.

As a section over $\mathscr{B}^0$ of $\mathrm{Fred}_0(\mathscr{E}^1, \mathscr{E}^0)$, H pulls back the first Stieffel-Whitney class, $\omega_1 \in H^1(\mathscr{B}^0; \mathbb{Z}/2)$. In this case, $\omega_1 = 0$, a fact which follows from Proposition 6.1. Let $[A]$, $[A'] \in \mathscr{B}^0$ be orbits where $\ker(H_{(\cdot)}) = \varnothing$. Then the relative class $\delta_H([A],[A']) \in H^0(\{[A],[A']\})$ is well defined.

The significance of δ_H in the present context stems from the following observation:

Proposition 5.6. *Let* $[A]$, $[A'] \in \mathscr{M}^0$. *Then* $\delta_H([A],[A'])$ *gives the sign difference between the intersection numbers of* $j_1^*\mathscr{M}_1'^0$ *with* $j_2^*\mathscr{M}_2'^0$ *in* $\mathscr{M}_0'^0$ *at the points* $j_1^* i_1^*[A]$ *and* $j_1^* i_1^*[A']$.

An appropriate interpretation of the intersection number will yield the proof of this proposition. Consider two smooth, oriented submanifolds X_1, X_2 in a third oriented manifold, X_0. Suppose that $\dim(X_1)+\dim(X_2) = \dim(X_0) \equiv d_0$. Let $p_0, p_1 \in X_1 \cap X_2$ be distinct points where the intersection is transverse. A calculation of the relative sign difference between the intersection numbers of $X_1 \cap X_2$ at points, p_0, p_1, can be made by choosing a smooth triple (λ, V, h); here, $\lambda: [0,1] \to X$ obeys $\lambda(0) = p_0$ and $\lambda(1) = p_1$, while $V \to [0,1]$ is a smooth, oriented vector bundle, and $h: V \to \lambda^* TX_0$ is a bundle homomorphism. Require that (λ, V, h) have the following two properties: (1) At $t = 0$ and $t = 1$, require that $h: V \approx \lambda^*(TX_1 \oplus TX_2)$ is an orientation preserving isomorphism. (2) Consider $h: V \to \lambda^* TX_0$. Let $\det(h)$ denote the corresponding section of the real line bundle $\Lambda^{d_0}(\lambda^* TX_0) \otimes \Lambda^{d_0} V$. Require that $\det(h)$ be transverse to the zero section.

It is not hard to construct such a triple. If X_2 is path connected, it may be convenient to take λ to lie in X_2. Then, take $V \equiv V' \oplus \lambda^* TX_2$ and $h \equiv h' \oplus \mathrm{id}$, with $h'|_{0,1}: V' \approx \lambda^* TX_1$ an orientation preserving isomorphism.

The two requirements on (λ, V, h) insure that $\det(h)^{-1}(0)$ is a finite set and that the relative orientation of $h(V) = \lambda^*(TX_1 \oplus TX_2)$ and $\lambda^* TX_0$ at $t = \{0,1\}$ differ by the mod(2) cardinality of $\det(h)^{-1}(0)$.

One more result is required before the proof of Proposition 5.6 can be presented. The lemma below summarizes results in [14]:

Lemma 5.7. *Fix an integer* $n \geq 0$, *and let* $\mathscr{F}_n$ *denote the space of real, bounded Fredholm operators of index n on a separable Hilbert space. Introduce* $\mathscr{F}_n^1 \subset \mathscr{F}_n$ *as the subspace of operators which have nonempty cokernel. Then,* $\mathscr{F}_n^1$ *is a subvariety of* $\mathscr{F}_n$ *whose nonsingular part has codimension* $n+1$, *and* $\mathscr{F}_n \backslash \mathscr{F}_n^1$ *is n-connected. Furthermore, let X be a smooth compact manifold of dimension* $\leq n$ *with boundary,* ∂X. *Let* $\varphi: X \to \mathscr{F}_n$ *be a smooth map which maps* ∂X *into* $\mathscr{F}_n \backslash \mathscr{F}_n^1$. *Then, given* $\varepsilon > 0$, *there exists a homotopy of* φ, $\operatorname{rel} \partial X$, *to a map* $\varphi': X \to \mathscr{F}_n \backslash \mathscr{F}_n^1$ *with* $\sup_X |\varphi - \varphi'|_{\mathrm{op}} < \varepsilon$ *and with* $\varphi - \varphi'$ *a smooth family of finite rank operators.*

Proof of Proposition 5.6. Choose a path $\lambda_0: [0,1] \to \mathscr{M}_0'^0$ between $j_1^* i_1^*[A]$ and $j_2^* i_2^*[A']$. Since $\mathscr{M}_2'^0$ is path connected, there is no loss in generality by assuming that $\lambda_0 = j_2^* \lambda_2$, with $\lambda_2: [0,1] \to \mathscr{M}_2'^0$. Using a cut-off function, it is straightforward to construct a path $\lambda: [0,1] \to \mathscr{B}_0$ which covers λ_2, i.e., such that $\lambda_2 = i_2^*\lambda$, $\lambda(0) = [A]$ and $\lambda(1) = [A']$. Set $\lambda_1 \equiv i_1^*\lambda$.

The operators $\nabla \not{f}_1'|_{i_1^*\lambda}$ define a section, φ, over $[0,1]$ of $\lambda^* i_1^* \mathscr{K}_1$ where $\mathscr{K}_1 \to \mathscr{B}_1^0$ is the Banach space bundle whose fiber over $[A]$ is the Banach space of bounded, index $3g-3$, Fredholm maps from $\mathscr{T}_{1A}$ to $\mathscr{L}_{1A}$. At $\{0,1\}$, Proposition 5.1 insures that the cokernels of $\varphi(0)$ and $\varphi(1)$ are trivial. If the cokernel of some $\varphi(t)$ is not trivial, use Lemma 5.7 to construct a small homotopy of φ (rel$\{0,1\}$) to obtain a section φ' for which the cokernel of each $\varphi'(t)$ is trivial.

Use Proposition 5.1 to identify the vector bundle $\ker(\nabla \not{f}_0'|_{j_1^* i_1^* \lambda})$ with $\lambda_0^* T \mathscr{M}_0'^0$ and to identify $\ker(\nabla \not{f}_2|_{i_2^*\lambda})$ with $\lambda_0^* j_{2*} T \mathscr{M}_2'^0$.

Set $V \to [0,1]$ to be the vector bundle $(\ker(\nabla \not{f}_1'|_{i_1^*\lambda}) \oplus \ker(\nabla \not{f}_2'|_{i_2^*\lambda})$. Let $\Pi_{(\cdot)}$ denote the L^2-orthogonal projection in $\lambda_0^* \mathscr{T}_{0(\cdot)}$ onto $\lambda_0^* T \mathscr{M}_0'^0$. Note that $\det(\Pi)^{-1}(0)$ is supported away from $\{0,1\}$. If $\det(\Pi)$ is not transverse, perturb Π (rel$\{0,1\}$) to Π' with $\det(\Pi')$ being transverse. Then (λ_0, V, Π') is a triple from which the relative intersection numbers of $\mathscr{M}_{1,2}'^0$ in $\mathscr{M}_0'^0$ can be computed, and it is apparent that this relative intersection number is equal to $\delta_H([A],[A'])$. Only the orientation of the vector bundle $\ker(\varphi') \to [0,1]$ remains unverified.

The said orientation may be verified by the following device: The space $\mathscr{G}_1 \equiv \operatorname{Aut} M_1 \times \mathrm{SU}(2)$ is homotopy equivalent to $C^\infty(M_1; \mathrm{SU}(2))$ and the latter is path connected. Therefore, the space $\mathscr{B}_1^0$ is simply connected. Choose a homotopy, F, (rel$\{0,1\}$) of λ_1 in $\mathscr{B}_1^0$ to a path, μ_1, which lies in $\mathscr{M}_1'^0$. A smooth section, s, of $F^*\mathscr{K}_1$ is defined on $\partial(\times_2[0,1])$ by $\mu_1^* \nabla \not{f}_1'$ and φ'. The cokernel of s is everywhere trivial. Use Lemma 5.7 to extend s over $\times_2[0,1]$ as a section, S, with trivial cokernel. Such an extension defines a smooth vector bundle, $\ker(S) \to \times_2[0,1]$. However, $\ker(S)|_{[1]\times[0,1]} \equiv \mu_1^* T \mathscr{M}_1'^0$ which shows that the orientation on $\ker(\varphi')$ is consistent.

6. Operators on M and Heegard splittings

Proposition 5.6 has translated a relative intersection number on $\mathscr{M}_0'^0$ into spectral data for a family of Fredholm operators. Proposition 5.2 requires a comparison of this data with the spectral flow data of §2. For this purpose, the family of operators $\nabla \not{f}'$ from §2 must be reconsidered vis-a-vis the Heegard splitting. Such a reconsideration assigns to each $[A] \in \mathscr{B}^0$ a new operator,

$$Q_A: \mathscr{E}_A^0 \to \mathscr{E}_A^1, \tag{6.1}$$

to compare with H_A in (4.3). The operator Q_A is constructed from operators $Q_{1,2A}: \mathcal{T}_{1A} \oplus \mathcal{T}_{2A} \to \mathcal{L}_{1,2A}$ and $Q_{0A}: \mathcal{T}_{0A} \to \mathcal{L}_{0A}$. With these operators defined, Q_A sends $\omega = (a_1, a_2, v_0)$ to

$$(6.2) \qquad Q_A\omega \equiv (Q_{1A}(a_1, a_2), Q_{2A}(a_1, a_2), Y(a_1, a_2) - (Q_{0A}^* v_0).$$

Proposition 6.1. *Let u and $f' \equiv f + du$ be as described in Proposition 5.1. Then for each $[A] \in \mathcal{B}^0$, there exist bounded operators $Q_{1,2A}: \mathcal{T}_{1A} \oplus \mathcal{T}_{2A} \to \mathcal{L}_{1,2A}$ and $Q_{0A}: \mathcal{T}_{0A} \to \mathcal{L}_{0A}$ which are such that $Q_A: \mathcal{E}_A^0 \to \mathcal{E}_A^1$ as given in* (6.2) *is an index* 0, *Fredholm operator. Furthermore, the following hold*:

(1) *$Q_A - H_A$ is a relatively compact operator.*

(2) *If $f'(A) = 0$, then $Q_A = H_A$.*

(3) *The assignment of A to Q_A is naturally $\mathcal{G}$-equivariant, and defines a smooth section, Q, over $\mathcal{B}^0$ of the Banach space bundle* $\operatorname{Fred}_0(\mathcal{E}^1, \mathcal{E}^0)$. *This section, and the section H of Proposition* 5.4, *are homotopic* $\operatorname{rel}(f'^{-1}(0))$.

(4) *Use the maps $I: \mathcal{B}^0 \to \mathcal{B}_1^0 \times \mathcal{B}_2^0$ and $j_1 \circ i_1: \mathcal{B}^0 \to \mathcal{B}_0^0$ of* §*4 to pull the bundles $\mathcal{T}_1 \oplus \mathcal{T}_2$, $\mathcal{T}_0$, and $\mathcal{L}_1 \oplus \mathcal{L}_2$, $\mathcal{L}_0$ back over $\mathcal{B}^0$. Use the same notation to denote the pulled back bundles. Then, the following diagram is a commutative diagram of Fredholm bundle maps* (*over* $\mathcal{B}^0$) :

$$\begin{array}{ccccccccc}
0 \to \mathcal{T} & \xrightarrow{\ \Phi\ } & \mathcal{T}_1 \oplus \mathcal{T}_2 & \xrightarrow{\ \Psi\ } & \mathcal{T}_0 \to 0 \\
\big\downarrow \nabla f' & & \big\downarrow Q_1 \oplus Q_2 & & \big\downarrow Q_0 \\
0 \to \mathcal{L} & \xrightarrow{\ \underline{\Phi}\ } & \mathcal{L}_1 \oplus \mathcal{L}_2 & \xrightarrow{\ \underline{\Psi}\ } & \mathcal{L}_0 \to 0,
\end{array}$$

where the rows are exact, and the Fredholm indices for $Q_1 \oplus Q_2$ and Q_0 are both equal to $6 \cdot g - 6$.

Observe that Proposition 6.1 has Proposition 5.2 as a corollary:

Proof of Proposition 5.2. It follows from assertions (2) and (3) and Proposition 5.6 that the $\mathbb{Z}/2\mathbb{Z}$ characteristic classes $\delta_Q([A], [A'])$ and $\delta_H([A], [A'])$ agree for any pair of orbits $[A], [A'] \in f'^{-1}(0) \cap \mathcal{B}^0$. Assertion (5) establishes that the $\mathbb{Z}/2\mathbb{Z}$ characteristic classes $\delta_Q([A], [A'])$ and $\delta_{\nabla f'}([A], [A'])$ agree for any pair of orbits $[A]$, $[A'] \in f'^{-1}(0) \cap \mathcal{B}^0$). Indeed, choose a splitting $\Theta: \mathcal{T}_0 \to \mathcal{T}_1 \oplus \mathcal{T}_2$ with the property that $\Psi \circ \Theta \equiv 1$. Likewise, choose a splitting $\underline{\Theta}: \mathcal{L}_0 \to \mathcal{L}_1 \oplus \mathcal{L}_2$ with the property that $\underline{\Psi} \circ \underline{\Theta} \equiv 1$. Since the rows in (4) are exact, $\Phi \oplus \Theta: \mathcal{T} \oplus \mathcal{T}_0 \to \mathcal{T}_1 \oplus \mathcal{T}_2$ and $\underline{\Phi} \oplus \underline{\Theta}: \mathcal{L} \oplus \mathcal{L}_0 \to \mathcal{L}_1 \oplus \mathcal{L}_2$ are isomorphisms. Under these isomorphisms, $(Q_1 \oplus Q_2) \circ \Phi \oplus \Theta) = (\underline{\Phi} \oplus \underline{\Theta}) \circ (\nabla f \oplus Q_0)$.

Choose a smooth path $[A(t)]_{t \in [0,1]}$ from $[A]$ to $[A']$. Due to assertion (2), $\operatorname{coker}(Q_0) = \{0\}$ at the endpoints of this path. Lemma 5.7 provides a homotopy (rel$\{0,1\}$) of the path $Q_{0A(t)}$ of index $6g-6$, Fredholm operators to a new path $Q'_{0A(t)}$ for which the cokernel is trivial for all t.

For each t, an index 0, Fredholm operator from $\mathscr{T}_{A(t)} \oplus \mathscr{T}_{0A(t)}$ to $\mathscr{L}_{A(t)} \oplus \mathscr{L}_{0A(t)} \oplus \mathbb{R}^{6g-6}$ is given by $\nabla\!\!\not{f}' \oplus Q_0 \oplus \Pi$, where Π is orthogonal projection onto $\ker(Q_{0A(t)})$. By construction, the δ invariant for this operator is the same as that for $\nabla\!\!\not{f}$.

A deformation (rel$\{0,1\}$) of the path of Fredholm operators $C(t) \equiv Q_{1A(t)} \oplus Q_{2A(t)}$ is defined by the operator $C'(t)\colon \mathscr{T}_{1A(t)} \oplus \mathscr{T}_{2A(t)} \to \mathscr{L}_{1A(t)} \oplus \mathscr{L}_{2A(t)}$ which is given by the formula $C'(t) \circ (\Phi \oplus \Theta) \equiv (\underline{\Phi} \oplus \underline{\Theta}) \circ (\nabla\!\!\not{f}'_{A(t)} \oplus Q'_{0A(t)})$. Then, a path of index 0, Fredholm operator with the same δ invariant as $\nabla\!\!\not{f}$ is defined by $C' \oplus \Pi \circ \Psi$.

This last operator has the same δ invariant as the path of index 0, Fredholm operators $C'' \equiv \{C''(t)\colon \mathscr{T}_{1A(t)} \oplus \mathscr{T}_{2A(t)} \oplus \mathscr{L}_{0A(t)} \to \mathscr{L}_{1A(t)} \oplus \mathscr{L}_{2A(t)} \oplus \mathscr{T}_{0A(t)}\}$ which is given by the formula $C''(t) \cdot (a_1, a_2, v_0) \equiv (C'(t) \cdot (a_1, a_2), Y(a_1, a_2) - Q^*_{0A(t)} v_0)$. Finally, C'' is homotopic (rel$\{0,1\}$) to the path $Q_{A(t)}$ which implies that they have the same δ invariants.

The proof of Proposition 5.2 is completed by the observation that $\delta_{\nabla\!\not{f}}([A],[A'])$ and the mod(2) spectral flow for the $\nabla\!\!\not{f}$ are the same.

Proof of Proposition 6.1. To define $Q_{1,2A}$ it is necessary to introduce a cut-off function, $\beta\colon [5/4, 7/4] \to [0,1]$. Require that $\beta \equiv 0$ on $[5/4, 11/8]$ and that $\beta \equiv 1$ on $[13/8, 7/4]$.

To save notation, write A for $i^*_{1,2}A$ and also for $j^*_1 i^*_1 A$. Associate to each pair $(a_1, a_2) \in \mathscr{T}_{1A} \oplus \mathscr{T}_{2A}$ the unique $\theta(a_1, a_2) \in \Omega^0(M) \times \mathrm{su}(2)$ which solves

$$d_{A^*} d_A \theta - d_{A^*} d_A(\beta \cdot \psi_1(a_1) + (1-\beta) \cdot \psi_2(a_2)) = 0. \tag{6.3}$$

Here, $\psi_{1,2}$ are defined in (4.4). With θ defined, introduce

$$\theta_1 \equiv \theta - \beta \cdot (\psi_1 - \psi_2), \qquad \theta_2 \equiv \theta - (1-\beta) \cdot (\psi_2 - \psi_1) \tag{6.4}$$

as elements of $\Omega^0(M_{1,2}) \times \mathrm{su}(2)$. (6.3) insures that θ_1 and θ_2 are harmonic on M_1 and M_2 respectively.

Next, define $w_1 \in \Omega^0(M_1) \times \mathrm{su}(2)$ to be the unique solution to

$$d_{A^*} d_A w_1 = -\beta \cdot (F_A \wedge \underline{a} - \underline{a} \wedge F_A + d_{A^*} \partial v[A] \cdot \underline{a}) \tag{6.5}$$

with $w_1|_{\partial M_1} = 0$. Here, $\underline{a} \equiv (a_2 + d_A\theta_2) - (a_1 + d_A\theta_1)$ and $\partial v[A] \cdot 1s$ defined in (4.10).

Introduce $w_2 \in \Omega^0(M_2) \times \mathrm{su}(2)$ as the unique solution to

$$d_{A^*} d_A w_2 = (1-\beta) \cdot (F_A \wedge \underline{a} - \underline{a} \wedge F_A + d_{A^*} \partial v[A] \cdot \underline{a}) \tag{6.6}$$

with $w_2|_{\partial M_2} = 0$.

Finally, define $w \in \Omega^0(M) \times \mathrm{su}(2)$ as the unique solution to

$$\begin{aligned} d_{A^*}d_A w = {} & F_A \wedge (\beta \cdot (a_2 + d_A\theta_2) + (1-\beta)\cdot(a_1 + d_A\theta_1)) \\ & - (\beta \cdot (a_2 + d_A\theta) + (1-\beta)\cdot(a_1 + d_A\theta)) \wedge F_A \\ & + \beta \cdot d_A * \partial v[A]\cdot(a_2 + d_A\theta_2) \\ & + (1-\beta)\cdot d_A * \partial v[A]\cdot(a_1 + d_A\theta_1). \end{aligned} \tag{6.7}$$

At this point $Q_{1,2A}$ can be defined:

$$\begin{aligned} Q_{1,2A}(a_1, a_2) \equiv {} & * d_A(a_{1,2} + d_A\theta_{1,2}) + \partial v[A]\cdot(a_{1,2} + d_A\theta_{1,2}) \\ & - d_A(w + w_{1,2}). \end{aligned} \tag{6.8}$$

The important features of $Q_{1,2A}$ are summarized in the following lemma.

Lemma 6.2. *Let $[A] \in \mathscr{B}^0$. Then the following hold:*

(1) *The operator $Q_{1A} \oplus Q_{2A}: \mathscr{T}_{1A} \oplus \mathscr{T}_{2A} \to \mathscr{L}_{1A} \oplus \mathscr{L}_{2A}$ defines a bounded, Fredholm operator.*

(2) *The difference $(Q_{1A} \oplus Q_{2A}) - (\nabla \not{f}'_{1A} \oplus \nabla \not{f}'_{2A})$ is a compact perturbation of $Q_{1A} \oplus Q_{2A}$.*

(3) *This difference vanishes when $\not{f}'(A) = 0$.*

(4) *Let $\underline{\Phi}: \mathscr{L}_A \to \mathscr{L}_{1A} \oplus \mathscr{L}_{2A}$ denote the restriction, $\underline{\Phi} f \equiv (f|_{M_1}, f|_{M_2})$. Let $\Phi: \mathscr{T}_A \to \mathscr{T}_{1A} \oplus \mathscr{T}_{2A}$ be as defined in* (4.7). *Then, $\underline{\Phi}\nabla \not{f}'_A(\cdot) = (Q_{1A} \oplus Q_{2A})\Phi(\cdot)$.*

(5) *The assignment $(Q_{1A} \oplus Q_{2A})$ to A defines a smooth section over $\mathscr{B}^0$ of the fiber bundle $I^*\,\mathrm{Fred}_{6\underline{g}-6}(\mathscr{T}_1 \oplus \mathscr{T}_2; \mathscr{L}_1 \oplus \mathscr{L}_2)$ whose fiber over $[A]$ is the Banach manifold of bounded, index $6g-6$, Fredholm operators from $\mathscr{T}_{1A} \oplus \mathscr{T}_{2A}$ to $\mathscr{L}_{1A} \oplus \mathscr{L}_{2A}$.*

Proof of Lemma 6.2. With Lemma A.4 from the Appendix, assertions (1), (2) and (5) are straightforward applications of the Sobolev inequalities and the Rellich lemma. The details are left to the reader. Assertion (4) requires the facts from Lemma 6.3, below. With Lemma 6.4, assertions (3) and (4) are simple exercises for the reader.

Lemma 6.3. *Let $[A] \in \mathscr{B}^0$ and let $a \in \mathscr{T}_A$. Introduce $\varphi_{1,2} \in \Omega^0(M_{1,2}) \times$* su(2) *as defined in* (4.3). *Then $\theta_{1,2}(\Phi(a)) = \varphi_{1,2}(a)$. In addition, $w_{1,2}(\Phi(a)) \equiv 0$ and $w(\Phi(a)) = u'(a)$ with u' defined in (4.11) using $\alpha = \varnothing$.*

Proof of Lemma 6.3. If $(a_1, a_2) = \Phi(a)$, then $a_1 - d_A\psi_1 = a_2 - d_A\psi_2$ on M_0. It follows that $a_1 + d_A\theta_1 = a_2 + d_A\theta_2$ on $[11/8, 13/8] \times \Sigma$. Define $a' \in \mathscr{T}_A$ by the following rule: On $M_1\backslash[13/8, 7/4]$, set $a' \equiv a_1 + d_A\theta_1$, and on $M_2\backslash[5/4, 11/8]$, set $a' \equiv a_2 + d_A\theta_2$. Note that $a' = a - d_A(\varphi_1 - \theta_1)$ on $M_1\backslash[13/8, 7/4]$, and that $a' = a - d_A(\varphi_2 - \theta_2)$ on $M_2\backslash[5/4, 11/8]$. Since A is irreducible on $[11/8, 13/8] \times \Sigma$, it follows that $\varphi_1 - \theta_1 \equiv \varphi_2 - \theta_2$ on $[11/8, 13/8] \times \Sigma$. Thus a global, harmonic section of $\Omega^0(M) \times \mathrm{su}(2)$ is defined by the pair $(\varphi_1 - \theta_1, \varphi_2 - \theta_2)$. As A is irreducible, this section must vanish. This fact and the unique continuation theorem in [3] imply that

$\varphi_1 \equiv \theta_1$ on M_1 and that $\varphi_2 \equiv \theta_2$ on M_2. The final assertion of the lemma is a direct consequence of the equality between $\theta_{1,2}$ and $\varphi_{1,2}$. q.e.d.

The inclusion of M_0 in $M_{1,2}$ induces the restriction maps, $j^*_{1,2}: \mathscr{L}_{1,2A} \to \mathscr{L}_{0A}$ and their difference $\underline{\Psi}: \mathscr{L}_{1A} \oplus \mathscr{L}_{2A} \to \mathscr{L}_{0A}$ which sends (f_1, f_2) to $|\underline{\Psi}(f_1, f_2) \equiv f_1 - f_2|_{M_0}$. The operator Q_{0A} is constructed so that $\underline{\Psi}(Q_{1A} \oplus Q_{2A}) = Q_{0A} \cdot \Psi$, where Ψ is defined in (4.7).

Some auxiliary constructs are required for the definition of Q_{0A}. Fix $a \in \mathscr{T}_{0A}$ and define $y_1(a) \in \Omega^0(M_1) \times \text{su}(2)$ to be the unique solution to the equation

(6.9) $d_A * d_A y_1 + \beta \cdot (F_A \wedge a - a \wedge F_A + d_A * \partial v[A] \cdot a) = 0$ with $y_1|_{\partial M_1} = 0$.

Likewise, define $y_2(a) \in \Omega^2(M_2) \times \text{su}(2)$ to be the unique solution to the equation

(6.10)

$d_A * d_A y_2 - (1 - \beta) \cdot (F_A \wedge a - a \wedge F_A + d_A * \partial v[A] \cdot a) = 0$ with $y_2|_{\partial M_2} = 0$.

The operator Q_{0A} sends a to

(6.11) $$Q_{0A} \cdot a \equiv (*d_A a + \partial v[A] \cdot a - d_A(y_2 - y_1),$$

The relevant properties of this operator are listed in

Lemma 6.4. *Let $[A] \in \mathscr{B}^0$. Then the following hold:*

(1) *The operator $Q_{0A}: \mathscr{T}_{0A} \to \mathscr{L}_{0A}$ defines a bounded, Fredholm operator.*

(2) *The difference $Q_{0A} - \nabla \not{f}'_{0A}$ is a compact perturbation of Q_{0A}.*

(3) *This difference vanishes when $\not{f}'(A) = 0$.*

(4) *Let $\underline{\Psi}: \mathscr{L}_{1A} \oplus \mathscr{L}_{2A} \to \mathscr{L}_{0A}$ denote the restriction $\underline{\Psi}(f_1, f_2) \equiv (f_1 - f_2)|_{M_0}$. Then, $\underline{\Psi}(Q_{1A} \oplus Q_{2A}) = Q_{0A}\Psi$.*

(5) *The assignment of Q_{0A} to A defines a smooth section over $\mathscr{B}^0$ of the fiber bundle $i_1^* j_1^* \text{Fred}_{6g-6}(\mathscr{T}_0, \mathscr{L}_0)$.*

Proof of Lemma 6.4. With Lemma A.4 from the Appendix, assertions (1), (2) and (5) are straightforward applications of the Sobolev inequalities and the Rellich lemma. The details are left to the reader. Assertion (3) follows from the fact that $u_0, y_{1,2}$ all vanish when $\not{f}'(A) = 0$. Assertion (4) is a direct calculation which is left to the reader. q.e.d.

With Q_A defined by (6.2), assertions (1)–(4) of Proposition 6.1 are all seen as immediate consequences of Lemmas 6.2, 6.3 and Lemma 4.3.

7. The signs of $\chi(\not{f})$ and $\lambda(M)$

A complete proof of the main theorem requires the comparison of the signs of $\chi(\not{f})$ and $\lambda(M)$. The purpose of this section is to establish that these signs agree.

For this purpose, take a Heegard splitting of M with large genus, g, and use this splitting to define $\lambda(M)$.

The sign of $\chi(\not{\ell})$ was defined by considering connections on $M \times \mathrm{SU}(2)$ of the form $A \equiv \Gamma + s \cdot a$, where Γ is the product connection, $0 < s < 1$ and $a \in \Omega^1 \times \mathrm{su}(2)$. No generality will be lost by assuming that $d_\Gamma^* a \equiv 0$. Assume that τ_0 of Definition 2.7 is not zero and that $[A] \in \mathscr{B}^0$ for all s. It follows from the discussion in §2 that for all small s, the operator K_A is nondegenerate. Such small values of s will henceforth be implicitly assumed.

Proposition 7.1. *Let M be an oriented homology 3-sphere with a Riemannian metric and large genus Heegard splitting. Choose $a \in \Omega^1 \times \mathrm{su}(2)$ for which $d_\Gamma^* a \equiv 0$. Introduce the 3×3 matrix τ_a and assume that it has nonzero determinant. Let $A \equiv \Gamma + s \cdot a$ for $s > 0$, but small and define the operators $Q_{\alpha A}$ using Proposition 6.1 with $\alpha \equiv 0$. Introduce the homomorphism Ψ of (4.4) and restrict it to $\Psi\colon \ker(Q_{1A} \oplus Q_{2A}) \to \ker(Q_{0A})$. Also, define a homomorphism $\Psi'\colon \ker(\nabla\not{\ell}_1|_A) \oplus \ker(\nabla\not{\ell}_2|_A) \to \ker(\nabla\not{\ell}_0|_A)$ to be Ψ followed by L^2-orthogonal projection onto $\ker(\nabla\not{\ell}_0|_A)$. Then the following hold:*

(1) *The $\mathrm{su}(2)$-valued 1-from, a, can be chosen so that for all $t \in [0, 1]$, the Fredholm operators $(1-t) \cdot (\nabla\not{\ell}_1|_A \oplus \nabla\mathscr{F}_2|_A) + t \cdot (Q_{1A} \oplus Q_{2A})$ and $(1-t) \cdot \nabla\not{\ell}_0|_A + t \cdot Q_{0A}$ have kernel dimension equal to $6 \cdot g - 6$.*

(2) *In addition, a can be chosen so that for all $t \in [0, 1]$, $(1-t) \cdot H_A + t \cdot Q_A\colon \mathscr{E}_A^0 \to \mathscr{E}_A^1$ is an isomorphism. In particular, both Ψ and Ψ' are isomorphisms.*

(3) *This interpolation of operators induces orientations on $\ker(Q_{1A} \oplus Q_{2A})$ and $\ker(Q_{0A})$ from those on $\ker(\nabla\not{\ell}_1|_A) \oplus \ker(\nabla\not{\ell}_2|_A)$ and $\ker(\nabla\not{\ell}_0|_A)$ respectively. These orientations have the property that $\mathrm{Sign}(\det(\Psi)) = \mathrm{sign}(\det(\Psi') = \mathrm{sign}(\tau_a)$.*

The preceding proposition will be proved shortly. It has the following corollary:

Proposition 7.2. *Let M be an oriented homology 3-sphere. Then $\mathrm{sign}(\lambda(M)) = \mathrm{sign}(\chi(\not{\ell}))$.*

Proof of Proposition 7.2. This is now an automatic consequence of Propositions 5.3 and 6.1. q.e.d.

Proposition 7.1 makes assertions about the kernels of certain elliptic operators. For s small, the operators in question (functorally defined with the connection $\Gamma + s \cdot a$) are perturbations of operators which are defined using the connection Γ. Analytic perturbation theory gives an effective method for computing $\Gamma + s \cdot a$ data in terms of data for Γ. To simplify the terminology for this perturbation theory, in the sequel, $\mathscr{O}(s)$ will be used to signify an s-dependent norm which is of the form $s \cdot u(s)$ with $u(s)$ bounded for s in some neighborhood of 0. (The bound may depend on the fixed 1-form a.)

Introduce the vector space of real-valued harmonic forms, $H^1(M_{0,1,2}) \equiv \{\omega \in \Omega^1(M_{0,1,2}): d\omega = 0, d^*\omega = 0 \text{ and } i^*_{\partial M_{0,1,2}}(*\omega) = 0\}$. There is a natural isomorphism $\Psi: H^1(M_1) \oplus H^1(M_2) \to H^1(M_0)$ which sends (v_1, v_2) to

$$\Psi(v_1, v_2) \equiv v_1 - d\eta_1 - v_2 + d\eta_2, \tag{7.1}$$

where $\eta_{1,2}$ is harmonic on M_0 for the scalar Laplacian, $d * d$, and obeys $i^*_{\partial M_0}(v_{1,2} - d\eta_{1,2})) = 0$. The vector spaces $H^1(M_0)$, $H^1(M_{1,2})$ have dimensions $2g$ and g, respectively.

Define a nondegenerate, symplectic pairing on $H^1(M_0)$ by

$$\langle v, v' \rangle \equiv \int_{M_0} dt \wedge (v \wedge v') = \tfrac{1}{2} \cdot \int_{\Sigma} (v \wedge v'). \tag{7.2}$$

Let $h_1 \equiv \Psi(H^1(M_1) \oplus 0)$ and $h_2 \equiv \Psi(0 \oplus H^1(M_2))$. Then each of $h_{1,2}$ is totally isotropic for the pairing above. (Use Stoke's theorem to prove this.)

To each $v \in H^1(M_0)$, there exists a unique $v^* \in H^1(M_0)$ with the property that for any $w \in H^1(M_0)$,

$$\langle w, v^* \rangle = \int_{M_0} w \wedge *v. \tag{7.3}$$

Note that $v \to v^*$ is an isometry, and $v^{**} = -v$. Since $h_{1,2}$ are totally isotropic, the symplectic dual, *, sends h_1 to h_2 and vice-versa. With the product metric on M_0, the identity

$$*v = -dt \wedge v^* \tag{7.4}$$

holds for any $v \in H^1(M_0)$. Use this product metric in the sequel.

The pairing above extends naturally to $H^1(M_0) \times \mathrm{su}(2)$ if it is defined to send (v_1, v_2) to

$$\langle v, v' \rangle \equiv -\operatorname{tr} \int_{M_0} dt \wedge (v \wedge v'). \tag{7.5}$$

The minus sign in (7.5) is due to the fact that the trace on su(2) is negative definite.

The 1-form a in Proposition 7.1 will be constructed after choosing elements $\omega_1 \in h_1 \otimes \mathrm{su}(2)$ and $\omega_2 \in h^2 \otimes \mathrm{su}(2)$. Write $\Psi^{-1}(\omega_1 - \omega_2) \equiv (\alpha_1, \alpha_2)$. Thus, α_1 restricts to M_0 as $\omega_1 + d_\Gamma \eta_1$ with η_1 harmonic; and α_2 restricts to M_0 as $\omega_2 + d_\Gamma \eta_2$ with η_2 harmonic. Set

$$a \equiv *d_\Gamma((1 - \beta) \cdot \alpha_1 + \beta \cdot \alpha_2) = - * \beta' \cdot (dt \wedge (\alpha_1 - \alpha_2)). \tag{7.6}$$

To compute $\det(\tau_a)$, choose $\sigma \in \mathrm{su}(2)$ with $\mathrm{tr}(\sigma^2) = 1$. Then,

(7.7)
$$\begin{aligned}\tau_a(\sigma,\sigma) &= -\int_{M_0} \mathrm{tr}([\sigma,(1-\beta)\cdot\alpha_1 + \beta\cdot\alpha_2]\wedge\beta'\cdot(dt\wedge[\sigma,\alpha_1-\alpha_2]))\\ &= \int_{M_0}\beta'\cdot dt\wedge\mathrm{tr}([\sigma,\alpha_1]\wedge[\sigma,\alpha_2]) = -2\cdot\langle[\sigma,\omega_1],[\sigma,\omega_2]\rangle.\end{aligned}$$

The last expression proves that τ_a is positive definite when $\omega_2 = -\omega_1^*$ for ω_1 chosen in general position.

Proposition 7.1 requires the consideration of certain families of elliptic operators. For this purpose, introduce

$$E \in \mathrm{Hom}(H^1(M_0)\times\mathrm{su}(2)\oplus H^1(M_0)\times\mathrm{su}(2),\mathrm{su}(2)^*)$$

as the pairing which sends (v, v', σ) to

$$E(v,v')\cdot\sigma \equiv 2\cdot\langle[\sigma,v],v'\rangle. \tag{7.8}$$

Note that $E(v,v') = E(v',v)$.

The following two lemmas will be proved at the end of this section.

Lemma 7.3. *Let a be as described in* (7.6). *Define $C, C'\colon h_1\times\mathrm{su}(2)\oplus h_2\times\mathrm{su}(2)\to\mathrm{su}(2)^*$ by sending (v_1,v_2) to*

$$C(v_1,v_2) \equiv E(\omega_2,v_1) + E(\omega_1,v_2),$$
$$C'(v_1,v_2) \equiv= -(E(\omega_1^*,v_1) + E(\omega_2^*,v_2)).$$

If C and C' are surjective, then for all $s > 0$, but sufficiently small, and for all $t \in [0,1]$,

$$\begin{aligned}\ker((1-t)\cdot\nabla\!\!\!/_0|_{\Gamma+s\cdot a} + t\cdot Q_{0\Gamma+s\cdot a}) &\approx \ker(C')\cap\ker(C)\\ &\subset H^1(M_0)\times\mathrm{su}(2).\end{aligned}$$

This isomorphism is obtained as follows: For each $v \in \ker((1-t)\cdot\nabla\!\!\!/_0|_{\Gamma+s\cdot a} + t\cdot Q_{0\Gamma+s\cdot a}))$, there exists a unique $\omega \in \ker(C')\cap\ker(C)$ with the property that $v-\omega$ is $O(s)$ using the L_1^2-norm on $\Omega^1(M_0)\times\mathrm{su}(2)$.

Lemma 7.4. *Let a be as described in* (7.6). *Define $C_1, C_2\colon h_{1,2}\times\mathrm{su}(2)\to\mathrm{su}(2)^*$ by sending v_1 to $C_1(v_1)\equiv E(\omega_2,v_1)$ and v_2 to $C_2(v_2)\equiv -E(\omega_1,v_2)$. For each $t\in[0,1]$, define a homomorphism $\Theta_t\colon h_1\times\mathrm{su}(2)\oplus h_2\times\mathrm{su}(2)\oplus\mathrm{su}(2)^*\to\bigoplus_3\mathrm{su}(2)$ by sending (v_1,v_2,σ) to*

$$\begin{aligned}&(1-t)\cdot(C_1(v_1)+C_2(v_2), C_1(v_1)-C_2(v_2),\sigma)\\ &\quad + t\cdot(C'(v_1,v_2), C(v_1,v_2), \tau_a\cdot\sigma + C_1(v_1)).\end{aligned}$$

If Θ_t is surjective for all t, then for all $s > 0$, but sufficiently small, and for all $t\in[0,1]$,

$$\ker((1-t)\cdot\nabla\!\!\!/_0|_{\Gamma+s\cdot a} + t\cdot Q_{0\Gamma+s\cdot a}) \approx \ker(\Theta_t) \subset H^1(M_0)\times\mathrm{su}(2).$$

This isomorphism is obtained as follows: For each $v \in \ker((1-t)\cdot\nabla\!\!\!/_0|_{\Gamma+s\cdot a}+t\cdot Q_{0\Gamma+s\cdot a})$, *there exists a unique* $\omega\in\ker(\Theta_t)$ *with the property that* $v-\omega$ *is* $O(s)$ *using the* L^2_1*-norm on* $\Omega^1(M_0)\times\mathrm{su}(2)$.

These formulas simplify when the $\omega_2\equiv-\omega_1^*$. Restrict to this case, and set $\omega\equiv\omega_1$. Then, $(C_1(v_1)+C_2(v_2), C_1(v_1)-C_2(v_2))=(C'(v_1,v_2), C(v_1,v_2))$. In this case, Lemma 7.4 specializes to

Lemma 7.5. *Let a be as described in* (7.6) *with* $\omega_1\equiv\omega$ *and* $\omega_2\equiv-\omega^*$. *Define a homomorphism* $\Theta^*\colon h_1\times\mathrm{su}(2)\oplus h_2\times\mathrm{su}(2)\to\bigoplus_2\mathrm{su}(2)$ *by sending* (v_1,v_2) *to* $(C'(v_1,v_2), C(v_1,v_2))$. *For generic* $\omega\in H_1\times\mathrm{su}(2)$, Θ^* *is surjective. For all* $s>0$, *but sufficiently small, and for all* $t\in[0,1]$,

$$\ker((1-t)\cdot\nabla\!\!\!/_0|_{\Gamma+s\cdot a}+t\cdot Q_{0\Gamma+s\cdot a})\approx\ker(\Theta^*)=\ker(C')\cap\ker(C)\subset H^1(M_0)\times\mathrm{su}(2).$$

This isomorphism is obtained as follows: For each $v\in\ker((1-t)\cdot\nabla\!\!\!/_0|_{\Gamma+s\cdot a}+t\cdot Q_{0\Gamma+s\cdot a})$, *there exists a unique* $\omega\in\ker(C')\cap\ker(C)$ *with the property that* $v-\omega$ *is* $\mathscr{O}(s)$ *using the* L^2_1*-norm on* $\Omega^1(M_0)\times\mathrm{su}(2)$.

Proof of Lemma 7.5. But for the assertion that Θ^* is surjective for generic ω, this is a direct consequence of Lemma 7.4. The proof of surjectivity is simple linear algebra which is omitted. q.e.d.

Lemmas 7.3 and 7.5 imply assertions (1) and (2) of Proposition 7.1. To prove assertion (3) of said proposition, the orientation on the vector spaces $\ker(\nabla\!\!\!/_1|_{\Gamma+s\cdot a})\oplus\ker(\nabla\!\!\!/_2|_{\Gamma+s\cdot a})$ must compare favorably with that on $\ker(\nabla\!\!\!/_0|_{\Gamma+s\cdot a})$. Here, one takes a as in Lemma 7.5. These vector spaces are oriented in the following way: When a connection, A, restricts to one or more of $M_{0,1,2}$ so that $[A]\in\mathscr{M}^0_{0,1,2}$, then $\ker(\nabla\!\!\!/_{0,1,2}|_A)$ is naturally identified with $T\mathscr{M}^0_{0,1,2}|_{[A]}$. Then, following a generic path from A to $\Gamma+s\cdot a$ will orient $\ker(\nabla\!\!\!/_{0,1,2}|_{\Gamma+s\cdot a})$.

This method for computing the orientation of $\ker(\nabla\!\!\!/_{0,1,2}|_{\Gamma+s\cdot a})$ is tractable because $[\Gamma]$ is a limit point of each of $\mathscr{M}^0_{0,1,2}$. Thus, one can find connections A of the form $\Gamma+s\cdot a'$, for small s, which restrict to one of $M_{0,1,2}$ so that $[A]\in\mathscr{M}^0_{0,1,2}$. Near the trivial connection, Γ, such a connection will have the form $A=\Gamma+s\cdot a'+\mathscr{O}(s^2)$ where the su(2) valued 1-form a' restricts to $M_{0,1,2}$ as $v+d_\Gamma\eta$ for $v\in H^1(M_{0,1,2})\times\mathrm{su}(2)$ and $\eta\in\Omega^0(M_{0,1,2})\times\mathrm{su}(2)$.

Calculations are clarified by first considering the general case where the connection $A=\Gamma+s\cdot a'$ with a' unrestricted. For fixed a', introduce the homomorphisms $G_{0,1,2}\colon H^1(M_{0,1,2})\times\mathrm{su}(2)\to\mathrm{su}(2)^*$ which are defined by

$$G_{0,1,2}(v)\cdot\sigma\equiv-\int_{M_{0,1,2}}\mathrm{tr}([\sigma,a']\wedge *v). \tag{7.9}$$

Also, introduce the homomorphism $G'\colon H^1(M_0) \times \mathrm{su}(2) \to \mathrm{su}(2)^*$ which is defined by

$$(7.10) \qquad G'(v) \cdot \sigma \equiv -\int_{M_0} dt \wedge \mathrm{tr}([\sigma, a'] \wedge v).$$

Lemma 7.6. *Let $a' \in \Omega^1(M) \times \mathrm{su}(2)$ be fixed. Suppose that $G_{0,1,2}$ and G' are all surjective. Then, for all s sufficiently small,*

$$\ker(\nabla\!\!\!/_0|_{\Gamma+s\cdot a'}) \approx \ker(G') \cap \ker(G_0) \subset H^1(M_0) \times \mathrm{su}(2),$$

$$\ker(\nabla\!\!\!/_{1,2}|_{\Gamma+s\cdot a'}) \approx \ker(G_{1,2}) \subset H^1(M_{1,2}) \times \mathrm{su}(2).$$

The isomorphisms are obtained as follows: For each $v_0 \in \ker(\nabla\!\!\!/_0|_{\Gamma+s\cdot a'})$, there exists a unique $\omega_0 \in \ker(G') \cap \ker(G_0)$, and for each $v_{1,2} \in \ker(\nabla\!\!\!/_{1,2}|_{\Gamma+s\cdot a'})$, there exists a unique $\omega_{1,2} \in \ker(G_{1,2})$; these are defined so that $v_{0,1,2}-\omega_{0,1,2}$ is L^2-orthogonal in $\Omega^1(M_{0,1,2})\times\mathrm{su}(2)$ to $H^1(M_{0,1,2})\times\mathrm{su}(2)$. Furthermore, $v_{0,1,2} - \omega_{0,1,2}$ is $\mathscr{O}(s)$ using the L^2_1-norm on $\Omega^1(M_0) \times \mathrm{su}(2)$.

This lemma will also be proved at the end of this section.

Orient $H^1(M_1) \approx h_1$ and $\mathrm{su}(2)$ so that when $[\Gamma+s\cdot a']$ restricts to M_1 to lie in m_1^0, then the orientation on $\ker(\nabla\!\!\!/_{1\Gamma+s\cdot a'}) \approx H^1(M_1)/\operatorname{Im}(G_1^*)$ agrees with the orientation which is obtained by identifying $\ker(\nabla\!\!\!/_{1\Gamma+s\cdot a'})$ with $T\mathscr{m}_1^0|_{\Gamma+s\cdot a'}$.

Via the pairing $\langle\ ,\ \rangle$, the orientation of $H^1(M_1)$ will induce orientations on $H^1(M_2)$ and $H^1(M_0)$. It is an exercise to check that this orientation on $H^1(M_2)$ has the property that when $[\Gamma + s\cdot a']$ restricts to M_2 to lie in $\mathscr{m}_2^0$, then the orientation on $\ker(\nabla\!\!\!/_{2\Gamma+s\cdot a'}) \approx H^1(M_2)/\operatorname{Im}(G_2^*)$ agrees with the orientation which is obtained by identifying $\ker(\nabla\!\!\!/_{2\Gamma+s\cdot a'})$ with $T\mathscr{m}_2^0|_{\Gamma+s\cdot a'}$.

Likewise, when $[\Gamma+s\cdot a']$ restricts to M_0 to lie in $\mathscr{m}_0^0$, then the orientation on $\ker(\nabla\!\!\!/_{0\Gamma+s\cdot a'}) \approx H^1(M_0)/\operatorname{Im}(G'^* \oplus G_0^*)$ agrees with the orientation which is obtained by identifying $\ker(\nabla\!\!\!/_{0\Gamma+s\cdot a'})$ with $T\mathscr{m}_0^0|_{\Gamma+s\cdot a'}$.

As a' varies in $\Omega^1(M) \times \mathrm{su}(2)$ keeping $G' \oplus G_0$ and $G_{1,2}$ surjective, the orientations on $\ker(\nabla\!\!\!/_{0,1,2\Gamma+s\cdot a'})$ are determined by the linear maps $G'\oplus G_0$ and $G_{1,2}$. For $a' \equiv a$ as given by (7.6) with $\omega_1 \equiv \omega$ and $\omega_2 \equiv -\omega^*$, one finds that $G_{1,2} \equiv C_{1,2}$, $G_0 \equiv C_0$ and $G' \equiv C'$.

The preceding paragraph and Lemmas 7.3 and 7.5 imply that the linear map Ψ is orientation preserving when a is given by (7.6) using $\omega_1 \equiv \omega$ and with $\omega_2 \equiv -\omega^*$. For such a, $\det(\tau_a) > 0$. These last two facts give assertion (3) of Proposition 7.1. (As a check, note that when a is given by (7.6), using $\omega_1 \equiv \omega$ and with $\omega_2 \equiv +\omega^*$ then Ψ is orientation reversing, but $\det(\tau_a) < 0$.)

Proposition 7.1 is complete with the proofs of Lemmas 7.3, 7.4 and 7.6. All of the assertions of Lemmas 7.3 and 7.6 are proved using standard techniques from linear perturbation theory. Generally, suppose that L_0 is a linear, Fredholm operator from a Hilbert space E_1 to a Hilbert space E_2, and suppose that u is a compact operator from E_1 to E_2. Define a linear map $h[u]\colon \ker(L_0) \to \operatorname{coker}(L_0)$ by sending v to the orthogonal projection of $u \cdot v$ onto $\operatorname{coker}(L_0)$. Suppose that this projection is surjective. Then, for all s sufficiently small, the kernel of $L_0 + s \cdot u$ is isomorphic to $h(u)^{-1}(0) \subset Ker(L_0)$. (This is another way of stating the Fredholm alternative.)

Proof of Lemma 7.6. Consider the situation for $\ker(\nabla\!\!\!/_{0,1,2}|_{\Gamma+s\cdot a'})$: Define the Hilbert spaces $E_1 \equiv \{(v, \varphi) \in L_1^2(\Omega^1(M_{0,1,2})) \times \mathrm{su}(2) \oplus L_1^2(M_{0,1,2}) \times \mathrm{su}(2)\colon i^*_{\partial M_{0,1,2}}(*v) = 0 = i^*_{\partial M_{0,1,2}}(\varphi)\}$ and $E_2 \equiv L^2(\Omega'(M_{0,1,2})) \times \mathrm{su}(2) \oplus L^2(M_{0,1,2}) \times \mathrm{su}(2)$. In all three cases, take for L_0:

$$\begin{matrix} *d_\Gamma & d_\Gamma \\ d_\Gamma^* & 0 \end{matrix}$$

Proof of Lemma 7.3. In this case, take E_1, E_2 and L_0 as described above for M_0. Both y_1 and y_2 of (5.9) and (5.10) have the form $s \cdot y'_{1,2} + s^2 \cdot y''_{1,2}$ with $y''_{1,2}$ being $\mathscr{O}(1)$ in L_2^2 as $s \to 0$. Each $y'_{1,2}$ obeys (5.9) or (5.10) with the operator $d^*_{\Gamma+s\cdot a} d_{\Gamma+s\cdot a}$ replaced by $d^*_\Gamma d_\Gamma$ and with $F_{\Gamma+s\cdot a}$ replaced by da. Using these facts, the lemma follows as an exercise in integrating by parts.

Proof of Lemma 7.4. The proof here requires a less trivial application of perturbation theory. This is because neither $[F_A, \theta_{1,2}]$ nor $d_A w$ in (5.8) need be $\mathscr{O}(s)$ as $s \to 0$. The following model applies: Let $E_{1,2}$ be Hilbert spaces and let $L_0\colon E_1 \to E_2$ again be a Fredholm operator. Introduce $\Pi_{L_0}\colon E_2 \to E_2$ to be the projection onto the cokernel of L_0.

Now, for $s > 0$, let $u(s)\colon E_1 \to E_2$ be a compact operator with smooth s-dependence, but suppose that $u(s)$ is singular at $s = 0$ in the following sense: Assume that there exist auxiliary Hilbert spaces E_3, E_4 and bounded maps $w_{10}, w_{11}\colon E_1 \to E_3$ and $w_2\colon E_4 \to E_2$, and that there exists a bounded, index 0, Fredholm operator $H_0\colon E_4 \to E_3$. Require that w_{10} maps into the orthogonal complement of the cokernel of H_0 and that w_2 maps the kernel of H_0 orthogonal to the cokernel of L_0. Introduce $\Pi_{H_0}\colon E_3 \to E_3$ to denote the orthogonal projections onto the cokernel of H_0.

For all $s \geq 0$, let $v_0, v_1\colon E_4 \to E_3$ be compact operators with v_0 annihilating $\ker(H_0)$. Allow for a smooth s dependence of v_1 for $s \geq 0$. Require that $H_0 + s \cdot v_0 + s^2 v_1(s)$ is invertible for all $s > 0$.

Finally, suppose that $u(s)$ can be factored as

$$u(s) = w_2 \cdot (H_0 + s \cdot v_0 + s^2 v_1(s))^{-1} \cdot (w_{10} + s \cdot w_{11}) + u',$$

where $u'(s): E_1 \to E_2$ is a compact operator which depends smoothly on $s \geq 0$.

Lemma 7.7. *Under a proviso, orthogonal projection in E_1 produces an isomorphism (and, to zeroth order in s, an isometry) between the kernel of $L_0 + s \cdot u(s)$ and the subset of elements $\{\psi + w_2 \cdot \sigma : (\psi, \sigma) \in \ker(L_0) \oplus \ker(H_0)\}$ which obey the following constraints:*

(1) $\Pi_{L_0} \cdot r \cdot (\psi + L_0^{-1} \cdot w_2 \cdot \sigma) = 0.$

(2) $\Pi_{H_0} \cdot (v_1(0) \cdot \sigma - w_{11} \cdot (\psi + L_0^{-1} \cdot w_2 \cdot \sigma)) = 0.$

Here, r is determined by the data $H_0, w_{10}^0, w_{11}^0, w_2, v_1, v_0$ and u'. The proviso requires the surjectivity of the linear map from $\ker(L_0) \oplus \ker(H_0)$ *to* $\operatorname{coker}(L_0) \oplus \operatorname{coker}(H_0)$ *which sends (ψ, σ) to*

$$(\Pi_{L_0} \cdot r \cdot (\psi + L_0^{-1} \cdot w_2 \cdot \sigma), \Pi_{H_0} \cdot (v \cdot \sigma - w_1 \cdot (\psi + L_0^{-1} \cdot w_2 \cdot \sigma))).$$

Proof of Lemma 7.7. This is a judicious application of the Fredholm alternative. q.e.d.

Lemma 7.6 is a direct, though computationally intensive, application of Lemma 7.7. The details are left as an exercise for the reader.

8. Perturbations

The purpose of this section is to study nondegenerate perturbations of the section $\not{f}$ of $T\mathscr{B}^{\#}$.

8a. Properties. The class of admissible functions on $\mathscr{B}$ was introduced in Definition 1.4. The purpose of this subsection is to explore the properties of the admissible functions.

To begin, fix a smoothly embedded loop in M, γ, and consider the function p_γ. The differentiability with the L_1^2-Hilbert manifold structure is not obvious. To consider this question, orient S^1, and introduce the matrix $P_\gamma[t, y; A] \in \mathrm{SU}(2)$ which defines the parallel transport from $\varphi_\gamma(t, y)$ to $\varphi_\gamma(t, y)$ around $\varphi_\gamma(\cdot, y)$ in the positive direction using the connection A.

The differential of $p_\gamma[A]$ is the linear form on $\Omega^1 \times \mathrm{su}(2)$ which assigns to $a \in \Omega^1 \times \mathrm{su}(2)$ the number

$$dp_\gamma|_{[A]}(a) = \int_{S^1 \times D^2} \mathrm{tr}(P_\gamma[t, y; A] \cdot a[t, y]) \cdot \eta(y) \cdot dt \cdot d^2 y. \tag{8.1}$$

Here, $a[t, y] \cdot dt \equiv \varphi(\cdot, y)^* a|_t$.

Observe that

$$|dp_\gamma|_{[A]}(a)| \leq \mathrm{const} \cdot ||a||_{L^1}, \tag{8.2}$$

so that p_γ is at least Lipschitz for the L_1^2-Hilbert manifold structure.

To define the nth derivatives of p_γ, introduce, for ordered pairs of points $(s,t) \in S^1$, the matrix $P_\gamma[t,s,y;A] \in \mathrm{SU}(2)$ which gives the parallel transport in the positive direction along $\varphi_\gamma(\cdot,y)$ from s to t using the connection A. The nth derivatives of p_γ define the n-linear form on $\Omega^1 \times \mathrm{su}(2)$ which sends $\{a_i\}_{i=1}^n$ to the number $\nabla^{(n)}p_\gamma|_{[A]}(\{a_i\}_{i=1}^n)$ which is given by the formula

$$\int_{(\times_n[0,1])\times D^2} \mathrm{tr}(P_\gamma[t_1,t_n,y;A]\cdot a[t_n,y]\cdots P_\gamma[t_2,t_1,y;A]\cdot a[t_1,y]) \cdot \left(\prod_{i=1}^n dt_i\right)\cdot \eta(y)\cdot d^2y, \tag{8.3}$$

which implies that

$$\left|\nabla^{(n)}p_\gamma\,|_{[A]}\,(\{a_i\})_{i=1}^n)\right| \le \int_{D^2}\left(\int_{S^1}|a[s,y]|\cdot ds\right)^n\cdot d^2y. \tag{8.4}$$

Since the map from $C^1(M)$ to $C^{0,1}(D^2)$ which sends f to

$$\int_{S^1} f(\varphi_\gamma(t,y))\cdot dt \tag{8.5}$$

extends to a bounded, linear map from $L_1^2(M)$ to $L_1^2(D^2)$, one obtains the uniform bound

$$\left|\nabla^{(n)}p_\gamma\,|_{[A]}\,(\{a_i\}_{i=1}^n)\right| \le \mathrm{const}\cdot\prod_{i=1}^n(\|\nabla_A a_i\|_{L^2}+\|a_i\|_{L^2}), \tag{8.6}$$

where the constant is independent of the connection A.

(8.6) implies that p_γ is infinitely differentiable with the L_1^2-Hilbert manifold structure.

One last fact: For $n = 2$, (8.6) can be bettered:

$$\nabla dp_\gamma\,|_{[A]}\,(a,b) \le \mathrm{const}\cdot\|a\|_{L^2}\cdot\|b\|_{L^2}, \tag{8.7}$$

with the constant being independent of A.

Select a base point x_0 in M and a finite set, Λ, of closed, embedded loops based at x_0. Require that a subset of Λ generate $\pi_1(M)$. For example, if M is furnished with a Heegard splitting along a genus g surface Σ, one could take for Λ a basis of loops on Σ which generated $H_1(\Sigma)$.

The set Λ of loops indexes the set of functions $\{p_\gamma : \gamma \in \Lambda\}$ on $\mathscr{B}$.

Lemma 8.1. *The set Λ can be chosen so that all of the following hold*:

(1) $(\bigoplus_{\gamma\in A}\ker(dp_\gamma|_{[A]}))\cap\ker(\nabla\not{f}|_{[A]}) = \{0\}$ *whenever* $[A]\in\not{f}^{-1}(0)$.

(2) *The set of functions $\{p_\gamma : \gamma\in\Lambda\}$ separate the points in $\not{f}^{-1}(0)$.*

(3) *Given a Heegard splitting of M, Λ contains only loops which lie in M_0.*

(4) *Given the Heegard splitting, the functions* $\{p_\gamma : \gamma \in \Lambda\}$ *separate the points of* $\mathcal{m}_0 \subset \mathscr{B}_0$ *and embed* $\mathcal{m}_0^0$ *in* $\times_{\gamma\in\Lambda} \mathbb{R}$.

(5) *The image of the reducible connections under* $\{p_\gamma : \gamma \in \Lambda\}$ *is disjoint from the image of* $\not{f}^{-1}(0) \cap \mathscr{B}^\#$.

Proof of Lemma 8.1. Consider assertion (1): Let $[A]$ be a flat connection, and let $v \in \ker(\nabla \not{f}|_A)$. Suppose that $dp_\gamma|_A \cdot v = 0$ for all loops γ in M. Fix a loop γ. Since A is flat, and $d_A v = 0$, the following integral is independent of the point $y \in D^2$:

$$s_\gamma[A](y) \cdot v \equiv \int_{S^1} \operatorname{tr}(P_\gamma[t, y; A] \cdot v[t, y]) \cdot dt. \tag{8.8}$$

Thus, $dp_\gamma|_{[A]}(v) = 0$ if and only if $s_\gamma[A] \cdot v \equiv s_\gamma[A](0) \cdot v = 0$.

If $s_\gamma[A] \cdot v = 0$ for all loops γ in M, it is not hard to construct $\varphi \in \Omega^0 \times \mathrm{su}(2)$ so that $v = d_A\varphi$. As A is irreducible, $v \equiv 0$.

This argument shows that for each orbit of a flat connection $[A] \in \mathscr{B}^\#$, there exists a finite set of loops, $\Lambda[A]$, such that the second assertion of the lemma holds for $[A]$ using the set $\Lambda[A]$. But, then this second assertion holds for all connections $[A']$ is an open neighborhood of $[A]$ in $\mathscr{B}^\#$. The first assertion of the lemma follows now because $\not{f}^{-1}(0)$ is compact.

To prove the second assertion of the lemma, consider distinct orbits, $[A]$, $[A']$, of flat connections. Each is determined by its holonomy; it is the holonomy which identifies $\not{f}^{-1}(0)$ with $\mathrm{Hom}(\pi_1(M); \mathrm{SU}(2))/\mathrm{Ad}(\mathrm{SU}(2))$. Thus, one can find a loop, $\gamma \equiv \gamma[A, A']$, with the property that p_γ distinguishes $[A]$ and $[A']$. By the continuity of p_γ, there will be an open neighborhood of $([A], [A'])$ in $\times_2 \mathscr{B}^\#$ on which $p_\gamma \times p_\gamma$ misses the diagonal in $\mathbb{R} \times \mathbb{R}$. Therefore, once a compact subset in the complement of the diagonal in $\times_2 \not{f}^{-1}(0)$ is fixed, there is a finite set Λ of loops with the property that for any $([A], [A'])$ in the said compact set, there is a loop in Λ for which the resulting p function takes distinct values on $[A]$ and $[A']$. This fact and the first assertion of the lemma imply the second assertion.

The third assertion of the lemma follows from the fact that the inclusion of M_0 in M induces a surjective group homomorphism $\pi_1(M_0) \to \pi_1(M)$. The fourth assertion of the lemma has a proof which mimics the arguments for assertions (1) and (2). The details are left to the reader.

The fifth assertion of the lemma is satisfied because $\pi_1(M)$ is equal to its own commutator subgroup. Thus, assertions (1), (2) and (5) can be satisfied with loops of the form $\gamma \equiv \gamma_1 \cdot \gamma_2 \cdot \gamma_1^{-1} \cdot \gamma_2^{-1}$, where $\gamma_i^{-1}(t) \equiv \gamma_i(-t)$ (for $t \in S^1$), and $\cdot$ is the composition law on the space of loops. On such a loop, the function p_γ has value 2 on any reducible connection. q.e.d.

To construct a perturbation of $\not{f}$, introduce the vector bundle $\mathscr{L} \to \mathscr{B}^\#$ of §4. Since the L_1^2-Hilbert space structure on $\mathscr{B}^\#$ is used, $\mathscr{L}$ differs from

$T\mathscr{B}^\#$ in that the L^2-Hilbert space structure is used on the fibers. As $\mathscr{L}$ is fiberwise its own dual, the forgetful functors give the continuous, injective bundle maps $T\mathscr{B}^\# \to \mathscr{L} \to (T\mathscr{B}^\#)^*$. As in §4, the bundle $\mathscr{L}$ is introduced because the section $\mathscr{f}$ of $(T\mathscr{B}^\#)^*$ is actually a smooth section of $\mathscr{L}$. And, it is a Fredholm section of $\mathscr{L}$ in that $\nabla\mathscr{f}: T\mathscr{B}^\# \to \mathscr{L}$ defines a continuous family of bounded, Fredholm maps.

Now, use Lemma 8.1 to provide the set Λ of loops; let N denote the number of loops in Λ. Choose a smooth function $f: \mathbb{R}^N \to \mathbb{R}$, and define a function on $\mathscr{B}$ by $\mathscr{u} \equiv f(\{p_\gamma\}_{\gamma\in\Lambda})$, and a section of $\mathscr{L} \to \mathscr{B}^\#$ by sending $[A]$ to $\mathscr{f}'|_{[A]} \equiv (\mathscr{f} + d\mathscr{u})|_{[A]}$.

The next three lemmas summarize important facts about $\mathscr{f}'$.

Lemma 8.2. *The section $\mathscr{f}'$ as defined above is a smooth section of $\mathscr{L}$ over $\mathscr{B}^\#$ which extends continuously to $\mathscr{B}$. Furthermore, the covariant derivative $\nabla\mathscr{f}': T\mathscr{B}^\# \to \mathscr{L}$ defines a smoothly varying family of Fredholm operators, and $\nabla\mathscr{f}' - \nabla\mathscr{f}$ defines a smoothly varying family of compact operators.*

The next lemma describes $\mathscr{f}'^{-1}(0)$.

Lemma 8.3. (1) *Suppose that $[A]$ is an L^2_1-connection on $M \times \mathrm{SU}(2)$ which obeys $f'_{[A]} \equiv 0$. Then $[A]$ is a smooth connection whose derivatives to any finite order obey a priori estimates which depend solely on the Riemannian structure on M and on certain norms of the derivatives of the function f.*

(2) *For any f, $\mathscr{f}'^{-1}(0)$ is compact.*

(3) *Given $\delta > 0$, there exists $\varepsilon > 0$ such that when the function f on $\mathbb{R}^N$ obeys $|df| < \varepsilon$, then $\mathscr{f}'^{-1}(0)$ lies within an L^2_1-distance δ of $\mathscr{f}^{-1}(0)$.*

Consider a specific choice for $\mathscr{u}$. Use the functions $\{p_\gamma\}_{\gamma\in\Lambda}$ to identity $\mathscr{f}^{-1}(0)$ with its range in $\mathbb{R}^N$. Introduce the set $\mathscr{M}^*$ as defined in Proposition 1.5. Let q be a smooth function on $\mathbb{R}^N$ which is identically 1 on a neighborhood of $\mathscr{f}^{-1}(0)\backslash\mathscr{M}^*$.

Define a section of $\mathscr{L} \to \mathscr{B}^\# \times \mathbb{R}^N$ by sending $([A], \{s_\gamma\}_{\gamma\in\Lambda})$ to

$$\mathscr{s}([A], \{s_\gamma\}) \equiv \left(\mathscr{f} + d\left(q(\{p_\gamma\}) \cdot \sum_{\gamma\in\Lambda} s_\gamma \cdot p_\gamma\right)\right)\Bigg|_{[A]}. \tag{8.9}$$

Let $B(\varepsilon)$ denote the ball of radius $\varepsilon > 0$ about the origin in $\mathbb{R}^N$.

Lemma 8.4. *There exists $\varepsilon > 0$ (which depends on the choice of q) such that the section $\mathscr{s}$ as defined above is transverse to the zero section of $\mathscr{L}$ when restricted $\mathscr{B}^\# \times B(\varepsilon)$.*

Proof of Lemma 8.2. The smoothness of $\mathscr{f}'$ is a consequence of (8.6). The behavior of $\nabla\mathscr{f}' - \nabla\mathscr{f}$ is a consequence of (8.7).

Proof of Lemma 8.3. The third assertion is a consequence of K. Uhlenbeck's compactness theorems in [18]. Indeed, if $[A] \subset \mathscr{f}'^{-1}(0)$, then (8.2) insures that $\|F_A\|_{L^2} < \text{const} \cdot \varepsilon$. The second assertion of the lemma follows from the first. The first assertion is proved by standard elliptic techniques. In fact, the equation $\mathscr{f}'_{[A]} = 0$ is an equation of the form $F_A = v[A]$. If A is an L_1^2-connection, then (8.2) asserts that $v[A]$ is uniformly bounded in L^∞. Thus, F_A is uniformly bounded in L^∞. Using Uhlenbeck's results in [18], this puts the connection in L_1^p for all p. Then, so is $v[A]$. But now bootstrapping, this puts F_A in L_1^p for all p and hence, the connection is in L_2^p for all p. Continuing in this vein completes the proof.

Proof of Lemma 8.4. Lemma 8.1 insures that $\mathscr{s}$ is transverse to the zero section at the points $\mathscr{f}^{-1}(0) \times \{0\} \subset \mathscr{B}^\# \times B(\varepsilon)$. Therefore, by continuity, there exist δ, $\varepsilon > 0$ such that this assertion holds for all points of the form $[A] \times \{s_\gamma\}$ with $[A]$ of L_1^2-distance less than δ from $\mathscr{f}^{-1}(0)$ and with $\{s_\gamma\}$ in $B(\varepsilon)$. Now appeal to Lemma 8.3. q.e.d.

Together, these three results have the following corollary:

Proposition 8.5. *Let q be a smooth function on $\mathbb{R}^N$, which is identically 1 on a neighborhood of $\mathscr{f}^{-1}(0)\backslash\mathscr{M}^*$. Then there exists $\varepsilon > 0$ such that for $\{s_\gamma\}$ in the complement of a set of measure zero in $B(\varepsilon)$, the section $\mathscr{f}' \equiv f + d(q(p) \cdot \sum_{\gamma\in\Lambda} s_\gamma \cdot p_\gamma)$ of $\mathscr{L}$ has a finite set of transverse zeros on $\mathscr{B}^\#$, each the orbit of a smooth connection. Furthermore, given $\delta > 0$, one can take ε sufficiently small so that every zero of $\mathscr{f}'$ is within an L_1^2-ball of radius δ of a zero of $\mathscr{f}$.*

Proof of Proposition 8.5. The first assertion is an application of the Smale-Sard theorem using Lemma 8.4. The second assertion was established in 8.3. q.e.d.

Proposition 8.5 establishes the existence of a nondegenerate perturbation of $\mathscr{f}$ as defined in §1.

Proof of Proposition 1.5. This is an immediate corollary of Lemma 8.1 and Proposition 8.5.

8b. Invariance under perturbations. The number $\chi(\mathscr{f})$ was defined in §2 after choosing a nondegenerate perturbation of $\mathscr{f}$ to $\mathscr{f}'$ as specified in Proposition 1.5. The number was asserted to be independent of the perturbation; the proof of this fact will be given here.

Let $\mathscr{u}_{0,1}$ be admissible functions on $\mathscr{B}$ as given in Definition 1.4. Consider the 1-parameter family of perturbations of $\mathscr{f}$ which sends $t \in \mathbb{R}$ to $\mathscr{f}_t \equiv \mathscr{f} + d\mathscr{u}_0 + t \cdot d\mathscr{u}_1$.

Suppose that $\mathscr{f}_{\pm 1}$ are nondegenerate perturbations of $\mathscr{f}$ in the sense of Definition 1.4. Propositions 2.3 and 2.11 assert that $\chi(\mathscr{f}_1)$ and $\chi(\mathscr{f}_{-1})$

agree. This fact is a straightforward consequence of the theory of nonlinear Fredholm maps (see [9]). The maps in question being $\not{f}_{\pm 1}$, thought of as maps from $\mathscr{B}^{\#}$ to $\mathscr{L}$ which commute with the vector bundle projection. Lemmas 1.2 and 1.3 also follow readily once this nonlinear Fredholm theory is set up.

The application of the abstract Fredholm theory to this gauge theoretic context is not complicated. There is a basic construction to reduce the problem to finite dimensions. Since all of the perturbations in question have compact zero sets (courtesy of Lemma 8.3), standard, finite dimensional degree theory then applies. The following is an outline of the reduction to finite dimensions.

To begin, fix attention on a zero $[A]$ of $\not{f}_t$ in $\mathscr{B}^{\#}$, and for notational convenience $t = 0$ will be taken. There exists $\delta > 0$ such that a neighborhood $[A]$ in $\mathscr{B}^{\#}$ is diffeomorphic (by construction) to

$$(8.10)\qquad \mathscr{N}(\delta) \equiv \{a \in L_1^2(\Omega^1 \times \mathrm{su}(2)) \colon d_A^* a = 0 \text{ and } \|a\|_A^2 \equiv \|\nabla_A a\|_{L^2}^2 + \|a\|_{L^2}^2 < \delta^2\}.$$

For $\delta > 0$, but sufficiently small, a trivialization of $\mathscr{L} \to \mathscr{N}(\delta)$ is defined by a map $l \colon \mathscr{L}|_{\mathscr{N}(\delta)} \to \mathscr{N}(\delta) \times \mathscr{L}_A$ which sends (a, f) to

$$(8.11)\qquad l(a) \cdot f \equiv f - d_A \chi,$$

where $\chi \equiv \chi(a, f) \in L_1^2(\Omega^0 \times \mathrm{su}(2))$ is the unique solution to the equation

$$(8.12)\qquad d_A * d_A \chi = *(a \wedge *f - *f \wedge a).$$

The Sobolev inequalities and Kato's inequality insure that $\|\chi\|_A \leq C[A] \cdot \|a\|_{L^4} \cdot \|f\|_{L^2} \leq C'[A] \cdot \|a\|_A \cdot \|f\|_{L^2}$ where c and c' are constants which depend on $[A]$ through the first eigenvalue of the operator $d_A^* d_A$ on $L^2(\Omega^0 \times \mathrm{su}(2))$.

Note that χ, being jointly linear in a and in f, has uniform estimates on the norms of its derivatives.

Let Π denote the L^2-orthogonal projection in $\mathscr{T}_A$ onto the kernel of $\nabla \not{f}_{0A}$. Let $v \in \ker(\nabla \not{f}_{0A})$. There exists $\delta_0 > 0$ such that if $v \in \mathscr{N}(\delta_0)$ and $|t| < \delta_0$, then there exists a unique $a'(v, t) \in \mathscr{N}(\delta)$ which solves the equation

$$(8.13)\qquad (1 - \Pi) \cdot l(v + a') \cdot (\not{f}_{0A+y+a'} + t \cdot d\omega_{1A+4+a'}) = 0 \in (1 - \Pi) \cdot \mathscr{L}_A.$$

This assertion is proved using the inverse function theorem applied to a map from $(1 - \Pi) \cdot \mathscr{L}_A$ to itself. The inverse function theorem insures that $a'(\cdot)$ depends smoothly on its arguments.

Note that $A + v + a'(v, t)$ is a zero of $\mathscr{f}_0 + t \cdot d\mathscr{u}$ if and only if v is a zero of the smooth map $S: \ker(\nabla \mathscr{f}_{0A}) \to \ker(\nabla \mathscr{f}_{0A})$ which sends v to

$$S(v) \equiv \Pi \cdot l(v + a'(v, t)) \cdot (\mathscr{f}_{0A+v+a'} + t \cdot d\mathscr{u}_{1A+v+a'}). \tag{8.14}$$

The finite dimensional map S is the local reduction of the original infinite dimensional system. Having constructed S, the proof of Proposition 2.3 is reduced to finite dimensional degree theory for the map S (see [9]). Note also that (8.14) establishes Lemma 1.2.

Lemma 1.3 is established by a similar argument, but one must deal with the singularity in the manifold structure on $\mathscr{B}$ which occurs at the trivial connection. To do this, fix a base point x_0 in M and introduce the group $\mathscr{G}_0$ as in §2. Then, construct an SO(3) equivariant version of S by working SO(3) equivariantly on $\mathscr{A}/\mathscr{G}_0$.

8c. Perturbations and Heegard splittings. Choose a set Λ of loops of size $N < \infty$. As allowed by Lemma 8.1, choose the loops to lie in the interior of M_0. In defining the functions $\{p_\gamma\}_{\gamma \in \Lambda}$, make sure the tubular neighborhood about each loop γ lies inside M_0. Set $\mathscr{u} \equiv f(\{p_\gamma\})$, where $f: \mathbb{R}^N \to \mathbb{R}$ is a smooth function. Set $\mathscr{f}' \equiv \mathscr{f} + d\mathscr{u}$. This is a perturbation of $\mathscr{f}$ as described in Definition 1.4; one which is compatible with the Heegard splitting in the sense of Definition 4.6.

Note that $d\mathscr{u}|_A$ defines, via (8.1), an L^2 and L^∞ section of $\Omega^1 \times \mathrm{su}(2)$; this section is denoted $v[A]$. A formula for $v[A]$ is provided in (4.9).

The assignment of $v[A]$ to $[A] \in \mathscr{B}_{0,1,2}$ defines a section, $d\mathscr{u}_{0,1,2}$, of the vector bundle $\mathscr{L}_{0,1,2}$ of (4.8). For $\alpha = 0, 1, 2$, introduce the section $\mathscr{f}'_\alpha \equiv \mathscr{f}_\alpha + d\mathscr{u}_\alpha$ of the vector bundle $\mathscr{L}_\alpha \to \mathscr{B}_\alpha$.

In each case $\alpha = 0, 1, 2$, the section $\mathscr{f}'_\alpha$ is a relatively mild deformation of $\mathscr{f}_\alpha$. Of particular interest is the behavior of $\mathscr{f}'^{-1}_\alpha(0)$.

Lemma 8.6. *Let M be a homology 3-sphere with Heegard splitting. Let $\mathscr{u}$ be admissible in the sense of Definition 1.4, and compatible with the Heegard splitting in the sense of Definition 4.6. For $\alpha = 0, 1, 2$, introduce the perturbation $\mathscr{f}'_\alpha \equiv \mathscr{f}_\alpha + d\mathscr{u}_\alpha$.*

(1) *Suppose that $[A]$ is an L^2_1-connection on $M_\alpha \times \mathrm{SU}(2)$ which obeys $\mathscr{f}'_\alpha|_{[A]} \equiv 0$. Then $[A]$ is a smooth connection whose derivatives to any finite order obey a priori estimates which depend solely on the Riemannian structure on M_α and certain norms of the derivatives of the function f.*

(2) *For any f, $\mathscr{f}'^{-1}_\alpha(0)$ is compact.*

(3) *Given $\delta > 0$, there exists $\varepsilon > 0$ such that when the function f on $\mathbb{R}^N$ obeys $|df| < \varepsilon$, then $\mathscr{f}'^{-1}_\alpha(0)$ lies within an L^2_1-distance δ of $\mathscr{f}^{-1}(0)$.*

Proof of Lemma 8.6. Except for potential complications due to the boundary of M_α, this lemma has virtually the same proof as Lemma 8.3.

However, for any connection A, the support of $v[A]$ is disjoint from ∂M_0, so any connection A which is a zero of $\mathscr{f}_\alpha'$ restricts to a neighborhood of ∂M_0 as a flat connection. Thus, the boundary question is moot.

The set $\mathscr{f}_\alpha'^{-1}(0) \subset \mathscr{B}_\alpha$ is a deformation of $\mathscr{f}_\alpha'^{-1}(0)$; the next lemma and proposition describe this set. To state the result, introduce the notation $\mathscr{m}_\alpha'^0$ for $\mathscr{f}_\alpha'^{-1}(0) \cap \mathscr{B}_\alpha^0$.

Proposition 8.7. *The set of loops Λ, a function $q: \mathbb{R}^N \to \mathbb{R}$, and, given $\delta > 0$, a number $\varepsilon > 0$ can be chosen so that for $\{s_\gamma\} \subset B(\varepsilon) \subset \mathbb{R}^N$, the function $\mathscr{u} \equiv q(\{p_\gamma\}) \cdot \sum_{\gamma\in\Lambda} s_\gamma \cdot p_\gamma$ has the following properties:*

(1) *The sets $\mathscr{f}_\alpha'^{-1}(0)$ and $\mathscr{f}_\alpha^{-1}(0)$ are identical on an open neighborhood of their intersections with $\mathscr{B}_\alpha \backslash \mathscr{B}_\alpha^0$, that is, on the complement of compact sets in $\mathscr{m}_\alpha'^0$ and $\mathscr{m}_\alpha^0$.*

(2) *The set $\mathscr{m}_\alpha'^0$ is a smoothly embedded submanifold of $\mathscr{B}_\alpha^0$, which is smoothly isotopic to $\mathscr{m}_\alpha^0$ by an ambient isotopy which is the identity on an open neighborhood of the intersection of $\mathscr{m}_\alpha'^0$ with $\mathscr{B}_\alpha \backslash \mathscr{B}_\alpha^0$.*

(3) *The L_1^2-distance moved by a point in $\mathscr{m}_\alpha'^0$ by the isotopy in* (2) *is less than δ.*

(4) *For an open, dense set $\{s_\gamma\} \subset B(\varepsilon)$, the conditions of Proposition* 1.5 *are met.*

Proof of Proposition 8.7. Use Lemma 8.1 to choose some $N < \infty$ loops Λ in the interior of M_0, so that the set of functions $\{p_\gamma\}_{\gamma\in\Lambda}$ separates the points of $\mathscr{m}_0$, and defines an embedding of $\mathscr{m}_0^0$ into $\mathbb{R}^N$.

Except for the orbit of the trivial connection, $\mathscr{m}_1$ intersects $\mathscr{m}_2$ in a compact subset of $\mathscr{m}_0^0$. This allows the function q to be chosen so that q vanishes on an open neighborhood of $\mathbb{R}^N$ of the points in $\mathscr{m}_0 \backslash \mathscr{m}_0^0$. It also allows for q to be chosen so that, vis-a-vis assertion (4), the requirements of Proposition 1.5 are satisfied for a dense set in $B(\varepsilon)$.

Assertions (1) and (2) are direct consequences of the implicit function theorem and Proposition 4.5 provided that (8.2), (8.6) and (8.7) are used to control the perturbation $d\mathscr{u}_\alpha$. The proofs are straightforward modifications of the proofs of Lemmas 8.2–8.4 and Proposition 8.5. See also the discussion is §8b, above. The details are left to the reader. Assertion (3) follows from Lemma 8.6.

Appendix

The purpose of this Appendix is to summarize the Fredholm properties of the various operators and families of operators which are needed for the proof of the equivalence of Casson's invariant and $\frac{1}{2} \cdot \chi(\mathscr{f})$. The proofs of the assertions below will be mostly brief, as they amount to straightforward applications of the Sobolev inequalities, the Rellich lemma and Kato's inequality (see [1], or in the gauge theory context, see the appendix in [12] and [17]). Proposition 2.1 is proved at the end of the Appendix.

To begin, let X be an oriented, Riemannian 3-manifold with boundary, ∂X. Introduce $\mathscr{A}$ to denote the space of smooth connections on $X \times \mathrm{SU}(2)$. Using the product connection, identify $\mathscr{A}$ with $\Omega^1(X) \times \mathrm{su}(2)$ and use the L^2_1-inner product on the latter space to give $\mathscr{A}$ the structure of a manifold, modelled on a pre-Hilbert space.

Part 1. The operators. Introduce $\mathscr{W}^1 \equiv \{a \in \Omega^1 \times \mathrm{su}(2) \colon i^*_{\partial X}(*a) \equiv 0\}$ and $\mathscr{W}^0 \equiv \{\lambda \in \Omega^0 \times \mathrm{su}(2) \colon i^*_{\partial X}(\lambda) \equiv 0\}$. Use the L^2_1-inner product to define a pre-Hilbert space structure on $\mathscr{W} \equiv \mathscr{W}^1 \oplus \mathscr{W}^0$. Let $\mathscr{J}^1 \equiv \Omega^1 \times \mathrm{su}(2)$ and $\mathscr{J}^0 \equiv \Omega^0 \times \mathrm{su}(2)$; use the L^2-inner product to define a pre-Hilbert space structure on $\mathscr{J} \equiv \mathscr{J}^1 \oplus \mathscr{J}^0$.

Introduce $\mathscr{G}$ to denote $C^\infty(X; \mathrm{SU}(2))$. Let $\mathscr{G}$ act on $\mathscr{W}$ and on $\mathscr{J}$ by conjugation, and let $\mathscr{G}$ act on $\mathscr{A}$ in the usual way. The products $\mathscr{A} \times \mathscr{W}$ and $\mathscr{A} \times \mathscr{J}$ then define smooth vector bundles over $\mathscr{A}$ to which the $\mathscr{G}$-action on $\mathscr{A}$ lifts. Since the L^2-inner product is being used to define the pre-Hilbert space structure on $\mathscr{J}$, this metric defines a smooth, $\mathscr{G}$-invariant fiber metric for the vector bundle $\mathscr{A} \times \mathscr{J}$.

To make a $\mathscr{G}$-invariant metric on $\mathscr{A} \times \mathscr{W}$, define, for each connection $A \in \mathscr{A}$, the metric $\langle \cdot, \cdot \rangle_A$ on $\mathscr{W}$ for which the inner product of $\psi, \psi' \in \mathscr{W}$ is given by

$$\langle \psi, \psi' \rangle_A \equiv \langle \nabla_A \psi, \nabla_A \psi' \rangle_{L^2} + \langle \psi, \psi' \rangle_{L^2}. \tag{A.1}$$

It is a standard exercise with the Sobolev inequalities to show that as A varies in $\mathscr{A}$, the metric above varies smoothly to define a $\mathscr{G}$-equivariant fiber metric on the vector bundle $\mathscr{A} \times \mathscr{W}$. That is, the metric on the fiber $\mathscr{W}$ of $\mathscr{A} \times \mathscr{W}$ over a connection A is the metric that is given in (A.1).

Let A be a connection on $P \equiv X \times \mathrm{SU}(2)$. (2.7) defines a bounded operator from $\mathscr{W}$ to $\mathscr{J}$, which will be denoted by K_A.

Lemma A.1. *Let X be a compact, connected, oriented, Riemannian 3-manifold with boundary. Let A be a smooth connection on $X \times \mathrm{SU}(2)$. Then $K_A \colon \mathscr{W} \to \mathscr{J}$ defines a bounded, Fredholm operator on the respective completions. The index of K_A is equal to $-\frac{3}{2} \cdot \chi_{\partial X}$, where $\chi_{\partial X}$ is the Euler characteristic of ∂X. By varying the connection, a smooth, $\mathscr{G}$-equivariant map, $K_{(\cdot)}$, is defined from $\mathscr{A}$ into the space of bounded, real Fredholm operators from $\mathscr{W}$ to $\mathscr{J}$.*

Proof of Lemma A.1. The assertion that K_A is Fredholm is a standard calculation (see [6] and [5]); K_A is the twisted signature operator on the

3-manifold with boundary. A change $A \to A + a$ in A changes K by a zeroth order term which is linear (over $C^\infty(X)$) in the components of the su(2) valued 1-form a. Thus, the index of K_A is independent of the connection A. For this reason, the index calculation reduces to DeRham theory; choose A to be the trivial connection and one is calculating Betti numbers of X. The last assertion of the lemma follows readily from the fact that K_A differs from K_{A+a} by a multiplication operator which is linear over $C^\infty(X)$ in the components of a. This is an application of the Sobolev inequalities. q.e.d.

Fix an irreducible connection A on P and fix $a \in \mathscr{W}^1$. Define $v \equiv v_A(a) \in \Omega^0(X) \times \mathrm{su}(2)$ to be the unique solution to the equation

$$(\mathrm{A}.2) \qquad d_A^* d_A v = d_A^* a, \qquad i_{\partial X}^*(*d_A v) \equiv 0.$$

Define $u \equiv u_A(a) \in \Omega^0 \times \mathrm{su}(2)$ to be the unique solution to the equation

$$(\mathrm{A}.3) \quad d_A^* d_A u = -*(F_A \wedge (a - d_A v) - (a - d_A v) \wedge F_A), \qquad i_{\partial X}^*(u) \equiv 0.$$

For $k \equiv 0, 1, 2$, use Ω_k^0 to denote the pre-Hilbert space whose point set is $\Omega^0(X) \times \mathrm{su}(2)$ and whose inner product is the L_k^2-inner product. Remember that $\mathscr{A}$ has its L_1^2-Hilbert space structure, and give $\Omega^1(X) \times \mathrm{su}(2)$ the L_1^2-Hilbert space structure. Then, the Sobolev inequalities insure that the assignment of (v, u) to (A, a) defines a smooth map from $\mathscr{A} \times \mathscr{W}^1$ to $\Omega_2^0 \times \Omega_1^0$. In addition, when A is fixed, this becomes a bounded, linear map from $\mathscr{W}^1$ into $\Omega_2^0 \times \Omega_1^0$.

Fix the irreducible connection A on $X \times \mathrm{SU}(2)$ and define $\underline{K}_A \colon \mathscr{W} \to \mathscr{J}$ by sending (a, φ) to

$$(\mathrm{A}.4) \qquad \underline{K}_A \omega \equiv (*d_A(a - d_A v_A(a)) - d_A u_A(a) + d_A \varphi, d_A^* a).$$

Lemma A.2. *Let X be a compact, connected, oriented, Riemannian 3-manifold with boundary. Let A be a smooth, irreducible connection on $X \times \mathrm{SU}(2)$. Then $\underline{K}_A \colon \mathscr{W} \to \mathscr{J}$ defines a bounded, Fredholm operator on the respective completions. The difference $\underline{K}_A - K_A$ is a compact operator. By varying the connection, a smooth, $\mathscr{G}$-equivariant map, $\underline{K}_{(\cdot)}$, is defined from $\mathscr{A}^\#$ into the space of bounded, real Fredholm operators from $\mathscr{W}$ to $\mathscr{J}$.*

Proof of Lemma A.2. Since the assignment of a to $(v_A(a), u_A(a))$ is smooth, $\underline{K}_A$ is bounded. The Rellich lemma insures that $\underline{K}_A - K_A$ is a compact operator. Thus, $\underline{K}_A$ is also a Fredholm operator. Since the assignment of (A, a) to $(v_A(a))$ is smooth, $\underline{K}_{(\cdot)}$ defines a smooth map from $\mathscr{A}^\#$ into the space of real, bounded Fredholm operators from $\mathscr{W}$ to $\mathscr{J}$. This map is $\mathscr{G}$-equivariant by inspection. q.e.d.

The operator $\underline{K}_{(\cdot)}$ is useful because it commutes with a natural Hodge-type decomposition of $\mathscr{A} \times \mathscr{W}^1$. To construct this decomposition, introduce the map $\Delta_1: \mathscr{A} \times \mathscr{W}^1 \to \Omega_0^0$ by sending (A, a) to

$$\Delta_1(A; a) \equiv d_A^* a. \tag{A.5}$$

Let $\mathscr{A}^\#$ denote the set of irreducible connections on $X \times \mathrm{SU}(2)$. Restricted to $A \in \mathscr{A}^\#$, $d_A^* d_A: \Omega_2^0 \to \Omega_0^0$ has bounded inverse with Neumann type boundary conditions. In fact, for each $\sigma \in \Omega_0^0$, there exists a unique $\sigma' \in \Omega_2^0$ which obeys $d_A^* d_A \sigma' = \sigma$ and $i_{\partial X}^*(*d_A\sigma') = 0$. The inverse function theorem insures that $\underline{\Delta}_1 \equiv \Delta_1^{-1}(0) \cap (\mathscr{A}^\# \times \mathscr{W}^1)$ is a smooth manifold with projection $\pi_1: \underline{\Delta}_1 \to \mathscr{A}^\#$ defining a smooth vector bundle. The group $\mathscr{G}$ acts naturally on $\underline{\Delta}_1$, and π is an equivariant map.

The fiber of this vector bundle over a connection A is the vector space

$$\mathscr{T}_A \equiv \{a \in \mathscr{W}^1 : d_A^* a = 0\}. \tag{A.6}$$

Let Ω_{1c}^0 denote the linear subspace $\{\varphi \in \Omega_1^0 : i^*(\varphi) = 0\}$. Note that the L_1^2-inner product defines a pre-Hilbert space structure on Ω_{1c}^0. Define a map $\Delta_0: \mathscr{A} \times \mathscr{J}^1 \to (\Omega_{1c}^0)^*$ by sending (A, a) to the linear functional on Ω_{1c}^0 which sends φ to

$$\Delta_0(A, a) \cdot \varphi \equiv -\int_X \operatorname{tr}(a \wedge *d_A\varphi). \tag{A.7}$$

Essentially a repeat of the preceding argument shows $\underline{\Delta}_0 \equiv \Delta_0^{-1}(0) \cap (\mathscr{A}^\# \times \mathscr{J}^1)$ is a smooth manifold with projection $\pi: \underline{\Delta}_0 \to \mathscr{A}^\#$ defining a smooth vector bundle. The fiber over a connection A is the vector space

$$\mathscr{L}_A \equiv \{a \in \mathscr{J}^1 : d_A^* a = 0\}. \tag{A.8}$$

The group $\mathscr{G}$ acts naturally on $\underline{\Delta}_1$, and π is again an equivariant map.

It is convenient to exhibit a cover of $\mathscr{A}^\#$ over which the bundles $\underline{\Delta}_{0,1}$ have trivializations. A set $\mathscr{U}(A; \varepsilon)$ in this cover is indexed by $A \in \mathscr{A}^\#$ and $\varepsilon > 0$, but small:

$$\mathscr{U}(A; \varepsilon) \equiv \{A + a : a \in \Omega^1 \times \mathrm{su}(2) \text{ obeys } ||a||_A < \varepsilon\}. \tag{A.9}$$

Define $l_1: \underline{\Delta}_1|_{\mathscr{U}} \to \mathscr{U} \times \mathscr{T}_A$ and $l_0: \underline{\Delta}_0|_{\mathscr{U}} \to \mathscr{U} \times \mathscr{L}_A$ by sending $(A + a, v)$ to $(A + a, v - d_A\kappa_{1,0})$, where $\kappa_{1,0} \equiv \kappa_{1,0}(a, v)$ obeys the equation

$$d_A^* d_A \kappa_{1,0} - *(a \wedge v + v \wedge a) = 0, \qquad i_{\partial X}^*(*d_A\kappa_1) = 0 = i_{\partial X}^*(\kappa_0). \tag{A.10}$$

The Sobolev inequalities insure that the $l_{1,0}$ are smooth vector bundle maps with smooth inverses as long as ε is small enough. Neither map is an isometry, but it is a straightforward exercise to establish the existence of a constant $z[A]$ such that ε is sufficiently small and $A + a \in \mathscr{U}(A;\varepsilon)$, then

$$\text{(A.11)} \quad \begin{aligned} &|\langle v_0, v_1\rangle_{A+a} - \langle v_0 - d_A\kappa_1(a, v_0), v_1 - d_A\kappa_1(a, v_0)\rangle_A| < z\cdot\varepsilon, \\ &|\langle w_0, w_1\rangle_{L^2} - \langle w_0 - d_A\kappa_0(a, w_0), w_1 - d_A\kappa_0(a, w_0)\rangle_{L^2}| < z\cdot\varepsilon, \end{aligned}$$

for all $(A + a, v_{0,1}, w_{0,1}) \in \underline{\Delta}_1 \otimes \underline{\Delta}_0|_{\mathscr{N}}$.

The vector bundle $\mathrm{Hom}(\underline{\Delta}_1, \underline{\Delta}_0) \to \mathscr{A}^\#$ is defined by taking the fiber over a connection A to be the Banach space of bounded, linear operators from $\mathscr{T}_A$ to $\mathscr{L}_A$. For each integer n, let $\mathrm{Fred}_n(\underline{\Delta}_1, \underline{\Delta}_0) \subset \mathrm{Hom}(\underline{\Delta}_1, \underline{\Delta}_0)$ denote the fiber bundle whose fiber over a connection A is the Banach manifold of bounded, linear, Fredholm operators from $\mathscr{T}_A$ and $\mathscr{L}_A$. The reader can use the trivializations in (A.9) and (A.10) and the Rellich lemma to check that these fiber bundles are well defined.

Let $T_A: \mathscr{W} \to \mathscr{T}_A$ and $L_A: \mathscr{J} \to \mathscr{L}_A$ denote the L^2-orthogonal projections. As the connection A is varied, $T_{(\cdot)}$ defines a smooth vector bundle map between the trivial bundle over $\mathscr{A}^\#$, $\mathscr{A}^\# \times \mathscr{W}^1$ and the bundle $\underline{\Delta}_1$. Likewise, $L_{(\cdot)}$ defines a smooth vector bundle map between $\mathscr{A}^\# \times \mathscr{J}^1$ and the bundle $\underline{\Delta}_0$.

By construction, the operator $\underline{K}_A$ obeys

$$\text{(A.12)} \qquad \underline{K}_A \circ T_A \equiv L_A \circ \underline{K}_A.$$

(A.12) and Lemma A.2 have the following immediate corollary:

Lemma A.3. *Let X be a compact, connected, oriented, Riemannian 3-manifold with boundary. Let A be a smooth, irreducible connection on $X \times \mathrm{SU}(2)$. Then $L_A \circ \underline{K}_A \circ T_A: \mathscr{T}_A \to \mathscr{L}_A$ defines a bounded, Fredholm operator on the respective completions. By varying the connection A, $L_A \circ \underline{K}_A \circ T_A: \mathscr{T}_A \to \mathscr{L}_A$ defines a smooth, $\mathscr{G}$-equivariant section, $\nabla \mathscr{f}$, over $\mathscr{A}^\#$ of the fiber bundle* $\mathrm{Fred}_d(\underline{\Delta}_1, \underline{\Delta}_0)$ *with $d = -\frac{3}{2}\cdot\chi(\partial X)$.*

A word about perturbations is in order: Let Λ be a finite set of embedded curves lying in the interior of X with parametrized tubular neighborhoods, also in the interior of X.

A section of $\mathrm{Fred}_d(\underline{\Delta}_1, \underline{\Delta}_0)$, $\nabla \mathscr{f}'$, is defined by assigning to a connection A the linear map which sends $a \in \mathscr{T}_A$ to

$$\text{(A.13)} \qquad \nabla \mathscr{f}'_A \cdot a \equiv *d_A a + \partial v[A]\cdot a - d_A u'_A(a).$$

Here, $v[A]$ is defined in (4.9), and $u'_A(a)$ is defined in (4.11).

Lemma A.4. *Let X be a compact, connected, oriented, Riemannian 3-manifold with boundary. Let A be a smooth, irreducible connection on $X \times \mathrm{SU}(2)$. Then $\nabla \mathscr{f}'_A: \mathscr{T}_A \to \mathscr{L}_A$ defines a bounded, Fredholm operator on the respective completions. By varying the connection A, $\nabla \mathscr{f}'_A: \mathscr{T}_A \to \mathscr{L}_A$*

defines a smooth, $\mathscr{G}$-equivariant section, $\nabla\not{f}'$, over $\mathscr{A}^\#$ of the fiber bundle $\mathrm{Fred}_d(\underline{\Delta}_1, \underline{\Delta}_0)$ *with* $d = -3/2 \cdot \chi_{\partial X}$.

Proof of Lemma A.4. The Rellich lemma and the Sobolev inequalities insure that the difference $\nabla\not{f}_A' - \nabla\not{f}_A$ is a compact operator. Smoothness as A varies is left as an exercise with the Sobolev inequalities.

Part 2. Adjoints. The adjoint $\nabla\not{f}_A'^*$ defines a bounded, linear map from the completion of $\mathscr{L}_A$ to the dual, $\mathscr{T}_A^*$, of the completion of $\mathscr{T}_A$. This operator assigns to $v \in \mathscr{L}_A$ the linear functional which sends $a \in \mathscr{T}_A$ to the real number

$$\text{(A.14)} \qquad \nabla\not{f}_A'^*(v) \cdot a \equiv \int_{\partial X} \operatorname{tr}(v \wedge a) - \int_X \operatorname{tr}(d_A v \wedge a + *\partial v[A] \cdot u \wedge a).$$

As $\mathscr{T}_A$ is a pre-Hilbert space with the inner product in (A.1), its completion and the dual to its completion are isomorphic. Using the Hilbert space metric $\langle\cdot,\cdot\rangle_A$, the operator $\nabla\not{f}_A'^*$ defines a bounded, Fredholm operator from $\mathscr{L}_A$ to the completion of $\mathscr{T}_A$ [13]. As A varies in $\mathscr{A}^\#$, $\nabla\not{f}_A'^*: \mathscr{L}_A \to \mathscr{T}_A$ defines a smooth, $\mathscr{G}$-equivariant section, $\nabla\not{f}'^*$, over $\mathscr{A}^\#$ of the fiber bundle $\mathrm{Fred}_d(\underline{\Delta}_0, \underline{\Delta}_1)$ with $d = \frac{3}{2} \cdot \chi_{\partial X}$.

Part 3. L^2-theory. For formally selfadjoint operators, an L^2-theory is useful. Begin again with the operator K_A and consider only the case $\partial X = \varnothing$. Let $\underline{\mathscr{W}}$ denote the L^2-completion of both $\mathscr{W}$ and $\mathscr{J}$. Lemma A.1 implies that K_A defines a closed, unbounded, essentially selfadjoint operator on $\underline{\mathscr{W}}$ with $\mathscr{W}$ as dense domain. The Weitzenboch formula for K_A shows that the norm on $\mathscr{W}$ which sends ψ to $\|K_A \cdot \psi\|_{L^2} + \|\psi\|_{L^2}$ is equivalent to the L_1^2-norm $\langle\cdot,\cdot\rangle_A$ on $\mathscr{W}$. Since the forgetful map from L_1^2 into L^2 is compact, this fact implies that the resolvent of K_A is compact. Thus, K_A has discrete spectrum without accumulation points.

Let $w: \underline{\mathscr{W}} \to \underline{\mathscr{W}}$ be a linear operator with dense domain $\mathscr{W}$ which is relatively compact with respect to K_A. Standard stability theorems [13] imply that $K_A + w$ has compact resolvent also. Lemma A.2 insures that $\underline{K}_A$ is such a perturbation of K_A. Thus $\underline{K}_A$ defines a closed, essentially selfadjoint operator on $\underline{\mathscr{W}}$ which has discrete spectrum without accumulation points.

Let $\underline{\mathscr{T}}_A$ denote the L^2-completion of $\mathscr{T}_A$ and of $\mathscr{L}_A$. Then, the preceding comments and (A.8) insure that $\nabla\not{f}_A$ defines a closed, essentially selfadjoint operator on $\underline{\mathscr{T}}_A$ which has discrete spectrum without accumulation points. Lemma A.4 asserts that $\nabla\not{f}_A'$ and $\nabla\not{f}_A$ differ by a compact operator, so $\nabla\not{f}_A'$ likewise defines a closed, essentially selfadjoint operator on $\underline{\mathscr{T}}_A$ which has discrete spectrum without accumulation points. These results are summarized by

Lemma A.5. *Let X be an oriented, compact, connected 3-manifold without boundary.*

(1) *The assignments of the operator K_A and $\underline{K}_A$ to a connection A on $X \times SU(2)$ define smooth maps from $\mathscr{A}$ and $\mathscr{A}^{\#}$ respectively into the space of closed, essentially selfadjoint Fredholm operators on the Hilbert space $\underline{\mathscr{W}}$; the difference, $K_A - \underline{K}_A$, defines a smooth map from $\mathscr{A}^{\#}$ into the space of K_A-relatively compact operators on the Hilbert space $\underline{\mathscr{W}}$.*

(2) *There is a smooth fiber bundle over $\mathscr{A}^{\#}$ whose fiber over a connection A is the space of closed, essentially selfadjoint, Fredholm operators on $\underline{\mathscr{T}}_A$ which restrict as bounded, Fredholm maps from $\mathscr{T}_A$ to $\mathscr{L}_A$.*

(3) *For each irreducible connection A, define the operator $\nabla f_A'$ by* (A.13). *The assignment of $\nabla f_A'$ to A defines smooth, $\mathscr{G}$-equivariant sections over $\mathscr{A}^{\#}$ of the fiber bundle in* (2). *The difference $\nabla f_A' - \nabla f_A$ is ∇f_A-relatively compact.*

Proof of Lemma A.5. Assertion (1) concerning $K_{(\cdot)}$ just summarizes the preceding discussion and Lemma A.1. For assertion (2), the existence of the asserted fiber bundle is proved by directly constructing it using the trivializations of $\underline{\Delta}_1$, which are given in (A.9) and (A.10). It suffices to remark that for $\varepsilon > 0$, but small, and for fixed $A + a \in \mathscr{N}(A; \varepsilon)$, the assignment of $v \in \Omega^1 \times su(2)$ to $d_A\chi(a, v) \in \Omega^1 \times su(2)$ extends to define a compact map from $L_1^2(\Omega^1 \times su(2))$ to itself, and to a compact map from $L^2(\Omega^1 \times su(2))$ to itself (use Weitzenboch formulas, and the Rellich lemma to show this). Assertion (3) follows from Lemma A.4. q.e.d.

Lemmas 2.4 and 2.6 are corollaries to Lemma A.5, and Lemma A.5 provides the framework for the proof of Proposition 2.1.

Proof of Proposition 2.1. The first assertion of the proposition is a standard consequence of assertion (3) of Lemma A.5. For the second assertion of the lemma, let $A(t)$, $t \in [0, 1]$, denote a path of connection with $A(0) = A_1$ and $A(1) = A_2$. As t varies, the eigenvalues of $\nabla f_{A(t)}'$ vary. If the zero crossings are not transverse, the family of operators $\nabla f_{A(t)}'$ ($t \in [0, 1]$) can be perturbed (rel$\{0, 1\}$) by adding to $\nabla f_{A(\cdot)}'$ a smooth family, $s(t)$ ($t \in [0, 1]$), of selfadjoint, relatively compact (with respect to $\nabla f_{A(\cdot)}'$) operators so that the perturbed family has its eigenvalues crossing zero transversely. This perturbation can be arbitrarily small in the following sense: Let T be a closed, linear operator between two Hilbert spaces. If an operator S has the same domain as T, there is the notion of the T-bound of the operator S (see [13]): this is the number

$$|S|_T \equiv \sup_{v \neq 0}(|S \cdot v|/(|T \cdot v| + |v|).$$

The perturbation family, $s(\cdot)$, above, may be taken to have arbitrarily small T-bound for $T \equiv \nabla\not{f}'_{A(\cdot)}$. The spectral flow for the perturbed family is well defined, and the discussion in [5] shows that this spectral flow is independent of the perturbation if the perturbation is sufficiently small. Use such a small perturbation to define the spectral flow along the path $A(t)$.

Change the path from $A(t)$ to a new path, $A'(t) \equiv A(t) + a(t)$, and a new family of operators, $\nabla\not{f}'_{A(\cdot)}$, is defined which differs from the old one by a relatively compact family of operators whose T-bound (for $T = \nabla\not{f}'_{A(\cdot)}$) is uniformly bounded by a constant multiple of the L^2_1-norm of $a(t)$. Use the trivializations of $\underline{\Delta}_0$ in (A.9) and (A.10) to show this. This latest fact implies that the spectral flow along a path from A_1 to A_2 is a constant function on the space of paths from A_1 to A_2. (The fact that $\mathscr{A}^\#$ is simply connected is implicit in this statement.)

The group $\mathscr{G}$ is not path connected. Let $g \in \mathscr{G}$ be a gauge transformation which is not homotopic to the identity. Let $A'_1 \equiv g \cdot A_1$. The spectral flow along a path from A'_1 to A_1 will not be the same as that from A_1 to A_1. The difference of the spectral flows along two paths is equal to the spectral flow around the closed loop in $\mathscr{B}^\#$ based at $[A_1]$ which, after lifting to $\mathscr{A}^\#$, is defined by going out on the first path, and coming back on the second.

The spectral flow around a closed loop of operators, $\nabla\not{f}'_{A(t)}$, is equal to the index of an operator on $M \times S^1$ [5]. The relevant operator on $M \times S^1$ has index divisible by 8. (This operator comes from the selfdual deformation complex for the mapping torus of the automorphism g, a principal SU(2) bundle over $M \times S^1$. The index is computed in [4].)

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