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ON THE GENERALIZATION OF KUMMER SURFACES

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1. Introduction

The subject of this paper is on the compact complex n-manifolds with trivial canonical bundle. Complex tori are of course the obvious examples which have been extensively studied. In this paper, we shall go into the other types of manifolds which have the zero first Betti number. The typical examples for complex surfaces are $K3$ surfaces. On the 3-dimensional case, aside from the geometric point of view, part of the purpose for the study of this type of 3-folds comes from particle physics in the finding of the Ricci flat Kähler 3-folds with zero first Chern class and 1st Betti number. The topological invariants of such manifolds, especially their Euler numbers and the fundamental groups, are interesting for physicists [2]. In a previous paper [5], we had constructed a subclass of such 3-manifolds. In this paper, the analogous construction in the higher dimensional case is considered. The following is the general problem:

Find the projective manifolds with trivial canonical bundle and zero first Betti number by the construction of resolving singularities of the quotient of a complex torus by a finite group. Compute their Euler numbers.

We shall only consider the case when acting on the torus, the group is abelian and all the elements in this group have at least one common fixed point. Consider the complex torus V to be a complex Lie group with 0 as the identity element. We may assume G to be a finite abelian group of Lie-automorphisms of V, and also the dualizing sheaf $\omega_{V/G}$ to be trivial. If x is an element of V fixed by some nontrivial element of G, the action of its isotropic subgroup G_x on V near x is isomorphic to that of some diagonal finite group g acting on $\mathbb{C}^n$ near the origin $\vec{0}$. Inside the affine algebraic variety $\mathbb{C}^n/g$, $\mathbb{C}^{*n}/g$ ($\mathbb{C}^* \doteqdot \mathbb{C}\backslash\{0\}$) is a Zariski-open set which has the structure of an algebraic torus, denoted by T. Then $\mathbb{C}^n/g$ is an equivariant affine embedding of T. A toroidal desingularization of $\mathbb{C}^n/g$ means a nonsingular equivariant embedding $\widehat{\mathbb{C}^n/g}$ of T together with a T-equivariant birational morphism $\pi\colon \widehat{\mathbb{C}^n/g} \to \mathbb{C}^n/g$. In this

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note, a desingularization of V/G always means a toroidal desingularization, i.e., a resolution $\rho\colon \widehat{V/G} \to V/G$ such that near each singular point q of V/G, $\rho\colon (\widehat{V/G}, \rho^{-1}(q)) \to (V/G, q)$ is isomorphic to a desingularization $\pi\colon (\widehat{\mathbb{C}^n/g}, \pi^{-1}(\vec{0})) \to (\mathbb{C}^n/g, \vec{0})$ (described before) as the germs of analytic spaces. We shall show that for every $\widehat{V/G}$ with trivial canonical bundle, the eigenvalues of the differential of the group action at a fixed point cannot be all equal when $\dim V \geq 4$. Therefore the actions with distinct eigenvalues should be studied for the higher dimensional generalization of Kummer surfaces. Some examples are discussed and a formula of the Euler number of such $\widehat{V/G}$ suggested by string theorists [1] can be derived.

This paper is organized as follows. In §2, we give a summary of the main results concerning toroidal compactification [3] needed for our purpose. The principal method here is to associate each toroidal embedding with a combinatorial data and to compute the topological invariants through this correspondence. In §3 we study the desingularization of quotients of complex tori to examine the possible generalization of the construction of Kummer surfaces on higher dimensional cases. In §4 we derive a formula to relate the Euler number of the desingularization $\widehat{V/G}$ with the fixed point set of G on V when the canonical bundle of $\widehat{V/G}$ is trivial. In fact, we work on a more general situation described in Theorem 2 below.

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Notation. For a finite group G acting faithfully on a compact complex manifold X, we shall use the following notation throughout this paper:

$$X^\varphi = \{x \in X \mid \varphi(x) = x\} \quad \text{for } \varphi \in G,$$
$$X_G = \bigcup \{X^\varphi \mid \varphi \in G, \varphi \neq \mathrm{id}\},$$
$$G_x = \{\varphi \in G \mid \varphi(x) = x\}, \quad \text{the isotropy subgroup at } x \in X.$$

2. Toroidal embeddings

First we recall a few useful results [3].

Let T be an algebraic n-torus over $\mathbb{C}$, $M = \mathrm{Hom}_{\mathrm{alg.group}}(T, G_m)$ be the group of characters of T, and $N = \mathrm{Hom}_{\mathrm{alg.group}}(\mathbb{G}_m, T)$ be the group of 1-parameter subgroups of T. If $r \in M$, we denote by $\mathfrak{X}^r$ the corresponding element of $\Gamma(T, \mathscr{O}_T^*)$. M and N are free abelian groups of rank n, related by a nonsingular canonical pair: $M \times N \to \mathbb{Z}$, $(r, a) \mapsto \langle r, a \rangle$. For a T-equivariant normal embedding X of T, or a T-space, there associates a finite rational partial polyhedral (f.r.p.p.) decomposition $\{\sigma_\alpha\}$ of $N_{\mathbf{R}}$ such that X is T-isomorphic to $X_{\{\sigma_\alpha\}}$, where $X_{\{\sigma_\alpha\}}$ is the union of affine open subsets X_{σ_α}'s, defined by $X_{\sigma_\alpha} = \operatorname{Spec} \mathbb{C}[\check{\sigma}_\alpha \cap M]$. There is a natural correspondence between the following two equivalent categories:

(A) The category of T-space:
object: T-space X,
morphism: T-equivariant algebraic morphism $\varphi\colon X \to Y$.

(B) The category of f.r.p.p. decomposition of $N_{\mathbf{R}}$:
object: f.r.p.p. decomposition,
morphism: Φ: $\{\sigma_\alpha\} \to \{\tau_\beta\}$ a continuous map Φ: $\bigcup \sigma_\alpha \to \bigcup \tau_\beta$ such that for each σ_α, $\Phi(\sigma_\alpha)$ is contained in some τ_β, and $\Phi_{\text{rest}}\colon \sigma_\alpha \to \tau_\beta$ is a linear map sending $\sigma_\alpha \cap N_{\mathbf{Q}}$ into $\tau_\beta \cap N_{\mathbf{Q}}$.

Denote the T-space associated to $\{\sigma_\alpha\}$ of the above correspondence by $X_{\{\sigma_\alpha\}}$. It is known that the nonsingular condition of $X_{\{\sigma_\alpha\}}$ is equivalent to the multiplicity of $\{\sigma_\alpha\}$ being equal to one, i.e. to say each σ_α is generated by a subset of a $\mathbb{Z}$-base of N.

For a T-space $X = X_{\{\sigma_\alpha\}}$, there is a one-to-one correspondence between the set of all complete T-invariant coherent sheafs $\mathscr{J}$ of fractional ideals over X and the set of functions $f\colon \bigcup \sigma_\alpha \to \mathbf{R}$ with the properties

(i) $f(\lambda x) = \lambda f(x), \qquad \lambda \in \mathbf{R}^+,\ x \in \cup\sigma_\alpha,$

(ii) f is continuous, piecewise linear,

(*) (iii) $f(N \cap \cup\sigma_\alpha) \subseteq \mathbb{Z}$,

(iv) For each σ_α, there exist $r_i \in M$ such that
$f(x) = \min_i \langle r_i, x\rangle$ for $x \in \sigma_\alpha$.

This is set up by the relation

$$\mathscr{J}_{X_{\sigma_\alpha}} = \bigoplus_{\substack{r \in M \\ r \geq f \text{ on } \sigma_\alpha}} \mathbb{C}\mathfrak{x}^r.$$

We shall denote by J_f the sheaf $\mathscr{J}$ corresponding to the above function f. Let $\{\sigma_1, \cdots, \sigma_N\}$ be the set of all 1-dimensional cones in $\{\sigma_\alpha\}$. For $1 \leq i \leq N$, let p_i be the primitive integral vector in σ_i, and D_i be the closure of the T-orbit corresponding to σ_i. The following facts are known:

(i) $X - T = \bigcup_{i=1}^N D_i$.

(ii) The Weil divisor $\mathscr{O}_X(\sum_{i=1}^N (-m_i) D_i)$ is equal to J_f, where f is the convex interpolation of the function $f_1\colon \bigcup_\alpha sk^1\sigma_\alpha \to \mathbf{R}$, sending p_i to m_i, i.e. to say the restriction of f on each σ_α is the least function satisfying (*) and $\geq f_1$ on $sk^1\sigma_\alpha$.

(iii) $\mathscr{O}_X(\sum_{i=1}^N(-m_i)D_i)$ is the trivial Cartier divisor $\Leftrightarrow$ the function f in (ii) is linear on $\bigcup \sigma_\alpha$.

(iv) The dualizing sheaf $\omega_{X_{\{\sigma_\alpha\}}} = \mathscr{O}_X(-\sum_{i=1}^N D_i) = J_\delta$, where δ is the convex interpolation of the function on $\bigcup sk^1\sigma_\alpha$, assigning p_i to 1.

We shall compute the Euler number of a T-space.

Lemma 1. *The Euler number of a T-space $X_{\{\sigma_\alpha\}}$ is equal to the number of n-dimensional polyhedral cones in $\{\sigma_\alpha\}$.*

Proof. We claim for a convex rational polyhedral cone σ in $N_{\mathbf{R}}$, the T-orbit $\mathscr{O}^\sigma$ in X_σ corresponding to σ is a strong deformation retract of X_σ. Consequently the Euler number of X_σ is equal to 0 or 1 according to $\dim\sigma < n$ or $= n$.

Take an element a in $N \cap [\sigma - \bigcup$ proper faces of $\sigma]$, and let λ_a be its associated 1-parameter subgroup of T. Then $\lambda_a(t)$ acts on X_σ for $t \in \mathbf{C}^*$. It is known that $\lim_{t\to 0}\lambda_a(\tau)\cdot x$ exists and lies in $\mathscr{O}^\sigma$ for all $x \in X_\sigma$. So $\{\lambda_a(t) | t \in \mathbf{C}\}$ gives the desired deformation retract.

Now we proceed to prove this lemma by the induction on $m := \max_{\sigma_\alpha} \dim\sigma_\alpha$. We want to show that

$$\chi(X_{\{\sigma_\alpha\}}) = \begin{cases} 0 & \text{if } m < n, \\ \#\{\sigma_\alpha | \dim\sigma_\alpha = n\} & \text{if } m = n, \end{cases}$$

where $\chi(X_{\{\sigma_\alpha\}})$ is the Euler number of $X_{\{\sigma_\alpha\}}$. It is obvious for $m = 0$.

Suppose that we are given a f.r.p.p. decomposition $\{\sigma_\alpha\}$ of $N_{\mathbf{R}}$ with $m > 1$. Let $\{\tau_1, \cdots, \tau_N\}$ be the set of σ_α with $\dim\sigma_\alpha = m$. Let $\{\sigma'_\alpha\}$ be the f.r.p.p. decomposition of $N_{\mathbf{R}}$ consisting of all the proper faces of τ_j's, and $\{\sigma''_\alpha\} := \{\sigma_\alpha\} - \{\tau_j | j = 1, \cdots, N\}$. Then

$$X_{\{\sigma_\alpha\}} = X_{\{\sigma''_\alpha\}} \cup \left(\bigcup_{i=1}^N X_{\tau_j}\right), \qquad X_{\{\sigma''_\alpha\}} \cap \left(\bigcup_{i=1}^N X_{\tau_j}\right) = X_{\{\sigma'_\alpha\}}.$$

By the induction hypothesis, $\chi(X_{\{\sigma'_\alpha\}}) = \chi(X_{\{\sigma''_\alpha\}}) = 0$ and $\chi(X_{\tau_j} \cap \bigcup_{j\neq i} X_{\tau_j}) = 0$. So

$$\chi(X_{\{\sigma_\alpha\}}) = \chi\left(\bigcup_{j=1}^N X_{\tau_j}\right) = \sum_{j=1}^N \chi(X_{\tau_j}).$$

Based on the fact claimed at the beginning of the proof, this number is equal to 0 or N according to $m < n$ or $m = n$. q.e.d.

Suppose that g is a finite diagonal group acting on the vector space $\mathbf{C}^n$. Assume the fixed point set of each nontrivial element in g has codim ≥ 2. Since g acts freely on $(\mathbf{C}^*)^n$, the quotient $(\mathbf{C}^*)^n/g$ has the structure of algebraic torus, denoted by T. Then $\mathbf{C}^n/g$ is a T-space of which the structure can be described as follows:

Let $\mathbb{R}^n$ be the vector space consisting of all $n \times 1$ column vectors, and let $\{e^1, \cdots, e^n\}$ be the standard base. Define

$$\exp\colon \mathbb{R}^n \to (\mathbb{C}^*)^n, \qquad \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \mapsto \begin{pmatrix} e^{2\pi\sqrt{-1}x_1} \\ \vdots \\ e^{2\pi\sqrt{-1}x_n} \end{pmatrix},$$

$$\mathrm{tr}\colon \mathbb{R}^n \to \mathbb{R}, \qquad \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \mapsto \sum_{i=1}^n x_i.$$

The group N of 1-parameter subgroups of T can be identified with the lattice $\exp^{-1}(g)$ in $\mathbb{R}^n$. It contains the standard lattice $\sum_{i=1}^n \mathbb{Z}e^i$, and $N/\sum_{i=1}^n \mathbb{Z}e^i$ is isomorphic to g. We can identify $N_{\mathbb{R}}$ with $\mathbb{R}^n$ by extending the embedding of N in $\mathbb{R}^n$. As T-spaces, $\mathbb{C}^n/g$ is isomorphic to the affine variety X_C, where C is the cone in $\mathbb{R}^n$ $(= N_{\mathbb{R}})$ defined by $C = \{\sum_{i=1}^n x_i e^i | x_i \geq 0\}$. The T-invariant divisors D_i are in one-to-one correspondence with all the 1-dimensional faces $\mathbb{R}_+ e^i$ of C, $1 \leq i \leq n$. From the condition of the fixed point set of g, each e^i is a primitive integral vector in N. The triviality of the dualizing sheaf of $\mathbb{C}^n/g$ is equivalent to the Weil divisor $\mathscr{O}_{X_C}(-\sum_{i=1}^n D_i)$ as the trivial Cartier divisor. This is also equivalent to $\mathrm{tr}(C \cap N) \subseteq \mathbb{Z}$. A desingularization of X_C is in fact a T-space $X_{\{\sigma_\alpha\}}$ with $\bigcup \sigma_\alpha = C$ and multiplicity of $\{\sigma_\alpha\} = 1$.

Denote $\Delta := \{x \in \mathbb{R}^n | \mathrm{tr}(x) = 1\} \cap C$, the triviality of the canonical bundle of $X_{\{\sigma_\alpha\}}$ is the same as requiring each σ_α to be a simplicial cone generated by vectors in $\Delta \cap N$. In this situation, for each σ_α of $\dim n$, we have

$$\begin{aligned} &\mathrm{vol}(\sigma_\alpha \cap \{x | \mathrm{tr}(x) \leq 1\}) : \mathrm{vol}(C \cap \{x | \mathrm{tr}(x) \leq 1\}) \\ &\quad = 1 : \left[N : \sum_1^n \mathbb{Z}e^i \right] = 1 : |g|, \end{aligned}$$

and therefore

$$|\{\sigma_\alpha | \dim \sigma_\alpha = n\}| = |g|.$$

This implies $\chi(X_{\{\sigma_\alpha\}}) = |g|$ by Lemma 1. Hence we have proved the following:

Proposition 1. *Let g be a finite diagonal group of $\mathbb{C}^n$ such that the dualizing sheaf of $\mathbb{C}^n/g$ is trivial. Let N, T, C, be the same as before. Then the following statements are equivalent:*

(i) *The T-space $X_{\{\sigma_\alpha\}}$ is a desingularization of $\mathbb{C}^n/g$ with trivial canonical bundle.*

(ii) *$\{\sigma_\alpha\}$ is a f.r.p.p. decomposition of $N_{\mathbb{R}}$ with $\bigcup \sigma_\alpha = C$, and each σ_α is a simplicial cone generated by elements in $N \cap \Delta$, which can be extended to a $\mathbb{Z}$-base of N.*

In this case, the Euler number of $X_{\{\sigma_\alpha\}}$ is equal to the order of g.

In general, the above $X_{\{\sigma_\alpha\}}$ may not exist for $n \geq 4$; but when $n = 2$ and 3, such desingularization always exists due to the following.

Proposition 2. *Let g be the same as in Proposition 1. For $n = 2$ and 3, $\{\sigma_\alpha\}$ is a f.r.p.p. decomposition of $N_{\mathbb{R}}$ with the following properties:*

(i) $\bigcup \sigma_\alpha = C$.

(ii) *Each σ_α is a simplicial cone.*

(iii) *$\{\mathbb{R}_+ p | p \in \Delta \cap N\}$ is the set of all 1-dimensional cones in $\{\sigma_\alpha\}$.*

Then $X_{\{\sigma_\alpha\}}$ is a desingularization of $\mathbb{C}^n/g$ with trivial canonical bundle.

Proof. From the triviality of the dualizing sheaf of $\mathbb{C}^n/g$, we know $\mathrm{tr}(C \cap N) \subset \mathbb{Z}$. If $\{\sigma_\alpha\}$ is a f.r.p.p. decomposition satisfying the assumption, all we have to show is that each σ_α of dimension n is generated by a $\mathbb{Z}$-base of N by Proposition 1.

When $n = 2$, a nontrivial element φ of g can be expressed by $\exp\binom{\alpha_\varphi}{\beta_\varphi}$, where $\alpha_\varphi, \beta_\varphi$ are some rational numbers less than 1 with $\alpha_\varphi + \beta_\varphi = 1$. Then

$$\Delta \cap N = \{e^1, e^2\} \amalg \{\tbinom{\alpha_\varphi}{\beta_\varphi} | \varphi \in g, \varphi \neq \mathrm{id}\}.$$

Every σ_α of dimension 2 is generated by two vectors p, p' in $\Delta \cap N$ such that the convex hull spanned by $\vec{0}, p, p'$ contains no other element of N. So p, p' form a $\mathbb{Z}$-base of N.

When $n = 3$, each nontrivial element φ in g can be expressed by

$$\exp\begin{pmatrix} \alpha_\varphi \\ \beta_\varphi \\ \gamma_\varphi \end{pmatrix}$$

for some nonnegative rational numbers $\alpha_\varphi, \beta_\varphi, \gamma_\varphi$ less than 1. Then $\alpha_\varphi + \beta_\varphi + \gamma_\varphi = 1$ or 2,

$$\Delta \cap N = \{e^i\}_{i=1}^3 \amalg \left\{ \begin{pmatrix} \alpha_\varphi \\ \beta_\varphi \\ \gamma_\varphi \end{pmatrix} \middle| \varphi \in g, \varphi \neq \mathrm{id}, \alpha_\varphi + \beta_\varphi + \gamma_\varphi = 1 \right\}.$$

If w^1, w^2, w^3 are three elements in $\Delta \cap N$ generating a 3-dimensional simplicial cone σ_α, denote the sublattice $\sum_{i=1}^3 \mathbb{Z} w^i$ of N by N_0. If $N \neq N_0$, there is an element $a = a_1 w^1 + a_2 w^2 + a_3 w^3$ in $N - N_0$ with $a_i \in \mathbb{Q}$, $0 \leq a_i < 1$. The value $\mathrm{tr}(a)$ must be 1 or 2 because $0 < \mathrm{tr}(a) < 3$, $\mathrm{tr}(N \cap C) \subseteq \mathbb{Z}$. If $\mathrm{tr}(a) = 2$, all a_i must be greater than 0. Replacing a by $(1 - a_1)w^1 + (1 - a_2)w^2 + (1 - a_3)w^3$, we may assume $\mathrm{tr}(a) = 1$. Therefore we can find an element a in $\{\sum x_i w^i | x_i \geq 0, \sum x_i = 1\} \cap (N - N_0)$, and hence in $\Delta \cap N$. But as $\{\sum x_i w^i | x_i \geq 0, \sum x_i = 1\}$ intersects $\Delta \cap N$ only at $\{w^i\}_{i=1}^3$, a has to be one of the w^i's, and that contradicts the assumption $a \notin N_0$. Therefore $N = N_0$, $\{w^i\}_{i=1}^3$ is a $\mathbb{Z}$-base of N. q.e.d.

The following lemma can provide some examples of $X_{\{\sigma_\alpha\}}$ with trivial canonical bundle in higher dimensional cases.

Lemma 2. *Let g_i $(i = 1, 2)$ be a finite diagonal linear group of $\mathbb{C}^{n_i}$, and g be the product group $g_1 \times g_2$ acting on $\mathbb{C}^n := \mathbb{C}^{n_1} \times \mathbb{C}^{n_2}$. Let N_i $(i = 1, 2)$ and N be the groups of 1-parameter subgroups of $(\mathbb{C}^*)^{n_i}/g_i$ and $(\mathbb{C}^*)^n/g$, respectively. If $\{\sigma_{\alpha^{(i)}}\}$ is a f.r.p.p. of $(N_i)_\mathbf{R}$ which satisfies condition* (ii) *in Proposition* 1 *for $i = 1, 2$, so does $\{\sigma_{\alpha^{(1)}} \times \sigma_{\alpha^{(2)}}\}$ for $N_\mathbf{R}$.*

Proof. Obvious.

3. Quotients of complex tori

Let V be a complex n-torus ($\simeq \mathbb{C}^n$/lattice) for $n \geq 2$. Consider V as a complex Lie group with 0 as the identity element. Let $\theta\colon V \to V$ be a (Lie) automorphism of order d, and $(d\theta)_0$ be the induced linear automorphism on the tangent space at 0.

Theorem 1. *Let θ be an automorphism of V with $(d\theta)_0 = \mu \cdot \mathrm{id}$, where μ is the dth primitive root of* 1. *If there exists a desingularization of $V/\langle\theta\rangle$ with trivial canonical bundle, then the dimension n of V is the same as the order d of θ, and it equals* 2 *or* 3.

In order to prove the above theorem, we need two lemmas.

Lemma 3. *Let θ be an order d automorphism as in Theorem* 1. *If θ has more than one fixed point on V, then $d = 2, 3$, or 4, and V^θ is a subset of the d-torsion part of V. Furthermore, for each $x \in V^\theta - \{0\}$ in the cases of $d = 3, 4$, there exists a (Lie) homomorphism $\lambda\colon E \to V$ from an elliptic curve E into V, together with an order d automorphism ρ of E, such that $\lambda \cdot \rho = \theta \cdot \lambda$, and $\lambda(E^\rho) =$ the subgroup of V generated by x.*

Proof. We may express

$$V = \mathbb{C}^n/L,$$

$$\begin{array}{ccc} \mathbb{C}^n & \xrightarrow{\ \psi\ } & \mathbb{C}^n \\ {\scriptstyle\pi}\downarrow & \circlearrowleft & \downarrow{\scriptstyle\pi} \\ V & \xrightarrow{\ \theta\ } & V, \end{array}$$

where L is a lattice in $\mathbb{C}^n$,

$$\pi\colon \mathbb{C}^n \to V, \qquad z \mapsto z + L,$$
$$\psi\colon \mathbb{C}^n \to \mathbb{C}^n, \quad z = \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix} \mapsto \begin{pmatrix} \mu z_1 \\ \vdots \\ \mu z_n \end{pmatrix}.$$

Let $x := \alpha + L$ be a fixed point of θ not equal to 0. The vector

$$\alpha = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}$$

is not zero in $\mathbb{C}^n$. Consider the linear map $\tilde{\lambda}$,

$$\tilde{\lambda}\colon \mathbb{C} \to \mathbb{C}^n, \qquad t \mapsto \begin{pmatrix} (\mu - 1)\alpha_1 t \\ \vdots \\ (\mu - 1)\alpha_n t \end{pmatrix}.$$

$\tilde{\lambda}$ is an injective map sending $l := \sum_{j=0}^{d-2} \mathbb{Z}\mu^j$ into L. When $d = 2$, the conclusion follows easily. So we need only consider the cases where $d \geq 3$. Since $L \cap \operatorname{Im} \tilde{\lambda}$ is a free $\mathbb{Z}$-module of rank ≤ 2, so l is a rank-2 lattice in $\mathbb{C}$. The multiplication by μ on $\mathbb{C}$ induces an order d Lie-automorphism ρ of the elliptic curve $E := \mathbb{C}/l$, so the only possibilities for d are 3, 4 and 6. The linear homomorphism $\tilde{\lambda}\colon \mathbb{C} \to \mathbb{C}^n$ induces a Lie-homomorphism $\lambda\colon E \to \mathbb{C}^n/L$ such that $\lambda\rho = \theta\lambda$. The element $(\mu - 1)^{-1} + l$ in E is fixed by ρ, and is sent to $\alpha + L$ by λ. Because of $\alpha + L \neq L$, $(\mu - 1)^{-1} + l$ is not equal to l in E. Hence the order of ρ cannot be 6, otherwise it would have only one fixed point. So $d = 3$ or 4, $(\mu - 1)^{-1} + l$ generates E^ρ, which is contained in the d-torsion part of E. This implies $d(\alpha + L) = 0$ in $\mathbb{C}^n/L$.

Lemma 4. *Let θ be an order d automorphism of a complex n-torus V. If all the eigenvalues of $(d\theta)_0$ are equal, then d has to be one of* 2, 3, 4, *or* 6. *Furthermore,*

$$d = 2, 3 \text{ or } 4 \quad \text{if and only if} \quad |V^\theta| \geq 2,$$
$$d = 6 \qquad\qquad \text{if and only if} \quad |V^\theta| = 1.$$

Proof. Assume $(d\theta)_0 = \mu \cdot \mathrm{id}$, where μ is the dth primitive root of 1. By Lemma 3, we know that d is one of 2, 3, or 4 when $|V^\theta| \geq 2$, so it suffices to show that d equals 6 under the assumption $|V^\theta| = 1$. Let p be a prime dividing d; the automorphism $\theta^{d/p}$ of V has the order p. Denote $\theta^{d/p}$ by θ'. Then

$$|V^{\theta'}| = -\chi(V - V^{\theta'}) = -p\chi((V - V^{\theta'})/\langle\theta'\rangle) \geq 2.$$

By Lemma 3, the order of θ' is equal to 2 or 3. This implies $d = 2^k 3^s$ for some nonnegative integers k, s. Claim: $k > 0, s > 0$. If not, we assume $d = 2^k$. Since $|V^\theta| = 1$, every $\langle\theta\rangle$-orbit in $V_{\langle\theta\rangle} - V^\theta$ has more than one element, and its cardinal number is a power of 2. So $|V_{\langle\theta\rangle}| \equiv 1 \pmod 2$. On the other hand,

$$|V_{\langle\theta\rangle}| = -\chi(V - V_{\langle\theta\rangle}) = -2^k\chi((V - V_{\langle\theta\rangle})/\langle\theta\rangle) \equiv 0 \pmod 2;$$

this gives the contradiction. Using the same argument, we can also show that $d \neq 3^s$. Therefore $d = 2^k 3^s$ for some k, s greater than 0. Let θ_1 be the order 3^s element in $\langle\theta\rangle$. Since $|V^{\theta_1}| \neq 1$, s has to be equal to 1 by Lemma 3. Let x be a point in $V^{\theta_1} - \{0\}$. From Lemma 3, we can find a homomorphism $\lambda: E \to V$ from an elliptic curve E into V with an order 3 automorphism ρ of E such that $\theta_1\lambda = \lambda\rho$ and λ sends a nontrivial fixed point of ρ to x. The image of λ is a θ_1-invariant 1-dimensional subtorus of V. The tangent space of this 1-dimensional subtorus at 0 is invariant under $(d\theta)_0$, so the restriction of θ induces an order-d automorphism on it. This implies $d = 6$. q.e.d.

Proof of Theorem 1. Let $\rho: \widehat{V/\langle\theta\rangle} \to V/\langle\theta\rangle$ be a desingularization of $V/\langle\theta\rangle$ with $K_{\widehat{V/\langle\theta\rangle}} =$ trivial. The dualizing sheaf of $V/\langle\theta\rangle$ has to be trivial, which implies $\det(d\theta)_0 = 1$. So $\mu^n = 1$ and n is divided by d. For $x \in V_{\langle\theta\rangle}$, the isotropic subgroup at x is a cyclic subgroup of $\langle\theta\rangle$ of order d_x (≥ 1). Let $[x]$ be the singular point of $V/\langle\theta\rangle$ corresponding to the orbit of x. By the requirement of the desingularization, $\rho: (\widehat{V/\langle\theta\rangle}, \rho^{-1}([x])) \to (V/\langle\theta\rangle, [x])$ is isomorphic to $\pi: (\widehat{\mathbb{C}^n/g}, \pi^{-1}(\vec{0})) \to (\mathbb{C}^n/g, \vec{0})$ as germs of analytic spaces, where g is the cyclic group generated by $\mu^{(d/d_x)} \cdot \mathrm{id}$, and $\widehat{\mathbb{C}^n/g}$ is a toroidal desingularization of $\mathbb{C}^n/g$. From Proposition 1, the triviality of the canonical bundle of $\widehat{\mathbb{C}^n/g}$ is equivalent to the condition $n = d_x$, because the lattice N in Proposition 1 in this case is generated by the standard base $e^1, \cdots, e^n$ and the vector

$$\begin{pmatrix} 1/d_x \\ \vdots \\ 1/d_x \end{pmatrix}.$$

The triviality of the canonical bundle of $\widehat{V/\langle\theta\rangle}$ is equivalent to $d_x = n$ for all $x \in V_{\langle\theta\rangle}$. Since the isotropic subgroup at the identity of V is $\langle\theta\rangle$, d has to equal n, which is one of 2, 3, 4, or 6 by Lemma 4. We have the expression of the Euler number of $V/\langle\theta\rangle$:

$$\chi(V/\langle\theta\rangle) = \sum_{i=0}^{2n}(-1)^i \dim H^i(V,\mathbb{C})^\theta,$$

$$\begin{aligned}\chi(V/\langle\theta\rangle) &= \chi((V - V_{\langle\theta\rangle})/\langle\theta\rangle) + \chi(V_{\langle\theta\rangle}/\langle\theta\rangle)\\ &= \frac{1}{d}\chi(V - V_{\langle\theta\rangle}) + \chi(V_{\langle\theta\rangle}/\langle\theta\rangle) = -\frac{1}{d}|V_{\langle\theta\rangle}| + |V_{\langle\theta\rangle}/\langle\theta\rangle|.\end{aligned}$$

If $d = 6$, $\frac{1}{6}|V_{\langle\theta\rangle}|$ is an integer, so $|V_{\langle\theta\rangle}| \geq 6$. From Lemma 4, $|V^\theta| = 1$. There is an element of $V_{\langle\theta\rangle}$ with the order of its isotropic subgroup less than 6. This contradicts the triviality of $K_{\widehat{V/\langle\theta\rangle}}$. If $d = 4$, it is easy to see that

$$\dim H^i(V,\mathbb{C})^\theta = \begin{cases} 1 & \text{for } i = 0, 8,\\ 16 & \text{for } i = 2, 6,\\ 38 & \text{for } i = 4,\\ 0 & \text{otherwise,}\end{cases}$$

$$|V_{\langle\theta\rangle}| = |V^{\theta^2}| = 2^8.$$

Therefore

$$\begin{aligned}72 = \chi(V/\langle\theta\rangle) &= -\tfrac{1}{4}|V_{\langle\theta\rangle}| + |V_{\langle\theta\rangle}/\langle\theta\rangle|\\ &= -2^6 + (|V^\theta| + (2^8 - |V^\theta|)/2),\end{aligned}$$

which implies that $|V^\theta| = 16$. We can find an element in $V_{\langle\theta\rangle}$ with its isotropic subgroup of order 2, and it contradicts the triviality of $K_{\widehat{V/\langle\theta\rangle}}$. Therefore, the only possibilities for d are 2 and 3, and the proof of Theorem 1 is complete. q.e.d.

Remark. If V and θ satisfy the condition of Theorem 1, then $\dim n$ of $V =$ the order of $\theta = 2$ or 3. When $n = 2$, θ is the involution, and $\widehat{V/\langle\theta\rangle}$ is the well-known Kummer surface. When $n = 3$, (V, θ) is classified in [5], which can be described as follows:

Denote

$$E = \mathbb{C}/\mathbb{Z} + \mathbb{Z}\omega, \qquad \omega = e^{2\pi\sqrt{-1}/3},$$

$$\sigma\colon E \to E, \qquad [z] \to [\omega z] \quad \text{for } z \in \mathbb{C},$$

$$\theta_\sigma\colon E \times E \times E \to E \times E \times E, \qquad (p_1, p_2, p_3) \mapsto (\sigma(p_1), \sigma(p_2), \sigma(p_3)).$$

The 3-torus V is equal to $(E \times E \times E)/K$ for some θ_σ-invariant finite subgroup K with $\text{order}(K) \equiv 1 \pmod 3$, and θ is the automorphism of V induced by θ_σ. The Betti numbers of $\widehat{V/\langle\theta\rangle}$ are known for both cases. In particular, their first Betti numbers vanish.

If we want to obtain the higher dimensional manifolds with trivial canonical bundles by resolving singularities of $V/\langle\theta\rangle$, there should be no restriction on the eigenvalues of $(d\theta)_0$ except $\det(d\theta)_0 = 1$ by the conclusion of Theorem 1. Even for the cases when $n = 2, 3$, the investigation of this general version will provide further examples of $K3$ surfaces and 3-folds with trivial canonical bundle. First, let us consider the situation when $|V_{\langle\theta\rangle}| < \infty$. For $n = 2$, the following simple result (a proof can be found in [4]) is known:

If V is a 2-torus acted on by θ with $\det(d\theta)_0 = 1$, then the order d of θ is one of 2, 3, 4, 6, and $|V_{\langle\theta\rangle}| = 16, 9, 16, 24$ according to $d = 2, 3, 4, 6$. Furthermore, the minimal desingularization of $V/\langle\theta\rangle$ is a $K3$ surface.

When $n = 3$, there are only two cases with $|V_{\langle\theta\rangle}| < \infty$ from the classification of the finite group action on 3-torus [4]. One of these is the order 3 group action described in the remark of Theorem 1. The other one is the action of order 7 automorphism on a 3-torus V with eigenvalues μ, μ^2, μ^4 ($\mu := e^{2\pi i/7}$) and $|V_{\langle\theta\rangle}| = |V^\theta| = 7$. The pair (V, θ) is unique up to the isogeneous relation. In fact, V is constructed from the cyclotomic field $\mathbb{Q}(\mu)$ with the three isomorphism φ_i, $1 \le i \le 3$, of $\mathbb{Q}(\mu)$ into $\mathbb{C}$ with $\varphi_1(\mu) = \mu, \varphi_2(\mu) = \mu^2, \varphi_3(\mu) = \mu^4$ [5]. Under the mapping

$$a \mapsto \begin{pmatrix} \varphi_1(a) \\ \varphi_2(a) \\ \varphi_3(a) \end{pmatrix} \in \mathbb{C}^3, \qquad a \in \mathbb{Q}(\mu),$$

$\mathbb{Q}(\mu)_{\mathbf{R}}$ can be identified with $\mathbb{C}^3$, and $\mathbb{Z}[\mu]$ corresponds to a lattice L in $\mathbb{C}^3$. Then V is equal to $\mathbb{C}^3/L$, and the automorphism θ of V is the one corresponding to the multiplication of the element μ on $\mathbb{Q}(\mu)_{\mathbf{R}}$. Furthermore, $V/\langle\theta\rangle$ has a desingularization with trivial canonical bundle. For the other dimension n with $2n+1$ as a prime number denoted by p, we imitate the above construction by choosing n distinct isomorphisms $\{\varphi_i\}_{i=1}^n$ of $\mathbb{Q}(e^{2\pi i/p})$ into $\mathbb{C}$ such that $(\mathbb{Q}(e^{2\pi i/p}), \{\varphi_i\}_{i=1}^n)$ is a field of C-M type. Then $\varphi_i(e^{2\pi i/p}) = (e^{2\pi i/p})^{m_i}$ for some $1 \le m_i < p$, and $\sum_{i=1}^n m_i \ge \sum_{i=1}^n i = \frac{1}{2}n(n+1)$. On the other hand, the existence of $\widehat{V/\langle\theta\rangle}$ with trivial canonical bundle requires $\sum_{i=1}^n m_i = p$ by Proposition 1. Therefore $n \le 3$, which are the cases we have discussed already. This suggests that for the higher dimensional cases, the condition $|V/\langle\theta\rangle| < \infty$ is still too restricted. We have to consider the situation when the dimension of $V/\langle\theta\rangle$ is positive. For $n = 3$, all such (V, θ) are classified in [4], and every such $V/\langle\theta\rangle$ has a desingularization with trivial canonical bundle from Proposition 2. Now we can use the following procedure to provide examples in the higher dimensional case: Let V_i ($i = 1, 2$) be n_i-dim complex torus acted by an order d_i automorphism θ_i. Then $\langle\theta_1\rangle \times \langle\theta_2\rangle$ acts on $V_1 \times V_2$. If $V_i/\langle\theta_i\rangle$ has a desingularization with trivial canonical bundle, so does $V_1 \times V_2/\langle\theta_1\rangle \times \langle\theta_2\rangle$

by patching the local construction in Lemma 2 suitably. In the case when d_1 and d_2 are relatively prime, $\langle\theta_1\rangle \times \langle\theta_2\rangle$ is a cyclic group $\langle\theta\rangle$; hence we obtain $\widehat{V/\langle\theta\rangle}$ with trivial canonical bundle for $\dim V \geq 4$. For all the examples we are able to obtain, the direct computation of the Euler number of $\widehat{V/G}$ can be tiresome except the cases when G is cyclic and $|V_G| < \infty$. In the next section we are going to derive a formula of the Euler number of $\widehat{V/G}$ in terms of V_G. In particular, we shall have

$$\chi(V_1 \times \widehat{V_2/\langle\theta_1\rangle \times \langle\theta_2\rangle}) = \chi(\widehat{V_1/\langle\theta_1\rangle})\chi(\widehat{V_2/\langle\theta_2\rangle}).$$

4. A formula for Euler numbers of "minimal" desingularizations of quotient varieties

When G is a finite abelian group of Lie automorphisms of a complex torus V, the triviality of $\omega_{V/G}$ is equivalent to (codim V_G in V) ≥ 2 and $\det(d\varphi)_0 = 1$ for $\varphi \in G - \{\mathrm{id}\}$. The purpose of this section is to express the Euler number of $\widehat{V/G}$ in terms of the fixed point sets V^φ, $\varphi \in G$, when $\widehat{V/G}$ has the trivial canonical bundle. In fact, we shall work in a more general setting, and hopefully the formula would be useful in some other situation.

Let G be a finite abelian group consisting of biholomorphic maps of a compact complex n-manifold X. It is known that near a point x of X^G, the action of G_x is equivalent to a linear action on $\mathbb{C}^n$ near $\vec{0}$. By a desingularization of X/G, we mean a proper analytic map $\rho\colon \widehat{X/G} \to X/G$ from a complex manifold $\widehat{X/G}$ onto X/G such that the following conditions hold:

(i) $\rho_{\mathrm{rest}}\colon \widehat{X/G} - \rho^{-1}(\mathrm{Sing}(X/G)) \to X/G - \mathrm{Sing}(X/G)$ is biregular.

(ii) For each singular point q of X/G, $\rho\colon (\widehat{X/G}, \rho^{-1}(q)) \to (X/G, q)$ is isomorphic to a toroidal desingularization $\pi\colon (\widehat{\mathbb{C}^n/g}, \pi^{-1}(\vec{0})) \to (\mathbb{C}^n/g, \vec{0})$ as germs of analytic spaces for some finite diagonal group g.

Theorem 2. *Let G be a finite abelian group of biholomorphic maps of the compact complex n-manifold X. Assume that X^φ is an analytic subspace of codimension ≥ 2 for each nontrivial element φ in G. If the dualizing sheaf $\omega_{X/G}$ is a locally free $\mathscr{O}_{X/G}$-sheaf, and $\rho\colon \widehat{X/G} \to X/G$ is a desingularization of X/G with $K_{\widehat{X/G}} \simeq \rho^*\omega_{X/G}$, then*

$$\chi(\widehat{X/G}) = \frac{1}{|G|} \sum_{(\varphi,\psi)\in G\times G} \chi(X^\varphi \cap X^\psi).$$

Proof. For $x \in X_G$, let $[x]$ be the orbit of x in X/G. $\omega_{X/G}$ is trivial near $[x]$ in X/G, and $K_{\widehat{X/G}}$ is the trivial bundle near $\rho^{-1}([x])$. $\rho\colon (\widehat{X/G}, \rho^{-1}([x])) \to (X/G, [x])$ is isomorphic to $\pi\colon (\widehat{\mathbb{C}^n/g}, \rho^{-1}(\vec{0})) \to (\mathbb{C}^n/g, \vec{0})$ for a diagonal group g with $|g| = |G_x|$. There is a neighborhood U of $[x]$ in X/G such that $\chi(\rho^{-1}(U)) = \chi(\widehat{\mathbb{C}^n/g}) = |G_x|$ by Proposition 1. Denote $\mathscr{S} = \{H | H$: a subgroup of $G\}$ and $X(H) = \{x \in X | G_x = H\}$ for $H \in \mathscr{S}$. $X(H)$ is G-invariant and the closure $\overline{X(H)}$ of $X(H)$ is equal to $\bigcap_{\varphi \in H} X^{\varphi}$. For $x \in \overline{X(H)}$, the action of G_x near x in X is equivalent to that of a diagonal group near $\vec{0}$ in $\mathbb{C}^n$. So $\overline{X(H)}$ is a closed complex submanifold of X, and $\overline{X(H)} - X(H)$ is the union of a finite number of proper submanifolds of $\overline{X(H)}$ intersecting normally. The quotient group G/H acts on $\overline{X(H)}$ and freely on $X(H)$. We can find a G-invariant neighborhood $N(H)$ of $\overline{X(H)} - X(H)$ in $\overline{X(H)}$ such that $(\overline{X(H)} - X(H))/G$ is a deformation retract of $N(H)/G$, and the boundary $\partial N(H)$ of $N(H)$ is a closed (odd-dimensional) real submanifold of X. Since $\rho_{\text{rest}}\colon \rho^{-1}(X(H)/G) \to X(H)/G$ is a topological fiber bundle and the Euler number of the fiber equals $|H|$, we have

$$\chi(\rho^{-1}(X(H)/G)) = \chi(X(H)/G)|H|,$$
$$\chi(\rho^{-1}(\partial N(H)/G)) = \chi(\partial N(H)/G)|H| = 0.$$

Find a sequence of G-invariant closed subspaces Y_i of X:

$$X = Y_0 \supsetneqq Y_1 \supsetneqq \cdots \supsetneqq Y_m \supsetneqq Y_{m+1} = \varnothing,$$

such that for each $0 \le i \le m$, $Y_i = Y_{i+1} \cup \overline{X(H_i)}$ and $Y_i - Y_{i+1} = X(H_i)$ for some $H_i \in \mathscr{S}$. So X is the disjoint union of $\{X(H_i) | 0 \le i \le m\}$, and this implies $\{H_i | 0 \le i \le m\} = \{H \in \mathscr{S} | X(H) \ne \varnothing\}$. Since

$$\begin{aligned}\chi(\rho^{-1}(Y_i/G)) &= \chi(\rho^{-1}(Y_{i+1}/G)) + \chi(\rho^{-1}(X(H_i)/G)) - \chi(\rho^{-1}(\partial N(H_i)/G)) \\ &= \chi(\rho^{-1}(Y_{i+1}/G)) + \chi(X(H_i)/G)|H_i|,\end{aligned}$$

we can prove by induction

$$\chi(\rho^{-1}(Y_j/G)) = \sum_{i=j}^{m} \chi(X(H_i)/G)|H_i| \quad \text{for } j = m, m-1, \cdots, 0.$$

If we set the Euler number of the empty set as zero, we have

$$\begin{aligned}\chi(\widehat{X/G}) = \chi(Y_0) &= \sum_{i=0}^{m} \chi(X(H_i)/G)|H_i| = \sum_{H \in \mathscr{S}} \chi(X(H)/G)|H| \\ &= \frac{1}{|G|} \sum_{H \in \mathscr{S}} \chi(X(H)/G)|G/H||H|^2 = \frac{1}{|G|} \sum_{H \in \mathscr{S}} \chi(X(H))|H|^2.\end{aligned}$$

On the other hand, for $(\varphi, \psi) \in G \times G$, $X^\varphi \cap X^\psi$ is the disjoint union of $\{X(H) | \varphi, \psi \in H \in \mathscr{S}\}$. Then we can also show that

$$\chi(X^\varphi \cap X^\psi) = \sum_{\substack{H \in \mathscr{S} \\ \varphi, \psi \in H}} \chi(X(H)).$$

Hence

$$\frac{1}{|G|} \sum_{(\varphi,\psi)\in G\times G} \chi(X^\varphi \cap X^\psi) = \frac{1}{|G|} \sum_{(\varphi,\psi)\in G\times G} \left[\sum_{\substack{H \in \mathscr{S} \\ \varphi, \psi \in H}} \chi(X(H)) \right]$$

$$= \frac{1}{|G|} \sum_{H \in \mathscr{S}} \sum_{(\varphi,\psi)\in H\times H} \chi(X(H)) = \frac{1}{|G|} \sum_{H \in \mathscr{S}} \chi(X(H))|H|^2.$$

Comparing this with the above formula for $\chi(\widehat{X/G})$, we obtain our result.

Corollary 1. *Let G_i $(i = 1, 2)$ be a finite abelian group acting on the compact complex manifold X_i, and G be the product group $G_1 \times G_2$ acting on the product manifold $X := X_1 \times X_2$. If $\widehat{X_i/G_i}$ $(i = 1, 2)$ and $\widehat{X/G}$ are the corresponding desingularizations satisfying the condition of Theorem 2, then*

$$\chi(\widehat{X/G}) = \chi(\widehat{X_1/G_1}) \cdot \chi(\widehat{X_2/G_2}).$$

Proof. For an element φ in G, $\varphi = \varphi_1 \times \varphi_2$ for $\varphi_i \in G_i$, and $X^\varphi = X_1^{\varphi_1} \times X^{\varphi_2}$. For $\varphi, \psi \in G$,

$$\chi(X^\varphi \cap X^\psi) = \chi((X_1^{\varphi_1} \cap X_1^{\psi_1}) \times (X_2^{\varphi_2} \cap X_2^{\psi_2}))$$
$$= \chi(X_1^{\varphi_1} \cap X_1^{\psi_1}) \cdot \chi(X_2^{\varphi_2} \cap X_2^{\psi_2}),$$

where $\varphi = \varphi_1 \times \varphi_2$ and $\psi = \psi_1 \times \psi_2$. By Theorem 2,

$$\chi(\widehat{X/G}) = \frac{1}{|G|} \sum_{\varphi_i, \psi_i \in G_i} \chi(X_1^{\varphi_1} \cap X_1^{\psi_1}) \chi(X_2^{\varphi_2} \cap X_2^{\psi_2})$$
$$= \frac{1}{|G_1||G_2|} \left\{ \sum_{\varphi_1, \psi_1 \in G_1} \chi(X_1^{\varphi_1} \cap X_1^{\psi_1}) \right\} \left\{ \sum_{\varphi_2, \psi_2 \in G_2} \chi(X_2^{\varphi_2} \cap X_2^{\psi_2}) \right\}$$
$$= \chi(\widehat{X_1/G_1}) \chi(\widehat{X_2/G_2}).$$

Corollary 2. *Let G be a finite abelian group of the biholomorphic maps of a complex n-torus V. If ρ: $\widehat{V/G} \to V/G$ is a desingularization of V/G with the $K_{\widehat{V/G}}$ = trivial, then*

$$\chi(\widehat{V/G}) = \frac{1}{|G|} \sum_{(\varphi,\psi)\in G\times G} |\{\textit{isolated points in } V^\varphi \cap V^\psi\}|.$$

Proof. Obviously, V, G and $\widehat{V/G}$ satisfy the assumption of the above theorem. For $(\varphi, \psi) \in G \times G$, each connected component of $V^{\varphi} \cap V^{\psi}$ with dimension ≥ 1 is a translation of a subtorus of V, and has the zero Euler number. Therefore our result follows immediately.

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