

MUTATION AND THE η -INVARIANT

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1. Introduction

This paper investigates the effect of a certain type of cutting and pasting operation on geometric invariants of a hyperbolic 3-manifold. The invariants which we discuss are the Chern-Simons invariant [4] and the η -invariant [2]. These are both defined for any closed Riemannian 3-manifold M ; the Chern-Simons invariant $CS(M)$ takes its values in the circle \mathbf{R}/\mathbf{Z} , while the η -invariant $\eta(M)$ is a real number. The two numbers are related as follows: $3CS(M) = 2\eta(M) \pmod{1}$. Hence the η -invariant determines the Chern-Simons invariant, modulo $1/2$. If M is a compact hyperbolic manifold, then Mostow's rigidity theorem [9] implies that both of these are in fact topological invariants of M . The Chern-Simons invariant is defined, modulo $1/2$, for finite-volume hyperbolic manifolds in [8]. The extension of Mostow rigidity to the finite volume case [11] implies that the Chern-Simons invariant is a topological invariant in this case as well.

If F is a surface embedded in a 3-manifold M , and φ is a diffeomorphism of F , then we can obtain a new manifold M^φ by cutting M along F and regluing via φ . For an arbitrary Riemannian manifold, this topological operation is likely to destroy any nice properties of the metric. In particular, there is no reason to expect any relation between invariants of the two manifolds. However, in certain special cases, the cutting and pasting may be done "geometrically." Let F be a surface of genus two and τ be the involution of F indicated in Figure 1. If F is embedded in a hyperbolic 3-manifold, then cutting and pasting via τ may be done geometrically:

Theorem 1.1 (*Ruberman* [12]). *Suppose the genus-2 surface F is embedded in the hyperbolic manifold M so that $\pi_1(F)$ injects into $\pi_1(M)$. Then the manifold M^τ is hyperbolic, and $\text{vol}(M^\tau) = \text{vol}(M)$.*

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FIGURE 1

The condition on the fundamental group is summarized by saying that F is incompressible. The theorem also applies to certain noncompact surfaces and involutions. These are most easily described as the subsurfaces of F : The once or twice punctured torus and the 3- or 4-punctured sphere all sit inside of F so that they are invariant under τ . If any of these open surfaces (or F itself) is properly embedded in a 3-manifold, then we will call the operation of cutting and pasting using τ a *mutation* of M . The more general version of the above theorem then states that a mutation M^τ of a hyperbolic 3-manifold M of finite volume is hyperbolic, and has the same volume as M . The terminology “mutation” comes from knot theory; in that context a 2-sphere hitting a knot in four points can be ‘mutated’ to produce a (potentially) different knot. This process is important in the combinatorial classification of knots via their projections [5].

The main results of this paper are the determination of how the Chern-Simons invariant and the η -invariant change under mutations of hyperbolic manifolds¹. The results concerning the η -invariant are valid for closed manifolds and mutations along a genus-two surface. A special case of the theorem concerning the η -invariant is:

Corollary 1.2. *Let F be an incompressible, separating genus-2 surface in the hyperbolic 3-manifold M , and M^τ the manifold resulting from mutation along F . Then $\eta(M^\tau) = \eta(M)$.*

In general (see Theorem 4.2) we show that the difference $\eta(M^\tau) - \eta(M)$ is equal to a topological invariant of the embedding of F in M called the signature of F in M . A corresponding result for the Chern-Simons invariant holds more generally for manifolds of finite volume. In order

¹The authors have recently computed the change in the η -invariant under general Riemannian cutting and pasting along closed surfaces. These results are in our preprint, *Cutting and pasting and the η -invariant*.

to establish this, we need a definition of the Chern-Simons invariant for manifolds with cusps. This is done *modulo* $1/2$ in [8] using the “torsion formula” developed in that paper. The torsion formula provides a method of computing the Chern-Simons invariant of $M \bmod 1/2$ using a framing of M defined in the complement of a link in terms of the torsion of the components of the link. In §3 of this paper, we enhance the definition of torsion in order to obtain a formula which gives the Chern-Simons invariant exactly, and prove the invariance of the Chern-Simons invariant under mutations of closed manifolds.

Using the enhanced torsion formula, we give a definition of the Chern-Simons invariant of 3-manifolds with cusps, with respect to certain homology classes in the cusps. For a knot complement, the homology class may be chosen to be a meridian, yielding a mod 1 Chern-Simons invariant of a knot. Using this definition, we show the invariance under mutation of the Chern-Simons invariant of knots.

2. Definitions and basic technique

Let M be a closed, oriented Riemannian 3-manifold. Then Chern and Simons [4] define a certain 3-form, Q , on the oriented frame bundle, $F(M)$. Any orientable 3-manifold is parallelizable, so we can use sections of the frame bundle to pull Q back to M . Integrating s^*Q over M produces a real number; this number depends, *a priori*, on the choice of section. Given one section, any other differs from it by a map from M to $SO(3)$; it is easily verified that the integral of the Chern-Simons form changes by $8\pi^2$ times the degree of this map. Hence we can define the Chern-Simons invariant:

$$\text{CS}(M) = \frac{1}{8\pi^2} \int_M s^*Q \pmod{1}.$$

The η -invariant is defined in a completely different manner, and it is surprising that the two invariants have anything to do with each other. The definition of the η -invariant was originally given for any odd-dimensional Riemannian manifold by Atiyah, Patodi, and Singer [2] in terms of the eigenvalues of the Laplace operator. They proved the following remarkable formula, which the reader can take as a definition of $\eta(M)$. If M is a 3-manifold, which is the boundary of a smooth 4-manifold W , then a framing α on M gives a relative Pontrjagin number $p_1(W)$. Define the signature defect $\delta(M, \alpha)$ to be the integer $\frac{1}{3}p_1(W)$ -signature (W). Then

$$\eta(M) = \frac{1}{12\pi^2} \int_M \alpha^*Q + \delta(M, \alpha).$$

From this formula we see the relationship between the η -invariant and the Chern-Simons invariant mentioned in the introduction.

For purposes of computation, however, the definitions of the Chern-Simons invariant and η -invariant are sometimes awkward to use, because of the necessity of keeping track of framings which may have little to do with the geometry of the manifold. For example, the cutting and pasting operation treated in this paper does not act particularly nicely with respect to framings. To get around this sort of difficulty, the first author introduced in his thesis [8] the device of using framings defined on the complement of a link L in M with prescribed behavior near L . We remind the reader that for any closed curve γ in a Riemannian manifold, there is the notion [13, volume 3] of the *torsion* of the curve, denoted $\tau(\gamma)$. If the curve is a geodesic, this is just the holonomy of an orthogonal vector field which is parallel along the curve.

Definition 2.1. A singular frame field on M with singularities along L is an orthonormal framing on $M - L$ which has the following local structure near each component K of L :

(i) In the limit, one vector (say e_1) is tangent to K .

(ii) The vectors e_2, e_3 determine an index ± 1 singularity in the disk transverse to K .

In a closed hyperbolic manifold, the following “torsion formula” [8] shows how to compute the Chern-Simons invariant (mod $1/2$) using a singular frame field:

$$\text{CS}(M) = \frac{1}{8\pi^2} \int_{M-L} s^* Q + \frac{1}{4\pi} \sum_{\gamma \subset L} \pm \tau(\gamma) \pmod{\frac{1}{2}},$$

where the sum is over the components γ of L , and the sign agrees with the index of the singularity along γ . If the manifold is not closed, but has finite volume, then there is still a way of defining a Chern-Simons invariant if one restricts the framing in the cusps to have a certain special form, called a *linear frame field* [8]. A (singular) frame field is linear in a horoball neighborhood of a cusp if the e_3 -vectors are perpendicular to the horospheres and point outwards, and the e_1 - and e_2 -vectors are parallel (in the Euclidean metric) on each horosphere.

In [8], it is proved that the formula above defines an invariant of a cusped hyperbolic manifold, provided that the singular framing is linear in the cusps.

Remark 2.2. It was remarked, though not used, in [8] that the same formula works if slightly more general frame fields are allowed at the cusps.

The frame field allowed will have e_3 -vectors as above, but on each horosphere, the e_1, e_2 frame is merely homotopic to one which is parallel. We will refer to these more general (singular) framings as homotopically linear frame fields. This extended definition is useful because it allows the e_1 -vectors to change direction in some region of a cusp (see Figure 2). To avoid awkward phrasing we will often suppress mention of linearity of a singular framing.

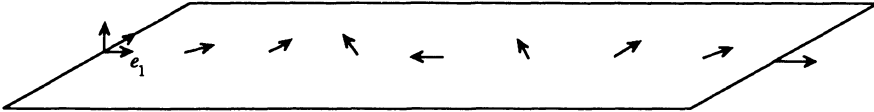


FIGURE 2. CHANGING e_1

To illustrate the utility of the torsion formula, we will use it to compute the effect of a mutation of a hyperbolic manifold on the Chern-Simons invariant, mod $1/2$. First we review the results of [12] which show how to do mutations geometrically, resulting in a hyperbolic metric on M^τ .

Let M_F be the covering space of M to which the embedding of F lifts; this is of course homotopy equivalent to F . One shows first that the involution τ on F can be realized by an isometry $\hat{\tau}$ of M_F , which will be an involution. Using least area surfaces, one finds a copy of F , isotopic to the original embedding, so that its lift \hat{F} to M_F is invariant under the isometry $\hat{\tau}$. Locally, \hat{F} looks like F , so the restriction of $\hat{\tau}$ to \hat{F} yields an isometry of F (with its induced metric). It follows easily from the construction that if one cuts along this copy of F and reglues via this isometry, then the resulting manifold, M^τ , has a complete hyperbolic metric. Moreover, any invariant, such as the volume, obtained by integrating a 3-form over M will have the same value on M^τ , since the two are isometric off a codimension-one subset.

It would follow easily that the Chern-Simons invariant is unchanged as well, except that we need a framing on M which can be “cut and pasted” to give a framing on M^τ . This would require a framing on M with the property that its restriction to $T_*(M)|_F \cong T_*(F) \oplus \varepsilon$ is invariant under the natural action of τ on $T_*(F) \oplus \varepsilon$. It is easily seen that there are no such framings, but we will show how to find a *singular* framing s with this invariance property. Moreover, the singular locus L will intersect F precisely at the fixed points of τ . Hence the singular framing s can be cut and pasted to give a singular framing s^τ on M^τ which agrees with s on

the complement of F . We perform the computation of the Chern-Simons invariant on M^τ in terms of this new framing s^τ .

Whether or not the Chern-Simons invariant changes under mutation depends on the number of fixed points of τ . Note that the number of fixed points of τ and the number of boundary components of F have the same parity. We will show (in Theorem 2.3) that the Chern-Simons invariant is unchanged if this number is even, while (cf. Theorem 2.4) it always changes if the number of fixed points is odd.

Theorem 2.3. *Let M be a cusped hyperbolic 3-manifold and F an incompressible, boundary incompressible 4-punctured sphere, 2-punctured torus, or genus-2 surface in M . If M^τ is the 3-manifold obtained by cutting and pasting M via the mutation τ , then $\text{CS}(M) = \text{CS}(M^\tau) \pmod{1/2}$.*

Proof. As discussed above, we show how to put a singular frame field on the surface F which is invariant under the mutation, and then extend it to get a singular frame field s on the rest of the manifold. The singular locus of s on the manifold M will intersect F precisely in the two fixed points of τ . We first give the argument in detail for the case when F is a 4-punctured sphere. The proof for the genus-2 surface is the same, except that one does not have to worry about the cusps. Some small modifications are needed to extend the argument to the case of a twice-punctured torus; these are discussed at the end of the proof.

On F , we know what the singular frame field should look like at the cusps and the fixed points (see Figure 3(a)). How can we extend this equivariantly over the rest of F ? Split F in half, as in Figure 3(b), to get a fundamental domain for the action of τ . The framing on the boundary, as drawn in Figure 3(b) represents the trivial element in $\pi_1(\text{SO}(3))$, and so extends over the fundamental domain. By construction, the framing on the boundary glues up to give an equivariant framing on F .

Now we have to extend this frame field to a singular framing on all of M . First we extend s in the cusps on M so that it is (homotopically) linear there. This is easily done, using Remark 2.2. Choose a curve $\gamma \subset M$ which hits the surface in exactly the fixed points of τ . This may be done by choosing an arc in F between the fixed points and pushing it off F in two directions to get a simple closed curve. The obstruction to extending this to a frame field on the rest of M is carried on a link L in the complement of $F \cup \gamma$. As in [8, §4.2] the frame field may then be extended to a singular frame field with singularities along $L \cup \gamma$.

Since the restriction of the singular framing s to F is invariant under τ , it gives a new singular framing s^τ on the mutated manifold M^τ . This framing is not necessarily homotopically linear at the cusps. As one goes

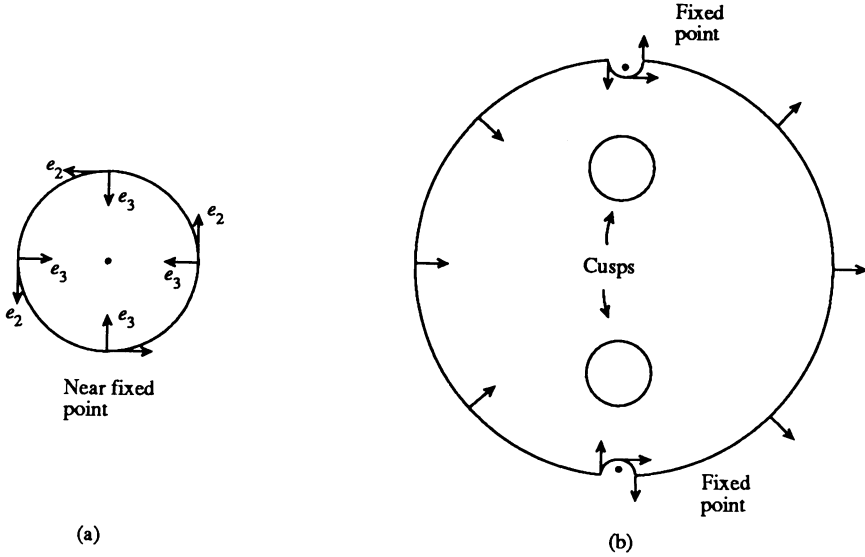


FIGURE 3. FRAMING ON F

along a curve in a cusp transverse to F , the e_1, e_2 -vectors rotate through a full turn relative to the linear field. This will happen precisely when there are two distinct cusps of M hitting F , and the mutation takes each cusp to itself (see Figure 4).

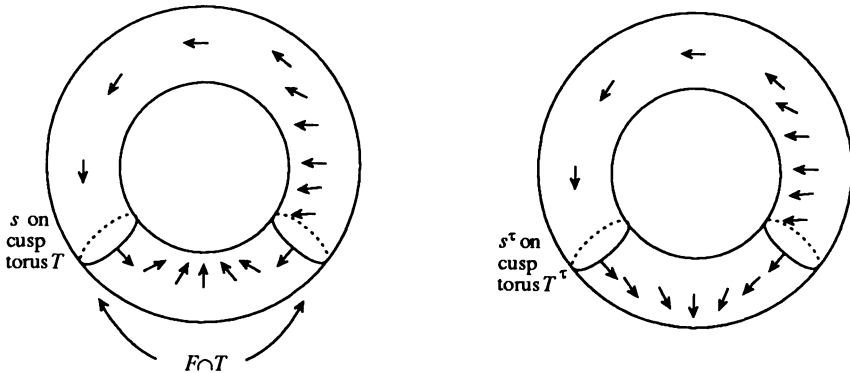


FIGURE 4. NONLINEAR FRAMING AT CUSP

View the cusp as $T^2 \times [0, \infty)$, and let t be the coordinate in the \mathbf{R} -direction. We may assume that this nonlinear framing is constant in the t -direction, which implies that the Chern-Simons integral on the cusp will be zero. We will now change the framing on the cusp so that it becomes

linear on $T^2 \times [1, \infty)$, and the cusp still contributes zero to the Chern-Simons invariant. To do this, note that there is a singular framing on $T^2 \times [0, 1]$ which is linear on $T^2 \times 1$, is our given framing on $T^2 \times 0$, and which has a single singular curve. The singular curve can be chosen as a geodesic in the Euclidean torus at height $\frac{1}{2}$; such a curve has zero torsion. The Chern-Simons integral over the rest of $T^2 \times [0, 1]$ can be computed to be zero, as in [8, §4.1].

The integral of the Chern-Simons form over the rest of M^τ is unchanged. The torsion term changes by $\pm\pi$ at each of the two fixed points. (For future reference we note that the signs are opposite at the two fixed points.) Therefore the torsion term does not change, mod $1/2$, which says that the Chern-Simons invariant is unchanged, mod $1/2$.

There are two difficulties which occur in extending the argument to apply to a 2-punctured torus. The first comes in the construction of the invariant singular framing on F . As above, we specify the framing on the boundary and cusps of a fundamental domain of τ , and then try to extend over the rest of the fundamental domain. An Euler characteristic argument shows that there is no such nonsingular extension for the twice-punctured torus. In this case, there is a singular framing on the fundamental domain with one more singular point in the interior of the fundamental domain. Thus there is still a τ -invariant singular framing on F . The singular set on F will now consist of the fixed points of τ , together with an additional pair of points which are interchanged by τ .

The other part of the argument which causes difficulty is the extension over the rest of M . For this, we need a link in M which passes through the singular set on F and does not meet F otherwise. But since there are an even number of singular points, this link can be chosen to be a trivial link lying near the surface with one component passing through the pair of singular points which are interchanged by τ . The proof proceeds as above, and we conclude that the Chern-Simons invariant does not change (mod $1/2$). q.e.d.

If there is an odd number of singular points as is the case for the 3-punctured sphere and punctured torus, then we get the surprising result that the Chern-Simons invariant *always* changes by $1/4$.

Theorem 2.4. *A mutation along an incompressible 3-punctured sphere or once-punctured torus in a hyperbolic 3-manifold changes the Chern-Simons invariant by $1/4$ (mod $1/2$).*

Proof. As in the previous theorem, there is an invariant singular framing on F for which the fixed points of τ are all singular points. (If F is the punctured torus, then the fixed set and singular set coincide, whereas if F

is the 3-punctured sphere, we must add a pair of singular points which are interchanged by τ .) To extend this framing over M , we need a link L in M hitting F in precisely the singular set. Since this set has an odd number of points, this is possible exactly when F does not separate M .

However, an incompressible surface with an odd number of boundary components in a manifold with torus boundary can never separate. This is readily seen by considering the intersection with the boundary. (We are indebted to Colin Adams for this remark.) Therefore, there is a simple closed curve hitting F exactly in one point and hence a collection of curves, each of which hits F precisely in a fixed point of τ . (In the case of a 3-punctured sphere, we add a component to L going through the pair of singular points interchanged by τ .) As in the proof of Theorem 2.3, the torsion term in the Chern-Simons invariant will change by $\pm\pi$ for each singular point. Since there is an odd number of singular points, the change in the Chern-Simons invariant is $\pi/(4\pi) = 1/4$. q.e.d.

Since the mod $1/2$ Chern-Simons invariant is a geometric (and therefore topological, by Mostow rigidity) invariant of cusped hyperbolic manifolds, we obtain:

Corollary 2.5. *In the situation above, the manifold M^τ is not homeomorphic to M , preserving orientation.*

For an interesting example, consider the Whitehead link. There is a disk spanning one component which hits the other one in two points, and thus results in a 3-punctured sphere in the link complement. A mutation along this sphere will turn a Whitehead link with a right-handed clasp into a left-handed one. The two link complements are homeomorphic via a reflection, which *reverses* orientation. However, the corollary implies that they are not homeomorphic preserving orientation, so we conclude that the two links are not isotopic. By combining our cutting and pasting arguments with some symmetry considerations, we can determine the Chern-Simons invariants of some hyperbolic link complements. As an example, we show how to calculate the Chern-Simons invariant of the complement of the Borromean rings.

Proposition 2.6. *The Chern-Simons invariant of the complement of the Borromean rings is zero.*

Proof. Let Wh_\pm denote the Whitehead link with a right- (or left-) handed clasp, and X_\pm be the respective link complements. Since X_+ and X_- are diffeomorphic via an orientation reversing diffeomorphism, $CS(X_+) \equiv -CS(X_-) \pmod{1/2}$. On the other hand, their Chern-Simons invariants differ by $1/4$, since the links are related by a mutation of a 3-punctured sphere. Therefore $CS(X_+) = 1/8$ or $3/8 \pmod{1/2}$.

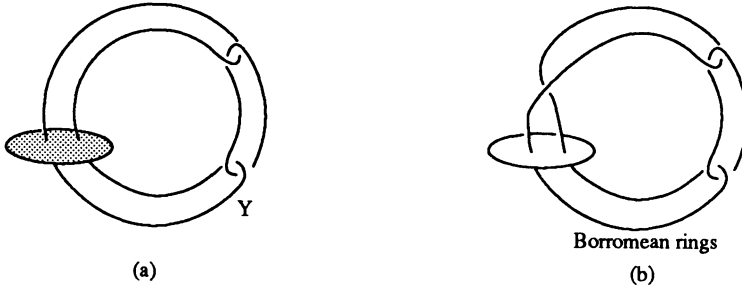


FIGURE 5

The complement of the Borromean rings is related to X_+ in the following simple way. Let Y be the two-fold cover of X_+ which is trivial over one component and nontrivial over the other. Then Y is the complement of the link drawn in Figure 5(a). The Borromean rings complement is obtained by a mutation of Y along the evident 3-punctured sphere. Now the Chern-Simons invariant multiplies under finite covers (even in the cusped case), so that $\text{CS}(Y) = 1/4$. By Theorem 2.4, the Chern-Simons invariant of the Borromean rings complement must then be zero.

Remark 2.7. In the next section, we will discuss the effect of mutation on the Chern-Simons invariant, *modulo 1*, for surfaces where τ fixes an even number of points. For use in those theorems, we observe that we could choose the singular locus a little more carefully than we did in Theorem 2.3. We do this by treating the problem of extending the invariant singular framing on F to a singular framing on all of M in two stages. Regard F as $F \times 0 \subset F \times [-1, 1]$, and try to extend the singular framing on F to $F \times [-1, 1]$, so that it is nonsingular on the boundary $F \times \{-1, 1\}$. It is not hard to show (cf. [8, §4.1]) that this can be done adding a single meridian curve to each component of the singular link L which passes through the singularities on F . Moreover, each component of L can be banded to its meridian (cf. [8, §3.3] or the proof of Theorem 3.1) to obtain a singular framing on $F \times [-1, 1]$ with singular locus precisely L . Now, as in the proof of Theorem 2.3, we can extend this singular framing on $F \times [-1, 1]$ over the rest of M , perhaps adding additional components to the singular locus. What we have achieved is to make the singular locus on M a split link, where the components which intersect F can be isotoped to lie in disjoint balls.

3. The extended torsion formula

We would like to extend the results of the previous section to give the invariance of the Chern-Simons invariant (mod 1) under mutation. It turns out that this stronger invariance holds for the genus-2 surface, and the twice-punctured torus and 4-punctured sphere in some special cases. These are exactly the surfaces where τ has an even number of fixed points.

The key point in our proof of Theorem 2.3 was that we could compute the Chern-Simons invariant mod $1/2$ using a singular framing. In this section, we give a stronger version of the torsion formula which computes the Chern-Simons invariant on the nose. For simplicity, we begin by describing this extended torsion formula in the closed case. The extension to the case of cusped manifolds has some complications which are discussed below.

One reason that the torsion formula of [8] only works mod $1/2$ is that the torsion of a curve is defined mod 2π , and the contribution of the torsion to the Chern-Simons invariant is $\pm \frac{1}{4\pi} \tau$. Thus we would like to extend the torsion to be defined modulo at least 4π . One way to define $\tau(\gamma)$ as a *real* number is to use an orthonormal frame field e with the property that restricted to γ , its e_1 -vector points along γ . Then $\tau_x(\gamma) = \int_\gamma e^* \omega_{23}$ is actually a real-valued invariant of γ and e , where we write ω_{ij} for the connection 1-forms in terms of the basis of the cotangent space dual to the e_i . This is the notion of torsion used by Yoshida [17] in his torsion formula (see Theorem 4.1 below). For the applications to cutting and pasting, however, this is not quite right as we lose the flexibility of a singular framing.

We compromise by using a frame field that is defined in a neighborhood of the singular locus of a singular frame field to get a torsion which is defined modulo 4π instead of as a real number. The frame field is defined by a surface that the singular locus bounds. As discussed in [8], the singular locus L of a singular framing on a 3-manifold M represents the obstruction (in $H_1(M; \mathbf{Z}_2) = H^2(M; \mathbf{Z}_2)$) to framing M . Since M is frameable, the obstruction vanishes and L is the boundary of a surface S .

We use S to give an orthonormal framing $\{e_i\}$ of M along L as follows. The e_1 -vector is tangent to L ; it does not matter which direction the e_1 's point along the components of L . The e_2 -vector will be tangent to S and point into S , and e_3 is determined by the previous two and the orientation of M . Note that this definition does not depend on S being orientable. Given a component γ of L , extend this framing on γ to a framing e on a

neighborhood of γ , and define

$$\tau_x(\gamma, S) = \int_{\gamma} \omega_{23}.$$

As in [13, volume 3], the extension is only used to compute the components ω_{ij} of the connection; the integral turns out to be independent of the extension. It is not hard to show that, modulo 4π , the sum

$$\sum_{\gamma \subset L} \pm \tau_x(\gamma, S)$$

does not depend on the choice of surface S ; we will usually eliminate the “ S ” in the notation for τ_x . Note that one can change this integral by multiples of 4π by choosing a surface which twists differently about γ . For example, a round circle in S^3 bounds both a disk and a Möbius band in S^3 . These give extended torsions which differ by 4π .

In the case of a cusped manifold with a linear frame field, the singular locus L is not necessarily a boundary. However, there is a surface with boundary the singular locus and some other curves which lie in the cusps. Such a surface may be extended to a surface with boundary L which goes out to infinity in the cusps. In contrast to the closed case, the surface may wind around the singular curves differently depending on how it goes out into the cusps. For instance, consider a linear frame field on a knot complement in S^3 which has singular curve a meridian of the knot. One surface with boundary the meridian and going out to ∞ in the cusp is a meridian disk which is punctured by the knot. Another such surface is a Seifert surface for the link formed by the knot and its meridian. In this surface, the pushoff of the singular curve links the singular curve once, and the extended torsions computed from these two surfaces differ by 2π .

One way out of this dilemma, which we adopt in this paper, is to choose, in each cusp C , a mod 2 homology class γ_C . (It may, of course be the zero class.) Then we define the extended torsion of the singular locus as above. However, we require the intersection of the surface with the cusp C to carry the homology class γ_C . Any two surfaces with this property will wind around the singular locus the same amount, mod 2. It is then straightforward to show that the extended torsion is well defined mod 4π . The essential point is that we can assume that the intersection of the surface with a cusp is a product over a geodesic in the Euclidean torus bounding the cusp; such a curve will have zero torsion. It is shown in [8, §4.3] that if M is the complement of a knot in S^3 , there is a singular framing with singular locus a meridian of the knot. In this case, the meridian disk provides the homology out into the cusp. Unless it is specified to the contrary,

if the cusped manifold is a knot complement, we will take the curve γ_C to be a meridian of the knot.

With this definition of the extended torsion, we can give an exact formula for the Chern-Simons invariant in the closed case, and define the Chern-Simons invariant (mod 1) for cusped manifolds, relative to some choice of $\{\gamma_C\}$.

Theorem 3.1. *Let s be a singular linear framing on a hyperbolic manifold M with singular locus L . Suppose that $L = \partial S$ where S hits the cusp C in the homology class $\{\gamma_C\}$. Then the number defined by the following formula is an invariant of M and the set $\{\gamma_C\}$:*

$$\frac{1}{8\pi^2} \int_{M-L} s^* Q + \frac{1}{4\pi} \sum_{\gamma \subset L} \pm(\tau_x(\gamma) - 2\pi) \pmod{1}.$$

Moreover, if M is closed, this is the Chern-Simons invariant of M .

Note. The fact that we need to subtract 2π in this formula stems from the fact that we can introduce extra components into L by the inverse of the “elimination” move of [8]. If K is a hyperbolic knot in S^3 , then we define the Chern-Simons invariant of K to be the (mod 1) Chern-Simons invariant of its complement, computed relative to the homology class of a meridian in the cusp by the above formula.

Proof. Following the pattern of proof of the torsion formula of [8], we show that the formula above is unchanged under three basic moves: (a) isotopy, (b) band move (=crossing move of [8]) and (c) removal. In the closed case, we can eliminate L (and get a nonsingular framing) by these moves, and so see that the formula is exactly the Chern-Simons invariant. In the cusped case, we can get from any L (and s) to another and so see that the formula gives a well-defined invariant of the manifold.

Isotopy move. The proof that the formula does not change under an isotopy of singular framings (and singular curves) is exactly as given in [8, §3.2], once we observe that a surface can be dragged along in an isotopy.

Band move. In [8], only oriented band moves are considered. In other words, band moves are done between components of L (or between a component and itself) which preserve the orientation of L derived from the e_1 -vectors of the singular framing. This is not sufficiently general to derive the torsion formula (even mod $1/2$). However the effect of an unoriented band move can be achieved by an oriented band move together with an additional move which flips the direction of e_1 on a single component γ of L . (We are using the isotopy move implicitly throughout.) This is illustrated in the sequence of pictures in Figure 6.

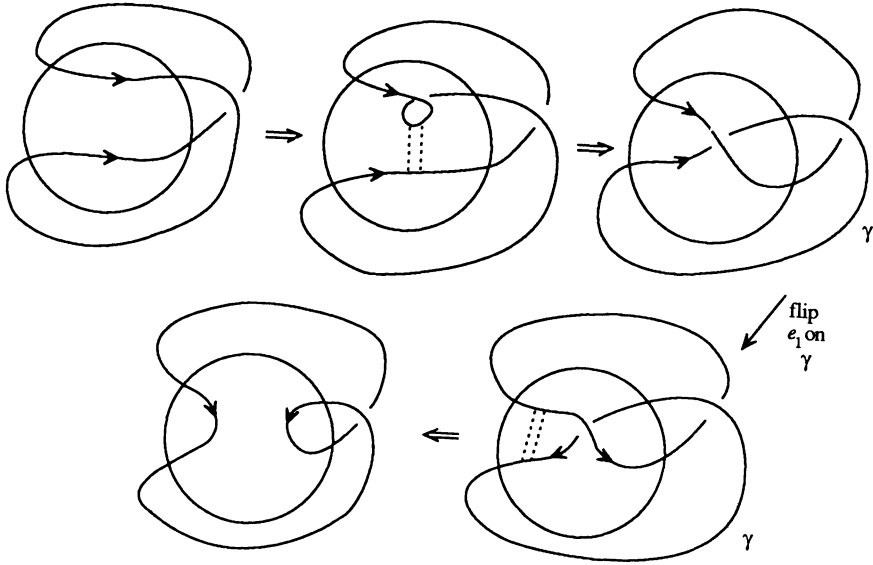


FIGURE 6. UNORIENTED MOVE

The proof that the oriented band move does not change the formula is similar to the proof in [8, §3.3]. In particular, it is shown there that

$$\frac{1}{8\pi^2} \int s^* \mathcal{Q}$$

changes by $1/2$ under an oriented band move. We may assume that the band move takes place in a small ball and that the intersection of S with that ball is a flat surface. Likewise, we may assume that the band is flat, so that after the band move, a new surface is created which coincides with the old one outside of the ball and is still flat inside the ball. Because the surfaces are all flat, the summation $\sum \pm \tau_x$ is unchanged. However, since the move was oriented, the number of components changes by 1. Because we subtract 2π for each component of L , the whole formula is unchanged (mod 1).

The flip move is performed by rotating the frame field by 180° (near the singular locus) in the plane perpendicular to the e_2 -vector. A calculation shows that the Chern-Simons integral over $M - L$ does not change during the course of this rotation. Therefore, the formula does not change.

Removal move. As in [8, §3.4] removing a small circle changes the Chern-Simons integral by $1/2$. But as in the discussion of the oriented band move, the number of components changes by one, so $(4\pi)^{-1} \sum \pm(\tau_x - 2\pi)$ changes by $1/2$ as well, and so the whole torsion formula is unchanged. q.e.d.

Using this extended torsion formula, we can now show the invariance of the Chern-Simons invariant under a mutation of a genus-2 surface.

Theorem 3.2. *Suppose F is a genus-2 incompressible surface in the closed hyperbolic manifold M . Then the Chern-Simons invariant of M^τ is the same as that of M , modulo 1.*

Proof. The idea is the same as in Theorem 2.3: one starts with a singular framing (as in that theorem) whose singular locus L hits F precisely in the fixed points of τ , and uses it to compute the Chern-Simons invariant both before and after cutting and pasting. The Chern-Simons integrand is the same after the mutation, so we must compute the change in the enhanced torsion.

Orient F and M , arbitrarily, so that there are \pm sides to F . Using Remark 2.7, we may assume that the only components of L which hit F are three curves C_1, C_2, C_3 . Each of these bounds a disk D_i which intersects F in an arc γ_i joining two fixed points of τ . The arc γ_i divides D_i into subdisks D_i^\pm , with boundaries $\partial D_i^\pm = C_i^\pm \cup \gamma_i$ (see Figure 7(a)). Label the subdisks so the D_i^+ 's are all on the (+) side of F .

By Remark 2.7, the other components of L will be disjoint from the disks D_i . Therefore there is a surface S with boundary L which consists of the D_i union some other components. The intersection Γ of S with F will thus be some closed curves, and the arcs γ_i .

After cutting and pasting, the singular locus of the singular framing s^τ is the link L^τ made from the pieces of L in M^τ . We need to find a surface S^τ in M^τ with boundary L^τ in order to compute the enhanced torsion. It is only important to see the surface in a neighborhood of L^τ , for that is where the surface defines the framing that is used to compute the torsion. In particular, an immersed surface which is embedded near L^τ will do as well for computing torsion, for it may be resolved into an embedded surface without affecting its boundary.

To start in constructing the surface S^τ , we have the pieces of S in M^τ , which hit F precisely in $\Gamma \cup \tau(\Gamma)$. Since τ acts trivially on mod 2 homology, each closed component in Γ is homologous to its image under τ . Therefore the closed components of Γ may be joined to their images under τ by subsurfaces of F , pushed slightly to the (-) side. (If this is done carefully, an embedded surface will result.) In finding the rest of S^τ , we may ignore these closed components.

To build the rest of the surface, we use the fact that each $\gamma_i \cup \tau(\gamma_i)$ is a closed curve, and that the three closed curves taken together bound the 3-punctured sphere in F which is drawn in Figure 7(b). After cutting and pasting, the pieces of S which hit the C_i consist of a homology from the

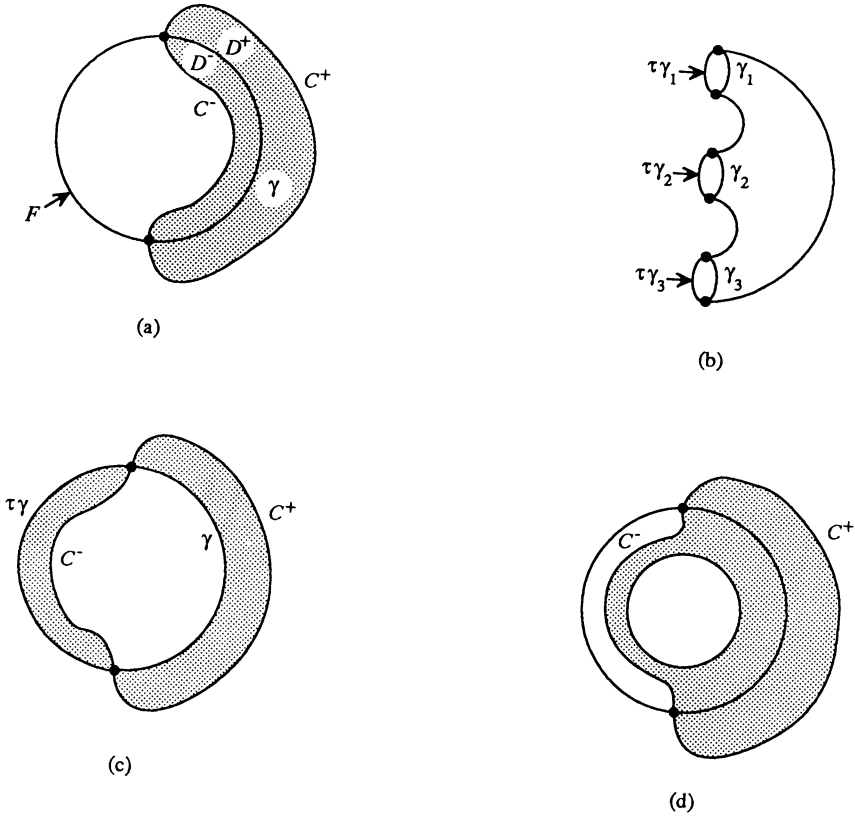


FIGURE 7

C_i^+ , say, to the γ_i , and the D_i^- which run from the C_i^- to the images $\tau(\gamma_i)$ in F (see Figure 7(c) above).

We build a new surface as follows. Near the C_i^- , build a surface, using the exponential map in the direction exactly opposite to the normal to C_i^- in D_i^- . Extend the surface near γ_i into the $(-)$ side of F . If the C_i were chosen to meet F perpendicularly, as is easy to arrange, these two pieces of surface fit together smoothly near C_i (see Figure 7(c)). The boundary of the resulting surface is L^τ (in M^τ), plus (for each i) a curve lying on the $(-)$ side of F and parallel to $\gamma_i \cup \tau(\gamma_i)$. But since

$$\bigcup_{i=1}^3 \gamma_i \cup \tau(\gamma_i)$$

bounds a subsurface of F , these extra boundary components can be filled in on one side of F using a pushoff of this subsurface.

The only possible change in the local framing near L under this operation comes from the last step in the construction of S^τ . But the induced framing near L^τ in M^τ is exactly the same as in M , because of how we made the surface S^τ lie on the other side of C_i^- . Therefore, the extended torsion of the singular locus is the same as measured in either M or M^τ . Thus the Chern-Simons invariant is unchanged. q.e.d.

The same considerations enable us to determine the effect of mutations on the Chern-Simons invariant in the cusped case as well. However, the result becomes awkward because of the necessity of specifying the homology classes γ_C before and after the mutation. One case where this is relatively canonical is the original operation of mutation of a knot in S^3 .

Given a knot K in S^3 , suppose that there is a 2-sphere meeting the K (transversally) in four points. The sphere bounds a ball, which may be removed and reglued by the involution τ of its boundary. The result is S^3 with a new knot, K^τ . There is a 4-punctured sphere F in the knot complement M ; it is easy to see that the complement of K^τ is exactly the manifold M^τ resulting from a mutation along F . Recalling that the Chern-Simons invariant of a hyperbolic knot is defined (mod 1), relative to its meridian as the curve γ_C in the cusp, we then have:

Theorem 3.3. *If K is a hyperbolic knot and K^τ a mutation of K along some 4-punctured sphere, then $\text{CS}(K) = \text{CS}(K^\tau) \pmod{1}$.*

Proof. The proof is similar to that of 3.2, except that we have to think a little more about the behavior of the framings and surfaces in the cusps. The knot will hit the 2-sphere in S^3 four times, so the cusp torus is divided by the mutating surface into four annuli. By rotating the e_1, e_2 vectors as in Theorem 2.3, it can be arranged that the cut and pasted framing s^τ is still (homotopically) linear in the cusp of M^τ .

The argument proceeds as in the previous theorem, until we get to the step where the circle $\gamma \cup \tau(\gamma)$ has to be filled in by a surface. This circle is of course essential in $H_1(F)$, but it is homologous to two of the cusps of F by a 3-punctured sphere.

Adjoining this homology to the rest of the surface gives a new surface S^τ with boundary L^τ . But since the two cusps of F homologous to $\gamma \cup \tau(\gamma)$ go out to the same cusp of M^τ , the intersection of S^τ with the cusp is still exactly a meridian of K^τ , as measured in \mathbf{Z}_2 -homology. Therefore we can use this surface to compute the extended torsion of L^τ , which is the same as that of L , as in Theorem 3.2. We conclude, as in that theorem, that the Chern-Simons invariant of K is unchanged by the mutation. q.e.d.

It is clear that further results about the Chern-Simons invariant of cusped manifolds can be obtained by this method. For instance, mutation

of links is defined exactly as for knots, and the Chern-Simons invariant is an invariant of that operation. We leave further generalizations to the untiring reader.

4. Mutation and the η -invariant

Our approach to the η -invariant will be through the formula of Yoshida. Suppose s is a singular framing with singular set L , and α is a framing with one vector, say e_1 , tangent to L . The difference between s and α defines a map from $M - L$ to $\text{SO}(3)$; since the two differ by an element of $\text{SO}(2)$ near L , one gets a difference degree $d(s, \alpha)$ as an element of $H_3(\text{SO}(3), \text{SO}(2)) \cong \mathbf{Z}$. Using the framing near L as in the previous section, Yoshida defines the torsion of L with respect to α , $\tau(L, \alpha)$ as a real number. Yoshida's formula extends the Atiyah-Patodi-Singer formula for the η -invariant by using a singular framing to compute the differential-geometric part:

Theorem 4.1 (Yoshida [17]). *Let M be an oriented Riemannian 3-manifold. Then*

$$(4.1) \quad \eta(M) = \frac{1}{12\pi^2} \int_{M-L} s^* Q - \frac{1}{6\pi} \tau(L, \alpha) + \frac{2}{3} d(s, \alpha) + \delta(M, \alpha).$$

Using this formula, we will determine how the η -invariant changes under mutation. The result may be stated in terms of the signature invariant of a surface in a 3-manifold defined by D. Cooper in his thesis [6]. For F a (closed) surface in a 3-manifold M , let K be the kernel of the inclusion of $H_1(F; \mathbf{R})$ in $H_1(M; \mathbf{R})$. Orient both F and M . Then K supports a "Seifert" form Θ defined by $\Theta(a, b) = \text{lk}(a, b^+)$. Here "lk" denotes linking number, and b^+ is the pushoff of b in the positive direction given by an orientation of F . The signature of F in M , $\sigma(F \subset M)$, is defined to be the signature of the symmetrized form $\Theta + \Theta'$. The main theorem proved in this section is then:

Theorem 4.2. *Let M be a closed, oriented hyperbolic 3-manifold containing an incompressible genus-2 surface F . Then the mutated manifold M^τ is hyperbolic, and $\eta(M^\tau) = \eta(M) - \sigma(F \subset M)$.*

We remark that the signature $\sigma(F \subset M)$ depends [6] only on the homology class carried by F . In particular, if F separates, then that homology class is trivial. Thus Corollary 1.2 of the introduction is a direct corollary of Theorem 4.2; however it will be established directly in the course of proving Theorem 4.2. It is also useful to note that the signature is independent of the orientation of F , and changes sign if the orientation of M is changed.

An important tool in identifying the difference $\eta(M^\tau) - \eta(M)$ as a signature is the “nonadditivity” formula of Wall [16]. This formula deals with the following situation: An oriented 4-manifold W is the union of two 4-manifolds W_\pm along a codimension-zero submanifold X_0 of their boundaries (see Figure 8). If X_0 were the full boundary of both W_+ and W_- (and all the orientations were correct), then the signature $\sigma(W)$ would simply be $\sigma(W_+) + \sigma(W_-)$. The nonadditivity formula gives the signature of W in terms of the signatures of W_\pm and another form defined in the next paragraph.

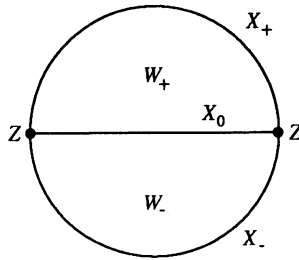


FIGURE 8

Write $\partial W_+ = X_+^3 \cup_Z X_0^3$ and $\partial W_- = X_-^3 \cup_Z X_0^3$. Wall [16] gives a convention for orienting the X 's and thus Z as well. Consider the subgroups A, B, C of $H_1(Z)$ (real coefficients understood) defined by

$$\begin{aligned} A &= \ker[H_1(Z) \rightarrow H_1(X_-)], \\ B &= \ker[H_1(Z) \rightarrow H_1(X_0)], \\ C &= \ker[H_1(Z) \rightarrow H_1(X_+)]. \end{aligned}$$

Let (\cdot) denote the intersection form of $H_1(Z)$. On the vector space $C \cap (A + B)$, Wall defines the form $\Psi(c, c') = c \cdot a'$, where a' is chosen so that $a' + c' \in B$. He shows that the form Ψ is symmetric, and proves the nonadditivity result:

Theorem 4.3. *In the situation above, the signature of W can be calculated as*

$$\sigma(W) = \sigma(W_+) + \sigma(W_-) - \sigma(\Psi).$$

Actually, since Ψ vanishes if c or c' is in the subspace $(C \cap A) + (C \cap B)$, Ψ induces a (nonsingular) form on the quotient

$$\frac{C \cap (A + B)}{(C \cap A) + (C \cap B)}$$

with the same signature.

As in the previous sections, a large part of the work in proving Theorem 4.2 goes into understanding what happens to the framings when we cut and paste. The involution τ acts on the tangent bundle of F , and hence on $T_*(F) \oplus \varepsilon$, preserving the section into ε . If F is embedded in a 3-manifold M , then a framing of the tangent bundle of M yields a framing of $T_*(F) \oplus \varepsilon$. The action of τ on the tangent bundle of M yields an action on framings, which we denote as $\alpha \rightarrow \tau^*(\alpha)$. We would like to compare α with $\tau^*\alpha$ on F .

For reference, take a copy of the genus 2 surface embedded in \mathbf{R}^3 . Choose a framing α_0 on \mathbf{R}^3 , say the one which is parallel in \mathbf{R}^3 to the framing at the origin, and restrict it to F . The framings α_0 and $\tau^*(\alpha_0)$ are not equal; however, they are homotopic. The two differ by a π -rotation in the xy -plane, so a homotopy between them is given by rotation of $\pi \cdot t$ in the xy -plane.

Any other framing of the tangent bundle of M restricted to F can be constructed from α_0 by multiplying the frame at each point by an element of $\text{SO}(3)$. Homotopy classes of maps F to $\text{SO}(3)$ are given by $H^1(F; \mathbf{Z}_2) = (\mathbf{Z}_2)^4$.

Lemma 4.4. *For any framing α of the tangent bundle of M restricted to F , $\tau^*(\alpha)$ is homotopic to α .*

The lemma follows easily from the facts that $\alpha_0 \simeq \tau^*(\alpha_0)$ and that τ^* acts by the identity on $H^1(F; \mathbf{Z}_2)$. However, to frame the cut and pasted manifold M^τ , and understand the effect on the terms in Yoshida's formula (4.1), we will need to exhibit specific homotopies.

Rather than enumerate the framings in terms of cohomology classes, we will use their Poincaré duals in $H_1(F; \mathbf{Z}_2)$. If $z \in H_1(F; \mathbf{Z}_2)$ is represented by a simple closed curve Z , then the corresponding framing α_Z can be drawn as follows. Choose a tubular neighborhood $U = Z \times \mathbf{I}$ of Z . Outside of U , the framing α_Z will agree with α_0 . As t goes from 0 to 1 in $Z \times \mathbf{I}$, rotate the framing in the xy -plane by $2\pi \cdot t$.

Now any homology class in $H_1(F; \mathbf{Z}_2)$ has a representative Z which is a simple closed curve. Moreover, it is possible to choose one which is invariant under τ . The curve Z will necessarily go through two of the fixed points of τ on F . A homotopy of framings α_s from α_Z to $\tau^*(\alpha_Z)$ is represented in the following pictures. Outside of U , the framing at time s will be α_Z ($= \alpha_0$) rotated by $\pi \cdot s$ in the xy -plane. Figure 9 shows $U \times \mathbf{I}$, with the framing α_Z on the inner surface $U \times 0$ and $\tau^*(\alpha_Z)$ on the outer surface $U \times 1$. The e_1 -vector will be vertical in $\partial(U \times \mathbf{I})$, and so is suppressed in

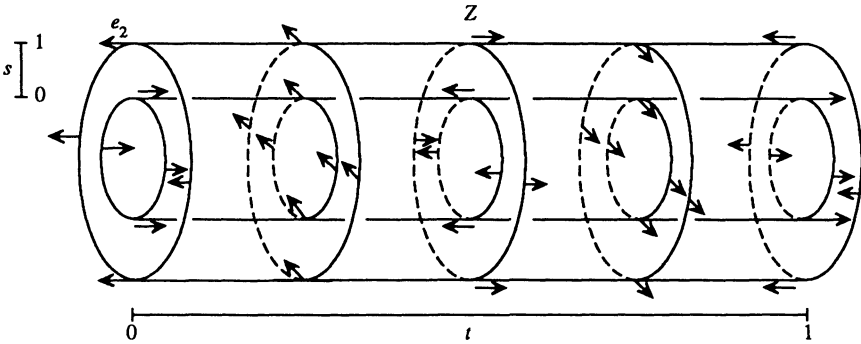


FIGURE 9

the picture. Also, since we have an oriented frame, it suffices to draw only one vector, say e_2 .

On $\partial U \times \mathbf{I}$, the framing must agree with the framing on the rest of F , so it is described by a rotation of $\pi \cdot s$.

It suffices to give an extension of the framing over the disk $D = z_0 \times \mathbf{I} \times \mathbf{I}$ for some point $z_0 \in Z$. For any other point (z', t, s) , we will choose the frame parallel (in \mathbf{R}^3) to that at (z_0, t, s) . Note that this is already true on the boundary of $U \times \mathbf{I}$. Choose z_0 to be a fixed point of τ on Z . Then we have the picture in Figure 10.

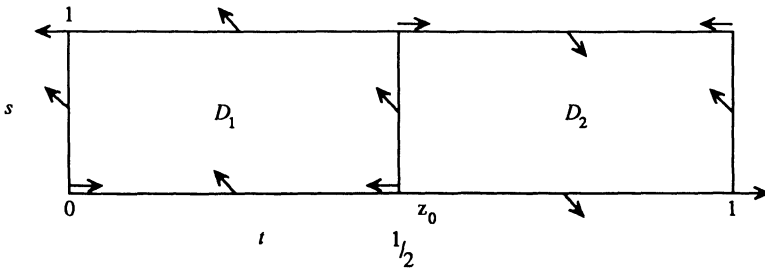


FIGURE 10

The disk D is divided into two subdisks, D_1 and D_2 , where t is less than or greater than $\frac{1}{2}$, respectively. On the line $L = z_0 \times \frac{1}{2} \times \mathbf{I}$, extend by rotating by $-\pi \cdot s$. The framing on ∂D_1 thus extends easily over D_1 by letting it be parallel along the slope 2 lines. On ∂D_2 , the framing rotates by $\pm 4\pi$ in the xy -plane as one goes around the boundary. This is twice the generator of $\pi_1(\text{SO}(3))$, and so is trivial. Hence the framing may be extended over D_2 as well; it is not important what the choice of extension is.

The homotopy of framings described by the above process has two important properties. One is that on the fixed set of τ , the e_1 -vector is always normal to F . The other is that except on $Z \times D_2$, the rotation of the framing is in the xy -plane throughout the homotopy.

With the homotopy α_t in hand, we are ready to calculate the change in the η -invariant under mutation.

Proof of Theorem 4.2. By [12], there is a genus-2 surface isotopic to F , with the property that cutting and pasting along F can be done geometrically. As in the proof of Theorem 2.3, choose a singular framing s whose restriction to F is invariant under τ . By construction, the singular locus L will hit F orthogonally in $\text{fix}(\tau)$, and the algebraic intersection number of L and F will be zero. Hence, as in 2.3, the singular framing on M may be cut and pasted to give a singular framing s^τ on M^τ .

Unfortunately, there is no way to arrange a nonsingular framing α so that it may be directly cut and pasted to give a framing on M^τ . This is remedied with the use of the homotopy described in the previous paragraphs. Let $N = F \times \mathbf{I}$ be a tubular neighborhood of F in M , which we may identify with a tubular neighborhood in \mathbf{R}^3 of some standard embedding of F . We may assume that the framing α on N is one of the framings α_z for $z \in H^1(F; \mathbf{Z}_2)$. By construction α has the property that its frames are parallel (in \mathbf{R}^3) along the \mathbf{I} -fibers in N . We may also assume that the intersection of L with N is $\text{fix}(\tau) \times \mathbf{I}$, by choosing the tubular neighborhood close enough to F .

Define the framing α^τ on M^τ as follows. On $M - N$, use the framing α . On N , replace the framing $\alpha = \alpha_z$ by the framing which is α_t on $F \times \{t\}$. Since $\alpha_1 = \tau^*(\alpha)$ and $\tau^2 = \text{Id}$, the framing on ∂N matches up with that on $\partial(M - N)$ to give a framing α^τ on M^τ . Since the homotopy α_t kept the e_1 -vector normal to F at all the fixed points of τ , the framing α^τ still has the property that its e_1 -vector is tangent to the singular locus for the singular framing s^τ . Hence we can use α^τ and s^τ in Yoshida's formula (4.1) to compute the η -invariant of M^τ . We will show that except for signature defect $\delta(M, \alpha)$, the value of each of the terms in his formula is unchanged when we replace M by M^τ , and the framings s and α by s^τ and α^τ , respectively. The signature defect changes by exactly $\sigma(F \subset M)$.

As in the proof of Theorem 2.3, the integral of s^{τ^*} of the Chern-Simons form over $M^\tau - L^\tau$ is the same as the corresponding integral over $M - L$. The torsion term is also easy to understand. As one goes from 0 to 1 in $L \cap N$, the framing α^τ rotates by π perpendicular to $L \cap N$ relative to α . Therefore, the torsion $\tau(L^\tau, \alpha^\tau)$ differs from $\tau(L, \alpha)$ by a sum $\sum \pm\pi$. There is one term in the sum for each component of $L \cap N$, and the sign

is determined by the orientation of L compared to the orientation of the fibers of N . But by construction each component of L hits F algebraically zero times, so the total change in the torsion term is zero.

The other two terms in Yoshida's formula are topological in nature. We discuss the "difference degree" $d(s, \alpha)$ first. This term is defined as follows. Off of the link L , the difference between s and α defines a map from $M - L$ to $\text{SO}(3)$. Since the e_1 -vector of α is tangent to L , and s has a special singularity near L , one can regard the difference as a map from $(M - \nu(L), \partial\nu(L))$ to $(\text{SO}(3), \text{SO}(2))$. (Here ν is a small tubular neighborhood of L .) The fundamental class of $(M - \nu, \partial\nu)$ thus defines a homology class in $H_3(\text{SO}(3), \text{SO}(2))$. Since this group is isomorphic to \mathbf{Z} , we get an integer, called the difference degree.

The change in the difference degree $d(s^\tau, \alpha^\tau) - d(s, \alpha)$ can be calculated on the part of M^τ where α^τ differs from α , i.e., on N . Since α^τ differs from α by an element of $\text{SO}(2)$ except on the solid torus $Z \times D_2$, an excision argument identifies the change in the difference degree with the element of $H_3(\text{SO}(3), \text{SO}(2))$ given by the difference between α^τ and α on $Z \times D_2$. But the difference between the two framings is constant in Z , by construction. Therefore the difference between the two factors through a map $(D_2, \partial D_2) \rightarrow (\text{SO}(3), \text{SO}(2))$. Thus its degree is zero, and the difference degree term in Yoshida's formula remains unchanged.

Finally, we come to the effect of cutting and pasting on the signature defect $\delta(M, \alpha)$. Since the Pontrjagin class is an invariant of the *stable* tangent bundle, one can show that the signature defect can be defined using a framing of the stable tangent bundle of the 3-manifold M . The calculation of the change in the signature defect divides into two cases, according to whether the surface F separates M or not. In the separating case we will show that the η -invariant is unchanged. Since the signature of a separating surface is zero, this implies the theorem in the separating case.

So suppose first that F separates M into two pieces A and B . We will view the tubular neighborhood N of F as being a collar of $F = \partial A$, with $F \times \{1\}$ corresponding to ∂A . According to [1], the 2-dimensional spin-cobordism group is \mathbf{Z}_2 , detected by an Arf invariant. Moreover, one can verify that any two spin-structures which are cobordant actually are related by a diffeomorphism. View F as embedded in S^3 in a standard way, with $S^3 - F$ the union of two unknotted handlebodies H_a and H_b . Now the spin structure on F obtained by viewing F as a submanifold of the framed manifold (M, α) is evidently null-bordant. Hence we can identify F (the

submanifold of M) with its copy in S^3 in such a way that the spin structure corresponding to α_0 extends over both of the handlebodies H_a and H_b .

Since the 3-dimensional spin-cobordism group is trivial, there is a spin 4-manifold W_a with boundary $A \cup_F H_a$, and similarly for B . This manifold should be regarded as a spin-cobordism, relative to $F \times \mathbf{I}$, from A to H_a . Take a handlebody decomposition for W_a which is a product along $F \times \mathbf{I}$. By surgery on W_a , if necessary, we may assume that there are only 2-handles in the handlebody decomposition.

Putting together the manifolds W_a and W_b along $F \times \mathbf{I}$, we obtain a spin-cobordism W from (M, α) to S^3 . Since W is spin, there is no obstruction to extending the framing α on M to a stable framing on W . We may also assume that on the submanifold $N \times \mathbf{I}$, the stable framing is identified with the framing $\alpha|_N$. The stable framing on W induces a stable framing on S^3 , which we will call β . By definition, the Pontrjagin class of W , relative to the stable framings on its boundary components, is trivial. Hence the signature defect of (M, α) is determined by the equation

$$\delta(M, \alpha) = \delta(S^3, \beta) - \sigma(W).$$

The manifold W may be cut and pasted by $\tau \times \text{id}_{\mathbf{I}}$ to give a cobordism W^τ from M^τ to the result of cutting and pasting S^3 . Since the involution τ extends over the handlebody H_a , it is easily verified that $(S^3)^\tau \cong S^3$. Since we assumed that on $N \times \mathbf{I}$ we had the framing $\alpha|_N$, the same technique that framed M^τ shows that W^τ will be stably framed. S^3 gets a stable framing β^τ by the exact same process. Hence the signature defect of (M^τ, α^τ) can be calculated from W^τ . In particular, we see that the defect of M changes by the formula

$$\delta(M, \alpha) - \delta(M^\tau, \alpha^\tau) = \delta(S^3, \beta) - \delta(S^3, \beta^\tau) + \sigma(W^\tau) - \sigma(W).$$

We will treat the signature terms and the $\delta(S^3, \beta)$ terms separately.

To understand why the signature of W^τ is the same as that of W , recall that by construction, the manifolds W_a and W_b are gotten from A and B by adding 2-handles. Hence W has a handlebody decomposition, starting from S^3 , with only 2-handles, where the attaching circles of the handles miss F . Since W^τ is obtained by cutting and pasting the cobordism W , it too has a handlebody decomposition of the same type. In fact, the attaching circles of the handles for W^τ can be viewed as follows. When we perform the diffeomorphism of $(S^3)^\tau$ back to S^3 , the circles which were inside the handlebody (say H_a) over which we extend τ get flipped over with H_a . These flipped over circles, plus the ones in H_b , form the attaching link for W^τ .

Orient the attaching circles, arbitrarily. The intersection form of W is given by the matrix of linking numbers between the attaching circles, with the framings on the diagonal. Now the extension of τ over H_a has the property that it acts by -1 on homology. It follows easily that the linking numbers between two circles which are both on one side of F is the same both before and after the mutation; the same applies to the framings of the circles. However, the linking number between circles on opposite sides of F gets its sign changed. Hence if we change the orientation of the circles in H_a , the linking matrix is unchanged. Therefore the signature of W^τ is the same as the signature of W . (This may also be seen by applying Wall nonadditivity (4.3) as in the nonseparating case below.)

The signature defect of a framing on S^3 can be understood as follows. Let β_0 be the stable framing of S^3 which extends over the ball B^4 . For any other stable framing β , the defect $\delta(S^3, \beta)$ is then just the difference degree $d(\beta_0, \beta) \in \pi_3(\text{SO}) \cong \mathbf{Z}$. Therefore the change in the signature defects between the stable framings β and β^τ is exactly their difference degree. But the proof that the other difference degree term ($d(s, \alpha)$) was unchanged under mutation applies, *mutatis mutandis*, to show that the difference degree is zero in this case.

The other case, in which the surface does not separate, is more complicated. Instead of a cobordism to S^3 , we will use a cobordism to $F \times S^1$. Regard M as the union of $F \times \mathbf{I}$ with a 3-manifold N . As above, since the 3-dimensional spin-cobordism group is zero, M is the boundary of a spin-manifold V . By making a corner at F , we may regard V as a spin-cobordism, relative to the boundary, from N to $F \times \mathbf{I}$. Gluing V together with $F \times \mathbf{I} \times \mathbf{I}$, we get a spin-cobordism W from M to $F \times S^1$, which contains a copy of $F \times \text{pt} \times \mathbf{I}$. As in the previous case, we will show that $\delta(F \times S^1, \alpha) = \delta((F \times S^1)^\tau, \alpha^\tau)$. This is complicated somewhat by the fact that $(F \times S^1)^\tau \neq F \times S^1$, but is instead $S^1 \times_\tau F$.

We will get around this problem by transferring to the double cover $F \times \widetilde{S^1} \cong F \times S^1 \cong (F \times \widetilde{S^1})^\tau$. Let g and h be the covering translations of $F \times \widetilde{S^1}$ over $F \times S^1$ and $F \times_\tau S^1$ respectively. Recall that a free involution t on a 3-manifold N has a ρ -invariant $\rho(N, t)$ defined by Atiyah and Singer [3]. If the involution extends to a free involution on a 4-manifold \tilde{V} , then

$$\rho(N, t) = 2\sigma(\tilde{V}) - \sigma(V).$$

For both involutions g and h , it is easy to see that $\rho = 0$. For in both cases, the involutions extend over $S^1 \times (\text{genus-2 handlebody})$, and one calculates directly that all the relevant signatures are 0.

Now suppose that $F \times S^1 = \partial(V)$, where the double cover extends over V . Then we may use V to calculate the signature defect of $F \times S^1$ with respect to the framing α , and \tilde{V} to calculate the defect with respect to the pulled-back framing $\tilde{\alpha}$. It is easy to see that the relative Pontrjagin class multiplies under covers, when we use the pull-back framing on the boundary upstairs. From this, we obtain

$$\delta(F \times S^1, \tilde{\alpha}) - 2\delta(F \times S^1, \alpha) = 2\sigma(V) - \sigma(\tilde{V}) = \rho(F \times S^1, g) = 0.$$

Similarly, we obtain that $\delta(F \times S^1, \tilde{\alpha}^\tau) = 2\delta(F \times_\tau S^1, \alpha^\tau)$. But one can verify directly that the framings $\tilde{\alpha}$ and $\tilde{\alpha}^\tau$ on $F \times S^1$ are isotopic, and therefore have the same δ . Thus we conclude that $\delta(F \times S^1, \alpha) = \delta((F \times S^1)^\tau, \alpha^\tau)$.

The more interesting part is that the signature of W changes under cutting and pasting by the formula $\sigma(W^\tau) = \sigma(W) - \sigma(F \subset M)$. As it turns out, this depends only on the fact that $\tau_* = -I$ on $H_1(F)$. In order to derive this formula, we use Wall's nonadditivity formula, Theorem 4.3. In Figure 11, we draw all the different boundary pieces.

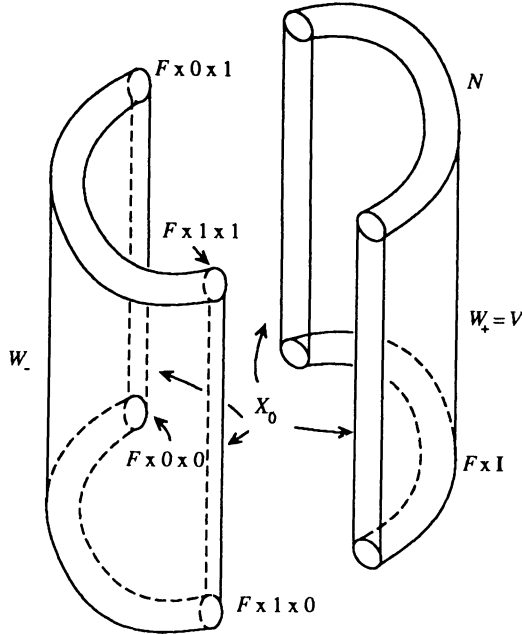


FIGURE 11

Write W as the union $W_+ \cup_{X_0} W_-$, where $W_- = F \times \mathbf{I} \times \mathbf{I}$ and $W_+ = V$. The second \mathbf{I} -coordinate represents height in Figure 11, and the first

coordinate represents front to back distance. Then $X_0 = F \times 0 \times \mathbf{I} \amalg F \times 1 \times \mathbf{I}$, and the rest of ∂W_- is $X_- = F \times \mathbf{I} \times 1 \amalg F \times \mathbf{I} \times 0$. Similarly X_+ is $F \times \mathbf{I} \amalg N$. The surface Z where the form Ψ lives is four disjoint copies of F :

$$Z = F \times 0 \times 0 \amalg F \times 1 \times 0 \amalg F \times 0 \times 1 \amalg F \times 1 \times 1.$$

Z is oriented so that $F \times 0 \times 0$ and $F \times 1 \times 1$ have the same orientation, and the other two components are oppositely oriented. With these identifications, we can compute $\sigma(\Psi)$.

From now on, all coefficients will be real. Let i_0 and i_1 be the inclusions $H_1(F) \rightarrow H_1(F \times 0 \times 1) \rightarrow H_1(N)$ and $H_1(F) \rightarrow H_1(F \times 1 \times 1) \rightarrow H_1(N)$ respectively. Then we can identify the subspaces A, B, C of the nonadditivity formula as

$$\begin{aligned} A &= \{(x, -x, y, -y): x, y \in H_1(F)\}, \\ B &= \{(x, y, -x, -y): x, y \in H_1(F)\}, \\ C &= \{(x, -x, y_0, y_1): x, y_0, y_1 \in H_1(F), i_0 y_0 = i_1 y_1\}. \end{aligned}$$

An easy exercise shows that $C \cap (A + B) \cong (C \cap A) + (C \cap B)$, so that for W , before we cut and paste, we have

$$\sigma(W) = \sigma(F \times \mathbf{I} \times \mathbf{I}) + \sigma(V) = \sigma(V).$$

After the mutation, however, the situation changes.

The cut and pasted manifold W^τ is obtained by gluing W_- to W_+ along X_0 , using the identity on $F \times 0 \times \mathbf{I}$, and τ on $F \times 1 \times \mathbf{I}$. Thus in computing $\sigma(W^\tau)$, the subspaces A and B are unchanged, but with our identification of $H_1(Z)$, the kernel $H_1(Z) \rightarrow H_1(X_+)$ becomes

$$C^\tau = \{(x, x, y_0, y_1): x \in H_1(F), i_0 y_0 = i_1 y_1\}.$$

To identify the space $C^\tau \cap (A + B)$, start with an element $c = (x, x, y_0, y_1) \in C^\tau$ and try to find a and b such that $c + a = b$. It is easy to see that a necessary condition is that $-2x = y_0 + y_1$. In fact this condition is sufficient as well, for we can set $a = (0, 0, \frac{1}{2}(y_0 - y_1), \frac{1}{2}(y_1 - y_0))$ and $b = (x, x, -x, -x)$. The form Ψ on $C \cap (A + B)$ may be identified in the following way: $C^\tau \cap (A + B)$ is isomorphic to the vector space

$$U = \{(y_0, y_1): i_0 y_0 = i_1 y_1\}.$$

Under this isomorphism, the form Ψ is

$$\begin{aligned}\Psi[(y_0, y_1), (y'_0, y'_1)] &= (x, x, y_0, y_1) \cdot \frac{1}{2}(0, 0, y'_0 - y'_1, y'_1 - y'_0) \\ &= \frac{1}{2}(y_1 \cdot (y'_1 - y'_0) - y_0 \cdot (y'_0 - y'_1)) \\ &= \frac{1}{2}((y_1 \cdot y'_1 - y_0 \cdot y'_0) + (y_0 \cdot y'_1 + y'_0 \cdot y_1)).\end{aligned}$$

(The signs come from the differing orientations of the components of Z .) The second term in the final equation is symmetric. Since the whole form Ψ is symmetric, the first term which is antisymmetric, must automatically vanish. (This may of course be verified directly.) Hence we obtain $2\Psi = (y_0 \cdot y'_1 + y'_0 \cdot y_1)$.

Recall that the signature of F in M is defined to be the signature of the symmetrized Seifert form $\Theta + \Theta'$ on $K = \ker[H_1(F) \rightarrow H_1(M)]$. Identify F in M with $F \times 1/2 \times 1$, then the \pm -pushoffs are given by i_0, i_1 respectively. Define a homomorphism $f: U \rightarrow K$ by $f(y_0, y_1) = y_1 - y_0$. An easy Mayer-Vietoris calculation shows that f is onto.

Claim. $(\Theta + \Theta')(f(c), f(c')) = \Psi(c, c')$.

Proof of Claim. By definition, $\Theta(f(c), f(c')) = \text{lk}(y_1 - y_0, (y'_1 - y'_0)^+)$, where $+$ is the pushoff corresponding to the 0-direction. Choose a homology D_0 from $i_0 y_0$ to $i_1 y_1$ in N . Then a null-homology D for $(y_1 - y_0)$ in M is composed of D_0 , together with a product tube T_0 running from $y_0 \subset F \times 1/2 \times 1$ to $F \times 0 \times 1$, and a tube T_1 starting at y_1 .

Using this null homology, we calculate the linking number:

$$\text{lk}(y_1 - y_0, (y'_1 - y'_0)^+) = D \cdot (y'_1 - y'_0)^+ = T_0 \cdot (y'_1 - y'_0)^+ = y_0 \cdot (y'_1 - y'_0).$$

Therefore

$$(\Theta + \Theta')(f(c), f(c')) = y_0 \cdot y'_1 + y'_0 \cdot y_1 = 2\Psi(c, c').$$

Finally, it is easy to see that for any $c \in \ker(f)$, we have $\Psi(c, c') = 0$ for all c' . It follows from this that $\sigma(\Psi) = \sigma(\Theta + \Theta') = \sigma(F \subset M)$. Therefore

$$\begin{aligned}\sigma(W^\tau) &= \sigma(F \times \mathbf{I} \times \mathbf{I}) + \sigma(V) - \sigma(\Psi) \\ &= \sigma(V) - \sigma(F \subset M) = \sigma(W) - \sigma(F \subset M).\end{aligned}$$

Thus the signature defect changes by $\sigma(F \subset M)$; since all the other terms in Yoshida's formula are unchanged, this completes the proof of Theorem 4.2.

As remarked in the introduction, the η -invariant of a 3-manifold M determines the Chern-Simons invariant modulo $1/2$. With a little more

information about M , one can determine $\text{CS}(M)$ from $\eta(M)$. Atiyah-Patodi-Singer define $S(M)$ to be the signature (modulo 2) of any spin-manifold which M bounds, and prove:

Proposition 4.5 [3]. *The Chern-Simons and η -invariants are related by the congruence $\frac{3}{2}\eta(M) \equiv \text{CS}(M) \pmod{1/2}$. Moreover, they are equal (mod 1) if and only if $S(M)$ is even. In other words,*

$$\frac{3}{2}\eta(M) \equiv \text{CS}(M) + \frac{1}{2}S(M) \pmod{1}.$$

They also remark that $S(M)$ may be calculated as the number of 2-primary summands of $H_1(M; \mathbf{Z})$. Using this proposition, it is easy to recover the result of §3 that the Chern-Simons invariant does not change under mutation along a closed genus-2 surface F .

Corollary 4.6. *If F is a genus-2 surface in the hyperbolic manifold M , then $\text{CS}(M^\tau) = \text{CS}(M)$.*

Proof. In the proof of Theorem 4.2, we constructed a spin-cobordism W of M to S^3 (or $S^1 \times F$) according to whether F separates M or not. By construction, cutting and pasting W gave a spin-cobordism of M^τ to S^3 (or $S^1 \times_\tau F$ respectively). It is easy to calculate that $S = 0$ for S^3 , $S^1 \times F$, and $S^1 \times_\tau F$. Therefore $S(M) = \sigma(W)$, and $S(M^\tau) \equiv \sigma(W^\tau) \pmod{2}$.

The proof of Theorem 4.2 shows that $\eta(M^\tau) = \eta(M) = \sigma(W) - \sigma(W^\tau)$ and therefore that $\eta(M^\tau) - \eta(M) \equiv S(M^\tau) - S(M) \pmod{2}$. Hence, by Proposition 4.5, the Chern-Simons invariant changes by

$$\text{CS}(M^\tau) - \text{CS}(M) \equiv \frac{3}{2}(\eta(M^\tau) - \eta(M)) + \frac{1}{2}(S(M^\tau) - S(M)) \equiv 0 \pmod{1}.$$

q.e.d.

To conclude, we illustrate these calculations with an example of how the η -invariant can change, leaving the Chern-Simons invariant unchanged.

Example 4.7. According to a theorem of Papadopoulos [10], any automorphism of the homology of a surface which preserves the intersection pairing is induced by a pseudo-Anosov [7] homeomorphism. Let φ be a pseudo-Anosov homeomorphism of a genus-2 surface whose action on homology is described by a block sum of matrices:

$$\varphi_* = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \oplus \text{Id}_{2 \times 2}.$$

Then $M =$ mapping torus of φ is a hyperbolic manifold [14], [15], and its first homology is $\mathbf{Z} \oplus \text{coker}[\varphi_* - I]$, which is easily calculated to be \mathbf{Z}^4 . Hence $S(M)$ is zero, and $\frac{3}{2}\eta(M)$ is the same (mod 1) as the Chern-Simons invariant. On the other hand, the homology of M^τ is $\mathbf{Z} \oplus \text{coker}[\varphi_* \tau_* - I]$.

Using the fact that $\tau_* = -I$, one calculates that $H_1(M^\tau) \cong \mathbf{Z} \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_4$. Therefore $S(M) = 1$, and $\frac{3}{2}\eta(M)$ differs (mod 1) from the Chern-Simons invariant.

On the other hand, we can compute the change in the η -invariant by using Theorem 4.2. It is straightforward to calculate the signature of F in M from the matrix description of the action of φ on homology. The kernel of the map on homology is isomorphic to \mathbf{Z} , and is generated by a single element y such that $\text{lk}(y, y^+) = 1$. Therefore the signature $\sigma(F \subset M)$ is 1, and the η -invariant changes by 1, while the Chern-Simons invariants of M and M^τ are the same.

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