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CYCLES AND RELATIVE CYCLES IN ANALYTIC K-HOMOLOGY

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0. Introduction

In this paper we continue the study of elliptic operators and K-homology, pursued by the first two authors in [5], [6], [7]. We particularly focus on the concept of relative cycles, their production from elliptic differential operators on manifolds with boundary, the behavior of such relative cycles under the boundary map in the exact sequence for K-homology, and implications of such calculations to various aspects of K-homology and index theory.

For orientation, we recall how (ordinary) cycles for $K_0(M)$ and $K_1(M)$, when M is a compact manifold (without boundary), are determined, respectively, by an elliptic pseudodifferential operator B on M or by an elliptic selfadjoint pseudodifferential operator A on M. Say B is of order m; we write $B \in \mathrm{OPS}^m(M)$, using the notation for pseudodifferential operators given in [28], [40]. Typically one has $m \geq 0$. If $m > 0$, replace B by $B_0 = B(1 + B^*B)^{-1/2} \in \mathrm{OPS}^0(M)$, so we can suppose $m = 0$. We suppose B maps sections of a vector bundle E_0 to sections of E_1, so (if $m = 0$) $B\colon L^2(M, E_0) \to L^2(M, E_1)$. The cycle determined by B is the pair (σ, B) where, for $f \in C(M)$, $\sigma(f) = \sigma_0(f) \oplus \sigma_1(f)$ is the operation of multiplication of sections of $E_0 \oplus E_1$ by the scalar function f (so σ_j is

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a representation of $C(M)$ on $L^2(M, E_j)$), and B almost intertwines these representations, in the sense that

$$\sigma_1(f)B - B\sigma_0(f) \in \mathscr{K} \tag{0.1}$$

for each $f \in C(M)$, where $\mathscr{K}$ denotes the space of compact operators (from $L^2(M, E_0)$ to $L^2(M, E_1)$). These properties were abstracted to define general cycles for $K_0(M)$ in Atiyah [1]. There remained the problem of putting an appropriate equivalence relation on the cycles to produce the correct group, which was solved independently in the works of Kasparov [29] and of Brown, Douglas, and Fillmore [11].

The way an elliptic selfadjoint pseudodifferential operator A acting on sections of a vector bundle E produces a cycle for $K_1(M)$ is the following. Let P be the orthogonal projection on $L^2(M, E)$ associated with the positive spectrum of A. Then P is a pseudodifferential operator; $P \in \mathrm{OPS}^0(M)$. We define a homomorphism

$$\sigma\colon C(M) \to \mathscr{Q}(H), \tag{0.2}$$

where H is the range of P and $\mathscr{Q}(H)$ is the Calkin algebra $\mathscr{L}(H)/\mathscr{K}(H)$, by the recipe

$$\sigma(f) = \pi(PM_f), \tag{0.3}$$

where M_f is the multiplication operator, $M_f u = fu$, and π is the natural quotient map of $\mathscr{L}(H)$ onto $\mathscr{Q}(H)$. Such a homomorphism defines a cycle for $K_1(M)$, denoted $\mathrm{Ext}(M)$ in [11], [24], where the appropriately generalized notion of cycle is set down, as well as the equivalence relation giving the group $K_1(M)$.

The groups $K_j(M)$, $j = 0$ or 1, are special cases of $K^j(\mathfrak{A})$ for a C^*-algebra with unit $\mathfrak{A}$, when $\mathfrak{A} = C(M)$. Cycles for $K^0(\mathfrak{A})$ in general are special cases of cycles for $K^0(\mathfrak{A}, \mathfrak{A}/\mathscr{J}) \approx KK(\mathscr{J}, \mathbb{C})$, described in §1, when $\mathscr{J} = \mathfrak{A}$, and cycles for $K^1(\mathfrak{A})$ are special cases of cycles for $KK^1(\mathscr{J}, \mathbb{C})$, with $\mathscr{J} = \mathfrak{A}$, described at the end of §2. In this paper we will be concerned exclusively with commutative C^*-algebras $\mathfrak{A} = C(X)$ for X a compact metric space, with particular emphasis on the case $X = \overline{M}$, a compact manifold with boundary.

Relative K-homology groups, the principal objects of study in this paper, play an important role in the study of K-homology, via their appearance in K-homology exact sequences, such as the segment

$$K_0(\partial M) \to K_0(\overline{M}) \to K_0(\overline{M}, \partial M) \xrightarrow{\partial} K_1(\partial M) \to K_1(\overline{M}). \tag{0.4}$$

The boundary map on $K_0(\overline{M}, \partial M)$ was defined in [8] and shown to be consistent with the boundary map of Kasparov [30] in the exact sequence

$$\begin{aligned} KK(\mathfrak{A}/\mathscr{J}, \mathbb{C}) \to KK(\mathfrak{A}, \mathbb{C}) \to KK(\mathscr{J}, \mathbb{C}) \\ \xrightarrow{\partial} KK^1(\mathfrak{A}/\mathscr{J}, \mathbb{C}) \to KK^1(\mathfrak{A}, \mathbb{C}) \end{aligned} \tag{0.5}$$

in the case $\mathfrak{A} = C(\overline{M})$ and $\mathscr{J} = C_0(M)$, the set of continuous functions vanishing on ∂M. As a consequence, as shown in [8], there follows the isomorphism

$$K_0(\overline{M}, \partial M) \approx KK(C_0(M), \mathbb{C}). \tag{0.6}$$

We will build upon these results in this paper, obtaining numerous consequences by studying relative cycles associated to first order elliptic differential operators on a compact manifold with boundary.

In §1, after recalling the definitions of cycles for $K_0(M, \partial M)$, and more generally for $K^0(\mathfrak{A}, \mathfrak{A}/\mathscr{J})$, and for the Kasparov groups $KK(\mathscr{J}, \mathbb{C})$, we present a general result on compactness of commutators which enables us to establish that various constructions involving (closed extensions of) first order elliptic differential operators do indeed produce relative cycles. This result, while fairly simple, plays a crucial role in the following sections.

§§2 and 3 contain the results associating an element of $KK(C_0(M), \mathbb{C})$ and of $K_0(M, \partial M)$, respectively, to a first order elliptic differential operator D on a manifold M. In §2, M can be any Riemannian manifold, and one can take any closed extension of D. A localization principle, exploiting finite propagation speed for solutions to symmetric hyperbolic equations, is used to establish that the element of $KK(C_0(M), \mathbb{C})$ so obtained is *independent* of the choice of such an extension. In §3, M is required to be a compact manifold with boundary, and some restrictions are placed on the class of closed extensions of D considered. Nevertheless, a wide variety of extensions are treated, including minimal and maximal extensions, and extensions defined by coercive local boundary conditions, when they exist. One important operator treated here is $D = \overline{\partial} + \mathfrak{D}$, satisfying the zero-order part of the $\overline{\partial}$-Neumann condition, in case M is a strongly pseudoconvex complex manifold, or more generally a weakly pseudoconvex domain for which there are appropriate subelliptic estimates. Using the results of §2, together with the isomorphism (0.6), we are able to draw the nontrivial conclusion that different boundary conditions on an elliptic differential operator D, within the class of those considered in §3, lead to the same element of $K_0(M, \partial M)$. The major results of the succeeding sections will depend heavily on this fact.

§4 is devoted to the study of the image under the boundary map ∂: $K_0(M, \partial M) \to K_1(\partial M)$ of the relative classes defined by first order elliptic differential operators. The map on the level of cycles depends explicitly on the choice of boundary condition on D. Utilizing the independence of the boundary condition of the class $[D]$, we thereby obtain useful identities amongst elements of $K_1(\partial M)$. When D is given its maximal extension, $\partial[D] \in K_1(\partial M)$ is seen to be the extension determined by the Calderon projector associated to D. In case D is a Dirac operator on M (provided M has a spin^c-structure), $\partial[D]$ coincides with the class in $K_1(\partial M)$ determined by the Dirac operator on ∂M; there is a generalization when D is an operator of "Dirac type". Study of other extensions of D leads to the identification of other cycles which are equivalent to this one in $K_1(\partial M)$.

One of the most important applications of this is the following. Let M be a compact complex manifold with boundary, and suppose M is strongly pseudoconvex. Then the operator $D = \overline{\partial} + \mathfrak{D}\colon \Lambda^{0,\text{even}}(\overline{M}) \to \Lambda^{0,\text{odd}}(\overline{M})$ can be regarded as giving an element of $K_0(M, \partial M)$ in two ways, either by taking the maximal extension or by assigning the 0-order part of the $\overline{\partial}$-Neumann boundary condition. The identity we obtain in this case is

$$[D_{\partial M}] = [\tau_M] \quad \text{in } K_1(\partial M), \tag{0.7}$$

where $D_{\partial M}$ is the Dirac operator on ∂M with its natural spin^c-structure, and $[\tau_M]$ is the Toeplitz extension of ∂M, associated with the "Bergmann projector" onto the space of L^2 holomorphic functions on M. We note that, given (0.7), Boutet de Monvel's index theorem for Toeplitz operators follows directly, as an application of the intersection product $K^1(\partial M) \times K_1(\partial M) \to K_0(\partial M)$, so (0.7) can be viewed as a refinement of this index theorem. Furthermore, (0.7) holds whenever M is a weakly pseudoconvex domain satisfying the condition that the $\overline{\partial}$-Neumann problem is subelliptic with loss of less than two derivatives on $(0,p)$-forms for $p \neq 0$. Thus Boutet de Monvel's index theorem is extended to this class of weakly pseudoconvex domain.

The ball bundle B^*X of a compact manifold can be given the structure of a strongly pseudoconvex domain. In that case, a construction of Boutet de Monvel [13], [14] gives easily

$$\mathscr{P}_X = [\tau_{B^*X}] \quad \text{in } K_1(S^*X), \tag{0.8}$$

where S^*X is the cosphere bundle of X and $\mathscr{P}_X$ is the "pseudodifferential operator extension" of S^*X, defined by $\tau\colon C(S^*X) \to \mathscr{Q}(L^2(X))$ where, for $p \in C^\infty(S^*X)$, $\tau(p)$ is an operator $P \in \mathrm{OPS}^0(X)$ with principal symbol equal to p, which is well defined mod $\mathscr{K}$. Comparing (0.7) and (0.8) then gives the important identity

$$\mathscr{P}_X = [D_{S^*X}], \tag{0.9}$$

which will yield further consequences in §6.

Applying results of §4 in conjunction with the Bott map, in §5 we obtain results on the boundary map $\partial : KK^1(C_0(M), \mathbb{C}) \to K_0(\partial M)$. Again, when D is a symmetric operator on M of Dirac type, we obtain

$$\partial[D] = [D^\#] \quad \text{in } K_0(\partial M), \tag{0.10}$$

for an associated operator $D^\#$ on ∂M of Dirac type. From the exactness of the sequence

$$KK^1(C_0(M), \mathbb{C}) \xrightarrow{\partial} K_0(\partial M) \to K_0(\overline{M}) \tag{0.11}$$

it follows that the image of $[D^\#]$ in $K_0(\overline{M})$ is zero; following this with the map $\overline{M} \to \text{pt.}$, giving $K_0(M) \to \mathbb{Z}$, we obtain the consequence

$$\operatorname{Index} D^\# = 0. \tag{0.12}$$

Thus the identity (0.10) is a refinement, in K-homology, of the cobordism invariance of the index of elliptic operators, which is the main content of Chapter XVII of the notes [35] on the Atiyah-Singer index theorem.

In §6 we make further contacts with index theory. To an elliptic operator $B \in \mathrm{OPS}^0(M)$ we associate a twisted Dirac operator D_E on M, the double of the ball bundle in T^*M, having the property that

$$\pi_*([D_E]) = [B] \quad \text{in } K_0(M). \tag{0.13}$$

In particular the operators B and D_E have the same index. The proof of (0.13) given in §6 makes use of the identity (0.9), together with the Bott map.

The boundary conditions studied in §3 and exploited in the later sections are all local boundary conditions. Nonlocal boundary conditions give rise to operators exhibiting different behavior. In Appendix A we make some brief comments on nonlocal boundary conditions of Atiyah-Patodi-Singer type.

We remark that many of the results on compactness of commutators have considerably sharper versions. Typically commutators shown here to be compact actually map $L^2(M)$ to the Sobolev space $H^1(M)$ (of course weaker results hold in cases associated with the $\overline{\partial}$-Neumann boundary condition). Proofs of these Sobolev space results are somewhat longer than the arguments given here. Such results are potentially of use in the production of p-summable Fredholm modules and may play a role in cyclic cohomology, but these sharper results are not needed for the applications to K-homology given here, so we have restricted attention to the cruder compactness results.

To close this introduction, we mention a notational convention we use for a pair of function spaces. Let M be a noncompact manifold. $C_0^\infty(M)$ denotes the space of all infinitely smooth functions on M with compact support, while $C_0(M)$ denotes the Banach space of continuous functions on M which tend to 0 at infinity.

1. A compactness principle

We begin this section by recalling the definitions of cycles for Kasparov groups and for relative K-homology groups, respectively. In each case a major ingredient is a compactness property of a commutator. We then present the main result of this section, a general principle for producing candidates for cycles for which such a compactness property holds.

In the following, $\mathcal{J}$ is a separable C^*-algebra, not necessarily with unit. We define a cycle giving an element of $KK(\mathcal{J}, \mathbb{C})$, confining attention to the case where $\mathcal{J}$ has no grading. For relative cycles, we assume $\mathfrak{A}$ is a separable C^*-algebra with unit and $\mathcal{J}$ a closed two-sided *-ideal; then we define a cycle for an element of $K^0(\mathfrak{A}, \mathfrak{A}/\mathcal{J})$. Now $K^0(\mathfrak{A}, \mathfrak{A}/\mathcal{J})$ is contravariant in the pair $(\mathfrak{A}, \mathcal{J})$; thus it is a K-cohomology group with respect to the algebras. The "cycles" mentioned above consequently might thus be called "cocycles". In the case $\mathfrak{A} = C(X)$ for X a compact metric space, $\mathcal{J} = C_Y(X) = \{f \in C(X): f = 0 \text{ on } Y\}$ for a closed subset Y of X, the groups

$$K_0(X, Y) = K^0(C(X), C(X)/C_Y(X))$$

are covariant in (X, Y), i.e., K-homology groups in these spaces. Since this includes the main cases of interest for this paper, we use the term "cycles".

A cycle for an element of $KK(\mathcal{J}, \mathbb{C})$ consists of a pair (σ, F), where σ is a *-representation of $\mathcal{J}$ on a sum of Hilbert space $H_0 \oplus H_1$ which leaves each H_j invariant; $\sigma = \sigma_0 \oplus \sigma_1$, and $F \in \mathcal{L}(H_0 \oplus H_1)$ is a bounded operator of the form

$$F = \begin{pmatrix} 0 & T^\# \\ T & 0 \end{pmatrix}, \tag{1.1}$$

where $T\colon H_0 \to H_1$ and $T^\#\colon H_1 \to H_0$ are bounded linear maps. The following three conditions are required to hold for all $a \in \mathcal{J}$:

$$[\sigma(a), F] \in \mathcal{K}(H_0 \oplus H_1) = \mathcal{K}, \tag{1.2}$$

where $\mathscr{K}$ is the space of *compact* operators,

$$\sigma(a)(F^2 - 1) \in \mathscr{K}, \tag{1.3}$$

$$\sigma(a)(F - F^*) \in \mathscr{K}. \tag{1.4}$$

We note that conditions equivalent to (1.2)–(1.4) are

$$\sigma_1(a)T - T\sigma_0(a) \in \mathscr{K}, \qquad \sigma_0(a)T^\# - T^\#\sigma_1(a) \in \mathscr{K}, \tag{1.5}$$

$$\sigma_0(a)(T^\# T - 1) \in \mathscr{K}, \qquad \sigma_1(a)(TT^\# - 1) \in \mathscr{K}, \tag{1.6}$$

$$\sigma_0(a)(T^\# - T^*) \in \mathscr{K}, \qquad (T^\# - T^*)\sigma_1(a) \in \mathscr{K} \tag{1.7}$$

for all $a \in \mathscr{J}$. In typical examples one takes $T^\# = T^*$ (i.e., $F = F^*$), and indeed it is known that any cycle is equivalent to one for which $F = F^*$, but there are uses for allowing the weaker hypothesis (1.4). In any case we note that, by (1.7), $T^\#$ can be replaced by T^* in (1.5)–(1.6). Furthermore, the second part of hypothesis (1.5) is redundant.

An element of $KK(\mathscr{J}, \mathbb{C})$ consists of an equivalence class of cycles of the form above. We refer to [10] for a full discussion of such equivalence relations, but mention one here which we will use in a crucial way. Namely, if $\sigma = \sigma_0 \oplus \sigma_1$ is as above and F_0, F_1 are two maps of the form (1.1), then $(\sigma, F_0) \sim (\sigma, F_1)$ provided there is a norm continuous map $t \mapsto F_t$ for $t \in [0,1]$, such that, for each t, the conditions (1.2)–(1.4) hold, with F replaced by F_t. This is the "operator homotopy" equivalence relation. In particular, if (σ, F_0) and (σ, F_1) define cycles for $KK(\mathscr{J}, \mathbb{C})$ and

$$\sigma(a)(F_0 - F_1) \in \mathscr{K} \tag{1.8}$$

for all $a \in \mathscr{J}$, since $\sigma(a)$ commutes with F_0 and F_1 mod $\mathscr{K}$, it follows that

$$F_t = tF_1 + (1-t)F_0 \tag{1.9}$$

produces an operator homotopy.

We next describe a cycle for an element of the relative group $K^0(\mathfrak{A}, \mathfrak{A}/\mathscr{J})$. This consists of a pair (σ, T), where $\sigma = \sigma_0 \oplus \sigma_1$ is a unital *-representation of $\mathfrak{A}$ on $H_0 \oplus H_1$, σ_j acting on H_j, and

$$T\colon H_0 \to H_1, \quad \text{bounded, with closed range, partial isometry mod } \mathscr{K}. \tag{1.10}$$

We require the following two additional conditions:

$$\sigma_1(a)T - T\sigma_0(a) \in \mathscr{K} \quad \text{for all } a \in \mathfrak{A}, \tag{1.11}$$

$$\sigma_j(a)P_j \in \mathscr{K} \quad \text{for all } a \in \mathscr{J}, \tag{1.12}$$

where P_0 is the orthogonal projection of H_0 onto $\ker T$ and P_1 the orthogonal projection of H_1 onto $\operatorname{coker} T$. Note that (1.11) holds for all $a \in \mathfrak{A}$, not just for the elements of $\mathcal{J}$. Equivalence relations among cycles to define a class in $K^0(\mathfrak{A}, \mathfrak{A}/\mathcal{J})$ are discussed in [8]. It can be shown that each cycle (σ, T) is equivalent to one for which T is a partial isometry. In such a case, one has

$$(1.13) \qquad I - T^*T = P_0, \qquad I - TT^* = P_1.$$

The examples of cycles we construct in this paper will all directly satisfy the property that T is equal to its partial isometry part modulo a compact operator, so that (1.13) holds $\bmod \mathcal{K}$. In that case, note that (1.11) implies

$$(1.14) \qquad [\sigma_j(a), P_j] \in \mathcal{K} \quad \text{for all } a \in \mathfrak{A}.$$

Clearly, if (σ, T) defines a cycle for an element of $K^0(\mathfrak{A}, \mathfrak{A}/\mathcal{J})$, such that (1.13) holds ($\bmod \mathcal{K}$), then the hypotheses (1.5)–(1.6) hold, with $T^\# = T^*$. In other words, with F of the form (1.1), $T^\# = T^*$, (σ, F) defines a cycle for the Kasparov group $KK(\mathcal{J}, \mathbb{C})$. It is shown in [8] that, if $\mathfrak{A}$ is a separable C^*-algebra with unit which is nuclear, in particular one of type I, most particularly one which is commutative, then this correspondence of cycles gives rise to a *group isomorphism*

$$(1.15) \qquad K^0(\mathfrak{A}, \mathfrak{A}/\mathcal{J}) \approx KK(\mathcal{J}, \mathbb{C}).$$

This isomorphism will play an important role in the present paper; it forms a double-edged tool. Advantages of considering $KK(\mathcal{J}, \mathbb{C})$ arise from the rich set of equivalence relations which arise naturally. In particular the operator homotopy relation mentioned above provides a very useful tool for showing that various cycles are equivalent in $KK(\mathcal{J}, \mathbb{C})$. Via the isomorphism (1.15) we obtain identities in $K^0(\mathfrak{A}, \mathfrak{A}/\mathcal{J})$ which are not trivial. Advantages of considering $K^0(\mathfrak{A}, \mathfrak{A}/\mathcal{J})$ arise for an opposite reason, from the extra structure provided by (1.11) (for $a \in \mathfrak{A}$) as opposed to the weaker condition (1.5) (for $a \in \mathcal{J}$). One particularly important manifestation of this is the computation of the boundary map $\partial : K^0(\mathfrak{A}, \mathfrak{A}/\mathcal{J}) \to K^1(\mathfrak{A}/\mathcal{J})$; under this map, identities in $K^0(\mathfrak{A}, \mathfrak{A}/\mathcal{J})$ alluded to above will give very important identities in $K^1(\mathfrak{A}/\mathcal{J})$, as we will see in §§4 and 5.

We now come to the main point of this section, the construction of a class of operators T for which the compactness condition (1.11) is satisfied. The operator T will be manufactured from an unbounded, closed, densely defined operator A from H_0 to H_1, as

$$(1.16) \qquad T = A(A^*A + 1)^{-1/2}.$$

We prepare to give the hypotheses relating A to the *-representation $\sigma = \sigma_0 \oplus \sigma_1$ of $\mathfrak{A}$. These hypotheses are motivated by the example of A as a first order elliptic differential operator on some compact manifold $\overline{M}$ with boundary, with $\mathfrak{A} = C(\overline{M})$. The hypotheses are related to those for an "unbounded Kasparov module" in Baaj and Julg [4], but differ in an important way, particularly in (1.19) below. It will be convenient to consider the closed, densely defined operator on $H_0 \oplus H_1$,

$$B = \begin{pmatrix} 0 & A^* \\ A & 0 \end{pmatrix} \tag{1.17}$$

which is selfadjoint. Let σ_j be *-representations of a C^*-algebra $\mathfrak{A}$ on H_j, giving $\sigma = \sigma_0 \oplus \sigma_1$ a *-representation of $\mathfrak{A}$ on $H_0 \oplus H_1$. We make the following hypotheses, where $\mathfrak{A}_0$ is some dense *-subalgebra of $\mathfrak{A}$:

(1.18) for $a \in \mathfrak{A}_0$, $\sigma(a)$ preserves $\mathscr{D}(B)$ and $[\sigma(a), B]$ extends to a bounded operator on $H_0 \oplus H_1$,

(1.19) $(B^2 + 1)^{-1}$ is compact on *either* H_0 or H_1.

Note that (1.19) says that either $(A^*A + 1)^{-1}$ is compact on H_0 or $(AA^* + 1)^{-1}$ is compact on H_1, since

$$B^2 = \begin{pmatrix} A^*A & 0 \\ 0 & AA^* \end{pmatrix}. \tag{1.20}$$

This phenomenon of compactness of the resolvent on one factor but not necessarily on the other is an important twist in our hypotheses; such a situation will prevail in the most subtle examples of cycles produced in subsequent sections.

The following is the main result of this section.

Proposition 1.1. *Granted the hypotheses* (1.18)–(1.19), *then*

(1.21) $[\sigma(a), B(B^2 + 1)^{-1/2}]$ *is compact on* $H_0 \oplus H_1$

for $a \in \mathfrak{A}$. Furthermore, B has closed range, i.e., A and A^* both have closed range. The range of B has finite codimension in H_j if $(B^2+1)^{-1}$ is compact on H_j.

Note that, if T is given by (1.16), then

$$B(B^2 + 1)^{-1/2} = \begin{pmatrix} 0 & T^* \\ T & 0 \end{pmatrix}, \tag{1.22}$$

so (1.11) follows from (1.21). Since the range of T is equal to the range of A, we also have (1.10), so once we prove Proposition 1.1 we can conclude that (σ, T) defines a relative cycle for $K^0(\mathfrak{A}, \mathfrak{A}/\mathscr{I})$ provided (1.12) can be verified.

Our technique for analyzing the commutator (1.21) uses an integral representation for $(B^2+1)^{-1/2}$ in terms of the resolvent of B^2, in a fashion similar to arguments of Cordes and Herman [22], Taylor [38], and Baaj and Julg [4]. The representation is

$$(1.23)\quad (B^2+1)^{-1/2} = \frac{1}{\pi}\int_0^\infty \frac{\lambda^{-1/2}}{B^2+1+\lambda}\,d\lambda = \frac{2}{\pi}\int_0^\infty (B^2+1+s^2)^{-1}\,ds.$$

In the course of the proof, we will suppose for definiteness that H_1 is the factor on which $(B^2+1)^{-1}$ is given to be compact. Note that it suffices to prove the compactness in (1.21) for a $a \in \mathfrak{A}_0$.

To begin, since $\sigma(a)$ preserves $\mathscr{D}(B)$ for $a \in \mathfrak{A}_0$, we can write

(1.24)
$$[\sigma(a), B(B^2+1)^{-1/2}] = [\sigma(a), B](B^2+1)^{-1/2} + B[\sigma(a), (B^2+1)^{-1/2}].$$

The first term on the right is clearly compact on H_1. Before looking at the last term on the right, we obtain the following result which gives some information on the behavior of the first term on H_0.

Lemma 1.2. *$(B^2+1)^{-1/2}$ is compact on the orthogonal complement of $\ker B$ (in H_0).*

Proof. Our assertion is the same as the claim that $(A^*A+1)^{-1/2}$ is compact on the orthogonal complement of $\ker A$ in H_0. What is behind this is the identity

$$(1.25)\qquad Ag(A^*A) = g(AA^*)A \quad \text{on } \mathscr{D}(A),$$

for any bounded continuous function g on R^+, which follows from the identity

$$(1.26)\qquad Bg(B^2) = g(B^2)B \quad \text{on } \mathscr{D}(B)$$

in light of (1.20). Since AA^* by hypothesis has compact resolvent, H_1 has an orthonormal basis of eigenvectors for AA^*, and we have

$$(1.27)\qquad \begin{aligned} A&\colon \mathrm{Eigen}(\lambda, A^*A) \to \mathrm{Eigen}(\lambda, AA^*), \\ A^*&\colon \mathrm{Eigen}(\lambda, AA^*) \to \mathrm{Eigen}(\lambda, A^*A). \end{aligned}$$

If $\lambda \neq 0$, these maps are inverses of each other, up to a factor λ, so are isomorphisms.

To prove the lemma, we first show that

$$(1.28)\qquad f \in C_0^\infty(\mathbf{R}),\quad f(0) = 0 \Rightarrow f(A^*A) \text{ is compact on } H_0.$$

Indeed, write $f(s)$ in the form

$$f(s) = sf_1(s)f_2(s)f_3(s), \qquad f_j \in C_0^\infty(\mathbf{R}).$$

Then

$$\begin{aligned} f(A^*A) &= A^*A(f_1f_2)(A^*A)f_3(A^*A) \\ &= A^*(f_1f_2)(AA^*)Af_3(A^*A) \\ &= A^*f_1(AA^*)f_2(AA^*)Af_3(A^*A), \end{aligned}$$

applying (1.25) with $g = f_1f_2$. Here, $Af_3(A^*A)$ is bounded from H_0 to H_1, $f_2(AA^*)$ is compact on H_1, and $A^*f_1(AA^*)$ is bounded from H_1 to H_0, so (1.28) is established. It follows that the spectrum of A^*A, which is contained in $[0, \infty)$, is discrete in any compact interval in $(0, \infty)$, of finite multiplicity. It remains to see that this spectrum cannot accumulate to 0. But the argument around (1.27) now shows that

$$\{0\} \cup \operatorname{spec} A^*A = \{0\} \cup \operatorname{spec} AA^*, \tag{1.29}$$

and since AA^* has compact resolvent, its spectrum does not accumulate to 0, so Lemma 1.2 is proved.

To proceed with the proof of Proposition 1.1, we next show that the last term in (1.24) is compact on H_1 and on the orthogonal complement of $\ker B$ in H_0. We use the integral representation (1.23), together with the following formula for $[\sigma(a), (B^2 + 1 + s^2)^{-1}]$. With $t = \sqrt{1 + s^2}$, we write

$$(B^2 + t^2)^{-1} = (B + it)^{-1}(B - it)^{-1}, \tag{1.30}$$

and hence

$$\begin{aligned} [\sigma(a), (B^2 + t^2)^{-1}] = [\sigma(a), (B + it)^{-1}](B - it)^{-1} \\ + (B + it)^{-1}[\sigma(a), (B - it)^{-1}]. \end{aligned} \tag{1.31}$$

Since, for $a \in \mathfrak{A}_0$, $\sigma(a)$ is assumed to preserve $\mathscr{D}(B)$, we can write

$$[\sigma(a), (B + it)^{-1}] = -(B + it)^{-1}[\sigma(a), B](B + it)^{-1}, \tag{1.32}$$

and hence (1.31) gives

$$\begin{aligned} [\sigma(a), (B^2 + t^2)^{-1}] = &-(B + it)^{-1}[\sigma(a), B](B^2 + t^2)^{-1} \\ &-(B^2 + t^2)^{-1}[\sigma(a), B](B - it)^{-1}. \end{aligned} \tag{1.33}$$

Consequently the last term in (1.24) is equal to

$$\begin{aligned} &-\frac{2}{\pi}\int_0^\infty B(B + i\sqrt{1 + s^2})^{-1}[\sigma(a), B](B^2 + 1 + s^2)^{-1}\, ds \\ &-\frac{2}{\pi}\int_0^\infty B(B^2 + 1 + s^2)^{-1}[\sigma(a), B](B - i\sqrt{1 + s^2})^{-1}\, ds \\ &= T_1 + T_2. \end{aligned} \tag{1.34}$$

In view of the operator norm estimates

$$\|B(B \pm it)^{-1}\| \le 1, \qquad \|(B \pm it)^{-1}\| \le 1/t \tag{1.35}$$

for $t \geq 0$, it follows that both integrals in (1.34) are convergent in the operator norm. Thus T_1 and T_2 are compact on any closed subspace of $H_0 \oplus H_1$ on which the integrands can be shown to be compact.

The integrand for T_1 is a product of bounded operators with the factor $(B^2 + 1 + s^2)^{-1}$, which we have seen is compact on H_1 and on $(\ker B)^{\perp}$; thus T_1 is compact there. The integrand in T_2 is a product of bounded operators and $B(B^2+1+s^2)^{-1}$. It follows from Lemma 1.2 that this factor is compact on all of $H_0 \oplus H_1$, so T_2 is in fact compact on $H_0 \oplus H_1$.

To complete the proof of the compactness of the commutator (1.21), it remains to show that this commutator is compact on $\ker B$ (in H_0) for $a \in \mathfrak{A}_0$. In such a case, we can write, in place of (1.24),

(1.36)
$$\begin{aligned}[\sigma(a), B(B^2+1)^{-1/2}] &= [\sigma(a), (B^2+1)^{-1/2}B] \quad (\text{on } \mathscr{D}(B)) \\ &= [\sigma(a), (B^2+1)^{-1/2}]B + (B^2+1)^{-1/2}[\sigma(a), B] \\ &\qquad (\text{on } \mathscr{D}(B)).\end{aligned}$$

This is equal to

$$(B^2+1)^{-1/2}[\sigma(a), B] \tag{1.37}$$

on $\ker B$. Now $[\sigma(a), B]$ maps H_0 to H_1, and $(B^2+1)^{-1/2}$ is compact on H_1, so (1.36) is compact on H_0. This completes the proof of the compactness assertion (1.21).

Finally, the proof of Lemma 1.2 shows that 0 is an isolated point of $\operatorname{spec} B^2$, hence of $\operatorname{spec} B$, so B has closed range. The proof of Proposition 1.1 is complete.

We make a few additional comments. In view of Lemma 1.2, we see that $T = A(A^*A+1)^{-1/2}$ is equal to the partial isometry part $A(A^*A)^{-1/2}$ (0 on $\ker A$) modulo $\mathscr{K}$. Thus the identities (1.13) hold mod $\mathscr{K}$, and hence the compactness result (1.14) is valid under the hypotheses of Proposition 1.1. Lemma 1.2 also implies that

$$\begin{pmatrix} P_0 & \\ & P_1 \end{pmatrix} = (B^2+1)^{-1} \qquad \operatorname{mod} \mathscr{K}. \tag{1.38}$$

Hence the remaining hypothesis (1.12) for (σ, T) to define a relative cycle for $K^0(\mathfrak{A}, \mathfrak{A}/\mathscr{J})$ is equivalent to

$$\sigma(a)(B^2+1)^{-1} \in \mathscr{K} \quad \text{for } a \in \mathscr{J}. \tag{1.39}$$

This hypothesis, and hypothesis (1.18) for $a \in \mathscr{J}_0$, a dense *-subalgebra of $\mathscr{J}$, form the hypotheses of Baaj and Julg [4] (see also [10]) that (σ, B) define an "unbounded Kasparov module", and consequently that (σ, F) define a cycle for $KK(\mathscr{J}, \mathbb{C})$. J. Rosenberg and A. Wasserman came upon the idea of using the commutator identity (1.31) to prove the Theorem of Baaj and Julg, and their comment to us on using this identity has been of use in the proof of Proposition 1.1. Note that, if F is given by (1.22), then

$$F^2 = I - (B^2+1)^{-1}, \tag{1.40}$$

so (1.39) is equivalent to (1.3).

We make a further remark on the commutator identity (1.31). This is a diagonal matrix, and, using

$$(B+i)^{-1} = \begin{pmatrix} -i(A^*A+1)^{-1} & A^*(AA^*+1)^{-1} \\ A(A^*A+1)^{-1} & -i(AA^*+1)^{-1} \end{pmatrix}, \tag{1.41}$$

a brief calculation of the upper left element in (1.31) gives the identity (1.42)

$$[\sigma_0(a), (A^*A+t^2)^{-1}] = -A^*L_1^{-1}[\sigma(a), A]L_0^{-1} - L_0^{-1}[\sigma(a), A^*]AL_0^{-1}$$

for $a \in \mathfrak{A}_0$, where $L_0 = A^*A + t^2$ and $L_1 = AA^* + t^2$.

We note the relation with the commutator identity

$$[\sigma_0(a), (A^*A+t^2)^{-1}] = -L_0^{-1}[\sigma_0(a), A^*A]L_0^{-1}, \tag{1.43}$$

which is valid under the hypothesis

$$\sigma_0(a) \text{ preserves } \mathscr{D}(A^*A) \text{ for } a \in \mathfrak{A}_0. \tag{1.44}$$

This is a much stronger sort of hypothesis than the hypothesis of Proposition 1.1 that $\sigma(a)$ preserve $\mathscr{D}(B)$. When both are valid, we can replace $[\sigma_0(a), A^*A]$ in (1.43) by

$$A^*[\sigma(a), A] + [\sigma(a), A^*]A,$$

and also use $L_0^{-1}A^* = A^*L_1^{-1}$ on $\mathscr{D}(A^*)$; then we obtain (1.42) again. However, it is very important to us to obtain (1.31) (and hence (1.42)) without making the hypothesis (1.44), a hypothesis which is not satisfied by our major examples produced in subsequent sections.

A final comment which we make on Proposition 1.1 is that the hypothesis (1.18) has only to be checked on one factor. More precisely, we have the following.

Proposition 1.3. *Assume that, for $a \in \mathfrak{A}_0$, $\sigma_0(a)$ preserves $\mathscr{D}(A)$, and that $W(a) = \sigma_1(a)A - A\sigma_0(a)$ extends from $\mathscr{D}(A)$ to a bounded operator from H_0 to H_1. Then $\sigma_1(a)$ preserves $\mathscr{D}(A^*)$, and* (1.18) *holds.*

Proof. Given $v \in \mathscr{D}(A^*)$ and $u \in \mathscr{D}(A)$, we have

$$\begin{aligned}(Au, \sigma_1(a)v) &= (\sigma_1(a^*)Au, v) = (A\sigma_0(a^*)u, v) + (W(a^*)u, v) \\ &= (\sigma_0(a^*)u, A^*v) + (W(a^*)u, v).\end{aligned}$$

This shows that $\sigma_1(a)v \in \mathscr{D}(A^*)$ and

$$A^*\sigma_1(a)v = \sigma_0(a)A^*v + W(a^*)^*v,$$

which implies (1.18).

2. Kasparov cycles defined by elliptic operators

In this section we consider how elements of the Kasparov group $KK(C_0(M), \mathbb{C})$ are defined by elliptic operators on a Riemannian manifold M (without boundary), where $C_0(M)$ consists of continuous functions vanishing at infinity. We emphasize that M is not assumed to be complete. Of particular interest will be the case where M is the interior of a compact manifold with boundary, but for now M is absolutely general. Consider an elliptic first order differential operator

$$D\colon C_0^\infty(M, E_0) \to C_0^\infty(M, E_1), \tag{2.1}$$

between sections of Hermitian vector bundles E_0 and E_1. Such an operator has a number of extensions to closed unbounded operators on $L^2(M, E_0)$. One is the "minimal extension", denoted $D_{\min}$, which is just the closure of D on $C_0^\infty(M, E_0)$. Another is the "maximal extension", $D_{\max}$, with domain

$$\mathscr{D}(D_{\max}) = \{u \in L^2(M, E_0)\colon Du \in L^2(M, E_1)\}, \tag{2.2}$$

where Du is computed a priori as a distribution on M. There is the well-known identity

$$D_{\max} = (D^t_{\min})^*, \tag{2.3}$$

where $D^t\colon C_0^\infty(M, E_1) \to C_0^\infty(M, E_0)$ is the formal adjoint of D, also a first order elliptic differential operator. Note that

$$H^1_{\text{comp}}(M, E_0) \subset \mathscr{D}(D_{\min}) \subset \mathscr{D}(D_{\max}) \subset H^1_{\text{loc}}(M, E_0), \tag{2.4}$$

$H^1_{\text{comp}}(M, E_0)$ being the space of compactly supported elements of the Sobolev space $H^1(M, E_0)$, etc. The last inclusion in (2.4) follows from elliptic regularity.

We will consider any closed extension D_e of D satisfying

$$D_{\min} \subset D_e \subset D_{\max}, \tag{2.5}$$

i.e., such that $\mathscr{D}(D_{\min}) \subset \mathscr{D}(D_e) \subset \mathscr{D}(D_{\max})$ and $D_e u = Du$, in the distributional sense, for $u \in \mathscr{D}(D_e)$. Associated to D_e we have the selfadjoint operator

$$B = \begin{pmatrix} 0 & D_e^* \\ D_e & 0 \end{pmatrix} \tag{2.6}$$

and the bounded operator

$$F_e = B(B^2+1)^{-1/2} = \begin{pmatrix} 0 & T^* \\ T & 0 \end{pmatrix} \tag{2.7}$$

on $H_0 \oplus H_1 = L^2(M, E_0 \oplus E_1)$, where

$$T = D_e(D_e^* D_e + 1)^{-1/2}, \qquad T^* = D_e^*(D_e D_e^* + 1)^{-1/2}.$$

We also have $*$-representations of $C_0(M)$ on H_0 and H_1, and $H_0 \oplus H_1$, given by scalar multiplication. We denote this action by M_f, $f \in C_0(M)$. The following is the main result of this section.

Proposition 2.1. *The pair (M, F_e) defines a cycle for $KK(C_0(M), \mathbb{C})$, which we denote $[D_e]$. Furthermore, if D_d is another closed extension of D satisfying* (2.5), *the pair (M, F_d) defines an equivalent cycle*:

$$[D_e] = [D_d] \in KK(C_0(M), \mathbb{C}). \tag{2.8}$$

Consequently, to D there is uniquely assigned an element

$$[D] \in KK(C_0(M), \mathbb{C}). \tag{2.9}$$

Note that, since $\mathscr{D}(D_e) \subset \mathscr{D}(D_{\max})$, in view of (2.4), $M_f \mathscr{D}(D_e) \subset H^1_{\mathrm{comp}}(M, E_0)$ provided $f \in C_0^\infty(M)$, so

$$f \in C_0^\infty(M) \Rightarrow M_f \quad \text{preserves } \mathscr{D}(D_e). \tag{2.10}$$

Also M_f preserves $\mathscr{D}(D_e^*)$ and hence $\mathscr{D}(B)$. Also,

$$(B^2+1)^{-1} \colon L^2(M, E_0 \oplus E_1) \to H^1_{\mathrm{loc}}(M, E_0 \oplus E_1), \tag{2.11}$$

and hence, by Rellich's theorem,

$$f \in C_0^\infty(M) \Rightarrow M_f(B^2+1)^{-1} \quad \text{is compact on } L^2(M, E_0 \oplus E_1). \tag{2.12}$$

Thus the fact that (M, F_e) defines a cycle for $KK(C_0(M), \mathbb{C})$ can be proved by the methods of Baaj and Julg [4]. We proceed to establish some localization properties for functions of the operator B, which make clear the operator homotopy between F_e and F_d for different extensions, and also gives sharp results on commutators which directly imply that (M, F_e) defines such a cycle.

We analyze functions of B via the formula

$$\varphi(B) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} \hat{\varphi}(t) e^{itB}\, dt, \tag{2.13}$$

which follows from the Fourier inversion formula and the spectral theorem if φ belongs to the Schwartz space $\mathscr{S}(\mathbb{R})$. This formula extends by limiting arguments to other classes of functions φ, and we will be able to use it for the class

$$S_1^0(\mathbb{R}) = \{\varphi \in C^\infty(\mathbb{R}) \colon |\varphi^{(j)}(\lambda)| \le C_j(1+|\lambda|)^{-j}\}, \tag{2.14}$$

in particular, for the function

$$\varphi(\lambda) = \lambda(\lambda^2+1)^{-1/2}. \tag{2.15}$$

This approach to functional calculus for elliptic self-adjoint operators has been used in [39] and in Chapter 12 of the book [40], and also in the paper [17] of Cheeger, Gromov, and Taylor.

If φ belongs to $S_1^0(\mathbb{R})$, it is easy to see that $\hat{\varphi}$, a priori a tempered distribution on $\mathbb{R}$, is smooth on $\mathbb{R}\backslash\{0\}$ and rapidly decreasing at infinity. In other words, given any $\varepsilon > 0$, for $\varphi \in S_1^0(\mathbb{R})$, we can write

$$\varphi = \varphi_1 + \varphi_2, \quad \operatorname{supp}\hat{\varphi}_1 \subset (-\varepsilon, \varepsilon), \quad \varphi_2 \in \mathscr{S}(\mathbb{R}). \tag{2.16}$$

Lemma 2.2. *For $f \in C_0(M)$, $M_f\varphi_2(B)$ and $\varphi_2(B)M_f$ are compact on $L^2(M, E_0 \oplus E_1)$.*

Proof. Write $\varphi_2(\lambda) = (1+\lambda^2)^{-N}\psi_N(\lambda)$, $\psi_N \in \mathscr{S}(\mathbb{R})$. Then

$$M_f\varphi_2(B) = M_f(B^2+1)^{-N}\psi_N(B). \tag{2.17}$$

The compactness then follows from (2.12). We note the more precise consequence of (2.17), which follows by elliptic regularity for B^2+1:

$$f \in C_0^\infty(M) \Rightarrow M_f\varphi_2(B) \colon L^2(M, E_0 \oplus E_1) \to C_0^\infty(M, E_0 \oplus E_1). \tag{2.18}$$

The function φ_1 belongs to $S_1^0(\mathbb{R})$, in particular is bounded and continuous, and $\varphi_1(B)$ is a bounded operator. Furthermore, φ_1 can be approximated by a sequence $\Phi_j \in \mathscr{S}(\mathbb{R})$ in such a fashion that $\hat{\Phi}_j \in C_0^\infty((-\varepsilon, \varepsilon))$, and $\varphi_1(B)$ is the strong limit of

$$\Phi_j(B) = (2\pi)^{-1/2}\int_{-\varepsilon}^{\varepsilon} \hat{\Phi}_j(t)e^{itB}\,dt. \tag{2.19}$$

To analyze such an integral, we exploit *finite propagation speed* for the operator e^{itB}, which is the solution operator to the symmetric hyperbolic system

$$\partial u/\partial t = iBu. \tag{2.20}$$

Accordingly, for any compact $K \subset M$ and any neighborhood $\mathscr{O}$ of K, there exists $\varepsilon > 0$ such that

$$\operatorname{supp} u \subset K, \qquad |t| < \varepsilon \Rightarrow e^{itB}u \quad \text{supported in } \mathscr{O}. \tag{2.21}$$

Furthermore if

$$B^{\#} = \begin{pmatrix} 0 & D_d^* \\ D_d & 0 \end{pmatrix}$$

for another closed extension D_d, we have

$$e^{itB}u = e^{itB^{\#}}u \quad \text{for } \operatorname{supp} u \subset K,\ |t| < \varepsilon. \tag{2.22}$$

Consequently, using (2.19) and passing to the limit, we obtain

$$\varphi_1(B)M_f = \varphi_1(B^{\#})M_f \quad \text{if } \operatorname{supp} f \subset K, \tag{2.23}$$

this operator having range in $\{u \in L^2(M, E_0 \oplus E_1)\colon \operatorname{supp} u \subset \mathcal{O}\}$. Taking adjoints, we also have $M_f\varphi_1(B) = M_f\varphi_1(B^{\#})$ in this case. This identity, together with Lemma 2.2, implies, for any $\varphi \in S_1^0(\mathbb{R})$,

$$M_f\varphi(B) = M_f\varphi(B^{\#}) \qquad \operatorname{mod} \mathcal{K} \tag{2.24}$$

if $f \in C_0^\infty(M)$, and a standard limiting argument also gives this for $f \in C_0(M)$. Taking φ to be given by (2.15), we have the "compact perturbation" relation (1.8), implying the operator homotopy relation which establishes the identity (2.8).

The finite propagation speed argument described above applies to the following more general situation. Let M' be another Riemannian manifold, with the property that M and M' are identified on $\mathcal{O}$, and let D_d be a closed extension of an elliptic first order differential operator on M', operating between sections of vector bundles E_0' and E_1', identified over $\mathcal{O}$ with E_0 and E_1, such that D_d coincides with D_e on $C_0^\infty(\mathcal{O}, E_0)$. Then (2.21)–(2.23) continue to hold, the operators in (2.23) having range consisting of functions supported in $\mathcal{O}$.

This observation leads to a sharp analysis of the commutator $[M_f, \varphi(B)]$ for $\varphi \in S_1^0(\mathbb{R})$ and $f \in C_0^\infty(M)$ as follows. Let $K = \operatorname{supp} f$, take $\tilde{f} \in C_0^\infty(\mathcal{O})$ for a relatively compact neighborhood $\mathcal{O}$ of K, such that $\tilde{f} = 1$ on a smaller neighborhood $\mathcal{O}^b$ of K. By Lemma 2.2, $[M_f, \varphi(B)] = [M_f, \varphi_1(B)]$ mod $\mathcal{K}$, and we concentrate on the analysis of this last commutator. By (2.21) we have

$$M_f\varphi_1(B) = M_f\varphi_1(B)M_{\tilde{f}}, \qquad \varphi_1(B)M_f = M_{\tilde{f}}\varphi_1(B)M_f. \tag{2.25}$$

Now we can produce a *compact* manifold M' identified with M on $\mathcal{O}$ (e.g., the double of a compact neighborhood of $\overline{\mathcal{O}}$ with smooth boundary) and a first order differential operator $D_d\colon C^\infty(M', E_0') \to C^\infty(M', E_1')$ coinciding with D_e on $C_0^\infty(\mathcal{O}, E_0)$ in the sense given above, and we have

$$\begin{aligned} M_f\varphi_1(B)M_{\tilde{f}} &= M_f\varphi_1(B^{\#})M_{\tilde{f}}, \\ M_{\tilde{f}}\varphi_1(B^{\#})M_f &= M_{\tilde{f}}\varphi_1(B)M_f, \end{aligned} \tag{2.26}$$

where $B^\#$ is a first order elliptic selfadjoint operator on the *compact* manifold M'. Consequently $\varphi_1(B^\#)$ is a pseudodifferential operator, in $\mathrm{OPS}^0_{1,0}(M')$, coinciding with $\varphi(B^\#)$ mod $\mathrm{OPS}^{-\infty}_{1,0}(M')$ (see [40, Chapter 12]). Therefore the standard pseudodifferential operator calculus implies that

$$M_f\varphi_1(B^\#)M_{\tilde{f}} = M_f\varphi_1(B^\#), \qquad M_{\tilde{f}}\varphi_1(B^\#)M_f = \varphi_1(B^\#)M_f \quad \mathrm{mod}\ \mathrm{OPS}^{-\infty}(M'), \tag{2.27}$$

$$[M_f, \varphi_1(B^\#)] \in \mathrm{OPS}^{-1}_{1,0}(M'). \tag{2.28}$$

In particular, this operator is compact. This implies

$$M_f\varphi_1(B)M_{\tilde{f}} - M_{\tilde{f}}\varphi_1(B)M_f \in \mathscr{K}, \tag{2.29}$$

and hence, by (2.25) and Lemma 2.2, we conclude that

$$[M_f, \varphi(B)] \in \mathscr{K} \tag{2.30}$$

for any $f \in C_0^\infty(M)$. In case $\varphi(\lambda) = \lambda(\lambda^2+1)^{-1/2}$, this is the compactness result which implies that (M, F_e) defines a cycle for $KK(C_0(M), \mathbb{C})$.

We now briefly discuss cycles for the Kasparov group $KK^1(C_0(M), \mathbb{C})$. The group $KK^1(\mathscr{I}, \mathbb{C})$ is defined for a (separable, nuclear) C^*-algebra $\mathscr{I}$, as

$$KK^1(\mathscr{I}, \mathbb{C}) = KK(\mathscr{I}, \mathbb{C}_1), \tag{2.31}$$

where $\mathbb{C}_1 = \mathbb{C} \oplus \mathbb{C}$ is the graded complex Clifford algebra with one generator. For a general pair of $\mathbb{Z}_2$-graded C^*-algebras $\mathscr{I}, \mathscr{J}$, the definition of $KK(\mathscr{I}, \mathscr{J})$ can be found in [30], [10]. In the case of (2.31), a cycle in the "Fredholm picture" is given by a pair (ψ, T), where ψ is a *-representation of $\mathscr{I}$ on a Hilbert space H and

$$T\colon H \to H \tag{2.32}$$

is a bounded operator, satisfying

$$(T - T^*)\psi(a) \in \mathscr{K}, \tag{2.33}$$

$$(T^2 - 1)\psi(a) \in \mathscr{K}, \tag{2.34}$$

$$[T, \psi(a)] \in \mathscr{K} \tag{2.35}$$

for all $a \in \mathscr{I}$ (cf. Blackadar [10, p. 184]).

Consider a first order elliptic differential operator

$$D\colon C_0^\infty(M, E) \to C_0^\infty(M, E) \tag{2.36}$$

(i.e., $E_0 = E_1 = E$ in (2.1)) which is formally symmetric:

$$D = D^t. \tag{2.37}$$

We aim to associate a class in $KK^1(C_0(M), \mathbb{C})$. Let D_e be any closed extension of D (not necessarily symmetric) satisfying (2.5), and set

$$T_e = D_e(D_e^* D_e + 1)^{-1/2}. \tag{2.38}$$

Let $M\colon C_0(M) \to \mathscr{L}(L^2(M, E))$ be defined by scalar multiplication, as before. Then the methods developed above also establish the following.

Proposition 2.3. *The pair (M, T_e) defines a cycle for $KK^1(C_0(M), \mathbb{C})$, which we denote $[D_e]$. Furthermore, if D_d is another closed extension of D satisfying* (2.5), *then*

$$(T_e - T_d)M_f \in \mathscr{K} \quad \textit{for } f \in C_0(M), \tag{2.39}$$

so the pair (M, T_d) defines an equivalent cycle:

$$[D_e] = [D_d] \quad \textit{in } KK^1(C_0(M), \mathbb{C}). \tag{2.40}$$

Consequently to a formally symmetric first order elliptic differential operator D there is uniquely associated an element

$$[D] \in KK^1(C_0(M), \mathbb{C}). \tag{2.41}$$

3. Relative cycles defined by elliptic operators

In this section we consider how elliptic operators on a Riemannian manifold M, under certain conditions, define relative cycles and hence elements of

$$K^0(C(\overline{M}), C(\overline{M})/C_0(M)) = K_0(M, \partial M).$$

Here M is a Riemannian manifold of a less general sort than in §2. We make the assumption that

(3.1) M is open in $\tilde{M}$, a smooth compact manifold with boundary.

Thus the closure $\overline{M}$ is compact in $\tilde{M}$, with boundary $\partial M = \overline{M} \backslash M$. Eventually we will focus attention on the case where ∂M is smooth, but at present we make no smoothness assumption on ∂M.

Let E_0, E_1 be smooth Hermitian vector bundles over $\overline{M}$, which we assume extend to smooth bundles over $\tilde{M}$. Suppose D is a first order differential operator from sections of E_0 to sections of E_1, which is elliptic over $\overline{M}$. We consider various closed extensions of

$$D\colon C_0^\infty(M, E_0) \to C_0^\infty(M, E_1). \tag{3.2}$$

As in §2, $D_{\min}$ and $D_{\max}$ are two such; also recall that (2.4) holds. We consider closed extensions D_B of D satisfying

$$D_{\min} \subset D_B \subset D_{\max} \tag{3.3}$$

and also a further condition that the domain of D_B be defined by a "local boundary condition". In its most general formulation, we can state this hypothesis as

$$M_f\colon \mathscr{D}(D_B) \to \mathscr{D}(D_B) \quad \text{for } f \in C^\infty(\overline{M}), \tag{3.4}$$

where $C^\infty(\overline{M})$ denotes the space of restrictions to $\overline{M}$ of functions in $C^\infty(\tilde{M})$. Since

$$[M_f, D]u = \sigma_D(x, df)u \tag{3.5}$$

for $u \in H^1_{\mathrm{loc}}(M, E_0)$, in particular for $u \in \mathscr{D}(D_B)$, $\sigma_D(x,\xi)$ denoting the symbol of D, this implies that the hypothesis of Proposition 1.3 holds, with $\mathfrak{A}_0 = C^\infty(\overline{M})$, $\sigma_0(f) = M_f$, and $A = D_B$. We also take $\sigma_1(f) = M_f$, and use M to denote $\sigma_0 \oplus \sigma_1$. Then the hypothesis (1.18) of Proposition 1.1 holds. We now have the following:

Proposition 3.1. *Suppose in addition that*

$$\text{either } D_B^* D_B \text{ or } D_B D_B^* \text{ has compact resolvent} \tag{3.6}$$

on $H_0 = L^2(M, E_0)$ *or* $H_1 = L^2(M, E_1)$. *Set*

$$T = D_B(D_B^* D_B + 1)^{-1/2}. \tag{3.7}$$

Then the pair (M, T) *defines a cycle for* $K_0(M, \partial M)$, *which we denote* $[D_B]$. *The extensions* $D_{\min}$ *and* $D_{\max}$ *both satisfy these hypotheses. Furthermore, all extensions satisfying our hypotheses give rise to equivalent cycles, so there is a uniquely defined element*

$$[D] \in K_0(M, \partial M). \tag{3.8}$$

Proof. Adding hypothesis (3.6) completes the hypotheses of Proposition 1.1. As remarked in (1.39), to show (M, T) defines a relative cycle it remains to show that

$$M_f(D_B^* D_B + 1)^{-1} \text{ and } M_f(D_B D_B^* + 1)^{-1} \text{ are compact for } f \in C_0(M). \tag{3.9}$$

It suffices to verify this for $f \in C_0^\infty(M)$; in such a case these operators map H_0 and H_1 respectively to $H^2_{\mathrm{comp}}(M, E_0)$ and $H^2_{\mathrm{comp}}(M, E_1)$, and the compactness of (3.9) follows from Rellich's theorem. Note that, by standard elliptic estimates,

$$\mathscr{D}(D_{\min}) = \overset{\circ}{H}{}^1(M, E_0), \tag{3.10}$$

the closure in $H^1(\tilde{M}, E_0)$ of $C_0^\infty(M, E_0)$. Since $D_{\max} = (D^t_{\min})^*$, we see that (3.4) holds for $D_{\min}$ and $D_{\max}$, and that $D^*_{\min} D_{\min}$ has compact resolvent, since

$$(3.11) \qquad (D^*_{\min} D_{\min} + 1)^{-1/2} \colon L^2(M, E_0) \to \overset{\circ}{H}{}^1(M, E_0).$$

Thus (3.6) holds for $D_{\min}$ and for $D_{\max}$. It remains to show that $[D_B] = [D_C]$ in $K_0(M, \partial M)$ for any two extensions D_B and D_C of D satisfying the hypotheses of Proposition 3.1. However, these extensions certainly satisfy the conditions of §2, and by Proposition 2.1 they define identical elements of $KK(C_0(M), \mathbf{C})$. The coincidence of $[D_B]$ and $[D_C]$ in $K_0(M, \partial M)$ then follows from the isomorphism

$$(3.12) \qquad K_0(M, \partial M) \approx KK(C_0(M), \mathbf{C})$$

mentioned in §1, which is proved in [8].

We now specialize to the case where M has smooth boundary, and discuss in further detail several types of local boundary conditions of interest. In such a case, if D is an elliptic first order differential operator as above, it is well known that the trace map

$$(3.13) \qquad \gamma(u) = u|_{\partial M}, \qquad \gamma \colon C^\infty(M, E_0) \to C^\infty(\partial M, E_0)$$

has a unique extension to a continuous map

$$(3.14) \qquad \gamma \colon \mathscr{D}(D_{\max}) \to H^{-1/2}(\partial M, E_0).$$

Very general results containing this are given in Theorems B.2.7–B.2.9 in vol. 3 of [28]. For later use we make note of the following.

Lemma 3.2. *If K is a bounded subset of $\mathscr{D}(D_{\max})$ which is relatively compact in $L^2(M, E_0)$, then $\gamma(K)$ is relatively compact in $H^{-1/2}(\partial M, E_0)$.*

Proof. Let $\mathscr{O}$ be a neighborhood of $\overline{M}$ in $\tilde{M}$ on which D is elliptic, with a parametrix E. For $u \in \mathscr{D}(D_{\max})$, extend $Du \in L^2(M, E_0)$ to be zero on $\mathscr{O} \backslash M$, and let Hu denote $EDu|_M$. Write $u = u_0 + u_1 = Hu + (1 - H)u$. Clearly we have $H \colon \mathscr{D}(D_{\max}) \to H^1(\overline{M}, E_0)$, so $\{Hu \colon u \in K\}$ is bounded in $H^1(M, E_0)$. By the trace theorem, $\{\gamma Hu \colon u \in K\}$ is bounded in $H^{1/2}(\partial M, E_0)$, hence compact in $H^{-1/2}(\partial M, E_0)$. On the other hand, $1 - H$ preserves $\mathscr{D}(D_{\max})$ and $D(1 - H) \colon \mathscr{D}(D_{\max}) \to C^\infty(\overline{M})$. It follows that $\{(1 - H)u \colon u \in K\}$ is compact in $\mathscr{D}(D_{\max})$, and (3.14) implies that $\{\gamma((1 - H)u) \colon u \in K\}$ is compact in $H^{-1/2}(\partial M, E_0)$.

The closed extensions of D we now consider are of the following form. Let a multiplication operator by

$$(3.15) \qquad B \in C^\infty(\partial M, \operatorname{Hom}(E_0, F_0))$$

be given, where F_0 is some smooth vector bundle over ∂M. Then D_B will denote the restriction of $D_{\max}$ to

$$\mathscr{D}(D_B) = \{u \in \mathscr{D}(D_{\max}) \colon B\gamma u = 0\}, \tag{3.16}$$

where applying B gives an operator from $H^{-1/2}(\partial M, E_0)$ to $H^{-1/2}(\partial M, F_0)$. It is clear that (3.4) is satisfied in this case, and (3.5) holds on $\mathscr{D}(D_B)$. Thus, by Proposition 3.1, with $T = D_B(D_B^* D_B + 1)^{-1/2}$, (M, T) defines a relative cycle for $[D_B] \in K_0(M, \partial M)$, as long as either $D_B^* D_B$ or $D_B D_B^*$ has compact resolvent, i.e., as long as either of the inclusions

$$\mathscr{D}(D_B) \hookrightarrow L^2(M, E_0), \qquad \mathscr{D}(D_B^*) \hookrightarrow L^2(M, E_1) \tag{3.17}$$

is compact. Note that, with $F_0 = E_0$, $D_B = D_{\min}$ if $B = \mathrm{id}$ and $D_B = D_{\max}$ if $B = 0$. We describe some other important classes of boundary conditions of the form (3.16).

One class of boundary conditions is the class of "coercive" boundary conditions. These are the ones for which

$$(D, B) \colon H^s(M, E_0) \to H^{s-1}(M, E_1) \oplus H^{s-1/2}(\partial M, F_0) \tag{3.18}$$

is Fredholm for $s \geq 1$. This property can be characterized algebraically, as a "Lopatinsky condition"; a standard discussion can be found in [28], [40], [42]. We note that, in such a case, $(D_B)^*$ is of the form $D^t_{B^\#}$ where D^t is the formal adjoint of D and $B^\# \in C^\infty(\partial M, \operatorname{Hom}(E_1, F_1))$ for a certain vector bundle $F_1 \to \partial M$. This adjoint boundary condition can be described as follows. By the generalized Green formula,

$$(Du, v)_{L^2(M)} - (u, D^t v)_{L^2(M)} = -\int_{\partial M} (Gu, v)\, dV,$$

where, for $x \in \partial M$, $G(x) = \sigma_D(x, \nu)$, ν being the unit normal to ∂M (see [35, p. 285]). Then $B^\#(x)$ is the orthogonal projection of E_{1x} onto $G(x)(\ker B(x)) = F_{1x}$. It is a standard result that, if (D, B) is coercive, so is $(D^t, B^\#)$.

Under this coerciveness condition,

$$\mathscr{D}(D_B) = \{u \in H^1(M, E_0) \colon B\gamma u = 0\}, \tag{3.19}$$

and $D_B \colon \mathscr{D}(D_B) \to L^2(M, E_1)$ is Fredholm. Hence the operator $T \colon H_0 \to H_1$ defined by (3.7) is Fredholm. Therefore, in this case $[D_B]$ actually defines an element of $K_0(\overline{M})$. As is well known, in many cases, D has no coercive local boundary condition; we will see in §4 that the image of the class $[D] \in K_0(M, \partial M)$ under the boundary map to $K_1(\partial M)$ provides an obstruction to the existence of such a boundary condition.

Another type of boundary problem which will play an important role in this paper is the following. Let $\overline{M}$ be a compact complex manifold, with smooth boundary, endowed with a Hermitian metric. We have the $\overline{\partial}$ operator

$$\overline{\partial}: \Lambda^{0,p}(\overline{M}) \to \Lambda^{0,p+1}(\overline{M}) \tag{3.20}$$

and its formal adjoint

$$\mathfrak{D}: \Lambda^{0,p}(\overline{M}) \to \Lambda^{0,p-1}(\overline{M}), \tag{3.21}$$

where $\Lambda^{0,p}(\overline{M})$ denotes the space of smooth $(0,p)$-forms on $\overline{M}$. In particular, consider

$$D = \overline{\partial} + \mathfrak{D}: \Lambda^{0,\text{even}}(\overline{M}) \to \Lambda^{0,\text{odd}}(\overline{M}). \tag{3.22}$$

Let

$$\mathscr{D}^p = \{u \in \Lambda^{0,p}(\overline{M}): (\overline{\partial} r) \rfloor u = 0 \text{ on } \partial M\}, \tag{3.23}$$

where r is a real valued function in $C^\infty(\overline{M})$ satisfying $r = 0$ on ∂M and $dr \neq 0$ on ∂M. Let D_N denote the closure of D restricted to $\bigoplus_{p\text{ even}} \mathscr{D}^p$. Then $\Box = D_N^* D_N$ is the complex Laplacian acting on $(0,p)$-forms for p even, likewise for $\Box = D_N D_N^*$ acting on $(0,p)$-forms for p odd. D_N and D_N^* define closed unbounded operators from H_0 to H_1, where

$$H_0 = L^2_{0,\text{even}}(M), \qquad H_1 = L^2_{0,\text{odd}}(M). \tag{3.24}$$

Typically, $D_N^* D_N$ does not have compact resolvent on H_0. In fact, for $p = 0$, the boundary condition (3.23) is vacuous, so $\ker D_N$ in $L^2_{0,0}(M)$ consists of the space of all holomorphic functions in $L^2(M)$, which is typically an infinite dimensional space. However, in many important cases the operator $\Box$ with $\overline{\partial}$-Neumann boundary conditions is known to have compact resolvent on $L^2_{0,p}(M)$ for all $p \neq 0$. The first result on this is due to Kohn [32]; he proved subelliptic estimates (with loss of one derivative) which imply such compactness whenever M is *strongly pseudoconvex*. Such compactness holds whenever the $\overline{\partial}$-Neumann problem is subelliptic with loss of less than two derivatives on $(0,p)$-forms for $p \neq 0$. In recent times a great deal has been learned about weakly pseudoconvex domains satisfying such a condition. Kohn [33] showed that, if M is a pseudoconvex domain in $\mathbb{C}^n$ with real analytic boundary, then such a regularity result holds provided that there is no germ of a (one complex dimensional) complex manifold contained within ∂M. Characterizations of such subellipticity in terms of "finite p-type" for ∂M smooth have come out of the work of d'Angelo [23] and Catlin [16].

It is clear that the locality condition (3.4) holds for D_N, and hence Proposition 3.1 applies. In view of the importance of this result for developments in later sections, we make a formal statement.

Proposition 3.3. *Let M be a complex domain, with a Hermitian metric, which is relatively compact with smooth boundary ∂M. Suppose the complex Laplacian $\Box$, with $\overline{\partial}$-Neumann boundary conditions, has compact resolvent on $L^2_{0,p}(M)$ for $p \neq 0$. Then, with $T = D_N(D_N^* D_N + 1)^{-1/2}$, the pair (M, T) defines a relative cycle for $K_0(M, \partial M)$. In particular, this occurs when M is strongly pseudoconvex. If we denote the class in $K_0(M, \partial M)$ by $[D_N]$, then*

$$[D_N] = [D_{\max}] \quad \text{in } K_0(M, \partial M), \tag{3.25}$$

$D_{\max}$ *denoting the maximal extension of the operator* (3.22).

We note that there are closed extensions of elliptic differential operators, defined by *nonlocal* boundary conditions, to which the results of this section do not apply, although the results of §2 do apply. See Appendix A for a brief discussion of some special cases.

4. The boundary map on $K_0(M, \partial M)$, and applications

We recall the characterization of the boundary map

$$\partial : K^0(\mathfrak{A}, \mathfrak{A}/\mathcal{J}) \to K^1(\mathfrak{A}/\mathcal{J}) \tag{4.1}$$

given in [5], [8]. Let (σ, T) be a cycle defining an element of $K^0(\mathfrak{A}, \mathfrak{A}/\mathcal{J})$, so (1.10)–(1.12) are satisfied. We also suppose T is a partial isometry, at least mod $\mathcal{K}$, so (1.14) holds. If K_j denotes the range of P_j and $\mathcal{Q}(K_j)$ the associated Calkin algebra, $\mathcal{L}(K_j)/\mathcal{K}$, we define

$$\tau_j : \mathfrak{A}/\mathcal{J} \to \mathcal{Q}(K_j) \tag{4.2}$$

by

$$\tau_j(f) = \pi(P_j \sigma_j(\tilde{f}) P_j), \tag{4.3}$$

where $\tilde{f} \in \mathfrak{A}$ is any preimage of f under $\mathfrak{A} \to \mathfrak{A}/\mathcal{J}$. The right side of (4.3) depends only on f, not on the choice of $\tilde{f}$, by (1.12), and (1.14) implies τ_j is a homomorphism of algebras. Thus each τ_j defines an element $[\tau_j] \in K^1(\mathfrak{A}/\mathcal{J})$, and we have

$$\partial[(\sigma, T)] = [\tau_0] - [\tau_1]. \tag{4.4}$$

It is proved in [8] that this agrees with the boundary map $\partial : KK(\mathcal{J}, \mathbb{C}) \to KK^1(\mathfrak{A}/\mathcal{J}, \mathbb{C})$ defined by Kasparov, under the isomorphism of relative and Kasparov K-homology groups. In particular, it is shown in [8] that the following diagram commutes, with the vertical arrows all representing isomorphisms, and horizontal rows exact:

(4.5)
$$\begin{array}{ccccccc} K^0(\mathfrak{A}) & \to & K^0(\mathfrak{A},\mathfrak{A}/\mathcal{J}) & \xrightarrow{\partial} & K^1(\mathfrak{A}/\mathcal{J}) & \to & K^1(\mathfrak{A}) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ KK(\mathfrak{A},\mathbb{C}) & \to & KK(\mathcal{J},\mathbb{C}) & \xrightarrow{\partial} & KK^1(\mathfrak{A}/\mathcal{J},\mathbb{C}) & \to & KK^1(\mathfrak{A},\mathbb{C}) \end{array}$$

We will examine the image under (4.1) of elements defined by elliptic operators considered in §3, in the case $\mathfrak{A} = C(\overline{M})$, $\mathcal{J} = C_0(M)$, M a manifold of the type considered in §3. In this case, we are looking at the boundary map

$$\partial\colon K_0(M,\partial M) \to K_1(\partial M). \tag{4.6}$$

Suppose D_B is a closed extension of an elliptic differential operator, with local boundary conditions, satisfying the hypotheses of Proposition 3.1. For definiteness, suppose $D_B D_B^*$ has compact resolvent. We define the element

$$[\ker D_B] \in K_1(\partial M) \tag{4.7}$$

to be the class of the cycle

$$\tau\colon C(\partial M) \to \mathcal{Q}(\ker D_B) \tag{4.8}$$

given by (4.3), with $j = 0$, P_0 the orthogonal projection of $L^2(M, E_0)$ onto $\ker D_B = \ker T$ and $\tilde{f}$ any element of $C(\overline{M})$ whose restriction to ∂M is f. In view of the discussion above, we have the following consequence of Proposition 3.1.

Proposition 4.1. *For D_B as above,*

$$\partial[D_B] = [\ker D_B] \quad \text{in } K_1(\partial M). \tag{4.9}$$

In particular, this result holds for $D_{\max}$. For any D_B satisfying the hypotheses here, we hence have

$$[\ker D_B] = [\ker D_{\max}] \quad \text{in } K_1(\partial M). \tag{4.10}$$

This result follows from (4.4) together with the fact that $[\tau_1] = 0$ in $K_1(\partial M)$ since τ_1 represents $C(\partial M)$ on $\mathcal{Q}(K_1)$ and K_1 is finite dimensional.

For Proposition 4.1, no smoothness of ∂M is required. We proceed to derive further consequences for the sorts of boundary problems considered in §3, where ∂M is smooth. First, note that if D_B is defined by coercive boundary conditions, we have $[\tau_0] = 0$ in $K_1(\partial M)$ as well as $[\tau_1] = 0$, since both Hilbert spaces κ_0 and κ_1 are finite dimensional in this case. Applying Proposition 4.1 to this case, we have the following conclusion.

Corollary 4.2. *If there exists a coercive local boundary condition for D, then $\partial[D] = 0$ in $K_1(\partial M)$. Consequently, in this case, one must have*

$$[\ker D_{\max}] = 0 \quad \text{in } K_1(\partial M). \tag{4.11}$$

Below we will see examples where (4.11) is clearly violated. In such examples it is well known that no coercive local boundary conditions exist.

When M has smooth boundary, it is desirable to have an intrinsic characterization of $[\ker D_{\max}] \in K_1(\partial M)$ in terms of a pseudodifferential operator on ∂M, the *Calderon projector* $Q \in \mathrm{OPS}^0(\partial M)$, acting on sections of $E_0|_{\partial M}$. This operator satisfies $Q^2 = Q, Q = Q^*$ on $L^2(\partial M, E_0)$, and also the following condition. For any $s \in \mathbb{R}$, consider

$$\mathfrak{p}_s = \{u \in H^s(M, E_0) : Du = 0\}. \tag{4.12}$$

Then the trace map γ takes $\mathfrak{p}_s$ into $H^{s-1/2}(\partial M, E_0)$, and

$$\gamma(\mathfrak{p}_s) = Q(H^{s-1/2}(\partial M, E_0)) \stackrel{\text{def}}{=} \mathscr{Q}_{s-1/2}. \tag{4.13}$$

The homomorphism $C(\partial M) \to \mathscr{Q}$ (Range Q) given by $\tau(f) = \pi(QM_f)$ defines a class

$$[Q] \in K_1(\partial M). \tag{4.14}$$

The construction of the Calderon projector can be found in several standard treatments of pseudodifferential operators; see [37], [40], [42]. We aim to prove the following.

Proposition 4.3. *If Q is the Calderon projector associated to the elliptic operator D, then*

$$[\ker D_{\max}] = [Q] \quad \text{in } K_1(\partial M). \tag{4.15}$$

Proof. We produce a Fredholm map from the Hilbert space $\mathfrak{p}_0$ to $\mathscr{Q}_0 = Q(L^2(\partial M, E_0)) = \gamma(\mathfrak{p}_{1/2})$ which gives rise to this equivalence. First, γ maps $\mathfrak{p}_0$ onto $\mathscr{Q}_{-1/2}$, with at most finite dimensional kernel. Now, let A_0 be any elliptic operator in $\mathrm{OPS}^{-1/2}(\partial M, E_0)$, with scalar principal symbol, and set

$$A = QA_0Q + (1 - Q)A_0(1 - Q). \tag{4.16}$$

This operator has the same principal symbol as A_0, and it commutes with Q, since $AQ = QA_0Q = QA$. Thus $A: \mathscr{Q}_{-1/2} \to \mathscr{Q}_0$ is Fredholm and the composition $A \circ \gamma : \mathfrak{p}_0 \to \mathscr{Q}_0$ is Fredholm. It remains to show that, for $f \in C(\partial M)$, $f = \tilde{f}|_{\partial M}$, and $\tilde{f} \in C(\overline{M})$,

$$(A \circ \gamma)PM_{\tilde{f}} - QM_f(A \circ \gamma) : \mathfrak{p}_0 \to \mathscr{Q}_0 \quad \text{is compact,} \tag{4.17}$$

where P is the orthogonal projection of $L^2(M, E_0)$ onto $\mathfrak{p}_0 = \ker D_{\max}$. Applying a parametrix for A which commutes with Q, we see that it suffices to prove

$$\gamma P M_{\tilde{f}} - Q M_f \gamma \colon \mathfrak{p}_0 \to \mathscr{Q}_{-1/2} \quad \text{is compact} \tag{4.18}$$

for $f = \tilde{f}|_{\partial M}$, $\tilde{f} \in C^\infty(\overline{M})$. Note that $\gamma M_{\tilde{f}} - M_f \gamma = 0$ on $\mathfrak{p}_0$, or equivalently

$$\gamma M_{\tilde{f}} P - M_f Q \gamma = 0 \quad \text{on } \mathfrak{p}_0. \tag{4.19}$$

In view of the standard compactness of $[M_f, Q] \in \mathrm{OPS}^{-1}(\partial M)$ on $H^{-1/2}(\partial M)$, to prove (4.18) it suffices to show that

$$\gamma \circ [M_{\tilde{f}}, P] \colon L^2(M, E_0) \to H^{-1/2}(\partial M, E_0) \quad \text{is compact} \tag{4.20}$$

for $\tilde{f} \in C^\infty(\overline{M})$. In fact, we clearly have

$$[M_{\tilde{f}}, P] \colon L^2(M, E_0) \to \mathscr{D}(D_{\max}), \tag{4.21}$$

bounded, while, as noted at the end of Proposition 1.1, (1.14) holds, so this commutator is compact on $L^2(M, E_0)$. Therefore (4.20) holds, by Lemma 3.2. This completes the proof of Proposition 4.3.

From Proposition 4.3 we can deduce what can be perceived as the "odd case" of the cobordism invariance of the index. Indeed, suppose $\overline{M}$ is a compact manifold with boundary, $\dim M$ even. Let $E_j \to \overline{M}$ be smooth Hermitian vector bundles, and $D \colon C^\infty(\overline{M}, E_1)$ a first order elliptic differential operator. Suppose M has a Riemannian metric and $\sigma_D(x, \xi) \colon E_{0x} \to E_{1x}$ is proportional to an isometry, in fact say $\|\sigma_D(x, \xi) v\| = \|\xi\| \cdot \|v\|$. Of course, $[D] \in K_0(M, \partial M)$, and Proposition 4.3 identifies $\partial[D] \in K_1(\partial M)$. We can reinterpret this result, as follows.

Let ν denote the unit (inward) normal field on ∂M. Then, for $x \in \partial M$,

$$\tau(x) = (1/i)\sigma_D(x, \nu) \colon F_{0x} \to F_{1x} \tag{4.22}$$

defines an isomorphism of the vector bundles F_0 and F_1, where $F_j \to \partial M$ is the restriction of E_j to ∂M. Using ν to define an injection $T^*_x(\partial M) \to T^*_x(M)$ for $x \in \partial M$, we see that

$$\tau(x)^{-1} \sigma_D(x, \xi) \colon F_{0x} \to F_{0x} \qquad (x \in \partial M,\ \xi \in T^*_x(\partial M)) \tag{4.23}$$

is the symbol of a first order elliptic differential operator $D^\#$ on ∂M:

$$D^\# \colon C^\infty(\partial M, F_0) \to C^\infty(\partial M, F_0). \tag{4.24}$$

Expanding the identity

$$\sigma_D(x, \xi + s\nu)^* \sigma_D(x, \xi + s\nu) = \|\xi\|^2 + s^2$$

for $x \in \partial M$ and $\xi \in T_x^*(\partial M)$ implies $\tau(x)^{-1} = \tau(x)^*$, and hence

$$\sigma_{D^\#}(x,\xi) = \sigma_{D^\#}(x,\xi)^*, \tag{4.25}$$

as well as

$$\|\sigma_{D^\#}(x,\xi)v\| = \|\xi\| \cdot \|v\|. \tag{4.26}$$

Furthermore it is clear that the principal symbol of the Calderon projector Q for D is the projection onto the sum of the positive eigenspaces of $\sigma_{D^\#}(x,\xi)$, so we have the following result.

Proposition 4.4. *With D and $D^\#$ as above,*

$$\partial[D] = [D^\#] \quad \text{in } K_1(\partial M). \tag{4.27}$$

A particular case of the situation described above is the following. Suppose M is an even dimensional spin^c manifold with boundary. If $E \to M$ is a smooth Hermitian vector bundle and D_E an associated ("twisted") Dirac operator (uniquely determined up to a zero order term), we have

$$[D_E] \in K_0(M, \partial M). \tag{4.28}$$

If $F \to \partial M$ is the restriction of $E \to M$, we have a Dirac operator D_F on ∂M and an element

$$[D_F] \in K_1(\partial M), \tag{4.29}$$

which coincides with $D_E^\#$ under the construction above. Therefore,

$$\partial[D_E] = [D_F]. \tag{4.30}$$

We will discuss the analogue in the "even case" in the next section, and show how it implies the well-known cobordism invariance of the index of elliptic operators.

The computation (4.30) provides examples where the conclusion (4.11) of Corollary 4.2 does not hold, since standard index calculations show that (4.30) is not generally zero in $K_1(\partial M)$. This reproduces the well-known fact that there are typically no coercive local boundary conditions for the Dirac operator.

We proceed now to the computation of $\partial[D_N]$ when $D_N = \overline{\partial} + \mathfrak{D}$ is the closed operator from $L^2_{0,\text{even}}(M)$ to $L^2_{0,\text{odd}}(M)$ with domain described in §3 (see (3.23)). M is a complex manifold with smooth boundary. We suppose M is a weakly pseudoconvex domain with the property:

(4.31) $\square = (\overline{\partial} + \mathfrak{D})^2$, with $\overline{\partial}$-Neumann boundary conditions, has compact resolvent on $L^2_{0,p}(M)$ for $p \neq 0$, which holds in particular if M is strongly pseudoconvex. In this case, Proposition 3.3 (as well as 1.1 and 3.1) applies. Hence $\ker D_N^*$ is finite dimensional and $\ker D_N$ is at most a finite dimensional perturbation of the summand

(4.32) $$H^+(M) = \ker D_N \cap L^2_{0,0}(M) = \{u \in L^2(M) : \overline{\partial} u = 0\},$$

the "Hardy space" of L^2 holomorphic functions on M. We have the "Toeplitz extension"

(4.33) $$[\tau_M] \in K_1(\partial M)$$

given by $\tau_M : C(\partial M) \to \mathscr{Q}(H^+(M))$ as

(4.34) $$\tau_M(f) = P_+(\tilde{f}u) \pmod{\mathscr{K}},$$

$\tilde{f}$ denoting any continuous extension of f to $\overline{M}$. Consequently

(4.35) $$\partial[D_N] = [\tau_M] \quad \text{in } K_1(\partial M)$$

under our hypotheses.

Now there is another way we can look at the operator $D = \overline{\partial} + \mathfrak{D}$, and that is as a *Dirac operator* on M, with spin^c-structure induced from its complex structure. Then $\Lambda^{0,\text{even}}$ and $\Lambda^{0,\text{odd}}$ make up the space of even and odd spinors. Consequently the formula (4.30) applies; we have

(4.36) $$\partial[\overline{\partial} + \mathfrak{D}] = [D_{\partial M}] \quad \text{in } K_1(\partial M),$$

where $D_{\partial M}$ is the Dirac operator on ∂M, with its naturally induced spin^c-structure. We collect our major conclusions in the following result, noting also that (4.35) and (4.36) must be equal.

Proposition 4.5. *If M is a complex manifold which is pseudoconvex and satisfies* (4.31), *then* $[\overline{\partial} + \mathfrak{D}] \in K_0(M, \partial M)$ *has image* $\partial[\overline{\partial} + \mathfrak{D}] \in K_1(\partial M)$ *given by* (4.35) *and* (4.36). *In particular, in this case,*

(4.37) $$[\tau_M] = [D_{\partial M}] \quad \textit{in } K_1(\partial M).$$

As noted in [6], the identity (4.37) immediately implies the index theorem [13] of Boutet de Monvel as a consequence of the Atiyah-Singer index theorem, as a special case of the intersection product

$$K^1(X) \times K_1(X) \to K_0(X), \qquad X = \partial M.$$

Thus (4.37) can be viewed as a refinement, in K-homology, of Boutet de Monvel's index theorem. That this refinement holds for a broader class of pseudoconvex domains than those which are strongly pseudoconvex also implies that Proposition 4.5 *extends* Boutet de Monvel's index theorem to the class of pseudoconvex domains for which (4.31) holds, e.g., the pseudoconvex domains considered by Kohn [33] and by Catlin [16]. Proposition 4.5 has additional uses in index theory, particularly via the identity (4.43) which we establish below. This identity will be applied in §6.

In parallel with Proposition 4.3, we want to relate the class of the Toeplitz extension $[\tau_M] \in K_1(\partial M)$ with the class given by the Szegö projector S, acting on $L^2(\partial M)$, defined to be the orthogonal projection of $L^2(\partial M)$ onto the set of boundary values of holomorphic functions in $H^{1/2}(M)$. In case M is strongly pseudoconvex, a detailed analysis of S has been given by Boutet de Monvel and Sjöstrand [15] (see also [41] for an alternative treatment). In that case, one knows that $S \in \mathrm{OPS}^0_{1/2,1/2}(\partial M)$. Thus, for any $A_0 \in \mathrm{OPS}^{-1/2}(\partial M)$, the commutator $[A_0, S]$ belongs to $\mathrm{OPS}^{-1}_{1/2,1/2}(\partial M)$, and $[M_f, S] \in \mathrm{OPS}^{-1/2}_{1/2,1/2}(\partial M)$ for $f \in C^\infty(\partial M)$. Thus the proof of Proposition 4.3 is easily modified to yield the following result.

Proposition 4.6. *When $\overline{M}$ is a compact complex manifold with boundary which is strongly pseudoconvex, we have*

$$[\tau_M] = [S] \quad \text{in } K_1(\partial M). \tag{4.38}$$

Consequently,

$$[S] = [D_{\partial M}] \quad \text{in } K_1(\partial M). \tag{4.39}$$

It is quite possible that such a result extends to the class of weakly pseudoconvex domains for which Proposition 4.5 holds. We remark that the identity (4.37) is in a sense more fundamental than (4.39), and (4.38)–(4.39) are only particularly useful when one has a rather precise hold on the Szegö projector S. Since at present one has a satisfactory understanding of S only when M is strongly pseudoconvex, we have not tried seriously to extend Proposition 4.6 beyond this case.

Our next result will apply Proposition 4.6 to a special class of strongly pseudoconvex domain. Namely, let X be a compact smooth manifold (without boundary); suppose X is endowed with a real analytic structure. Then, as is well known, the ball bundle B^*X can be given the structure of a strongly pseudoconvex domain (in a noncanonical fashion). In this case, $\partial(B^*X) = S^*X$ is the sphere bundle in T^*X. In addition to the Toeplitz extension $[\tau_{B^*X}] \in K_1(S^*X)$, there is the element

$$\mathscr{P}_X \in K_1(S^*X), \tag{4.40}$$

the "pseudodifferential operator extension",

$$0 \to \mathscr{K}(H) \to \overline{\mathrm{OPS}^0(X)} \to C(S^*X) \to 0, \tag{4.41}$$

where $H = L^2(X)$. We have the following important identities.

Proposition 4.7. *For any compact smooth manifold X,*

$$\mathscr{P}_X = [S] \quad \textit{in } K_1(S^*X). \tag{4.42}$$

Consequently, if D is the Dirac operator on S^*X determined by its spin^c-*structure,*

$$\mathscr{P}_X = [D] \quad \textit{in } K_1(S^*X). \tag{4.43}$$

Proof. The identity (4.42) is proved using a construction previewed by Boutet de Monvel in [13] and carried out in the appendix to [14] of a Fourier integral operator F with complex phase mapping $\mathscr{D}'(X)$ to $\mathscr{D}'(S^*X)$, satisfying (modulo smoothing operators)

$$F^*F = I, \qquad FF^* = S, \tag{4.44}$$

providing a Fredholm map from $L^2(X)$ onto the range of S in $L^2(S^*X)$. If σ_A in $C^\infty(S^*X)$ is the principal symbol of $A \in \mathrm{OPS}^0(X)$, then the operator calculus developed in [14] implies

$$A = F^*(SM_{\sigma_A})F \qquad \bmod \mathrm{OPS}^{-1/2}_{1/2,1/2}(X), \tag{4.45}$$

so the extension $C(S^*X) \to \mathscr{Q}(L^2(X))$ and the extension $C(S^*(X)) \to \mathscr{Q}$ (Range S) are seen to coincide in $K_1(S^*X)$. This proves (4.42), and (4.43) follows from Proposition 4.6.

As we have mentioned, Proposition 4.7 has further applications to index theory, which will be discussed in §6.

5. The boundary map on $KK^1(C_0(M), \mathbb{C})$

Part of the exact sequence of Kasparov K-theory is the segment

(5.1)

$$KK^1(C(M), \mathbb{C}) \to KK^1(C_0(M), \mathbb{C}) \xrightarrow{\partial} KK(C(\partial M), \mathbb{C}) \to KK(C(M), \mathbb{C}),$$

where M is a compact manifold with boundary. We have the identification

$$KK(C(\partial M), \mathbb{C}) = K_0(\partial M). \tag{5.2}$$

We want as explicit as possible an identification of the boundary map ∂, on the level of cycles. The formula given here will be derived from that of §4, i.e., the formula for

$$\partial\colon K_0(M, \partial M) \to K_1(\partial M), \tag{5.3}$$

via use of the Bott map. The Bott map provides isomorphism:

$$\begin{array}{ccc} KK^1(C_0(M),\mathbb{C}) & \xrightarrow{\quad\partial\quad} & KK(C(\partial M),\mathbb{C}) \\ \otimes_b \big\downarrow \approx & & \otimes_b \big\downarrow \approx \\ KK(C_0(I)\otimes C_0(M),\mathbb{C}) & & KK^1(C_0(I)\otimes C(\partial M),\mathbb{C}) \end{array} \tag{5.4}$$

These vertical isomorphisms are given by Kasparov products. We take

$$KK^1(C_0(M),\mathbb{C}) = KK(C_0(M),\mathbb{C}_1). \tag{5.5}$$

The Kasparov product

$$KK(C_0(I),\mathbb{C}_1)\otimes KK(C_0(M),\mathbb{C}_1) \to KK(C_0(I)\otimes C_0(M),\mathbb{C}_1\otimes\mathbb{C}_1) \tag{5.6}$$

produces an isomorphism

$$\otimes_b\colon KK(C_0(M),\mathbb{C}_1) \overset{\approx}{\to} KK(C_0(I)\otimes C_0(M),\mathbb{C}_1\otimes\mathbb{C}_1) \tag{5.7}$$

upon applying the element

$$b \in KK(C_0(I),\mathbb{C}_1) = KK^1(C_0(I),\mathbb{C}) \tag{5.8}$$

defined by any closed extension of the symmetric operator

$$id/dx \quad \text{on } C_0^\infty(I). \tag{5.9}$$

One follows this with the natural isomorphism

$$KK(\mathscr{A},\mathbb{C}_1\otimes\mathbb{C}_1) \approx KK(\mathscr{A},\mathbb{C}) \tag{5.10}$$

(see Blackadar [10]).

One natural closed extension to take of (5.9) is via *periodic* boundary conditions. This also produces

$$b' \in KK^1(C(S^1),\mathbb{C}). \tag{5.11}$$

In fact, (5.8) and (5.11) correspond under the natural isomorphism

$$KK^1(C(S^1),\mathbb{C}) \overset{\approx}{\to} KK^1(C_0(I),\mathbb{C}), \tag{5.12}$$

which follows from the Kasparov exact sequence associated to the short exact sequence

$$0 \to C_0(I) \to C(S^1) \to \mathbb{C} \to 0. \tag{5.13}$$

This is a special case (for M = pt.) of the cohomology exact sequence arising from the split short exact sequence

$$0 \to C_0(I\times X) \overset{j}{\to} C(S^1\times X) \underset{\kappa}{\overset{\rho}{\rightleftarrows}} C(X) \to 0 \tag{5.14}$$

for compact X (e.g., $X = \partial M$). The Kasparov exact sequence is:
(5.15)

$$\begin{array}{ccccc} KK(C_0(I \times X), \mathbb{C}) & \xrightarrow{\partial} & KK^1(C(X), \mathbb{C}) & \underset{\kappa^*}{\overset{\rho^*}{\rightleftarrows}} & KK^1(C(S^1 \times X), \mathbb{C}) \\ j^* \uparrow & & & & \downarrow j^* \\ KK(C(S^1 \times X), \mathbb{C}) & \underset{\kappa^*}{\overset{\rho^*}{\leftrightarrows}} & KK(C(X), \mathbb{C}) & \xleftarrow{\partial} & KK^1(C_0(I) \otimes C(X), \mathbb{C}) \end{array}$$

One has $\kappa^* \rho^* = \mathrm{id}$, since $\rho\kappa = \mathrm{id}$ on $C(X)$. Hence ρ^* is injective, which implies $\partial = 0$ in this case, and consequently gives the split short exact sequence

$$(5.16) \quad 0 \to KK^1(C(X), \mathbb{C}) \underset{\kappa^*}{\overset{\rho^*}{\rightleftarrows}} KK^1(C(S^1 \times X), \mathbb{C}) \xrightarrow{j^*} KK^1(C_0(I) \otimes C(X), \mathbb{C}) \to 0$$

and its complement. Note that $KK^1(C(X), \mathbb{C}) = K_1(X)$, which is 0 if X is a point; this proves (5.12). We also see that

$$(5.17) \qquad \otimes_b : KK(C(X), \mathbb{C}) \to KK(C_0(I) \otimes C(X), \mathbb{C}_1)$$

factors through $\otimes_{b'}$:

$$(5.18) \qquad \begin{array}{ccc} KK(C(X), \mathbb{C}) & \xrightarrow{\otimes_{b'}} & KK^1(C(S^1 \times X), \mathbb{C}) \\ & \searrow^{\approx} & \downarrow j^* \\ & & KK^1(C_0(I) \otimes C(X), \mathbb{C}) \end{array}$$

In particular, we deduce that

$$(5.19) \qquad \otimes_{b'} \text{ is injective.}$$

From (5.16) we see that $\otimes_{b'}$ provides an isomorphism of $KK(C(X), \mathbb{C}) = K_0(X)$ with

$$(5.20) \qquad \{u \in KK^1(C(S^1 \times X), \mathbb{C}) = K_1(S^1 \times X) : \kappa^* u = 0\},$$

where κ^* is defined by the projection $\pi_X : S^1 \times X \to X; \kappa^* = \pi_{X^*}$. This result is also given in [11]; there the map $K_0(X) \to K_1(S^1 \times X)$ is explicitly defined as follows. If (σ, T) defines a cycle in $K_0(X)$, consider the commutative C^*-algebra in $\mathscr{L}/\mathscr{K} = \mathscr{Q}$ generated by $\sigma(a)$, $a \in C(X)$, and by the unitary part of $\pi(T) \in \mathscr{Q}$, provided $\sigma_0 = \sigma_1$ on $H_0 = H_1$. This defines a *-homomorphism $C(S^1 \times X) \to \mathscr{Q}$, which gives the desired element of $K_1(S^1 \times X)$.

We have a similar result replacing $C(X)$ by $C_0(M)$. We obtain from

$$(5.21) \qquad 0 \to C_0(I \times M) \xrightarrow{j} C_0(S^1 \times M) \underset{\kappa}{\overset{\rho}{\rightleftarrows}} C_0(M) \to 0$$

the short exact sequence

$$(5.22)\quad 0 \to KK^1(C_0(M),\mathbb{C}) \underset{\kappa^*}{\overset{\rho^*}{\rightleftarrows}} KK^1(C_0(S^1\times M),\mathbb{C}) \xrightarrow{j^*} KK^1(C_0(I\times M),\mathbb{C}) \to 0.$$

The reason we bring out these facts is that, in order to analyze ∂ in (5.4), it is convenient to factor through:

$$(5.23)\quad \begin{array}{ccc} KK^1(C_0(M),\mathbb{C}) & \xrightarrow{\partial} & KK(C(\partial M),\mathbb{C}) \\ {\scriptstyle \otimes_{b'}}\downarrow & & \downarrow{\scriptstyle \otimes_{b'}} \\ KK(C_0(S^1\times M),\mathbb{C}) & \xrightarrow{\partial} & KK^1(C(S^1\times\partial M),\mathbb{C}) \\ {\scriptstyle j^*}\downarrow & & \downarrow{\scriptstyle j^*} \\ KK(C_0(I\times M),\mathbb{C}) & & KK^1(C_0(I\times\partial M),\mathbb{C}) \end{array}$$

This is a commutative diagram, by the associativity of the Kasparov product. The point is that the *second* ∂ in (5.23) has been evaluated in §4, since

$$(5.24)\qquad KK(C_0(S^1\times M),\mathbb{C}) \approx K_0(S^1\times M, S^1\times\partial M).$$

The injectivity (5.19) implies that this will be an effective tool for specifying the top ∂ in (5.23).

One application we make of these constructions is the following. Suppose $\overline{M}$ is a compact spin^c manifold with boundary, $E \to \overline{M}$ a smooth Hermitian vector bundle. We have Dirac operators

$$(5.25)\qquad [D_E] \in KK^j(C_0(M),\mathbb{C}), \qquad [D_F] \in KK^{j+1}(C(\partial M),\mathbb{C}),$$

where $F = E|_{\partial M}$ is the restricted bundle; $F \to \partial M$. Here $j = 0$ if $\dim M$ is even, $j = 1$ if $\dim M$ is odd, and $j+1$ is computed mod 2. As shown in §4, if M has even dimension, and we consider

$$(5.26)\qquad \begin{array}{ccc} \partial\colon KK(C_0(M),\mathbb{C}) & \to & KK^1(C(\partial M),\mathbb{C}) \\ \| & & \| \\ K_0(M,\partial M) & & K_1(\partial M) \end{array}$$

then

$$(5.27)\qquad \partial[D_E] = [D_F].$$

Our result here is:

Proposition 5.1. *If M has odd dimension, and we consider*

$$(5.28)\qquad \begin{array}{ccc} \partial\colon KK^1(C_0(M),\mathbb{C}) & \longrightarrow & KK(C(\partial M),\mathbb{C}) \\ & & \| \\ & & K_0(\partial M) \end{array}$$

then

$$\partial[D_E] = [D_F]. \tag{5.29}$$

In view of our analysis via (5.23), the following is a key ingredient in the proof.

Lemma 5.2. *The map*

$$\otimes_{b'}: KK^j(C_0(M), \mathbb{C}) \to KK^{j+1}(C_0(S^1 \times M), \mathbb{C}) \tag{5.30}$$

with $j+1$ *computed* mod 2, *has the property that*

$$\otimes_{b'}[D_E] = [D_G] \tag{5.31}$$

where G *is the pull back of* $E \to M$, *so* $G \to S^1 \times M$. *In* (5.31), *one uses* (5.30) *with* $j = 0$ *if* $\dim M$ *is even, and* $j = 1$ *if* $\dim M$ *is odd.*

Proof. This follows from the fact that D_E and D_G define unbounded Kasparov modules for $C_0(M)$ and $C_0(S^1 \times M)$, together with the result of Baaj and Julg [4] which implies that the Kasparov product of D_E with $[id/dx] \in KK^1(C(S^1), \mathbb{C})$ is given by $D_I \hat{\otimes} I + I \hat{\otimes} D_E$, where D_I is the operator associated to id/dx on the graded $(C(S^1), \mathbb{C}_1)$-bimodule, and $\hat{\otimes}$ is the appropriate graded tensor product. The identification of this with D_G is standard Clifford algebra.

The proof of Proposition 5.1 is now immediate. By commutativity of (5.23), and by (5.27),

$$\otimes_{b'} \circ \partial[D_E] = [D_G], \tag{5.32}$$

while, by Lemma 5.2, we have

$$\otimes_{b'}[D_F] = [D_G], \tag{5.33}$$

where F is the restriction of E to ∂M and G the pull back of F to $S^1 \times \partial M$. The injectivity (5.19) hence implies Proposition 5.1.

From commutativity and exactness in

$$\begin{array}{ccccc} KK^1(C_0(M), \mathbb{C}) \to & KK(C(\partial M), \mathbb{C}) \to & KK(C(M), \mathbb{C}) \\ & \| & \| \\ & K_0(\partial M) & K_0(M) \\ & & \downarrow \\ & & K_0(\text{pt.}) \end{array} \tag{5.34}$$

we deduce the following spin^c-cobordism invariance of the index of Dirac operators.

Corollary 5.3. *If* $Y = \partial M$ *is an even dimensional compact* spin^c *manifold, and* $F \to Y$ *is* spin^c*-cobordant to* 0, *then*

$$\operatorname{Index} D_F = 0. \tag{5.35}$$

Consequently, if Y_1 and Y_2 are even dimensional compact spin^c *manifolds and $F_1 \to Y_1$ and $F_2 \to Y_2$ are* spin^c*-cobordant Hermitian vector bundles, then*

$$\operatorname{Index} D_{F_1} = \operatorname{Index} D_{F_2}. \tag{5.36}$$

The same sort of arguments produce the following "even" analogue of Proposition 4.4. Let $E \to M$ be a Hermitian vector bundle over M ($\dim M$ odd), endowed with a Riemannian metric, and suppose $D\colon C^\infty(M,E) \to C^\infty(M,E)$ is a first order elliptic differential operator satisfying

$$\sigma_D(x,\xi)^* = -\sigma_D(x,\xi),\ \sigma_D(x,\xi)^2 = -\|\xi\|^2. \tag{5.37}$$

Consequently D defines an element $[D] \in K_1(M,\partial M)$. Define bundles $E^+, E^- \to M$ by

$$E_x^{\pm} = \{e \in E_x\colon \sigma_D(x,\nu)e = \pm ie\}, \qquad x \in \partial M, \tag{5.38}$$

where ν is the unit inward normal to ∂M. Then $E|_{\partial M} = E^+ \oplus E^-$, and, for $\xi \in T_x^*(\partial M)$, $\sigma_D(x,\xi)\colon E_x^+ \to E_x^-$ isomorphically, so it defines the principal symbol of a first order elliptic differential operator $D^\#$ on ∂M:

$$D^\#\colon C^\infty(\partial M, E^+) \to C^\infty(M, E^-). \tag{5.39}$$

Generalizing Proposition 5.1 and Corollary 5.3, we have:

Proposition 5.4. *Under the hypotheses above,*

$$\partial[D] = [D^\#] \quad \text{in } K_0(\partial M). \tag{5.40}$$

Consequently,

$$\operatorname{Index}[D^\#] = 0. \tag{5.41}$$

The conclusion (5.41) is the principal result of Chapter XVII of the notes [35] on the Atiyah-Singer index theorem. In particular, it applies to signature operators. These exist on arbitrary Riemannian manifolds, without extra structure like a spin^c-structure, and this provides a more flexible tool for proving index theorems than Corollary 5.3, since the ordinary cobordism ring (with vector bundles) has a much simpler structure than the spin^c-cobordism ring. This cobordism invariance of the index was the major analytic point in the proof of the Atiyah-Singer index theorem described in [35].

6. Further remarks on index theory

In §5 we derived the spin^c-cobordism invariance of the index of Dirac operators. This, together with the elementary multiplicativity of the index of Dirac operators, is one of the main analytical tools in a proof of the index theorem for Dirac operators, on a spin^c manifold M, which states

$$\text{(6.1)} \qquad \operatorname{Index} D_E = (ch(E) \cup Td(TM))[M],$$

where D_E is the Dirac operator on $S \otimes E$, S being the bundle of spinors. Of course, the bordism invariance implied by Proposition 5.1 and the exact sequence (5.34), and its analogue for KK proved in §4, (4.27), are also key analytic results for proving the isomorphism

$$\text{(6.2)} \qquad K_j^t(X) \xrightarrow{\mu} K_j^a(X),$$

where $K_j^a(X) = KK^j(C(X), \mathbb{C})$ is the analytic K-homotopy group and $K_j^t(X)$ the topological K-homology group. A map on the level of cycles, which is proposed to lead to the isomorphism (6.2), has been described in [5], [6]. As pointed out in these papers, one can derive the index theorem for Dirac operators, indeed for general pseudodifferential operators on a compact manifold, from the existence and commutativity of the diagram

$$\text{(6.3)} \qquad \begin{tikzcd} & K^0(\hat{M}) \arrow[dl, "i_t"'] \arrow[dr, "i_a"] & \\ K_0^t(M) \arrow[rr, "\mu"] & & K_0^a(M) \end{tikzcd}$$

for a general compact smooth manifold M. Here, $\hat{M}$ is the double of the unit ball bundle of M (endowed with a Riemannian metric), with a spin^c structure, arising from its natural almost complex structure. The map i_a assigns to a vector bundle over $\hat{M}$ an elliptic symbol whereby such a bundle is created from two bundles over M by "clutching", and thence the element in $K_0^a(M)$ determined by an associated elliptic pseudodifferential operator.

The map $\mu \circ i_t$ can be described as follows. The spin^c structure on $\hat{M}$ together with a vector bundle $E \to \hat{M}$ determines a Dirac operator D_E on sections of $S \otimes E \to \hat{M}$ and hence an element of $K_0^a(\hat{M})$; $\mu \circ i_t([E])$ is the image of $[D_E]$ under the map $K_0^a(\hat{M}) \to K_0(M)$ defined by the natural projection $\pi \colon \hat{M} \to M$. Consequently, a major ingredient in the proof of commutativity of (6.3) is the following result (a special case of such commutativity), which can be stated without specifically bringing topological K-homology.

Proposition 6.1. *We have a commutative diagram:*

$$\text{(6.4)} \qquad \begin{tikzcd} K^0(\hat{M}) \arrow[d, "\cap[D]"'] \arrow[dr, "i_a"] & \\ K_0(\hat{M}) \arrow[r, "\pi_*"] & K_0(M) \end{tikzcd}$$

Before we give a detailed proof of Proposition 6.1, let us note that it immediately reduces the general problem of computing the index of a pseudodifferential operator to the computation of the index of Dirac operators. If E_0 and E_1 are complex vector bundles over M, and

$$P: C^\infty(M, E_0) \to C^\infty(M, E_1) \tag{6.5}$$

is an elliptic pseudodifferential operator, whose principal symbol provides a clutching map to define a vector bundle

$$E \to \hat{M}, \tag{6.6}$$

and if

$$D_E: C^\infty(\hat{M}, S_+ \otimes E) \to C^\infty(\hat{M}, S_- \otimes E) \tag{6.7}$$

is the associated Dirac operator, then the content of Proposition 6.1 is that

$$\pi_*[D_E] = [P] \in K_0(M). \tag{6.8}$$

Since the index is obtained via the unique map to a point:

$$\begin{tikzcd} K_0(\hat{M}) \arrow[r, "\pi_*"] \arrow[dr] & K_0(M) \arrow[d] \\ & K_0(\text{pt.}) = \mathbb{Z} \end{tikzcd}$$

then (6.8) implies

$$\operatorname{Index} P = \operatorname{Index} D_E. \tag{6.9}$$

Formula (6.1) for the index of a Dirac operator gives the general index formula

$$\operatorname{Index} P = (ch(E) \cup Td(T\hat{M}))[\hat{M}]. \tag{6.10}$$

The reducibility of the general index problem to that for Dirac operators is well known, but it seems to be accomplished most directly via Proposition 6.1. This analysis also explains why the general index formula has such a close formal resemblance to the index formula for Dirac operators, as opposed say to signature operators. It is a happy coincidence that very simple and accessible direct proofs of the index theorem for Dirac operators have recently become available [26], [27], [9].

Our proof of Proposition 6.1 will make use of Lemma 4.13, proved in [7], which states the following. Consider

$$\mathscr{P}_M \in K_1(S^*M), \tag{6.11}$$

the pseudodifferential operator extension,

$$0 \to \mathscr{K}(H) \to \overline{\mathrm{OPS}^0(M)} \to C(S^*M) \to 0 \tag{6.12}$$

where $H = L^2(M)$. Then there is a natural commutative diagram

$$\begin{tikzcd} K^0(S^*M) \arrow[d, "\cap\mathscr{P}_M"'] \arrow[dr, "\rho_*"] & \\ K_1(S^*M) \arrow[r, "\pi_*"] & K_1(M) \end{tikzcd} \tag{6.13}$$

having the following property. If $A \in \mathrm{OPS}^m(M)$ is an elliptic selfadjoint operator defining a class $[A] \in K_1(M)$, $A\colon C^\infty(M, E_0) \to C^\infty(M, E_0)$, and if $E_+ \to S^*M$ is the vector subbundle of $\pi^*E_0 \to S^*M$ which is the direct sum of the positive eigenspaces of the symbol of A, then

$$\pi_*(E_+ \cap \mathscr{P}_M) = [A] \quad \text{and} \quad \rho_*([E_+]) = [A]. \tag{6.14}$$

In view of the identity (4.43) proved in Proposition 4.7, we can replace $\mathscr{P}_M$ by $[D]$ in (6.13), where D is the Dirac operator on S^*M, with its natural spin^c-structure. It is in this form that we will use (6.13) below.

We can prove Proposition 6.1 by replacing M by $S^1 \times M$ in (6.13), using the injection

$$K_0(M) \overset{\otimes_{b'}}{\hookrightarrow} K_1(S^1 \times M) \tag{6.15}$$

described in §5, and its analogue with M replaced by $\hat{M}$, together with the identification

$$S^*(S^1 \times M) = S^1 \times \hat{M}.$$

Then (6.13) yields the commutative diagram:

$$\begin{tikzcd} K^0(S^1 \times \hat{M}) \arrow[d, "\cap[D_{S^1 \times \hat{M}}]"'] \arrow[dr, "\rho_*"] & \\ K_1(S^1 \times \hat{M}) \arrow[r, "\pi_*"] & K_1(S^1 \times M) \end{tikzcd}$$

We also have the following commutative diagrams, with $j\colon S^1 \times \hat{M} \to \hat{M}$ the natural projection:

$$\begin{tikzcd} K^0(\hat{M}) \arrow[r, "j^*"] \arrow[d, "\cap[D_{\hat{M}}]"'] & K^0(S^1 \times \hat{M}) \arrow[d, "\cap[D_{S^1 \times \hat{M}}]"] \\ K_0(\hat{M}) \arrow[r, "\otimes_{b'}"] & K_1(S^1 \times \hat{M}) \end{tikzcd}$$

Furthermore,

$$\begin{tikzcd} K^0(\hat{M}) \arrow[r, "j^*"] \arrow[d, "i_a"'] & K^0(S^1 \times \hat{M}) \arrow[d, "\rho_*"] \\ K_0(M) \arrow[r, "\otimes_{b'}"] & K_1(S^1 \times M) \end{tikzcd}$$

and also

$$\begin{tikzcd}
K_0(\hat{M}) \arrow[r, "\otimes_{b'}"] \arrow[d, "\pi_*"'] & K_1(S^1 \times \hat{M}) \arrow[d, "\pi_*"] \\
K_0(M) \arrow[r, "\otimes_{b'}"] & K_1(S^1 \times M)
\end{tikzcd}$$

A straightforward diagram chase shows that the diagram

$$\begin{tikzcd}
K^0(\hat{M}) \arrow[r, "i_a"] \arrow[d, "\cap[D_{\hat{M}}]"'] & K_0(M) \arrow[dr, "\otimes_{b'}"] & \\
 & & K_1(S^1 \times M) \\
K_0(\hat{M}) \arrow[r, "\pi_*"] & K_0(M) \arrow[ur, "\otimes_{b'}"'] &
\end{tikzcd}$$

is commutative, and the injectivity of (6.15) completes the proof of Proposition 6.1.

We also point out that Proposition 3.8 of Connes-Skandalis [21] gives a result related to Proposition 6.1; in fact their result applies also to families of elliptic operators.

A. Boundary conditions of Atiyah-Patodi-Singer type

As we indicated at the end of §3, there are boundary conditions defining closed extensions of elliptic differential operators which satisfy all the hypotheses of Proposition 3.1, *except* that the boundary conditions are not local. In such a case the compactness result of Proposition 3.1 need not hold. Since there are important examples of this phenomenon which occur naturally, we give some further discussion here. Before discussing details we emphasize that the closed extensions considered here do fall within the framework of §2, giving cycles in $KK(C_0(M), \mathbb{C})$.

The operators we consider here are closed extensions of an elliptic operator of first order D on M, a compact manifold with smooth boundary, given by a boundary condition $Qu = 0$ on ∂M, where Q is an orthogonal projection in $\mathrm{OPS}^0(\partial M)$ which has the same principal symbol as the Calderon projector associated to D. This class of boundary problems includes those, which we denote D_{APS}, investigated by Atiyah, Patodi, and Singer [2], who pointed out the role of a nonlocal invariant, the eta invariant, associated to an operator on ∂M, in the formula for the index of D_{APS}. The prescription of §3 does not generally define a cycle for $K_0(M, \partial M)$ in this case. If it did, the Fredholm property would actually produce an element of $K_0(M)$, and $\partial[D_{\mathrm{APS}}]$ would equal 0 in $K_1(\partial M)$.

Since $[D_{\mathrm{APS}}] = [D_{\max}]$ in $KK(C_0(M), \mathbb{C}) \approx K_0(M, \partial M)$, we see this is impossible whenever it can be verified that $\partial[D_{\max}] \neq 0$ in $K_1(\partial M)$, a result that frequently occurs, as we have noted in §4. On the other hand, D_{APS} does produce a cycle for $K^0(\mathfrak{A}, \mathfrak{A}/C_0(M))$, where $\mathfrak{A}$ is $C_0(M)$ with the identity adjoined (as a constant function on M), and hence an element of $K_0(M^\#)$, where $M^\#$ is obtained from M by collapsing ∂M to a point; $M^\#$ is thus typically a space with a conic-type singularity. We denote the associated element of $K_0(M^\#)$ by $\{D_{\mathrm{APS}}\}$.

As an example of this last phenomenon, we consider in detail a family of operators on the unit disc $\Omega = \{z \in \mathbb{C}\colon |z| \leq 1\}$, defining a family of homology classes in $K_0(S^2)$. Take

$$D = \partial/\partial\bar{z} \quad \text{on } \Omega. \tag{A.1}$$

Then $[D] = [D_{\max}] \in K_0(\Omega, \partial\Omega)$, with $\partial\Omega = S^1$. Define operators $D_{(k)}$ by

$$\operatorname{Dom}(D_{(k)}) = \left\{ u \in H^1(\Omega)\colon u|_{S^1} = \sum_{n \leq k} a_n e^{in\theta} \right\}. \tag{A.2}$$

Thus each $D_{(k)}$ is defined by a nonlocal boundary condition like that of Atiyah-Patodi-Singer, and

$$\operatorname{Index} D_{(k)} = k + 1. \tag{A.3}$$

In this case, $\Omega^\# = \Omega/\partial\Omega = S^2$, and we have

$$\{D_{(k)}\} \in K_0(S^2). \tag{A.4}$$

We can get an explicit hold on these elements by considering the following commutative diagram with exact rows:

$$\begin{tikzcd}
 & K_0(S^1) \arrow[r, "\alpha", "\approx"'] & K_0(\Omega) \arrow[r, "\beta=0"] \arrow[d, "\gamma"'] & K_0(\Omega, S^1) \arrow[r, "\partial", "\approx"'] \arrow[d, "\approx", "{[D]}"'] & K_1(S^1) \arrow[r] & 0 \\
0 \arrow[r] & K_0(\mathrm{pt.}) \arrow[ur, "\approx"] \arrow[r, shift left, "\varphi"] & K_0(S^2) \arrow[l, shift left, "\mathrm{ind}"] \arrow[r, "\eta"] & K_0(S^2, \mathrm{pt.}) \arrow[rr, "\partial"] & & 0 \\
 & & \{D_{(k)}\} \arrow[u, phantom, sloped, "\in"] & & &
\end{tikzcd}$$

One has an isomorphism

$$\begin{aligned} \mathrm{ind} \oplus \eta\colon K_0(S^2) \to {} & K_0(\mathrm{pt.}) \oplus K_0(S^2, \mathrm{pt.}), \\ & \quad \| \\ & \quad \mathbb{Z} \end{aligned} \tag{A.5}$$

$$(\mathrm{ind} \oplus \eta)\{D_{(k)}\} = (k+1, [D_{\max}]). \tag{A.6}$$

Note also that, if $\mathscr{D}$ denotes the Dirac operator on S^2, associated to its complex structure, defining

$$[\mathscr{D}] \in K_0(S^2), \tag{A.7}$$

then

$$\eta[\mathscr{D}] = [D] \in K_0(\Omega, S^1) \approx K_0(S^2, \text{pt.}). \tag{A.8}$$

Since Index $\mathscr{D} = 1$, we have the identity

$$[\mathscr{D}] = \{D_{(0)}\} \quad \text{in } K_0(S^2). \tag{A.9}$$

In this case, $\mathscr{D} = \overline{\partial}$, taking 0-forms to $(0,1)$-forms, and Index $\mathscr{D} = 1$ is a special case of the Riemann-Roch theorem. The sphere S^2 also has a *spin* structure, with associated spin^c structure differing from the one above by a factor of a line bundle whose square is the canonical bundle. The associated Dirac operator $\mathscr{D}'$ then satisfies Index $\mathscr{D}' = 0$, so $[\mathscr{D}'] = \{D_{(-1)}\}$ in $K_0(S^2)$.

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