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SCHWARZ'S LEMMA FOR CIRCLE PACKINGS. II

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To Kotaro Oikawa on his 60th birthday

Introduction

In [6] it was shown that the Riemann mapping function is the limit of circle packing isomorphisms as the circles shrink to points. In [5] and the present paper this connection between conformal mapping and circle packing is investigated further. We are able to estimate the rate of convergence of the circle packing isomorphisms and to obtain information about the relationship of the derivative of the Riemann mapping function to the ratio of the radii of image circle to preimage circle under these isomorphisms.

In [6] the following question was raised: Under the circle packing isomorphisms, does the ratio of the radius of an image circle to that of its preimage converge to the modulus of the derivative of the Riemann mapping function? It is known that the answer is affirmative if the convergence is taken in the L^p norm on compact subsets for some $p > 2$ (see §1). In §4 we prove that the convergence actually holds in the norm of Bounded Mean Oscillation (BMO) on compact subsets (Theorem 6). A key ingredient of the proof is the Schwarz Lemma analog for circle packings [5]. As another application of this Schwarz Lemma analog—specifically, the BMO estimates—we obtain a sufficient condition for the convergence to be uniform. This condition, which is much weaker than the previously known condition, refers to the hexagonal packing constants s_n introduced in [6].

For $n \geq 2$ the constant s_n is defined as the smallest number with the property that, for any circle packing which is combinatorially equivalent to n generations of the regular hexagonal circle packing, the ratio ρ of the radius of a circle of generation 1 to the radius of the circle of generation 0 satisfies $|1 - \rho| \leq s_n$. It is shown in [6] that $s_n \to 0$ as $n \to \infty$.

It had been known previously (see §5) that if $s_n = O(1/n)$ as $n \to \infty$, then the convergence of the ratio-of-radii function to the modulus of the derivative of the Riemann mapping function is uniform on compact

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subsets. In Theorem 8 of the present paper it is shown that the weaker condition $s_n = o(1/\log^{4+\varepsilon} n)$ is sufficient for the convergence to be uniform on compact subsets. The proof uses the BMO convergence as well as results from §3 on the rate of convergence of the circle packing isomorphisms.

In §3 we obtain estimates on the rate of convergence of the circle packing isomorphisms to the Riemann mapping function. Theorem 5 shows that this rate is of order at most $(\sqrt{s_{[1/2\varepsilon^{1/2}]}} + \varepsilon^{1/4})^{1/p}$, where ε is the size of the preimage circles, s_n is the packing constant defined above and $p > 2$. Lemma 3.2 is a combinatorial-geometric result on circle packings of a disk. Its Corollary 3.3 represents a strengthening of the Length Area Lemma [6] from an estimate of order $1/\log(1/\varepsilon)$ to one of order $\varepsilon^{\pi^2/24}$. This corollary can be considered as an analog for circle packings of the fact that a conformal mapping onto the disk sends points of the domain whose distance from the boundary is of order ε to points whose distance from the boundary of the disk is at most of order $\sqrt{\varepsilon}$.

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1. Circle packing and conformal mapping

In this section and the next we recall the notation and some results from [5], [6]. A *circle packing* in the plane is a collection of circles whose interiors are disjoint. The *nerve* of a circle packing is the graph with a vertex for each circle in the packing; two vertices are joined by an edge if and only if the circles which correspond to the vertices are tangent. Two circle packing are said to be *combinatorially equivalent* if the associated nerves are isomorphic graphs. There is a natural imbedding in the plane of the nerve of a circle packing—the vertices become the centers of the circles, and the edges become the line segments joining tangent circles.

Let Ω be a bounded region in the plane and let z_0, z_1 be points of Ω. For $\varepsilon > 0$ let HCP(ε), the *regular hexagonal circle packing of the plane by circles of radius* ε, consist of circles of radius ε centered at the points of the lattice $\{2\varepsilon(m + n\omega) : m, n \in \mathbf{Z}\}$, where $\omega = e^{i\pi/3}$.

If ε is sufficiently small we "approximate" Ω by a collection Ω_ε of circles from HCP(ε) defined as follows. Let c_0 be a circle from HCP(ε) whose distance from z_0 is less than ε. Form chains $c_0, c_1, c_2, \cdots, c_j$ of circles from HCP(ε) such that each one is tangent to its predecessor, and each one together with its six tangent neighbors lies in Ω. The set of circles which appear in such chains is denoted I_ε and called the set of *inner circles*. The circles of $\mathrm{HCP}_\varepsilon - I_\varepsilon$ which are tangent to inner circles form B_ε, the set of *border circles*. The line segments connecting the centers of pairs of tangent border circles form a closed Jordan polygon which is everywhere within a distance 3ε from the boundary of Ω. All of the inner circles lie inside this polygon. We define $\Omega_\varepsilon = I_\varepsilon \cup B_\varepsilon$.

According to an idea of W. Thurston, one can apply his version of Andreev's Theorem to obtain a circle packing Ω'_ε, which is combinatorially equivalent to the circle packing Ω_ε and is a *packing of the unit disk* $\mathbf{D}$; that is, the circles of Ω'_ε lie in $\mathbf{D}$ and those which correspond to the border circles of Ω_ε are internally tangent to $\mathbf{D}$ ([6]; see also [4] and [7, Theorem 13.7.1]). Let $c \to c'$ denote the isomorphism which maps a circle of Ω_ε to the corresponding circle of Ω'_ε. Recall that c_0 was a circle of Ω_ε which was within a distance ε from z_0; let c_1 be a circle of Ω_ε which is within a distance ε from z_1. By performing a Möbius transformation of the disk, we may require that the circle c'_0 is centered at the origin and that the circle c'_1 is centered on the real axis.

In [6] it is shown that the correspondence $c \to c'$ of $\Omega_\varepsilon \to \Omega'_\varepsilon$ may be viewed as an approximate mapping of Ω into the unit disk and that as $\varepsilon \to 0$ this approximate mapping converges to the Riemann mapping function normalized to send z_0 to the origin and z_1 to the positive real axis. Let us state this result in more detail.

Consider the complex T_ε of triangles formed by line segments joining the centers of each pair of tangent circles in Ω_ε. There is a corresponding complex T'_ε associated to the circle packing Ω'_ε. The isomorphism $c \to c'$ of $\Omega_\varepsilon \to \Omega'_\varepsilon$ gives a mapping of the vertices of the complexes T_ε and T'_ε. This mapping extends to a piecewise linear mapping $f_\varepsilon \colon |T_\varepsilon| \to |T'_\varepsilon|$, where $|T_\varepsilon|$, the *carrier* of T_ε, denotes the union of the triangles of T_ε, and similarly for $|T'_\varepsilon|$. In [6] the proof is given for

Theorem 1. *The piecewise linear mappings $f_\varepsilon \colon |T_\varepsilon| \to |T'_\varepsilon|$ are uniformly k-quasiconformal. As $\varepsilon \to 0$, f_ε converges uniformly on compact subsets of Ω to the Riemann mapping function $f \colon \Omega \to \mathbf{D}$ normalized by $f(z_0) = 0$, $f(z_1) > 0$.*

Consider again the isomorphism $c \to c'$ of $\Omega_\varepsilon \to \Omega'_\varepsilon$. If z is the center of c, define

$$r_\varepsilon(z) = (\text{radius of } c')/(\text{radius of } c).$$

This defines a map r_ε on the vertices of T_ε, and it is natural to ask if r_ε tends to $|f'|$ as $\varepsilon \to 0$. It will be convenient to extend r_ε to a continuous map on all of $|T_\varepsilon|$,

$$r_\varepsilon \colon |T_\varepsilon| \to \mathbf{R}, \tag{1.1}$$

for example, by taking the piecewise linear barycentric extension. In [6] the following problem was raised.

Question. As $\varepsilon \to 0$, does $r_\varepsilon\colon |T_\varepsilon| \to \mathbf{R}$ converge to $|f'|\colon \Omega \to \mathbf{R}$ uniformly on compact subsets of Ω?

Such questions about the convergence of $r_\varepsilon \to |f'|$ can be reduced to questions about the convergence of $|\partial f_\varepsilon/\partial z| \to |f'|$ by means of (2.4) below. In [5] it was shown that the complex dilatation

$$\mu_\varepsilon \equiv (\partial f_\varepsilon/\partial \bar{z})/(\partial f_\varepsilon/\partial z) \tag{1.2}$$

of f_ε tends to zero in the L^∞ norm on compacta of Ω. A well-known result [3, Theorem V.5.3] states: *If w_n is a sequence of k-quasiconformal mappings of a region Ω, w_n converges uniformly on compacta to a quasiconformal mapping w, and the complex dilatations of w_n converge a.e. to the complex dilatation of w, then for some $p > 2$ the derivatives $\partial w_n/\partial z$ and $\partial w_n/\partial \bar{z}$ converge to $\partial w/\partial z$ and $\partial w/\partial \bar{z}$ in the L^p norm on compacta of Ω.* Therefore, for some $p > 2$,

$$\frac{\partial f_\varepsilon}{\partial z} \to \frac{\partial f}{\partial z}, \qquad \frac{\partial f_\varepsilon}{\partial \bar{z}} \to 0 \tag{1.3}$$

and, using (2.4),

$$r_\varepsilon \to |f'| \tag{1.4}$$

in the L^p norm on compacta. In §4 we prove a result which is stronger than this L^p convergence although not as strong as uniform convergence; namely, the result that $r_\varepsilon \to |f'|$ in the BMO norm on compacta of Ω as $\varepsilon \to 0$.

2. The Schwarz Lemma for circle packings

In [5, Theorem 5.1] the following analog of Schwarz's Lemma was proved.

Theorem 2. *There is an absolute constant a with the following property. For any $N \geq 1$ let HCP_N be N generations of the regular hexagonal circle packing, and D the smallest disk which contains HCP_N. Let HCP'_N be any circle packing combinatorially equivalent to HCP_N and also contained in D. Then $R'_0 \leq aR_0$, where R_0 and R'_0 are the radii of the generation zero circles in HCP_N and HCP'_N respectively.*

Fix a compact subset K in Ω and then consider a sufficiently small $\varepsilon > 0$. Let $d = \mathrm{dist}(K, \partial\Omega)$ and $N = [d/2\varepsilon] - 1$ (*greatest integer* notation). Then every $c \in \Omega_\varepsilon$ with $c \cap K \neq \varnothing$ is the center of an HCP_N configuration contained in Ω_ε (the symbols HCP_N and HCP'_N are defined in Theorem

2 above). The isomorphism $\Omega_\varepsilon \to \Omega'_\varepsilon$ makes this HCP_N configuration correspond to an HCP'_N configuration with center c' contained in the unit disk. If Theorem 2 is applied (after rescaling the HCP_N configuration by a scaling factor approximately equal to $1/d$), we can conclude that the ratio of the radii of c' to c is bounded above by a constant independent of ε. This proves

Theorem 3. *As $\varepsilon \to 0$ the mappings $r_\varepsilon\colon |T_\varepsilon| \to \mathbf{R}$ of (1.1) are uniformly bounded on compact subsets of Ω.*

The ratio-of-radii function r_ε in (1.1) can be compared with the partial derivative $\partial f_\varepsilon/\partial z$ of the quasiconformal mapping f_ε. With the help of the Schwarz Lemma analog (Theorem 2) this will lead to the result that $\partial f_\varepsilon/\partial \overline{z} \to 0$ in L^∞ on compact subsets of Ω, a fact which does not follow from the weaker result [6] that $\mu_\varepsilon \to 0$ in L^∞ on compact subsets.

Suppose three mutually tangent ε-circles c_1, c_2, c_3 in Ω_ε correspond to c'_1, c'_2, c'_3 under the isomorphism $\Omega_\varepsilon \to \Omega'_\varepsilon$. Let ρ_i be the radius of c'_i. The map f_ε sends the center of c_i to the center of c'_i and is linear inside the triangle of centers. An explicit computation (Lemma 6.4 of [5]) shows that inside this triangle

$$(2.1)\qquad 2\varepsilon^2\left|\frac{\partial f_\varepsilon}{\partial z}\right|^2 = \frac{1}{12}[(\rho_1+\rho_2)^2+(\rho_2+\rho_3)^2+(\rho_3+\rho_1)^2] + \frac{1}{\sqrt{3}}[\rho_1\rho_2\rho_3(\rho_1+\rho_2+\rho_3)]^{1/2},$$

$$(2.2)\qquad 2\varepsilon^2\left|\frac{\partial f_\varepsilon}{\partial \overline{z}}\right|^2 = \frac{1}{12}[(\rho_1+\rho_2)^2+(\rho_2+\rho_3)^2+(\rho_3+\rho_1)^2] - \frac{1}{\sqrt{3}}[\rho_1\rho_2\rho_3(\rho_1+\rho_2+\rho_3)]^{1/2}.$$

Let K be the triangle formed by the centers of c_1, c_2, c_3, let $d = \operatorname{dist}(K, \partial\Omega)$, and let $N = [d/2\varepsilon] - 1$. As we saw earlier, each c'_i is the center of an HCP'_N configuration. Therefore, by definition of the packing constants s_N,

$$(2.3)\qquad \rho_2 = \rho_1(1+O(s_N)), \qquad \rho_3 = \rho_1(1+O(s_N)),$$

where $|O(s_N)/s_N|$ is bounded (in fact by 1) as $\varepsilon \to 0$ and $N = N(\varepsilon) \to \infty$. When (2.1) and (2.2) are simplified by means of (2.3) we find that

$$\left|\frac{\partial f_\varepsilon}{\partial z}\right|^2 = r_\varepsilon^2(1+O(s_N)), \qquad \left|\frac{\partial f_\varepsilon}{\partial \overline{z}}\right|^2 = r_\varepsilon^2 O(s_N),$$

where $r_\varepsilon = \rho_1/\varepsilon$ is the ratio-of-radii function evaluated at any point of K. If we make use of the boundedness of r_ε on compacta (Theorem 3), we can write these equations as

$$\left|\frac{\partial f_\varepsilon}{\partial z}\right|^2 = r_\varepsilon^2 + O_K(s_N), \qquad \left|\frac{\partial f_\varepsilon}{\partial \bar{z}}\right|^2 = O_K(s_N),$$

where $O_K(\cdot)$ denotes a bounded function whose bound depends on K.

Theorem 4. *Let $K \Subset \Omega$ and $n = [\mathrm{dist}(K, \partial\Omega)/2\varepsilon]$. Then the following equations hold on the interiors of the triangles of T_ε which meet K:*

$$\left|\frac{\partial f_\varepsilon}{\partial z}\right| = r_\varepsilon(1 + O(s_n)) = r_\varepsilon + O_K(s_n), \tag{2.4}$$

$$\left|\frac{\partial f_\varepsilon}{\partial \bar{z}}\right| = r_\varepsilon O(\sqrt{s_n}) = O_K(\sqrt{s_n}), \tag{2.5}$$

$$|\mu_\varepsilon| = O(\sqrt{s_n}), \tag{2.6}$$

where $O(\cdot)$ denotes a function which is bounded by an absolute constant, and $O_K(\cdot)$ denotes a bounded function whose bound depends on K.

3. The rate of convergence of $f_\varepsilon \to f$

Let $W_\varepsilon = |T_\varepsilon|$ and $D_\varepsilon = |T'_\varepsilon|$. Consider the Riemann mapping functions $G_{W_\varepsilon}: W_\varepsilon \to \mathbf{D}$ and $G_{D_\varepsilon}: D_\varepsilon \to \mathbf{D}$, which are to be normalized by the conditions $G_{W_\varepsilon}(z_0) = 0$, $G_{W_\varepsilon}(z_1) > 0$ and $G_{D_\varepsilon}(0) = 0$, $G_{D_\varepsilon}(f_\varepsilon(z_1)) > 0$. We shall estimate $\|f_\varepsilon - f\|_K$ by three comparisons according to the right-hand side of the inequality

(3.1)

$$|f(z) - f_\varepsilon(z)| \le |f(z) - G_{W_\varepsilon}(z)| + |G_{W_\varepsilon}(z) - G_{D_\varepsilon} \circ f_\varepsilon \circ G_{W_\varepsilon}^{-1}(G_{W_\varepsilon}(z))| + |G_{D_\varepsilon}(f_\varepsilon(z)) - f_\varepsilon(z)|.$$

The first term on the right compares the Riemann map f of Ω with the Riemann map of a domain W_ε very near to Ω. The second term compares $G_{D_\varepsilon} \circ f_\varepsilon \circ G_{W_\varepsilon}^{-1}$, a quasiconformal selfmap of $\mathbf{D}$ with small dilatation on compacta, with the identity map. The third term compares the Riemann map $G_{D_\varepsilon}(\zeta)$ of the near-disk D_ε with the identity map ζ. To estimate the first term we can appeal to a classical result on Riemann maps of nearby domains (Lemma 3.1). To estimate the third term we will need Lemma 3.1 as well as a result (Lemma 3.2 and its Corollary 3.3) which provides an estimate of how close ∂D_ε is to $\partial \mathbf{D}$ (an estimate of this kind, although not sharp enough for our purposes here, can be obtained from the Length Area Lemma of [6]). To estimate the second term we use a result (Lemma 3.4) on estimating quasiconformal selfmaps of $\mathbf{D}$.

Lemma 3.1. *Let Ω be a bounded simply connected region containing z_0 and z_1, let $K \Subset \Omega$ and let $f: \Omega \to \mathbf{D}$ be the Riemann map normalized by $f(z_0) = 0$ and $f(z_1) > 0$. Then there is a constant C with the following property. If W is any simply connected region containing z_0 and z_1, $K \Subset W \subset \Omega$, and $\operatorname{dist}(z, \partial\Omega) \leq \delta$ for all $z \in \partial W$, then*

$$\|f(z) - G_W(z)\|_K \leq C\sqrt{\delta}, \tag{3.2}$$

where $G_W: W \to \mathbf{D}$ is the Riemann map of W normalized by $G_W(z_0) = 0$, $G_W(z_1) > 0$. If Ω is the unit disk, and z_1 is real, then (3.2) can be improved to

$$\|z - G_W(z)\|_K \leq C\delta. \tag{3.3}$$

Lemma 3.1 follows from Lemmas 3 and 5 of Warschawski [8].

The next lemma is a strengthening of the Length Area Lemma of [6]. It refers to the circle packing Ω'_ε of the unit disk $\mathbf{D}$ and will be used to show that the distance from D_ε to $\partial\mathbf{D}$ is at most of order $O(\varepsilon^{1/4})$. A *border circle* of Ω'_ε is a circle c which is tangent (internally) to the unit circle. If n is not too large, the circles in Ω'_ε of generation n about a border circle c will form a *cross cut chain* which separates c from the origin. There are at most $6n$ circles in the cross cut chain of generation n about c. The *geometric length* of a cross cut chain is the sum of the diameters of the circles in the chain.

Lemma 3.2. *There is an absolute constant C with the following property. Let c be a border circle of Ω'_ε, and suppose that the cross cut chains about c of generations $1, 2, \cdots, N$ separate c from the origin. Then the minimum s of the geometric lengths of the cross cuts of generations $k, k+1, \cdots, 2k-1$ (where $2k \leq N$) satisfies*

$$s \leq C(k/N)^{\pi^2/24}. \tag{3.4}$$

Proof. Consider a border circle c. Let s_i be the geometric length of the ith generation cross cut chain about c. Let n_i be the number of circles in the ith cross cut chain ($n_i \leq 6i$), let r_{ij} be the radius of the jth circle in the ith cross cut chain ($1 \leq j \leq n_i$), and let $a_i = \sum_{j=1}^{n_i} \pi r_{ij}^2$ be the area of the ith cross cut chain. Then

$$s_i^2 = \left[\sum_{j=1}^{n_i} 2r_{ij}\right]^2 \leq 4n_i \sum_{j=1}^{n_i} r_{ij}^2 \leq \frac{24}{\pi} i a_i. \tag{3.5}$$

Consequently, for any subset I of $\{1, \cdots, N\}$,

$$\min\{s_i^2 : i \in I\} \sum_{i \in I} \frac{1}{i} \leq \frac{24}{\pi} \sum_{i \in I} a_i \leq 24. \tag{3.6}$$

The inequality between the first and last terms above is the basis of the Length Area Lemma in [6]. It shows that when N is large there will exist small s_i. We will obtain a substantial improvement of that estimate by using the isoperimetric inequality (amounting to an improvement from $O(1/\log(1/\varepsilon))$ to $O(\varepsilon^{\pi^2/24})$; see Corollary 3.3 below).

Note that (3.6) yields $s^2 \le 24/(k^{-1}+\cdots+(2k)^{-1}) = O(1)$; thus (3.4) is easy to prove in case k/N is bounded away from zero. For the remainder of the proof we may therefore assume k/N is as small as necessary. Note that from (3.6) we also obtain $\min\{s_i^2 : \alpha N < i \le N\} \le O(1/\log(1/\alpha))$. Choose $\alpha > 0$ so small that this last term $O(1/\log(1/\alpha))$ is less than 1. We can now be sure that, for some $i > \alpha N$, the ith cross cut chain will have length less than 1 and hence be completely contained in a half disk of $\mathbf{D}$. The consequence we use is that for any border circle c, the associated cross cut chains of generations $1, \cdots, [\alpha N]$ will all lie in a common half disk D_c of $\mathbf{D}$. We will henceforth assume that $k/N < \alpha/2$.

Perform the linear fractional transformation which sends $\mathbf{D}$ onto the upper half plane $\tilde{\mathbf{D}}$ and carries D_c onto the half disk $\tilde{D}_c = \{re^{it} : r \le 1$ and $0 \le t \le \pi\}$. The packing Ω'_ε of $\mathbf{D}$ is carried[1] to a packing $\tilde{\Omega}'_\varepsilon$ of $\tilde{\mathbf{D}}$. The quantities s_i, r_{ij}, and a_i for Ω'_ε correspond to similar quantities for $\tilde{\Omega}'_\varepsilon$ which will be denoted $\tilde{s}_i$, $\tilde{r}_{ij}$ and $\tilde{a}_i$.

As in (3.5) we have

$$\tilde{s}_i^2 = \left[\sum_{j=1}^{n_i} 2\tilde{r}_{ij}\right]^2 \le 4n_i \sum_{j=1}^{n_i} \tilde{r}_{ij}^2 \le \frac{24}{\pi} i\tilde{a}_i. \tag{3.7}$$

By reflection, the classical isoperimetric inequality implies the following statement: If a cross cut of length L in a half plane, together with a portion of the boundary of the half plane, bounds a finite area A, then $A \le L^2/2\pi$. For this reason we have, for $i \le [\alpha N]$,

$$\tilde{a}_0 + \tilde{a}_1 + \cdots + \tilde{a}_{i-1} \le \frac{1}{2\pi}\tilde{s}_i^2.$$

By means of (3.7) this gives

$$\tilde{a}_0 + \tilde{a}_1 + \cdots + \tilde{a}_{i-1} \le \frac{12}{\pi^2} i\tilde{a}_i. \tag{3.8}$$

It follows from (3.8) that the quantity $(\tilde{a}_0+\tilde{a}_1+\cdots+\tilde{a}_{i-1})^{12/\pi^2}/i$ increases with i for $i = 1, \cdots, [\alpha N]$. {*Proof.*[2] Let $\tilde{A}_i = \tilde{a}_0 + \tilde{a}_1 + \cdots + \tilde{a}_{i-1}$. Note

[1]The purpose of working in $\tilde{\mathbf{D}}$ rather than in $\mathbf{D}$ is that the isoperimetric inequality with free boundary which we are going to use takes a simpler form in $\tilde{\mathbf{D}}$. This simplification, in the context of conformal mapping, is due to Warschawski [9].

[2]The proof is due to Roger W. Barnard (oral communication).

that for $0 < \beta \le 1$ the function $x^\beta - \beta x$ has a maximum at $x = 1$; hence $x^\beta \le \beta x - \beta + 1$. If we take $x = 1 + (1/i)$ and $\beta = \pi^2/12$ this last inequality yields $(1 + (1/i))^{\pi^2/12} \le 1 + (\pi^2/12i)$. From (3.8) we have $\pi^2/12i \le \tilde{a}_i/\tilde{A}_i$. Thus $(1 + (1/i))^{\pi^2/12} \le 1 + (\tilde{a}_i/\tilde{A}_i)$ which is equivalent to $\tilde{A}_i^{12/\pi^2}/i \le (\tilde{A}_i + \tilde{a}_i)^{12/\pi^2}/(i+1)$. Hence $\tilde{A}_i^{12/\pi^2}/i$ increases with i.} Thus

$$\begin{aligned}(\tilde{a}_0 + \tilde{a}_1 + \cdots + \tilde{a}_{i-1})^{12/\pi^2}/i &\le (\tilde{a}_0 + \tilde{a}_1 + \cdots + \tilde{a}_{[\alpha N]-1})^{12/\pi^2}/[\alpha N] \\ &\le (\pi/2)^{12/\pi^2}/[\alpha N],\end{aligned}$$

or

$$\tilde{a}_0 + \tilde{a}_1 + \cdots + \tilde{a}_{i-1} \le C\left(\frac{i}{N}\right)^{\pi^2/12}. \tag{3.9}$$

From (3.7) we obtain

$$\min\{\tilde{s}_i^2 : i \in I\} \sum_{i \in I} \frac{1}{i} \le \frac{24}{\pi} \sum_{i \in I} \tilde{a}_i. \tag{3.10}$$

We apply (3.9) to (3.10) and obtain

$$\tilde{s}^2 = \min\{\tilde{s}_i^2 : k \le i < 2k\} \sum_{i=k}^{2k-1} \frac{1}{i} \le \frac{24}{\pi} \sum_{i=0}^{2k-1} \tilde{a}_i \le C\left(\frac{2k}{N}\right)^{\pi^2/12},$$

or $\tilde{s} \le C(k/N)^{\pi^2/24}$. Since lengths in $\tilde{D}_c$ and D_c differ only by a uniformly bounded factor, this proves (3.4).

We apply Lemma 3.2 with $k = 1$ and note that N can be taken as $O(1/\varepsilon)$, since Ω_ε contains approximately $\operatorname{dist}(z_0, \partial\Omega)/2\varepsilon$ generations about z_0. We conclude that the generation 1 cross cut chain about the border circle c has length $s = O(\varepsilon^{\pi^2/24})$. Since s is an upper bound for the radius of c we obtain

Corollary 3.3. *There is a constant C_Ω (depending on the distance from z_0 to $\partial\Omega$) such that for all sufficiently small $\varepsilon > 0$, each border circle of Ω'_ε has radius at most $C_\Omega \varepsilon^{\pi^2/24}$. Consequently, each point of ∂D_ε is at a distance less than $C_\Omega \varepsilon^{\pi^2/24}$ from $\partial\mathbf{D}$.*

The proof of the next lemma relies heavily on the treatment of the Beltrami equation in Ahlfors-Bers [1]. We shall make use of their notation and conventions without additional explanation.

Lemma 3.4. *Given $0 < \rho_0 < 1$ and $k < 1$, there are constants c and $p > 2$ with the following properties. If g is a μ-conformal homeomorphism ($|\mu| \le k$) of the unit disk $\mathbf{D}$ onto itself normalized by $g(0) = 0$, $g(\rho) > 0$ for some $\rho \in (\rho_0, 1)$, then for any $\delta > 0$ and $z \in \mathbf{D}$*

$$|g(z) - z| \le c(\|\mu\|_{\infty, 1-\delta} + \delta)^{1/p}, \tag{3.11}$$

where the norm is the L^∞ norm on the disk $\{|z| \le 1 - \delta\}$.

Proof. Extend g by reflection to all of $\mathbf{C} \cup \{\infty\}$. Denote the extended map by $\tilde{g}$ and let $\tilde{\mu}$ denote its dilatation. Set $\tilde{\mu} = \mu_1 + \mu_2$, where $\mu_1(z)$ is defined to be $\tilde{\mu}(z)$ for $\{|z| \leq 1/(1-\delta)\}$ and zero elsewhere. Thus $\|\mu_2\|_\infty = \|\mu\|_{\infty,1-\delta}$.

One has $w^{\tilde{\mu}} = w^\lambda \circ w^{\mu_2}$, where $|\lambda| = |\mu_1/(1-\tilde{\mu}\overline{\mu}_2)| \circ (w^{\mu_2})^{-1}$. We shall use the following inequalities:

$$\|w^{\mu_2}(z) - z\|_{\infty,R} \leq c_R\|\mu_2\|_\infty = c_R\|\mu\|_{\infty,1-\delta}, \tag{3.12}$$

$$\|w^\lambda(z) - z\|_{\infty,R} \leq c_R\|\lambda\|_p. \tag{3.13}$$

(3.12) follows immediately from Lemma 17 of Ahlfors-Bers [1]. (3.13) with $p > 2$ follows from the proof of Lemma 17 in [1].

For $|z| \leq 1/(1-\delta)$ use (3.12) and (3.13) with $R = 2$ (we may assume δ and $\|\mu\|_{\infty,1-\delta}$ to be as small as we like) to obtain

$$\begin{aligned} |w^{\tilde{\mu}}(z) - z| &\leq |w^\lambda(w^{\mu_2}(z)) - w^{\mu_2}(z)| + |w^{\mu_2}(z) - z| \\ &\leq c(\|\lambda\|_p + \|\mu\|_{\infty,1-\delta}). \end{aligned} \tag{3.14}$$

Since $w^{\tilde{\mu}}$ and $\tilde{g}$ are both $\tilde{\mu}$-conformal selfmaps of $\mathbf{D}$ which fix 0 they differ by a rotation, $\tilde{g} = e^{i\theta}w^{\tilde{\mu}}$. Furthermore, we find that

$$\sin\theta \leq |w^{\tilde{\mu}}(\rho) - \rho|/|w^{\tilde{\mu}}(\rho)| \leq c(\|\lambda\|_p + \|\mu\|_{\infty,1-\delta}).$$

Thus for $|z| \leq 1$

$$\begin{aligned} |\tilde{g}(z) - z| &\leq |e^{i\theta}w^{\tilde{\mu}}(z) - w^{\tilde{\mu}}(z)| + |w^{\tilde{\mu}}(z) - z| \\ &\leq c(\|\lambda\|_p + \|\mu\|_{\infty,1-\delta}). \end{aligned} \tag{3.15}$$

We estimate $\|\lambda\|_p$ by

$$\begin{aligned} \|\lambda\|_p^p &\leq \iint_{w^{\mu_2}(|z|\leq 1-\delta)} \{|\mu_1/(1-\mu\overline{\mu}_2)| \circ (w^{\mu_2})^{-1}\}^p\,du\,dv \\ &\quad + \iint_{w^{\mu_2}(|z|>1-\delta)} \{|\mu_1/(1-\tilde{\mu}\overline{\mu}_p)| \circ (w^{\mu_2})^{-1}\}^p\,du\,dv \\ &\leq c\|\mu_1\|_{\infty,1-\delta}^p + \iint_{w^{\mu_2}(1-\delta<|z|<1/(1-\delta))} du\,dv. \end{aligned}$$

By (3.12) w^{μ_2} maps $1-\delta \leq |z| \leq 1/(1-\delta)$ into $a \leq |w| \leq b$, where $a = 1-\delta - c\|\mu\|_{\infty,1-\delta}$ and $b = 1/(1-\delta) + c\|\mu\|_{\infty,1-\delta}$. Therefore $\|\lambda\|_p^p \leq c(\|\mu\|_{\infty,1-\delta} + \delta)$.

Theorem 5. *Given $z_0, z_1 \in \Omega$ and $K \Subset \Omega$, there exist constants c and $p > 2$ such that for all $\varepsilon > 0$*

$$\|f - f_\varepsilon\|_K \leq c\left(\sqrt{s_{[1/2\varepsilon^{1/2}]}} + \varepsilon^{1/4}\right)^{1/p}, \tag{3.16}$$

where the s_n are the hexagonal packing constants.

Proof. Recall inequality (3.1) (and the definitions immediately preceding it):

(3.17)
$$|f(z) - f_\varepsilon(z)| \le |f(z) - G_{W_\varepsilon}(z)| + |G_{W_\varepsilon}(z) - G_{D_\varepsilon} \circ f_\varepsilon \circ G_{W_\varepsilon}^{-1}(G_{W_\varepsilon}(z))| + |G_{D_\varepsilon}(f_\varepsilon(z)) - f_\varepsilon(z)|.$$

By Lemma 3.1, if $z \in K$ then

$$|f(z) - G_{W_\varepsilon}(z)| \le c\varepsilon^{1/2}. \tag{3.18}$$

Note that there is a fixed compact subset $K_1 \Subset \mathbf{D}$ such that $f_\varepsilon(K) \subset K_1$ for all sufficiently small $\varepsilon > 0$ (by the uniform convergence of $f_\varepsilon \to f$ on K) and that each point of ∂D_ε is at a distance less than $c\varepsilon^{\pi^2/24}$ from $\partial\mathbf{D}$ (Corollary 3.3). Therefore, by the second part of Lemma 3.1, if $z \in K$ then

$$|G_{D_\varepsilon}(f_\varepsilon(z)) - f_\varepsilon(z)| \le c\varepsilon^{\pi^2/24} < c\varepsilon^{1/4}. \tag{3.19}$$

By (2.6), the complex dilatation of f_ε satisfies $|\mu_\varepsilon(z)| \le c\sqrt{s_n}$, where $n = [\mathrm{dist}(z, \partial\Omega)/2\varepsilon]$. Let $h_\varepsilon = G_{D_\varepsilon} \circ f_\varepsilon \circ G_{W_\varepsilon}^{-1}$ and denote the complex dilatation of h_ε by ν_ε. In order to apply Lemma 3.4 we will estimate $\|\nu_\varepsilon\|_{\infty,1-\delta}$ for a convenient choice of δ. Let $K_\varepsilon = \{z \in \Omega: \mathrm{dist}(z, \partial\Omega) \ge \varepsilon^{1/2}\}$. Then $G_{W_\varepsilon}(K_\varepsilon)$ contains the subdisk $|z| \le 1 - c\varepsilon^{1/4}$ (Lemma 3.1) and so for $\delta = c\varepsilon^{1/4}$, $\|\nu_\varepsilon\|_{\infty,1-\delta} \le \|\mu_\varepsilon\|_{\infty,K_\varepsilon} \le c\sqrt{s_m}$, where $m = [\mathrm{dist}(K_\varepsilon, \partial\Omega)]/2\varepsilon] = [1/2\varepsilon^{1/2}]$. Now (3.11) applied to $|h_\varepsilon(z) - z|$ gives

$$|G_{W_\varepsilon}(z) - G_{D_\varepsilon} \circ f_\varepsilon \circ G_{W_\varepsilon}^{-1}(G_{W_\varepsilon}(z))| \le c\left(\sqrt{s_{[1/2\varepsilon^{1/2}]}} + \varepsilon^{1/4}\right)^{1/p}. \tag{3.20}$$

The estimate (3.16) follows from (3.17)–(3.20).

4. Convergence of r_ε in BMO norm

Let K be a compact subset of Ω which is the closure of a subregion of Ω. If g is a function defined on K, then $\|g\|_{K,p}$ denotes the L^p norm of g over K ($1 \le p \le \infty$). Let Σ denote any square in Ω, and $|\Sigma|$ its area. The mean value of g over Σ is denoted by

$$g_\Sigma = |\Sigma|^{-1} \int\int_\Sigma g\,dx\,dy. \tag{4.1}$$

The BMO norm of g over K is defined by

$$\|g\|_{K,\mathrm{BMO}} = \sup_{\Sigma \subset K} |\Sigma|^{-1} \int\int_\Sigma |g - g_\Sigma|\,dx\,dy. \tag{4.2}$$

Lemma 4.1. *Let w_n be a sequence of quasiconformal mappings which converges uniformly on compacta of Ω to a quasiconformal mapping w. Suppose that $\partial w_n/\partial \bar{z} \to \partial w/\partial \bar{z}$ in the L^∞ norm on compacta of Ω. Then $\partial w_n/\partial z \to \partial w/\partial z$ in the norm of BMO on compacta of Ω.*

Proof. Let $\varphi \in C^\infty$ have compact support in Ω. Set $g_n = \varphi w_n$ and $g = \varphi w$. Then $\partial g_n/\partial \bar{z} = (\partial \varphi/\partial \bar{z}) w_n + \varphi(\partial w_n/\partial \bar{z}) \to \partial g/\partial \bar{z}$ in $L^\infty(\mathbf{C})$. We have

$$\frac{\partial(g_n - g)}{\partial z} = S\left(\frac{\partial(g_n - g)}{\partial \bar{z}}\right), \tag{4.3}$$

where S is the Hilbert transform

$$S\omega(z) = -\frac{1}{\pi}\int\int_{\mathbf{C}} \frac{\omega(\zeta)}{(\zeta - z)^2}\, d\xi\, d\eta, \qquad (\zeta = \xi + i\eta),$$

(see [3, Lemma 7.2]). Now S transforms L^∞ to BMO continuously (Fefferman-Stein [2]), i.e.,

$$\|Sh\|_{\mathbf{C},\mathrm{BMO}} \le C\|h\|_{\mathbf{C},\infty}. \tag{4.4}$$

Therefore

$$\begin{aligned} \left\|\frac{\partial(g_n - g)}{\partial z}\right\|_{\mathbf{C},\mathrm{BMO}} &= \left\|S\left(\frac{\partial(g_n - g)}{\partial \bar{z}}\right)\right\|_{\mathbf{C},\mathrm{BMO}} \\ &\le C\left\|\frac{\partial(g_n - g)}{\partial \bar{z}}\right\|_{\mathbf{C},\infty} \to 0. \end{aligned} \tag{4.5}$$

If $K \Subset \Omega$, then the assertion of the lemma,

$$\left\|\frac{\partial(w_n - w)}{\partial z}\right\|_{K,\mathrm{BMO}} \to 0,$$

follows by taking $\varphi \equiv 1$ on K.

Theorem 6. *As $\varepsilon \to 0$, the ratio-of-radii function $r_\varepsilon : |T_\varepsilon| \to \mathbf{R}$ converges to the modulus of the derivative of the Riemann mapping function $|f'| : \Omega \to \mathbf{R}$ in the norm of Bounded Mean Oscillation (BMO) on compact subsets of Ω.*

Proof. By Theorems 1, 3 and 4, the hypotheses of Lemma 4.1 are satisfied for $f_\varepsilon \to f$. We conclude that $\partial f_\varepsilon/\partial z \to f'$, and therefore $|\partial f_\varepsilon/\partial z| \to |f'|$, in the BMO norm on compacta. It now follows from Theorem 4 that $r_\varepsilon \to |f'|$ in the BMO norm on compacta.

We record the following fact from the proof of Theorem 6.

Corollary. *As $\varepsilon \to 0$, $\partial f_\varepsilon / \partial z$ converges to f' in the norm of BMO on compact subsets of Ω.*

5. Sufficiency of $s_n = O(1/n)$

HCP_n denotes n generations of the regular hexagonal circle packing in the plane. Thus HCP_n consists of $1+6+12+\cdots+6n = 3n^2+3n+1$ circles of equal radii. Let HCP'_n denote a circle packing combinatorially equivalent to HCP_n. Let ρ denote the ratio of the radii of a generation 1 circle and the generation 0 circle in HCP'_n. Let s_n denote the supremum of $|1-\rho|$ over all possible packings HCP'_n and all choices of generation 1 circles in these packings. In [6] it is shown that $s_n \to 0$ as $n \to \infty$ (consequently, any packing combinatorially equivalent to an infinite regular hexagonal packing must consist of circles of equal radius, i.e., is itself an infinite regular hexagonal packing).

The rate of convergence $s_n \to 0$ of the packing constants s_n is not known. The importance of such information is illustrated by the following result which had been known earlier.

Proposition 1. *Consider the situation described in Theorem* 1. *Let $r_\varepsilon \colon |T_\varepsilon| \to \mathbf{R}$ be the mapping defined in* (1.1). *Thus r_ε determines the ratio of the radii of corresponding circles in the isomorphism $\Omega_\varepsilon \to \Omega'_\varepsilon$, and $|f'|$ is the modulus of the normalized Riemann mapping function of Ω. A sufficient condition that r_ε converges to $|f'|$ uniformly on compact subsets of Ω as $\varepsilon \to 0$ is that $s_n = O(1/n)$ as $n \to \infty$.*

Proof. Fix $z \in \Omega$. For ε, $\delta > 0$ let $H_{\varepsilon,\delta}$ be a hexagon of diameter 2δ centered at an ε-circle of the circle packing Ω_ε which is closest to z. As $\varepsilon \to 0$ let $H_{\varepsilon,\delta}$ approach a limiting hexagon $H_{0,\delta}$ centered at z. Then by Theorem 1 we have

$$|f'(z)|^2 = \lim_{\delta\to 0} |f(H_{0,\delta})|/|H_{0,\delta}| = \lim_{\delta\to 0}\lim_{\varepsilon\to 0} |f_\varepsilon(H_{\varepsilon,\delta})|/|H_{\varepsilon,\delta}|, \tag{5.1}$$

where $|H_{\varepsilon,\delta}|$ and $|f_\varepsilon(H_{\varepsilon,\delta})|$ denote the areas of $H_{\varepsilon,\delta}$ and $f_\varepsilon(H_{\varepsilon,\delta})$.

Suppose now that $H_{\varepsilon,\delta}$ is the convex hull of the centers of N generations of circles from the circle packing Ω_ε, so $\delta = 2N\varepsilon$. Assume $\mathrm{dist}(z, \partial\Omega) > d \gg \delta \gg \varepsilon > 0$. Then the packing Ω_ε will contain at least $[d/2\varepsilon] = [dN/\delta]$ generations of the hexagonal packing centered about any circle of Ω_ε which meets $H_{\varepsilon,\delta}$. In fact, this estimate of $[d/2\varepsilon]$ generations is locally uniform in z.

Now $f_\varepsilon(H_{\varepsilon,\delta})$ is the union of triangles formed by the nerve of an HCP'_N configuration of circles from Ω'_ε; let R be the radius of the generation 0 circle. Every circle from Ω'_ε which lies in $f_\varepsilon(H_{\varepsilon,\delta})$ can be considered as the generation 0 circle of a hexagonal configuration of $[dN/\delta]$ generations of circles from Ω'_ε. Therefore, by definition of the packing constants s_n, the radius of any circle from Ω'_ε which lies in $f_\varepsilon(H_{\varepsilon,\delta})$ is at least $R(1-s_{[dN/\delta]})^N$ and at most $R(1+s_{[dN/\delta]})^N$.

We use these bounds to estimate the area of $f_\varepsilon(H_{\varepsilon,\delta})$ and find that

$$(5.2)\qquad (R/\varepsilon)^2(1-s_{[dN/\delta]})^{2N} \le |f_\varepsilon(H_{\varepsilon,\delta})|/|H_{\varepsilon,\delta}| \le (R/\varepsilon)^2(1+s_{[dN/\delta]})^{2N}.$$

By hypothesis, $0 \le s_n \le B/n$ for some constant B. Consequently, for δ fixed,

$$\begin{aligned}\lim_{\varepsilon\to 0}(1+s_{[dN/\delta]})^{2N} &= \lim_{N\to\infty}(1+s_{[dN/\delta]})^{2N}\\ &\le \lim_{N\to\infty}(1+B/[dN/\delta])^{2N} = e^{2B\delta/d},\end{aligned}$$

$$\begin{aligned}\lim_{\varepsilon\to 0}(1-s_{[dN/\delta]})^{2N} &= \lim_{N\to\infty}(1-s_{[dN/\delta]})^{2N}\\ &\le \lim_{N\to\infty}(1+B/[dN/\delta])^{2N} = e^{-2B\delta/d},\end{aligned}$$

From (5.2) we obtain

$$(5.3)\qquad \limsup_{\varepsilon\to 0}\left(\frac{R}{\varepsilon}\right)^2 e^{-2B\delta/d} \le |f(H_{0,\delta})|/|H_{0,\delta}| \le \liminf_{\varepsilon\to 0}\left(\frac{R}{\varepsilon}\right)^2 e^{2B\delta/d}.$$

Let $\delta \to 0$ in (5.3). The center term converges to $|f'(z)|^2$. We conclude that $\lim_{\varepsilon\to 0}(R/\varepsilon) = |f'(z)|$ and since this limit exists it must be the same as $\lim_{\varepsilon\to 0} r_\varepsilon(z)$ because the functions r_ε are piecewise linear. We already noted that these estimates are locally uniform in z. Thus $r_\varepsilon(z) \to |f'(z)|$ uniformly on compact subsets of Ω.

6. Sufficiency of $s_n = o(1/\log^q n)$

Proposition 1 showed that the convergence rate $s_n = O(1/n)$ is a sufficient condition for the convergence of $r_\varepsilon \to |f'|$ as $\varepsilon \to 0$ to be uniform on compact subsets of Ω. We shall now show that this result can be improved if we make use of the BMO convergence established in Theorem 6.

Theorem 7. *In the context of Proposition 1, a sufficient condition that $r_\varepsilon \to |f'|$ uniformly on compact subsets of Ω as $\varepsilon \to 0$ is that $s_n = o(1/\log^q n)$ as $n \to \infty$ where $q > 4$.*

Theorem 7 follows, via (2.4), from

Theorem 8. *A sufficient condition that $\partial f_\varepsilon/\partial z \to f'$ in L^∞ on compact subsets of Ω is that $s_n = o(1/\log^q n)$ where $q > 4$.*

Proof. Fix a compact subset K of Ω which is the closure of a subregion of Ω. We seek a sufficient condition that

$$\|f_{\varepsilon,z} - f'\|_{K,\infty} \to 0 \quad \text{as } \varepsilon \to 0, \tag{6.1}$$

where $f_{\varepsilon,z}$ is an alternate notation for $\partial f_\varepsilon/\partial z$. Let K_ε be the intersection with K of the interiors of the triangles of the triangulation T_ε. Then (6.1) will follow if we prove that

$$\sup\{|f_{\varepsilon,z}(\zeta) - f'(\zeta)| : \zeta \in K_\varepsilon\} \to 0 \quad \text{as } \varepsilon \to 0. \tag{6.2}$$

Suppose $\zeta \in \Delta \subset K_\varepsilon$, where Δ is the interior of one of the triangles in T_ε which meets K. Since $f_{\varepsilon,z}$ is constant in Δ, it is clear that

$$\begin{aligned} f_{\varepsilon,z}(\zeta) - f'(\zeta) &= \frac{1}{|\Delta|}\int\int_\Delta (f_{\varepsilon,z} - f'(\zeta))\,dx\,dy \\ &= \frac{1}{|\Delta|}\int\int_\Delta (f_{\varepsilon,z} - f')\,dx\,dy + o(1), \end{aligned} \tag{6.3}$$

where $o(1)$ will be used to denote a function which depends on the parameter ε, is defined on K_ε, and tends uniformly to 0 as $\varepsilon \to 0$. Set $g \equiv f_{\varepsilon,z} - f'$; we have reduced (6.1) to the assertion that

$$g_\Delta \equiv \frac{1}{|\Delta|}\int_\Delta g\,dx\,dy = o(1). \tag{6.4}$$

Let H_δ denote a regular hexagon of diameter 2δ centered at one of the vertices of Δ and having sides parallel to Δ. Fix d such that $0 < d < \operatorname{dist}(K, \partial\Omega)$. Then there is a compact set K_1 contained in Ω and containing K such that for all ε sufficiently small, H_d will be contained in K_1, independently of the particular $\Delta \subset K_\varepsilon$.

Note that, in general, if Γ is an equilateral triangle or regular hexagon, and Σ is a sufficiently small square containing Γ, then

$$\begin{aligned} |\psi_\Gamma - \psi_\Sigma| &= \frac{1}{|\Gamma|}\left|\int\int_\Gamma (\psi - \psi_\Sigma)\,dx\,dy\right| \\ &\le \frac{|\Sigma|}{|\Gamma|}\frac{1}{|\Sigma|}\int\int_\Sigma |\psi - \psi_\Sigma|\,dx\,dy \le 3\|\psi\|_{\mathrm{BMO}}. \end{aligned}$$

Similarly, if $\Sigma \subset \Sigma_1$ are squares of sides s and $2s$, then $|\psi_\Sigma - \psi_{\Sigma_1}| \le 4\|\psi\|_{\mathrm{BMO}}$. Consequently, if $2\varepsilon \le 2\delta \le d$, then

$$|g_{H_{2\delta}} - g_{H_\delta}| \le 10\|g\|_{K_1,\mathrm{BMO}}, \tag{6.5}$$

and (recall that Δ is an equilateral triangle of side 2ε)

$$|g_{H_{2\varepsilon}} - g_\Delta| \le 10\|g\|_{K_1,\mathrm{BMO}}. \tag{6.6}$$

Now let m be the integer which satisfies $2^m \leq (d/\varepsilon) < 2^{m+1}$. Then

$$(6.7)\qquad \begin{aligned} |g_{H_{2^m\varepsilon}} g_\Delta| &\leq |g_{H_{2^m\varepsilon}} - g_{H_{2^{m-1}\varepsilon}}| + \cdots + |g_{H_{2\varepsilon}} - g_\Delta| \\ &\leq 10m\|g\|_{K_1,\mathrm{BMO}} \leq C\log(d/\varepsilon)\|g\|_{K_1,\mathrm{BMO}}. \end{aligned}$$

Choose $\varphi \in C^\infty$ with $\varphi \equiv 1$ on K_1 and such that φ has support in a compact subset $K_2 \subset \Omega$. By (4.3), (4.4), (3.16), and (2.5) we have

$$(6.8)\qquad \begin{aligned} \|g\|_{K_1,\mathrm{BMO}} &= \|f_{\varepsilon,z} - f'\|_{K_1,\mathrm{BMO}} = \|S((\varphi f_\varepsilon - \varphi f)_{\bar{z}})\|_{K_1,\mathrm{BMO}} \\ &\leq C\|(\varphi(f_\varepsilon - f))_{\bar{z}}\|_{\mathrm{C},\infty} \leq C(\|f_\varepsilon - f\|_{K_2,\infty} + \|f_{\varepsilon,\bar{z}}\|_{K_2,\infty}) \\ &\leq C\left(\left(\sqrt{s_{[1/2\varepsilon^{1/2}]}} + \sqrt[4]{\varepsilon}\right)^{1/p} + \sqrt{s_n}\right), \end{aligned}$$

where $2 < p < q/2$ and $n = [\mathrm{dist}(K_2, \partial\Omega)/(2\varepsilon)]$. Under the hypothesis that $s_n = o(1/\log^q n)$ the last term above is $o(1/\log(1/\varepsilon))$. From (6.7) and (6.8) we therefore obtain

$$(6.9)\qquad |g_{H_{2^m\varepsilon}} - g_\Delta| \leq o(1).$$

Let $\varepsilon \to 0$ in (6.9). Then $g_{H_{2^m\varepsilon}} \to 0$; indeed, $d/2 < 2^m\varepsilon \leq d$ so

$$|g_{H_{2^m\varepsilon}}| \leq \frac{1}{|H_{d/2}|}\int\int_{H_d} |f_{\varepsilon,z} - f'|\,dx\,dy,$$

and this double integral converges to 0 as $\varepsilon \to 0$ because of the L^p (and therefore L^1) convergence referred to at the end of §1. We conclude that $g_\Delta = o(1)$. Therefore (6.4), and hence the theorem, follows.

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