J. DIFFERENTIAL GEOMETRY
30 (1989) 643-688

ENTIRE SOLUTIONS OF THE MINIMAL SURFACE EQUATION

LEON SIMON

A solution u of the minimal surface equation

$$\Delta u - \sum_{i,j}^{n} \frac{D_i u D_j u}{1+|Du|^2} D_i D_j u = 0$$

is said to be *entire* if it is C^2 on all of $\mathbf{R}^n$, and *exterior* if it is C^2 on $\mathbf{R}^n \sim U$, where U is a bounded open subset of $\mathbf{R}^n$. In this paper we present a number of new results concerning asymptotic behavior of exterior and entire solutions, and in addition we establish the existence of many new examples of nonlinear entire solutions. Prior to this work, only the example constructed by Bombieri, De Giorgi and Giusti [3] in each even dimension $n \geq 8$ was known. (To be precise, these were the only nonlinear examples known modulo the additional examples obtained by scaling, rigid motions and adding "redundant variables", i.e., writing $\tilde{u}(x^1, \cdots, x^{n+k}) \equiv u(x^1, \cdots, x^n)$ to get a solution $\tilde{u}$ on $\mathbf{R}^{n+k}$ from a solution u on $\mathbf{R}^n$.)

The main results about asymptotic behavior are given in §5, and include some fairly precise upper estimates on rate of growth (given in Theorem 5) under suitable a priori restrictions on the "tangent cylinder at ∞"; the notion of tangent cylinder at ∞ is discussed in §4. We also obtain corresponding lower growth estimates in Theorem 5. Caffarelli, Nirenberg and Spruck [7], Nitsche [18], and Ecker and Huisken [8] were already able to obtain lower growth estimates for entire solutions; while their estimates are not sharp in general (see the discussion in Remark (5.8) below), they have the advantage of being applicable to all nonlinear entire solutions, without a priori restrictions on the tangent cylinder at ∞.

The new examples referred to above are constructed in §6. It is proved that for each of the codimension 1 minimizing cones C in the list of Lawson [14] or in the list of Ferus and Karcher [10], there is an entire solution of the minimal surface equation which has $C \times R$ as tangent

Received May 18, 1987 and, in revised form, July 26, 1988. The author's research was supported in part by a National Science Foundation Grant at Stanford University.

cylinder at ∞. It is interesting that here we need some of the growth estimates from §5 in showing that the existence procedure, which, like the procedure of [3], involves solving the Dirichlet problem on an expanding sequence of domains, actually does yield entire solutions.

1. Preliminary discussion

We begin by considering a method (proposed by M. Miranda) of constructing examples of nonlinear entire solutions of the minimal surface equation in $\mathbf{R}^n$. To fix ideas, we first take the simplest example (a much more general discussion appears in §6): take C to be the minimizing cone

$$\{(x,y)\in \mathbf{R}^p\times\mathbf{R}^p\colon |x|=|y|\}\subset \mathbf{R}^n \qquad (n=2p,\ p\geq 4)$$

(more general cones are considered in §6) and let

$$A_\pm = \{(x,y)\colon |x| > (<)\, |y|\}\cap \partial B_1(0),$$

respectively. Let u_k be the bounded $C^2(B_1(0)\cup A_+\cup A_-)$ solution of the minimal surface equation with boundary data

$$u_k = \begin{cases} +k & \text{on } A_+, \\ -k & \text{on } A_-. \end{cases}$$

Such a u_k exists, and is unique (see, e.g., [12, Chapter 16]); of course u_k is discontinuous at $\partial A_+ = \partial A_- \subset \partial B_1(0)$. By the symmetries of C and uniqueness of u_k it is easy to check that

$$\begin{cases} |Du_k(0)| = 0, \qquad u_k\equiv 0 \text{ on } \overline{C}\cap B_1(0), \\ u_k(x,y)\equiv -u_k(y,x)\equiv u_k(-x,-y). \end{cases}$$

As $k\to\infty$ it is easy to check that $\sup_{B_\sigma(0)}|Du_k|\to\infty$ for each fixed $\sigma>0$ (see the more general discussion in §6). Therefore we can select $0<\sigma_k<1$ such that

$$\sup_{B_{\sigma_k}(0)}|Du_k| = 1, \qquad \sigma_k\downarrow 0.$$

We then let $\tilde{u}_k(x) = \sigma_k^{-1}u_k(\sigma_k x)$, $x\in B_{\sigma_k^{-1}}(0)$. As $k\to\infty$ we can use standard compactness theory for minimal graphs (see the detailed discussion in §6 below) to ensure that there are an open domain $\Omega\supset\overline{C}$ and a subsequence of $\{u_k\}$ converging to a $C^2(\Omega)$ solution u of the minimal surface equation, where either $\Omega=\mathbf{R}^n$ or $\partial\Omega$ has two components $F_\pm\subset\mathbf{R}^n_\pm$ respectively, where $\mathbf{R}^n_\pm$ are the two components of $\mathbf{R}^n\sim\overline{C}$, and where $u\to\pm\infty$ on approach to $F_\pm$.

Thus the point is this: While the above direct procedure always gives a complete minimal graph, we are left with the possibility that it may not be entire; that is, that $u \in C^2(\Omega)$ with $\Omega \neq \mathbf{R}^n$. This difficulty was pointed out by M. Miranda.

We now briefly explain the approach (carried out in detail in §6 for a general class of examples) to prove that $\Omega = \mathbf{R}^n$. Assuming on the contrary that $\Omega \neq \mathbf{R}^n$, we consider the components $F_\pm$ of $\partial\Omega$ introduced above. Both $F_\pm$ are minimizing hypersurfaces (see Lemma (4.3) below); indeed by the result of [13], the $F_\pm$ have no singularities and some homothety of F_+ agrees with the smooth minimizing hypersurface in $\mathbf{R}^n$ constructed explicitly in [3] by Bombieri, De Giorgi, and Giusti. Then if $\omega_\pm$ are given points of $S^{n-1} \cap \mathbf{R}^n_\pm$ respectively, the rays $\{\lambda\omega_\pm\}$ intersect $F_\pm$ respectively in just one point. Further, $F_\pm$ approach C asymptotically near infinity at a known rate: specifically,

$$(1) \qquad F_\pm \sim B_{R_1}(0) = \operatorname{graph}_C v_\pm,$$

where $v_\pm \in C^2(C \sim K)$ for suitable $R_1 > 0$ and bounded $K \subset C$, and where

$$(2) \qquad r^{\gamma_1} v_\pm(r\omega) \to c \neq 0 \quad \text{as } r \to \infty$$

with

$$\gamma_1 = \frac{n-3}{2} - \sqrt{\left(\frac{n-3}{2}\right)^2 - (n-2)} > 0;$$

here $\operatorname{graph} v_\pm = \{\xi + v_\pm(\xi)\eta_\pm(\xi) \colon \xi \in C \sim B_{R_2}(0)\}$, with $\eta_\pm(\xi)$ $(\equiv \eta_\pm(\xi/|\xi|))$ the unit normal of C pointing into $\mathbf{R}^n_\pm$ respectively. These facts are extended to general minimizing cones in [13]—such a generalization is needed in §6, and in earlier sections of the present paper; see (5.5) below. Notice that (1) in particular means that the part of $\operatorname{graph} u$ lying outside the cylinder $B_R(0) \times \mathbf{R}$ is pointwise close to the cylinder $C \times \mathbf{R}$ for suitably large R; precisely, for suitable $R_1 > 0$ and bounded $K \subset C$,

$$(3) \qquad \operatorname{graph} u \sim (B_{R_1}(0) \times \mathbf{R}) = \operatorname{graph}_{C\times\mathbf{R}} w,$$

where $w \in C^2(C \sim (K \times \mathbf{R}))$ and

$$\operatorname{graph}_{C\times\mathbf{R}} w = \{(r\omega, y) + w(r\omega, y)\eta_+(r\omega) \colon (r\omega, y) \in (C \sim K) \times \mathbf{R}\},$$

with η_+ the unit normal of C as above.

Of course since $\operatorname{graph} u$ is a minimal hypersurface in $\mathbf{R}^{n+1}$, then so is $\operatorname{graph}_{C\times\mathbf{R}} w$, and hence w satisfies the Euler-Lagrange equation corresponding to the area functional for graphs of functions over domains in $C \times \mathbf{R}$. That is, for suitable $\rho > 0$, we have

$$(4) \qquad \mathscr{M}_{C\times\mathbf{R}} w = 0 \quad \text{on } G \equiv \{(r\omega, y) \colon r > \rho/2, \omega \in C \cap S^{n-1}, y \in \mathbf{R}\},$$

where $\mathscr{M}_{C\times\mathbf{R}}$ is the minimal surface operator for C^2 functions over domains in $C \times R$. Furthermore because of (1) we have some additional a priori information: Viz., by (1), (2) and the fact that graph u is C^2 close to $F_+ \times \mathbf{R}$ near points (ξ, y) of $F_+ \times \mathbf{R}$ with y sufficiently large, we know that there exist $\rho > 0$ such that for any $\varepsilon, \delta > 0$ there is $y_1 = y_1(\delta) \geq 1$ such that

$$w(r\omega, y) \geq c\left(\frac{r}{\rho}\right)^{-\gamma_1-\varepsilon} w(\rho\omega, y) \tag{5}$$

for $\rho < r < \delta^{-1}\rho$, $y \geq y_1$, $\omega \in C \cap S^{n-1}$. It turns out that solutions of (4) which satisfy a priori growth bounds of the form (5) automatically satisfy certain lower growth bounds along vertical rays $\{(\rho\omega, y): y \geq y_1\}$. Specifically, in Theorem 1 of §2 below we prove a general theorem about solutions of equations of the form (4) over domains in $C \times \mathbf{R}$, subject to growth bounds of the form (5); the main result applies in the special case above to establish that

$$\rho^{\gamma_1} y^{\varepsilon} \frac{\partial w}{\partial y}(\rho\omega, y) \geq c \quad \text{for } y \geq y_1, \tag{6}$$

with $c > 0$ independent of y. Now this evidently gives information about the gradient of the original solution u defined over $\Omega \subset \mathbf{R}^n$. Specifically we note that for any $\varepsilon > 0$

$$\frac{\partial w}{\partial y}(\rho\omega, y) = \frac{1}{|Du(\xi)|} \tag{7}$$

for given $(\rho\omega, y) \in G$, where $y = u(\xi)$ and $\xi \in \Omega$ is the point

$$\xi = (\rho\omega, y) + w(\rho\omega, y)\eta_+(\rho\omega)$$

with η_+ the unit normal of C as described above. Notice that y approaches ∞ as ξ approaches a point of F_+ $(= \partial\Omega \cap \mathbf{R}^n_+)$. However (6) and (7) imply $|D(u(\xi))^{1-\varepsilon}|$ is bounded in a neighborhood of any point of F_+, hence u itself is bounded in such a neighborhood, contradicting the fact that $u \to \infty$ on approach to points of F_+. Thus we conclude that $\Omega = \mathbf{R}^n$ and hence that u is an entire solution. Since by construction $\sup_{B_1(0)} |Du| = 1$ and u vanishes on C, we know that u is nonlinear; indeed by construction it has $C \times \mathbf{R}$ as its "tangent cylinder" at ∞. See §4 for a discussion of tangent cylinders at ∞.

Hopefully the above discussion provides adequate motivation for the rather technical considerations of quasilinear equations over cylindrical domains in the next section. In actual fact, considerations such as those in the next section lead to general growth bounds for a class of entire and exterior solutions discussed in §5, in addition to their role indicated above in the construction of the new examples of §6.

2. Quasilinear operators on cylindrical domains

Let C be an $(n-1)$-dimensional cone in $\mathbf{R}^n$ of the form $C = \{\lambda\omega\colon \omega \in \Sigma, \lambda > 0\}$, where Σ is a smooth compact embedded $(n-2)$-dimensional submanifold of S^{n-1}. We are going to consider, on domains $\Omega \subset C \times \mathbf{R}$ with points x represented $(r\omega, y)$, $r > 0$, $\omega \in \Sigma$, $y \in \mathbf{R}$, a quasilinear divergence-form operator of the form

$$\mathscr{M}u = \operatorname{div}_{C\times\mathbf{R}} A(\omega, u/r, \nabla u) + r^{-1}B(\omega, u/r, \nabla u). \tag{2.1}$$

Here $A = (A^1, \cdots, A^{n+1})$, and $A^j = A^j(\omega, z, p)$ and $B = B(\omega, z, p)$ are smooth functions of $(\omega, z, p) \in \Sigma \times \mathbf{R} \times \mathbf{R}^{n+1}$. We assume that $\mathscr{M}(0) = 0$ and that the linearization $Lu \equiv \frac{d}{ds}\mathscr{M}(su)|_{s=0}$ has the special form

$$Lu = \Delta u + r^{-2}q(\varepsilon)u, \tag{2.2}$$

where q is smooth on Σ, and Δ is the Laplacian on $C \times \mathbf{R}$.

(2.3) Remark. If $\mathscr{M}$ is the minimal surface operator, i.e., the Euler-Lagrange operator of the area functional, on $C \times \mathbf{R}$, then $\mathscr{M}$ is as in (2.1) and in this case (2.2) holds with $q(\omega) = |A(\omega)|^2$, the squared length of the second fundamental form $A(\omega)$ of $\Sigma \subset S^{n-1}$. For the example discussed in §1 we have $|A(\omega)|^2 \equiv n-2$ (see, e.g., [29]).

We are going to assume that there exist positive solutions of the equation $\Delta_C u + r^{-2}qu = 0$ ($\Delta_C u =$ Laplace-Beltrami operator on C) over the whole cone C. This is equivalent to

$$-\frac{(n-3)^2}{4} \le \lambda_1, \tag{2.4}$$

where, here and subsequently, λ_1 is the minimum eigenvalue of the operator $L_\Sigma = \Delta_\Sigma + q$ on Σ. In fact if (2.4) holds, we get positive solutions of the form $u = r^{-\gamma_1}\phi_1$, where $\phi_1 > 0$ is the eigenfunction of L_Σ corresponding to λ_1, and

$$\gamma_1 = \frac{n-3}{2} - \sqrt{\left(\frac{n-3}{2}\right)^2 + \lambda_1};$$

this notation will be used subsequently. We shall also use the notation $\beta = 2\sqrt{((n-3)/2)^2 + \lambda_1}$.

The main theorem of this section concerns solutions u of $\mathscr{M}u = 0$ which are defined over domains of the form

$$G = \{(r\omega, y)\colon r > \rho(y)/2, \omega \in \Sigma\} \subset C \times \mathbf{R}, \tag{2.5}$$

where ρ is a given positive Lipschitz function on $\mathbf{R}$ satisfying

$$\text{(2.6)}\qquad \text{(i) } \operatorname{Lip}\rho < \delta, \quad \text{and} \quad \text{(ii) } \operatorname{sgn}(y)\rho'(y) \geq 0 \text{ a.e. } y \in \mathbf{R},$$

where $\delta \in (0, 1]$ is to be specified later.

Concerning the solution u we assume always that, again for $\delta \in (0, 1]$ to be specified,

(2.7)

$$r^{-1}|u(r\omega, y)| + |\nabla u(r\omega, y)| + r|\nabla^2 u(r\omega, y)| \leq \begin{cases} \delta & \text{if } \beta > 0, \\ \delta(r/\rho(y))^{-\theta} & \text{if } \beta = 0, \end{cases}$$

where $(r\omega, y) \in G$, $\theta \in (0, 1)$, and

$$\text{(2.8)}\qquad \frac{\partial u}{\partial y}(r\omega, y) > 0 \quad \text{in } G.$$

Theorem 1. *Given any $\alpha, \varepsilon, \mu, \theta \in (0, 1)$, with $\alpha < \varepsilon < \beta$ in case $\beta > 0$ and $0 < \alpha < \varepsilon < 1$ in case $\beta = 0$, there is $\delta = \delta(\varepsilon, \mathscr{M}, \mu, \theta) > 0$ such that if $\mathscr{M}u = 0$, if (2.2), (2.4), (2.6), (2.7), and (2.8) hold, if $h(r\omega, y) = u(r\omega, y) - u(r\omega, -y)$, if $\rho_*(y) = \max\{\rho(y), \rho(-y)\}$, and if there is $y_1 = y_1(\delta) \geq 1$ so that*

$$(*)\qquad \begin{cases} h(\delta^{-1}\rho_*(y)\omega, y) \geq \begin{cases} \delta^{\gamma_1+\alpha} h(\rho_*(y)\omega, y) & \text{for } \beta > 0, \\ (\log\delta^{-1})^{\alpha}\delta^{\gamma_1} h(\rho_*(y)\omega, y) & \text{for } \beta = 0, \end{cases} \\ \inf_{z \geq y} h(\rho_*(z)\omega, z) \geq \mu h(\rho_*(y)\omega, y) \end{cases}$$

for all $y \geq y_1$ and all $\omega \in \Sigma$, then

$$(**)\qquad |y|^{\varepsilon} r^{\gamma_1} v(r\omega, y) \geq c \quad \text{for all } y \in \mathbf{R},\ \omega \in \Sigma,\ y \geq r \geq \rho(y),$$

where $v(x, y) = \dfrac{\partial u}{\partial y}(x, y)$ and where $c > 0$ is independent of y and r.

Furthermore δ can be chosen so that there is a sequence $y_j \to \infty$ with

$$\lim_{j\to\infty} |y_j|^{-\varepsilon} \rho_*(y_j)^{\gamma_1} v(\rho_*(y_j)\omega, y_j) = 0,$$
$$\lim_{j\to\infty} |y_j|^{-\varepsilon-1} \rho_*(y_j)^{\gamma_1} h(\rho_*(y_j)\omega, y_j) = 0.$$

(In particular the lower bound in $(**)$ is best possible up to a factor $|y|^{2\varepsilon}$ in case $r = \rho_*(y)$, $y > y_1$.)

(2.9) Remarks. (1) If $\rho \equiv \text{const}$, then the theorem gives

$$v(\rho\omega, y) \geq c|y|^{-\varepsilon},$$

so that for $\varepsilon < 1$

$$u(\rho\omega, y) \geq c|y|^{1-\varepsilon}$$

for all $y \geq y_1$. (In particular, if $\varepsilon < 1$ and $\rho = \text{constant}$, then u cannot be bounded.)

(2) Note that if C is the minimizing cone considered in §1, and $\mathscr{M}$ is the minimal surface operator as in Remark (2.3) above, then $\lambda_1 \equiv -(n-2)$ and hence γ_1 is as in §1 (and $\beta > 0$). One should keep this in mind when checking that hypothesis $(*)$ of the above theorem is satisfied in the application already discussed in §1. In later applications of the above theorem to the minimal surface equation (discussed in §5 and §6) condition $(*)$ causes us no problem, provided the "tangent cylinder at ∞" of the graph of the solution is $C \times \mathbf{R}$ with $\operatorname{sing} C = \{0\}$, and with C strictly minimizing and strictly stable. (See the discussion in §5.)

(3) Unfortunately it is not possible to do *without* hypothesis $(*)$ of the theorem. For example if $n = 8$, C is the cone of §1 with $p = 4$ (i.e., $C = \{x = (x_1, \cdots, x_8)\colon \sum_{i=1}^4 (x_i)^2 = \sum_{i=5}^8 (x_i)^2\}$), $\rho \equiv 1$, and $\mathscr{M}u = \Delta u + 6u/r^2$ (this is just the linearization of the minimal surface operator on $C \times \mathbf{R}$; note $\phi_1 \equiv 1$, $\lambda_1 = -6$, $\gamma_1 = 2$ in this case), then we have solutions

$$u = \alpha r^{-3} \arctan(y/r),$$

$\alpha > 0$ being an arbitrary constant, so that $v = \alpha r^{-2}/(r^2 + y^2)$, hence $v(\rho(y), y) = \alpha/(1 + y^2)$, thus violating the conclusion of the theorem. It is worth noting that in this case we can write down a large family of solutions of $\mathscr{M}u = 0$; in fact if ψ is any bounded (or sufficiently slow growth) measurable function on $\mathbf{R}$, then

$$u = r^{-2} \int_{-\infty}^{\infty} \frac{\psi(\zeta)}{r^2 + (y - \zeta)^2}\, d\zeta \tag{2.10}$$

is a solution of $\mathscr{M}u = 0$. This general formula comes as an application of the Poisson integral formula for the Laplacian operator in the half space; see the discussion in §3 below. One can write down a similar general Poisson formula (with kernel depending on β) in the general case.

For the proof of Theorem 1 we shall need two technical lemmas concerning solutions of linear equations on subdomains of $C \times \mathbf{R}$.

We in fact need to look at solutions w of linear equations of the form

$$Lw = \nabla_i(a_{ij}(x,y)\nabla_j w) + r^{-1}a_i(x,y)\nabla_i w + r^{-2}a(x,y)w \tag{2.11}$$

on open subsets Ω of G, where $\nabla_i = e_i \cdot \nabla$ and

$$r|\nabla_i a_{ij}(x,y)| + |a_{ij}(x,y)| + |a_i(x,y)| + |a(x,y)| \le \delta \tag{2.12}$$

($\delta \in (0,1]$ a parameter to be specified) for all $(x,y) = (r\omega, y) \in \Omega$.

(2.13) Remark. Notice of course that u itself and also the functions v and h of Theorem 1 satisfy an equation of the form (2.11) over G, and that for these choices (2.7) implies (2.12) with $c\delta$ in place of δ, where c

depends only on $\mathscr{M}$. Before beginning the proof of Theorem 1 we need to record the following lemmas; the proofs will be given in §3.

Lemma 1. *Suppose $\varepsilon > 0$. There are constant $\delta = \delta(\varepsilon) \in (0, 1/2)$ and $b = (\varepsilon, L) \geq 2$ such that if (2.6)(i) holds, $y_0 \in \mathbf{R}$, (2.11) and (2.12) hold on the region $\Omega = G \cap \{(r\omega, y) : r < R, |y - y_0| < R\}$ for some $R \geq 2b\delta^{-1}\rho(y_0)$, and $w > 0$ in this region, then*

$$w(r\omega, y_0) \leq b^{-\gamma_1+\varepsilon} w(b^{-1}r\omega, y_0), \quad \forall r \in [b\delta^{-1}\rho(y_0), R/2], \ \omega \in \Sigma.$$

For the second lemma we need a slight strengthening of (2.12) for $\beta = 0$ (in line with condition (2.7)); we replace (2.12) by

$$(2.12)' \qquad r|\nabla_i a_{ij}| + |a_{ij}| + |a_i| + |a| \leq \begin{cases} \delta & \text{in case } \beta > 0, \\ \delta(r/\rho(y))^{-\theta} & \text{in case } \beta = 0 \end{cases}$$

on the domain Ω.

Lemma 2. *Suppose $M, K \geq 4$, $\alpha, \varepsilon > 0$ are given with $\alpha < \varepsilon < \beta$ if $\beta > 0$ and $\alpha < \varepsilon < 1$ if $\beta = 0$. There are $\delta = \delta(\varepsilon, \alpha, K, M, \mu, \theta, \beta) > 0$ and $b = b(\varepsilon, L) \geq 2$ such that if (2.6) holds, $\rho_*(y) = \max\{\rho(y), \rho(-y)\}$, z satisfies $z \geq b\delta^{-1}\rho_*(z)$, and (2.11) and (2.12)′ hold in the domain $\Omega = \{(x, y) : \rho_*(y)/2 < r < Kz, |y| < Kz\}$ together with the additional conditions*

$$\text{(i)} \qquad \frac{\partial w}{\partial y} > 0, \qquad w(x, -y) = -w(x, y) \quad \text{on } \Omega,$$

$$\text{(ii)} \qquad w(\delta^{-1}\rho_*(z)\omega, z) \geq \begin{cases} \delta^{\gamma_1+\alpha} w(\rho_*(z)\omega, z) & \text{if } \beta > 0, \\ (\log \delta^{-1})^{\alpha} \delta^{\gamma_1} w(\rho_*(z)\omega, z) & \text{if } \beta = 0 \end{cases}$$

for each $\omega \in \Sigma$;

$$\text{(iii)} \qquad \rho_*(y)^{\gamma_1} w(\rho_*(y)\omega, y) \leq M\rho_*(z)^{\gamma_1} w(\rho_*(z)\omega, z)$$

for each $y \in [z, Kz]$ and $\omega \in \Sigma$, and

$$\text{(iv)} \qquad \rho_*(Kz) \leq M\rho_*(z),$$

then

$$w(r\omega, z) \geq b^{-\gamma_1-\varepsilon} w(b^{-1}r\omega, z) \quad \forall r \in [b\delta^{-1}\rho_*(z), Kz/2], \ \omega \in \Sigma.$$

(2.14) Remark. Note that by iterating the inequalities of Lemmas 1 and 2 and using Harnack's inequality (see (2.15) below) we get the inequalities

$$w(r_2\omega, y_0) \leq c\left(\frac{r_2}{r_1}\right)^{-\gamma_1+\varepsilon} w(r_1\omega, y_0), \qquad \delta^{-1}\rho(y_0) < r_1 < r_2 < R, \ \omega \in \Sigma,$$

$$w(r_2\omega, z) \geq c\left(\frac{r_2}{r_1}\right)^{-\gamma_1-\varepsilon} w(r_1\omega, z), \qquad \delta^{-1}\rho_*(z) < r_1 < r_2 < Kz/2, \ \omega \in \Sigma,$$

respectively; by applying the lemmas with $\varepsilon/2$ in place of ε and modifying the choice of δ accordingly, we can always arrange to get $c = 1$ in case $r_2 \geq br_1$ for suitable $b = b(\varepsilon) \geq 2$ in the above inequalities.

(2.15) Remark. For later reference we here make some remarks about estimates for solutions of equations of the form (2.11) and (2.12). Specifically if (2.6), (2.11) and (2.12) hold with $\delta \leq 1$ small enough to ensure uniform ellipticity of the equation, then for any $\sigma > 0$, by scaling $(x, y) \mapsto \sigma^{-1}(x, y)$ and applying standard Harnack theory (e.g. [12, Chapter 8]), we have

$$\sup_{B_\sigma(x_0,y_0)} w \leq c \inf_{B_\sigma(x_0,y_0)} w, \qquad c = c(L),$$

for any positive solution of (2.11) on $B_{2\sigma}(x_0, y_0)$, provided $B_{2\sigma}(x_0, y_0) \subset \overline{G}$. In particular, by connecting any point (x_0, y_0) with a point (x, y_0) via a sequence of balls $B_{\sigma_j}(x_j, y_0) \subset G$ with $\sigma_0 = \sigma$, $\sigma_{j+1} \geq c\sigma_j$ ($c > 1$ fixed) and $x_j \in B_{\sigma_{j-1}}(x_{j-1}, y_0) \cap l$, l the line segment joining x_0 to x, we deduce that, if $|x_0| \leq |x|$, then

$$w(x, y_0) \leq c \left(\frac{|x|}{|x_0|}\right)^Q w(x_0, y_0), \qquad Q = Q(L),$$

provided w is a positive solution of (2.11), (2.12) on all of G. Thus given any pair of points $(x_0, y_0), (x, y) \in G$ with $\rho(y_0) < |x_0|, \rho(y) < |x|$, and $|x_0| < |x|$ we have

$$w(x, y) \leq c \left(\frac{|x| + |x_0| + |y - y_0|}{\min\{|x_0|, |x|\}}\right)^Q w(x_0, y_0)$$

for suitable $Q = Q(L) \geq 1$, provided again that w is positive on all of G.

Proof of Theorem 1. Take any $\eta > 0$ with $\alpha + \eta < \beta$ in case $\beta > 0$ and $\alpha + \eta < 1$ in case $\beta = 0$, and $P, K \geq 2$ also arbitrary for the moment, and let $\delta_1 = \delta_1(\varepsilon, \alpha, \eta, \theta, K, P, L, \beta, \mu)$ be the smaller of the constants δ of Lemmas 1 and 2, in case $\varepsilon = \eta$ in Lemma 1 and in case $M = K^P, \varepsilon = \alpha + \eta$ in Lemma 2. Also let $\delta_2 > 0$ be small enough to ensure the inequalities of Remark (2.15) in case (2.12) holds with δ_2 in place of δ. For the remainder of the proof, take $\delta = \min\{\delta_1, \delta_2\}$, so that $\delta = \delta(\varepsilon, \alpha, \eta, \theta, K, P, L, \beta, \mu)$. Assume $y_1 = y_1(\delta)$ is such that $(*)$ holds with this choice of δ. Define

$$\sigma(y) \equiv \tilde{h}(r, y) = \int_\Sigma h(r\omega, y)\phi_1(\omega)\, d\omega, \qquad \tilde{v}(r, y) = \int_\Sigma v(r\omega, y)\phi_1(\omega)\, d\omega$$

and define sequences $\{y_j, \sigma_j, \tau_j\}$ inductively as follows: y_1 as above and for $j = 1, 2, \cdots$ define

$$\sigma(y) = \rho_*(y)^{\gamma_1}\tilde{h}(\rho_*(y), y), \quad \sigma_j = \sigma(y_j), \quad \tau_j = \rho_*(y_j),$$

a $y_{j+1} = Ky_j$ if $\sigma(Ky_j) < K^P\sigma(y_j)$ and $\rho_*(Ky_j) < K^P\rho_*(y_j)$, and $y_{j+1} = \inf\{y > y_j : \sigma(y) \geq K^P\sigma_j$ or $\rho_*(y) \geq K^P\tau_j\}$ otherwise. Thus either

(i) $\quad y_{j+1} = Ky_j \quad$ and both $\quad \sigma(Ky_j) < K^P\sigma(y_j),\ \rho_*(Ky_j) < K^P\rho_*(y_j)$,

or

(ii) $\quad y_{j+1} < Ky_j \quad$ and either $\quad \sigma_{j+1} = K^P\sigma_j \quad$ or $\quad \tau_{j+1} = K^P\tau_j$.

Now by Remark (2.15), we have $h(y_{j+1}\omega, y_j) \geq K^{-Q}h(y_j\omega, y_j)$. Then in case of alternative (ii) we have

$$\tilde{h}(y_{j+1}, y_j) \geq \left(\frac{\sigma_{j+1}}{\sigma_j}\right)^{-\eta}\left(\frac{\tau_{j+1}}{\tau_j}\right)^{-\eta}\tilde{h}(y_j, y_j) \tag{1}$$

for any $\eta > 0$, provided $P = P(\eta, L, \mu)$ is sufficiently large, which we subsequently assume; notice that here we used $\sigma_{j+1} \geq \mu\sigma_j$ from hypothesis $(*)$. In case of alternative (i) we can use Lemma 2 (and in particular the remark following it) with $z = y_j$ and $\varepsilon' = \alpha + \eta$ in place of ε to give

$$\tilde{h}(y_{j+1}, y_j) \geq \left(\frac{y_{j+1}}{y_j}\right)^{-\gamma}\tilde{h}(y_j, y_j), \tag{1$'$}$$

where $\gamma = \gamma_1 + \varepsilon'$. So in any case, regardless of which alternative holds, if $K = K(\eta, L)$ is sufficiently large we get

$$\tilde{h}(y_{j+1}, y_j) \geq \left(\frac{\tilde{\sigma}_{j+1}}{\tilde{\sigma}_j}\right)^{-\eta}\left(\frac{y_{j+1}}{y_j}\right)^{-\gamma}\tilde{h}(y_j, y_j), \tag{2}$$

where $\tilde{\sigma}_j = \sigma_j\tau_j$. On the other hand by Remark (2.15) we have

$$c^{-1}\tilde{v}(r,0) \leq \frac{\tilde{h}(r, y_j)}{y_j} \leq c\tilde{v}(r,0) \quad \text{for } r \geq y_j,\ c = c(L);$$

hence, choosing $K = K(\eta, L)$ sufficiently large, we see that (2) implies

$$\tilde{v}(y_{j+1}, 0) \geq \left(\frac{\tilde{\sigma}_{j+1}}{\tilde{\sigma}_j}\right)^{-\eta'}\left(\frac{y_{j+1}}{y_j}\right)^{-\gamma'}\tilde{v}(y_j, 0),$$

where $\gamma' = \gamma_1 + \alpha + 2\eta$, $\eta' = 2\eta$. Iterating (2) for $j = 1, 2, \cdots$, we conclude

$$\tilde{v}(y_j, 0) \geq \left(\frac{\tilde{\sigma}_j}{\tilde{\sigma}_1}\right)^{-\eta'}\left(\frac{y_j}{y_1}\right)^{-\gamma'}\tilde{v}(y_1, 0) \quad \forall j \geq 1.$$

By hypothesis $(*)$ and Remark (2.15), we then conclude that for each $r \geq y_1$

$$\tilde{v}(r,0) \geq c\left(\frac{\rho_*(r)^{\gamma_1+1}\tilde{h}(\rho_*(r), r)}{\rho_*(y_1)^{\gamma_1+1}\tilde{h}(\rho_*(y_1), y_1)}\right)^{-\eta'}\left(\frac{r}{y_1}\right)^{-\gamma'}\tilde{v}(y_1, 0), \tag{3}$$

where $c > 0$ is independent of r.

Now on the other hand by Remark (2.15) and Lemma 1 we have

$$\tilde{v}(r,0) \le cv(r,r) \le c(r/s)^{-\gamma_1+\eta}\tilde{v}(s,r), \qquad d\rho(r) \le s \le r,$$

where $d = d(\eta)$. Applying Harnack again this gives

$$\tilde{v}(r,0) \le c(r/s)^{-\gamma_1+\eta}\tilde{v}(s,r), \qquad c = c(\eta),\ \rho(r) \le s \le r.$$

Similarly

$$\tilde{v}(r,0) \le c(r/s)^{-\gamma_1+\eta}\tilde{v}(s,-r), \qquad \rho(-r) \le s \le r.$$

In view of the arbitrariness of η, and the fact that $\rho_*(y) \le \rho(0) + \delta|y|$ by (2.6) and $|\tilde{h}(\rho_*(y),y)| \le 2\delta|y|$ by (2.7), the proof is now completed by combining inequality (3) and the last pair of inequalities. (We choose $\eta = (\varepsilon - \alpha)/4$.)

For upper bounds, we let $z_k \to \infty$ be a sequence satisfying

$$\text{(4)} \qquad \sup_{y\in[z_k,Kz_k]} \rho_*(y)^{\gamma_1}\tilde{h}(\rho_*(y),y) \le K^P\rho_*(z_k)^{\gamma_1}\tilde{h}(\rho_*(z_k),z_k),$$
$$\rho_*(Kz_k) \le K^P\rho_*(z_k).$$

Notice that there must be such a sequence for K sufficiently large, otherwise we would deduce that $\rho_*(y)^{\gamma_1+1}\tilde{h}(\rho_*(y),y) \ge cy^{P/2}$ for all sufficiently large y, thus contradicting the fact that $\tilde{h}(\rho_*(y),y) \le cy^Q$ by Remark (2.15).

Then for sufficiently large k, Lemma 2 applies to give

$$\tilde{h}(z_k,2z_k) \ge \left(\frac{z_k}{\rho_*(2z_k)}\right)^{-\gamma}\tilde{h}(\rho_*(2z_k),2z_k),$$

where $\gamma = \gamma_1 + \alpha + \eta$, while on the other hand (2.15) and Lemma 1 tell us that for any $\eta > 0$ and sufficiently large k

$$\frac{\tilde{h}(z_k,2z_k)}{z_k} \le c\tilde{v}(z_k,0) \le cz_k^{-\gamma_1+\eta}$$

with c independent of k.

Hence

$$\text{(5)} \qquad \rho_*(2z_k)^{\gamma_1}\tilde{h}(\rho_*(2z_k),2z_k) \le cz_k^{\varepsilon'+1},$$

where $\varepsilon' = \alpha + 2\eta$. Thus, writing $y = z_k$ we have

$$y^{-1}(\tilde{h}(\rho_*(2y),2y) - \tilde{h}(\rho_*(2y),y)) \le cy^{\varepsilon'}\rho_*(2y)^{-\gamma_1}$$

and hence by the mean value theorem

$$\tilde{v}(\rho_*(2y),\theta y) \le cy^{\varepsilon'}\rho_*(2y)^{-\gamma_1}$$

for some $\theta \in (1,2)$. Since $\rho_*(2y) \le K^P\rho_*(y) \le K^P\rho_*(\theta y)$ we get, by Remark (2.15),

$$\tilde{v}(\rho_*(y'),y') \le c(y')^{\varepsilon'}\rho_*(y')^{-\gamma_1},$$

where $y' = \theta y$.

Finally by combining (4) and (5) and using $\rho_*(y)^{\gamma_1}\tilde{h}(\rho_*(y),y) \le c\rho_*(2y)^{\gamma_1}\tilde{h}(\rho_*(2y),2y)$ (by (2.15) and (4)) we deduce that

$$\rho_*(y')^{\gamma_1}\tilde{h}(\rho_*(y'),y') \le c(y')^{\alpha+2\eta+1}.$$

In view of the arbitrariness of η, the final limit statements of the theorem are now proved.

3. Proof of Lemmas 1 and 2

Before beginning the proofs, we need some preliminary observations. First note that in terms of the coordinates $(r,\omega,y) \in (0,\infty)\times\Sigma\times\mathbf{R}$ we can express the operator L as

$$Lu = r^{2-n}\frac{\partial}{\partial r}\left(r^{n-2}\frac{\partial u}{\partial r}\right) + \frac{\partial^2 u}{\partial y^2} + r^{-2}L_\Sigma u,$$

where L_Σ is as in §2. Then letting

$$\tilde{u}(r,y) = \int_\Sigma u(r\omega,y)\phi_1(\omega)\,d\omega, \qquad \tilde{f}(r,y)\int_\Sigma f(r\omega,y)\phi_1\,d\omega,$$

where $\phi_1 > 0$ is the first eigenfunction of L_Σ, we see that if $Lu = f$ then

$$r^{2-n}\frac{\partial}{\partial r}\left(r^{n-2}\frac{\partial \tilde{u}}{\partial r}\right) + \frac{\partial^2 \tilde{u}}{\partial y^2} - r^{-2}\lambda_1\tilde{u} = \tilde{f},$$

so that if $w(r,y) = r^\gamma\tilde{u}$, then after a straightforward computation we obtain

$$(3.1)\quad r^{-(n-2)+2\gamma}\frac{\partial}{\partial r}\left(r^{n-2-2\gamma}\frac{\partial w}{\partial r}\right) + \frac{\partial^2 w}{\partial y^2} + r^{-2}(\gamma^2-(n-3)\gamma-\lambda_1)w = r^\gamma\tilde{f}.$$

In particular, if $\gamma = \gamma_1$ as in §2, and β is also as in §2, then

$$(3.2)\qquad r^{-1-\beta}\frac{\partial}{\partial r}\left(r^{1+\beta}\frac{\partial w}{\partial r}\right) + \frac{\partial^2 w}{\partial y^2} = r^{\gamma_1}\tilde{f}.$$

In view of these facts it is not surprising that solutions of the equation

$$(3.3)\qquad r^{-1-\beta}\frac{\partial}{\partial r}\left(r^{1+\beta}\frac{\partial w}{\partial r}\right) + \frac{\partial^2 w}{\partial y^2} = 0,$$

where $\beta \ge 0$ is a given constant, will play an important role in the proofs of Lemmas 1 and 2. Note that $w \equiv 1$ and $w \equiv r^{-\beta}$ are solutions of this equation. Concerning solutions of (3.3) we need the following lemmas:

(3.4) Lemma. (1) *If w is a positive C^2 solution of* (3.3) *on $0 < r < 2, |y| < 1$, then*

$$w(t,0) \leq cw(s,0), \qquad 0 < s < t \leq 1,$$

where c depends only on β. In case $\int_{0<r<1,|y|<1} |Dw|^2 r^{1+2\beta}\,dr\,dy < \infty$ the conclusion holds for all t, s, $0 < t, s \leq 1$.

(2) *If $\dfrac{\partial w}{\partial y} > 0$, $w(r,y) = -w(r,-y)$, $0 < r < 2$, $|y| < 2$, and in addition $\int_{0<r<1,|y|<1} |Dw|^2 r^{1+2\beta} < \infty$, then*

$$\frac{w(t,y)}{y} \leq c\frac{w(s,z)}{z}, \qquad 0 < t, s \leq 1,\ 0 < |y|, |z| \leq 1.$$

Proof. First note that if w is as in (2) then $\delta^{-1}(w(r, y+\delta) - w(r,y))$ is a positive solution on $0 < r < 2$, $|y| < 2 - \delta$ for $0 < \delta \leq 1$, and so after a change of scale we can apply the first part of the lemma. One readily checks that this leads to the required inequality. Thus it remains to prove (1). For this we need the monotonicity result of the following lemma.

(3.5) Lemma. *Let P denote the operator on the left of* (3.3). *Then $\int_{D_1(0)} |Dw|^2 r^{1+2\beta}\,dr\,dy < \infty$ and $Pw \geq 0$ (≤ 0) in $D_1(0)$ imply that $\rho^{-3-\beta}\int_{D_\rho(\zeta)} w r^{1+\beta}$ is an increasing (resp. decreasing) function of ρ, $0 < \rho < 1 - |\zeta|$. Here $D_\rho(\zeta) = \{(r,y) : r > 0, (r-\xi)^2 + (y-\eta)^2 < \rho^2\}$, $\zeta = (\xi, \eta)$ with $\xi \leq 0, \geq 0$ according as $Pw \geq 0, \leq 0$ respectively.*

Proof. First note that

$$Pw \geq 0\ (\leq 0) \Leftrightarrow -\int \left(\frac{\partial w}{\partial r}\frac{\partial v}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial v}{\partial y}\right) r^{1+\beta}\,dr\,dy \geq 0\ (\leq 0)$$

for each nonnegative C^2 function v on $0 < r < 1$, $|y| < 1$, with v vanishing near $|y| = 1$ and near $r = 0, 1$. Using a sequence of v approximating the characteristic function of $D_\rho(\zeta)$ in the appropriate sense, we get

$$\int_\Gamma \frac{\partial w}{\partial \eta} r^{1+\beta} \geq 0 \ (\leq 0 \text{ resp.}),$$

where $\partial w/\partial \eta$ denotes differentiation in the outward unit normal direction of Γ, $\Gamma = \partial D_\rho \cap \{r > 0\}$. Notice that there is no boundary term over $\{r = 0\}$ because of the assumption $\int_{D_1(0)} |Dw|^2 r^{1+2\beta} < \infty$.

Now using polar coordinates ρ, θ with $r = \xi + \rho\cos\theta$, $y = \eta + \rho\sin\theta$, we see that this last inequality can be written as

$$\int_{-\omega(\rho)}^{\omega(\rho)} \frac{\partial w}{\partial \rho}(\xi + \rho\cos\theta)^{\beta+1} \geq 0\ (\leq 0 \text{ resp.}), \qquad \omega(\rho) = \cos^{-1}(-\xi/\rho) \in [0, \pi],$$

which implies (since $\operatorname{sign}\omega'(\rho) = -\operatorname{sign}\xi$)

$$\frac{\partial}{\partial\rho}\int_{-\omega(\rho)}^{\omega(\rho)} w(\rho\varepsilon^{i\theta})(\xi+\rho\cos\theta)^{1+\beta}\,d\theta - (1+\beta)\rho^{-1} \times \int_{-\omega(\rho)}^{\omega(\rho)} w(\rho e^{i\theta})(\xi+\rho\cos\theta)^{1+\beta}\,d\theta \geq 0 \ (\leq 0 \text{ resp.}),$$

which in turn can be written

$$\frac{\partial}{\partial\rho}\left(\rho^{-2-\beta}\int_\Gamma w(r,y)r^{1+\beta}\right) \geq 0 \ (\leq 0 \text{ resp.}).$$

Thus $\rho^{-2-\beta}\int_\Gamma w(r,y)r^{1+\beta}$ is increasing (respectively decreasing) as a function of ρ on $(0, 1-|\zeta|)$, and the required monotonicity follows from this by integration.

Proof of Lemma (3.4)(1). By the usual Harnack theory for uniformly elliptic equations in $\mathbf{R}^2$ there is a constant $\kappa \geq 1$ such that for any fixed $s \in (0,1), |y| < 1/2$

$$(*) \qquad \kappa w(s,\zeta) \geq w(r,y) \geq \kappa^{-1}w(s,\zeta), \qquad |y-\zeta| < s/3,\ |r-s| < s/2.$$

Let $w_K = \min\{w, Kw(s,0)\}$, $K \geq 1$. Since $\min\{t,K\}$ is a concave increasing function of t, we have that w_K satisfies $Pw_K \leq 0$ in the appropriate weak sense, and $\int_{D_\rho(0)}|Dw_K|^2r^{1+\beta} < \infty$ for $\rho < 1$, so by Lemma (3.5)

$$\text{(i)} \qquad w(s,0) \geq s^{-3-\beta}\int_{D_s(s,0)} w_K r^{1+\beta}\,dr\,dy \geq t^{-3-\beta}\int_{D_t(s,0)} w_K r^{1+\beta}$$

for $s < t < 1$. Now substituting t in place of s everywhere in $(*)$, we have

$$t^{-3-\beta}\int_{D_t(s,0)} w_K r^{1+\beta}\,dr\,dy \geq cw(t,0)$$

for sufficiently large K, which by (i) gives the required result. This completes the proof of the first part of Lemma (3.4)(1).

To prove the second inequality of Lemma (3.4)(1), we first note, by $(*)$ above, for $s \in (0, 1/2), |z| < 1/2$,

$$\text{(ii)} \qquad w(s,z) \leq cs^{-3-\beta}\int_{D_s(0,z)} wr^{1+\beta}\,dr\,dy \leq ct^{-3-\beta}\int_{D_t(0)} wr^{1+\beta}\,dr\,dy$$

(by (3.5)) for $0 < t < 1$ (and in particular $\sup_{D_{1/2}(0)} w < \infty$). On the other hand, for any $\theta \in (0,1), t \in (0,1)$,

$$\text{(iii)} \qquad \begin{aligned} t^{-3-\beta}\int_{D_t(0)} wr^{1+\beta}\,dr\,dy &\leq c\theta\sup_{D_t(0)} w + t^{-3-\beta}\int_{D_t\cap\{r>\theta t\}} wr^{1+\beta}\,dr\,dy \\ &\leq c\theta\sup_{D_t(0)} w + c'w(t,0) \quad \text{by } (*), \end{aligned}$$

where c' depends only on θ and β. The proof now follows directly from (ii), (iii), and $(*)$.

Proof of Lemma 1. Evidently it is enough to prove the lemma with $y_0 = 0$ and $\rho(0) = 1$. If the lemma is false for a given b, ε, by (2.15) we get solutions w_k of (2.11), (2.12), with $\delta = k^{-1}$ and R_k in place of R, such that $w_k(r_k\omega, 0) > cb^{-\gamma_1+\varepsilon}w_k(b^{-1}r_k\omega, 0)$ for some r_k satisfying $bk < r_k < R_k/2$ with $c = c(L)$. Then let $\tilde{w}_k(r,y) = \int_\Sigma w_k(r\omega, y)\phi_1\,d\omega$ and

$$\psi_k(s,\eta) = s^{\gamma_1}\frac{\tilde{w}_k(r_ks, r_k\eta)}{\tilde{w}_k(r_k, 0)}.$$

By the Harnack inequality and the related continuity estimates a subsequence (henceforth denoted simply ψ_k) converges locally uniformly to a positive solution ψ of

$$s^{-1-\beta}\frac{\partial}{\partial s}\left(s^{1+\beta}\frac{\partial\psi}{\partial s}\right) + \frac{\partial^2\psi}{\partial\eta^2} = 0, \qquad 0 < s < 2,\ |\eta| < 2,$$

satisfying

$$\psi(1,0) \ge cb^{\varepsilon}\psi(b^{-1}, 0).$$

However this contradicts the first part of Lemma (3.4)(1) for suitably chosen $b = b(\varepsilon, L)$.

Proof of Lemma 2. We first consider the case $\beta > 0$. Let $\alpha, \varepsilon, K, M$ be given to satisfy the conditions stated. Suppose there are $b \ge 4$ and solutions w_k of (2.11), (2.12)$'$ such that hypotheses (i)–(iii) of Lemma 2 hold with $\delta = k^{-1}$ and

$$\Omega = \Omega_k = \{(r\omega, y)\colon \rho_*(y)/2 < r < Kz_k, |y| < Kz_k\},$$

where $bk\rho_*(z_k) \le z_k$, and ρ_* depending on k satisfies (2.6) with $\delta = k^{-1}$. Further, let

$$\begin{aligned}
&U_k = \{(r,y)\colon \rho_*(y)/2 < r < Kz_k, |y| < Kz_k\} \subset \mathbf{R}^2,\\
&\tilde{w}_k(r,y) = r^{\gamma}\int_\Sigma w_k(r\omega, y)\phi_1\,d\omega, \qquad (r,y) \in U_k,\ \gamma = \gamma_1 + \varepsilon,\\
&\hat{w}_k = (\tilde{w}_k - \varepsilon_k)_+,
\end{aligned}$$

where $\varepsilon_k = M^{1+\varepsilon}\tilde{w}_k(\rho_*(z_k), z_k)$. Notice that then $\hat{w}_k = 0$ in a neighborhood of the segment $\{r = \rho_*(y), |y| < Kz_k\}$ by hypotheses (iii), (iv). Also, let $\{y_k\}$ be a given sequence in $[-Kz_k/2, Kz_k/2]$ and for $R > 0, z \in \mathbf{R}$

$$D_R(z) = \{(r,y)\colon |y - z| < R, \rho_*(y) < r < R\},$$

and abbreviate $D_{R,k} = D_R(y_k)$.

Let $y_k \in [-Kz_k/2, Kz_k/2]$, $0 < R_k \le z_k$. First note that by (3.1) and Remark (2.15) $\tilde{w}_k$ satisfies on $D_{R_k,k}$ an equation of the form

$$(1)\qquad -\mu r^{-2}\tilde{w}_k + r^{-1-\beta'}\frac{\partial}{\partial r}\left(r^{1+\beta'}\frac{\partial \tilde{w}_k}{\partial r}\right) + \frac{\partial^2 \tilde{w}_k}{\partial y^2} = r^{-2}a_k\tilde{w}_k,$$

where $\mu = -(\gamma^2 - (n-3)\gamma - \lambda_1) \equiv \varepsilon(\beta - \varepsilon) > 0$, where $\beta' = \beta - 2\varepsilon$, and where $|a_k| \le ck^{-1}$. Then we take a C^1 function ζ with $\zeta \equiv 1$ on $D_{R_k/2,k}$, $\zeta \equiv 0$ outside $D_{R_k,k}$, $0 \le \zeta \le 1$, and $|D\zeta| \le cR_k^{-1}$, and substitute into the weak version of the equation. This gives

$$(2)\qquad \int_{D_{R_k/2,k}} (\mu r^{-1+\beta'}\hat{w}_k^2 + r^{1+\beta'}|D\hat{w}_k|^2) \le c\int_{D_{R_k,k}} \tilde{w}_k^2 r^{1+\beta'}|D\zeta|^2\,dr\,dy.$$

Replacing ζ by ζ^Q and using Young's inequality, this leads directly to

$$\begin{aligned}\int_{D_{R_k/2,k}} & r^{-1+\beta'}(\hat{w}_k^2 + r^2|D\hat{w}_k|^2)\\ &\le c\int_{D_{R_k,k}} r^{-1+\beta'+2Q}\tilde{w}_k^2|D\zeta|^{2Q} + c\varepsilon_k^2 R_k^{1+\beta'}\end{aligned}$$

with $c = c(Q)$ and hence, using Remark (2.15) again, we conclude

$$(3)\qquad \int_{D_{R_k/2,k}} r^{-1+\beta'}(\hat{w}_k^2 + r^2|D\hat{w}_k|^2) \le cR_k^{1+\beta'}(\varepsilon_k^2 + \tilde{w}_k^2(R_k, z_k)),\quad c = c(K).$$

Now let $r_k \in [bk\rho_*(z_k), Kz_k/2]$ be the first value of r such that strict inequality in $\tilde{w}_k(r, z_k) \ge \tilde{w}_k(b^{-1}r, z_k)$ fails to hold. We assume such r_k exists and proceed to get a contradiction. Then by definition of r_k and by hypothesis (ii) of Lemma 2 we obtain

$$(4)\qquad \begin{aligned}\tilde{w}_k(r_k, z_k) &= \tilde{w}_k(b^{-1}r_k, z_k) \ge \tilde{w}_k(k\rho_*(z_k), z_k)\\ &\ge ck^{\varepsilon-\alpha}\tilde{w}_k(\rho_*(z_k), z_k) \equiv ck^{\varepsilon-\alpha}M^{-1-\varepsilon}\varepsilon_k.\end{aligned}$$

Hence if we use $R_k = 2r_k/K$ and $y_k \in [-Kz_k/2, Kz_k/2]$ in the above computations, we conclude from (2.15) and (4) that

$$(5)\qquad \tilde{w}_k(R_k, z_k) \le c(K)\tilde{w}_k(r_k, z_k)\quad\text{and}\quad \frac{\varepsilon_k}{\tilde{w}_k(R_k, z_k)} \to 0,$$

so that (2.15) and (3) give, for sufficiently large k,

$$(6)\qquad \int_{D_{r_k,k}} r^{-1+\beta'}(\hat{w}_k^2 + r^2|D\hat{w}_k|^2) \le c(M,K)\tilde{w}_k^2(r_k, z_k)r_k^{1+\beta'}.$$

Define $\psi_k(s,\eta) = \tilde{w}_k(r_k s, r_k\eta)/\tilde{w}_k(r_k, z_k)$, and note by (4) that $\psi_k(1, z_k/r_k) = \psi_k(b^{-1}, z_k/r_k)$; also note that then ψ_k is defined over

$\rho_*(r_k\eta)/(2r_k) < s < Kz_k/(2r_k)$, $|\eta| < Kz_k/r_k$, and, by (5),

$$(6)' \qquad \int_{|\eta - y_k/r_k|<1,\ \rho_*(r_k\eta)/r_k<s<1} (s^{-1+\beta'}\hat{\psi}_k^2 + s^{1+\beta'}|D\hat{\psi}_k|^2) \le c(M,K),$$

where $\hat{\psi}_k = (\psi_k - \varepsilon_k/\tilde{w}_k(r_k, z_k))_+$. Note that $\varepsilon_k/\tilde{w}_k(r_k, z_k) \to 0$ by (5) and (2.15).

Then define $\zeta = \liminf_{k\to\infty} z_k/r_k \in [2K^{-1}, \infty]$, and consider the cases:

$$\text{(a) } \zeta < \infty, \qquad \text{(b) } \zeta = \infty.$$

In case (a) we define

$$\tilde{\psi}_k(s,\eta) = \psi_k(s\zeta, \eta\zeta),$$

while in case (b) we define

$$\tilde{\psi}_k(s,\eta) = \psi_k(s, \eta + z_k/r_k).$$

In either case (by (6)′) we get a subsequence of $\tilde{\psi}_k$ converging to a solution ψ of

$$\frac{\mu}{s^2}\psi + s^{-1-\beta'}\frac{\partial}{\partial s}\left(s^{1+\beta'}\frac{\partial\psi}{\partial s}\right) + \frac{\partial^2\psi}{\partial\eta^2} = 0$$

with

$$(7) \qquad \left.\begin{aligned} \psi(s,-\eta) &= -\psi(s,\eta),\ \partial\psi/\partial\eta > 0 \\ \psi(b^{-1},1) &= \psi(1,1) \end{aligned}\right\} \quad \text{on } s < K/2,\ |\eta| < K$$

in case (a), and

$$(7)' \qquad \left.\begin{aligned} \psi &> 0 \\ \psi(b^{-1},0) &= \psi(1,0) \end{aligned}\right\} \quad \text{on } 0 < s < \infty,\ \eta \in \mathbf{R}$$

in case (b), and in either case, again by (6)′,

$$\int_{D_1(z)} |D\psi|^2 s^{1+\beta'}\, ds\, d\eta < \infty,$$

valid for all $z \in \mathbf{R}$ in case (b) and for all $|z| < K/2$ in case (a). But then by (3.1), $\tilde{\psi} = s^{-\varepsilon}\psi$ satisfies

$$s^{-1-\beta}\frac{\partial}{\partial s}\left(s^{1+\beta}\frac{\partial\tilde{\psi}}{\partial s}\right) + \frac{\partial^2\tilde{\psi}}{\partial\eta^2} = 0, \qquad \int_{D_1(z)} |D\tilde{\psi}|^2 s^{1+\beta} < \infty,$$

and (7) and (7)′ contradict the result of Lemma (3.4) for $b = b(\varepsilon, L)$ sufficiently large. This completes the proof in case $\beta > 0$.

To handle the case $\beta = 0$ we first note that the obvious modifications of the argument leading to (3) above (using $\beta' = 0$, $\gamma = \gamma_1$ in all arguments) establish only that

$$\int_{D_{R_k,k}} r|D\hat{w}_k|^2\zeta^2 \le cR_k(\varepsilon_k^2 + \tilde{w}_k^2(R_k, z_k)) + ck^{-1}\int_{D_{R_k,k}} \zeta^2 r^{-1}(r/\rho_k)^{-\theta}\tilde{w}_k\hat{w}_k, \tag{8}$$

where $c = c(M, K)$ and $\rho_k = \rho_*(Kz_k)$. Since $\partial w_k/\partial y > 0$ and $\tilde{w}_k < \varepsilon_k$ in a neighborhood of the boundary segment $r = \rho_*(y), -Kz_k < y < Kz_k$, we have

$$D_{R_k,k} \cap \{(r,y)\colon \hat{w}_k(r,y) > 0\} \subset D_{R_k,k} \cap \{(r,y)\colon r > \rho_k\}.$$

Using the fact that

$$\int r^{-1}\left(\frac{r}{\rho_k}\right)^{-\theta} f\,dr\,dy \equiv \frac{1}{\theta}\int \left(\frac{r}{\rho_k}\right)^{-\theta}\frac{\partial f}{\partial r}\,dr\,dy$$

for any f with compact support in the region $r > 0$, we thus conclude (using this with $f = (\hat{w}_k\zeta)^2$) that

$$\int_{D_{R_k,k}} r^{-1}\left(\frac{r}{\rho_k}\right)^{-\theta}\hat{w}_k^2\zeta^2 \le c\int_{D_{R_k,k}} r\left(\frac{r}{\rho_k}\right)^{-\theta}|D\hat{w}_k|^2\zeta^2 + c\int_{D_{R_k,k}} r\left(\frac{r}{\rho_k}\right)^{-\theta}\hat{w}_k^2|D\zeta|^2.$$

Then by (8) and Young's inequality we obtain

$$\int_{D_{R_k/2,k}} r|D\hat{w}_k|^2 \le cR_k(\varepsilon_k^2 + \tilde{w}_k^2(R_k, z_k)) \tag{9}$$

(cf. (3)). Now in particular this guarantees that, by selecting $y_k = z_k$,

$$\int_{\rho_k<r<R_k/4,|y-z_k|<R_k/4} r|D\hat{w}_k|^2 \le c(M,K)(\varepsilon_k^2 + \tilde{w}_k^2(R_k, z_k))R_k. \tag{10}$$

Suppose now that $R_k \ge k\rho(z_k)$ ($\ge K^{-P}k\rho_k$) and

$$\liminf_{k\to\infty}\frac{\tilde{w}_k(R_k, z_k)}{\varepsilon_k} = M_1 < \infty. \tag{11}$$

(We give a barrier argument to contradict (11).) First note that by (10) we can select ζ_k with $z_k < \zeta_k < z_k + R_k/4$ such that

$$\int_{\rho_k}^{R_k/4} r|D\hat{w}_k(r,\zeta_k)|\,dr \le c(\varepsilon_k + \tilde{w}_k(R_k, z_k))R_k, \qquad c = c(M,K). \tag{12}$$

Let Λ_k denote the region of $\mathbf{R}^2$ defined by

$$\Lambda_k = \{(r,y)\colon \rho_k < r < R_k/4, z_k - R_k/4 < y < \zeta_k\}.$$

By (2.15) and the definition of ε_k we would have $d = d(M, M_1, K) \geq 1$ such that

$$\text{(13)} \qquad \tilde{w}_k(r,y) \leq d\varepsilon_k \quad \text{on } r = \rho_k,\ R_k/4,\ z_k - R_k/4 < y < \zeta_k.$$

Keeping in mind (11), (12) and the fact that $\partial w_k/\partial y > 0$, it is then evidently possible to select ϕ_k on $0 < r < R_k/4, z_k - R_k/4 < y < \zeta_k$ such that $\min \phi_k = M\varepsilon_k; \phi_k \geq \tilde{w}_k$ on $0 < r < R_k/4$, $y = z_k - R_k/4$, ζ_k, $\phi_k = cM_1\varepsilon_k R_k \geq \tilde{w}_k$ on $\{r = R_k/4, z_k - R_k/4 < y < \zeta_k\}$; $\partial\phi_k(r,y)/\partial r \leq 0$ for $0 < r < R_k/4, y = \zeta_k, z_k - R_k/4$, and

$$\int_{\rho_k}^{R_k/4} r(\phi_k(r,\zeta_k) + \phi_k(r, z_k - R_k/4)) \leq c\varepsilon_k R_k^2, \qquad c = c(M,K).$$

Then since the operator

$$\text{(14)} \qquad Pu \equiv r^{-1}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{\partial^2 u}{\partial y^2}$$

can be interpreted as the Laplacian operator on $\mathbf{R}^3$ when applied to functions expressed in terms of the cylindrical coordinates (r,y), where $r = |x'|$, $y = x^3$ for $x = (x^1, x^2, x^3) \equiv (x', x^3) \in \mathbf{R}^3$, we can use a Green's function representation to obtain a solution v of the equation $Pv \equiv 0$ on $\{(r,y)\colon 0 < r < z_k, z_k - R_k/4 < y < \zeta_k\}$ such that the following hold: $v = v(r,y)$, $v \geq d\varepsilon_k$ everywhere, $(r\partial v/\partial r)(0_+, y) = 0$, and $v = \phi_k$ on $\{r > 0\} \cap \partial\{(r,y)\colon 0 < r < z_k, z_k - R_k/4 < y < \zeta_k\}$.

Notice that, since $r\partial v/\partial r$ satisfies the same equation as v, by the maximum principle we have $\partial v/\partial r \leq 0$. Then, by direct computation, for sufficiently large k the function $\tilde{v} \equiv (2 - (r/\rho_k)^{-\theta})v$ is a supersolution on Λ_k of the equation (having the form (1) with $a_k = -c\theta^2(r/\rho_k)^{-\theta}$). Then by the appropriate version of the maximum principle (see e.g., [20, p. 73]) we conclude that $\tilde{w}_k \leq \tilde{v}$ on Λ_k for all sufficiently large k, and in particular $\tilde{w}_k(r, z_k) \leq c\varepsilon_k$ for $\rho_k < r < R_k$, with c independent of k, thus contradicting the hypothesis (ii) of Lemma 2 (with $\delta = k^{-1}$). Hence we have established (contrary to (11)) that $\varepsilon_k/\tilde{w}_k(R_k, z_k) \to 0$. The remainder of the proof is now as for the case $\beta > 0$.

Note. The above proof (for the case $\beta = 0$) would be valid if in hypothesis (ii) we replace $(\log\delta^{-1})^\alpha$ by any function $\phi(\delta) \uparrow \infty$ as $\delta \downarrow 0$;

the choice $(\log\delta^{-1})^\alpha$ happens to be convenient from the point of view of applications to exterior solutions in §5 and §6.

4. Tangent cylinders at ∞ for exterior solutions

The minimal surface operator $\mathscr{M}$ on $\Omega \subset \mathbf{R}^n$ is the Euler-Lagrange operator of the area functional $\int_\Omega \sqrt{1+|Du|^2}$, so that

$$\mathscr{M}u = \sum_{i=1}^{n} D_i\left(\frac{D_i u}{\sqrt{1+|Du|^2}}\right), \tag{4.1}$$

or, alternatively,

$$\mathscr{M}u = (1+|Du|^2)^{-1/2}\left(\Delta u - \sum_{i,j=1}^{n}\frac{D_i u D_j u}{1+|Du|^2}D_i D_j u\right).$$

Our main purpose in this section is to derive a sufficient condition for the existence of a unique tangent cylinder at ∞ for exterior solutions of $\mathscr{M}u = 0$. The terminology is explained below. This will be of fundamental importance in our discussion in the next section of asymptotic behavior of exterior and entire solutions.

To begin we need some technical preliminaries concerning solutions of (4.1), including some compactness results slightly extending previous work of Miranda [16].

(4.2) Lemma. *Suppose $n \geq 2$, and $G_k = \operatorname{graph} u_k$ is any sequence of graphs of solutions $u_k \in C^2(\Omega_k)$ of the minimal surface equation, with Ω_k open in $\mathbf{R}^n$ and either*

(i) $\qquad \Omega_k \supset B_{R_k}(0) \sim B_{\rho_k}(0)$ *with $R_k \uparrow \infty$ and $\rho_k \downarrow 0$*

or

(ii) $\quad G_k$ *is minimizing (as a current) and closed (as a set) in $\mathbf{R}^{n+1}$.*

Then there are a subsequence $G_{k'}$ and a multiplicity 1 minimizing current H in $\mathbf{R}^{n+1}$, with $G_{k'} \rightharpoonup H$ in the weak sense of currents in $\mathbf{R}^{n+1} \sim \{0\}\times\mathbf{R}$ and $G_{k'} \to \operatorname{spt} H$ locally in the Hausdorff distance sense on $\mathbf{R}^{n+1} \sim \{0\}\times\mathbf{R}$, where $\operatorname{spt} H$ denotes the support of H. Furthermore, for any H obtained in this way, either

$(*)$ $\qquad H = H_1 \times \mathbf{R}$ *with $H_1 = \partial[\![E_1]\!]$ minimizing in $\mathbf{R}^n$*

for some open $E_1 \subset \mathbf{R}^n$, or

$(**)$ $\qquad \operatorname{reg} H = \operatorname{graph} w,$

where $w \in C^2(\Omega)$, Ω *an open connected subset of* $\mathbf{R}^n$, *with* graph w *a closed subset of* $\mathbf{R}^{n+1}$ (*so* $\operatorname{sing} H = \varnothing$ *in this latter case*), *and* $|u_{k'}| \to \infty$ *uniformly on compact subsets of* $\mathbf{R}^n \sim (\overline{H}_1 \cup \{0\})$ *in case* $(*)$, *and uniformly on compact subsets of* $\mathbf{R}^n \sim (\overline{\Omega} \cup \{0\})$ *in case* $(**)$.

If $y_{k'} \in \mathbf{R}^n$ *is such that* $\{u_{k'}(y_{k'})\}$ *is bounded and* $y_{k'} \to y \in \mathbf{R}^n \sim \{0\}$, *then, in case* $(*)$, $y \in \operatorname{spt} H_1$ *and*

$$\lim_{k' \to \infty} \sup_{B_\rho(y)} |Du_{k'}| = \infty \quad \textit{for each } \rho > 0,$$

while if $(**)$ *holds then* $y \in \Omega$ *and*

$$\limsup_{k' \to \infty} \sup_{B_\rho(y)} |Du_{k'}| < \infty \quad \textit{for some } \rho > 0.$$

Here and subsequently $[\![E]\!]$ denotes the current obtained by integration over the set E of smooth n-forms with compact support in $\mathbf{R}^n$.

Remarks. (1) Notice that hypothesis (i) allows the possibility that each Ω_k omits a neighborhood of the origin; in case each Ω_k contains all of B_{R_k}, the convergence of $G_{k'}$ to H is actually in all of $\mathbf{R}^{n+1}$, as it is in the case of hypothesis (ii). We emphasize that in any case H is a current in $\mathbf{R}^{n+1}$ which minimizes in $\mathbf{R}^{n+1}$.

(2) It is an open question whether or not it must automatically be true that each $\Omega_k = \mathbf{R}^n$ in case (ii); notice that in case (ii) we must necessarily have $u \to +\infty$ or $u \to -\infty$ on approach to any point of $\partial\Omega_k$ by virtue of the closedness of G_k and the openness of Ω_k.

(3) Case $(*)$ includes the possibility $H \equiv 0$, which will be the case if $\inf u_k \to \infty$ for example.

The following lemma gives us precise information concerning Ω in case w is as in Lemma (4.2).

(4.3) Lemma. *Suppose* $n \geq 2$ *and* $w \in C^2(\Omega)$ *is such that* graph w *is closed in* $\mathbf{R}^{n+1}$ (*as a set*) *and minimizing in* $\mathbf{R}^{n+1}$ (*as a current*). *Then* Ω *is connected and exactly one of the following three alternatives holds*:

(i) $\Omega = \mathbf{R}^n$.

(ii) $\mathbf{R}^n \sim \overline{\Omega}$ *has exactly one component*, $\partial[\![\Omega]\!]$ *is minimizing in* $\mathbf{R}^n$, $\operatorname{spt}\partial[\![\Omega]\!] = \partial\Omega$, *and either* $\lim_{x\to y, x\in\Omega} w(x) = +\infty\ \forall y \in \partial\Omega$ *or* $\lim_{x\to y, x\in\Omega} w(x) = -\infty\ \forall y \in \Omega$.

(iii) $\mathbf{R}^n \sim \overline{\Omega}$ *has exactly two components* F_+, F_- *with* $T_+ = \partial[\![F_+]\!]$ *and* $T_- = \partial[\![F_-]\!]$ *minimizing in* $\mathbf{R}^n$, $\partial[\![\Omega]\!] = -T_+ - T_-$, $\operatorname{spt}\partial[\![\Omega]\!] = \partial\Omega = \operatorname{spt} T_+ \cup \operatorname{spt} T_-$, $\operatorname{spt} T_+ \cap \operatorname{spt} T_- = \varnothing$, *and* $\lim_{x\to y, x\in\Omega} w(x) = +\infty\ \forall y \in \operatorname{spt} T_+$, $\lim_{x\to y, x\in\Omega} w(x) = -\infty\ \forall y \in \operatorname{spt} T_-$, *where* $\operatorname{spt} T$ *denotes support of* T.

Proof of Lemma (4.2). In case (ii) we are assuming that G_k is minimizing, hence

$$(1) \qquad |G_k \cap B_\rho(\xi)| \le c\rho^n \quad \forall \xi \in \mathbf{R}^{n+1},\ \rho > 0.$$

In case (i) we need to recall (see, e.g., [25]) that each G_k minimizes in $\Omega_k \times \mathbf{R}$ so that (1) holds for $\rho < \operatorname{dist}\{\xi', \partial\Omega_k\}/2$, ξ' denoting the projection of ξ onto its first n-coordinates. Define, for $\rho > 0$,

$$u_\rho = \begin{cases} \rho & \text{if } u > \rho, \\ u & \text{if } |u| \le \rho, \\ -\rho & \text{if } u < -\rho. \end{cases}$$

Multiplying by u_ρ in (4.1), and using the divergence theorem over the set $B_\rho(0) \sim B_{\rho_k}(0)$, we obtain

$$|G_k \cap \{(x,y)\colon \rho_k < |x| < \rho, |y| < \rho\}| \le c\rho^n$$

for sufficiently large k. Thus in case (i) we get

$$(2) \qquad |\tilde{G}_k \cap B_\rho(\xi)| \le c\rho^n \quad \forall \xi \in \mathbf{R}^{n+1},\ \rho > 0,$$

where $\tilde{G}_k = G_k \cap \{(x,y) \in \mathbf{R}^n \times \mathbf{R}\colon \rho_k < |x| < R_k\}$ with $\rho_k \downarrow 0, R_k \uparrow \infty$. Of course (2) is trivially implied by (1) in case (ii), so we may use (2) in both case (i) and case (ii). Then by the standard compactness and regularity theory for codimension 1 minimizers (see, e.g., [24], [9]) there is a subsequence $G_{k'}$ such that $G_{k'} \rightharpoonup H$ in $\mathbf{R}^{n+1} \sim \{0\} \times \mathbf{R}$, where H is minimizing on $\mathbf{R}^{n+1}$, $G_{k'} \to \operatorname{spt} H$ locally in the Hausdorff distance sense in $\mathbf{R}^{n+1} \sim \{0\} \times \mathbf{R}$, and $H = \partial [\![E]\!]$ for some open $E \subset \mathbf{R}^{n+1}$. Actually, we first get that H is only minimizing on $\mathbf{R}^{n+1} \sim \{0\} \times \mathbf{R}$, but in view of the bounds (2) it is straightforward to check that H is minimizing on all of $\mathbf{R}^{n+1}$ and that

$$(3) \qquad \mathbf{M}_{B_\rho(y)} H \le c\rho^n \quad \forall \rho > 0,\ y \in \mathbf{R}^{n+1},$$

where $\mathbf{M}_B$ denotes mass taken in B; we emphasize that (3) also holds for $y \in \{0\} \times \mathbf{R}$.

Next we recall [5] that $\operatorname{reg} H$ is connected. Since the G_k are graphs, $\operatorname{reg} H$ has smooth unit normal ν such that $\nu \cdot e_{n+1} \ge 0$. But since $\Delta\nu \cdot e_{n+1} + |A|^2 \nu \cdot e_{n+1} = 0$ on $\operatorname{reg} H$, where A denotes the second fundamental form of $\operatorname{reg} H$ (see, e.g., [4]), we then have by the Hopf maximum principle and connectivity of $\operatorname{reg} H$ that *either* $\nu \cdot e_{n+1} \equiv 0$ *or* $\nu \cdot e_{n+1} > 0$ everywhere on $\operatorname{reg} H$. In the former case (since $\partial H = 0$), the homotopy formula for currents gives $(*)$ as required. In the latter case, since $G_{k'}$ converges in the C^2 sense locally near points of $\operatorname{reg} H \sim \{0\} \times \mathbf{R}$ and the $G_{k'}$ are graphs, we see that $\operatorname{reg} H = \operatorname{graph} w$ for some $w \in C^2(\Omega)$, $\Omega \subset \mathbf{R}^n$ open. Note that in either case the required uniform convergence of $|u_{k'}|$ follows directly from the fact that $G_{k'}$ converges to $\operatorname{spt} H$ in the Hausdorff distance sense on $\mathbf{R}^{n+1} \sim \{0\} \times \mathbf{R}$.

To prove graph w is closed, take $x_0 \in \partial\Omega$ arbitrary, and assume w is bounded in a component Ω_σ of $B_\sigma(x_0) \cap \Omega$ for some $\sigma > 0$. If also

$$\mathscr{H}^{n-1}(B_\sigma(x_0) \cap \partial\Omega_\sigma) > 0,$$

then we would have $\mathscr{H}^{n-1}(\operatorname{sing} H) > 0$ thus contradicting the regularity theory for codimension 1 minimizing currents. On the other hand $\mathscr{H}^{n-1}(B_\sigma(x_0) \cap \partial\Omega_\sigma) = 0$ together with boundedness of w on Ω_σ implies (by the Poincaré inequality) that Ω_σ is the unique component of $B_\sigma(x_0) \cap \Omega$ and (by a well-known argument based on an idea of R. Finn [11]–see, e.g., the appendix of [22]) that w extends to give a C^2 solution of the minimal surface equation on all of $B_\sigma(x_0)$, contradicting the fact that $x_0 \notin \operatorname{reg} H$. Thus w is unbounded on Ω_σ for each $\sigma > 0$. Now we can show that there is no sequence $y_j \to x_0$, $y_j \in \Omega$ with $w(y_j)$ bounded. If there were such a sequence, the unboundedness of w shown above would imply that there are sequences $\sigma_j \downarrow 0$, $z_j \to x_0$ with

$$|u(z_j) - u(y_j)| \geq 2 \quad \text{and} \quad z_j, y_j \in A_j,$$

where A_j is the component of $B_{\sigma_j}(x_0) \cap \Omega$ containing y_j. Since $\sigma_j \downarrow 0$, it follows that $\overline{H}$ contains a line segment of length 1 in the vertical line $\{(x_0, 0) + \lambda e_{n+1} : \lambda \in \mathbf{R}\}$. We evidently have $\inf_{B_\sigma(z) \cap \operatorname{reg} H} \nu \cdot e_{n+1} = 0$ for each z in this line segment. But then, since $\Delta\nu \cdot e_{n+1} \leq 0$ (as we already mentioned above) the Harnack theory of [5] implies $\nu \cdot e_{n+1} \equiv 0$, thus contradicting the fact that $\nu \cdot e_{n+1} > 0$ on $\operatorname{reg} H$. Hence there is no sequence $y_j \to x_0$ with $w(y_j)$ bounded, and we conclude that graph w is closed, as required.

Finally, suppose that $\{u_{k'}(y_{k'})\}$ is bounded, with $y_{k'} \to y \neq 0$. By virtue of the convergence of $G_{k'}$ to $\operatorname{spt} H$ in the Hausdorff distance sense locally in $\mathbf{R}^n \sim \{0\} \times \mathbf{R}$, we evidently have $y \in \operatorname{spt} H_1$ in case $(*)$ and $y \in \Omega$ in case $(**)$. Further in case $(**)$ the Allard-De Giorgi regularity theorem implies that $G_{k'}$ converges in the C^1 sense to $\operatorname{reg} H$ $(= \operatorname{graph} w)$ locally in $\Omega \times \mathbf{R} \sim \{0\} \times \mathbf{R}$. Thus in particular $\limsup_{k' \to \infty} \sup_{B_\rho(y)} |Du_{k'}| < \infty$ for some $\rho > 0$ as required in this case. In case $(*)$ we must have $\liminf_{k' \to \infty} \sup_{B_\rho(y)} |Du_{k'}| = \infty$ for any $\rho > 0$, otherwise $\sup_{B_\rho(y)} |Du_{k'}|$ is bounded for some $\rho > 0$, and then by the regularity theory for quasilinear equations, some subsequence of $G_{k'}$ converges in the C^2 sense to a C^2 graph near a point of $\{y\} \times \mathbf{R}$, thus contradicting $(*)$.

Proof of Lemma (4.3). Let $H = \operatorname{graph} w$. H and hence Ω are connected by [5], so $\mathbf{R}^{n+1} \sim H$ has exactly two components H_+, H_-, where H_+ contains $\{(x,y): x \in \Omega, y > w(x)\}$ and H_- contains $\{(x,y): x \in \Omega, y < w(x)\}$. Indeed assuming H is appropriately oriented, we have $\partial[\![H_+]\!] = -H = -\partial[\![H_-]\!]$. Likewise if $H_\pm(t) = \{(x, y \pm te_{n+1}): (x,y) \in H_\pm\}$, then $\partial[\![H_\pm(t)]\!] = \pm\operatorname{graph}(w \pm t)$. Also $H_+(t)$ evidently converges to $H_+ \sim \Omega \times \mathbf{R}$ as $t \to \infty$ in the L^1_{loc} sense. Thus by the usual compactness theory for minimizing currents, $H_+ \sim \Omega \times \mathbf{R} = \varnothing$ or else $\partial[\![H_+ \sim \Omega \times \mathbf{R}]\!]$ is minimizing on $\mathbf{R}^{n+1}$. Likewise $H_- \sim \Omega \times \mathbf{R} = \varnothing$ or else $\partial[\![H_- \sim \Omega \times \mathbf{R}]\!]$ is minimizing on $\mathbf{R}^{n+1}$. (Notice that here we use the area estimates $|H \cap B_\rho(\xi)| \le c\rho^n$ $\forall \xi \in \mathbf{R}^{n+1}$, $\rho > 0$, which holds since $\operatorname{graph} w$ is minimizing in $\mathbf{R}^{n+1}$.)

Notice also that $H_\pm \sim \Omega \times \mathbf{R}$ are cylinders $F_\pm \times \mathbf{R}$ where $F_\pm \subset \mathbf{R}^n$, each $F_\pm$ being a closed connected subset of $\mathbf{R}^n \sim \Omega$ and $F_+ \cap F_- \subset \partial\Omega$. Then $T_\pm \equiv \partial[\![F_\pm]\!]$ are minimizing in $\mathbf{R}^n$ and, since $\operatorname{graph} w$ is closed,

$$\operatorname{spt} T_\pm = \partial F_\pm, \qquad \partial\Omega = \operatorname{spt} T_+ \cup \operatorname{spt} T_-.$$

Since T_+ lies on one side of T_- in the sense that the interiors of $F_\pm$ are disjoint, by [28] we also have

$$F_+ \cap F_- = \varnothing.$$

Throughout this discussion we allow the possibility that one or both of $F_\pm$ are empty; in this case of course we do not need the result of [28]. The remaining claims of (4.3) now follow directly from the definition of $F_\pm$.

Next we recall (see [27], or [16] for the case $U = \varnothing$) that the graph of any $C^2(\mathbf{R}^n \sim U)$ solution u of $\mathscr{M}u = 0$, U bounded and open, $n \ge 3$, *either* has "tangent cylinders" $C \times \mathbf{R}$ at ∞, where C is a multiplicity 1 minimizing cone in $\mathbf{R}^n$ with singular vertex at 0, *or* is asymptotic to a plane at ∞. The latter possibility occurs if and only if u has bounded gradient on $\mathbf{R}^n \sim U$; see, e.g., [27] for a discussion. The precise meaning of the former alternative, when $|Du|$ is unbounded, is as follows: For each $\lambda > 0$ let $(\lambda): \mathbf{R}^{n+1} \to \mathbf{R}^{n+1}$ denote the homothety $x \mapsto \lambda x$. Then for each sequence $\lambda_k \downarrow 0$ there is a subsequence $\lambda_{k'}$ such that

$$(\lambda_{k'})_\# \operatorname{graph} u \rightharpoonup C \times \mathbf{R}, \tag{4.4}$$

where C is minimizing, $C = \partial[\![E]\!]$ for some open E in $\mathbf{R}^n$, C is a cone (i.e., $(\lambda)_\# C = C$ $\forall \lambda > 0$), $0 \in \operatorname{sing} C$. The convergence in (4.4) is in the weak sense of currents, where $\operatorname{graph} u$ is oriented with its upward unit normal $(-Du, 1)/\sqrt{1 + |Du|^2}$ and is of course assigned multiplicity 1. (4.4) is a direct consequence of Lemma (4.2) and the monotonicity formula for minimal hypersurfaces. The fact that $0 \in \operatorname{sing} C$, i.e., that C is not a hyperplane, requires a little more argument, at least in the case of exterior solutions. See, e.g., [27] for details.

Notice that it is not clear that C is independent of the choice of sequences $\lambda_k, \lambda_{k'}$; the following theorem says that it *is* so independent if $\operatorname{sing} C = \{0\}$.

Theorem 2. *If* $\operatorname{graph} u$ *has at least one tangent cylinder* $C \times \mathbf{R}$ *with* $\operatorname{sing} C = \{0\}$, *then* $C \times \mathbf{R}$ *is the unique tangent cylinder of* $\operatorname{graph} u$ *at* ∞; *that is,*

$$(\lambda)_\# \operatorname{graph} u \rightharpoonup C \times \mathbf{R} \quad \text{as } \lambda \downarrow 0.$$

Remark. Of course this theorem is useful in that it already gives a fairly precise picture of the geometric shape of graph u near ∞. This picture will be made considerably more precise in Theorem 4 of the next section.

Proof. The proof is based on Lemma 1 of §2 and on the theory of unique asymptotic limits for elliptic evolution equations developed in [23] and [26]. Let $G = \operatorname{graph} u$ be equipped with multiplicity 1 and oriented via the upward unit normal, so that in the sense of currents $\operatorname{spt} \partial G \subset \partial U \times \mathbf{R}$. For $0 < \rho < R \leq \infty$ let

$$U(\rho, R) = \{(x, y) \in \mathbf{R}^{n+1} \colon \rho < |x| < R, |y| < R\},$$
$$U_0(\rho, R) = \{x \in \mathbf{R}^n \colon \rho < |x| < R\}.$$

We identify $U_0(\rho, R)$ with $U(\rho, R) \cap \mathbf{R}^n \times \{0\}$.

The Allard-De Giorgi regularity theorem (see, e.g., [24] or [1]) guarantees that weak convergence, in the sense of (4.4) above, implies C^2 convergence near $\operatorname{reg} C \times \mathbf{R}$; that is, letting $\mu_k \downarrow 0$ be such that $(\mu_k)_\# G \rightharpoonup C \times \mathbf{R}$ and letting $R_k = \mu_k^{-1}$, we have that there is a sequence $\theta_k \downarrow 0$ such that

$$G \cap U(R_k/2, \theta_k^{-1} R_k) \subset \operatorname{graph} w_k \subset G,$$

where $w_k \in C^2(U(R_k/4, 2\theta_k^{-1} R_k) \cap (C \times \mathbf{R}))$ with

$$\sup(|x|^{-1}|w_k(x, y)| + |\nabla w_k(x, y)| + |x|\,|\nabla^2 w_k(x, y)|) \leq \theta_k.$$

Now let S be the slice of G by $y = 0$; that is,

$$S = G \cap (\mathbf{R}^n \times \{0\}), \tag{1}$$

and for constants $\alpha \in (0, 1)$ and $\delta > 0$, and for $k \geq k(\delta, \alpha)$, let $\psi_k \in C^2(U_0(R_k/4, \overline{R}_k) \cap C)$, with $2\theta_k^{-1} R_k \leq \overline{R}_k \leq \infty$, be the maximal C^2 extension of $w_k | U_0(R_k/4, 2\theta_k^{-1} R_k) \cap C$ satisfying the restrictions

$$\tag{2} \begin{aligned} &S \cap U_0(R_k/2, \overline{R}_k/2) \subset \operatorname{graph} \psi_k \subset S, \\ &\sup(|x|^{-1}|\psi_k(x)| + |\nabla \psi_k(x)| + |x|\,|\nabla^2 \psi_k|) \leq \delta^\alpha. \end{aligned}$$

By virtue of (2) and the fact that for *each* sequence $\mu_j \downarrow 0$ there is a subsequence $\mu_{j'} \downarrow 0$ such that $(\mu_{j'})_\# G$ converges weakly to some *vertical cylinder* $\tilde{C} \times \mathbf{R}$ and also the fact that the convergence is C^2 near points of the regular set of $\tilde{C} \times \mathbf{R}$, we note that for each $\theta > 0$ and fixed $k = k(\theta, \delta, \alpha)$, we can find

$$(3)\quad \tilde{w}_k \in C^2(\Omega_k),\qquad \Omega_k = \{(x,y)\colon R_k/3 + \theta|y| < |x| < 3\overline{R}_k, |y| < 3\overline{R}_k\} \cap (C \times \mathbf{R})$$

with

$$G \cap \Omega_k' \subset \operatorname{graph} \tilde{w}_k \subset G,$$
$$(4)\quad \Omega_k' = \{(x,y)\colon R_k/2 + 2\theta|y| < |x| < 2\overline{R}_k, |y| < 2\overline{R}_k\},$$
$$\sup(|x|^{-1}|\tilde{w}_k(x,y)| + |\nabla \tilde{w}_k(x,y)| + |x|\,|\nabla^2 \tilde{w}_k(x,y)|) \le \eta(\delta),$$

with $\eta(\delta) \downarrow 0$ as $\delta \downarrow 0$. Notice that

$$(5)\qquad \tilde{w}_k | \Omega_k \cap C \text{ is an extension } \tilde{\psi}_k \text{ of } \psi_k$$

with $\tilde{\psi}_k \in C^2(\Omega_k \cap C)$ and

$$(6)\qquad \sup(|x|^{-1}|\tilde{\psi}_k(x)| + |\nabla \tilde{\psi}_k(x)| + |x|\,|\nabla^2 \tilde{\psi}_k(x)|) \le \eta(\delta).$$

Furthermore since G is a graph over $\mathbf{R}^n \sim U$, and $\Sigma = C \cap S^{n-1}$ is connected, after replacing u by $-u$ if necessary, it follows that

$$(7)\qquad v_k \equiv \partial \tilde{w}_k / \partial y > 0 \quad \text{on } \Omega_k.$$

Indeed we notice that in fact

$$(8)\qquad v_k = 1/|Du|(\xi),$$

where $\xi \in \mathbf{R}^n \sim U$ is such that $\xi + u(\xi)e_{n+1} = (x,y) + \tilde{w}_k(x,y)\nu_C(x)$, ν_C being the unit normal for C.

Since G is minimal, i.e., stationary with respect to the area functional, we know that $\tilde{w}_k$ satisfies $\mathscr{M}_{C\times\mathbf{R}}\tilde{w}_k = 0$ on Ω_k, where $\mathscr{M}_{C\times\mathbf{R}}$ is the minimal surface operator on $C \times \mathbf{R}$. Thus by (4) we deduce that both $\tilde{w}_k$ and v_k (as in (7)) satisfy equations of the form (2.11), (2.12) on Ω_k, with $c\eta(\delta)$ in place of δ. Then in view of (7) we can apply Lemma 1 to deduce that for small enough δ and θ, with θ as in (4), and for $k = k(\delta, \theta, \alpha)$

$$(9)\qquad v_k(x,0) \le c\theta_k (r/R_k)^{-\mu}, \qquad R_k/2 < r = |x| < 2\overline{R}_k,$$

for some $\mu = \mu(L) \in (0,1)$. Furthermore by (6) and the Schauder estimates, together with (2.15), we have that

$$(10)\qquad |x|^{-1} v_k(x,y) + |\nabla v_k(x,y)| + |x||\nabla^2 v_k(x,y)| \le c\theta_k (|x|/R_k)^{-\mu},$$

$R_k/2 < |x| < 2\overline{R}_k$, $|y| < 2\overline{R}_k$, where $c = c(\theta)$. Thus, along $y = 0$, the equation $\mathscr{M}_{C\times\mathbf{R}}\tilde{w}_k = 0$ can be written, after splitting off the y-derivatives and using (10) together with interior Schauder estimates,

$$\mathscr{M}_C\tilde{\psi}_k(x) = |x|^{-2}f(x), \qquad x \in C \cap \Omega_k', \tag{11}$$

where

$$|x|^{-1}|f(x)| + |\nabla f(x)| + |x|\,|\nabla^2 f(x)| \le c\theta_k(|x|/R_k)^{-\mu}, \quad x \in C \cap \Omega_k'. \tag{12}$$

Next we need some estimates on the quantity $\xi \cdot \nu(\xi)$, $\xi \in G$, where ν is the unit normal of G at ξ. First note that by the monotonicity formula,

$$\int_G \frac{(\xi\cdot\nu)^2}{|\xi|^{n+2}} < \infty,$$

and therefore in particular for sufficiently large k $(\ge k(\delta))$

$$\int_{G\cap U(R_k,\infty)} \frac{(\xi\cdot\nu)^2}{|\xi|^{n+2}} < \delta. \tag{13}$$

Now at the point $\xi = (x,y) + \tilde{w}_k(x,y)\nu_C(x)$ we have (see, e.g., [26, p. 219])

$$\xi\cdot\nu(\xi) = (1 + |\nabla\tilde{w}_k|^2)^{-1/2}\frac{\partial}{\partial\lambda}\tilde{w}^{(\lambda)}(x,y)|_{\lambda=1}, \tag{14}$$

where $w^{(\lambda)}(x,y) = \lambda^{-1}w(\lambda x, \lambda y)$. On the other hand by homogeneity

$$\mathscr{M}_{C\times\mathbf{R}}w^{(\lambda)}(\xi) = \lambda\mathscr{M}_{C\times\mathbf{R}}w(\lambda\xi),$$

so that $\hat{w}_k = \frac{\partial}{\partial\lambda}\tilde{w}_k^{(\lambda)}(x,y)|_{\lambda=1}$, $(x,y) \in \Omega_k'$, satisfies an equation of the form

$$\mathscr{L}\hat{w}_k = \frac{1}{|x|^2}\frac{\partial}{\partial\lambda}f^{(\lambda)}(x,y)|_{\lambda=1} \equiv |x|^{-2}(-f(x,y) + (x,y)\cdot\nabla f(x,y)),$$

where $\mathscr{L}$ is the linearization of $\mathscr{M}_{C\times\mathbf{R}}$ at $\tilde{w}_k$, so that $\mathscr{L} = 0$ has the general form of (2.11), (2.12) with $\eta(\delta)$ in place of δ. Hence the usual sup estimates for divergence-form equations guarantee that if $(x_0, y_0) \in C\times\mathbf{R}$ with

$$A_\rho \equiv \{(x,y) \in C\times\mathbf{R}\colon |x - x_0| < \rho, |y - y_0| < \rho\} \subset \Omega_k'',$$

where

$$\Omega_k'' \equiv \{(x,y) \in C\times\mathbf{R}\colon \tfrac{3}{4}R_k + 3\theta|y| < |x| < \tfrac{3}{2}\overline{R}_k,\ |y| < \tfrac{3}{2}\overline{R}_k\},$$

then, since $\mathscr{M}_{C\times\mathbf{R}}w^{(\lambda)} = 0$,

$$\sup_{A_{\rho/2}}\hat{w}_k^2 \le c\rho^{-n}\int_{A_\rho}\hat{w}_k^2.$$

In view of (14) this guarantees that if $\xi_0 \in G$ and $\{\xi \in G\colon |\xi - \xi_0| < \rho\} \subset \operatorname{graph}(\tilde{w}_k|\Omega'_k)$, then

$$\sup_{\xi\in G, |\xi-\xi_0|<\rho/2} (\xi\cdot\nu(\xi))^2 \le c\rho^{-n}\int_{G\cap|\xi-\xi_0|<\rho} (\xi\cdot\nu(\xi))^2,$$

and if in addition $\xi_0 \in S$, then by (8) and (9)

$$\begin{aligned}\rho^{-2}\sup_{S\cap|\xi-\xi_0|<\rho/2} (\xi\cdot\nu'(\xi))^2 \le c\rho^{-n-2}&\int_{G\cap|\xi-\xi_0|<\rho} (\xi\cdot\nu(\xi))^2 \\ &+ c\theta_k(|\xi_0|/R_k)^{-\mu},\end{aligned}\tag{15}$$

where $\nu' = (Du, 0)/|Du|$ is the unit normal of S in $\mathbf{R}^n \times \{0\}$.

By combining (13) and (15) we get

$$\begin{aligned}&\sup_{S\cap U_0(3R_k/4, 5\overline{R}_k/4)} (|\xi|^{-1}\xi\cdot\nu'(\xi))^2 \le c\delta, \\ &\int_{S\cap U_0(3R_k/4,5\overline{R}_k/4)} \frac{(\xi\cdot\nu'(\xi))^2}{|\xi|^{n+1}} \le c\delta.\end{aligned}\tag{16}$$

On the other hand since $|x|^{-2}f$ in (11) is geometrically just the mean curvature of $S\cap\Omega'_k$, we deduce, from (12) and the monotonicity formula,

$$\begin{aligned}&r^{2-n}\int_{S\cap\{|\xi|=r\}} |\nabla|\xi|| - R_k^{2-n}\int_{S\cap\{|\xi|=R_k\}} |\nabla|\xi|| \\ &= \int_{S\cap\{R_k<|\xi|<r\}} \frac{(\xi\cdot\nu')^2}{|\xi|^{n+1}} + E, \qquad |E| \le c\left(\frac{r}{R_k}\right)^{-\mu}\theta_k\end{aligned}\tag{17}$$

for $R_k \le r \le 5\overline{R}_k/4$, which (since $|\nabla|\xi||^2 = 1 - (\xi\cdot\nu'(\xi))^2/|\xi|^2$) gives us from (16) that

$$r^{2-n}|S\cap\{|\xi| = r\}| \le |\Sigma| + c\delta, \qquad R_k \le r \le \tfrac{5}{4}\overline{R}_k.\tag{18}$$

We now want to show that we can apply the theory developed in [23] and [26] to show that $\overline{R}_k = \infty$ and $\tilde{\psi}_k(x)/|x| \to 0$. (Of course this establishes the required uniqueness of $C\times\mathbf{R}$ as required by virtue of (3) above, for example.) We fix $k = k(\delta, \theta, \alpha)$ so that the above estimates are valid, and let $\phi_k \in C^2(C\cap U_0(R_k, 5\overline{R}_k/4))$ be such that

$$\begin{aligned}&\operatorname{graph}\tilde{\psi}_k|U_0\left(\frac{3}{4}R_k, \overline{R}_k\right)\cap C \\ &\subset\left\{\frac{x+\phi_k(x)\nu_C(x)}{\sqrt{1+(\phi_k(x)/|x|)^2}}\colon x\in U_0\left(R_k, \frac{5}{4}\overline{R}_k\right)\cap C\right\} \\ &\subset \operatorname{graph}\tilde{\psi}_k|U_0\left(\frac{5}{4}R_k, \frac{3}{2}\overline{R}_k\right)\cap C.\end{aligned}\tag{19}$$

Then by (11) and (12), ϕ_k satisfies

$$\tilde{\mathcal{M}}_C(\phi_k) = |x|^{-2}\tilde{f},$$

$$(20) \qquad |x|^{-1}|\tilde{f}(x)| + |\nabla\tilde{f}(x)| + |x|\,|\nabla^2\tilde{f}(x)| \le c\theta_k\left(\frac{|x|}{R_k}\right)^{-\mu}$$

for $x \in U_0(R_k, 5\overline{R}_k/4) \cap C$, where $\tilde{\mathcal{M}}_C$ is the minimal surface operator relative to the "spherical graphical representation" of S given by ϕ_k as in (19) (cf. [23] and [26]).

Note that by virtue of (6)

$$(21) \qquad \sup_{U_0(R_k/2, 3\overline{R}_k/2)\cap C} (|x|^{-1}|\phi_k(x)| + |\nabla\phi_k(x)| + |x|\,|\nabla^2\phi_k(x)|) \le c\eta(\delta)$$

for small enough δ and $k = k(\delta, \theta, \alpha)$. Also by definition of $\overline{R}_k$, *either*

$$(22) \qquad \sup_{U_0(R_k, 5\overline{R}_k/4)\cap C} (|x|^{-1}|\phi_k(x)| + |\nabla\phi_k(x)| + |\nabla^2\phi_k(x)|) \ge c\delta^\alpha$$

or

$$(23) \qquad \overline{R}_k = \infty.$$

In view of (18), (19), (20), (21), (22), and (23), after a change of variable $t = (\log|x|/R_k)$, $\omega = x/|x|$, we can apply Theorem 5.5 of [26, Part II] in the case $m = -(n-1) < 0$. The reader should note that the term $\delta e^{-\varepsilon t}$ can be replaced simply by δ in inequality 5.3 of [26, Part II]. Indeed by using the monotonicity (17), which can be written in the form of inequality 2.4 on p. 243 of [26], and inequality (18), which gives an inequality like 5.3 of [26] with δ in place of $\delta e^{-\varepsilon t}$, and by minor modifications of the relevant arguments on pp. 245–247 of [26], we get an inequality like 2.22 on p. 247 of [26], provided the function v there has $\|v\|_{(\rho,\rho+2)} \ge \delta$; we can arrange this by working on suitable intervals $\tau_1 < t < \tau_2$, with $\|v\|_{(\rho,\rho+2)} \ge \delta \forall \rho \in [\tau_1, \tau_2 - 2]$ and with $\|v\|_{(\tau_2-2,\tau_2)} \ge \rho^\alpha$. The negative exponential in inequality 5.3 of [26, Part II] is not needed in the remaining arguments of [26] either. Note also that the extension property 5.4 on p. 266 of [26] is also valid here by virtue of the argument which we used above to show that (2) implies (5) and (6). It is of course alternatively possible to modify the arguments of [23] to the present setting.

Thus we establish that $\overline{R}_k = \infty$ and that $\lim_{|x|\to\infty} \tilde{\psi}_k(x)/|x|$ exists. However since $\lim_{j\to\infty} \tilde{\psi}_k(R_j\omega)/R_j = 0$, we then deduce that $\lim_{|x|\to\infty} \tilde{\psi}_k(x)/|x| = 0$ as required.

Note. The reader should be aware that in 2.17, 2.23, 5.3, and 5.9 of [26], and also in $(*)$ on p. 272 of [26], the quantity $\mathscr{F}_\Sigma(u(t))$ should be $\int_{\Sigma\times\{t\}}(F - \dot{u}\cdot F_{\dot{u}})$ in case $m > 0$. This causes no difficulty in the rest of the discussion of [26], and in any case only the case $m < 0$ is relevant to our present discussion.

5. Main theorems concerning exterior solutions

Let U be a bounded open subset of $\mathbf{R}^n$, $n \geq 2$, and consider an exterior solution of the minimal surface equation. That is, let $u \in C^2(\mathbf{R}^n \sim U)$ satisfy the minimal surface equation

$$(5.1) \qquad \Delta u - \sum_{i,j=1}^{n} \frac{D_i u D_j u}{1 + |Du|^2} D_i D_j u = 0$$

on $\mathbf{R}^n \sim U$. Since we are only interested in asymptotic behavior of u near ∞, we could assume without loss of generality that U is an open ball centered at 0.

We first recall the following theorem, proved in [27] and extending previous results for entire solutions in [29], [16]. The terminology is as in the previous section.

Theorem 3. *If u is as above, then either $Du(x)$ is bounded and has a limit as $x \to \infty$, or else all tangent cones of* graph u *at ∞ are cylinders of the form $C \times \mathbf{R}$, where C is an $(n-1)$-dimensional minimizing cone in $\mathbf{R}^n$ with $C = \partial[\![E]\!]$ for some open $E \subset \mathbf{R}^n$ and $0 \in \operatorname{sing} C$.*

In particular, since the standard regularity theory for minimizing currents guarantees that no such cones C can exist for $n \leq 7$, we conclude

Corollary 1 [27]. *If $n \leq 7$, then $Du(x)$ is bounded and has a limit as $|x| \to \infty$.*

This extends a well-known result of L. Bers [2] for the case $n = 2$.

In view of the above theorem and corollary, we henceforth assume $n \geq 8$ and that $|Du|$ is unbounded near ∞.

We recall that if C is a minimizing cone in $\mathbf{R}^n$ with $\operatorname{sing} C \subset \{0\}$, then $\mathbf{R}^n \sim \overline{C}$ has exactly two connected components E_+, E_-, and there are smooth embedded complete minimizing hypersurfaces

$$(5.2) \qquad S_+ \subset E_+, \quad S_- \subset E_-, \quad \operatorname{dist}(S_\pm, 0) = 1, \quad \partial S_\pm = 0.$$

Furthermore $S_\pm$ approach C asymptotically near infinity in the sense that there is $R_0 > 0$ such that

$$(5.3) \qquad S_\pm \sim B_{R_0} \subset \operatorname{graph} v_\pm \subset S_\pm,$$

where $v_\pm$ are positive C^2 functions on $C \sim B_{R_0/2}$, and

$$(5.4) \qquad v_\pm(r\omega) \leq c r^{-\alpha}, \qquad r > R_0,$$

for some $\alpha > 0$. Here $\operatorname{graph} v_{\pm} = \{r\omega \pm v_{\pm}(r\omega)\nu_C(\omega)\}$, where $\nu_C(\omega)$ is the unit normal of C pointing into E_+. Also $S_{\pm}$ are, up to homothety, the unique minimizing hypersurfaces without boundary, which are different from C and have support contained in $\overline{E}_{\pm}$ respectively. For later reference we also note that if C is *strictly minimizing*, then

$$(5.5)\qquad v_{\pm}(r\omega) \sim \begin{cases} cr^{-\gamma_1}\phi_1 & \text{as } r \uparrow \infty \text{ in case } \beta > 0, \\ cr^{-(n-3)/2}(\log r)\phi_1 & \text{as } r \uparrow \infty \text{ in case } \beta = 0. \end{cases}$$

Here γ_1, β are given by

$$(5.6)\qquad \gamma_1 = \frac{n-3}{2} - \sqrt{\left(\frac{n-3}{2}\right)^2 + \lambda_1}, \qquad \beta = 2\sqrt{\left(\frac{n-3}{2}\right)^2 + \lambda_1},$$

where λ_1 is the minimum eigenvalue of the operator $\Delta_\Sigma + |A(\omega)|^2$ with $A(\omega)$ the second fundamental form of $\Sigma \subset S^{n-1}$; thus γ_1, β are as in Theorem 1 with $q(\omega) \equiv |A(\omega)|^2$. Notice that in this case $\lambda_1 < 0$ and $\lambda_1 \geq -((n-3)/2)^2$ by virtue of stability of the cone C, so that (2.4) holds. If C is strictly stable, we have strict inequality $\lambda_1 > -((n-3)/2)^2$. See [6] for a discussion.

For further discussion and proofs of (5.2)–(5.5), we refer the reader to [13].

It will be convenient to introduce the terminology that if S_1, S_2 are embedded hypersurfaces and $\varepsilon > 0$, then S_2 is within ε of S_1 in the C^2_* sense if

$$(5.7)\qquad S_2 = \operatorname{graph}_{S_1} v \equiv \{x + v(x)\nu_1(x) \colon x \in S_1\},$$

where ν_1 is a smooth unit normal for S_1, and $v \in C^2(S_1)$ with

$$|v|_{C^2_*} < \varepsilon, \qquad |v|_{C^2_*} \equiv \sup_{x \in S_1 \sim \{0\}} (|x|^{-1}|v(x)| + |\nabla v(x)| + |x|\,|\nabla^2 v(x)|).$$

In the following theorem, and subsequently, we let

$$S_y = S_y(u) = \{x \in \mathbf{R}^n \sim U \colon u(x) = y\}.$$

Theorem 4. *Suppose U is a bounded open subset of $\mathbf{R}^n$, $u \in C^2(\mathbf{R}^n \sim U)$ satisfies the minimal surface equation on $\mathbf{R}^n \sim U$, and $G = \operatorname{graph} u$ has a tangent cylinder $C \times \mathbf{R}$ at ∞ with $\operatorname{sing} C \subset \{0\}$. Then $C \times \mathbf{R}$ is the unique tangent cylinder of G at ∞, and the two components of $\mathbf{R}^n \sim \overline{C}$ can be labelled $E_{\pm}$ such that*

$$\text{(i)}\qquad \lim_{r \uparrow \infty} \frac{u(r\omega)}{r} = \begin{cases} +\infty & \text{if } \omega \in E_+ \cap S^{n-1}, \\ -\infty & \text{if } \omega \in E_- \cap S^{n-1}, \end{cases}$$

where the convergence is uniform for compact subsets of $E_{\pm} \cap S^{n-1}$.

Furthermore $|Du(x)| \to \infty$ *as* $|x| \to \infty$; *i.e.,*

(ii) $$\lim_{r\uparrow\infty} \inf_{\omega\in S^{n-1}} |Du(r\omega)| = \infty,$$

and for any given $\varepsilon > 0$ *there is* $y(\varepsilon) > 0$ *such that if* $y > y(\varepsilon)$ $(y < -y(\varepsilon))$, *then, with* $\lambda_y = \operatorname{dist}(S_y, \{0\})$,

(iii) $(\lambda_y^{-1})S_y$ *is within* ε *of* S_+ *(resp.* S_-*) in the* C^2_* *sense of* (5.7).

Finally the second fundamental form A *of* G *has length* $|A|$ *satisfying*

(iv) $$|A(x, u(x))| \le c/|x|, \qquad x \in \mathbf{R}^n \sim U,$$

and the gradient function $v = \sqrt{1 + |Du(x)|^2}$ *satisfies*

(v) $$\sup_{G_\rho(x_0)} v \le c \inf_{G_\rho(x_0)} v$$

for any $x_0 \in \mathbf{R}^n$ *and* $\rho > 0$ *such that* $|x_0| > 2\rho + \operatorname{diam} U$, *where*

$$G_\rho(x_0) = \{x \in \mathbf{R}^n \sim U : |x - x_0|^2 + |u(x) - u(x_0)|^2 < \rho^2\}.$$

In (iv) *and* (v), *c is a constant depending only on u and not on* x, x_0, *and* ρ.

Remarks. (1) The theorem evidently gives us a rather precise picture of how G looks near ∞.

(2) The hypothesis that G has a tangent cylinder $C \times \mathbf{R}$ with $\operatorname{sing} C \subset \{0\}$ is automatically satisfied in case $n = 8$ (unless Du is bounded and has a limit at ∞) by Theorem 2, because for $n = 8$ all minimizing cones have $\operatorname{sing} C \subset \{0\}$ by the standard regularity theory.

(3) In general, for any $n \ge 8$ and any u with $|Du|$ unbounded, the existence of such a tangent cylinder with $\operatorname{sing} C \subset \{0\}$ is implied by (and hence equivalent to, by the theorem) an estimate of the form (iv). Thus in place of the hypothesis that there is a tangent cylinder $C \times \mathbf{R}$ at ∞ with $\operatorname{sing} C \subset \{0\}$, we can alternatively require that G is "regular near ∞" in the sense that $\sup_{x\in\mathbf{R}^n\sim U} |x|\,|A(x, u(x))| < \infty$.

(4) In interpreting (v) one should keep in mind that if we let $\tilde{v}$ be the function on G such that $\tilde{v}(x, u(x)) = v(x)$, then (v) simply says

$$\sup_{B_\rho(X_0)\cap G} \tilde{v} \le c \inf_{B_\rho(X_0)\cap G} \tilde{v},$$

where $X_0 = (x_0, u(x_0))$.

Proof of Theorem 4. The uniqueness of the tangent cylinder is guaranteed by Theorem 2, and then the limit statements in (i) follow directly from the fact that $(\lambda)_\# G$ converges to $\operatorname{spt} C \times \mathbf{R}$ in the Hausdorff distance sense in $\mathbf{R}^{n+1} \sim \{0\} \times \mathbf{R}$.

Next we prove (iv) and (v). Suppose first that (iv) is false. Then we can find a sequence $\{x_k\} \subset \mathbf{R}^n \sim U$ with $|x_k| \to \infty$ and

$$(1) \qquad |x_k|\,|A(x_k, u(x_k))| \to \infty.$$

Let $\lambda_k = |x_k|$, $\mu_k = u(x_k)$, and

$$G_k = (\lambda_k^{-1})(G - \mu_k e_{n+1}) \quad (\equiv \{\lambda_k^{-1}(X - \mu_k e_{n+1}) \colon X \in G\}).$$

By Lemma (4.2) we have that some subsequence of G_k converges to a minimizer H. Also, since G has tangent cylinder $C \times \mathbf{R}$, we have $H = C \times \mathbf{R}$ in case μ_k is bounded. Also, since the convergence of $(\lambda)_\# G$ is C^2 near points of $C \times \mathbf{R}$ by the Allard-De Giorgi theorem and elliptic regularity theory, we know that for sufficiently large R_0 the set $S_0 \sim B_{R_0}$ is a smooth complete hypersurface with boundary in ∂B_{R_0} and $(\lambda)_\#(S_0 \sim B_{R_0})$ converges locally near points of C in the C^2 sense to C as $\lambda \downarrow 0$. Hence $\mathbf{R}^n \sim (S_0 \cup \overline{B}_{R_0})$ has exactly two unbounded components $V_\pm$ with $(\lambda)_\# V_\pm$ converging with respect to Lebesgue measure to $E_\pm$ respectively. Note that each level set S_y also has tangent cone C at ∞, and since Σ is connected, there is only one unbounded component of S_y. But then by the Hopf maximum principle S_y is connected for all y with $|y| > \sup_{\partial U} |u|$. Thus for each such y we have $S_y \subset V_\pm$ according as $\pm y > 0$. Then, in case $\mu_k \to \infty$, which we may assume without loss of generality if μ_k is not bounded, we evidently must have

$$(2) \qquad \operatorname{spt} H \subset \overline{E}_+ \times \mathbf{R}.$$

By construction, since $G_{k'} \to \operatorname{spt} H$ locally in the Hausdorff distance sense in $\mathbf{R}^{n+1} \sim \{0\} \times \mathbf{R}$, we know that there is a point $x \in S^{n-1} \times \{0\} \cap \operatorname{spt} H$ with $|x_j|^{-1} x_j \to x$ for some subsequence $\{j\}$ of $\{k\}$. Furthermore as in Lemma (4.2) H is either a vertical cylinder $H_1 \times \mathbf{R}$ with $H_1 = \partial[\![E]\!]$ for some open $E \subset \mathbf{R}^n$, or else has the form $H = \operatorname{graph} w$, where $w \in C^2(\Omega)$ and $\operatorname{graph} w$ is a closed subset of $\mathbf{R}^{n+1}$. In the first case, H_1 is C or a homothety of S_+ in view of (2) and the uniqueness property of S_+ mentioned prior to Theorem 4. Thus in any case, $x \in \operatorname{reg} H$, and then we must have that for some $\rho > 0$, $G_{k'} \cap B_\rho(x)$ converges to $H \cap B_\rho(x)$ in the C^2 sense by the Allard-De Giorgi regularity theorem. But this contradicts (1), because $(x_j, u(x_j)) \in G_j \cap B_\rho(x)$. Thus (iv) is established.

Now (v) follows easily from (iv) and the Harnack inequality for solutions of uniformly elliptic equations on domains in $\mathbf{R}^n$, because $\phi \equiv (1 + |Du|^2)^{-1/2} \equiv e_{n+1} \cdot \nu$ satisfies the equation $\Delta\phi + |A|^2\phi = 0$ on G. Here we also need the fact that there is a $\theta \in (0,1)$ with the property that if $|x_0| > \rho + \operatorname{diam} U$, then $G_{\theta\rho}(x_0)$ is connected—this follows, for example, from (iv) and the fact that G is minimizing in $\mathbf{R}^{n+1} \sim U \times \mathbf{R}$.

Now let $\omega_\pm$ be points of $S^{n-1} \cap E_\pm$ respectively, and let $\varepsilon > 0$. Then there exists $K = K(\varepsilon, u, \omega_\pm)$ such that

$$(3) \qquad |Du(r\omega_\pm)| \geq K \Rightarrow \begin{cases} \operatorname{dist}(S_{u(r\omega_\pm)}, 0)^{-1} S_{u(r\omega_\pm)} \text{ is within } \varepsilon \text{ of } S_\pm, \\ |Du| \geq \varepsilon^{-1} \text{ at each point of } S_{u(r\omega_\pm)}, \end{cases}$$

where the first implication is to be interpreted in the C^2_* sense of (5.7). Indeed if (3) is false for ω_+ say, then there exist $\varepsilon > 0$ and a sequence $r_k \uparrow \infty$, with $|Du(r_k\omega_+)| \geq k$ and such that at least one of the conclusions in (3) is false with $r = r_k$. In this case let

$$u_k(x) = r_k^{-1}(u(r_kx) - u(r_k\omega_+)), \qquad x \in \mathbf{R}^n \sim (r_k^{-1})U,$$
$$G_k = \operatorname{graph} u_k, \qquad S_y^k = \{x \colon u_k(x) = y\}.$$

Then we have

$$(*) \qquad Du_k(x) \equiv (Du)(r_kx), \qquad x \in \mathbf{R}^n \sim (r_k^{-1})U,$$
$$(**) \qquad S_0^k = (r_k^{-1})S_{u(r_k\omega_+)},$$

and, by the same argument as we used in the proof of (iv) above, some subsequence of G_k converges to a minimizing current H with $\operatorname{spt} H \subset \overline{E}_+$.

Thus by Lemma (4.2) (keeping in mind that $|Du_k(\omega_+)| \to \infty$ by construction) and the uniqueness result of [13] we have

$$H = (\lambda)S_+ \times \mathbf{R},$$

where $\lambda > 0$ is such that $\lambda\omega_+ \in S_+$. Therefore we have shown that a subsequence of G_k converges, locally in the C^2 sense, to $(\lambda)S_+ \times \mathbf{R}$, and hence for any sequence $\lambda_k \downarrow 0$, $(\lambda_k)_\# G_k$ converges to $C \times \mathbf{R}$ in the C^2 sense near points of $C \times \mathbf{R}$. In view of (v) and the invariance of the C^2_* norm under changes of scale, it then also follows directly that $\inf_{S_0^k} |Du_k| \to \infty$. By $(*)$ and $(**)$ these facts contradict our definition of r_k. Thus (3) is proved.

Now to prove (ii) we argue as follows. In view of (iv) and (v), if $|Du(x_k)| \leq c$ with $|x_k| \to \infty$, then there are $\theta \in (0, 1)$ and neighborhoods $B_k = B_{\theta|x_k|}(x_k)$ on which $\sup_{B_k} |Du|$ is bounded, so we would deduce that there is a ray $\{r\omega_0 \colon r > 0\}$ with $\omega_0 \in S^{n-1} \sim C$ such that

$$\liminf_{r \to \infty} |Du(r\omega_0)| < \infty.$$

Of course

$$\limsup_{r \to \infty} |\partial u(r\omega_0)/\partial r| = \infty,$$

otherwise $|u(r\omega_0)| \le cr$, thus contradicting (i). Hence for each $K > \liminf_{r\to\infty} |Du(r\omega_0)|$ and each $k = 1, 2, \cdots$ we can select $r_k < s_k$ with $r_k \to \infty$,

$$|Du(r_k\omega_0)| = K, \qquad |Du(s_k\omega_0)| = k,$$
$$|Du(\lambda\omega_0)| \ge K, \qquad r_k < \lambda < s_k.$$

Now suppose without loss of generality that $\omega_0 \in E_+ \cap S^{n-1}$ (rather than in $E_- \cap S^{n-1}$), and let $\varepsilon > 0$ be given. For K large enough (depending on ε), (3) above ensures that a homothety of the set $S_{u(r\omega_0)}$ is within ε of S_+ in the sense described in (5.7) for any $r_k < r < s_k$. Let $u_k(x) = r_k^{-1}(u(r_k x) - u(r_k\omega_0))$ and apply Lemma (4.2) again, together with the bound (v) which we already established above. Then $\psi \in C^2(\Omega)$, graph ψ is closed in $\mathbf{R}^{n+1}$, $\omega_0 \in \Omega$, $\psi(\omega_0) = 0$, $\psi(r\omega_0) > 0$ for $r > 1$, and $|D\psi(r\omega_0)| \ge K$ for $r \ge 1$, so long as $r\omega_0 \in \Omega$. Furthermore $\Omega \subset E_+$ because if $t \in \mathbf{R}$ and $\psi(x) > t$, then $r_k^{-1}(u(r_k x) - u(r_k\omega_0)) \ge t$, so that $u(r_k x)/r_k \ge t + u(r_k\omega_0)/r_k > 0$ for all sufficiently large k. Thus $r_k x \in \{\xi \colon u(\xi) > 0\}$ for sufficiently large k and hence $x \in \overline{E}_+$ by (i).

Now by Lemma (4.3) and the uniqueness result of [13] we see that there are only the following four possibilities for Ω:

(a) $\Omega = E_+$.

(b) Ω is the component of $\mathbf{R}^n \sim (\lambda)S_+$ not containing C for some $\lambda > 0$.

(c) Ω is the region between $(\lambda)S_+$ and $(\mu)S_+$ for some $\mu > \lambda > 0$.

(d) Ω is the region between $\overline{C}$ and $(\lambda)S_+$ for some $\lambda > 0$.

Further, in case (b) we evidently have $\psi \to -\infty$ on approach to $(\lambda)S_+$ from Ω, while in case (c) we have $\psi \to \pm\infty$ on approach to $(\mu)S_+$, $(\lambda)S_+$ respectively, and in case (d) $\psi \to -\infty$ on approach to $\overline{C}$, and $\psi \to +\infty$ on approach to $(\lambda)S_+$. In all cases we have

$$(4) \qquad \lim_{x\to y} |D\psi(x)| = \infty \quad \text{uniformly for } y \text{ in compact subsets of } \partial\Omega,$$
$$|D\psi(x)| \ge K \quad \text{for } |x| \ge R = R(K).$$

These facts are easily checked using (3) and properties (iv) and (v) for u.

In cases (a)–(d), by first extending $D\psi/\sqrt{1+|D\psi|^2}$ to $\overline{\Omega} \sim \{0\}$ by continuity, which is possible by virtue of the curvature estimates (iv), and then extending it to be constant along the connected segments of the sets $\{\lambda\omega \colon \lambda > 0, \lambda\omega \in E_+ \sim \Omega\}$, we get a (weakly) divergence-free vector field ν on E_+ with $\nu = D\psi/\sqrt{1+|D\psi|^2}$ in Ω (so that $|\nu| < 1$ in Ω), and $\nu \to$ the unit normal of C on approach to points of C. We claim that this implies that C is *strictly minimizing* on the side $\overline{E}_+$; C is said to be strictly minimizing on the side $\overline{E}_+$ if there is $\theta > 0$ such that for each $R \geq 1$

$$|C_R| \leq |S| - \theta \qquad (C_R = C \cap B_R(0))$$

whenever S is a hypersurface with $\partial S = \partial C_R$, $S - C_R = \partial[\![U_R]\!]$ for some open $U_R \subset E_+$ and $S \cap B_1(0) = \varnothing$. Of course for such a C the proof of (5.5) given in [13] applies without change to v_+, so we can (and we shall) use (5.5) for v_+.

To prove the strict minimizing of C on the side $\overline{E}_+$ take a minimizer S_R among all surfaces S as described above. We know such S_R exists by the compactness theory for codimension 1 integer multiplicity rectifiable currents; note that $\operatorname{spt} S_R \cap S^{n-1} \neq \varnothing$, otherwise S_R is locally minimizing, and then we get a contradiction, because a suitable homothety of S_+ can be made to lie on one side of $\operatorname{spt} S_R$ and to touch $\operatorname{spt} S_R$ in at least one point. Now if C is not strictly minimizing we have

$$|S_R| - |C_R| \to 0 \quad \text{as } R \to \infty. \tag{5}$$

By virtue of (5) and the fact that S_+ is (up to homothety) the unique minimizer contained in E_+ by [13], we can select a sequence $R_j \to \infty$ such that $S_{R_j} \to S_+$ both in the weak sense of currents and in the Hausdorff distance sense. Now let $\omega \in S_+ \cap S^{n-1}$. By rescaling if necessary, we may assume that $\omega \in \Omega$. Then by the divergence theorem (with S_{R_j} as above, keeping in mind that S_{R_j} contains $\omega_j, \omega_j \to \omega$) we have

$$\int_{S_{R_j}} \nu_j^R \cdot \nu = \int_{C_{R_j}} \nu \cdot \nu = |C_{R_j}|,$$

while $|\nu^{R_j} \cdot \nu| \leq 1 - \varepsilon$ in a fixed neighborhood of ω independent of j, thus

$$\begin{aligned}|C_{R_j}| &\leq (1-\varepsilon)|S_{R_j} \cap B_\rho(\omega)| + |S_{R_j} \sim B_\rho(\omega)| \\ &\leq |S_{R_j}| - \varepsilon|S_{R_j} \cap B_\rho(\omega)| \leq |S_{R_j}| - c\varepsilon\rho^n\end{aligned}$$

for fixed constant $c > 0$. This contradicts (5), hence we have proved the strict minimizing of C in $\overline{E}_+$ as required. Using this we want now to show that Theorem 1 can be applied to establish (ii).

First consider the possibilities (c) and (d). In this case we can take $\rho \equiv \rho_0 > 0$ where ρ_0 is a constant. By virtue of (3) and (4) it is clear that there is a bounded positive function $w \in C^2(\{(x,y) \in C \times \mathbf{R}\colon |x| > \rho_0/2\})$ such that $\operatorname{graph}_{C\times\mathbf{R}} w \subset \operatorname{graph} \psi$ provided K and ρ_0 are sufficiently large. Furthermore in view of (3), (4), and (5.5), it is clear that, for any given $\delta > 0$ and $\varepsilon < \min\{\beta, 1\}$ or $0 < \varepsilon < 1$ in case $\beta = 0$, the hypotheses of Theorem 1 hold with $q(\omega) = |A(\omega)|^2$ and $\rho \equiv \rho_0$, again provided K and ρ_0 are sufficiently large. Notice that in case $\beta = 0$ we need to use Lemma

1 to check hypothesis (2.7). However this contradicts Remark (2.10)(1), so the proof of (ii) is complete in this case.

Next we consider the possibilities (a) and (b). Take $\omega_0 \in E_+$, let $T = (\lambda)S_+$ in case (b) and $T = C$ in case (a), and let

$$\tilde{S}_y = \{x \in \Omega \colon \psi(x) = y\}. \tag{6}$$

Also let $\rho_0(y) > 0$ be such that $\rho_0(y)\omega_0 \in \tilde{S}_y$. In view of (iv) and (v), the weak convergence of graph u_k to graph ψ is actually C^2 convergence locally near points of graph ψ. Since $\psi(\omega_0) = 0$ and $|D\psi(r\omega_0)| \geq K$ for $r \geq 1$ by construction of ψ, using (3) and (4) it follows that if $\varepsilon > 0$ is given and $K \geq K_0$, $K_0 = K_0(\varepsilon)$, then

(7) $\rho_0(y)^{-1}\tilde{S}_y$ is within ε of S_+ in the C^2_* sense of (5.7),

$$\inf_{\tilde{S}_y} |D\psi| \geq \varepsilon^{-1} \tag{8}$$

for all $y \geq 0$. Using this facts in combination with (5.5), we can again check that there is a positive $w \in C^2(\{(x,y) \in C \times \mathbf{R} \colon |x| > \rho(y)/2\})$ with

$$\operatorname{graph}_{C\times\mathbf{R}} w \subset \operatorname{graph} \psi,$$

and the hypotheses of Theorem 1 are satisfied for any $\delta > 0$ provided we take $\rho(y) = \mu\rho_0(\max\{y, 0\})$ and provided we select y_1 and μ sufficiently large. We again need to use Lemma 1 here in the case $\beta = 0$ in order to check (2.7). Then, since $\partial w/\partial y = 1/|D\psi|$, Theorem 1 implies that, for suitable fixed $\omega_0 \in E_+$, $|D\psi(r\omega_0)| \leq cr^{\gamma_1}|\psi|^{\varepsilon}$, which implies $|D|\psi|^{1-\varepsilon}| \leq cr^{\gamma_1}$. For $\varepsilon < 1$ this contradicts the fact that $\psi(r\omega_0) \to -\infty$ as $r \downarrow 0$ in case (a) and as $r \downarrow \lambda_0$ in case (b), where λ_0 is such that $\lambda_0\omega_0 \in (\lambda)S_+$. ($\lambda$ as in (b).) This completes the proof of (ii).

Finally we note that (iii) follows directly from (ii) and (3) above. This completes the proof of Theorem 4.

To conclude this section we want to establish some *growth estimates* for exterior solutions u. For this we need to assume that graph u has tangent cylinder $C \times \mathbf{R}$ at ∞ with $\operatorname{sing} C \subset \{0\}$ as in Theorem 4 above, and in addition we must here assume that C is *strictly minimizing* in the sense of [13] and strictly stable in the sense that the strict inequality $\lambda_1 > -(n-3)^2/4$ (or equivalently $\beta > 0$) holds. All presently known examples of codimension 1 minimizing cones C with $\operatorname{sing} C = \{0\}$ are strictly minimizing and strictly stable. See the discussion in the final section, where many new examples of nonlinear entire solutions are discussed.

Theorem 5. *Suppose U is a bounded open subset of $\mathbf{R}^n$, $u \in C^2(\mathbf{R}^n \sim U)$ satisfies the minimal surface equation on $\mathbf{R}^n \sim U$, and suppose* graph u *has tangent cylinder* $C \times \mathbf{R}$ *at* ∞, *with* $\operatorname{sing} C \subset \{0\}$ *and* C *strictly minimizing and strictly stable. Then for each* $\varepsilon > 0$ *there are constants* $c = c(\varepsilon, u) > 0$ *and* $R_0 = R_0(\varepsilon, u)$ *such that*

$$|Du(x)| \le c|x|^{\gamma_1+\varepsilon}, \qquad |u(x)| \le c|x|^{\gamma_1+1+\varepsilon}$$

for all $|x| \ge R_0$, *and there is a sequence* $\{x_j\}$ *with* $|x_j| \to \infty$ *and*

$$|Du(x_j)| \ge c^{-1}|x_j|^{\gamma_1-\varepsilon}, \qquad |u(x_j)| \ge c^{-1}|x_j|^{\gamma_1+1-\varepsilon}$$

for each $j = 1, 2, \cdots$.

The latter inequalities guarantee that the upper growth bounds are best possible, modulo factors of order $|x|^{\varepsilon}$.

(5.8) Remark. By a result of J. Simons [29, 6.1.7], we have always, since C cannot be a hyperplane by Theorem 3, that $\lambda_1 \le -(n-2)$. This means that the growth exponent γ_1 of $|Du|$ given in the above theorem satisfies

$$\gamma_1 \ge \frac{n-3}{2} - \sqrt{\left(\frac{n-3}{2}\right)^2 - (n-2)}.$$

Some of the examples of entire solutions constructed in the next section have "minimum growth"

$$\frac{n-3}{2} - \sqrt{\left(\frac{n-3}{2}\right)^2 - (n-2)},$$

but some have faster growth. Notice also that

$$\begin{aligned}\frac{n-3}{2} - \sqrt{\left(\frac{n-3}{2}\right)^2 - (n-2)} &> \frac{n-3}{2} - \sqrt{\left(\frac{n-3}{2}\right)^2 - (n-4)} \\ &= \frac{n-3}{2} - \frac{n-5}{2} = 1,\end{aligned}$$

so that the lower growth bounds of [7], [18], [8] are never sharp in the case considered here (when the tangent cylinder $C \times \mathbf{R}$ satisfies $\operatorname{sing} C \subset \{0\}$ and C strictly minimizing and strictly stable).

Proof of Theorem 5. Let $\varepsilon > 0$. By Theorem 4(iii) we know that if $\omega_\pm \in S^{n-1} \cap E_\pm$ respectively, then there is $y_0 = y_0(\varepsilon, \omega_\pm) > 0$ such that for each $y \in \mathbf{R}$ with $\pm y \ge y_0$ the ray $\{\lambda\omega_\pm : \lambda > 0\}$ intersects S_y at a unique point $\rho_0(y)\omega_\pm$ respectively, and

$$\begin{aligned}(\rho_0(y))^{-1}S_y \text{ is within } \varepsilon \text{ of some homothety of } S_\pm \\ \text{in the } C^2_* \text{ sense of (5.7),}\end{aligned} \tag{1}$$

according as $\pm y > y_0$ respectively, and by Theorem 4(ii)

$$|Du| \to \infty \quad \text{as } |x| \to \infty. \tag{2}$$

By virtue of (1) and (2) we then have

$$\text{(3)} \qquad \operatorname{Lip} \rho_0|(z,\infty),\ \operatorname{Lip} \rho_0|(-\infty,-z) \to 0 \quad \text{as } z \to \infty,$$

and for sufficiently large μ and y_0 writing $\tilde{\rho}_0(y) = \max\{\rho_0(y), \rho_0(y_0), \rho_0(-y_0)\}$

$$\text{(4)} \qquad G \cap \{(x,y) \in \mathbf{R}^{n+1} \colon |x| > \mu\tilde{\rho}_0(y)/4\} = \operatorname{graph}_{C\times\mathbf{R}} w,$$

where $w \in C^1(V)$ for some open set $V \subset C \times \mathbf{R}$ with $V \supset \{(x,y) \in C \times \mathbf{R} \colon |x| > (\mu/2)\tilde{\rho}_0(y)\}$ (w, V depending on μ) and

$$\text{(5)} \qquad \begin{aligned} &\lim_{\mu\to\infty} |w|_{C^2_*} = 0, \\ &\qquad \text{where } |w|_{C^2_*} = \sup(|x|^{-1}|w(x)| + |\nabla w(x)| + |x|\,|\nabla^2 w(x)|). \end{aligned}$$

Notice that we are able to assert that the C^2_*-norm, rather than merely the C^1_*-norm, is small by virtue of the standard interior regularity theory for uniformly elliptic quasilinear equations.

In view of (3), (4), and (5), for any given $\delta > 0$ we can select μ and y_0 so that if ρ is defined by $\rho(y) = \mu\tilde{\rho}_0(y)$, then

$$\text{(6)} \qquad \operatorname{Lip} \rho < \delta, \quad y \operatorname{sgn} \rho'(y) \geq 0 \quad \text{a.e. } y \in \mathbf{R},$$

$$\text{(7)} \qquad G \cap \{(x,y) \colon |x| > \rho(y)/4\} \subset \operatorname{graph} w \subset G, \qquad |w|_{C^2_*} < \delta.$$

By virtue of (6), (7), and (5.5), after selecting y_1 sufficiently large, we can apply Theorem 1 with w in place of u and $v = \frac{\partial w}{\partial y}$. Notice that

$$v(x,y) = \frac{1}{|Du(\xi)|},$$

where, for given $(x,y) \in V, \xi \in \mathbf{R}^n \sim U$ is such that $y = u(\xi)$ and $\xi = (x,y) + \nu_C(x) w(x,y)$, with ν_C the unit normal of C pointing into E_+, so that the first conclusion of Theorem 1 implies

$$\text{(8)} \qquad |Du(\xi)| \leq c|u(\xi)|^{\varepsilon}|\xi|^{\gamma_1},$$

provided $|u(\xi)|$ is sufficiently large, and $|\xi| \geq \mu \operatorname{dist}(S_{u(\xi)}, \{0\})$ for a sufficiently large constant μ. But by virtue of parts (iii) and (v) of Theorem 4 this gives

$$\text{(9)} \qquad |Du(x)| \leq c(1 + |u(x)|)^{\varepsilon}|x|^{\gamma_1}, \qquad |x| \text{ sufficiently large.}$$

This can be written

$$|D(1 + |u(x)|)^{1-\varepsilon}| \leq c|x|^{\gamma_1}, \qquad |x| \text{ sufficiently large,}$$

and by integration this yields

$$|u(x)|^{1-\varepsilon} \leq c|x|^{\gamma_1+1}.$$

By substituting this back into (9) we then have the required upper bounds.

Finally, the required lower bounds follow in a similar way from the final two limit statements in Theorem 1.

6. Examples of entire solutions

In this section we use the terminology that a cone C is an *isoparametric* cone if $\Sigma \equiv C \cap S^{n-1}$ is minimal and part of a smooth family of isoparametric hypersurfaces in S^{n-1}; see, e.g., [17], [10] for a discussion and terminology. Any homogeneous codimension 1 minimal cone C with $\operatorname{sing} C = \{0\}$ (for a list of such see [14]) is automatically such a cone. Some examples of nonhomogeneous minimizing isoparametric cones with $\operatorname{sing} C = \{0\}$ are given in [10].

We also note here that all the examples of minimizing cones given in [14] are also *strictly minimizing*. See the discussion in [13]. The list in [14] includes all *homogeneous* isoparametric minimal cones. Concerning which of these cones are minimizing, most cases are settled in [14]; note that the classes 6, 7, 9, 10 of Table 1 of [14] are all unstable, hence not minimizing. Also, to be compatible with the text of [14], V^2 for class 5 of Table 1 should be written $((xy)^5(x^2-y^2)^4)^2$. Simoes [30] proved that the cone over $S^1 \times S^5$, which is not settled in [14], is not minimizing, and F. H. Lin [15] proved that the cone over $S^2 \times S^4$ is strictly minimizing. Of the remaining cases not explicitly settled in [14], each is either unstable or strictly minimizing. (Private communication of B. Solomon.) Thus all homogeneous minimizing cones C with $\operatorname{sing} C = \{0\}$ are automatically strictly minimizing. Furthermore, by virtue of the alternate characterization of strictly minimizing given in [13, Theorem 3.2(v)], the argument used in [10] to prove minimizing is easily modified to prove strictly minimizing; that is, the isoparametric cones shown in [10] to be minimizing are all strictly minimizing. We note also that since any isoparametric minimal hypersurface $\Sigma \subset S^{n-1}$ has second fundamental form of constant length whose square is given by $p(n-2)$ where $p = 0, 1, 2, 3, 5$ (see, e.g., [19] or [17]), and since there are no integer solutions $n \geq 3$ of the equation $p(n-2) = (n-3)^2/4$ for the cases $p = 1, 2, 3, 5$, we thus see that all stable isoparametric cones C with $\operatorname{sing} C = \{0\}$ are automatically strictly stable. In particular this means that all the minimizing isoparametric cones C with $\operatorname{sing} C = \{0\}$ are strictly stable. Hence in particular all the examples of minimizing cones in [14] and [10] are both strictly stable and strictly minimizing.

Our main aim here is to prove the following:

Theorem 6. *Suppose C is a strictly minimizing isoparametric cone in $\mathbf{R}^n$ with* $\operatorname{sing} C = \{0\}$. *Then there is an entire solution u of the minimal surface equation in $\mathbf{R}^n$ having $C \times \mathbf{R}$ as tangent cylinder at ∞. Furthermore it can be arranged that* graph u *inherits all the symmetries of C; that is, $u \circ g = u$ for each isometry g of $\mathbf{R}^n$ with $g(C) = C$.*

Remarks. (1) In view of the discussion preceding the theorem we thus show that entire solutions with tangent cylinder $C \times \mathbf{R}$ exist for any homogeneous minimizing cone C with $\operatorname{sing} C = \{0\}$, and for any of the isoparametric minimizing cones C shown to exist in [10].

(2) Of course the growth estimates of Theorem 5 apply to all these examples because they are automatically strictly stable by the discussion preceding the theorem.

We shall need the following technical result.

(6.1) Lemma. *Suppose C is strictly minimizing and strictly stable, with* $\operatorname{sing} C = \{0\}$, *and let $w \in C^2(\Omega)$ be as in Lemma* (4.3) *with $0 \in \Omega$ and $\partial\Omega \cap \overline{C} = \emptyset$. Then $\Omega = \mathbf{R}^n$ (so that the alternatives* (ii) *and* (iii) *of Lemma* (4.3) *cannot occur).*

Remark. The above lemma suffices for our present purposes, but it is perhaps worth mentioning that with only minor modifications of the argument below we could prove the same conclusion without the hypothesis $0 \in \Omega$, provided we assume a priori that either $\partial\Omega \cap \overline{C} = \emptyset$ or $\overline{C}$ is a component of $\partial\Omega$. Also, in this more general case it is enough to assume that C is merely minimizing in case $\Omega \subset$ one of the components of $\mathbf{R}^n \sim \overline{C}$, because in this case we can use an argument as in the proof of Theorem 4(ii) to deduce that C is automatically strictly minimizing on one side of C; recall that this argument did not require strict stability.

Proof of Lemma (6.1). The proof involves an application of Theorem 1 similar to that in the proof of Theorem 4(ii). Assume $\Omega \neq \mathbf{R}^n$. Without loss of generality we can assume that $\partial\Omega \cap E_+ \neq \emptyset$. Since $\partial\Omega \cap \overline{C} = \emptyset$ and $0 \in \Omega$, in view of the uniqueness result of [13] there are only the two possibilities:

(1) Ω is the region between $(\mu)S_-$ and $(\lambda)S_+$ for some $\lambda, \mu > 0$;

(2) Ω is the component of $\mathbf{R}^n \sim (\lambda)S_+$ containing C.

Replacing w by $-w$ if necessary, we may also assume that $w \to +\infty$ (rather than $-\infty$) on approach to $(\lambda)S_+$. Notice that in either case (1) or case (2) $C \times \mathbf{R}$ is the unique tangent cylinder for graph u at ∞. Then an examination of the proof of Theorem 4 will show that, with only minor modifications to the arguments (applying Lemma (4.2) under hypothesis (ii) instead of (i)) for each $\varepsilon > 0$ and each $\omega_\pm \in E_\pm$ there is $K = K(\varepsilon, w, \omega_\pm)$ such that

$$(3) \qquad |Dw(r\omega_\pm)| \ge K \Rightarrow \begin{cases} (\operatorname{dist}(S_y, 0))^{-1} S_y \text{ is within } \varepsilon \text{ of } S_\pm, \\ |Dw| \ge \varepsilon^{-1} \text{ everywhere on } S_y, \end{cases}$$

whenever $r\omega_\pm \in \Omega$ and $y = w(r\omega_\pm)$ respectively. (Cf. (3) in the proof of Theorem 4.) Also by making the appropriate minor modifications to the proof of Theorem 4(ii), we have

$$(4) \qquad |Dw(x)| \to \infty \quad \text{as } |w(x)| + |x| \to \infty, \ x \in \Omega.$$

Note that by combining (3) and (4) we deduce that for each $\varepsilon > 0$ there is $y_0 = y_0(\varepsilon)$ such that

$$(5) \qquad S_y \text{ is within } \varepsilon \text{ of } S_\pm \text{ in case } \pm y > y_0 \text{ respectively.}$$

Then let $\rho_0(y)$ be such that $\rho_0(y)\omega_\pm \in S_y$ whenever $\pm y > y_0$ respectively, and note that (4) and (5) imply that

$$(6) \qquad \operatorname{Lip} \rho_0|(z, \infty), \operatorname{Lip} \rho_0|(-\infty, -z) \to 0 \quad \text{as } z \to \infty,$$

and hence, with $\tilde{\rho}_0(y) = \max\{\rho_0(y), \rho_0(y_0), \rho_0(-y_0)\}$, for any given $\delta > 0$ we can choose μ such that

$$(7) \qquad G \sim \{(x, y) \colon |x| > 2\mu\tilde{\rho}_0(y)\} \subset \operatorname{graph} \phi \subset G$$

for some ϕ on $\{(x, y) \in C \times \mathbf{R} \colon |x| > \mu\rho_0(y)\}$ with $|\phi|_{C^2_*} < \delta$. Then, in view of (4), (5), (6), and (5.5), we can apply Theorem 1 with $w = \phi$ and $\rho = \mu\tilde{\rho}_0$ as in the proof of Theorem 4(ii) to contradict the fact that $w \to \infty$ on approach to $\partial\Omega$. Thus $\Omega = \mathbf{R}^n$ as required.

Proof of Theorem 6. Since it is slightly simpler, we first consider the proof for the case where C is a homogeneous minimizing cone with $\operatorname{sing} C = \{0\}$, so that C is either isometric to the cone over $S^2 \times S^4$ or else isometric to one of the examples of [14]. In this case the proof begins by constructing a special sequence of solutions u_k^0 of the minimal surface equation on the unit ball.

Specifically, using the usual notation $\Sigma = C \cap S^{n-1}$ so that Σ is a connected compact embedded submanifold of S^{n-1}, define u_k^0 to be the solution of the minimal surface equation on $B_1(0)$ with boundary data

$$(1) \qquad u_k^0 = \begin{cases} +k & \text{on } A_+, \\ -k & \text{on } A_-, \end{cases}$$

where $A\pm$ are the components of $S^{n-1} \sim \Sigma$. It is standard that such $u_k^0 \in C^2(B_1(0) \cup A_+ \cup A_-)$ exists and is unique (see, e.g., [12, Chapter 16]). By uniqueness u_k^0 inherits all symmetries of C.

By the argument of [21, pp. 248–249] we have

$$(2)\qquad \begin{cases} k - |u_k^0(0)| \to \infty & \text{as } k \to \infty, \\ \sup\limits_{B_\rho(0)} |Du_k^0| \to \infty & \text{as } k \to \infty \end{cases}$$

for each fixed $\rho > 0$.

By examining the list of [14] one readily checks that all codimension 1 homogeneous minimizing cones are invariant under the isometry $x \mapsto -x$. Hence either $-A_+ = A_-$ or $-A_+ = A_+$. In these cases we have respectively, using again the uniqueness of the solution u_k^0,

$$(3)\qquad \begin{cases} \text{either} & u_k^0(-x) = -u_k^0(x)\ \forall x \in B_1(0), \\ \text{or} & u_k^0(-x) = u_k^0(x)\ \forall x \in B_1(0). \end{cases}$$

If the first alternative in (3) holds, then it is straightforward to check, using the invariance of u_k^0 under the isometries which leave C invariant, that $u_k^0(x) \equiv 0$ on $C \cap B_1(0)$, and hence that $Du_k^0(0) = 0$ by virtue of the fact that 0 is a singular point of C. In case the latter alternative in (3) holds we have $Du_k^0(x) \equiv -Du_k^0(-x)$ $\forall x \in B_1(0)$, and hence again $Du_k^0(0) = 0$. Thus in any event we have $Du_k^0(0) = 0$, and we can choose $0 < \rho_k < 1$ such that

$$(4)\qquad \sup_{B_{\rho_k}(0)} |Du_k^0| = 1 \quad \text{and} \quad |Du_k^0| < 1 \quad \text{at each point of } B_{\rho_k}(0).$$

Notice that $\rho_k \to 0$ by (2).

We now define

$$u_k(x) = \rho_k^{-1}(u_k^0(\rho_k x) - u_k^0(0)), \qquad x \in B_{\rho_k^{-1}}.$$

Of course u_k satisfies the minimal surface equation on $B_{\rho_k^{-1}}$, and

$$(5)\qquad u_k(0) = 0, \quad Du_k(0) = 0, \quad \sup_{B_1(0)} |Du_k| = 1.$$

Since $\rho_k \to 0$, by (5) and Lemma (4.2) we can find a subsequence $\{k'\}$ (henceforth denoted simply $\{k\}$), a connected domain $\Omega \subset \mathbf{R}^n$, and a $C^2(\Omega)$ solution ψ of the minimal surface equation on Ω with the properties $u_k \to \psi$ locally in $C^2(\Omega)$, graph ψ is closed, and

$$(6)\qquad \overline{B}_1(0) \subset \Omega, \quad \psi(0) = 0, \quad D\psi(0) = 0, \quad \max_{\overline{B}_1(0)} |D\psi| = 1.$$

We claim that Ω satisfies $\partial\Omega \cap \overline{C} = \varnothing$. Indeed this is clear because by construction Ω is invariant under the set of all isometries which leave C invariant (this is a transitive set of isometries of Σ), and because by Lemma

(4.2) each component of $\partial\Omega$ is a minimizing hypersurface. Then the fact that $0 \notin \partial\Omega$ would tell us that if $\partial\Omega \cap \overline{C} \neq \emptyset$ there would be a minimizing hypersurface different than C but having the same boundary as $C \cap B_R(0)$ for some $R > 0$, thus contradicting the fact that $C \cap \partial B_R(0)$ is a boundary of uniqueness for minimizing hypersurfaces of multiplicity 1. Hence, by Lemma (6.1), $\Omega = \mathbf{R}^n$. Finally, we have to show that $\pm C \times \mathbf{R}$ is the tangent cylinder for graph ψ at ∞. This is a consequence of the fact that ψ is invariant under all isometries holding C fixed, together with the fact that ψ is not linear—because $\psi(0) = 0$, $D\psi(0) = 0$, and $\max_{\overline{B}_1(0)} |D\psi| = 1$ by construction of ψ. Thus (see the discussion at the beginning of §5) $|D\psi|$ is unbounded and graph ψ has tangent cylinders at ∞. Then let $\tilde{C} \times \mathbf{R}$ be any tangent cylinder of graph ψ. $\tilde{C}$ is invariant under all the isometries which leave C invariant, and $\operatorname{reg}\tilde{C}$ is connected by [5], so it follows that either $\tilde{C} = \pm C$ or else $\operatorname{spt}\tilde{C} \cap \operatorname{spt} C = \{0\}$. However since both $\tilde{C}$ and C are minimizing and $\operatorname{sing} C = \{0\}$, it is standard that the latter alternative is impossible (see, e.g., [5]). Thus $\tilde{C} = \pm C$ as required, and the proof is complete.

In the general case when C is a strictly minimizing isoparametric cone with $\operatorname{sing} C = \{0\}$, we first let $A_\pm$ be the two components of $S^{n-1} \sim \Sigma$ as before, and

$$d(\omega) = \pm\operatorname{dist}(\omega, \Sigma), \qquad \omega \in A_\pm \text{ resp.},$$

where distance is geodesic distance measured in S^{n-1}. Then (see, e.g., the discussion in [17], [10]) the image of d is an interval $[\alpha_-, \alpha_+]$, and $\Gamma_\pm \equiv d^{-1}(\alpha_\pm)$ are "focal submanifolds" of dimension $\leq n-3$. The transformation $T\colon x \mapsto y = re^{i\theta}$, where $r = |x|$ and $\theta = d(x/|x|)$, takes $B_1(0)$ to the sector

$$D = \{re^{i\theta}\colon 0 < r < 1, \alpha_- < \theta < \alpha_+\},$$

and the standard Euclidean metric $dy_1^2 + dy_2^2$ for D pulls back to the standard Euclidean metric $dx_1^2 + \cdots + dx_n^2$ for $B_1(0)$. Thus in particular if $u = v \circ T$, where $v \in C^1(D)$, and if $V(y) = |y|^{n-2}\mathscr{H}^{n-2}(T^{-1}(y/|y|))$, then

$$\int_D \sqrt{1 + |Dv|^2} V(y)\, dy = \int_{B_1(0)} \sqrt{1 + |Du|^2}. \tag{7}$$

Thus let v_k be a bounded $C^\infty(D)$ solution of the equation

$$\operatorname{div} \frac{VDv}{\sqrt{1 + |Du|^2}} = 0$$

satisfying boundary data

$$v(e^{-\theta}) = \begin{cases} -k & \text{if } \alpha_- < \theta < 0, \\ +k & \text{if } 0 < \theta < \alpha_+. \end{cases}$$

We get such a solution by applying standard quasilinear existence theory for the Dirichlet problem on an increasing sequence of proper subdomains. Then in view of (7), the function $u_k^0 \equiv v_k \circ T$ is a bounded solution of the minimal surface equation on $B_1(0) \sim (K_+ \cup K_-)$ satisfying the boundary conditions (1), where $K_\pm$ denote the cones over $\Gamma_\pm$. Since v_k is bounded and $\mathscr{H}^{n-1}K_\pm = 0$, it is easy to show that u_k^0 extends to give a $C^2(B_1(0))$ solution (still denoted u_k^0) satisfying (1). (See, e.g., the discussion in the Appendix of [22].) Next we need to note that any isoparametric cone is invariant under the isometry $x \mapsto -x$ by virtue of the characterization given in [17, Satz C]; note particularly that, with q as in [17], if g is odd then Σ must be given as the *zero* set of the relevant polynomial. In view of these facts we can directly modify the argument from the homogeneous case to show that this solution has vanishing gradient at 0. The remainder of the argument is similar to the homogeneous case; the proof that the limit function ψ has domain Ω satisfying $\partial\Omega \cap \overline{C} = \varnothing$ follows easily from the fact that $\psi = \phi \circ T$ (by construction) for suitable ϕ in D. So again we can apply Lemma (6.1) to prove $\Omega = \mathbf{R}^n$. In fact that graph ψ has tangent cylinder $C \times \mathbf{R}$ follows essentially as in the homogeneous case, except that we use the fact that ψ can be written in the form $\phi \circ T$ in place of the previous invariance under a transitive group of isometries.

References

[1] W. Allard, *On the first variation of a varifold*, Ann. of Math. (2) **95** (1972) 417–491.

[2] L. Bers, *Isolated singularities of minimal surfaces*, Ann. of Math. (2) **53** (1951) 364–386.

[3] E. Bombieri, E. De Giorgi & E. Giusti, *Minimal cones and the Berstein problem*, Invent. Math. **7** (1969) 243–268.

[4] E. Bombieri, E. De Giorgi & M. Miranda, *Una maggiorazzione a priori relativa alle ipersuperfici minimali non parametrici*, Arch. Rational Mech. Anal. **32** (1969) 255–267.

[5] E. Bombieri & E. Giusti, *Harnack's inequality for elliptic equations on minimal surfaces*, Invent. Math. **15** (1972) 24–46.

[6] L. Caffarelli, R. Hardt & L. Simon, *Minimal surfaces with isolated singularities*, Manuscripta Math. **48** (1984) 1–18.

[7] L. Caffarelli, L. Nirenberg & J. Spruck, *On a form of Berstein's Theorem*, preprint.

[8] K. Ecker & G. Huisken, *A Berstein result for minimal graphs of controlled growth*, preprint.

[9] H. Federer, *Geometric measure theory*, Springer, Berlin, 1969.

[10] D. Ferus & H. Karcher, *Non-rotational minimal spheres and minimizing cones*, Comment. Math. Helv. **60** (1985) 247–269.

[11] R. Finn, *Isolated singularities of non-linear partial differential equations*, Trans. Amer. Math. Soc. **73** (1953) 383–404.

[12] D. Gilbarg & N. Trudinger, *Elliptic partial differential equations of second order*, 2nd ed., Springer, Berlin, 1983.

[13] R. Hardt & L. Simon, *Area minimizing hypersurfaces with isolated singularities*, J. Reine Angew. Math. **362** (1985) 102–129.

[14] H. Lawson, *The equivariant Plateau problem and interior regularity*, Trans. Amer. Math. Soc. **173** (1973) 231–149.
[15] F.-H. Lin, *Minimality and stability of minimal hypersurfaces on* $\mathbf{R}^n$, preprint.
[16] M. Miranda, *Superfici minime illimitate*, Ann. Scuola Norm. Sup. Pisa **4** (1977) 313–322.
[17] H. Münzner, *Isoparametrische Hyperflächen in Sphären*, Math. Ann. **251** (1980) 57–71.
[18] J. C. C. Nitsche, to appear.
[19] C. K. Peng & C. L. Terng, *Minimal hypersurfaces of spheres with constant scalar curvature*, Seminar on minimal submanifolds (E. Bombieri, ed.), Ann. of Math. Studies, No. 103, Princeton University Press, Princeton, NJ, 1983.
[20] M. Protter & H. Weinberger, *Maximum principles in differential equations*, Prentice-Hall, Englewood Cliffs, NJ, 1967.
[21] R. Schoen, L. Simon & F. Almgren, *Regularity and singularity estimates on hypersurfaces minimizing parametric elliptic variational integrals*, Acta. Math. **139** (1977) 217–265.
[22] L. Simon, *On some extensions of Bernstein's Theorem*, Math. Z. **154** (1977) 265–273.
[23] ——, *Asymptotics for a class of non-linear evolution equations, with applications to geometric problems*, Ann. of Math. (2) **118** (1983) 525–571.
[24] ——, *Lectures on geometric measure theory*, Proc. Centre for Math. Analysis, No. 3, Australian National University, 1983.
[25] ——, *Survey lectures on minimal submanifolds* (E. Bombieri, ed.), Ann. of Math. Studies, No. 103, Princeton University Press, Princeton, NJ, 1983, 3–52.
[26] ——, *Isolated singularities of extrema of geometric variational problems*, Harmonic Mappings and Minimal Immersions, Lecture Notes in Math., Vol. 1161, Springer, Berlin, 1985, 206–277.
[27] ——, *Asymptotic behaviour of minimal graphs over exterior domains*, Ann. Inst. H. Poincaré **4** (1987) 231–242.
[28] ——, *A strict maximum principle for area minimizing hypersurfaces*, to appear.
[29] J. Simons, *Minimal varieties in Riemannian manifolds*, Ann. of Math. (2) **88** (1968) 62–105.
[30] P. Simoes, *On a class of minimal cones in* $\mathbf{R}^n$, Bull. Amer. Math. Soc. **80** (1974) 488–489.

STANFORD UNIVERSITY