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DEFORMING THE METRIC ON COMPLETE RIEMANNIAN MANIFOLDS

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1. Introduction

In his paper [3] R. S. Hamilton introduced the evolution equation method which has proved to be very useful in the research of differential geometrical problems.

Using the evolution equation to deform the metric on any n-dimensional Riemannian manifold (M, g_{ij}):

$$\frac{\partial}{\partial t} g_{ij} = -2R_{ij}, \tag{1}$$

where R_{ij} is the Ricci curvature of M, the first important thing which we have to consider is the short-time existence of the solution of the evolution equation (1). In the case where M is a compact Riemannian manifold, Hamilton in [3] proved that for any given initial data metric g_{ij} on M the evolution equation (1) always has a unique solution for a short time. Therefore the short time existence problem of the evolution equation (1) was solved completely in the case when M is compact.

In the case where M is a noncompact complete Riemannian manifold, the short time existence problem of the evolution equation (1) is more difficult than the same problem for the compact case. Actually one cannot prove the short time existence of the evolution equation (1) for an arbitrary complete noncompact Riemannian manifold M; it is easy to find a complete noncompact Riemannian manifold (M, g_{ij}) on which the evolution equation (1) does not have any solution for an arbitrarily small time interval. Therefore to get the short time existence we have to make some assumptions on the curvature of M.

For a Riemannian manifold M with metric

$$ds^2 = g_{ij}(x)\, dx^i\, dx^j > 0,$$

we use $\{R_{ijkl}\}$ to denote the Riemannian curvature tensor of M and let

$$R_{ij} = g^{kl} R_{ikjl} \quad \text{and} \quad R = g^{ij} R_{ij} = g^{ij} g^{kl} R_{ikjl}$$

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to be the Ricci curvature and scalar curvature respectively, where $(g^{ij}) = (g_{ij})^{-1}$.

For any tensors such as $\{T_{ijkl}\}$, $\{U_{ijkl}\}$ defined on M, we have the inner product

$$\langle T_{ijkl}, U_{ijkl}\rangle = g^{i\alpha}g^{j\beta}g^{k\gamma}g^{l\delta}T_{ijkl}U_{\alpha\beta\gamma\delta},$$

and the norm of $\{T_{ijkl}\}$ is defined as follows:

$$|T_{ijkl}|^2 = \langle T_{ijkl}, T_{ijkl}\rangle.$$

We use ∇T_{ijkl} to denote the covariant derivatives of the tensor $\{T_{ijkl}\}$ with respect to the metric ds^2, $\nabla^m T_{ijkl}$ all of the mth covariant derivatives of $\{T_{ijkl}\}$, and $\mathrm{inj}(M)$ the injectivity radius of M.

Under these notations, the main theorem which we will prove in this paper is the following:

Theorem 1.1. *Let $(M, g_{ij}(x))$ be an n-dimensional complete noncompact Riemannian manifold with its Riemannian curvature tensor $\{R_{ijkl}\}$ satisfying*

$$|R_{ijkl}|^2 \le k_0 \quad \text{on } M, \tag{2}$$

where $0 < k_0 < +\infty$ is a constant. Then there exists a constant $T(n, k_0) > 0$ depending only on n and k_0 such that the evolution equation

$$\begin{aligned} &\frac{\partial}{\partial t} g_{ij}(x,t) = -2R_{ij}(x,t) \quad \text{on } M, \\ &g_{ij}(x,0) = g_{ij}(x) \quad \forall x \in M \end{aligned} \tag{3}$$

has a smooth solution $g_{ij}(x,t) > 0$ for a short time $0 \le t \le T(n,k_0)$, and satisfies the following estimates: For any integer $m \ge 0$, there exist constants $c_m > 0$ depending only on n, m and k_0 such that

$$\sup_{x\in M} |\nabla^m R_{ijkl}(x,t)|^2 \le C_m/t^m, \qquad 0 \le t \le T(n,k_0). \tag{4}$$

In Theorem 1.1 if we consider the new metric

$$d\tilde{s}^2 = g_{ij}(x,T)\,dx^i\,dx^j > 0$$

on the manifold M, then we get the following theorem immediately:

Theorem 1.2. *Let $(M, g_{ij}(x))$ be an n-dimensional complete noncompact Riemannian manifold with its Riemannian curvature tensor $\{R_{ijkl}\}$ satisfying*

$$|R_{ijkl}(x)|^2 \le k_0 \quad \forall x \in M,$$

where $0 < k_0 < +\infty$ is a constant. Then there exists another metric

$$d\tilde{s}^2 = \tilde{g}_{ij}(x)\,dx^i\,dx^j > 0$$

on M and constants $c > 0$, $\tilde{c}_m > 0$ $(m = 0, 1, 2, 3, \cdots)$ depending only on n and k_0 such that

$$\frac{1}{c} g_{ij}(x) \le \tilde{g}_{ij}(x) \le c g_{ij}(x) \quad \forall x \in M, \tag{5}$$
$$|\tilde{\nabla}^m \tilde{R}_{ijkl}(x)|^2 \le \tilde{c}_m \quad \forall x \in M, m \ge 0,$$

where $\tilde{\nabla}^m \tilde{R}_{ijkl}$ denotes the mth covariant derivatives of the curvature tensor $\{\tilde{R}_{ijkl}(x)\}$ with respect to the metric $d\tilde{s}^2$.

Proof of Theorem 1.2. We let $T = T(n, k_0)$ and

$$\tilde{g}_{ij}(x) = g_{ij}(x, T) \quad \forall x \in M,$$

where $g_{ij}(x, t)$ is the solution of the evolution equation (3) in Theorem 1.1. Since $T > 0$ depends only on n and k_0, from (4) we know that for any integer $m \ge 0$ one has

$$|\tilde{\nabla}^m \tilde{R}_{ijkl}(x)|^2 \le \tilde{c}_m(n, k_0) \quad \forall x \in M, \tag{6}$$

where $0 < \tilde{c}_m(n, k_0) < +\infty$ are constants depending only on n and k_0.

From the equation

$$\frac{\partial}{\partial t} g_{ij}(x, t) = -2R_{ij}(x, t), \qquad 0 \le t \le T,$$

it follows that

$$\left|\frac{\partial}{\partial t} g_{ij}\right|^2 \le 4|R_{ij}|^2 \le 4n^2 |R_{ijkl}|^2, \qquad 0 \le t \le T. \tag{7}$$

Using (4) we have

$$|R_{ijkl}|^2 \le c_0, \qquad 0 \le t \le T. \tag{8}$$

From (7) and (8) we get

$$\left|\frac{\partial}{\partial t} g_{ij}\right|^2 \le 4n^2 c_0, \qquad 0 \le t \le T,$$
$$\left|\frac{\partial}{\partial t} g_{ij}\right| \le 2n\sqrt{c_0}, \qquad 0 \le t \le T,$$
$$-2n\sqrt{c_0} g_{ij} \le \frac{\partial}{\partial t} g_{ij} \le 2n\sqrt{c_0} g_{ij}, \qquad 0 \le t \le T.$$

This implies

$$e^{-2n\sqrt{c_0}t} g_{ij}(x, 0) \le g_{ij}(x, t) \le e^{2n\sqrt{c_0}t} g_{ij}(x, 0) \quad \forall x \in M, 0 \le t \le T. \tag{9}$$

Let $t = T$ and $c = e^{2n\sqrt{c_0}T}$. From (9) we get

$$\frac{1}{c} g_{ij}(x) \le g_{ij}(x, T) \le c g_{ij}(x);$$

that is

$$\frac{1}{c} g_{ij}(x) \le \tilde{g}_{ij}(x) \le c g_{ij}(x), \tag{10}$$

which together with (6) shows that (5) is true; thus we have completed the proof of Theorem 1.2.

In the remainder of this paper we will prove Theorem 1.1.

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2. Modified equation and zero order estimates

In the remainder of the paper we will assume that M is an n-dimensional complete noncompact Riemannian manifold with metric

$$d\tilde{s}^2 = \tilde{g}_{ij}(x)\, dx^i\, dx^j > 0, \tag{1}$$

and that its Riemannian curvature tensor $\{R_{ijkl}\}$ satisfies

$$|\tilde{R}_{ijkl}|^2 \le k_0 \quad \text{on } M. \tag{2}$$

Fix a point $x_0 \in M$ and let $B(x_0, \gamma)$ be the geodesic ball of radius γ centered at x_0. For any integer $k > 0$, let

$$D_k = B(x_0, k). \tag{3}$$

Then we get a family of open subsets $\{D_k\}$ such that

$$\begin{aligned} &D_k \subseteq D_{k+1}, \\ &\overline{D}_k \text{ is a compact subset of } M, \\ &M = \bigcup_{k=1}^{\infty} D_k, \end{aligned} \tag{4}$$

where $\overline{D}_k = D_k \cup \partial D_k$ denotes the closure of D_k on M.

To obtain a solution of the evolution equation

$$\frac{\partial}{\partial t} \hat{g}_{ij}(x,t) = -2\hat{R}_{ij}(x,t), \qquad \hat{g}_{ij}(x,0) = \tilde{g}_{ij}(x) \tag{5}$$

for a short time $0 \le t \le T$, we try to solve the Dirichlet boundary problem

$$\begin{aligned} &\frac{\partial}{\partial t} \hat{g}_{ij}(k,x,t) = -2\hat{R}_{ij}(k,x,t), \qquad x \in D_k, \\ &\hat{g}_{ij}(k,x,0) = \tilde{g}_{ij}(x), \qquad x \in D_k, \\ &\hat{g}_{ij}(k,x,t) \equiv \tilde{g}_{ij}(x), \qquad x \in \partial D_k,\ 0 \le t \le T, \end{aligned} \tag{6}$$

for each open set D_{ℓ}, and then we let $\ell \to +\infty$. If the limit metric $\hat{g}_{ij}(x,t) = \lim_{\ell \to +\infty} \hat{g}_{ij}(\ell, x, t)$ exists, we get a solution of (5) for a short time $0 \le t \le T$.

The Dirichlet boundary problem (6) may not have any solutions because the evolution equation (6) is not a strictly parabolic system, and is only a weak parabolic system. For the proof of weak parabolicity of (6), one can see R. S. Hamilton [3].

Therefore, instead of considering the weak parabolic system (5) we consider a modified evolution equation which is strictly parabolic so that we can get a solution of it for at least a short time by solving the corresponding Dirichlet boundary problems. The solution of system (5) then comes from the solution of the modified equation.

Suppose the metrics

$$d\hat{s}_t^2 = \hat{g}_{ij}(x,t)\,dx^i\,dx^j > 0, \qquad 0 \le t \le T, \tag{7}$$

are the solution of (5) for $0 \le t \le T$, and $\varphi_t: M \to M$ $(0 \le t \le T)$ is a family of diffeomorphisms of M. Let

$$ds_t^2 = \varphi_t^*\,d\hat{s}_t^2, \qquad 0 \le t \le T, \tag{8}$$

be the pull-back metrics. Then we want to find the evolution equation for the metrics ds_t^2.

For any coordinate system $x = \{x^1, x^2, \cdots, x^n\}$ on M, let

$$ds_t^2 = g_{ij}(x,t)\,dx^i\,dx^j > 0, \tag{9}$$

$$y(x,t) = \varphi_t(x) = \left\{y^1(x,t), y^2(x,t), \cdots, y^n(x,t)\right\}. \tag{10}$$

Then by (7), (8) and (9) we have

$$g_{ij}(x,t) = \frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\beta}{\partial x^j}\hat{g}_{\alpha\beta}(y,t) \tag{11}$$

and by the assumption $\hat{g}_{\alpha\beta}(x,t)$ satisfies the following equations:

$$\begin{aligned} &\frac{\partial}{\partial t}\hat{g}_{\alpha\beta}(x,t) = -2\hat{R}_{\alpha\beta}(x,t), \qquad 0 \le t \le T, \\ &\hat{g}_{\alpha\beta}(x,0) = \tilde{g}_{\alpha\beta}(x). \end{aligned} \tag{12}$$

We use R_{ij}, $\hat{R}_{ij}$, $\tilde{R}_{ij}$; Γ_{ij}^k, $\hat{\Gamma}_{ij}^k$, $\tilde{\Gamma}_{ij}^k$; ∇, $\hat{\nabla}$, $\tilde{\nabla}$ to denote the Ricci curvature, the Christoffel symbols, and the covariant derivatives with respect to g_{ij}, $\hat{g}_{ij}$,

$\tilde{g}_{ij}$, respectively. Then from (11) it follows that

$$\begin{aligned}\frac{\partial}{\partial t}g_{ij}(x,t) &= \frac{\partial}{\partial t}\left[\frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\beta}{\partial x^j}\hat{g}_{\alpha\beta}(y,t)\right] \\ &= \frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\beta}{\partial x^j}\frac{\partial}{\partial t}\hat{g}_{\alpha\beta}(y,t) + \frac{\partial}{\partial x^i}\left(\frac{\partial y^\alpha}{\partial t}\right)\frac{\partial y^\beta}{\partial x^j}\hat{g}_{\alpha\beta}(y,t) \\ &\quad + \frac{\partial y^\alpha}{\partial x^i}\frac{\partial}{\partial x^j}\left(\frac{\partial y^\beta}{\partial t}\right)\hat{g}_{\alpha\beta}(y,t).\end{aligned} \tag{13}$$

Using (12) we have

$$\frac{\partial}{\partial t}\hat{g}_{\alpha\beta}(y,t) = -2\hat{R}_{\alpha\beta}(y,t) + \frac{\partial \hat{g}_{\alpha\beta}}{\partial y^\gamma}\frac{\partial y^\gamma}{\partial t}, \tag{14}$$

and therefore

$$\begin{aligned}\frac{\partial}{\partial t}g_{ij}(x,t) = &-2\frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\beta}{\partial x^j}\hat{R}_{\alpha\beta}(y,t) \\ &+ \frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\beta}{\partial x^j}\frac{\partial \hat{g}_{\alpha\beta}}{\partial y^\gamma}\frac{\partial y^\gamma}{\partial t} + \frac{\partial}{\partial x^i}\left(\frac{\partial y^\alpha}{\partial t}\right)\frac{\partial y^\beta}{\partial x^j}\hat{g}_{\alpha\beta}(y,t) \\ &+ \frac{\partial y^\alpha}{\partial x^i}\frac{\partial}{\partial x^j}\left(\frac{\partial y^\beta}{\partial t}\right)\hat{g}_{\alpha\beta}(y,t), \qquad 0 \le t \le T.\end{aligned} \tag{15}$$

It is easy to see that

$$R_{ij}(x,t) = \frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\beta}{\partial x^j}\hat{R}_{\alpha\beta}(y,t). \tag{16}$$

If we choose a coordinate system $\{x^i\}$ such that at one point

$$\Gamma_{ij}^k = 0, \tag{17}$$

then

$$\frac{\partial g_{ij}}{\partial x^k} = 0. \tag{18}$$

From (11) we have

$$\hat{g}_{\alpha\beta} = \frac{\partial x^k}{\partial y^\alpha}\frac{\partial x^l}{\partial y^\beta}g_{kl}, \tag{19}$$

and therefore

$$\begin{aligned}\frac{\partial y^\beta}{\partial x^j}\hat{g}_{\alpha\beta}(y,t) &= \frac{\partial x^k}{\partial y^\alpha}g_{jk}, \\ \frac{\partial}{\partial x^i}\left(\frac{\partial y^\alpha}{\partial t}\right)\frac{\partial y^\beta}{\partial x^j}\hat{g}_{\alpha\beta}(y,t) &= \frac{\partial}{\partial x^i}\left(\frac{\partial y^\alpha}{\partial t}\right)\cdot\frac{\partial x^k}{\partial y^\alpha}g_{jk},\end{aligned} \tag{20}$$

which together with (18) implies

(21)

$$\frac{\partial}{\partial x^i}\left(\frac{\partial y^\alpha}{\partial t}\right)\frac{\partial y^\beta}{\partial x^j}\hat{g}_{\alpha\beta}(y,t)=\frac{\partial}{\partial x^i}\left(\frac{\partial y^\alpha}{\partial t}\frac{\partial x^k}{\partial y^\alpha}g_{jk}\right)-\frac{\partial y^\alpha}{\partial t}\frac{\partial}{\partial x^i}\left(\frac{\partial x^k}{\partial y^\alpha}\right)\cdot g_{jk}.$$

For the same reasoning we get

(22)

$$\frac{\partial y^\alpha}{\partial x^i}\frac{\partial}{\partial x^j}\left(\frac{\partial y^\beta}{\partial t}\right)\hat{g}_{\alpha\beta}(y,t)=\frac{\partial}{\partial x^j}\left(\frac{\partial y^\beta}{\partial t}\frac{\partial x^k}{\partial y^\beta}g_{ik}\right)-\frac{\partial y^\beta}{\partial t}\frac{\partial}{\partial x^j}\left(\frac{\partial x^k}{\partial y^\beta}\right)\cdot g_{ik}.$$

From (19) we have

$$\frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\beta}{\partial x^j}\frac{\partial y^\gamma}{\partial t}\frac{\partial \hat{g}_{\alpha\beta}}{\partial y^\gamma}=\frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\beta}{\partial x^j}\frac{\partial y^\gamma}{\partial t}\frac{\partial}{\partial y^\gamma}\left(\frac{\partial x^k}{\partial y^\alpha}\frac{\partial x^l}{\partial y^\beta}g_{kl}\right).$$

Since

$$\frac{\partial}{\partial y^\gamma}g_{kl}=0 \tag{23}$$

by (18), the above equation becomes

$$\begin{aligned}\frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\beta}{\partial x^j}\frac{\partial y^\gamma}{\partial t}\frac{\partial \hat{g}_{\alpha\beta}}{\partial y^\gamma}&=\frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\beta}{\partial x^j}\frac{\partial y^\gamma}{\partial t}g_{kl}\frac{\partial}{\partial y^\gamma}\left(\frac{\partial x^k}{\partial y^\alpha}\frac{\partial x^l}{\partial y^\beta}\right)\\&=\frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\gamma}{\partial t}\frac{\partial}{\partial y^\gamma}\left(\frac{\partial x^k}{\partial y^\alpha}\right)g_{jk}\\&\quad+\frac{\partial y^\beta}{\partial x^j}\frac{\partial y^\gamma}{\partial t}\frac{\partial}{\partial y^\gamma}\left(\frac{\partial x^k}{\partial y^\beta}\right)g_{ik}.\end{aligned}\tag{24}$$

Substituting (16), (21), (22), and (24) into (15) gives

$$\begin{aligned}\frac{\partial}{\partial t}g_{ij}(x,t)=&-2R_{ij}(x,t)+\frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\gamma}{\partial t}\frac{\partial}{\partial y^\gamma}\left(\frac{\partial x^k}{\partial y^\alpha}\right)g_{jk}\\&+\frac{\partial y^\beta}{\partial x^j}\frac{\partial y^\gamma}{\partial t}\frac{\partial}{\partial y^\gamma}\left(\frac{\partial x^k}{\partial y^\beta}\right)g_{ik}\\&-\frac{\partial y^\alpha}{\partial t}\frac{\partial}{\partial x^i}\left(\frac{\partial x^k}{\partial y^\alpha}\right)g_{jk}-\frac{\partial y^\beta}{\partial t}\frac{\partial}{\partial x^j}\left(\frac{\partial x^k}{\partial y^\beta}\right)\cdot g_{ik}\\&+\frac{\partial}{\partial x^i}\left(\frac{\partial y^\alpha}{\partial t}\frac{\partial x^k}{\partial y^\alpha}g_{jk}\right)+\frac{\partial}{\partial x^j}\left(\frac{\partial y^\beta}{\partial t}\frac{\partial x^k}{\partial y^\beta}g_{ik}\right).\end{aligned}\tag{25}$$

On the other hand,

(26)

$$-\frac{\partial y^\alpha}{\partial t}\frac{\partial}{\partial x^i}\left(\frac{\partial x^k}{\partial y^\alpha}\right)g_{jk}=-\frac{\partial y^\alpha}{\partial t}\frac{\partial^2 x^k}{\partial y^\gamma\partial y^\alpha}\frac{\partial y^\gamma}{\partial x^i}g_{jk}=-\frac{\partial y^\alpha}{\partial x^i}\frac{\partial^2 x^k}{\partial y^\gamma\partial y^\alpha}\frac{\partial y^\gamma}{\partial t}g_{jk},$$

or

$$\frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\gamma}{\partial t}\frac{\partial}{\partial y^\gamma}\left(\frac{\partial x^k}{\partial y^\alpha}\right)g_{jk}-\frac{\partial y^\alpha}{\partial t}\frac{\partial}{\partial x^i}\left(\frac{\partial x^k}{\partial y^\alpha}\right)g_{jk}=0.$$

We also have

$$(27)\qquad \frac{\partial y^\beta}{\partial x^j}\frac{\partial y^\gamma}{\partial t}\frac{\partial}{\partial y^\gamma}\left(\frac{\partial x^k}{\partial y^\beta}\right)g_{ik} - \frac{\partial y^\beta}{\partial t}\frac{\partial}{\partial x^j}\left(\frac{\partial x^k}{\partial y^\beta}\right)g_{ik} = 0.$$

Combining (25), (26), and (27) we get

(28)

$$\frac{\partial}{\partial t}g_{ij}(x,t) = -2R_{ij}(x,t) + \frac{\partial}{\partial x^i}\left(\frac{\partial y^\alpha}{\partial t}\frac{\partial x^k}{\partial y^\alpha}g_{jk}\right) + \frac{\partial}{\partial x^j}\left(\frac{\partial y^\beta}{\partial t}\frac{\partial x^k}{\partial y^\beta}g_{ik}\right).$$

Since $\Gamma^k_{ij} = 0$, from (28) one has

$$(29)\quad \frac{\partial}{\partial t}g_{ij}(x,t) = -2R_{ij}(x,t) + \nabla_i\left(\frac{\partial y^\alpha}{\partial t}\frac{\partial x^k}{\partial y^\alpha}g_{jk}\right) + \nabla_j\left(\frac{\partial y^\alpha}{\partial t}\frac{\partial x^k}{\partial y^\alpha}g_{ik}\right).$$

If we define $y(x,t) = \varphi_t(x)$ by the equations

$$(30)\qquad \frac{\partial y^\alpha}{\partial t} = \frac{\partial y^\alpha}{\partial x^k}g^{\beta\gamma}\left(\Gamma^k_{\beta\gamma} - \tilde{\Gamma}^k_{\beta\gamma}\right), \qquad y^\alpha(x,0) = x^\alpha,$$

then (30) is a quasilinear first order system:

$$(31)\qquad \begin{gathered}\frac{\partial y^\alpha}{\partial t} = \frac{\partial y^\alpha}{\partial x^k}g^{\beta\gamma}g^{ik}\cdot\frac{1}{2}\left(\tilde{\nabla}_\beta g_{i\gamma} + \tilde{\nabla}_\gamma g_{i\beta} - \tilde{\nabla}_i g_{\beta\gamma}\right),\\ y^\alpha(x,0) = x^\alpha.\end{gathered}$$

From (29) we get

$$(32)\qquad \begin{aligned}\frac{\partial}{\partial t}g_{ij}(x,t) = &-2R_{ij}(x,t) + \nabla_i[g_{jk}g^{\beta\gamma}(\Gamma^k_{\beta\gamma} - \tilde{\Gamma}^k_{\beta\gamma})]\\ &+ \nabla_j[g_{ik}g^{\beta\gamma}(\Gamma^k_{\beta\gamma} - \tilde{\Gamma}^k_{\beta\gamma})].\end{aligned}$$

Since $y^\alpha(x,0) = x^\alpha$, from (11) it follows that

$$(33)\qquad g_{ij}(x,0) = \hat{g}_{ij}(x,0) = \tilde{g}_{ij}(x).$$

If we define a tensor

$$(34)\qquad V_i = g_{ik}g^{\beta\gamma}(\Gamma^k_{\beta\gamma} - \tilde{\Gamma}^k_{\beta\gamma}),$$

then using (32) and (33) we get the evolution equation for the pull-back metric $g_{ij}(x,t)$:

$$(35)\qquad \begin{gathered}\frac{\partial}{\partial t}g_{ij}(x,t) = -2R_{ij}(x,t) + \nabla_i V_j + \nabla_j V_i,\\ g_{ij}(x,0) = \tilde{g}_{ij}(x).\end{gathered}$$

System (35) is the modified evolution equation. In this paper we consider system (35) instead of the original evolution equation (5) because (35) is a strictly parabolic system, while (5) is only a weak parabolic system.

Lemma 2.1. *The modified evolution equation* (35) *is a strictly parabolic system. Actually we have*

$$\begin{aligned}\frac{\partial}{\partial t}g_{ij} = g^{\alpha\beta}\tilde\nabla_\alpha\tilde\nabla_\beta g_{ij} - g^{\alpha\beta}g_{ip}\tilde g^{pq}\tilde R_{j\alpha q\beta} - g^{\alpha\beta}g_{jp}\tilde g^{pq}\tilde R_{i\alpha q\beta}\\ + \frac{1}{2}g^{\alpha\beta}g^{pq}\left(\tilde\nabla_i g_{p\alpha}\cdot\tilde\nabla_j g_{q\beta} + 2\tilde\nabla_\alpha g_{jp}\cdot\tilde\nabla_q g_{i\beta} - 2\tilde\nabla_\alpha g_{jp}\cdot\tilde\nabla_\beta g_{iq}\right.\\ \left. - 2\tilde\nabla_j g_{p\alpha}\cdot\tilde\nabla_\beta g_{iq} - 2\tilde\nabla_i g_{p\alpha}\cdot\tilde\nabla_\beta g_{jq}\right).\end{aligned}$$

Proof. By the definition of the Christoffel symbols and the Riemannian curvature tensor we have

$$\begin{aligned}R_{ijkl} &= g_{pk}R^p_{ijl},\\ R^k_{ijl} &= \frac{\partial}{\partial x^i}\Gamma^k_{jl} - \frac{\partial}{\partial x^j}\Gamma^k_{il} + \Gamma^k_{ip}\Gamma^p_{jl} - \Gamma^k_{jp}\Gamma^p_{il},\\ \Gamma^k_{ij} &= \frac{1}{2}g^{kl}\left(\frac{\partial g_{il}}{\partial x^j} + \frac{\partial g_{jl}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^l}\right);\end{aligned} \tag{36}$$

$$\begin{aligned}R_{ijkl} = \frac{1}{2}&\left(\frac{\partial^2 g_{jk}}{\partial x^i\partial x^l} + \frac{\partial^2 g_{il}}{\partial x^j\partial x^k} - \frac{\partial^2 g_{jl}}{\partial x^i\partial x^k} - \frac{\partial^2 g_{ik}}{\partial x^j\partial x^l}\right)\\ &+ \frac{1}{4}g^{pq}\left(\frac{\partial g_{pk}}{\partial x^j}\frac{\partial g_{iq}}{\partial x^l} + \frac{\partial g_{pk}}{\partial x^j}\frac{\partial g_{lq}}{\partial x^i} - \frac{\partial g_{pk}}{\partial x^j}\frac{\partial g_{il}}{\partial x^q}\right.\\ &\qquad - \frac{\partial g_{pk}}{\partial x^i}\frac{\partial g_{jq}}{\partial x^l} - \frac{\partial g_{pk}}{\partial x^i}\frac{\partial g_{lq}}{\partial x^j} + \frac{\partial g_{pk}}{\partial x^i}\frac{\partial g_{jl}}{\partial x^q}\\ &\qquad + \frac{\partial g_{ik}}{\partial x^p}\frac{\partial g_{jq}}{\partial x^l} + \frac{\partial g_{ik}}{\partial x^p}\frac{\partial g_{lq}}{\partial x^j} - \frac{\partial g_{ik}}{\partial x^p}\frac{\partial g_{jl}}{\partial x^q}\\ &\qquad - \frac{\partial g_{ip}}{\partial x^k}\frac{\partial g_{jq}}{\partial x^l} - \frac{\partial g_{ip}}{\partial x^k}\frac{\partial g_{lq}}{\partial x^j} + \frac{\partial g_{ik}}{\partial x^p}\frac{\partial g_{jl}}{\partial x^q}\\ &\qquad - \frac{\partial g_{jk}}{\partial x^p}\frac{\partial g_{iq}}{\partial x^l} - \frac{\partial g_{jk}}{\partial x^p}\frac{\partial g_{lq}}{\partial x^i} + \frac{\partial g_{jk}}{\partial x^p}\frac{\partial g_{il}}{\partial x^q}\\ &\qquad \left. + \frac{\partial g_{jp}}{\partial x^k}\frac{\partial g_{iq}}{\partial x^l} + \frac{\partial g_{jp}}{\partial x^k}\frac{\partial g_{lq}}{\partial x^i} - \frac{\partial g_{jp}}{\partial x^k}\frac{\partial g_{il}}{\partial x^q}\right).\end{aligned} \tag{37}$$

We still have

$$\Gamma^j_{kl} - \tilde\Gamma^j_{kl} = \tfrac{1}{2}g^{jp}(\tilde\nabla_k g_{pl} + \tilde\nabla_l g_{pk} - \tilde\nabla_p g_{kl}). \tag{38}$$

By definition, $V_i = g_{ij}g^{kl}(\Gamma^j_{kl} - \tilde\Gamma^j_{kl})$, so

$$V_i = \tfrac{1}{2}g^{kl}\left(\tilde\nabla_k g_{il} + \tilde\nabla_l g_{ik} - \tilde\nabla_i g_{kl}\right). \tag{39}$$

Since $\nabla_k g_{ij} = 0$ and $\nabla_k g^{ij} = 0$, from (39) we have

$$\begin{aligned}\nabla_j V_i + \nabla_i V_j = \tfrac{1}{2}g^{kl}&\left(\nabla_j\tilde\nabla_k g_{il} + \nabla_j\tilde\nabla_l g_{ik} - \nabla_j\tilde\nabla_i g_{kl}\right.\\ &\left.+\nabla_i\tilde\nabla_k g_{jl} + \nabla_i\tilde\nabla_l g_{jk} - \nabla_i\tilde\nabla_j g_{kl}\right).\end{aligned} \tag{40}$$

If we choose a coordinate system $\{x^i\}$ such that at one point

$$\tilde{\Gamma}_{ij}^k = 0, \tag{41}$$

then from (38) it follows that

$$\Gamma_{ij}^k = \tfrac{1}{2} g^{kl}\left(\tilde{\nabla}_i g_{jl} + \tilde{\nabla}_j g_{il} - \tilde{\nabla}_l g_{ij}\right). \tag{42}$$

By the definition of covariant derivative, we have

$$\begin{aligned} \nabla_j \tilde{\nabla}_k g_{il} &= \frac{\partial}{\partial x^j} \tilde{\nabla}_k g_{il} - \Gamma_{jk}^p \tilde{\nabla}_p g_{il} - \Gamma_{ji}^p \tilde{\nabla}_k g_{pl} - \Gamma_{jl}^p \tilde{\nabla}_k g_{ip} \\ &= \tilde{\nabla}_j \tilde{\nabla}_k g_{il} - \Gamma_{jk}^p \tilde{\nabla}_p g_{il} - \Gamma_{ji}^p \tilde{\nabla}_k g_{pl} - \Gamma_{jl}^p \tilde{\nabla}_k g_{ip}. \end{aligned} \tag{43}$$

Substituting (42) into (43) yields

$$\begin{aligned} &\tfrac{1}{2} g^{kl} \nabla_j \tilde{\nabla}_k g_{il} = \tfrac{1}{2} g^{kl} \tilde{\nabla}_j \tilde{\nabla}_k g_{il} \\ &+ \tfrac{1}{4} g^{kl} g^{pq} \left(\tilde{\nabla}_p g_{il} \cdot \tilde{\nabla}_q g_{jk} - \tilde{\nabla}_p g_{il} \cdot \tilde{\nabla}_j g_{qk} - \tilde{\nabla}_p g_{il} \cdot \tilde{\nabla}_k g_{jq}\right. \\ &\qquad + \tilde{\nabla}_k g_{pl} \cdot \tilde{\nabla}_q g_{ij} - \tilde{\nabla}_j g_{qi} \cdot \tilde{\nabla}_k g_{pl} - \tilde{\nabla}_k g_{pl} \cdot \tilde{\nabla}_i g_{jq} \\ &\qquad \left.+ \tilde{\nabla}_k g_{ip} \cdot \tilde{\nabla}_q g_{jl} - \tilde{\nabla}_k g_{ip} \cdot \tilde{\nabla}_j g_{ql} - \tilde{\nabla}_k g_{ip} \cdot \tilde{\nabla}_l g_{jq}\right). \end{aligned} \tag{44}$$

Substituting (44) and the similar formulas into (40), we get

(45)

$$\begin{aligned} &\nabla_i V_j + \nabla_j V_i \\ &= \frac{1}{2} g^{kl} \left(\tilde{\nabla}_j \tilde{\nabla}_k g_{il} + \tilde{\nabla}_j \tilde{\nabla}_l g_{ik} - \tilde{\nabla}_j \tilde{\nabla}_i g_{kl}\right. \\ &\qquad \left. + \tilde{\nabla}_i \tilde{\nabla}_k g_{jl} + \tilde{\nabla}_i \tilde{\nabla}_l g_{jk} - \tilde{\nabla}_i \tilde{\nabla}_j g_{kl}\right) \\ &\quad + \frac{1}{2} g^{kl} g^{pq} \left(\tilde{\nabla}_p g_{il} \cdot \tilde{\nabla}_q g_{jk} - \tilde{\nabla}_p g_{il} \cdot \tilde{\nabla}_j g_{qk} - \tilde{\nabla}_p g_{il} \cdot \tilde{\nabla}_k g_{qj}\right. \\ &\qquad + \tilde{\nabla}_k g_{pl} \cdot \tilde{\nabla}_q g_{ij} - \tilde{\nabla}_j g_{qi} \cdot \tilde{\nabla}_k g_{pl} - \tilde{\nabla}_k g_{pl} \cdot \tilde{\nabla}_i g_{qj} \\ &\qquad + \tilde{\nabla}_k g_{ip} \cdot \tilde{\nabla}_q g_{jl} - \tilde{\nabla}_k g_{ip} \cdot \tilde{\nabla}_j g_{ql} - \tilde{\nabla}_k g_{ip} \cdot \tilde{\nabla}_l g_{qj} \\ &\qquad + \tilde{\nabla}_p g_{jl} \cdot \tilde{\nabla}_q g_{ik} - \tilde{\nabla}_p g_{jl} \cdot \tilde{\nabla}_i g_{qk} - \tilde{\nabla}_p g_{jl} \cdot \tilde{\nabla}_k g_{qi} \\ &\qquad + \tilde{\nabla}_k g_{pl} \cdot \tilde{\nabla}_q g_{ij} - \tilde{\nabla}_k g_{pl} \cdot \tilde{\nabla}_i g_{qj} - \tilde{\nabla}_k g_{pl} \cdot \tilde{\nabla}_j g_{qi} \\ &\qquad \left.+ \tilde{\nabla}_k g_{jp} \cdot \tilde{\nabla}_q g_{il} - \tilde{\nabla}_k g_{jp} \cdot \tilde{\nabla}_i g_{ql} - \tilde{\nabla}_k g_{jp} \cdot \tilde{\nabla}_l g_{qi}\right) \\ &\quad + \frac{1}{4} g^{kl} g^{pq} \left(\tilde{\nabla}_p g_{kl} \cdot \tilde{\nabla}_j g_{qi} + \tilde{\nabla}_p g_{kl} \cdot \tilde{\nabla}_i g_{qj} - \tilde{\nabla}_p g_{kl} \cdot \tilde{\nabla}_q g_{ij}\right. \\ &\qquad + \tilde{\nabla}_j g_{qk} \cdot \tilde{\nabla}_i g_{pl} + \tilde{\nabla}_i g_{pl} \cdot \tilde{\nabla}_k g_{qj} - \tilde{\nabla}_i g_{pl} \cdot \tilde{\nabla}_q g_{kj} \\ &\qquad + \tilde{\nabla}_i g_{pk} \cdot \tilde{\nabla}_j g_{ql} + \tilde{\nabla}_i g_{pk} \cdot \tilde{\nabla}_l g_{qj} - \tilde{\nabla}_i g_{pk} \cdot \tilde{\nabla}_q g_{jl} \\ &\qquad + \tilde{\nabla}_p g_{kl} \cdot \tilde{\nabla}_i g_{qj} + \tilde{\nabla}_p g_{kl} \cdot \tilde{\nabla}_j g_{qi} - \tilde{\nabla}_p g_{kl} \cdot \tilde{\nabla}_q g_{ij} \\ &\qquad + \tilde{\nabla}_i g_{qk} \cdot \tilde{\nabla}_j g_{pl} + \tilde{\nabla}_j g_{pl} \cdot \tilde{\nabla}_k g_{qi} - \tilde{\nabla}_j g_{pl} \cdot \tilde{\nabla}_q g_{ki} \\ &\qquad \left.+ \tilde{\nabla}_j g_{pk} \cdot \tilde{\nabla}_i g_{ql} + \tilde{\nabla}_j g_{pk} \cdot \tilde{\nabla}_l g_{qi} - \tilde{\nabla}_j g_{pk} \cdot \tilde{\nabla}_q g_{il}\right). \end{aligned}$$

Since $R_{ij} = g^{kl}R_{ikjl}$ and $\tilde{\Gamma}^k_{ij} = 0$ at one point, from (37) it follows that

(46)
$$\begin{aligned}
-2R_{ij} = g^{kl}&\left(\frac{\partial^2 g_{ij}}{\partial x^k \partial x^l} + \frac{\partial^2 g_{kl}}{\partial x^i \partial x^j} - \frac{\partial^2 g_{jk}}{\partial x^i \partial x^l} - \frac{\partial^2 g_{il}}{\partial x^k \partial x^j}\right) \\
&+ \frac{1}{2} g^{kl} g^{pq} \big(\tilde{\nabla}_k g_{pj} \cdot \tilde{\nabla}_q g_{il} - \tilde{\nabla}_k g_{pj} \cdot \tilde{\nabla}_l g_{qi} - \tilde{\nabla}_k g_{pj} \cdot \tilde{\nabla}_i g_{ql} \\
&\qquad + \tilde{\nabla}_i g_{pj} \cdot \tilde{\nabla}_l g_{qk} + \tilde{\nabla}_i g_{pj} \cdot \tilde{\nabla}_k g_{ql} - \tilde{\nabla}_i g_{pj} \cdot \tilde{\nabla}_q g_{kl} \\
&\qquad + \tilde{\nabla}_p g_{ij} \cdot \tilde{\nabla}_q g_{kl} - \tilde{\nabla}_p g_{ij} \cdot \tilde{\nabla}_l g_{qk} - \tilde{\nabla}_p g_{ij} \cdot \tilde{\nabla}_k g_{ql} \\
&\qquad + \tilde{\nabla}_j g_{ip} \cdot \tilde{\nabla}_l g_{qk} + \tilde{\nabla}_j g_{ip} \cdot \tilde{\nabla}_k g_{ql} - \tilde{\nabla}_j g_{ip} \cdot \tilde{\nabla}_q g_{kl} \\
&\qquad + \tilde{\nabla}_p g_{jk} \cdot \tilde{\nabla}_l g_{qi} + \tilde{\nabla}_p g_{jk} \cdot \tilde{\nabla}_i g_{ql} - \tilde{\nabla}_p g_{jk} \cdot \tilde{\nabla}_q g_{il} \\
&\qquad + \tilde{\nabla}_j g_{pk} \cdot \tilde{\nabla}_q g_{il} - \tilde{\nabla}_j g_{pk} \cdot \tilde{\nabla}_l g_{qi} - \tilde{\nabla}_j g_{pk} \cdot \tilde{\nabla}_i g_{ql}\big).
\end{aligned}$$

By definition,

$$\tilde{\nabla}_l g_{ij} = \frac{\partial g_{ij}}{\partial x^l} - \tilde{\Gamma}^p_{il} g_{pj} - \tilde{\Gamma}^p_{jl} g_{ip}.$$

But since $\tilde{\Gamma}^k_{ij} = 0$, we have

$$\begin{aligned}
\tilde{\nabla}_k \tilde{\nabla}_l g_{ij} = \frac{\partial}{\partial x^k} \tilde{\nabla}_l g_{ij} &= \frac{\partial}{\partial x^k}\left(\frac{\partial g_{ij}}{\partial x^l} - \tilde{\Gamma}^p_{il} g_{pj} - \tilde{\Gamma}^p_{jl} g_{ip}\right) \\
&= \frac{\partial^2 g_{ij}}{\partial x^k \partial x^l} - g_{jp} \frac{\partial}{\partial x^k} \tilde{\Gamma}^p_{il} - g_{ip} \frac{\partial}{\partial x^k} \tilde{\Gamma}^p_{jl},
\end{aligned}$$

and therefore the following formula:

$$g^{kl} \frac{\partial^2 g_{ij}}{\partial x^k \partial x^l} = g^{kl} \tilde{\nabla}_k \tilde{\nabla}_l g_{ij} + g^{kl} g_{jp} \frac{\partial}{\partial x^k} \tilde{\Gamma}^p_{il} + g^{kl} g_{ip} \frac{\partial}{\partial x^k} \tilde{\Gamma}^p_{jl}. \tag{47}$$

Similarly,

$$\begin{aligned}
g^{kl} \frac{\partial^2 g_{kl}}{\partial x^i \partial x^j} &= g^{kl} \tilde{\nabla}_i \tilde{\nabla}_j g_{kl} + \frac{\partial}{\partial x^i} \tilde{\Gamma}^k_{kj} + \frac{\partial}{\partial x^i} \tilde{\Gamma}^l_{lj}, \\
g^{kl} \frac{\partial^2 g_{jk}}{\partial x^i \partial x^l} &= g^{kl} \tilde{\nabla}_i \tilde{\nabla}_l g_{jk} + \frac{\partial}{\partial x^i} \tilde{\Gamma}^l_{lj} + g^{kl} g_{jp} \frac{\partial}{\partial x^i} \tilde{\Gamma}^p_{kl}, \\
g^{kl} \frac{\partial^2 g_{il}}{\partial x^k \partial x^j} &= g^{kl} \tilde{\nabla}_k \tilde{\nabla}_j g_{il} + \frac{\partial}{\partial x^k} \tilde{\Gamma}^k_{ij} + g^{kl} g_{ip} \frac{\partial}{\partial x^k} \tilde{\Gamma}^p_{jl}.
\end{aligned} \tag{48}$$

From (47) and (48) it follows that

$$\begin{aligned}
&g^{kl}\left(\frac{\partial^2 g_{ij}}{\partial x^k \partial x^l} + \frac{\partial^2 g_{kl}}{\partial x^i \partial x^j} - \frac{\partial^2 g_{jk}}{\partial x^i \partial x^l} - \frac{\partial^2 g_{il}}{\partial x^k \partial x^j}\right) \\
&= g^{kl}\left(\tilde{\nabla}_k \tilde{\nabla}_l g_{ij} + \tilde{\nabla}_i \tilde{\nabla}_j g_{kl} - \tilde{\nabla}_i \tilde{\nabla}_l g_{jk} - \tilde{\nabla}_k \tilde{\nabla}_j g_{il}\right) \\
&\quad + g^{kl} g_{jp} \frac{\partial}{\partial x^k} \tilde{\Gamma}^p_{il} - g^{kl} g_{jp} \frac{\partial}{\partial x^i} \tilde{\Gamma}^p_{kl} + \frac{\partial}{\partial x^i} \tilde{\Gamma}^k_{kj} - \frac{\partial}{\partial x^k} \tilde{\Gamma}^k_{ij}.
\end{aligned} \tag{49}$$

Since $\tilde{\Gamma}^k_{ij} = 0$, we have

$$\frac{\partial}{\partial x^k}\tilde{\Gamma}^p_{il} - \frac{\partial}{\partial x^i}\tilde{\Gamma}^p_{kl} = \tilde{R}^p_{kil} = \tilde{g}^{pq}\tilde{R}_{kiql},$$

$$g^{kl}g_{jp}\frac{\partial}{\partial x^k}\tilde{\Gamma}^p_{il} - g^{kl}g_{jp}\frac{\partial}{\partial x^i}\tilde{\Gamma}^p_{kl} = g^{kl}g_{jp}\tilde{g}^{pq}\tilde{R}_{kiql}, \tag{50}$$

$$\frac{\partial}{\partial x^i}\tilde{\Gamma}^k_{kj} - \frac{\partial}{\partial x^k}\tilde{\Gamma}^k_{ij} = \tilde{R}^k_{ikj} = \tilde{g}^{kl}\tilde{R}_{iklj} = -\tilde{R}_{ij}. \tag{51}$$

Combining (49), (50), and (51) we get

$$\begin{aligned} & g^{kl}\left(\frac{\partial^2 g_{ij}}{\partial x^k \partial x^l} + \frac{\partial^2 g_{kl}}{\partial x^i \partial x^j} - \frac{\partial^2 g_{jk}}{\partial x^i \partial x^l} - \frac{\partial^2 g_{il}}{\partial x^k \partial x^j}\right) \\ & = g^{kl}\left(\tilde{\nabla}_k\tilde{\nabla}_l g_{ij} + \tilde{\nabla}_i\tilde{\nabla}_j g_{kl} - \tilde{\nabla}_i\tilde{\nabla}_l g_{jk} - \tilde{\nabla}_k\tilde{\nabla}_j g_{il}\right) \\ & \quad + g^{kl}g_{jp}\tilde{g}^{pq}\tilde{R}_{iklq} - \tilde{R}_{ij}. \end{aligned} \tag{52}$$

Substituting (52) into (46) gives

$$\begin{aligned} -2R_{ij} = {} & g^{kl}\left(\tilde{\nabla}_k\tilde{\nabla}_l g_{ij} + \tilde{\nabla}_i\tilde{\nabla}_j g_{kl} - \tilde{\nabla}_i\tilde{\nabla}_l g_{jk} - \tilde{\nabla}_k\tilde{\nabla}_j g_{il}\right) \\ & - g^{kl}g_{jp}\tilde{g}^{pq}\tilde{R}_{ikql} - \tilde{R}_{ij} \\ & + \frac{1}{2}g^{kl}g^{pq}\left(\tilde{\nabla}_k g_{jp}\cdot\tilde{\nabla}_q g_{il} - \tilde{\nabla}_k g_{jp}\cdot\tilde{\nabla}_l g_{qi} - \tilde{\nabla}_k g_{jp}\cdot\tilde{\nabla}_i g_{ql}\right. \\ & \qquad + \tilde{\nabla}_i g_{jp}\cdot\tilde{\nabla}_l g_{qk} + \tilde{\nabla}_i g_{jp}\cdot\tilde{\nabla}_k g_{ql} - \tilde{\nabla}_i g_{jp}\cdot\tilde{\nabla}_q g_{kl} \\ & \qquad + \tilde{\nabla}_p g_{ij}\cdot\tilde{\nabla}_q g_{kl} - \tilde{\nabla}_p g_{ij}\cdot\tilde{\nabla}_l g_{qk} - \tilde{\nabla}_p g_{ij}\cdot\tilde{\nabla}_k g_{ql} \\ & \qquad + \tilde{\nabla}_j g_{ip}\cdot\tilde{\nabla}_l g_{qk} + \tilde{\nabla}_j g_{ip}\cdot\tilde{\nabla}_k g_{ql} - \tilde{\nabla}_j g_{ip}\cdot\tilde{\nabla}_q g_{kl} \\ & \qquad + \tilde{\nabla}_p g_{jk}\cdot\tilde{\nabla}_l g_{qi} + \tilde{\nabla}_p g_{jk}\cdot\tilde{\nabla}_i g_{ql} - \tilde{\nabla}_p g_{jk}\cdot\tilde{\nabla}_q g_{il} \\ & \qquad \left. + \tilde{\nabla}_j g_{pk}\cdot\tilde{\nabla}_q g_{il} - \tilde{\nabla}_j g_{pk}\cdot\tilde{\nabla}_l g_{qi} - \tilde{\nabla}_j g_{pk}\cdot\tilde{\nabla}_i g_{ql}\right). \end{aligned} \tag{53}$$

Substituting (45) and (53) into (35), simplifying and collecting the terms of the same type, and using the following formula to switch the second covariant derivatives:

$$\tilde{\nabla}_i\tilde{\nabla}_j g_{kl} = \tilde{\nabla}_j\tilde{\nabla}_i g_{kl} + \tilde{g}^{pq}\tilde{R}_{ijkp}g_{ql} + \tilde{g}^{pq}\tilde{R}_{ijlp}g_{qk}. \tag{54}$$

Finally,

$$\begin{aligned} \frac{\partial}{\partial t}g_{ij} = {} & g^{kl}\tilde{\nabla}_k\tilde{\nabla}_l g_{ij} - g^{kl}g_{ip}\tilde{g}^{pq}\tilde{R}_{jkql} - g^{kl}g_{jp}\tilde{g}^{pq}\tilde{R}_{ikql} \\ & + \frac{1}{2}g^{kl}g^{pq}\left(\tilde{\nabla}_i g_{pk}\cdot\tilde{\nabla}_j g_{ql} + 2\tilde{\nabla}_k g_{jp}\cdot\tilde{\nabla}_q g_{il} - 2\tilde{\nabla}_k g_{jp}\cdot\tilde{\nabla}_l g_{iq}\right. \\ & \qquad \left. -2\tilde{\nabla}_j g_{pk}\tilde{\nabla}_l g_{iq} - 2\tilde{\nabla}_i g_{pk}\cdot\tilde{\nabla}_l g_{jq}\right). \end{aligned} \tag{55}$$

Hence we have completed the proof of the lemma.

From Lemma 2.1 we know that the modified evolution equation (35) is a strictly parabolic system, therefore we can consider the corresponding Dirichlet boundary problem in any domain $D \subseteq M$.

Suppose $D \subseteq M$ is a domain with boundary ∂D a compact C^∞ $(n-1)$-dimensional submanifold of M, and assume that the closure $\overline{D} = D \cup \partial D$ is a compact subset of M. We will solve the following Dirichlet boundary problem:

$$\begin{aligned} &\frac{\partial}{\partial t} g_{ij}(x,t) = -2R_{ij}(x,t) + \nabla_i V_j + \nabla_j V_i, \qquad x \in D, \\ &g_{ij}(x,0) = \tilde{g}_{ij}(x), \qquad x \in D, \\ &g_{ij}(x,t) \equiv \tilde{g}_{ij}(x), \qquad x \in \partial D,\ 0 \le t \le T. \end{aligned} \tag{56}$$

In this section we want to establish the zero order estimates for the solution of (56). The existence theorem for the solution of (56) will be proved in the next section.

First, we have the following lemma.

Lemma 2.2. *Suppose $g_{ij}(x,t) > 0$ is a solution of (56), and $m > 0$ is an integer, and define*

$$\varphi(x,t) = g^{\alpha_1\beta_1}\tilde{g}_{\beta_1\alpha_2} g^{\alpha_2\beta_2}\tilde{g}_{\beta_2\alpha_3} g^{\alpha_3\beta_3}\tilde{g}_{\beta_3\alpha_4} \cdots g^{\alpha_m\beta_m}\tilde{g}_{\beta_m\alpha_1}, \qquad x \in D,\ 0 \le t \le T.$$

Then

$$\frac{\partial \varphi}{\partial t} \le g^{\alpha\beta}\tilde{\nabla}_\alpha\tilde{\nabla}_\beta\varphi + 2mn\sqrt{k_0}\varphi^{1+1/m}, \qquad x \in D, \tag{57}$$

$$\varphi(x,0) \equiv n, \qquad x \in D, \tag{58}$$

$$\varphi(x,t) \equiv n, \qquad x \in \partial D,\ 0 \le t \le T. \tag{59}$$

Proof. Using the initial and boundary value conditions in (56) we get (58) and (59) immediately. From Lemma 2.1 it follows that

$$\begin{aligned} \frac{\partial}{\partial t} g^{ij} = &- g^{ik}g^{jl}\frac{\partial}{\partial t}g_{kl} \\ = &- g^{\alpha\beta}g^{ik}g^{jl}\tilde{\nabla}_\alpha\tilde{\nabla}_\beta g_{kl} + g^{\alpha\beta}g^{ik}g^{jl}g_{kp}\tilde{g}^{pq}\tilde{R}_{l\alpha q\beta} \\ &+ g^{\alpha\beta}g^{ik}g^{jl}g_{pl}\tilde{g}^{pq}\tilde{R}_{k\alpha q\beta} + \frac{1}{2}g^{\alpha\beta}g^{pq}g^{ik}g^{jl} \\ &\cdot \big(2\tilde{\nabla}_\alpha g_{pl}\cdot\tilde{\nabla}_\beta g_{qk} + 2\tilde{\nabla}_l g_{p\alpha}\cdot\tilde{\nabla}_\beta g_{qk} + 2\tilde{\nabla}_k g_{p\alpha}\cdot\tilde{\nabla}_\beta g_{ql} \\ &\qquad -2\tilde{\nabla}_\alpha g_{pl}\cdot\tilde{\nabla}_q g_{\beta k} - \tilde{\nabla}_k g_{p\alpha}\cdot\tilde{\nabla}_l g_{q\beta}\big). \end{aligned} \tag{60}$$

Since $g^{ip}g_{jp} = \delta^i_j$, we have

$$\tilde{\nabla}_\beta(g^{ip}g_{jp}) = 0,$$
$$g_{jp}\tilde{\nabla}_\beta g^{ip} + g^{ip}\tilde{\nabla}_\beta g_{jp} = 0,$$
$$\tilde{\nabla}_\beta g^{ij} = -g^{ip}g^{jq}\tilde{\nabla}_\beta g_{pq}, \tag{61}$$
$$\tilde{\nabla}_\alpha\tilde{\nabla}_\beta g^{ij} = -g^{ip}g^{jq}\tilde{\nabla}_\alpha\tilde{\nabla}_\beta g_{pq} - \tilde{\nabla}_\alpha(g^{ip}g^{jq})\cdot\tilde{\nabla}_\beta g_{pq},$$
$$g^{\alpha\beta}\tilde{\nabla}_\alpha\tilde{\nabla}_\beta g^{ij} = -g^{\alpha\beta}g^{ik}g^{jl}\tilde{\nabla}_\alpha\tilde{\nabla}_\beta g_{kl} - g^{\alpha\beta}\tilde{\nabla}_\alpha(g^{ip}g^{jq})\cdot\tilde{\nabla}_\beta g_{pq}.$$

Substituting (61) into (60) yields

$$\begin{aligned}\frac{\partial}{\partial t}g^{ij} = {} & g^{\alpha\beta}\tilde{\nabla}_\alpha\tilde{\nabla}_\beta g^{ij} + g^{\alpha\beta}g^{ik}g^{jl}g_{kp}\tilde{g}^{pq}\tilde{R}_{l\alpha q\beta} \\ & + g^{\alpha\beta}g^{ik}g^{jl}g_{pl}\tilde{g}^{pq}\tilde{R}_{k\alpha q\beta} + g^{\alpha\beta}g^{ip}\tilde{\nabla}_\alpha g^{jq}\cdot\tilde{\nabla}_\beta g_{pq} \\ & + g^{\alpha\beta}g^{jq}\tilde{\nabla}_\alpha g^{ip}\tilde{\nabla}_\beta g_{pq} + \frac{1}{2}g^{\alpha\beta}g^{pq}g^{ik}g^{jl} \\ & \cdot\big(2\tilde{\nabla}_\alpha g_{pl}\cdot\tilde{\nabla}_\beta g_{qk} + 2\tilde{\nabla}_l g_{p\alpha}\cdot\tilde{\nabla}_\beta g_{qk} + 2\tilde{\nabla}_k g_{p\alpha}\cdot\tilde{\nabla}_\beta g_{ql} \\ & \quad -2\tilde{\nabla}_\alpha g_{pl}\cdot\tilde{\nabla}_q g_{\beta k} - \tilde{\nabla}_k g_{p\alpha}\cdot\tilde{\nabla}_l g_{q\beta}\big).\end{aligned} \tag{62}$$

If we choose a coordinate system $\{x^i\}$ such that at one point

$$(\tilde{g}_{ij}) = \begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & & \ddots & \\ & & & 1\end{pmatrix}, \qquad (g_{ij}) = \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots & \\ & & & \lambda_n\end{pmatrix}, \tag{63}$$

then

$$(g^{ij}) = \begin{pmatrix} \lambda_1^{-1} & & 0 \\ & \lambda_2^{-1} & \\ 0 & & \ddots & \\ & & & \lambda_n^{-1}\end{pmatrix}. \tag{64}$$

From (61) it follows that

$$\widetilde{\nabla}_\beta g^{ij} = -\frac{1}{\lambda_i\lambda_j}\widetilde{\nabla}_\beta g_{ij}. \tag{65}$$

Substituting (63), (64), and (65) into (62), we get

$$\begin{aligned}\frac{\partial}{\partial t}g^{ij} = {} & g^{\alpha\beta}\widetilde{\nabla}_\alpha\widetilde{\nabla}_\beta g^{ij} + \frac{1}{\lambda_i\lambda_q}\widetilde{R}_{iqjq} + \frac{1}{\lambda_j\lambda_q}\widetilde{R}_{jqiq} \\ & - \frac{2}{\lambda_\alpha\lambda_q\lambda_i\lambda_j}\widetilde{\nabla}_\alpha g_{jq}\cdot\widetilde{\nabla}_\alpha g_{iq} + \frac{1}{2\lambda_i\lambda_j\lambda_k\lambda_q} \\ & \cdot(2\widetilde{\nabla}_k g_{qj}\cdot\widetilde{\nabla}_k g_{iq} + 2\widetilde{\nabla}_j g_{qk}\cdot\widetilde{\nabla}_k g_{iq} + 2\widetilde{\nabla}_i g_{qk}\cdot\widetilde{\nabla}_k g_{jq} \\ & \quad - 2\widetilde{\nabla}_k g_{qj}\cdot\widetilde{\nabla}_q g_{ik} - \widetilde{\nabla}_i g_{qk}\cdot\widetilde{\nabla}_j g_{qk}).\end{aligned} \tag{66}$$

By the definition of $\varphi(x,t)$ we have

$$\varphi(x,t) = \sum_{k=1}^{n} \left(\frac{1}{\lambda_k}\right)^m > 0; \tag{67}$$

thus

$$\frac{\partial \varphi}{\partial t} = m\left(\frac{1}{\lambda_i}\right)^{m-1} \frac{\partial}{\partial t} g^{ii}.$$

Using (66) we find

$$\begin{aligned} \frac{\partial \varphi}{\partial t} &= m\left(\frac{1}{\lambda_i}\right)^{m-1} g^{\alpha\beta}\tilde{\nabla}_\alpha\tilde{\nabla}_\beta g^{ii} + \frac{2m}{\lambda_i^m\lambda_q}\tilde{R}_{iqiq} \\ &\quad - \frac{m}{\lambda_i^{m+1}\lambda_q\lambda_k}\tilde{\nabla}_k g_{iq}\cdot\tilde{\nabla}_k g_{iq} - \frac{m}{2\lambda_i^{m+1}\lambda_k\lambda_q} \\ &\quad \cdot\left(\tilde{\nabla}_i g_{qk}\cdot\tilde{\nabla}_i g_{qk} + 2\tilde{\nabla}_k g_{iq}\cdot\tilde{\nabla}_q g_{ik} - 4\tilde{\nabla}_i g_{qk}\cdot\tilde{\nabla}_k g_{iq}\right), \\ \frac{\partial \varphi}{\partial t} &= m\left(\frac{1}{\lambda_i}\right)^{m-1} g^{\alpha\beta}\tilde{\nabla}_\alpha\tilde{\nabla}_\beta g^{ii} + \frac{2m}{\lambda_i^m\lambda_q}\tilde{R}_{iqiq} \\ &\quad - \frac{m}{2\lambda_q\lambda_k\lambda_i^{m+1}}\left(\tilde{\nabla}_k g_{iq} + \tilde{\nabla}_q g_{ik} - \tilde{\nabla}_i g_{qk}\right)^2. \end{aligned} \tag{68}$$

From (67) it follows that

(69)

$$\begin{aligned} g^{\alpha\beta}\tilde{\nabla}_\alpha\tilde{\nabla}_\beta\varphi &= m\left(\frac{1}{\lambda_i}\right)^{m-1} g^{\alpha\beta}\tilde{\nabla}_\alpha\tilde{\nabla}_\beta g^{ii} + m g^{\alpha\beta}\tilde{\nabla}_\alpha g^{ij} \\ &\quad \cdot\tilde{\nabla}_\beta g^{ij}\left[\left(\frac{1}{\lambda_i}\right)^{m-2} + \left(\frac{1}{\lambda_i}\right)^{m-3}\left(\frac{1}{\lambda_j}\right) + \cdots + \left(\frac{1}{\lambda_j}\right)^{m-2}\right]. \end{aligned}$$

Substituting (69) into (68) and using (65) we have

$$\begin{aligned} \frac{\partial \varphi}{\partial t} &= g^{\alpha\beta}\tilde{\nabla}_\alpha\tilde{\nabla}_\beta\varphi + \frac{2m}{\lambda_i^m\lambda_q}\tilde{R}_{iqiq} \\ &\quad - m\left(\frac{1}{\lambda_\alpha}\right)\left[\left(\frac{1}{\lambda_i}\right)^m\left(\frac{1}{\lambda_j}\right)^2 + \left(\frac{1}{\lambda_i}\right)^{m-1}\left(\frac{1}{\lambda_j}\right)^3\right. \\ &\quad \left. + \cdots + \left(\frac{1}{\lambda_i}\right)^2\left(\frac{1}{\lambda_j}\right)^m\right](\tilde{\nabla}_\alpha g_{ij})^2 \\ &\quad - \frac{m}{2\lambda_i^{m+1}\lambda_q\lambda_k}(\tilde{\nabla}_k g_{iq} + \tilde{\nabla}_q g_{ik} - \tilde{\nabla}_i g_{qk})^2; \end{aligned} \tag{70}$$

thus

$$\frac{\partial \varphi}{\partial t} \le g^{\alpha\beta}\tilde{\nabla}_\alpha\tilde{\nabla}_\beta\varphi + \frac{2m}{\lambda_i^m\lambda_q}\tilde{R}_{iqiq}. \tag{71}$$

By assumption (2), $|\widetilde{R}_{iqiq}| \leq \sqrt{k_0}$, and, in consequence of (67) and (71),

$$\frac{\partial \varphi}{\partial t} \leq g^{\alpha\beta}\widetilde{\nabla}_\alpha \widetilde{\nabla}_\beta \varphi + 2m\sqrt{k_0}\left(\sum_{q=1}^{n} \frac{1}{\lambda_q}\right) \cdot \varphi.$$

It is easy to see that

$$\sum_{q=1}^{n} \frac{1}{\lambda_q} \leq n\varphi^{1/m},$$

so that

$$\frac{\partial \varphi}{\partial t} \leq g^{\alpha\beta}\widetilde{\nabla}_\alpha \widetilde{\nabla}_\beta \varphi + 2mn\sqrt{k_0}\varphi^{1+1/m},$$

which completes the proof of Lemma 2.2.

Lemma 2.3. *Suppose $g_{ij}(x,t) > 0$ is a solution of* (56). *Then for any $\delta > 0$, we have*

$$g_{ij}(x,t) \geq (1-\delta)\tilde{g}_{ij}(x),$$

where $x \in D$, $0 \leq t \leq (1/(2\sqrt{k_0}))(\frac{1}{n})^{1+1/m}[1-(\frac{1}{2})^{1/m}]$, $m > 0$ is an integer, and

$$\frac{\log 2n}{\log(1/(1-\delta))} \leq m < \frac{\log 2n}{\log(1/(1-\delta))} + 1. \tag{72}$$

Proof. Choose an integer $m > 0$ which satisfies (72), and let $\varphi(x,t)$ be the function defined in Lemma 2.2. Since $\overline{D} \subseteq M$ is compact, we can define

$$\varphi(t) = \max_{x \in \overline{D}} \varphi(x,t). \tag{73}$$

Using the maximal principle on $\overline{D}$, from (57), (58), (59), and (73) we get

$$\frac{d\varphi(t)}{dt} \leq 2mn\sqrt{k_0}\varphi(t)^{1+1/m}, \qquad \varphi(0) = n. \tag{74}$$

Thus we have

$$\varphi(t) \leq \frac{n}{[1-2n^{1+1/m}\sqrt{k_0}t]^m}, \tag{75}$$

$$\varphi(x,t) \leq \frac{n}{[1-2n^{1+1/m}\sqrt{k_0}t]^m} \quad \forall x \in \overline{D}. \tag{76}$$

If

$$0 \leq t \leq \frac{1}{2\sqrt{k_0}}\left(\frac{1}{n}\right)^{1+1/m}\left[1-\left(\frac{1}{2}\right)^{1/m}\right],$$

then from (76),

$$\varphi(x,t) \leq 2n \quad \forall x \in \overline{D};$$

that is, $\sum_{k=1}^{n}(1/\lambda_k)^m \leq 2n$.

Since $0 < (1/\lambda_k)^m \leq 2n\ \forall k$, $\lambda_k \geq (1/2n)^{1/m}$, $k = 1, 2, \cdots, n$. From (63) it follows that

$$g_{ij}(x,t) \geq \left(\frac{1}{2n}\right)^{1/m} \tilde{g}_{ij}(x) \quad \forall x \in \overline{D}.$$

By (72) we have $(1/2n)^{1/m} \geq 1 - \delta$, and therefore

$$g_{ij}(x,t) \geq (1-\delta)\tilde{g}_{ij}(x) \quad \forall x \in D \tag{77}$$

if

$$0 \leq t \leq \frac{1}{2\sqrt{k_0}}\left(\frac{1}{n}\right)^{1+1/m}\left[1 - \left(\frac{1}{2}\right)^{1/m}\right].$$

In Lemma 2.3 we obtained the lower bound of $g_{ij}(x,t)$; now we want to estimate the upper bound of $g_{ij}(x,t)$.

Using the notation of (63), from Lemma 2.1 we get

$$\begin{aligned}\frac{\partial}{\partial t}g_{ij} = g^{\alpha\beta}\widetilde{\nabla}_\alpha\widetilde{\nabla}_\beta g_{ij} &- \frac{1}{\lambda_i\lambda_\alpha}\widetilde{R}_{j\alpha i\alpha} - \frac{1}{\lambda_j\lambda_\alpha}\widetilde{R}_{i\alpha j\alpha} \\ &+ \frac{1}{2\lambda_k\lambda_q}(\widetilde{\nabla}_i g_{kq}\cdot\widetilde{\nabla}_j g_{kq} + 2\widetilde{\nabla}_k g_{jq}\cdot\widetilde{\nabla}_q g_{ik} - 2\widetilde{\nabla}_k g_{jq}\cdot\widetilde{\nabla}_k g_{iq} \\ &\qquad - 2\widetilde{\nabla}_j g_{kq}\cdot\widetilde{\nabla}_k g_{iq} - 2\widetilde{\nabla}_i g_{kq}\cdot\widetilde{\nabla}_k g_{jq}).\end{aligned} \tag{78}$$

Supposing $\varepsilon > 0$ is a constant and $m > 0$ is an integer, we define a function

$$F(x,t) = \frac{1}{1 - [1/(n+\varepsilon)](\lambda_1^m + \lambda_2^m + \cdots + \lambda_n^m)}. \tag{79}$$

Then from (56) we know that

$$F(x,0) \equiv (n+\varepsilon)/\varepsilon, \qquad x \in D, \tag{80}$$

$$F(x,t) \equiv (n+\varepsilon)/\varepsilon, \qquad x \in \partial D,\ 0 \leq t \leq T. \tag{81}$$

By definition we have

$$\frac{\partial F}{\partial t} = \left(1 - \frac{1}{n+\varepsilon}\sum_{k=1}^{n}\lambda_k^m\right)^{-2} \cdot \frac{m\lambda_i^{m-1}}{n+\varepsilon}\frac{\partial}{\partial t}g_{ii}$$

$$= \left(1 - \frac{1}{n+\varepsilon}\sum_{k=1}^{n}\lambda_k^m\right)^{-2} \cdot \frac{m\lambda_i^{m-1}}{n+\varepsilon} \cdot g^{\alpha\beta}\widetilde{\nabla}_\alpha\widetilde{\nabla}_\beta g_{ii}$$

$$+ \left(1 - \frac{1}{n+\varepsilon}\sum_{k=1}^{n}\lambda_k^m\right)^{-2} \cdot \frac{m\lambda_i^{m-1}}{n+\varepsilon} \tag{82}$$

$$\cdot \left[-\frac{2}{\lambda_i\lambda_\alpha}\widetilde{R}_{i\alpha i\alpha} + \frac{1}{2\lambda_k\lambda_q}(\widetilde{\nabla}_i g_{qk} \cdot \widetilde{\nabla}_i g_{qk} + 2\widetilde{\nabla}_k g_{qi} \cdot \widetilde{\nabla}_q g_{ik} - 2\widetilde{\nabla}_k g_{qi} \cdot \widetilde{\nabla}_k g_{qi} - 4\widetilde{\nabla}_i g_{qk} \cdot \widetilde{\nabla}_k g_{qi})\right].$$

For any $\delta > 0$, from Lemma 2.3 we know that there exists a constant $T(\delta, n, k_0) > 0$ depending only on δ, n, and k_0 such that

$$g_{ij}(x,t) \geq (1-\delta)\tilde{g}_{ij}(x), \qquad 0 \leq t \leq T(\delta, n, k_0). \tag{83}$$

Thus

$$\lambda_k \geq 1-\delta, \qquad k = 1, 2, \cdots, n,\ 0 \leq t \leq T(\delta, n, k_0). \tag{84}$$

Furthermore if $0 \leq t \leq T(\delta, n, k_0)$ and $F(x,t) < +\infty$, then

$$\lambda_k \geq 1-\delta, \qquad 1 - \frac{1}{n+\varepsilon}\sum_{k=1}^{n}\lambda_k^m > 0. \tag{85}$$

Using (85) and the fact $|\widetilde{R}_{i\alpha i\alpha}| \leq \sqrt{k_0}$ we get

$$\frac{m\lambda_i^{m-1}}{n+\varepsilon}\left[-\frac{2}{\lambda_i\lambda_\alpha}\widetilde{R}_{i\alpha i\alpha} + \frac{1}{2\lambda_k\lambda_q}(\widetilde{\nabla}_i g_{qk} \cdot \widetilde{\nabla}_i q_{qk} + 2\widetilde{\nabla}_k g_{qi} \cdot \widetilde{\nabla}_q g_{ik} - 2\widetilde{\nabla}_k g_{qi} \cdot \widetilde{\nabla}_k g_{qi} - 4\widetilde{\nabla}_i g_{qk} \cdot \widetilde{\nabla}_k g_{qi})\right] \tag{86}$$

$$\leq \frac{m}{(1-\delta)^3}\left(n^2\sqrt{k_0} + 4\widetilde{\nabla}_k g_{ij} \cdot \widetilde{\nabla}_k g_{ij}\right).$$

Substituting (86) into (82) yields

(87)

$$\frac{\partial F}{\partial t} \leq \left(1 - \frac{1}{n+\varepsilon}\sum_{k=1}^{n}\lambda_k^m\right)^{-2} \cdot \frac{m\lambda_i^{m-1}}{n+\varepsilon} \cdot g^{\alpha\beta}\widetilde{\nabla}_\alpha\widetilde{\nabla}_\beta g_{ii} + \frac{mF^2}{(1-\delta)^3}\left(n^2\sqrt{k_0} + 4\widetilde{\nabla}_k g_{ij} \cdot \widetilde{\nabla}_k g_{ij}\right), \qquad 0 \leq t \leq T(\delta, n, k_0).$$

On the other hand,

$$\begin{aligned} g^{\alpha\beta}\widetilde{\nabla}_\alpha\widetilde{\nabla}_\beta F = &\left(1-\frac{1}{n+\varepsilon}\sum_{k=1}^{n}\lambda_k^m\right)^{-2}\cdot\frac{m\lambda_i^{m-1}}{n+\varepsilon}\cdot g^{\alpha\beta}\widetilde{\nabla}_\alpha\widetilde{\nabla}_\beta g_{ii} \\ &+\left(1-\frac{1}{n+\varepsilon}\sum_{k=1}^{n}\lambda_k^m\right)^{-3}\cdot\frac{2m^2\lambda_i^{m-1}\lambda_j^{m-1}}{(n+\varepsilon)^2}\cdot g^{\alpha\beta}\widetilde{\nabla}_\alpha g_{ii}\cdot\widetilde{\nabla}_\beta g_{jj} \\ &+\left(1-\frac{1}{n+\varepsilon}\sum_{k=1}^{n}\lambda_k^m\right)^{-2}\cdot g^{\alpha\beta}\widetilde{\nabla}_\alpha g_{ij}\cdot\widetilde{\nabla}_\beta g_{ij} \\ &\cdot\frac{m}{n+\varepsilon}(\lambda_i^{m-2}+\lambda_i^{m-3}\lambda_j+\cdots+\lambda_j^{m-2}). \end{aligned}$$

Since

$$\left(1-\frac{1}{n+\varepsilon}\sum_{k=1}^{n}\lambda_k^m\right)^{-3}\cdot\frac{2m^2\lambda_i^{m-1}\lambda_j^{m-1}}{(n+\varepsilon)^2}\cdot g^{\alpha\beta}\widetilde{\nabla}_\alpha g_{ii}\cdot\widetilde{\nabla}_\beta g_{jj}\geq 0,$$

one has

$$\begin{aligned} g^{\alpha\beta}\widetilde{\nabla}_\alpha\widetilde{\nabla}_\beta F \geq &\left(1-\frac{1}{n+\varepsilon}\sum_{k=1}^{n}\lambda_k^m\right)^{-2}\cdot\frac{m\lambda_i^{m-1}}{n+\varepsilon}\cdot g^{\alpha\beta}\widetilde{\nabla}_\alpha\widetilde{\nabla}_\beta g_{ii} \\ &+\left(1-\frac{1}{n+\varepsilon}\sum_{k=1}^{n}\lambda_k^m\right)^{-2}\cdot g^{\alpha\beta}\widetilde{\nabla}_\alpha g_{ij}\cdot\widetilde{\nabla}_\beta g_{ij} \\ &\cdot\frac{m}{n+\varepsilon}[\lambda_i^{m-2}+\lambda_i^{m-3}\lambda_j+\cdots+\lambda_j^{m-2}]. \end{aligned}$$

Substituting this into (87) gives

$$\begin{aligned} \frac{\partial F}{\partial t}\leq\ & g^{\alpha\beta}\widetilde{\nabla}_\alpha\widetilde{\nabla}_\beta F+\frac{mF^2}{(1-\delta)^3}(n^2\sqrt{k_0}+4\widetilde{\nabla}_k g_{ij}\cdot\widetilde{\nabla}_k g_{ij}) \\ &-\frac{mF^2}{n+\varepsilon}(\lambda_i^{m-2}+\lambda_i^{m-3}\lambda_j+\cdots+\lambda_j^{m-2}) \\ &\cdot\frac{1}{\lambda_k}\widetilde{\nabla}_k g_{ij}\cdot\widetilde{\nabla}_k g_{ij},\qquad 0\leq t\leq T(\delta,n,k_0). \end{aligned}\tag{88}$$

From (85) we have $1-\delta\leq\lambda_k\leq(n+\varepsilon)^{1/m}$; thus from (88) we get

$$\begin{aligned} \frac{\partial F}{\partial t}\leq\ & g^{\alpha\beta}\widetilde{\nabla}_\alpha\widetilde{\nabla}_\beta F+\frac{mF^2}{(1-\delta)^3}(n^2\sqrt{k_0}+4\widetilde{\nabla}_k g_{ij}\cdot\widetilde{\nabla}_k g_{ij}) \\ &-m(m-1)\left(\frac{1}{n+\varepsilon}\right)^{1+1/m}(1-\delta)^{m-2}\widetilde{\nabla}_k g_{ij}\cdot\widetilde{\nabla}_k g_{ij}\cdot F^2, \\ &\qquad 0\leq t\leq T(\delta,n,k_0). \end{aligned}\tag{89}$$

Lemma 2.4. *Suppose $g_{ij}(x,t) > 0$ is a solution of* (56). *Then for a* *$\theta > 0$ there exists a constant $c(\theta, n, k_0) > 0$ depending only on θ, n, and* *such that*

$$g_{ij}(x,t) \le (1+\theta)\tilde{g}_{ij}(x), \qquad x \in D,\ 0 \le t \le c(\theta, n, k_0).$$

Proof. In (79) we let $\varepsilon = n$, and m be an integer such that

$$20n^2 + \frac{\log 2n}{\log(1+\theta)} \le m < \frac{\log 2n}{\log(1+\theta)} + 20n^2 + 1. \tag{90}$$

Then

$$(2n)^{1/m} \le 1 + \theta, \tag{91}$$

$$(m-1)\left(\frac{1}{2n}\right)^2 \ge \frac{9}{2}; \tag{92}$$

thus we can find a constant $\delta > 0$ depending only on m such that

$$(m-1)\left(\frac{1}{2n}\right)^2 \ge \frac{4}{(1-\delta)^{m+1}} \tag{93}$$

so that

$$(m-1)\left(\frac{1}{2n}\right)^{1+1/m} \ge \frac{4}{(1-\delta)^{m+1}},$$

$$m(m-1)\left(\frac{1}{2n}\right)^{1+1/m}(1-\delta)^{m-2} \ge \frac{4m}{(1-\delta)^3}. \tag{94}$$

From (89) we have

$$\begin{aligned}\frac{\partial F}{\partial t} \le g^{\alpha\beta}\tilde{\nabla}_\alpha\tilde{\nabla}_\beta F &+ \frac{mn^2\sqrt{k_0}}{(1-\delta)^3}F^2 \\ &+ \left(\frac{4m}{(1-\delta)^3} - m(m-1)\left(\frac{1}{2n}\right)^{1+1/m}(1-\delta)^{m-2}\right)F^2 \\ &\cdot\tilde{\nabla}_k g_{ij}\cdot\tilde{\nabla}_k g_{ij}, \qquad 0 \le t \le T(\delta, n, k_0),\end{aligned}$$

which can be reduced to, in consequence of (94),

$$\frac{\partial F}{\partial t} \le g^{\alpha\beta}\tilde{\nabla}_\alpha\tilde{\nabla}_\beta F + \frac{mn^2\sqrt{k_0}}{(1-\delta)^3}F^2, \qquad 0 \le t \le T(\delta, n, k_0). \tag{95}$$

Since from (80) and (81) we have

$$F(x,0) \equiv 2, \quad x \in D, \qquad F(x,t) \equiv 2, \quad x \in \partial D, \tag{96}$$

using the maximal principle, from (95) and (96) it follows that

$$F(x,t) \le 2\left[1 - \frac{mn^2\sqrt{k_0}}{(1-\delta)^3}t\right]^{-1}, \qquad 0 \le t \le T(\delta, n, k_0). \tag{97}$$

If we let

$$0 \le t \le \min\left[T(\delta, n, k_0), \frac{(1-\delta)^3}{2mn^2\sqrt{k_0}}\right], \tag{98}$$

then we know that $F(x,t) \le 4$ by (97), and that

$$\sum_{k=1}^{n} \lambda_k^m \le 2n, \tag{99}$$

by (79). From (91) and (99) it follows that

$$\lambda_k \le (2n)^{1/m} \le 1 + \theta, \qquad k = 1, 2, \cdots, n. \tag{100}$$

Thus if we define

$$c(\theta, n, k_0) = \min\left(T(\delta, n, k_0), \frac{(1-\delta)^3}{2mn^2\sqrt{k_0}}\right),$$

then

$$g_{ij}(x,t) \le (1+\theta)\tilde{g}_{ij}(x), \qquad x \in D,\ 0 \le t \le c(\theta, n, k_0).$$

A combination of Lemmas 2.3 and 2.4 gives readily

Theorem 2.5. *Suppose $g_{ij}(x,t) > 0$ is a solution of (56). Then for any $\delta > 0$ there exists a constant $T(\delta, n, k_0) > 0$ depending only on $\delta, n,$ and k_0 such that*

$$(1-\delta)\tilde{g}_{ij}(x) \le g_{ij}(x,t) \le (1+\delta)\tilde{g}_{ij}(x), \qquad x \in D,\ 0 \le t \le T(\delta, n, k_0).$$

3. Solving the Dirichlet boundary problem

As in the previous section, we assume that $D \subseteq M$ is a domain with boundary ∂D a compact C^∞, $(n-1)$-dimensional submanifold (not necessarily connected) of M, and its closure $\overline{D}$ is a compact subset of M.

In this section we want to prove the existence theorem for the solution of the following Dirichlet boundary problem:

$$\begin{aligned} &\frac{\partial}{\partial t} g_{ij}(x,t) = -2R_{ij}(x,t) + \nabla_i V_j + \nabla_j V_i, \qquad x \in D, \\ &g_{ij}(x,0) = \tilde{g}_{ij}(x), \qquad x \in D, \\ &g_{ij}(x,t) \equiv \tilde{g}_{ij}(x), \qquad x \in \partial D,\ 0 \le t \le T. \end{aligned} \tag{1}$$

If we define a new tensor

$$h_{ij}(x,t) = g_{ij}(x,t) - \tilde{g}_{ij}(x), \tag{2}$$

then from (1) and Lemma 2.1 we get

$$\begin{aligned} &\frac{\partial}{\partial t}h_{ij} = g^{\alpha\beta}\widetilde{\nabla}_\alpha\widetilde{\nabla}_\beta h_{ij} + A_{ij}, \qquad x \in D,\\ &h_{ij}(x,0) \equiv 0, \qquad x \in D,\\ &h_{ij}(x,t) \equiv 0, \qquad x \in \partial D,\ 0 \le t \le T, \end{aligned} \tag{3}$$

where we use the property $\widetilde{\nabla}\tilde{g}_{ij}(x) \equiv 0$ and

$$\begin{aligned} A_{ij} = &- g^{\alpha\beta}g_{ip}\tilde{g}^{pq}\widetilde{R}_{j\alpha q\beta} - g^{\alpha\beta}g_{jp}\tilde{g}^{pq}\widetilde{R}_{i\alpha q\beta}\\ &+ \tfrac{1}{2}g^{\alpha\beta}g^{pq}(\widetilde{\nabla}_i h_{p\alpha}\cdot\widetilde{\nabla}_j h_{q\beta} + 2\widetilde{\nabla}_\alpha h_{jp}\cdot\widetilde{\nabla}_q h_{i\beta} - 2\widetilde{\nabla}_\alpha h_{jp}\cdot\widetilde{\nabla}_\beta h_{iq}\\ &- 2\widetilde{\nabla}_j h_{p\alpha}\cdot\widetilde{\nabla}_\beta h_{iq} - 2\widetilde{\nabla}_i h_{p\alpha}\cdot\widetilde{\nabla}_\beta h_{jq}). \end{aligned} \tag{4}$$

If $\frac{1}{2}\tilde{g}_{ij}(x) \le g_{ij}(x,t) \le 2\tilde{g}_{ij}(x)$, from (4) it follows that

$$\begin{aligned} (-8n\sqrt{k_0} - 20|\widetilde{\nabla}_\alpha h_{\beta\gamma}|^2)\tilde{g}_{ij}(x) &\le A_{ij}(x,t)\\ &\le (8n\sqrt{k_0} + 20|\widetilde{\nabla}_\alpha h_{\beta\gamma}|^2)\tilde{g}_{ij}(x). \end{aligned} \tag{5}$$

Lemma 3.1. *There exists a constant $\delta = \delta(n, k_0) > 0$ depending only on n and k_0 such that if $g_{ij}(x,t) > 0$ is a smooth solution of* (1) *and*

$$(1-\delta)\tilde{g}_{ij}(x) \le g_{ij}(x,t) \le (1+\delta)\tilde{g}_{ij}(x) \tag{6}$$

holds for all $(x,t) \in \overline{D} \times [0,T]$, then for any integer $m \ge 0$, there exist constants $C_m(n,\theta_0,k_0,T,\tilde{g}_{ij},D) > 0$ depending only on $n,\theta_0,k_0,T,\tilde{g}_{ij}$, and D such that

$$|\widetilde{\nabla}^m g_{ij}(x,t)|^2 \le c_m(n,\theta_0,k_0,T,\tilde{g}_{ij},D) \tag{7}$$

for all $(x,t) \in \overline{D} \times [0,T]$, where $\theta_0 = \inf_{x\in\overline{D}} \mathrm{inj}(x) > 0$ is the lower bound of $\mathrm{inj}(x)$ on $\overline{D}$.

Proof. From (2) and (6) it follows that

$$-\delta\tilde{g}_{ij}(x) \le h_{ij}(x,t) \le \delta\tilde{g}_{ij}(x). \tag{8}$$

If we let

$$H_{ij}(x,t) = \tfrac{1}{\delta}h_{ij}(x,t), \tag{9}$$

then by (3) we have

$$\begin{aligned} &\frac{\partial}{\partial t}H_{ij} = g^{\alpha\beta}\widetilde{\nabla}_\alpha\widetilde{\nabla}_\beta H_{ij} + B_{ij}, \qquad x \in D,\ 0 \le t \le T,\\ &H_{ij}(x,0) \equiv 0, \qquad x \in D,\\ &H_{ij}(x,t) \equiv 0, \qquad x \in \partial D,\ 0 \le t \le T, \end{aligned} \tag{10}$$

where $B_{ij} = \frac{1}{\delta} A_{ij}$. From (5), (8), (6) we get, respectively,

$$\begin{aligned}(11)\qquad &-\left(\frac{8n\sqrt{k_0}}{\delta} + 20\delta|\tilde{\nabla}_\alpha H_{\beta\gamma}|^2\right)\tilde{g}_{ij}(x) \le B_{ij}(x,t)\\ &\le \left(\frac{8n\sqrt{k_0}}{\delta} + 20\delta|\tilde{\nabla}_\alpha H_{\beta\gamma}|^2\right)\tilde{g}_{ij}(x), \qquad (x,t)\in D\times[0,T],\end{aligned}$$

$$(12)\qquad -\tilde{g}_{ij}(x) \le H_{ij}(x,t) \le \tilde{g}_{ij}(x), \qquad (x,t)\in \overline{D}\times[0,T],$$

$$(13)\qquad \frac{1}{1+\delta}\tilde{g}^{\alpha\beta}(x) \le g^{\alpha\beta}(x,t) \le \frac{1}{1-\delta}\tilde{g}^{\alpha\beta}(x).$$

Furthermore, we still have

$$\begin{aligned}\tilde{\nabla}_i g^{\alpha\beta} &= -g^{\alpha k} g^{\beta l}\tilde{\nabla}_i g_{kl}\\ &= -g^{\alpha k} g^{\beta l}\tilde{\nabla}_i h_{kl} = -g^{\alpha k} g^{\beta l}\cdot\delta\tilde{\nabla}_i H_{kl}.\end{aligned}$$

Thus

$$(14)\qquad |\tilde{\nabla}_i g^{\alpha\beta}|^2 \le \frac{\delta^2}{(1-\delta)^4}|\tilde{\nabla}_i H_{kl}|^2 \quad \text{on } D\times[0,T].$$

Using (10)–(14) and exactly the same arguments as in the proof of Theorem 6.1 in [4, §6, Chapter VII], we know that if $\delta > 0$ is small enough compared to n and k_0, then we can find a constant $\tilde{c}_1(n, \theta_0, k_0, \tilde{g}_{ij}, D)$, $0 < \tilde{c}_1 < +\infty$, such that

$$(15)\qquad \max_{(x,t)\in\overline{D}\times[0,T]} |\tilde{\nabla} H_{ij}(x,t)|^2 \le \tilde{c}_1(n, \theta_0, k_0, \tilde{g}_{ij}, D).$$

Since

$$\tilde{\nabla}_k g_{ij} = \tilde{\nabla}_k h_{ij} = \delta\tilde{\nabla}_k H_{ij},$$

we get

$$(16)\qquad \max_{(x,t)\in\overline{D}\times[0,T]} |\tilde{\nabla} g_{ij}(x,t)|^2 \le c_1(n, \theta_0, k_0, \tilde{g}_{ij}, D).$$

Using (10)–(15) and the same arguments as in [4, Chapter IV, §§5–9], we know that for any integer $m \ge 2$ we have

$$(17)\qquad \max_{(x,t)\in\overline{D}\times[0,T]} |\tilde{\nabla}^m H_{ij}(x,t)|^2 \le \tilde{c}_m(n, \theta_0, k_0, T, \tilde{g}_{ij}, D).$$

But $\tilde{\nabla}^m g_{ij} = \delta\tilde{\nabla}^m H_{ij}$, so we get

$$(18)\qquad \max_{(x,t)\in\overline{D}\times[0,T]} |\tilde{\nabla}^m g_{ij}(x,t)|^2 \le c_m(n, \theta_0, k_0, T, \tilde{g}_{ij}, D),$$

which completes the proof of the lemma.

As soon as we established the prior estimates in (7), using Theorem 2.5 and the same arguments as in the proof of Theorem 7.1 in [4, §7, Chapter VII], we have the following existence theorem.

Theorem 3.2. *There exists a constant $T(n, k_0) > 0$ depending only on n and k_0 such that the Dirichlet boundary problem* (1) *has a unique smooth solution $g_{ij}(x,t) > 0$ on $0 \le t \le T(n, k_0)$.*

4. Local estimates and convergence

In the last section we get a solution $g_{ij}(x,t) > 0$ on the domain $D \subseteq M$ by solving the Dirichlet boundary problem. To get a solution $g_{ij}(x,t) > 0$ on the whole manifold M by letting ∂D go to infinity on M we need to estimate $g_{ij}(x,t)$ locally; that means to control the derivatives of $g_{ij}(x,t)$ only in terms of $\tilde{g}_{ij}(x)$ and independent of D.

Fix a point $x_0 \in M$ and let $B(x_0, \gamma)$ be the geodesic ball of radius γ centered at x_0 with respect to the metric $\tilde{g}_{ij}$. Then we have the following lemma.

Lemma 4.1. *Suppose $0 < \gamma, \delta, T < +\infty$ are some constants, and $g_{ij}(x,t) > 0$ is a solution of the following equation*:

$$(1)\quad \begin{aligned} &\frac{\partial}{\partial t} g_{ij}(x,t) = -2R_{ij}(x,t) + \nabla_i V_j + \nabla_j V_i \\ &\qquad\qquad\qquad \textit{for } (x,t) \in B(x_0, \gamma+\delta) \times [0,T], \\ &g_{ij}(x,0) = \tilde{g}_{ij}(x), \qquad x \in B(x_0, \gamma+\delta). \end{aligned}$$

We also assume that on $B(x_0, \gamma+\delta) \times [0,T]$ we have

$$(2)\quad \left(1 - \frac{1}{256000n^{10}}\right) \tilde{g}_{ij}(x) \le g_{ij}(x,t) \le \left(1 + \frac{1}{256000n^{10}}\right) \tilde{g}_{ij}(x).$$

Then there exists $c(n, \gamma, \delta, T, \tilde{g}_{ij}) > 0$ depending only on n, γ, δ, T, and $\tilde{g}_{ij}$ such that

$$(3)\quad |\tilde{\nabla} g_{ij}(x,t)|^2 \le c(n, \gamma, \delta, T, \tilde{g}_{ij})$$

for all $(x,t) \in B(x_0, \gamma + \delta/2) \times [0,T]$.

Proof. Differentiating the equation in Lemma 2.1, we get

$$(4)\quad \begin{aligned} \frac{\partial}{\partial t}\tilde{\nabla} g_{ij} = {} & g^{\alpha\beta}\tilde{\nabla}_\alpha \tilde{\nabla}_\beta(\tilde{\nabla} g_{ij}) + \widetilde{\mathrm{Rm}} * g^{-1} * \tilde{\nabla} g + g^{-1} * g * \tilde{\nabla}\widetilde{\mathrm{Rm}} \\ & + \widetilde{\mathrm{Rm}} * g^{-1} * g^{-1} * g * \tilde{\nabla} g + g^{-1} * g^{-1} * \tilde{\nabla} g * \tilde{\nabla}\tilde{\nabla} g \\ & + g^{-1} * g^{-1} * g^{-1} * \tilde{\nabla} g * \tilde{\nabla} g * \tilde{\nabla} g, \end{aligned}$$

where we have used g, g^{-1}, $\tilde{\nabla}^k g$ and $*$ to denote respectively the tensor g_{ij}, the tensor g^{ij}, the tensor $\tilde{\nabla}^k g_{ij}$, and the tensor product. From (4) it follows that

(5)
$$\begin{aligned}\frac{\partial}{\partial t}|\tilde{\nabla} g_{ij}|^2 = {} & g^{\alpha\beta}\tilde{\nabla}_\alpha\tilde{\nabla}_\beta|\tilde{\nabla} g_{ij}|^2 - 2g^{\alpha\beta}\tilde{\nabla}_\alpha(\tilde{\nabla} g_{ij})\cdot\tilde{\nabla}_\beta(\tilde{\nabla} g_{ij}) \\ & + \widetilde{\mathrm{Rm}} * g^{-1} * g^{-1} * g * \tilde{\nabla} g * \tilde{\nabla} g + g^{-1} * g * \tilde{\nabla}\widetilde{\mathrm{Rm}} * \tilde{\nabla} g \\ & + g^{-1} * g^{-1} * \tilde{\nabla} g * \tilde{\nabla} g * \tilde{\nabla}\tilde{\nabla} g \\ & + g^{-1} * g^{-1} * g^{-1} * \tilde{\nabla} g * \tilde{\nabla} g * \tilde{\nabla} g * \tilde{\nabla} g.\end{aligned}$$

Since the closure $\overline{B(x_0, \gamma+\delta)}$ is compact, there exists a constant $c_0(\tilde{g}_{ij}) > 0$ such that

$$|\tilde{\nabla}\widetilde{\mathrm{Rm}}| \le c_0(\tilde{g}_{ij}) \quad \text{on } \overline{B(x_0, \gamma+\delta)}. \tag{6}$$

From (2) we have

$$\tfrac{1}{2}\tilde{g}_{ij}(x) \le g_{ij}(x,t) \le 2\tilde{g}_{ij}(x) \quad \text{on } B(x_0, \gamma+\delta)\times[0,T]. \tag{7}$$

Thus

$$\begin{aligned}&\widetilde{\mathrm{Rm}} * g^{-1} * g^{-1} * g * \tilde{\nabla} g * \tilde{\nabla} g \le c_0|\tilde{\nabla} g|^2, \\ &g^{-1} * g * \tilde{\nabla}\widetilde{\mathrm{Rm}} * \tilde{\nabla} g \le c_0|\tilde{\nabla} g|,\end{aligned} \tag{8}$$

where the constant $c_0 > 0$ depends only on n and $\tilde{g}_{ij}$, and is not necessarily the same as the constant in (6).

Estimating the last two terms in (5) yields

$$\begin{aligned}&g^{-1} * g^{-1} * \tilde{\nabla} g * \tilde{\nabla} g * \tilde{\nabla}\tilde{\nabla} g \le 72n^5|\tilde{\nabla} g|^2\cdot|\tilde{\nabla}\tilde{\nabla} g|, \\ &g^{-1} * g^{-1} * g^{-1} * \tilde{\nabla} g * \tilde{\nabla} g * \tilde{\nabla} g * \tilde{\nabla} g \le 160n^6|\tilde{\nabla} g|^4,\end{aligned} \tag{9}$$

where we have to use (7) and check carefully the number of terms in the equation of Lemma 2.1.

From (7) we also have

$$g^{\alpha\beta}\tilde{\nabla}_\alpha(\tilde{\nabla} g_{ij})\cdot\tilde{\nabla}_\beta(\tilde{\nabla} g_{ij}) \ge \tfrac{1}{2}|\tilde{\nabla}\tilde{\nabla} g|^2. \tag{10}$$

Substituting (8), (9), and (10) into (5) gives

$$\begin{aligned}\frac{\partial}{\partial t}|\tilde{\nabla} g|^2 \le {} & g^{\alpha\beta}\tilde{\nabla}_\alpha\tilde{\nabla}_\beta|\tilde{\nabla} g|^2 - |\tilde{\nabla}^2 g|^2 + c_0|\tilde{\nabla} g|^2 \\ & + c_0|\tilde{\nabla} g| + 72n^5|\tilde{\nabla} g|^2\,|\tilde{\nabla}^2 g| + 160n^6|\tilde{\nabla} g|^4,\end{aligned} \tag{11}$$

where the norm $|\tilde{\nabla} g|$ is with respect to the metric $\tilde{g}_{ij}$.

It is easy to see that

$$72n^5|\tilde{\nabla} g|^2\cdot|\tilde{\nabla}^2 g| + 160n^6|\tilde{\nabla} g|^4 \le \tfrac{1}{2}|\tilde{\nabla}^2 g|^2 + 3200n^{10}|\tilde{\nabla} g|^4;$$

thus from (11) we get

$$(12)\qquad \frac{\partial}{\partial t}|\widetilde{\nabla} g|^2 \le g^{\alpha\beta}\widetilde{\nabla}_\alpha\widetilde{\nabla}_\beta|\widetilde{\nabla} g|^2 - \tfrac{1}{2}|\widetilde{\nabla}^2 g|^2 + 3200n^{10}|\widetilde{\nabla} g|^4 + c_0|\widetilde{\nabla} g|^2 + c_0.$$

If we let $\varepsilon = 1/256000n^{10}$ and use the notation in (63) of §2, then from (2) and (7) we have

$$(13)\qquad 1-\varepsilon \le \lambda_k \le 1+\varepsilon, \quad \tfrac{1}{2} \le \lambda_k \le 2, \qquad k = 1, 2, \cdots, n.$$

Now let

$$(14)\qquad m = 25600n^{10}, \qquad a = 6400n^{10},$$

and define a function:

$$(15)\qquad \varphi(x,t) = a + \sum_{k=1}^{n} \lambda_k^m \quad \forall (x,t) \in B(x_0, \gamma+\delta) \times [0, T].$$

Then by definition and Lemma 2.1 we have

$$(16)\qquad \begin{aligned} \frac{\partial \varphi}{\partial t} &= m\lambda_k^{m-1}\frac{\partial}{\partial t} g_{kk} \\ &= m\lambda_k^{m-1} g^{\alpha\beta}\widetilde{\nabla}_\alpha\widetilde{\nabla}_\beta g_{kk} \\ &\quad + m\lambda_k^{m-1} * (\widetilde{\mathrm{R}}\mathrm{m} * g^{-1} * g + g^{-1} * g^{-1} * \widetilde{\nabla} g * \widetilde{\nabla} g). \end{aligned}$$

Using the same reasoning as before and (13) and (14) we get

$$(17)\qquad \begin{aligned} &m\lambda_k^{m-1} * \widetilde{R}\mathrm{m} * g^{-1} * g \le c_0, \\ &m\lambda_k^{m-1} * g^{-1} * g^{-1} * \widetilde{\nabla} g * \widetilde{\nabla} g \le 10n^3 m(1+\varepsilon)^{m-1}|\widetilde{\nabla} g|^2. \end{aligned}$$

Substituting (17) into (16) yields

$$(18)\qquad \frac{\partial \varphi}{\partial t} \le m\lambda_k^{m-1} g^{\alpha\beta}\widetilde{\nabla}_\alpha\widetilde{\nabla}_\beta g_{kk} + c_0 + 10n^3 m(1+\varepsilon)^{m-1}|\widetilde{\nabla} g|^2.$$

On the other hand, we have

$$\begin{aligned} g^{\alpha\beta}\widetilde{\nabla}_\alpha\widetilde{\nabla}_\beta \varphi &= g^{\alpha\beta}\widetilde{\nabla}_\alpha\widetilde{\nabla}_\beta\left(\sum_{k=1}^{n}\lambda_k^m\right) \\ &= m\lambda_k^{m-1} g^{\alpha\beta}\widetilde{\nabla}_\alpha\widetilde{\nabla}_\beta g_{kk} + m(\lambda_i^{m-2} + \lambda_i^{m-3}\lambda_j + \cdots + \lambda_j^{m-2}) \\ &\quad \cdot g^{\alpha\beta}\widetilde{\nabla}_\alpha g_{ij} \cdot \widetilde{\nabla}_\beta g_{ij}, \end{aligned}$$

which implies, in consequence of (13),

$$g^{\alpha\beta}\widetilde{\nabla}_\alpha\widetilde{\nabla}_\beta \varphi \ge m\lambda_k^{m-1} g^{\alpha\beta}\widetilde{\nabla}_\alpha\widetilde{\nabla}_\beta g_{kk} + \frac{m(m-1)}{2}(1-\varepsilon)^{m-2}|\widetilde{\nabla} g|^2.$$

Substituting this into (18) gives

$$\frac{\partial\varphi}{\partial t} \le g^{\alpha\beta}\tilde{\nabla}_\alpha\tilde{\nabla}_\beta\varphi - \frac{m(m-1)}{2}(1-\varepsilon)^{m-2}|\tilde{\nabla}g|^2 + c_0 + 10n^3m(1+\varepsilon)^{m-1}|\tilde{\nabla}g|^2. \tag{19}$$

By the definition of ε and m we have

$$10mn^3(1+\varepsilon)^{m-1} \le \frac{m^2}{16}, \tag{20}$$

$$(1-\varepsilon)^{m-2} \ge \frac{3}{4}; \tag{21}$$

thus

$$\frac{m(m-1)}{2}(1-\varepsilon)^{m-2} \ge \frac{m^2}{4}(1-\varepsilon)^{m-2} \ge \frac{3}{16}m^2. \tag{22}$$

Substituting (20) and (22) into (19) yields

$$\frac{\partial\varphi}{\partial t} \le g^{\alpha\beta}\tilde{\nabla}_\alpha\tilde{\nabla}_\beta\varphi + c_0 - \frac{m^2}{8}|\tilde{\nabla}g|^2. \tag{23}$$

From (12) and (23) it follows that

(24)

$$\frac{\partial}{\partial t}(\varphi\cdot|\tilde{\nabla}g|^2) \le g^{\alpha\beta}\tilde{\nabla}_\alpha\tilde{\nabla}_\beta(\varphi\cdot|\tilde{\nabla}g|^2) - 2g^{\alpha\beta}\tilde{\nabla}_\alpha\varphi\tilde{\nabla}_\beta|\tilde{\nabla}g|^2 - \frac{\varphi}{2}|\tilde{\nabla}^2g|^2 + 3200n^{10}\varphi|\tilde{\nabla}g|^4 + c_0\varphi|\tilde{\nabla}g|^2 + c_0\varphi + c_0|\tilde{\nabla}g|^2 - \frac{m^2}{8}|\tilde{\nabla}g|^4.$$

From (15) we have

$$a + n(1-\varepsilon)^m \le \varphi(x,t) \le a + n(1+\varepsilon)^m, \tag{25}$$

which together with (14) implies

$$\begin{gathered} 3200n^{10}\varphi \le 3200n^{10}[6400n^{10} + n(1+\varepsilon)^m] \le \frac{m^2}{16}, \\ 3200n^{10}\varphi|\tilde{\nabla}g|^4 \le \frac{m^2}{16}|\tilde{\nabla}g|^4. \end{gathered} \tag{26}$$

Using (24), (25), and (26) we find that

$$\begin{aligned} \frac{\partial}{\partial t}(|\tilde{\nabla}g|^2\varphi) \le{}& g^{\alpha\beta}\tilde{\nabla}_\alpha\tilde{\nabla}_\beta(|\tilde{\nabla}g|^2\varphi) - 2g^{\alpha\beta}\tilde{\nabla}_\alpha\varphi\cdot\tilde{\nabla}_\beta|\tilde{\nabla}g|^2 \\ &- \frac{\varphi}{2}|\tilde{\nabla}^2g|^2 - \frac{m^2}{16}|\tilde{\nabla}g|^4 + c_0|\tilde{\nabla}g|^2\varphi + c_0, \end{aligned} \tag{27}$$

but we also know that

$$(28)\quad \begin{aligned} -2g^{\alpha\beta}\tilde{\nabla}_\alpha\varphi\cdot\tilde{\nabla}_\beta|\tilde{\nabla}g|^2 &= -2g^{\alpha\beta}\tilde{\nabla}_\alpha\left(\sum_{k=1}^{n}\lambda_k^m\right)\cdot\tilde{\nabla}_\beta|\tilde{\nabla}g|^2 \\ &= -4g^{\alpha\beta}\cdot m\lambda_k^{m-1}\cdot\tilde{\nabla}_\alpha g_{kk}\cdot\tilde{\nabla}g\cdot\tilde{\nabla}\tilde{\nabla}g \\ &\le 8mn^5(1+\varepsilon)^{m-1}|\tilde{\nabla}g|^2\cdot|\tilde{\nabla}\tilde{\nabla}g| \\ &\le 16mn^5|\tilde{\nabla}g|^2\cdot|\tilde{\nabla}\tilde{\nabla}g| \\ &\le \frac{\varphi}{2}|\tilde{\nabla}^2g|^2+\frac{200m^2n^{10}}{\varphi}|\tilde{\nabla}g|^4. \end{aligned}$$

Combining (27) and (28) gives

$$(29)\quad \begin{aligned} \frac{\partial}{\partial t}(|\tilde{\nabla}g|^2\varphi) \le g^{\alpha\beta}\tilde{\nabla}_\alpha\tilde{\nabla}_\beta(|\tilde{\nabla}g|^2\varphi)+\frac{200m^2n^{10}}{\varphi}|\tilde{\nabla}g|^4 \\ -\frac{m^2}{16}|\tilde{\nabla}g|^4+c_0|\tilde{\nabla}g|^2\varphi+c_0. \end{aligned}$$

Since $\varphi(x,t)\ge a=6400n^{10}$, we have

$$\frac{200m^2n^{10}}{\varphi}\le\frac{m^2}{32},$$

and therefore, in consequence of (29),

$$(30)\quad \begin{aligned} \frac{\partial}{\partial t}(|\tilde{\nabla}g|^2\varphi) \le g^{\alpha\beta}\tilde{\nabla}_\alpha\tilde{\nabla}_\beta(|\tilde{\nabla}g|^2\varphi)-\frac{m^2}{32}|\tilde{\nabla}g|^4 \\ +c_0|\tilde{\nabla}g|^2\varphi+c_0. \end{aligned}$$

Since

$$\frac{m^2}{32}|\tilde{\nabla}g|^4=\frac{m^2}{32\varphi^2}|\tilde{\nabla}g|^4\varphi^2\ge\frac{m^2}{32[a+n(1+\varepsilon)^m]^2}|\tilde{\nabla}g|^4\varphi^2,$$

using (14) we get

$$\frac{m^2}{32}|\tilde{\nabla}g|^4\ge\frac{1}{8}|\tilde{\nabla}g|^4\varphi^2.$$

Thus

$$(31)\quad \begin{aligned} \frac{\partial}{\partial t}(|\tilde{\nabla}g|^2\varphi) \le g^{\alpha\beta}\tilde{\nabla}_\alpha\tilde{\nabla}_\beta(|\tilde{\nabla}g|^2\varphi)-\frac{1}{8}|\tilde{\nabla}g|^4\varphi^2 \\ +c_0|\tilde{\nabla}g|^2\varphi+c_0. \end{aligned}$$

If we define a function

$$(32)\quad \psi(x,t)=|\tilde{\nabla}g|^2\varphi(x,t),$$

then

$$(33)\quad \frac{\partial\psi}{\partial t}\le g^{\alpha\beta}\tilde{\nabla}_\alpha\tilde{\nabla}_\beta\psi-\frac{1}{16}\psi^2+c_0(n,\tilde{g}_{ij})$$

on $B(x_0, \gamma+\delta) \times [0, T]$. For any $x \in M$, we use $\gamma(x, x_0)$ to denote the distance between x_0 and x with respect to the metric $\tilde{g}_{ij}$. Then we have

$$|\tilde{\nabla}\gamma(x, x_0)| \leq 1, \quad \forall x \in M. \tag{34}$$

Since $|\tilde{R}_{ijkl}|^2 \leq k_0$ on M, using the Hession comparison theorem in Riemannian geometry we know that there exists a constant $c(\gamma, \delta, k_0) > 0$ depending only on γ, δ, and k_0 such that

$$\tilde{\nabla}_\alpha \tilde{\nabla}_\beta \gamma(x, x_0) \leq c(\gamma, \delta, k_0) \tilde{g}_{\alpha\beta}(x) \tag{35}$$

for all $x \in B(x_0, \gamma+\delta) \backslash B(x_0, \gamma)$.

Choose a cut-off function $\eta(x) \in C^\infty(\mathbf{R})$ such that

$$\begin{aligned} &\eta(x) \equiv 1, \quad x \leq 0, \\ &1 \geq \eta(x) \geq 0, \quad 0 \leq x \leq 1, \\ &\eta(x) \equiv 0, \quad x \geq 1 \end{aligned} \tag{36}$$

and that

$$\begin{aligned} &\eta'(x) \leq 0 \quad \forall x \in \mathbf{R}, \\ &|\eta''(x)| \leq 8 \quad \forall x \in \mathbf{R}, \\ &|\eta'(x)|^2/\eta(x) \leq 16, \quad x \leq 1; \end{aligned} \tag{37}$$

it is easy to see that such a function $\eta(x)$ exists. We define $\xi(x) \in c_0^\infty(M)$ as

$$\xi(x) = \eta\left(\frac{\gamma(x, x_0) - (\gamma + \delta/2)}{\delta/4}\right), \quad x \in M. \tag{38}$$

From (36), (37) and (38) we have

$$\begin{aligned} &\xi(x) \equiv 1, \quad x \in B(x_0, \gamma + \delta/2), \\ &\xi(x) \equiv 0, \quad x \in M \backslash B(x_0, \gamma + 3\delta/4), \\ &0 \leq \xi(x) \leq 1, \quad x \in M, \end{aligned} \tag{39}$$

$$|\tilde{\nabla}\xi(x)|^2 \leq \frac{16^2}{\delta^2} |\tilde{\nabla}\gamma(x, x_0)|^2 \cdot \xi(x), \quad x \in M,$$

which is reduced to, by use of (34),

$$|\tilde{\nabla}\xi(x)|^2 \leq \frac{16^2}{\delta^2} \xi(x), \quad x \in M. \tag{40}$$

On the other hand, we have

$$\begin{aligned} \tilde{\nabla}_\beta \xi(x) &= \frac{4}{\delta} \eta' \cdot \tilde{\nabla}_\beta \gamma(x, x_0), \\ \tilde{\nabla}_\alpha \tilde{\nabla}_\beta \xi(x) &= \frac{4}{\delta} \eta' \cdot \tilde{\nabla}_\alpha \tilde{\nabla}_\beta \gamma(x, x_0) + \frac{16}{\delta^2} \eta'' \cdot \tilde{\nabla}_\alpha \gamma(x, x_0) \cdot \tilde{\nabla}_\beta \gamma(x, x_0). \end{aligned} \tag{41}$$

From (36) and (37) it follows that

$$0 \geq \eta'(x) \geq -4\eta(x)^{1/2} \geq -4, \qquad x \in \mathbb{R}. \tag{42}$$

Thus we get

$$\frac{4}{\delta}\eta' \cdot \tilde{\nabla}_\alpha \tilde{\nabla}_\beta \gamma(x, x_0) \geq -\frac{16}{\delta} c(\gamma, \delta, k_0)\tilde{g}_{\alpha\beta}(x) \quad \forall x \in M \tag{43}$$

from (35) and

$$\frac{16}{\delta^2}\eta'' \cdot \tilde{\nabla}_\alpha \gamma(x, x_0) \cdot \tilde{\nabla}_\beta \gamma(x, x_0) \geq -\frac{128}{\delta^2}\tilde{g}_{\alpha\beta}(x), \qquad x \in M \tag{44}$$

from (34) and (37). Combining (41), (44), and (43) yields

$$\tilde{\nabla}_\alpha \tilde{\nabla}_\beta \xi(x) \geq -c_0(\gamma, \delta, k_0)\tilde{g}_{\alpha\beta}(x), \qquad x \in M. \tag{45}$$

Now consider the function

$$F(x,t) = \xi(x)\psi(x,t), \qquad (x,t) \in B(x_0, \gamma+\delta) \times [0,T]. \tag{46}$$

Then

$$F(x,t) = \xi(x)\varphi(x,t) \cdot |\tilde{\nabla} g|^2 \geq 0. \tag{47}$$

Since $|\tilde{\nabla} g|^2(x,0) \equiv 0$, it follows that

$$F(x,0) \equiv 0, \qquad x \in B(x_0, \gamma+\delta). \tag{48}$$

Using (39) we have

$$F(x,t) \equiv 0, \qquad (x,t) \in (M \backslash B(x_0, \gamma + 3\delta/4)) \times [0,T]. \tag{49}$$

From (47), (48), and (49) we know that there exists a point $(x_0, t_0) \in B(x_0, \gamma + 3\delta/4) \times [0,T]$ such that

$$F(x_0, t_0) = \max_{B(x_0, \gamma+\delta)\times[0,T]} F(x,t), \tag{50}$$

$$t_0 > 0, \tag{51}$$

which imply the following:

$$\begin{aligned} &\frac{\partial F}{\partial t}(x_0, t_0) \geq 0, \\ &\tilde{\nabla} F(x_0, t_0) = 0, \\ &g^{\alpha\beta}\tilde{\nabla}_\alpha \tilde{\nabla}_\beta F(x_0, t_0) \leq 0. \end{aligned} \tag{52}$$

Thus

$$\xi(x_0)\frac{\partial \psi}{\partial t}(x_0, t_0) \geq 0. \tag{53}$$

Since $\xi \geq 0$, from (33), (53), (52), (54) and (55) we get

$$\xi \cdot g^{\alpha\beta}\widetilde{\nabla}_\alpha\widetilde{\nabla}_\beta\psi - \tfrac{1}{16}\xi\psi^2 + c_0\xi \geq 0 \quad \text{at } (x_0, t_0), \tag{54}$$

$$\xi \cdot g^{\alpha\beta}\widetilde{\nabla}_\alpha\widetilde{\nabla}_\beta\psi + \psi \cdot g^{\alpha\beta}\widetilde{\nabla}_\alpha\widetilde{\nabla}_\beta\xi + 2g^{\alpha\beta}\widetilde{\nabla}_\alpha\xi \cdot \widetilde{\nabla}_\beta\psi \leq 0, \tag{55}$$

$$\tfrac{1}{16}\xi\psi^2 \leq c_0\xi - 2g^{\alpha\beta}\widetilde{\nabla}_\alpha\xi \cdot \widetilde{\nabla}_\beta\psi - \psi \cdot g^{\alpha\beta}\widetilde{\nabla}_\alpha\widetilde{\nabla}_\beta\xi. \tag{56}$$

But $\widetilde{\nabla}F(x_0, t_0) = 0$, so we have

$$\begin{gathered} \xi \cdot \widetilde{\nabla}_\alpha\psi + \psi \cdot \widetilde{\nabla}_\alpha\xi = 0, \\ -2g^{\alpha\beta}\widetilde{\nabla}_\alpha\xi \cdot \widetilde{\nabla}_\beta\psi = \frac{2\psi}{\xi}g^{\alpha\beta}\widetilde{\nabla}_\alpha\xi \cdot \widetilde{\nabla}_\beta\xi. \end{gathered} \tag{57}$$

From (56) and (57) it follows that

$$\frac{1}{16}\xi\psi^2 \leq c_0\xi + \frac{2\psi}{\xi}g^{\alpha\beta}\widetilde{\nabla}_\alpha\xi \cdot \widetilde{\nabla}_\beta\xi - \psi \cdot g^{\alpha\beta}\widetilde{\nabla}_\alpha\widetilde{\nabla}_\beta\xi \quad \text{at } (x_0, t_0). \tag{58}$$

Using (13), (40) and (45) we find

$$\frac{2\psi}{\xi}g^{\alpha\beta}\widetilde{\nabla}_\alpha\xi \cdot \widetilde{\nabla}_\beta\xi \leq 1024\psi, \tag{59}$$

$$-\psi \cdot g^{\alpha\beta}\widetilde{\nabla}_\alpha\widetilde{\nabla}_\beta\xi \leq 2\tilde{c}_0\psi. \tag{60}$$

But $0 \leq \xi \leq 1$, so from (58), (59), and (60) we get

$$\tfrac{1}{16}\xi\psi^2 \leq c_0 + 1024\psi + 2\tilde{c}_0\psi \quad \text{at } (x_0, t_0), \tag{61}$$

where c_0, $\tilde{c}_0 > 0$ depend only on n, γ, δ, T, and $\tilde{g}_{ij}(x)$. Since

$$\tfrac{1}{16}(\xi\psi)^2 \leq c_0\xi + (2\tilde{c}_0 + 1024)(\xi\psi) \quad \text{at } (x_0, t_0),$$

and $\xi \leq 1$, we have

$$\tfrac{1}{16}F(x_0, t_0)^2 \leq c_0 + (2\tilde{c}_0 + 1024)F(x_0, t_0). \tag{62}$$

Thus

$$F(x_0, t_0) \leq c(n, \gamma, \delta, T, \tilde{g}_{ij}), \tag{63}$$

which together with (50) implies

$$F(x, t) \leq c(n, \gamma, \delta, T, \tilde{g}_{ij}) \quad \text{on } B(x_0, \gamma + \delta) \times [0, T]. \tag{64}$$

Since $\xi(x) \equiv 1$ on $B(x_0, \gamma + \delta/2)$, from (46) and (64) we get

$$\begin{gathered} \psi(x, t) \leq c(n, \gamma, \delta, T, \tilde{g}_{ij}) \quad \text{on } B(x_0, \gamma + \delta/2) \times [0, T], \\ |\widetilde{\nabla}g|^2\varphi(x, t) \leq c(n, \gamma, \delta, T, \tilde{g}_{ij}) \quad \text{on } B(x_0, \gamma + \delta/2) \times [0, T]. \end{gathered} \tag{65}$$

From (25), $\varphi(x,t) \geq a = 6400n^{10}$; thus using (65) we have

$$|\tilde{\nabla} g_{ij}(x,t)|^2 \leq \frac{1}{6400n^{10}} c(n,\gamma,\delta,T,\tilde{g}_{ij}) \quad \text{on } B(x_0,\gamma+\delta/2)\times[0,T],$$

which completes the proof of Lemma 4.1.

The function $\xi(x)$ defined in (38) may not be smooth at some points of M, but as P. Li and S. T. Yau mentioned in their paper [5], this does not affect our using the maximal principle on $\xi(x)\psi(x,t)$.

Lemma 4.2. *Under the assumptions in Lemma 4.1, for any integer $m \geq 0$, there exist constants $c(n,m,\gamma,\delta,T,\tilde{g}_{ij}) > 0$ depending only on n,m,γ,δ,T, and $\tilde{g}_{ij}$ such that*

$$|\tilde{\nabla}^m g_{ij}(x,t)|^2 \leq c(n,m,\gamma,\delta,T,\tilde{g}_{ij}) \tag{66}$$

for all $(x,t) \in B(x_0,\gamma+\delta/(m+1))\times[0,T]$, where the norm in (66) *is with respect to the metric $\tilde{g}_{ij}(x)$.*

Proof. We prove this lemma by induction. If $m=0$, using (7) we have

$$|g_{ij}(x,t)|^2 \leq 4n \tag{67}$$

for all $(x,t) \in B(x_0,\gamma+\delta)\times[0,T]$. Therefore the lemma is true for the case $m=0$.

If $m=1$, from Lemma 4.1 we know that (66) is also true.

Suppose for $k = 0,1,2,\cdots,m-1$ we have

$$|\tilde{\nabla}^k g_{ij}(x,t)|^2 \leq c(k,n,\gamma,\delta,T,\tilde{g}_{ij}) \tag{68}$$

for all $(x,t) \in B(x_0,\gamma+\delta/(k+1))\times[0,T]$.

Now we consider the case $k = m$ and assume $m \geq 2$. First, differentiating the equation in Lemma 2.1 m times, we get

$$\begin{aligned}\frac{\partial}{\partial t}\tilde{\nabla}^m g_{ij} &= g^{\alpha\beta}\tilde{\nabla}_\alpha\tilde{\nabla}_\beta(\tilde{\nabla}^m g_{ij}) \\ &+ \sum_{\substack{0\leq k_1,k_2,\cdots,k_{m+2}\leq m+1\\ k_1+k_2+\cdots+k_{m+2}\leq m+2}} \tilde{\nabla}^{k_1} g * \tilde{\nabla}^{k_2} g * \cdots * \tilde{\nabla}^{k_{m+2}} g * P_{k_1k_2\cdots k_{m+2}},\end{aligned} \tag{69}$$

where

$$P_{k_1k_2\cdots k_{m+2}} = P_{k_1k_2\cdots k_{m+2}}(g, g^{-1}, \widetilde{\mathrm{Rm}}, \tilde{\nabla}\widetilde{\mathrm{Rm}}, \tilde{\nabla}^2\widetilde{\mathrm{Rm}}, \cdots, \tilde{\nabla}^m\widetilde{\mathrm{Rm}})$$

is a polynomial of $g, g^{-1}, \widetilde{\mathrm{Rm}}, \tilde{\nabla}\widetilde{\mathrm{Rm}}, \tilde{\nabla}^2\widetilde{\mathrm{Rm}}, \cdots, \tilde{\nabla}^m\widetilde{\mathrm{Rm}}$.

From (69) we get

$$\begin{aligned}\frac{\partial}{\partial t}|\tilde{\nabla}^m g_{ij}|^2 &= g^{\alpha\beta}\tilde{\nabla}_\alpha\tilde{\nabla}_\beta|\tilde{\nabla}^m g_{ij}|^2 - 2g^{\alpha\beta}\tilde{\nabla}_\alpha(\tilde{\nabla}^m g_{ij})\cdot\tilde{\nabla}_\beta(\tilde{\nabla}^m g_{ij}) \\ &+ \sum_{\substack{0\leq k_1,\cdots,k_{m+2}\leq m+1\\ k_1+\cdots+k_{m+2}\leq m+2}} \tilde{\nabla}^{k_1} g * \cdots * \tilde{\nabla}^{k_{m+2}} g * \tilde{\nabla}^m g * P_{k_1k_2\cdots k_{m+2}}.\end{aligned} \tag{70}$$

Since the closure $\overline{B(x_0, \gamma + \delta)}$ is compact, there exist constants $c(\ell, \tilde{g}_{ij}) > 0$ such that

$$|\widetilde{\nabla}^k \widetilde{\mathrm{Rm}}|^2 \leq c(\ell, \tilde{g}_{ij}) \quad \text{on } \overline{B(x_0, \gamma + \delta)} \tag{71}$$

for all integers $k = 0, 1, 2, \cdots$.

From (7) and (71) we get

$$|P_{k_1 k_2 \cdots k_{m+2}}| \leq c(m, n, \tilde{g}_{ij}) \quad \text{on } B(x_0, \gamma + \delta) \times [0, T], \tag{72}$$

$$g^{\alpha\beta} \widetilde{\nabla}_\alpha (\widetilde{\nabla}^m g_{ij}) \cdot \widetilde{\nabla}_\beta (\widetilde{\nabla}^m g_{ij}) \geq \tfrac{1}{2} |\widetilde{\nabla}^{m+1} g_{ij}|^2. \tag{73}$$

Substituting (68), (72), and (73) into (70) yields

$$\begin{aligned} \frac{\partial}{\partial t} |\widetilde{\nabla}^m g_{ij}|^2 \leq{} & g^{\alpha\beta} \widetilde{\nabla}_\alpha \widetilde{\nabla}_\beta |\widetilde{\nabla}^m g_{ij}|^2 - |\widetilde{\nabla}^{m+1} g_{ij}|^2 \\ & + c_0(m, n, \gamma, \delta, T, \tilde{g}_{ij}) [|\widetilde{\nabla}^m g| \cdot |\widetilde{\nabla}^{m+1} g| \cdot (1 + |\widetilde{\nabla} g|) \\ & + |\widetilde{\nabla}^m g|^2 \cdot (1 + |\widetilde{\nabla} g|^2 + |\widetilde{\nabla}^2 g|) + |\widetilde{\nabla}^m g|] \end{aligned} \tag{74}$$

on $B(x_0, \gamma + \delta/m) \times [0, T]$, where $c_0(m, n, \gamma, \delta, T, \tilde{g}_{ij}) > 0$ means some constant depending only on m, n, γ, δ, T, and $\tilde{g}_{ij}$.

Since $m \geq 2$, for $m = 2$ from (68) and (74) we get

$$\begin{aligned} \frac{\partial}{\partial t} |\widetilde{\nabla}^2 g|^2 \leq{} & g^{\alpha\beta} \widetilde{\nabla}_\alpha \widetilde{\nabla}_\beta |\widetilde{\nabla}^2 g|^2 - |\widetilde{\nabla}^3 g|^2 \\ & + c_0(|\widetilde{\nabla}^2 g| \cdot |\widetilde{\nabla}^3 g| + |\widetilde{\nabla}^2 g|^2 + |\widetilde{\nabla}^2 g|^3 + |\widetilde{\nabla}^2 g|), \\ \frac{\partial}{\partial t} |\widetilde{\nabla}^2 g|^2 \leq{} & g^{\alpha\beta} \widetilde{\nabla}_\alpha \widetilde{\nabla}_\beta |\widetilde{\nabla}^2 g|^2 - \tfrac{1}{2} |\widetilde{\nabla}^3 g|^2 + c_0 |\widetilde{\nabla}^2 g|^3 + c_0 \end{aligned} \tag{75}$$

on $B(x_0, \gamma + \delta/2) \times [0, T]$.

If $m \geq 3$, from (68) and (74) we get

$$\frac{\partial}{\partial t} |\widetilde{\nabla}^m g|^2 \leq g^{\alpha\beta} \widetilde{\nabla}_\alpha \widetilde{\nabla}_\beta |\widetilde{\nabla}^m g|^2 - \tfrac{1}{2} |\widetilde{\nabla}^{m+1} g|^2 + c_0 |\widetilde{\nabla}^m g|^2 + c_0 \tag{76}$$

on $B(x_0, \gamma + \delta/m) \times [0, T]$.

By combining (75) and (76), for $m \geq 2$ we always have

$$\frac{\partial}{\partial t} |\widetilde{\nabla}^m g|^2 \leq g^{\alpha\beta} \widetilde{\nabla}_\alpha \widetilde{\nabla}_\beta |\widetilde{\nabla}^m g|^2 - \tfrac{1}{2} |\widetilde{\nabla}^{m+1} g|^2 + c_0 |\widetilde{\nabla}^m g|^3 + c_0 \tag{77}$$

on $B(x_0, \gamma + \delta/m) \times [0, T]$. If we replace m by $m - 1$ in (74) and use (68), we get

$$\frac{\partial}{\partial t} |\widetilde{\nabla}^{m-1} g|^2 \leq g^{\alpha\beta} \widetilde{\nabla}_\alpha \widetilde{\nabla}_\beta |\widetilde{\nabla}^{m-1} g|^2 - |\widetilde{\nabla}^m g|^2 + c_0(|\widetilde{\nabla}^m g| + |\widetilde{\nabla}^2 g| + 1) \tag{78}$$

on $B(x_0, \gamma + \delta/m - 1) \times [0, T]$. Since $m \geq 2$, from (78) we get

$$\frac{\partial}{\partial t} |\widetilde{\nabla}^{m-1} g|^2 \leq g^{\alpha\beta} \widetilde{\nabla}_\alpha \widetilde{\nabla}_\beta |\widetilde{\nabla}^{m-1} g|^2 - \tfrac{1}{2} |\widetilde{\nabla}^m g|^2 + c_0 \tag{79}$$

on $B(x_0, \gamma + \delta/(m-1)) \times [0, T]$.

We define a function

$$\psi(x,t) = (a + |\widetilde{\nabla}^{m-1} g|^2) \cdot |\widetilde{\nabla}^m g|^2, \tag{80}$$

where $a > 0$ is a constant to be determined later. Then from (77) and (79) we have

$$\frac{\partial \psi}{\partial t} = [a + |\widetilde{\nabla}^{m-1} g|^2] \frac{\partial}{\partial t} |\widetilde{\nabla}^m g|^2 + |\widetilde{\nabla}^m g|^2 \frac{\partial}{\partial t} |\widetilde{\nabla}^{m-1} g|^2,$$

$$\begin{aligned} \frac{\partial \psi}{\partial t} \leq\ & g^{\alpha\beta} \widetilde{\nabla}_\alpha \widetilde{\nabla}_\beta \psi - 2 g^{\alpha\beta} \widetilde{\nabla}_\alpha |\widetilde{\nabla}^{m-1} g|^2 \cdot \widetilde{\nabla}_\beta |\widetilde{\nabla}^m g|^2 \\ & - \frac{1}{2}(a + |\widetilde{\nabla}^{m-1} g|^2)|\widetilde{\nabla}^{m+1} g|^2 + c_0 (a + |\widetilde{\nabla}^{m-1} g|^2)|\widetilde{\nabla}^m g|^3 \\ & + c_0 (a + |\widetilde{\nabla}^{m-1} g|^2) - \tfrac{1}{2}|\widetilde{\nabla}^m g|^4 + c_0 |\widetilde{\nabla}^m g|^2 \end{aligned} \tag{81}$$

on $B(x_0, \gamma + \delta/m) \times [0, T]$.

On the other hand from (68) it follows that

$$a \leq a + |\widetilde{\nabla}^{m-1} g|^2 \leq a + c(m-1, n, \gamma, \delta, T, \tilde{g}_{ij}) \tag{82}$$

on $B(x_0, \gamma + \delta/m) \times [0, T]$. Thus from (81) we know that

$$\begin{aligned} \frac{\partial \psi}{\partial t} \leq\ & g^{\alpha\beta} \widetilde{\nabla}_\alpha \widetilde{\nabla}_\beta \psi - 2 g^{\alpha\beta} \widetilde{\nabla}_\alpha |\widetilde{\nabla}^{m-1} g|^2 \cdot \widetilde{\nabla}_\beta |\widetilde{\nabla}^m g|^2 - \frac{1}{2}|\widetilde{\nabla}^m g|^4 \\ & - \frac{1}{2}(a + |\widetilde{\nabla}^{m-1} g|^2)|\widetilde{\nabla}^{m+1} g|^2 + c_0 |\widetilde{\nabla}^m g|^3 + c_0, \end{aligned}$$

where we have used (82), and therefore that

$$\begin{aligned} \frac{\partial \psi}{\partial t} \leq\ & g^{\alpha\beta} \widetilde{\nabla}_\alpha \widetilde{\nabla}_\beta \psi - 2 g^{\alpha\beta} \widetilde{\nabla}_\alpha |\widetilde{\nabla}^{m-1} g|^2 \cdot \widetilde{\nabla}_\beta |\widetilde{\nabla}^m g|^2 - \frac{1}{4}|\widetilde{\nabla}^m g|^4 \\ & - \frac{1}{2}(a + |\widetilde{\nabla}^{m-1} g|^2)|\widetilde{\nabla}^{m+1} g|^2 + c_0(a, m, n, \gamma, \delta, T, \tilde{g}_{ij}) \\ & \qquad \text{on } B(x_0, \gamma + \delta/m) \times [0, T], \end{aligned} \tag{83}$$

$$\begin{aligned} & - 2 g^{\alpha\beta} \widetilde{\nabla}_\alpha |\widetilde{\nabla}^{m-1} g|^2 \cdot \widetilde{\nabla}_\beta |\widetilde{\nabla}^m g|^2 \\ & \quad = -8 g^{-1} * \widetilde{\nabla}^{m-1} g * \widetilde{\nabla}^m g * \widetilde{\nabla}^m g * \widetilde{\nabla}^{m+1} g, \\ & - 2 g^{\alpha\beta} \widetilde{\nabla}_\alpha |\widetilde{\nabla}^{m-1} g|^2 \cdot \widetilde{\nabla}_\beta |\widetilde{\nabla}^m g|^2 \\ & \quad \leq 16 \cdot |\widetilde{\nabla}^{m-1} g| \cdot |\widetilde{\nabla}^m g|^2 \cdot |\widetilde{\nabla}^{m+1} g|. \end{aligned} \tag{84}$$

Using induction hypothesis (68) and (84) we get

$$\begin{aligned} & - 2 g^{\alpha\beta} \widetilde{\nabla}_\alpha |\widetilde{\nabla}^{m-1} g|^2 \cdot \widetilde{\nabla}_\beta |\widetilde{\nabla}^m g|^2 \\ & \quad \leq \tilde{c}_0(m, n, \gamma, \delta, T, \tilde{g}_{ij}) \cdot |\widetilde{\nabla}^m g|^2 |\widetilde{\nabla}^{m+1} g| \\ & \quad \leq \frac{1}{2} a |\widetilde{\nabla}^{m+1} g|^2 + \frac{1}{2a} \tilde{c}_0(m, n, \gamma, \delta, T, \tilde{g}_{ij})^2 |\widetilde{\nabla}^m g|^4. \end{aligned} \tag{85}$$

Substituting (85) into (83) yields

$$\begin{aligned}\frac{\partial \psi}{\partial t} \leq g^{\alpha\beta}\widetilde{\nabla}_\alpha\widetilde{\nabla}_\beta \psi + \frac{1}{2a}\tilde{c}_0(m,n,\gamma,\delta,T,\tilde{g}_{ij})^2|\widetilde{\nabla}^m g|^4 \\ - \frac{1}{4}|\widetilde{\nabla}^m g|^4 + c_0(a,m,n,\gamma,\delta,T,\tilde{g}_{ij}).\end{aligned} \tag{86}$$

If we choose

$$a = 4[\tilde{c}_0(m,n,\gamma,\delta,T,\tilde{g}_{ij})^2 + 1], \tag{87}$$

then by (86) we get

$$\frac{\partial \psi}{\partial t} \leq g^{\alpha\beta}\widetilde{\nabla}_\alpha\widetilde{\nabla}_\beta \psi - \frac{1}{8}|\widetilde{\nabla}^m g|^4 + c_0 \tag{88}$$

on $B(x_0, \gamma + \delta/m) \times [0,T]$,

$$\frac{\partial \psi}{\partial t} \leq g^{\alpha\beta}\widetilde{\nabla}_\alpha\widetilde{\nabla}_\beta \psi - \frac{\psi^2}{8[a + |\widetilde{\nabla}^{m-1} g|^2]^2} + c_0. \tag{89}$$

From (82) it follows that there exists a constant $c_1 > 0$ depending only on m, n, γ, δ, T, and $\tilde{g}_{ij}$ such that

$$\frac{\partial \psi}{\partial t} \leq g^{\alpha\beta}\widetilde{\nabla}_\alpha\widetilde{\nabla}_\beta \psi - c_1\psi^2 + c_0(m,n,\gamma,\delta,T,\tilde{g}_{ij}) \tag{90}$$

on $B(x_0, \gamma + \delta/m) \times [0,T]$.

By means of the maximal principle on (90) and the same reasoning as we used for (33)–(65) we know that there exists a constant $c_2(m,n,\gamma,\delta,T,\tilde{g}_{ij}) > 0$ such that

$$\psi(x,t) \leq c_2(m,n,\gamma,\delta,T,\tilde{g}_{ij}) \quad \text{on } B(x_0, \gamma + \delta/(m+1)) \times [0,T]. \tag{91}$$

From (80) and (87) it follows that

$$\psi(x,t) \geq a|\widetilde{\nabla}^m g|^2 \geq 4|\widetilde{\nabla}^m g|^2, \tag{92}$$

which together with (91) implies

$$|\widetilde{\nabla}^m g|^2 \leq \tfrac{1}{4}c_2(m,n,\gamma,\delta,T,\tilde{g}_{ij}) \quad \text{on } B(x_0, \gamma + \delta/(m+1)) \times [0,T].$$

Thus Lemma 4.2 is true for the case $k = m$ and hence for all integers $m \geq 0$ by induction.

Now we are going to construct the solution of the modified evolution equation (1) on the whole manifold M. Fix a point $x_0 \in M$ and choose a family of domains $\{D_k | k = 1,2,3,\cdots\}$ on M such that for each k, ∂D_k is a compact C^∞, $(n-1)$-dimensional submanifold of M and

$$\begin{aligned}&\overline{D}_k = D_k \cup \partial D_k \text{ is a compact subset of } M, \\ &B(x_0, k) \subseteq D_k,\end{aligned} \tag{93}$$

where $\partial D_{\mathscr{k}}$ is not necessarily connected.

Using Theorem 3.2 and Theorem 2.5 we know that there exists a constant $T(n,k_0) > 0$ depending only on n and k_0 such that the system

$$\begin{aligned} &\frac{\partial}{\partial t} g_{ij}(\mathscr{k},x,t) = -2R_{ij}(\mathscr{k},x,t) + \nabla_i V_j + \nabla_j V_i, \qquad x \in D_{\mathscr{k}}, \\ &g_{ij}(\mathscr{k},x,0) = \tilde{g}_{ij}(x), \qquad x \in D_{\mathscr{k}}, \\ &g_{ij}(\mathscr{k},x,t) \equiv \tilde{g}_{ij}(x), \qquad x \in \partial D_{\mathscr{k}},\ 0 \le t \le T \end{aligned} \tag{94}$$

has an unique smooth solution $g_{ij}(\mathscr{k},x,t) > 0$ on the time interval $0 \le t \le T(n,k_0)$ for each $\mathscr{k}$. We still have

$$(1 - \frac{1}{256000n^{10}})\tilde{g}_{ij}(x) \le g_{ij}(\mathscr{k},x,t) \le (1 + \frac{1}{256000n^{10}})\tilde{g}_{ij}(x) \tag{95}$$

for all $(x,t) \in D_{\mathscr{k}} \times [0, T(n,k_0)]$ and all integers $\mathscr{k} \ge 1$ by Theorem 2.5.

For any integer $q \ge 1$, from (93) it follows that

$$\begin{aligned} &B(x_0,q) \subseteq D_{\mathscr{K}} \qquad \text{if } \mathscr{k} \ge q, \\ &B(x_0,q+1) \subseteq D_{\mathscr{k}} \quad \text{if } \mathscr{k} \ge q+1. \end{aligned} \tag{96}$$

Using Lemma 4.2 and (95) we know that for any integer $m \ge 0$ there exist constants $c(m,n,q,k_0,\tilde{g}_{ij}) > 0$ depending only on m, n, q, k_0, and $\tilde{g}_{ij}$ such that

$$|\widetilde{\nabla}^m g_{ij}(\mathscr{k},x,t)|^2 \le c(m,n,q,k_0,\tilde{g}_{ij}) \tag{97}$$

for all $(x,t) \in B(x_0,q) \times [0, T(n,k_0)]$ and $\mathscr{k} \ge q+1$.

Also from (93) we have

$$M = \bigcup_{\mathscr{k}=1}^{\infty} D_{\mathscr{k}}. \tag{98}$$

Since the constants $c(m,n,q,k_0,\tilde{g}_{ij})$ in (97) are independent of $\mathscr{k}$, by (97) the derivatives of $g_{ij}(\mathscr{k},x,t)$ are uniformly bounded on any compact subset of M. Let $\mathscr{k} \to +\infty$. From (97) and (98) it follows that there exists a smooth metric $g_{ij}(x,t) > 0$ on $M \times [0, T(n,k_0)]$ such that

$$g_{ij}(\mathscr{k},x,t) \xrightarrow{C^\infty} g_{ij}(x,t) \quad \text{as } \mathscr{k} \to +\infty. \tag{99}$$

This means the metrics $g_{ij}(\mathscr{k},x,t)$ and all of their derivatives converge uniformly to the metric $g_{ij}(x,t)$ and its derivatives respectively on any compact subset of M as $\mathscr{k} \to +\infty$. Thus from (94) and (95) we get the following theorem.

Theorem 4.3. *There exists a constant* $T(n,k_0) > 0$ *depending only on* n *and* k_0 *such that the modified evolution equation*

$$\begin{aligned} &\frac{\partial}{\partial t} g_{ij}(x,t) = -2R_{ij}(x,t) + \nabla_i V_j + \nabla_j V_i, \qquad x \in M, \\ &g_{ij}(x,0) = \tilde{g}_{ij}(x), \qquad x \in M, \end{aligned} \tag{100}$$

has a smooth solution $g_{ij}(x,t) > 0$ *on* $0 \le t \le T(n,k_0)$, *and satisfies the estimate*

$$\left(1 - \frac{1}{256000n^{10}}\right)\tilde{g}_{ij}(x) \le g_{ij}(x,t) \le \left(1 + \frac{1}{256000n^{10}}\right)\tilde{g}_{ij}(x) \tag{101}$$

for all $(x,t) \in M \times [0, T(n,k_0)]$.

5. First derivative estimate

Suppose $g_{ij}(x,t) > 0$ is the smooth solution obtained in Theorem 4.3 on $M \times [0, T(n,k_0)]$. In this section we are going to estimate the first covariant derivatives of g_{ij} with respect to the metric $\tilde{g}_{ij}$ on the whole manifold M.

If we choose $T(n,k_0) > 0$ small enough, then from Theorem 2.5 it follows that

$$[1 - \delta(n,k_0)]\tilde{g}_{ij}(x) \le g_{ij}(x,t) \le [1 + \delta(n,k_0)]\tilde{g}_{ij}(x) \tag{1}$$

for all $(x,t) \in M \times [0, T(n,k_0)]$, where $\delta(n,k_0) > 0$ is the constant in Lemma 3.1. Actually (1) comes from Theorem 2.5 and (99) in §4.

Using the notation of §3, let

$$h_{ij}(x,t) = g_{ij}(x,t) - \tilde{g}_{ij}(x), \qquad H_{ij}(x,t) = \tfrac{1}{\delta}h_{ij}(x,t). \tag{2}$$

Then we have

$$\begin{aligned} &\frac{\partial}{\partial t}H_{ij} = g^{\alpha\beta}\tilde{\nabla}_\alpha\tilde{\nabla}_\beta H_{ij} + B_{ij} \quad \text{on } M \times [0, T(n,k_0)], \\ &H_{ij}(x,0) \equiv 0 \quad \forall x \in M, \end{aligned} \tag{3}$$

where $B_{ij} = \frac{1}{\delta}A_{ij}$ was defined in (4) of §3. From (11) of §3 it follows

$$\begin{aligned} &-\left(\frac{8n\sqrt{k_0}}{\delta} + 20\delta|\tilde{\nabla}_\alpha H_{\beta\gamma}|^2\right)\tilde{g}_{ij}(x) \le B_{ij}(x,t) \\ &\le \left(\frac{8n\sqrt{k_0}}{\delta} + 20\delta|\tilde{\nabla}_\alpha H_{\beta\gamma}|^2\right)\tilde{g}_{ij}(x) \quad \forall (x,t) \in M \times [0, T(n,k_0)]. \end{aligned} \tag{4}$$

By using (12), (13) and (14) of §3 we still have

$$\begin{aligned} -\tilde{g}_{ij}(x) \le H_{ij}(x,t) &\le \tilde{g}_{ij}(x), \\ \frac{1}{1+\delta}\tilde{g}^{\alpha\beta}(x) \le g^{\alpha\beta}(x,t) &\le \frac{1}{1-\delta}\tilde{g}^{\alpha\beta}(x), \\ |\tilde{\nabla}_i g^{\alpha\beta}|^2 &\le \frac{\delta^2}{(1-\delta)^4}|\tilde{\nabla}_i H_{kl}|^2 \end{aligned} \tag{5}$$

on $M \times [0, T(n, k_0)]$. Let $\gamma_0 = \frac{1}{8}(1/k_0)^{1/4}$. If the injectivity radius of M at some fixed point $x_0 \in M$ satisfies

$$\text{inj}(x_0) \geq \pi(1/k_0)^{1/4}, \tag{6}$$

then the geodesic ball $B(x_0, \gamma_0) \subseteq M$ is roughly the same as a ball in the Euclidean space $\mathbf{R}^n$. Thus using (3)–(6) and the same arguments as in the proof of Theorem 6.1 in [4, §6, Chapter VII] we know that if $\delta(n, k_0) > 0$ is small enough compared to n and k_0, then we can find a constant $\tilde{c}(n, k_0) > 0$ depending only on n and k_0 such that

$$\sup_{(x,t) \in B(x_0, \gamma_0) \times [0, T(n,k_0)]} |\widetilde{\nabla} H_{ij}(x, t)|^2 \leq \tilde{c}(n, k_0). \tag{7}$$

We need condition (6) to prove (7) since in the proof one needs to use the Poincaré inequality, the Sobolev inequality and integral estimates. The constants in these inequalities depend on the injectivity radius.

If (6) is not true at x_0, we consider the ball

$$\widehat{B}(0, \pi(1/k_0)^{1/4}) \subseteq T_{x_0}M \tag{8}$$

of radius $\pi(1/k_0)^{1/4}$ in the tangent space at $x_0 \in M$. Since $|\tilde{R}_{ijkl}|^2 \leq k_0$, using the comparison theorem we know that

$$\exp_{x_0}\colon \widehat{B}(0, \pi(1/k_0)^{1/4}) \to M \tag{9}$$

is nonsingular; therefore we can use this exponential map to pull everything back from M to $\widehat{B}(0, \pi(1/k_0)^{1/4})$ and do the analysis on $\widehat{B}(0, \pi(1/k_0)^{1/4})$. For the ball $\widehat{B}(0, \gamma_0) \subseteq T_{x_0}M$ of radius $\gamma_0 = \frac{1}{8}(1/k_0)^{1/4}$ by the same reason as (7) we get

$$\sup_{(x,t) \in \hat{B}(0, \gamma_0) \times [0, T(n,k_0)]} |\widetilde{\nabla} H_{ij}(x, t)|^2 \leq \tilde{c}(n, k_0). \tag{10}$$

Pushing back to M from (10) we know that (7) is also true for the case when (6) does not hold.

Since $x_0 \in M$ is arbitrary, from (7) it follows that

$$\sup_{(x,t) \in M \times [0, T(n,k_0)]} |\widetilde{\nabla} H_{ij}(x, t)|^2 \leq \tilde{c}(n, k_0). \tag{11}$$

Using (2) and (8) we hence have

Theorem 5.1. *There exists a constant $T(n, k_0) > 0$ depending only on n and k_0 such that*

$$\sup_{(x,t) \in M \times [0, T(n,k_0)]} |\widetilde{\nabla} g_{ij}(x, t)|^2 \leq c(n, k_0), \tag{12}$$

where $c(n, k_0) > 0$ depending only on n and k_0.

Note. Theorem 5.1 is true only for the solution $g_{ij}(x,t) > 0$ constructed in Theorem 4.3, and not for all of the solutions of the modified evolution equation. For the general solution of the modified evolution equation, (1) may not be true.

6. Second derivative estimate

In §§4 and 5 we obtained a smooth solution $g_{ij}(x,t) > 0$ on $M \times [0, T(n,k_0)]$ of the modified evolution equation

$$(1) \qquad \begin{aligned} &\frac{\partial}{\partial t} g_{ij}(x,t) = -2R_{ij}(x,t) + \nabla_i V_j + \nabla_j V_i \\ &g_{ij}(x,0) = \tilde{g}_{ij}(x) \quad \forall x \in M \end{aligned}$$

satisfying the following inequalities:

$$(2) \qquad \left. \begin{aligned} &\tfrac{1}{2}\tilde{g}_{ij}(x) \le g_{ij}(x,t) \le 2\tilde{g}_{ij}(x) \\ &|\widetilde{\nabla} g_{ij}(x,t)|^2 \le c(n,k_0) \end{aligned} \right. \quad \text{on } M \times [0, T(n,k_0)].$$

In this section we want to estimate the second covariant derivative $\widetilde{\nabla}\widetilde{\nabla} g_{ij}$ on M; usually the upper bound of the whole second derivative $|\widetilde{\nabla}\widetilde{\nabla} g_{ij}|^2$ depends not only on n and k_0, but also on the first derivative $\widetilde{\nabla}\tilde{R}_{ijkl}$ of the curvature tensor of the initial metric $\tilde{g}_{ij}$. Therefore instead of estimating $|\widetilde{\nabla}\widetilde{\nabla} g_{ij}|^2$, we want to estimate $|R_{ijkl}|^2$ and $|\nabla_i V_j|^2$ in terms of n and k_0. First, we want to find the evolution equation for R_{ijkl} and $\nabla_i V_j$.

If we define $y(x,t) = \varphi_t(x)$ by the equation

$$(3) \qquad \frac{\partial y^\alpha}{\partial t} = \frac{\partial y^\alpha}{\partial x^k} g^{\beta\gamma}(\Gamma^k_{\beta\gamma} - \widetilde{\Gamma}^k_{\beta\gamma}), \qquad y^\alpha(x,0) = x^\alpha,$$

then φ_t: $M \to M$ is a family of diffeomorphism (at least locally). Let

$$(4) \qquad g_{ij}(x,t) = \frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\beta}{\partial x^j}\hat{g}_{\alpha\beta}(y,t),$$

$$(5) \qquad d\hat{s}_t^2 = \hat{g}_{ij}(x,t)\,dx^i\,dx^j > 0, \quad ds_t^2 = g_{ij}(x,t)\,dx^i\,dx^j > 0.$$

Then from (5), (7), (8), (9), (11), (30) and (35) of §2 it follows that

$$(6) \qquad ds_t^2 = \varphi_t^*\,d\hat{s}_t^2,$$

$$(7) \qquad \frac{\partial}{\partial t}\hat{g}_{ij}(x,t) = -2\hat{R}_{ij}(x,t), \qquad \hat{g}_{ij}(x,0) = \tilde{g}_{ij}(x).$$

By (6) we have

$$R_{ijkl}(x,t) = \frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\beta}{\partial x^j}\frac{\partial y^\gamma}{\partial x^k}\frac{\partial y^\delta}{\partial x^l}\widehat{R}_{\alpha\beta\gamma\delta}(y,t), \tag{8}$$

and therefore

$$\begin{aligned}\frac{\partial}{\partial t}R_{ijkl}(x,t) &= \frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\beta}{\partial x^j}\frac{\partial y^\gamma}{\partial x^k}\frac{\partial y^\delta}{\partial x^l}\frac{d}{dt}\widehat{R}_{\alpha\beta\gamma\delta}(y,t)\\ &\quad+\frac{\partial}{\partial t}\left(\frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\beta}{\partial x^j}\frac{\partial y^\gamma}{\partial x^k}\frac{\partial y^\delta}{\partial x^l}\right)\cdot\widehat{R}_{\alpha\beta\gamma\delta}(y,t)\\ &= \frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\beta}{\partial x^j}\frac{\partial y^\gamma}{\partial x^k}\frac{\partial y^\delta}{\partial x^l}\frac{\partial}{\partial t}\widehat{R}_{\alpha\beta\gamma\delta}(y,t)\\ &\quad+\frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\beta}{\partial x^j}\frac{\partial y^\gamma}{\partial x^k}\frac{\partial y^\delta}{\partial x^l}\frac{\partial y^\theta}{\partial t}\frac{\partial}{\partial y^\theta}\widehat{R}_{\alpha\beta\gamma\delta}(y,t)\\ &\quad+\frac{\partial}{\partial t}\left(\frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\beta}{\partial x^j}\frac{\partial y^\gamma}{\partial x^k}\frac{\partial y^\delta}{\partial x^l}\right)\cdot\widehat{R}_{\alpha\beta\gamma\delta}(y,t).\end{aligned} \tag{9}$$

From Theorem 7.1 of [3] it follows that

$$\frac{\partial}{\partial t}\widehat{R}_{ijkl} = \hat{g}^{\alpha\beta}\widehat{\nabla}_\alpha\widehat{\nabla}_\beta\widehat{R}_{ijkl} + \hat{\psi}_{ijkl}, \tag{10}$$

where

$$\begin{aligned}\hat{\psi}_{ijkl} &= 2(\hat{B}_{ijkl} - \hat{B}_{ijlk} - \hat{B}_{iljk} + \hat{B}_{ikjl})\\ &\quad - \hat{g}^{pq}(\widehat{R}_{pjkl}\widehat{R}_{qi} + \widehat{R}_{ipkl}\widehat{R}_{qj} + \widehat{R}_{ijpl}\widehat{R}_{qk} + \widehat{R}_{ijkp}\widehat{R}_{ql}),\\ \widehat{B}_{ijkl} &= \hat{g}^{p\gamma}\hat{g}^{qs}\widehat{R}_{piqj}\widehat{R}_{\gamma ksl}.\end{aligned}$$

Thus

$$\begin{aligned}&\frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\beta}{\partial x^j}\frac{\partial y^\gamma}{\partial x^k}\frac{\partial y^\delta}{\partial x^l}\frac{\partial}{\partial t}\widehat{R}_{\alpha\beta\gamma\delta}(y,t)\\ &\quad= \frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\beta}{\partial x^j}\frac{\partial y^\gamma}{\partial x^k}\frac{\partial y^\delta}{\partial x^l}(\hat{g}^{\lambda\eta}\widehat{\nabla}_\lambda\widehat{\nabla}_\eta\widehat{R}_{\alpha\beta\gamma\delta} + \hat{\psi}_{\alpha\beta r\delta}).\end{aligned} \tag{11}$$

By (6) we have

$$\frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\beta}{\partial x^j}\frac{\partial y^\gamma}{\partial x^k}\frac{\partial y^\delta}{\partial x^l}\hat{g}^{\lambda\eta}\widehat{\nabla}_\lambda\widehat{\nabla}_\eta\widehat{R}_{\alpha\beta\gamma\delta} = g^{pq}\nabla_p\nabla_q R_{ijkl}. \tag{12}$$

If we let $\Delta = g^{pq}\nabla_p\nabla_q$ be the Laplacian operator of the metric g_{ij}, then from (11) and (12) it follows that

$$\frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\beta}{\partial x^j}\frac{\partial y^\gamma}{\partial x^k}\frac{\partial y^\delta}{\partial x^l}\frac{\partial}{\partial t}\widehat{R}_{\alpha\beta\gamma\delta}(y,t) = \Delta R_{ijkl} + \psi_{ijkl}, \tag{13}$$

where

$$(14)\qquad \begin{aligned} \psi_{ijkl} &= 2(B_{ijkl} - B_{ijlk} - B_{iljk} + B_{ikjl}) \\ &\quad - g^{pq}(R_{pjkl}R_{qi} + R_{ipkl}R_{qj} + R_{ijpl}R_{qk} + R_{ijkp}R_{ql}), \\ B_{ijkl} &= g^{p\gamma} g^{qs} R_{piqj} R_{\gamma ksl}. \end{aligned}$$

Since

$$\frac{\partial}{\partial y^\theta}\widehat{R}_{\alpha\beta\gamma\delta}(y,t) = \frac{\partial x^p}{\partial y^\theta}\frac{\partial}{\partial x^p}\widehat{R}_{\alpha\beta\gamma\delta}(y,t),$$

we have

$$(15)\qquad \begin{aligned} &\frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\beta}{\partial x^j}\frac{\partial y^\gamma}{\partial x^k}\frac{\partial y^\delta}{\partial x^l}\frac{\partial y^\theta}{\partial t}\frac{\partial}{\partial y^\theta}\widehat{R}_{\alpha\beta\gamma\delta}(y,t) \\ &\quad = \frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\beta}{\partial x^j}\frac{\partial y^\gamma}{\partial x^k}\frac{\partial y^\delta}{\partial x^l}\frac{\partial y^\theta}{\partial t}\frac{\partial x^p}{\partial y^\theta}\frac{\partial}{\partial x^p}\widehat{R}_{\alpha\beta\gamma\delta}(y,t) \\ &\quad = \frac{\partial y^\theta}{\partial t}\frac{\partial x^p}{\partial y^\theta}\frac{\partial}{\partial x^p}\left(\frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\beta}{\partial x^j}\frac{\partial y^\gamma}{\partial x^k}\frac{\partial y^\delta}{\partial x^l}\widehat{R}_{\alpha\beta\gamma\delta}(y,t)\right) \\ &\quad\quad - \frac{\partial y^\theta}{\partial t}\frac{\partial x^p}{\partial y^\theta}\frac{\partial}{\partial x^p}\left(\frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\beta}{\partial x^j}\frac{\partial y^\gamma}{\partial x^k}\frac{\partial y^\delta}{\partial x^l}\right)\cdot \widehat{R}_{\alpha\beta\gamma\delta}(y,t). \end{aligned}$$

By (3) and the definition of V_i,

$$(16)\qquad V_i = g_{ij} g^{\beta\gamma}(\Gamma^j_{\beta\gamma} - \widetilde{\Gamma}^j_{\beta\gamma}),$$

we have

$$(17)\qquad \frac{\partial y^\alpha}{\partial t} = \frac{\partial y^\alpha}{\partial x^k} V_l \cdot g^{kl};$$

thus

$$(18)\qquad \frac{\partial y^\theta}{\partial t}\frac{\partial x^p}{\partial y^\theta} = g^{pq} V_q.$$

From (8), (15), and (18) it follows that

$$\begin{aligned} &\frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\beta}{\partial x^j}\frac{\partial y^\gamma}{\partial x^k}\frac{\partial y^\delta}{\partial x^l}\frac{\partial y^\theta}{\partial t}\frac{\partial}{\partial y^\theta}\widehat{R}_{\alpha\beta r\delta}(y,t) \\ &\quad = g^{pq} V_q \cdot \frac{\partial}{\partial x^p} R_{ijkl} - g^{pq}\cdot V_q \frac{\partial}{\partial x^p}\left(\frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\beta}{\partial x^j}\frac{\partial y^\gamma}{\partial x^k}\frac{\partial y^\delta}{\partial x^l}\right)\cdot \widehat{R}_{\alpha\beta\gamma\delta}. \end{aligned}$$

Using (8) again, we get

$$\begin{aligned}(19)\quad &\frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\beta}{\partial x^j}\frac{\partial y^\gamma}{\partial x^k}\frac{\partial y^\delta}{\partial x^l}\frac{\partial y^\theta}{\partial t}\frac{\partial}{\partial y^\theta}\widehat{R}_{\alpha\beta\gamma\delta}(y,t) = g^{pq}V_q\cdot\frac{\partial}{\partial x^p}R_{ijkl}\\ &\quad - g^{pq}V_q\left(\frac{\partial^2 y^\alpha}{\partial x^p\partial x^i}\frac{\partial x^s}{\partial y^\alpha}R_{sjkl}+\frac{\partial^2 y^\beta}{\partial x^p\partial x^j}\frac{\partial x^s}{\partial y^\beta}R_{iskl}\right.\\ &\quad\quad\left.+\frac{\partial^2 y^\gamma}{\partial x^p\partial x^k}\frac{\partial x^s}{\partial y^\gamma}R_{ijsl}+\frac{\partial^2 y^\delta}{\partial x^p\partial x^l}\frac{\partial x^s}{\partial y^\delta}R_{ijks}\right),\end{aligned}$$

$$\begin{aligned}&\frac{\partial}{\partial t}\left(\frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\beta}{\partial x^j}\frac{\partial y^\gamma}{\partial x^k}\frac{\partial y^\delta}{\partial x^l}\right)\cdot\widehat{R}_{\alpha\beta\gamma\delta}(y,t)\\ &\quad=\frac{\partial}{\partial x^i}\left(\frac{\partial y^\alpha}{\partial t}\right)\cdot\frac{\partial x^s}{\partial y^\alpha}R_{sjkl}+\frac{\partial}{\partial x^j}\left(\frac{\partial y^\beta}{\partial t}\right)\cdot\frac{\partial x^s}{\partial y^\beta}R_{iskl}\\ &\quad+\frac{\partial}{\partial x^k}\left(\frac{\partial y^\gamma}{\partial t}\right)\cdot\frac{\partial x^s}{\partial y^\gamma}R_{ijsl}+\frac{\partial}{\partial x^l}\left(\frac{\partial y^\delta}{\partial t}\right)\cdot\frac{\partial x^s}{\partial y^\delta}R_{ijks}.\end{aligned}$$

By means of (17) we obtain

$$\begin{aligned}&\frac{\partial}{\partial t}\left(\frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\beta}{\partial x^j}\frac{\partial y^\gamma}{\partial x^k}\frac{\partial y^\delta}{\partial x^l}\right)\cdot\widehat{R}_{\alpha\beta\gamma\delta}(y,t)\\ &\quad= g^{pq}V_q\left(\frac{\partial^2 y^\alpha}{\partial x^i\partial x^p}\frac{\partial x^s}{\partial y^\alpha}R_{sjkl}+\frac{\partial^2 y^\beta}{\partial x^j\partial x^p}\frac{\partial x^s}{\partial y^\beta}R_{iskl}\right.\\ (20)\quad &\quad\quad\left.+\frac{\partial^2 y^\gamma}{\partial x^k\partial x^p}\frac{\partial x^s}{\partial y^\gamma}R_{ijsl}+\frac{\partial^2 y^\delta}{\partial x^l\partial x^p}\frac{\partial x^s}{\partial y^\delta}R_{ijks}\right)\\ &\quad\quad+R_{pjkl}\frac{\partial}{\partial x^i}(g^{pq}V_q)+R_{ipkl}\frac{\partial}{\partial x^j}(g^{pq}V_q)\\ &\quad\quad+R_{ijpl}\frac{\partial}{\partial x^k}(g^{pq}V_q)+R_{ijkp}\frac{\partial}{\partial x^l}(g^{pq}V_q).\end{aligned}$$

Substituting (13), (19), and (20) into (9) yields

$$\begin{aligned}&\frac{\partial}{\partial t}R_{ijkl}=\Delta R_{ijkl}+\psi_{ijkl}+g^{pq}V_q\cdot\frac{\partial}{\partial x^p}R_{ijkl}\\ (21)\quad &\quad+R_{pjkl}\frac{\partial}{\partial x^i}(g^{pq}V_q)+R_{ipkl}\frac{\partial}{\partial x^j}(g^{pq}V_q)\\ &\quad+R_{ijpl}\frac{\partial}{\partial x^k}(g^{pq}V_q)+R_{ijkp}\frac{\partial}{\partial x^l}(g^{pq}V_q).\end{aligned}$$

If we choose a coordinate system $\{x^i\}$ such that at one point

$$(22)\quad \frac{\partial g_{ij}}{\partial x^k}=0,$$

then from (21) we get

(23)
$$\frac{\partial}{\partial t}R_{ijkl} = \Delta R_{ijkl} + \psi_{ijkl} + g^{pq}V_q \cdot \nabla_p R_{ijkl} + g^{pq}(R_{pjkl}\nabla_i V_q + R_{ipkl}\nabla_j V_q + R_{ijpl}\nabla_k V_q + R_{ijkp}\nabla_l V_q),$$

where ψ_{ijkl} was defined by (14).

From (22) it follows that at one point $\Gamma^k_{ij} = 0$, and therefore that

$$\begin{aligned}
\frac{\partial}{\partial t}(\Gamma^p_{kl} - \tilde{\Gamma}^p_{kl}) &= \frac{\partial}{\partial t}\Gamma^p_{kl} = \frac{1}{2}\frac{\partial}{\partial t}\left[g^{pq}\left(\frac{\partial g_{qk}}{\partial x^l} + \frac{\partial g_{ql}}{\partial x^k} - \frac{\partial g_{kl}}{\partial x^q}\right)\right] \\
&= \frac{1}{2}g^{pq}\left[\frac{\partial}{\partial x^l}\left(\frac{\partial}{\partial t}g_{qk}\right) + \frac{\partial}{\partial x^k}\left(\frac{\partial}{\partial t}g_{ql}\right) - \frac{\partial}{\partial x^q}\left(\frac{\partial}{\partial t}g_{kl}\right)\right] \\
&= \frac{1}{2}g^{pq}\left[\nabla_l\left(\frac{\partial}{\partial t}g_{qk}\right) + \nabla_k\left(\frac{\partial}{\partial t}g_{ql}\right) - \nabla_q\left(\frac{\partial}{\partial t}g_{kl}\right)\right] \\
&= \frac{1}{2}g^{pq}[\nabla_l(-2R_{qk} + \nabla_q V_k + \nabla_k V_q) \\
&\qquad + \nabla_k(-2R_{ql} + \nabla_q V_l + \nabla_l V_q) \\
&\qquad - \nabla_q(-2R_{kl} + \nabla_k V_l + \nabla_l V_k])] \\
&= g^{pq}(\nabla_q R_{kl} - \nabla_k R_{ql} - \nabla_l R_{qk}) \\
&\quad + \frac{1}{2}g^{pq}(\nabla_l\nabla_q V_k \\
&\quad + \nabla_l\nabla_k V_q + \nabla_k\nabla_l V_q + \nabla_k\nabla_q V_l - \nabla_q\nabla_k V_l - \nabla_q\nabla_l V_k),
\end{aligned} \tag{24}$$

$$\begin{aligned}
g^{kl}g_{ip}\frac{\partial}{\partial t}(\Gamma^p_{kl} - \tilde{\Gamma}^p_{kl}) &= g^{kl}(\nabla_i R_{kl} - \nabla_k R_{il} - \nabla_l R_{ik}) \\
&\quad + \frac{1}{2}g^{kl}(\nabla_l\nabla_i V_k + \nabla_l\nabla_k V_i + \nabla_k\nabla_l V_i + \nabla_k\nabla_i V_l \\
&\qquad - \nabla_i\nabla_k V_l - \nabla_i\nabla_l V_k).
\end{aligned}$$

From the Bianchi identity we have

$$g^{kl}\nabla_i R_{kl} = \nabla_i R, \quad g^{kl}\nabla_k R_{il} = \tfrac{1}{2}\nabla_i R, \quad g^{kl}\nabla_l R_{ik} = \tfrac{1}{2}\nabla_i R,$$

and therefore

$$g^{kl}(\nabla_i R_{kl} - \nabla_k R_{il} - \nabla_l R_{ik}) = 0. \tag{25}$$

The following formulas are well known:

$$\begin{aligned}
\nabla_k\nabla_i V_l - \nabla_i\nabla_k V_l &= g^{pq}R_{kilp}V_q, \\
\nabla_l\nabla_i V_k - \nabla_i\nabla_l V_k &= g^{pq}R_{likp}V_q.
\end{aligned} \tag{26}$$

Substituting (25) and (26) into (24) gives

$$g^{kl}g_{ip}\frac{\partial}{\partial t}(\Gamma^p_{kl} - \tilde{\Gamma}^p_{kl}) = g^{kl}\nabla_k\nabla_l V_i + g^{pq}R_{ip}V_q, \tag{27}$$

$$\begin{aligned}\frac{\partial}{\partial t}V_i &= \frac{\partial}{\partial t}[g^{kl}g_{ip}(\Gamma^p_{kl}-\tilde{\Gamma}^p_{kl})]\\ &= g^{kl}g_{ip}\frac{\partial}{\partial t}(\Gamma^p_{kl}-\tilde{\Gamma}^p_{kl}) + g^{kl}\left(\frac{\partial}{\partial t}g_{ip}\right)\cdot(\Gamma^p_{kl}-\tilde{\Gamma}^p_{kl})\\ &\quad + \left(\frac{\partial}{\partial t}g^{kl}\right)\cdot g_{ip}(\Gamma^p_{kl}-\tilde{\Gamma}^p_{kl})\\ &= g^{kl}\nabla_k\nabla_l V_i + g^{kl}R_{ik}V_l + \left(\frac{\partial}{\partial t}g_{ip}\right)\cdot g^{kl}(\Gamma^p_{kl}-\tilde{\Gamma}^p_{kl})\\ &\quad + \left(\frac{\partial}{\partial t}g^{kl}\right)\cdot g_{ip}(\Gamma^p_{kl}-\tilde{\Gamma}^p_{kl}).\end{aligned}$$

Since $g^{kl}\nabla_k\nabla_l V_i = \Delta V_i$ and $g^{kl}(\Gamma^p_{kl}-\tilde{\Gamma}^p_{kl}) = g^{pq}V_q$,

$$(28)\qquad \begin{aligned}\frac{\partial}{\partial t}V_i &= \Delta V_i + g^{kl}R_{ik}V_l + \left(\frac{\partial}{\partial t}g_{ip}\right)\cdot g^{pq}V_q\\ &\quad + \left(\frac{\partial}{\partial t}g^{kl}\right)\cdot g_{ip}(\Gamma^p_{kl}-\tilde{\Gamma}^p_{kl}).\end{aligned}$$

We still use g, g^{-1}, Rm, ∇^kRm, and $*$ to denote, respectively, the tensors g_{ij}, g^{ij}, R_{ijkl}, the kth covariant derivative of Rm, and the tensor product. Let

(29)
$$g^2 = g*g, g^3 = g*g*g, \cdots, g^{-2} = g^{-1}*g^{-1}, g^{-3} = g^{-1}*g^{-1}*g^{-1}, \cdots .$$

Since $R_{ij} = g^{kl}R_{ikjl}$, Ricci curvature can be denoted as $g^{-1}*\mathrm{Rm}$.

Since $\frac{\partial}{\partial t}g^{ij} = -g^{ik}g^{jl}\frac{\partial}{\partial t}g_{kl}$, we have

$$(30)\qquad \frac{\partial}{\partial t}g^{-1} = g^{-2}*\frac{\partial}{\partial t}g.$$

From (1) it follows that

$$(31)\qquad \frac{\partial}{\partial t}g = g^{-1}*\mathrm{Rm} + \nabla V,$$

where V denotes the tensor $\{V_i\}$. From (39) of §2 and (14) we know that

$$(32)\qquad V = g^{-1}*\tilde{\nabla}g,$$

$$(33)\qquad \psi_{ijkl} = g^{-2}*\mathrm{Rm}*\mathrm{Rm}.$$

Thus using (23) we get

$$(34)\qquad \begin{aligned}\frac{\partial}{\partial t}R_{ijkl} &= \Delta R_{ijkl} + g^{-2}*\mathrm{Rm}*\mathrm{Rm} + g^{-2}*V*\nabla\mathrm{Rm}\\ &\quad + g^{-1}*\mathrm{Rm}*\nabla V.\end{aligned}$$

From (28) it follows that

$$\frac{\partial}{\partial t}V_i = \Delta V_i + g^{-2} * \mathrm{Rm} * V + g^{-1} * V * \frac{\partial g}{\partial t} + g^{-1} * \tilde{\nabla} g * \frac{\partial}{\partial t} g^{-1} * g. \tag{35}$$

Substituting (30), (31), and (32) into (35) hence yields

Lemma 6.1. *The following equations hold:*

$$\frac{\partial}{\partial t}R_{ijkl} = \Delta R_{ijkl} + g^{-2} * \mathrm{Rm} * \mathrm{Rm} + g^{-1} * V * \nabla \mathrm{Rm} + g^{-1} * \mathrm{Rm} * \nabla V, \tag{36}$$

$$\frac{\partial}{\partial t}V_i = \Delta V_i + g^{-3} * \tilde{\nabla} g * \mathrm{Rm} + g^{-2} * \tilde{\nabla} g * \nabla V. \tag{37}$$

To estimate the curvature R_{ijkl} and $\nabla_i V_j$ we need the integral estimate of them. Define the volume element

$$dw_t = \sqrt{\det(g_{ij}(x,t))}\,dx^1 dx^2 \cdots dx^n. \tag{38}$$

Then

$$\begin{aligned} \frac{\partial}{\partial t} dw_t &= \frac{1}{2\sqrt{\det(g_{ij})}} \cdot \det(g_{ij}) \cdot g^{ij} \cdot \left(\frac{\partial}{\partial t} g_{ij}\right) \cdot dx^1 dx^2 \cdots dx^n \\ &= \frac{1}{2}\left(g^{ij}\frac{\partial}{\partial t} g_{ij}\right) dw_t = \frac{1}{2}(-2R_{ij} + \nabla_i V_j + \nabla_j V_i) \cdot g^{ij} \cdot dw_t, \\ \frac{\partial}{\partial t} dw_t &= (-R + g^{ij}\nabla_i V_j)\, dw_t. \end{aligned} \tag{39}$$

In this section we use $|\ |^2$ and $|\ |_0^2$ to denote the norms with respect to the metrics $g_{ij}(x,t)$ and $\tilde{g}_{ij}(x)$ respectively. Using (2) we know that these two norms are equivalent to each other.

For any point $x_0 \in M$, we denote by $B(x_0,\gamma)$ the geodesic ball, centered at x_0, of radius γ with respect to the metric $\tilde{g}_{ij}$. Let $T = T(n,k_0)$ be the number in (2). Then we have the following lemma.

Lemma 6.2. *For any $x_0 \in M$ and $0 < \gamma < +\infty$ we have*

$$\int_0^T \int_{B(x_0,\gamma)} |\tilde{\nabla}\tilde{\nabla} g_{ij}(x,t)|_0^2\, dw_0\, dt \le c_0(n,k_0,\gamma),$$

where $0 < c_0(n,k_0,\gamma) < +\infty$ is some constant depending only on n, k_0, and γ.

Proof. Similar to (39) and (40) of §4, using the mollifier technique we can find a function $\xi(x) \in C_0^\infty(M)$ such that

$$|\tilde{\nabla}\xi(x)|_0 \le 8, \qquad x \in M, \tag{40}$$

$$\begin{aligned} &\xi(x) \equiv 1, && x \in B(x_0, \gamma), \\ &\xi(x) \equiv 0, && x \in M \backslash B(x_0, \gamma + \tfrac{1}{2}), \\ &0 \leq \xi(x) \leq 1, && x \in M. \end{aligned} \tag{41}$$

Let

$$\Omega = B(x_0, \gamma + 1). \tag{42}$$

From Lemma 2.1 we have

$$\begin{aligned} \frac{\partial}{\partial t}\tilde{\nabla}_k g_{ij} &= g^{\alpha\beta}\tilde{\nabla}_\alpha\tilde{\nabla}_\beta(\tilde{\nabla}_k g_{ij}) + g^{-1} * g * \tilde{\nabla}\widetilde{\mathrm{Rm}} \\ &\quad + g^{-2} * g * \tilde{\nabla} g * \widetilde{\mathrm{Rm}} + g^{-2} * \tilde{\nabla} g * \tilde{\nabla}\tilde{\nabla} g \\ &\quad + g^{-3} * \tilde{\nabla} g * \tilde{\nabla} g * \tilde{\nabla} g, \end{aligned} \tag{43}$$

and therefore

$$\begin{aligned} &\frac{\partial}{\partial t}\int_\Omega |\tilde{\nabla}_k g_{ij}(x,t)|_0^2 \xi(x)^2\, dw_0 \\ &= 2\int_\Omega \tilde{\nabla}_k g_{ij} \cdot \frac{\partial}{\partial t}\tilde{\nabla}_k g_{ij} \cdot \xi(x)^2\, dw_0 \\ &= 2\int_\Omega \tilde{\nabla}_k g_{ij} \cdot g^{\alpha\beta}\tilde{\nabla}_\alpha\tilde{\nabla}_\beta(\tilde{\nabla}_k g_{ij}) \cdot \xi(x)^2\, dw_0 \\ &\quad + \int_\Omega g^{-1} * g * \tilde{\nabla} g * \tilde{\nabla}\widetilde{\mathrm{Rm}} \cdot \xi(x)^2\, dw_0 \\ &\quad + \int_\Omega \tilde{\nabla} g * [g^{-2} * g * \tilde{\nabla} g * \widetilde{\mathrm{Rm}} + g^{-2} * \tilde{\nabla} g * \tilde{\nabla}\tilde{\nabla} g \\ &\qquad\qquad + g^{-3} * \tilde{\nabla} g * \tilde{\nabla} g * \tilde{\nabla} g]\xi(x)^2\, dw_0. \end{aligned} \tag{44}$$

Using (2) we have

$$\tfrac{1}{2}\tilde{g} \leq g \leq 2\tilde{g}, \quad |\tilde{\nabla} g|_0^2 \leq c(n, k_0) \quad \text{on } M \times [0, T], \tag{45}$$

using (45) and the condition $|\widetilde{\mathrm{Rm}}|_0^2 \leq k_0$ we get

$$\tilde{\nabla} g * (g^{-2} * g * \tilde{\nabla} g * \widetilde{\mathrm{Rm}} + g^{-2} * \tilde{\nabla} g * \tilde{\nabla}\tilde{\nabla} g + g^{-3} * \tilde{\nabla} g * \tilde{\nabla} g * \tilde{\nabla} g) \leq c_0 + c_0|\tilde{\nabla}\tilde{\nabla} g|_0,$$

$$\begin{aligned} &\int_\Omega \tilde{\nabla} g * (g^{-2} * g * \tilde{\nabla} g * \widetilde{\mathrm{Rm}} + g^{-2} * \tilde{\nabla} g * \tilde{\nabla}\tilde{\nabla} g \\ &\qquad + g^{-3} * \tilde{\nabla} g * \tilde{\nabla} g * \tilde{\nabla} g)\xi(x)^2\, dw_0 \\ &\qquad \leq c_0 \int_\Omega (1 + |\tilde{\nabla}\tilde{\nabla} g|_0)\xi(x)^2\, dw_0. \end{aligned} \tag{46}$$

On the other hand, a use of (41), (42) and the assumption $|\widetilde{\mathrm{Rm}}|_0^2 \le k_0$ gives

$$\int_\Omega \xi(x)^2\,dw_0 \le \int_\Omega dw_0 = \int_{B(x_0,\gamma+1)} dw_0 \le c_0(k_0,\gamma). \tag{47}$$

Combining (46) and (47) we have

$$\begin{aligned}\int_\Omega \tilde\nabla g * (g^{-2} * g * \tilde\nabla g * \widetilde{\mathrm{Rm}} + g^{-2} * \tilde\nabla g * \tilde\nabla\tilde\nabla g \\ + g^{-3} * \tilde\nabla g * \tilde\nabla g * \tilde\nabla g)\xi(x)^2\,dw_0 \\ \le c_0 + c_0 \int_\Omega |\tilde\nabla\tilde\nabla g|_0 \xi(x)^2\,dw_0,\end{aligned} \tag{48}$$

where c_0 means some constant depending only on n, k_0, and γ; they may not be the same as each other.

By integration by parts, we get

$$\begin{aligned}&2\int_\Omega \tilde\nabla_k g_{ij}\cdot g^{\alpha\beta}\tilde\nabla_\alpha\tilde\nabla_\beta(\tilde\nabla_k g_{ij})\cdot\xi(x)^2\,dw_0 \\ &= -2\int_\Omega \tilde\nabla_\beta(\tilde\nabla_k g_{ij})\cdot\tilde\nabla_\alpha[g^{\alpha\beta}\cdot\tilde\nabla_k g_{ij}\cdot\xi(x)^2]\,dw_0 \\ &= -2\int_\Omega g^{\alpha\beta}\tilde\nabla_\beta\tilde\nabla_k g_{ij}\cdot\tilde\nabla_\alpha\tilde\nabla_k g_{ij}\cdot\xi(x)^2\,dw_0 \\ &\quad + \int_\Omega \tilde\nabla\tilde\nabla g * \tilde\nabla g^{-1} * \tilde\nabla g\cdot\xi(x)^2\,dw_0 \\ &\quad + \int_\Omega g^{-1} * \tilde\nabla\tilde\nabla g * \xi(x) * \tilde\nabla\xi * \tilde\nabla g\cdot dw_0.\end{aligned} \tag{49}$$

On the other hand, using (40), (41), and (45) we have

$$\begin{aligned}&\int_\Omega g^{-1} * \tilde\nabla\tilde\nabla g * \xi(x) * \tilde\nabla\xi * \tilde\nabla g\cdot dw_0 \\ &\le c_0\int_\Omega |\tilde\nabla\tilde\nabla g|_0\xi(x)\,dw_0.\end{aligned} \tag{50}$$

Since $\tilde\nabla g^{-1} = g^{-2} * \tilde\nabla g$, from (45) it follows that

$$|\tilde\nabla g^{-1}|_0^2 \le c_0; \tag{51}$$

thus

$$\begin{aligned}&\int_\Omega \tilde\nabla\tilde\nabla g * \tilde\nabla g^{-1} * \tilde\nabla g * \xi(x)^2\,dw_0 \\ &\le c_0\int_\Omega |\tilde\nabla\tilde\nabla g|_0\cdot\xi(x)^2\,dw_0 \le c_0\int_\Omega |\tilde\nabla\tilde\nabla g|_0\cdot\xi(x)\,dw_0.\end{aligned} \tag{52}$$

From (45) we know that

$$g^{\alpha\beta}\tilde{\nabla}_\beta\tilde{\nabla}_k g_{ij}\cdot\tilde{\nabla}_\alpha\tilde{\nabla}_k g_{ij} \geq \tfrac{1}{2}|\tilde{\nabla}\tilde{\nabla}g|_0^2. \tag{53}$$

A use of (49), (50), (52), and (53) yields

$$\begin{aligned} 2\int_\Omega &\tilde{\nabla}_k g_{ij}\cdot g^{\alpha\beta}\tilde{\nabla}_\alpha\tilde{\nabla}_\beta(\tilde{\nabla}_k g_{ij})\cdot\xi(x)^2\,dw_0 \\ &\leq -\int_\Omega|\tilde{\nabla}\tilde{\nabla}g|_0^2\xi(x)^2\,dw_0 + c_0\int_\Omega|\tilde{\nabla}\tilde{\nabla}g|_0\cdot\xi(x)\,dw_0 \\ &\leq -\tfrac{1}{2}\int_\Omega|\tilde{\nabla}\tilde{\nabla}g|_0^2\xi(x)^2\,dw_0 + c_0. \end{aligned} \tag{54}$$

By using integration by parts again, we have

$$\begin{aligned} \int_\Omega & g^{-1}*g*\tilde{\nabla}g*\tilde{\nabla}\tilde{\mathrm{R}}\mathrm{m}\cdot\xi(x)^2\,dw_0 \\ &= -\int_\Omega\tilde{\mathrm{R}}\mathrm{m}*\tilde{\nabla}(g^{-1}*g*\tilde{\nabla}g*\xi(x)^2)\,dw_0 \\ &= \int_\Omega\tilde{\mathrm{R}}\mathrm{m}*[\tilde{\nabla}g^{-1}*g*\tilde{\nabla}g*\xi(x) + g^{-1}*\tilde{\nabla}g*\tilde{\nabla}g*\xi(x) \\ &\qquad + g^{-1}*g*\tilde{\nabla}\tilde{\nabla}g*\xi(x) + g^{-1}*g*\tilde{\nabla}g*\tilde{\nabla}\xi]\xi(x)\,dw_0. \end{aligned} \tag{55}$$

Using $|\tilde{\mathrm{R}}\mathrm{m}|_0^2 \leq k_0$ and (40), (41), (45), and (51) we get

$$\begin{aligned} \tilde{\mathrm{R}}\mathrm{m} &* (\tilde{\nabla}g^{-1}*g*\tilde{\nabla}g*\xi(x) + g^{-1}*\tilde{\nabla}g*\tilde{\nabla}g*\xi(x) \\ &\quad + g^{-1}*g*\tilde{\nabla}\tilde{\nabla}g*\xi(x) + g^{-1}*g*\tilde{\nabla}g*\tilde{\nabla}\xi) \\ &\leq c_0 + c_0|\tilde{\nabla}\tilde{\nabla}g|_0. \end{aligned} \tag{56}$$

Substituting (56) into (55) gives

$$\begin{aligned} \int_\Omega & g^{-1}*g*\tilde{\nabla}g*\tilde{\nabla}\tilde{\mathrm{R}}\mathrm{m}\cdot\xi(x)^2\,dw_0 \\ &\leq c_0\int_\Omega\xi(x)\,dw_0 + c_0\int_\Omega|\tilde{\nabla}\tilde{\nabla}g|_0\xi(x)\,dw_0, \end{aligned} \tag{57}$$

which together with (47) implies

$$\int_\Omega g^{-1}*g*\tilde{\nabla}g*\tilde{\nabla}\tilde{\mathrm{R}}\mathrm{m}\cdot\xi(x)^2\,dw_0 \leq c_0 + c_0\int_\Omega|\tilde{\nabla}\tilde{\nabla}g|_0\cdot\xi(x)\,dw_0. \tag{58}$$

Substituting (48), (54), and (58) into (44), we get

$$\begin{aligned}
&\frac{\partial}{\partial t}\int_\Omega |\widetilde{\nabla}_k g_{ij}(x,t)|_0^2 \xi(x)^2\, dw_0 \\
&\quad \le -\frac{1}{2}\int_\Omega |\widetilde{\nabla}\widetilde{\nabla} g|_0^2 \xi(x)^2\, dw_0 \\
&\qquad + c_0 \int_0 |\widetilde{\nabla}\widetilde{\nabla} g|_0 \cdot \xi(x)\, dw_0 + c_0 \int_\Omega |\widetilde{\nabla}\widetilde{\nabla} g|_0 \xi(x)^2\, dw_0 + c_0 \\
&\quad \le -\frac{1}{2}\int_\Omega |\widetilde{\nabla}\widetilde{\nabla} g|_0^2 \xi(x)^2\, dw_0 + c_0 \int_\Omega |\widetilde{\nabla}\widetilde{\nabla} g|_0 \xi(x)\, dw_0 + c_0,
\end{aligned}$$

$$\frac{\partial}{\partial t}\int_\Omega |\widetilde{\nabla}_k g_{ij}(x,t)|_0^2 \xi(x)^2\, dw_0 \le -\frac{1}{4}\int_\Omega |\widetilde{\nabla}\widetilde{\nabla} g|_0^2 \xi(x)^2\, dw_0 + c_0. \tag{59}$$

Since

$$\int_\Omega |\widetilde{\nabla}_k g_{ij}(x,0)|_0^2 \xi(x)^2\, dw_0 = 0, \tag{60}$$

integrating (59) from 0 to T yields

$$\int_\Omega |\widetilde{\nabla}_k g_{ij}(x,T)|_0^2 \xi(x)^2\, dw_0 + \frac{1}{4}\int_0^T \int_\Omega |\widetilde{\nabla}\widetilde{\nabla} g|_0^2 \xi(x)^2\, dw_0\, dt \le c_0,$$

$$\int_0^T \int_\Omega |\widetilde{\nabla}\widetilde{\nabla} g|_0^2 \xi(x)^2\, dw_0\, dt \le c_0.$$

But on $B(x_0,\gamma)$, $\xi(x) \equiv 1$; thus

$$\int_0^T \int_{B(x_0,\gamma)} |\widetilde{\nabla}\widetilde{\nabla} g|_0^2\, dw_0\, dt \le c_0, \tag{61}$$

which completes the proof of the lemma.

Lemma 6.3. *We still have the following inequalities*:

$$\int_0^T \int_{B(x_0,\gamma)} |\widetilde{\nabla}\widetilde{\nabla} g|^2\, dw_t\, dt \le c_0(n,k_0,\gamma), \tag{62}$$

$$\int_0^T \int_{B(x_0,\gamma)} |\nabla\widetilde{\nabla} g|^2\, dw_t\, dt \le c_0(n,k_0,\gamma). \tag{63}$$

Proof. Using (45) we get

$$\begin{aligned} &|\widetilde{\nabla}\widetilde{\nabla} g|^2 \le 16|\widetilde{\nabla}\widetilde{\nabla} g|_0^2 \\ &dw_t \le 2^{n/2} dw_0 \end{aligned} \quad \text{on } M \times [0,T]; \tag{64}$$

thus

$$\int_0^T \int_{B(x_0,\gamma)} |\widetilde{\nabla}\widetilde{\nabla} g|^2\, dw_t\, dt \le 2^{n/2+4}\int_0^T \int_{B(x_0,\gamma)} |\widetilde{\nabla}\widetilde{\nabla} g|_0^2\, dw_0\, dt.$$

From (61) we know that (62) is true. But one has

$$\nabla\widetilde{\nabla}g = \widetilde{\nabla}\widetilde{\nabla}g + g^{-1} * \widetilde{\nabla}g * \widetilde{\nabla}g,$$

$$|\nabla\widetilde{\nabla}g|^2 \le 2|\widetilde{\nabla}\widetilde{\nabla}g|^2 + 2|g^{-1} * \widetilde{\nabla}g * \widetilde{\nabla}g|^2. \tag{65}$$

From (45) and (65) we get

$$|\nabla\widetilde{\nabla}g|^2 \le 2|\widetilde{\nabla}\widetilde{\nabla}g|^2 + c_0, \tag{66}$$

which together with (62) shows that (63) is true.

Lemma 6.4. *For any $x_0 \in M$, $t \in [0, T]$, and $0 < \gamma < +\infty$ we have*

$$\int_{B(x_0,\gamma)} \{|R_{ijkl}(x,t)|^2 + |\nabla_i V_j|^2\}\, dw_t \le c_0(n, k_0, \gamma),$$

where $0 < c_0(n, k_0, \gamma) < +\infty$ depends only on n, k_0, and γ.

Proof. Suppose $\xi(x) \in C_0^\infty(M)$ is the function satisfying (40) and (41), and let $\Omega = B(x_0, \gamma+1)$. Since $|R_{ijkl}|^2 = g^{i\alpha}g^{j\beta}g^{k\gamma}g^{l\delta}R_{ijkl}R_{\alpha\beta\gamma\delta}$, we have

$$\frac{\partial}{\partial t}|R_{ijkl}|^2 = 2R_{ijkl}\frac{\partial}{\partial t}R_{ijkl} + \mathrm{Rm} * \mathrm{Rm} * \frac{\partial}{\partial t}g^{-1} * g^{-3}, \tag{67}$$

where we have assumed $g_{ij} = \delta_{ij}$ at one point. Thus

(68)

$$\begin{aligned}
\int_\Omega |R_{ijkl}(x,t)|^2\xi(x)^2\, dw_t
&= \int_\Omega |R_{ijkl}(x,0)|_0^2\xi(x)^2\, dw_0 \\
&\quad + \int_0^t \frac{\partial}{\partial t}\int_\Omega |R_{ijkl}(x,t)|^2\xi(x)^2\, dw_t\, dt \\
&= \int_\Omega |\widetilde{R}_{ijkl}(x)|_0^2\xi(x)^2\, dw_0 \\
&\quad + \int_0^t\int_\Omega \left(2R_{ijkl}\frac{\partial}{\partial t}R_{ijkl} + g^{-3} * \mathrm{Rm} * \mathrm{Rm} * \frac{\partial}{\partial t}g^{-1}\right)\xi(x)^2\, dw_t\, dt \\
&\quad + \int_0^t\int_\Omega |R_{ijkl}|^2\xi(x)^2 \cdot \frac{\partial}{\partial t}\, dw_t\, dt.
\end{aligned}$$

Since $|\widetilde{R}_{ijkl}(x)|_0^2 \le k_0$, we have

$$\begin{aligned}
\int_\Omega |\widetilde{R}_{ijkl}(x)|_0^2\xi(x)^2\, dw_0 &\le k_0\int_\Omega \xi(x)^2\, dw_0 \\
&\le k_0\int_{B(x_0,\gamma+1)} dw_0 \le c_0.
\end{aligned} \tag{69}$$

From (39) it follows that

$$\frac{\partial}{\partial t} dw_t = (g^{-2} * \mathrm{Rm} + g^{-1} * \nabla V)\, dw_t, \tag{70}$$

$$\begin{aligned} &\int_0^t \int_\Omega |R_{ijkl}|^2 \xi(x)^2 \cdot \frac{\partial}{\partial t} dw_t\, dt \\ &\quad = \int_0^t \int_\Omega g^{-4} * \mathrm{Rm} * \mathrm{Rm} * (g^{-2} * \mathrm{Rm} + g^{-1} * \nabla V) \xi(x)^2\, dw_t\, dt. \end{aligned} \tag{71}$$

Using (30), (31) and Lemma 6.1 we get

$$\frac{\partial}{\partial t} g^{-1} = g^{-3} * \mathrm{Rm} + g^{-2} * \nabla V, \tag{72}$$

$$\begin{aligned} &\int_0^t \int_\Omega g^{-3} * \mathrm{Rm} * \mathrm{Rm} * \frac{\partial}{\partial t} g^{-1} \cdot \xi(x)^2\, dw_t\, dt \\ &\quad = \int_0^t \int_\Omega g^{-4} * \mathrm{Rm} * \mathrm{Rm} * \{g^{-2} * \mathrm{Rm} + g^{-1} * \nabla V\} \xi(x)^2\, dw_t\, dt, \end{aligned} \tag{73}$$

$$\begin{aligned} 2 &\int_0^t \int_\Omega R_{ijkl} \frac{\partial}{\partial t} R_{ijkl} \cdot \xi(x)^2\, dw_t\, dt \\ &= 2 \int_0^t \int_\Omega R_{ijkl} \cdot \Delta R_{ijkl} \cdot \xi(x)^2\, dw_t\, dt \\ &\quad + \int_0^t \int_\Omega \{g^{-6} * \mathrm{Rm} * \mathrm{Rm} * \mathrm{Rm} + g^{-5} * \mathrm{Rm} * \mathrm{Rm} * \nabla V \\ &\qquad + g^{-5} * V * \mathrm{Rm} * \nabla \mathrm{Rm}\} \xi(x)^2\, dw_t\, dt. \end{aligned} \tag{74}$$

From (68), (69), (71), (73), and (74) it follows that

$$\begin{aligned} &\int_\Omega |R_{ijkl}(x,t)|^2 \xi(x)^2\, dw_t \\ &\quad \le c_0 + 2 \int_0^t \int_\Omega R_{ijkl} \cdot \Delta R_{ijkl} \cdot \xi(x)^2\, dw_t\, dt \\ &\qquad + \int_0^t \int_\Omega \{g^{-6} * \mathrm{Rm} * \mathrm{Rm} * \mathrm{Rm} + g^{-5} * \mathrm{Rm} * \mathrm{Rm} * \nabla V \\ &\qquad\quad + g^{-5} * V * \mathrm{Rm} * \nabla \mathrm{Rm}\} \xi(x)^2\, dw_t\, dt. \end{aligned} \tag{75}$$

Integrating by parts yields

$$
\begin{aligned}
(76)\qquad & 2\int_0^t \int_\Omega R_{ijkl}\cdot \Delta R_{ijkl}\cdot \xi(x)^2\,dw_t\,dt \\
&= 2\int_0^t \int_\Omega R_{ijkl}\cdot g^{\alpha\beta}\nabla_\alpha\nabla_\beta R_{ijkl}\cdot \xi(x)^2\,dw_t\,dt \\
&= -2\int_0^t \int_\Omega g^{\alpha\beta}\nabla_\beta R_{ijkl}\cdot \nabla_\alpha[R_{ijkl}\cdot \xi(x)^2]\,dw_t\,dt \\
&= -2\int_0^t \int_\Omega |\nabla R_{ijkl}|^2\xi(x)^2\,dw_t\,dt \\
&\quad + \int_0^t \int_\Omega g^{-5} * \mathrm{Rm} * \nabla\mathrm{Rm} * \xi(x) * \nabla\xi(x)\,dw_t\,dt.
\end{aligned}
$$

But since $\xi(x)\in C_0^\infty(M)$ is a function, we have

$$\nabla\xi(x) = \widetilde{\nabla}\xi(x),$$

and by (40),

$$
\begin{aligned}
(77)\qquad & |\nabla\xi(x)| = |\widetilde{\nabla}\xi(x)| \le \sqrt{2}|\widetilde{\nabla}\xi(x)|_0 \le 12, \\
& \int_0^t \int_\Omega g^{-5} * \mathrm{Rm} * \nabla\mathrm{Rm} * \xi(x) * \nabla\xi(x)\,dw_t\,dt \\
&\quad \le c_0\int_0^t \int_\Omega |\mathrm{Rm}|\cdot|\nabla\mathrm{Rm}|\cdot\xi(x)\,dw_t\,dt \\
&\quad \le \frac{1}{2}\int_0^t \int_\Omega |\nabla\mathrm{Rm}|^2\xi(x)^2\,dw_t\,dt + c_0\int_0^t \int_\Omega |\mathrm{Rm}|^2\,dw_t\,dt.
\end{aligned}
$$

Substituting this into (76) gives

$$
\begin{aligned}
(78)\qquad & 2\int_0^t \int_\Omega R_{ijkl}\cdot \Delta R_{ijkl}\cdot \xi(x)^2\,dw_t\,dt \\
&\quad \le -\frac{3}{2}\int_0^t \int_\Omega |\nabla\mathrm{Rm}|^2\xi(x)^2\,dw_t\,dt + c_0\int_0^t \int_\Omega |\mathrm{Rm}|^2\,dw_t\,dt.
\end{aligned}
$$

Now we use (32) and (45) to get

$$(79)\qquad |V_i|_0^2 \le c_0,\quad |V_i|^2 \le 2|V_i|_0^2 \le c_0,\quad \text{on } M\times[0,T],$$

so that

(80)
$$\begin{aligned}
&\int_0^t \int_\Omega g^{-5} * V * \mathrm{Rm} * \nabla\mathrm{Rm} \cdot \xi(x)^2 \, dw_t \, dt \\
&\quad \le c_0 \int_0^t \int_\Omega |\mathrm{Rm}| \cdot |\nabla \mathrm{Rm}| \cdot \xi(x)^2 \, dw_t \, dt \\
&\quad \le \frac{1}{4} \int_0^t \int_\Omega |\nabla \mathrm{Rm}|^2 \xi(x)^2 \, dw_t \, dt + c_0 \int_0^t \int_\Omega |\mathrm{Rm}|^2 \xi(x)^2 \, dw_t \, dt \\
&\quad \le \frac{1}{4} \int_0^t \int_\Omega |\nabla \mathrm{Rm}|^2 \xi(x)^2 \, dw_t \, dt + c_0 \int_0^t \int_\Omega |\mathrm{Rm}|^2 \, dw_t \, dt,
\end{aligned}$$

where we have used (41) and (45) again.

Integrating by parts yields

$$\begin{aligned}
&\int_0^t \int_\Omega g^{-5} * \mathrm{Rm} * \mathrm{Rm} * \nabla V \cdot \xi(x)^2 \, dw_t \, dt \\
&\quad = -\int_0^t \int_\Omega g^{-5} * V * \nabla[\mathrm{Rm} * \mathrm{Rm} * \xi(x)^2] \, dw_t \, dt \\
&\quad = \int_0^t \int_\Omega g^{-5} * V * [\mathrm{Rm} * \nabla \mathrm{Rm} * \xi(x)^2 \\
&\qquad\qquad + \mathrm{Rm} * \mathrm{Rm} * \xi(x) * \nabla \xi(x)] \, dw_t \, dt.
\end{aligned}$$

By (41), (45), (77), and (79) we get

(81)
$$\begin{aligned}
&\int_0^t \int_\Omega g^{-5} * \mathrm{Rm} * \mathrm{Rm} * \nabla V \cdot \xi(x)^2 \, dw_t \, dt \\
&\quad \le c_0 \int_0^t \int_\Omega |\mathrm{Rm}| \cdot |\nabla \mathrm{Rm}| \cdot \xi(x) \, dw_t \, dt \\
&\qquad + c_0 \int_0^t \int_\Omega |\mathrm{Rm}|^2 \, dw_t \, dt \\
&\quad \le \frac{1}{8} \int_0^t \int_\Omega |\nabla \mathrm{Rm}|^2 \xi(x)^2 \, dw_t \, dt + c_0 \int_0^t \int_\Omega |\mathrm{Rm}|^2 \, dw_t \, dt.
\end{aligned}$$

Substituting (78), (80), and (81) into (75) gives

(82)
$$\begin{aligned}
\int_\Omega |R_{ijkl}(x,t)|^2 \xi(x)^2 \, dw_t \le& -\frac{9}{8} \int_0^t \int_\Omega |\nabla \mathrm{Rm}|^2 \xi(x)^2 \, dw_t \, dt \\
&+ \int_0^t \int_\Omega g^{-6} * \mathrm{Rm} * \mathrm{Rm} * \mathrm{Rm} \cdot \xi(x)^2 \, dw_t \, dt \\
&+ c_0 \int_0^t \int_\Omega |\mathrm{Rm}|^2 \, dw_t \, dt + c_0.
\end{aligned}$$

By means of (37) of §2 it is easy to see that

$$R_{ijkl} = g_{pi}\tilde{g}^{pq}\tilde{R}_{qjkl} + \tilde{\nabla}\tilde{\nabla}g + g^{-1} * \tilde{\nabla}g * \tilde{\nabla}g;$$

thus

$$\mathrm{Rm} = \widetilde{\mathrm{Rm}} * \tilde{g}^{-1} * g + \tilde{\nabla}\tilde{\nabla}\tilde{g} + g^{-1} * \tilde{\nabla}g * \tilde{\nabla}g. \tag{83}$$

Since $\nabla\tilde{\nabla}g = \tilde{\nabla}\tilde{\nabla}g + g^{-1} * \tilde{\nabla}g * \tilde{\nabla}g$, we have

$$\mathrm{Rm} = \widetilde{\mathrm{Rm}} * \tilde{g}^{-1} * g + \nabla\tilde{\nabla}g + g^{-1} * \tilde{\nabla}g * \tilde{\nabla}g, \tag{84}$$

$$\begin{aligned}
&\int_0^t \int_\Omega g^{-6} * \mathrm{Rm} * \mathrm{Rm} * \mathrm{Rm} \cdot \xi(x)^2 \, dw_t \, dt \\
&\quad = \int_0^t \int_\Omega g^{-6} * \mathrm{Rm} * \mathrm{Rm} * (\widetilde{\mathrm{Rm}} * \tilde{g}^{-1} * g + \nabla\tilde{\nabla}g \\
&\qquad + g^{-1} * \tilde{\nabla}g * \tilde{\nabla}g)\xi(x)^2 \, dw_t \, dt.
\end{aligned} \tag{85}$$

By using (41), (45), and $|\widetilde{\mathrm{Rm}}|_0^2 \le k_0$ we find

$$\begin{aligned}
&\int_0^t \int_\Omega g^{-6} * \mathrm{Rm} * \mathrm{Rm} * (\widetilde{\mathrm{Rm}} * \tilde{g}^{-1} * g + g^{-1} * \tilde{\nabla}g * \tilde{\nabla}g)\xi(x)^2 \, dw_t \, dt \\
&\qquad \le c_0 \int_0^t \int_\Omega |\mathrm{Rm}|^2 \, dw_t \, dt.
\end{aligned} \tag{86}$$

Integrating by parts yields

$$\begin{aligned}
&\int_0^t \int_\Omega g^{-6} * \mathrm{Rm} * \mathrm{Rm} * \nabla\tilde{\nabla}g \cdot \xi(x)^2 \, dw_t \, dt \\
&\quad = -\int_0^t \int_\Omega g^{-6} * \tilde{\nabla}g * \nabla(\mathrm{Rm} * \mathrm{Rm} * \xi(x)^2) \, dw_t \, dt, \\
&\int_0^t \int_\Omega g^{-6} * \mathrm{Rm} * \mathrm{Rm} * \nabla\tilde{\nabla}g \cdot \xi(x)^2 \, dw_t \, dt \\
&\quad = \int_0^t \int_\Omega g^{-6} * \tilde{\nabla}g * (\mathrm{Rm} * \nabla\mathrm{Rm} * \xi(x)^2 \\
&\qquad + \mathrm{Rm} * \mathrm{Rm} * \xi(x) * \nabla\xi(x)) \, dw_t \, dt.
\end{aligned}$$

Using (41), (45), and (77) we get

$$\begin{aligned}
&\int_0^t \int_\Omega g^{-6} * \mathrm{Rm} * \mathrm{Rm} * \nabla\tilde{\nabla}g \cdot \xi(x)^2 \, dw_t \, dt \\
&\quad \le c_0 \int_0^t \int_\Omega |\mathrm{Rm}| \cdot |\nabla\mathrm{Rm}| \cdot \xi(x) \, dw_t \, dt + c_0 \int_0^t \int_\Omega |\mathrm{Rm}|^2 \, dw_t \, dt \\
&\quad \le \frac{1}{8} \int_0^t \int_\Omega |\nabla\mathrm{Rm}|^2 \xi(x)^2 \, dw_t \, dt + c_0 \int_0^t \int_\Omega |\mathrm{Rm}|^2 \, dw_t \, dt.
\end{aligned} \tag{87}$$

Then it follows that

$$\begin{aligned}\int_0^t \int_\Omega g^{-6} * \mathrm{Rm} * \mathrm{Rm} * \mathrm{Rm} \cdot \xi(x)^2 \, dw_t \, dt \\ \leq \frac{1}{8} \int_0^t \int_\Omega |\nabla \mathrm{Rm}|^2 \xi(x)^2 \, dw_t \, dt + c_0 \int_0^t \int_\Omega |\mathrm{Rm}|^2 \, dw_t \, dt,\end{aligned} \tag{88}$$

from (85), (86), and (87), and that

$$\begin{aligned}\int_\Omega |R_{ijkl}(x,t)|^2 \xi(x)^2 \, dw_t \leq -\int_0^t \int_\Omega |\nabla \mathrm{Rm}|^2 \xi(x)^2 \, dw_t \, dt \\ + c_0 \int_0^t \int_\Omega |\mathrm{Rm}|^2 \, dw_t \, dt + c_0\end{aligned} \tag{89}$$

from (82) and (88). Using (45), $|\widetilde{\mathrm{Rm}}|_0^2 \leq k_0$, and (84) we get

$$|\mathrm{Rm}|^2 \leq c_0 + |\nabla \widetilde{\nabla} g|^2 \cdot c_0; \tag{90}$$

thus

$$\begin{aligned}\int_0^t \int_\Omega |\mathrm{Rm}|^2 \, dw_t \, dt &\leq c_0 \int_0^T \int_\Omega dw_t \, dt \\ &\quad + c_0 \int_0^T \int_\Omega |\nabla \widetilde{\nabla} g|^2 \, dw_t \, dt \\ &\leq c_0 + c_0 \int_0^T \int_{B(x_0, \gamma+1)} |\nabla \widetilde{\nabla} g|^2 \, dw_t \, dt.\end{aligned} \tag{91}$$

Use of Lemma 6.3 with γ replaced by $\gamma + 1$ yields

$$\int_0^T \int_{B(x_0, \gamma+1)} |\nabla \widetilde{\nabla} g|^2 \, dw_t \, dt \leq c_0,$$

which together with (91) implies

$$\int_0^t \int_\Omega |\mathrm{Rm}|^2 \, dw_t \, dt \leq c_0. \tag{92}$$

Substituting (92) into (89), we get

$$\int_\Omega |R_{ijkl}(x,t)|^2 \xi(x)^2 \, dw_t \leq -\int_0^t \int_\Omega |\nabla \mathrm{Rm}|^2 \xi(x)^2 \, dw_t \, dt + c_0. \tag{93}$$

Since

$$\nabla_i V_j = \frac{\partial}{\partial x^i} V_j - \Gamma_{ij}^k V_k, \tag{94}$$

we have

$$\frac{\partial}{\partial t} \nabla_i V_j = \frac{\partial}{\partial x^i} \left(\frac{\partial}{\partial t} V_j \right) - \Gamma_{ij}^k \frac{\partial}{\partial t} V_k - V_k \frac{\partial}{\partial t} \Gamma_{ij}^k.$$

Suppose (22) is true at one point. Then

$$\frac{\partial}{\partial t}\nabla_i V_j = \nabla_i\left(\frac{\partial}{\partial t}V_j\right) - V_k\frac{\partial}{\partial t}\Gamma_{ij}^k, \tag{95}$$

$$\begin{aligned}\frac{\partial}{\partial t}\Gamma_{ij}^k &= \frac{1}{2}g^{kl}\left[\frac{\partial}{\partial x^i}\left(\frac{\partial}{\partial t}g_{jl}\right) + \frac{\partial}{\partial x^j}\left(\frac{\partial}{\partial t}g_{il}\right) - \frac{\partial}{\partial x^l}\left(\frac{\partial}{\partial t}g_{ij}\right)\right]\\ &= \frac{1}{2}g^{kl}\left[\nabla_i\left(\frac{\partial}{\partial t}g_{jl}\right) + \nabla_j\left(\frac{\partial}{\partial t}g_{il}\right) - \nabla_l\left(\frac{\partial}{\partial t}g_{ij}\right)\right],\end{aligned} \tag{96}$$

and therefore

$$\frac{\partial}{\partial t}\Gamma_{ij}^k = g^{-1} * \nabla\left(\frac{\partial g}{\partial t}\right).$$

From (31) it follows that

$$\begin{aligned}\frac{\partial}{\partial t}\Gamma_{ij}^k &= g^{-1} * \nabla(g^{-1} * \mathrm{Rm} + \nabla V),\\ \frac{\partial}{\partial t}\Gamma_{ij}^k &= g^{-2} * \nabla\mathrm{Rm} + g^{-1} * \nabla\nabla V.\end{aligned} \tag{97}$$

Substituting (97) into (95) yields

$$\frac{\partial}{\partial t}\nabla_i V_j = \nabla_i\left(\frac{\partial}{\partial t}V_j\right) + g^{-2} * V * \nabla\mathrm{Rm} + g^{-1} * V * \nabla\nabla V. \tag{98}$$

Using (32) and (37) we know that

$$\begin{aligned}\frac{\partial}{\partial t}\nabla_i V_j &= \nabla_i(\Delta V_j + g^{-3} * \tilde{\nabla} g * \mathrm{Rm} + g^{-2} * \tilde{\nabla} g * \nabla V)\\ &\quad + g^{-3} * \tilde{\nabla} g * \nabla\mathrm{Rm} + g^{-2} * \tilde{\nabla} g * \nabla V,\\ \frac{\partial}{\partial t}\nabla_i V_j &= \Delta(\nabla_i V_j) + g^{-3} * \nabla\tilde{\nabla} g * \mathrm{Rm} + g^{-3} * \tilde{\nabla} g * \nabla\mathrm{Rm}\\ &\quad + g^{-2} * \nabla\tilde{\nabla} g * \nabla V + g^{-2} * \tilde{\nabla} g * \nabla\nabla V,\end{aligned} \tag{99}$$

where we have used the interchange formula of two covariant derivatives to claim that

$$\nabla_i(\Delta V_j) = \Delta(\nabla_i V_j) + g^{-3} * \nabla\tilde{\nabla} g * \mathrm{Rm} + g^{-3} * \tilde{\nabla} g * \nabla\mathrm{Rm}.$$

From the definition we have

$$|\nabla_i V_j|^2 = g^{-2} * \nabla V * \nabla V;$$

thus

$$\frac{\partial}{\partial t}|\nabla_i V_j|^2 = 2\nabla_i V_j \cdot \frac{\partial}{\partial t}\nabla_i V_j + g^{-1} * \frac{\partial g^{-1}}{\partial t} * \nabla V * \nabla V. \tag{100}$$

Using (30) and (31) we get

$$\frac{\partial}{\partial t}|\nabla_i V_j|^2 = 2\nabla_i V_j \cdot \frac{\partial}{\partial t}\nabla_i V_j + g^{-4} * \mathrm{Rm} * \nabla V * \nabla V + g^{-3} * \nabla V * \nabla V * \nabla V,$$

(101)
$$\int_\Omega |\nabla_i V_j|^2 \xi(x)^2\, dw_t = \int_\Omega |\nabla_i V_j|_0^2 \xi(x)^2\, dw_0 + \int_0^t \frac{\partial}{\partial t}\int_\Omega |\nabla_i V_j|^2 \xi(x)^2\, dw_t\, dt.$$

Since $\nabla_i V_j \equiv 0$ at the time $t = 0$,

$$\int_\Omega |\nabla_i V_j|_0^2 \xi(x)^2\, dw_0 = 0,$$

(102)
$$\int_\Omega |\nabla_i V_j|^2 \xi(x)^2\, dw_t = \int_0^t \int_\Omega \frac{\partial}{\partial t}|\nabla_i V_j|^2 \cdot \xi(x)^2\, dw_t\, dt + \int_0^t \int_\Omega |\nabla_i V_j|^2 \xi(x)^2 \cdot \frac{\partial}{\partial t} dw_t\, dt.$$

Substituting (70), (99), and (101) into (102) gives

(103)
$$\begin{aligned}\int_\Omega &|\nabla_i V_j|^2 \xi(x)^2\, dw_t \\ &= 2\int_0^t \int_\Omega \nabla_i V_j \cdot \Delta(\nabla_i V_j) \cdot \xi(x)^2\, dw_t\, dt \\ &\quad + \int_0^t \int_\Omega (g^{-5} * \nabla\tilde{\nabla} g * \mathrm{Rm} * \nabla V + g^{-5} * \tilde{\nabla} g * \nabla \mathrm{Rm} * \nabla V \\ &\qquad + g^{-4} * \nabla\tilde{\nabla} g * \nabla V * \nabla V + g^{-4} * \tilde{\nabla} g * \nabla\nabla V * \nabla V \\ &\qquad + g^{-4} * \mathrm{Rm} * \nabla V * \nabla V + g^{-3} * \nabla V * \nabla V * \nabla V)\xi(x)^2\, dw_t\, dt.\end{aligned}$$

By integrating by parts, we get

$$\begin{aligned}2\int_0^t &\int_\Omega \nabla_i V_j \cdot \Delta(\nabla_i V_j) \cdot \xi(x)^2\, dw_t\, dt \\ &= 2\int_0^t \int_\Omega \nabla_i V_j \cdot g^{\alpha\beta}\nabla_\alpha \nabla_\beta(\nabla_i V_j) \cdot \xi(x)^2\, dw_t\, dt \\ &= -2\int_0^t \int_\Omega \nabla_\beta(\nabla_i V_j) \cdot \nabla_\alpha\{g^{\alpha\beta}\nabla_i V_j \cdot \xi(x)^2\}\, dw_t\, dt \\ &= -2\int_0^t \int_\Omega |\nabla\nabla V|^2 \xi(x)^2\, dw_t\, dt \\ &\quad + \int_0^t \int_\Omega g^{-3} * \nabla\nabla V * \nabla V * \xi(x) * \nabla\xi(x)\, dw_t\, dt.\end{aligned}$$

Using (77) and (45) yields

$$(104)\quad \begin{aligned} 2\int_0^t\int_\Omega \nabla_i V_j\cdot\Delta(\nabla_i V_j)\cdot\xi(x)^2\,dw_t\,dt &\le -2\int_0^t\int_\Omega |\nabla\nabla V|^2\xi(x)^2\,dw_t\,dt\\ &\quad + c_0\int_0^t\int_\Omega |\nabla\nabla V|\cdot|\nabla V|\cdot\xi(x)\,dw_t\,dt,\\ 2\int_0^t\int_\Omega \nabla_i V_j\cdot\Delta(\nabla_i V_j)\cdot\xi(x)^2\,dw_t\,dt &\le -\frac{15}{8}\int_0^t\int_\Omega |\nabla\nabla V|^2\xi(x)^2\,dw_t\,dt\\ &\quad + c_0\int_0^t\int_\Omega |\nabla V|^2\,dw_t\,dt, \end{aligned}$$

which together with (103) implies

$$(105)\quad \begin{aligned} \int_\Omega |\nabla_i V_j|^2\xi(x)^2\,dw_t &\le -\frac{15}{8}\int_0^t\int_\Omega |\nabla\nabla V|^2\xi(x)^2\,dw_t\,dt\\ &\quad + \int_0^t\int_\Omega (g^{-5}*\nabla\tilde{\nabla}g*\mathrm{Rm}*\nabla V + g^{-5}*\tilde{\nabla}g*\nabla\mathrm{Rm}*\nabla V\\ &\quad + g^{-4}*\nabla\tilde{\nabla}g*\nabla V*\nabla V + g^{-4}*\tilde{\nabla}g*\nabla\nabla V*\nabla V\\ &\quad + g^{-4}*\mathrm{Rm}*\nabla V*\nabla V + g^{-3}*\nabla V*\nabla V*\nabla V)\xi(x)^2\,dw_t\,dt\\ &\quad + c_0\int_0^t\int_\Omega |\nabla V|^2\,dw_t\,dt. \end{aligned}$$

Using (41) and (45) we get

$$(106)\quad \begin{aligned} \int_0^t\int_\Omega (g^{-5}*\tilde{\nabla}g*\nabla\mathrm{Rm}*\nabla V + g^{-4}*\tilde{\nabla}g*\nabla\nabla V*\nabla V)\cdot\xi(x)^2\,dw_t\,dt &\le c_0\int_0^t\int_\Omega (|\nabla\mathrm{Rm}|\cdot|\nabla V| + |\nabla\nabla V|\cdot|\nabla V|)\xi(x)^2\,dw_t\,dt\\ &\le c_0\int_0^t\int_\Omega (|\nabla\mathrm{Rm}|\cdot|\nabla V| + |\nabla\nabla V|\cdot|\nabla V|)\xi(x)\,dw_t\,dt\\ &\le \frac{1}{8}\int_0^t\int_\Omega (|\nabla\mathrm{Rm}|^2 + |\nabla\nabla V|^2)\xi(x)^2\,dw_t\,dt\\ &\quad + c_0\int_0^t\int_\Omega |\nabla V|^2\,dw_t\,dt. \end{aligned}$$

Integrating by parts yields

$$\begin{aligned}
&\int_0^t \int_\Omega g^{-5} * \nabla\tilde{\nabla} g * \mathrm{Rm} * \nabla V \cdot \xi(x)^2\, dw_t\, dt \\
&\quad = -\int_0^t \int_\Omega g^{-5} * \tilde{\nabla} g * \nabla[\mathrm{Rm} * \nabla V \cdot \xi(x)^2]\, dw_t\, dt \\
&\quad = \int_0^t \int_\Omega g^{-5} * \tilde{\nabla} g * [\nabla \mathrm{Rm} * \nabla V * \xi(x)^2 + \mathrm{Rm} * \nabla\nabla V * \xi(x)^2 \\
&\qquad\qquad + \mathrm{Rm} * \nabla V * \xi(x) * \nabla\xi(x)]\, dw_t\, dt,
\end{aligned}$$

which together with (41), (45) and (77) gives

(107)

$$\begin{aligned}
&\int_0^t \int_\Omega g^{-5} * \nabla\tilde{\nabla} g * \mathrm{Rm} * \nabla V \cdot \xi(x)^2\, dw_t\, dt \\
&\quad \le c_0 \int_0^t \int_\Omega (|\nabla \mathrm{Rm}| \cdot |\nabla V| + |\mathrm{Rm}| \cdot |\nabla\nabla V| + |\mathrm{Rm}| \cdot |\nabla V|)\xi(x)\, dw_t\, dt \\
&\quad \le \frac{1}{8} \int_0^t \int_\Omega (|\nabla \mathrm{Rm}|^2 + |\nabla\nabla V|^2)\xi(x)^2\, dw_t\, dt \\
&\qquad + c_0 \int_0^t \int_\Omega |\nabla V|^2\, dw_t\, dt + c_0 \int_0^t \int_\Omega |\mathrm{Rm}|^2\, dw_t\, dt.
\end{aligned}$$

From (32) it follows that

$$\nabla V = g^{-1} * \nabla\tilde{\nabla} g; \tag{108}$$

thus $g^{-3} * \nabla V * \nabla V * \nabla V = g^{-4} * \nabla\tilde{\nabla} g * \nabla V * \nabla V$. Substituting this and (84) into (105) yields

(109)

$$\begin{aligned}
&\int_0^t \int_\Omega (g^{-4} * \nabla\tilde{\nabla} g * \nabla V * \nabla V + g^{-4} * \mathrm{Rm} * \nabla V * \nabla V \\
&\qquad\qquad + g^{-3} * \nabla V * \nabla V * \nabla V)\xi(x)^2\, dw_t\, dt \\
&\quad = \int_0^t \int_\Omega (g^{-4} * \nabla\tilde{\nabla} g * \nabla V * \nabla V + \widetilde{\mathrm{Rm}} * \tilde{g}^{-1} * g * g^{-4} * \nabla V * \nabla V \\
&\qquad\qquad + g^{-5} * \tilde{\nabla} g * \tilde{\nabla} g * \nabla V * \nabla V)\xi(x)^2\, dw_t\, dt.
\end{aligned}$$

Using (41), (45), and $|\widetilde{\mathrm{Rm}}|_0^2 \le k_0$ we get

$$\begin{aligned}
&\int_0^t \int_\Omega (\widetilde{\mathrm{Rm}} * \tilde{g}^{-1} * g * g^{-4} * \nabla V * \nabla V \\
&\qquad + g^{-5} * \tilde{\nabla} g * \tilde{\nabla} g * \nabla V * \nabla V)\xi(x)^2\, dw_t\, dt \\
&\quad \le c_0 \int_0^t \int_\Omega |\nabla V|^2\, dw_t\, dt.
\end{aligned} \tag{110}$$

By integrating by parts, we find

$$\begin{aligned}\int_0^t \int_\Omega g^{-4} * \nabla\widetilde{\nabla} g * \nabla V * \nabla V * \xi(x)^2\, dw_t\, dt \\ = -\int_0^t \int_\Omega g^{-4} * \widetilde{\nabla} g * \nabla[\nabla V * \nabla V * \xi(x)^2]\, dw_t\, dt \\ = \int_0^t \int_\Omega g^{-4} * \widetilde{\nabla} g * (\nabla V * \nabla\nabla V * \xi(x)^2 + \\ + \nabla V * \nabla V * \xi(x) * \nabla\xi(x))\, dw_t\, dt,\end{aligned}$$

and therefore, in consequence of (41), (45), and (77),

$$\begin{aligned}\int_0^t \int_\Omega g^{-4} * \nabla\widetilde{\nabla} g * \nabla V * \nabla V * \xi(x)^2\, dw_t\, dt \\ \leq c_0 \int_0^t \int_\Omega (|\nabla V| \cdot |\nabla\nabla V| + |\nabla V|^2) \xi(x)\, dw_t\, dt \\ \leq \frac{1}{8} \int_0^t \int_\Omega |\nabla\nabla V|^2 \xi(x)^2\, dw_t\, dt + c_0 \int_0^t \int_\Omega |\nabla V|^2\, dw_t\, dt. \end{aligned} \tag{111}$$

Substituting (110) and (111) into (109) gives

$$\begin{aligned}\int_0^t \int_\Omega (g^{-4} * \nabla\widetilde{\nabla} g * \nabla V * \nabla V + g^{-4} * \mathrm{Rm} * \nabla V * \nabla V \\ + g^{-3} * \nabla V * \nabla V * \nabla V) \xi(x)^2\, dw_t\, dt \\ \leq \frac{1}{8} \int_0^t \int_\Omega |\nabla\nabla V|^2 \xi(x)^2\, dw_t\, dt + c_0 \int_0^t \int_\Omega |\nabla V|^2\, dw_t\, dt,\end{aligned} \tag{112}$$

and substituting (106), (107), and (112) into (105) gives

$$\begin{aligned}\int_\Omega |\nabla_i V_j|^2 \xi(x)^2\, dw_t \\ \leq -\frac{3}{2} \int_0^t \int_\Omega |\nabla\nabla V|^2 \xi(x)^2\, dw_t\, dt + \frac{1}{4} \int_0^t \int_\Omega |\nabla \mathrm{Rm}|^2 \xi(x)^2\, dw_t\, dt \\ + c_0 \int_0^t \int_\Omega |\mathrm{Rm}|^2\, dw_t\, dt + c_0 \int_0^t \int_\Omega |\nabla V|^2\, dw_t\, dt.\end{aligned} \tag{113}$$

If we replace γ by $\gamma + 1$ in Lemma 6.3, from (63) it follows that

$$\int_0^T \int_\Omega |\nabla\widetilde{\nabla} g|^2\, dw_t\, dt \leq c_0. \tag{114}$$

Using (108) and (114) we know that

$$\int_0^T \int_\Omega |\nabla V|^2\, dw_t\, dt \leq c_0. \tag{115}$$

Substituting (92) and (115) into (113), we have

$$\tag{116} \begin{aligned} \int_\Omega |\nabla_i V_j|^2 \xi(x)^2\, dw_t \le & -\frac{3}{2}\int_0^t \int_\Omega |\nabla\nabla V|^2 \xi(x)^2\, dw_t\, dt \\ & + \frac{1}{4}\int_0^t \int_\Omega |\nabla \mathrm{Rm}|^2 \xi(x)^2\, dw_t\, dt + c_0, \end{aligned}$$

which together with (93) yields

$$\tag{117} \begin{aligned} &\int_\Omega (|R_{ijkl}(x,t)|^2 + |\nabla_i V_j|^2)\xi(x)^2\, dw_t \\ &\quad + \frac{3}{4}\int_0^t \int_\Omega (|\nabla \mathrm{Rm}|^2 + |\nabla\nabla V|^2)\xi(x)^2\, dw_t\, dt \le c_0. \end{aligned}$$

Since $\xi(x) \equiv 1$ on $B(x_0, \gamma)$, from (117) it follows that

$$\tag{118} \max_{0\le t\le T} \int_{B(x_0,\gamma)} (|R_{ijkl}(x,t)|^2 + |\nabla_i V_j|^2)\, dw_t \le c_0,$$

and hence Lemma 6.4.

If we let $t = T$, from (117) we get

$$\tag{119} \int_0^T \int_{B(x_0,\gamma)} \{|\nabla \mathrm{Rm}|^2 + |\nabla\nabla V|^2\}\, dw_t\, dt \le c_0(n, k_0, \gamma),$$

where $0 < c_0(n, k_0, \gamma) < +\infty$ depends only on n, k_0, and γ.

Lemma 6.5. *For any $x_0 \in M$, $0 < \gamma < +\infty$, and integer $m \ge 1$, we have*

$$\int_0^T \int_{B(x_0,\gamma)} [|R_{ijkl}(x,t)|^2 + |\nabla_i V_j|^2]^{m-1} |\nabla\tilde{\nabla} g|^2\, dw_t\, dt \le c(n, m, k_0, \gamma),$$

$$\max_{0\le t\le T} \int_{B(x_0,\gamma)} [|R_{ijkl}(x,t)|^2 + |\nabla_i V_j|^2]^m\, dw_t \le c(n, m, k_0, \gamma),$$

$$\begin{aligned} &\int_0^T \int_{B(x_0,\gamma)} [|R_{ijkl}(x,t)|^2 + |\nabla_i V_j|^2]^{m-1} \cdot [|\nabla \mathrm{Rm}|^2 + |\nabla\nabla V|^2]\, dw_t\, dt \\ &\quad \le c(n, m, k_0, \gamma), \end{aligned}$$

where $0 < c(n, m, k_0, \gamma) < +\infty$ depends only on n, m, k_0, and γ.

Proof. We prove this lemma by induction. In the case $m = 1$ from Lemma 6.4, Lemma 6.3 and (119) it follows that Lemma 6.5 is true.

Suppose for $s = 1, 2, \cdots, m-1$ we have

(120)

$$\int_0^T \int_{B(x_0,\gamma)} [|R_{ijkl}(x,t)|^2 + |\nabla_i V_j|^2]^{s-1} |\nabla \tilde{\nabla} g|^2 \, dw_t \, dt \le c(n,s,k_0,\gamma),$$

$$\max_{0 \le t \le T} \int_{B(x_0,\gamma)} [|R_{ijkl}(x,t)|^2 + |\nabla_i V_j|^2]^s \, dw_t \le c(n,s,k_0,\gamma),$$

$$\int_0^T \int_{B(x_0,\gamma)} [|R_{ijkl}(x,t)|^2 + |\nabla_i V_j|^2]^{s-1} (|\nabla \mathrm{Rm}|^2 + |\nabla\nabla V|^2) \, dw_t \, dt \le c(n,s,k_0,\gamma)$$

for any $x_0 \in M$ and $0 < \gamma < +\infty$.

In the case $s = m$, suppose $\xi(x) \in C_0^\infty(M)$ is the function defined by (40) and (41), and let $\Omega = B(x_0, \gamma + 1)$.

Using the induction hypothesis (120) and the same arguments as in the proof of Lemma 6.3 and (117) we get

(121)

$$\int_0^T \int_\Omega [|R_{ijkl}(x,t)|^2 + |\nabla_i V_j|^2]^{m-1} |\nabla \tilde{\nabla} g|^2 \xi(x)^2 \, dw_t \, dt \le c(n,m,k_0,\gamma),$$

$$\int_\Omega [|R_{ijkl}(x,t)|^2 + |\nabla_i V_j|^2]^m \xi(x)^2 \, dw_t + \frac{3}{4} \int_0^t \int_\Omega [|R_{ijkl}(x,t)|^2 + |\nabla_i V_j|^2]^{m-1} \cdot [|\nabla \mathrm{Rm}|^2 + |\nabla\nabla V|^2] \xi(x)^2 \, dw_t \, dt \le c(n,m,k_0,\gamma)$$

for all $t \in [0, T]$. Since $\xi(x) \equiv 1$ on $B(x_0, \gamma)$, we have

$$\int_0^T \int_{B(x_0,\gamma)} [|R_{ijkl}(x,t)|^2 + |\nabla_i V_j|^2]^{m-1} |\nabla \tilde{\nabla} g|^2 \, dw_t \, dt \le c(n,m,k_0,\gamma),$$

(122)

$$\max_{0 \le t \le T} \int_{B(x_0,\gamma)} [|R_{ijkl}(x,t)|^2 + |\nabla_i V_j|^2]^m \, dw_t \le c(n,m,k_0,\gamma),$$

$$\int_0^T \int_{B(x_0,\gamma)} [|R_{ijkl}(x,t)|^2 + |\nabla_i V_j|^2]^{m-1} \cdot (|\nabla \mathrm{Rm}|^2 + |\nabla\nabla V|^2) \, dw_t \, dt \le c(n,m,k_0,\gamma).$$

Thus the lemma is also true in the case $s = m$.

Theorem 6.6. *There exists a constant $c(n, k_0) > 0$ depending only on n and k_0 such that*

(123)

$$\sup_{M \times [0,T]} |R_{ijkl}(x,t)|^2 \le c(n,k_0), \qquad \sup_{M \times [0,T]} |\nabla_i V_j|^2 \le c(n,k_0).$$

Proof. From Lemma 6.1 we know that

$$\frac{\partial}{\partial t}R_{ijkl} = \Delta R_{ijkl} + g^{-2} * \mathrm{Rm} * \mathrm{Rm} + g^{-1} * V * \nabla \mathrm{Rm} + g^{-1} * \mathrm{Rm} * \nabla V.$$

Since

$$g^{-1} * V * \nabla \mathrm{Rm} = \nabla(g^{-1} * V * \mathrm{Rm}) - g^{-1} * \mathrm{Rm} * \nabla V,$$

$$\frac{\partial}{\partial t}R_{ijkl} = \Delta R_{ijkl} + \nabla(g^{-1} * V * \mathrm{Rm}) + g^{-2} * \mathrm{Rm} * \mathrm{Rm} + g^{-1} * \mathrm{Rm} * \nabla V,$$

which can be written as

$$\frac{\partial}{\partial t}R_{ijkl} = \Delta R_{ijkl} + \nabla P + Q, \tag{124}$$

where $P = g^{-1} * V * \mathrm{Rm}$, and $Q = g^{-2} * \mathrm{Rm} * \mathrm{Rm} + g^{-1} * \mathrm{Rm} * \nabla V$. Using (37) and (98) we get

$$\begin{aligned}\frac{\partial}{\partial t}\nabla_i V_j = {}& \nabla_i(\Delta V_j) + \nabla_i(g^{-3} * \tilde{\nabla} g * \mathrm{Rm} + g^{-2} * \tilde{\nabla} g * \nabla V) \\ & + g^{-2} * V * \nabla \mathrm{Rm} + g^{-1} * V * \nabla\nabla V,\end{aligned} \tag{125}$$

and

$$\nabla_i(\Delta V_j) = \Delta(\nabla_i V_j) + g^{-2} * \mathrm{Rm} * \nabla V + g^{-2} * V * \nabla \mathrm{Rm} \tag{126}$$

by means of the interchange formula of two covariant derivatives. Substituting (126) into (125) yields

$$\begin{aligned}\frac{\partial}{\partial t}\nabla_i V_j = {}& \Delta(\nabla_i V_j) + \nabla_i(g^{-3} * \tilde{\nabla} g * \mathrm{Rm} + g^{-2} * \tilde{\nabla} g * \nabla V) \\ & + g^{-2} * V * \nabla \mathrm{Rm} + g^{-1} * V * \nabla\nabla V + g^{-2} * \mathrm{Rm} * \nabla V.\end{aligned} \tag{127}$$

Since

$$\begin{aligned} g^{-2} * V * \nabla \mathrm{Rm} &= \nabla(g^{-2} * V * \mathrm{Rm}) - g^{-2} * \mathrm{Rm} * \nabla V \\ &= \nabla(g^{-3} * \tilde{\nabla} g * \mathrm{Rm}) - g^{-2} * \mathrm{Rm} * \nabla V, \\ g^{-1} * V * \nabla\nabla V &= \nabla(g^{-1} * V * \nabla V) - g^{-1} * \nabla V * \nabla V \\ &= \nabla(g^{-2} * \tilde{\nabla} g * \nabla V) - g^{-1} * \nabla V * \nabla V,\end{aligned}$$

from (127) it follows that

$$\begin{aligned}\frac{\partial}{\partial t}\nabla_i V_j = {}& \Delta(\nabla_i V_j) + \nabla(g^{-3} * \tilde{\nabla} g * \mathrm{Rm} + g^{-2} * \tilde{\nabla} g * \nabla V) \\ & + g^{-2} * \mathrm{Rm} * \nabla V + g^{-1} * \nabla V * \nabla V,\end{aligned} \tag{128}$$

which can be written as

$$\frac{\partial}{\partial t}\nabla_i V_j = \Delta(\nabla_i V_j) + \nabla F + G,$$

where $F = g^{-3} * \widetilde{\nabla} g * \mathrm{Rm} + g^{-2} * \widetilde{\nabla} g * \nabla V$ and $G = g^{-2} * \mathrm{Rm} * \nabla V + g^{-1} * \nabla V * \nabla V$.

Let $\gamma_0 = \frac{1}{8}(1/k_0)^{1/4}$. For any $x_0 \in M$, from (45), (79), (124), (129), and Lemma 6.5 it follows that for any integer $m \geq 1$ we can find constants $c(n, m, k_0) > 0$ depending only on n, m, and k_0 such that

$$(130)\qquad \begin{aligned} \max_{0 \leq t \leq T} \int_{B(x_0,\gamma_0)} |P|^m \, dw_t \leq c(n, m, k_0), \\ \max_{0 \leq t \leq T} \int_{B(x_0,\gamma_0)} |Q|^m \, dw_t \leq c(n, m, k_0); \end{aligned}$$

$$(131)\qquad \begin{aligned} \max_{0 \leq t \leq T} \int_{B(x_0,\gamma_0)} |F|^m \, dw_t \leq c(n, m, k_0), \\ \max_{0 \leq t \leq T} \int_{B(x_0,\gamma_0)} |G|^m \, dw_t \leq c(n, m, k_0). \end{aligned}$$

If the injectivity radius of M at x_0 satisfies

$$(132)\qquad \operatorname{inj}(x_0) \geq \pi(1/k_0)^{1/4},$$

then the geodesic ball $B(x_0, \gamma_0) \subseteq M$ basically is the same as a ball in Euclidean n-space $\mathbb{R}^n$. Thus using (118), (119), (130), (131), (124), (129), and the same arguments as in the proof of Theorem 8.1 in [4, §8, Chapter III] we know that there exists a constant $c(n, k_0) > 0$ depending only on n and k_0 such that

$$(133)\qquad \begin{aligned} \sup_{B(x_0,\gamma_0/2)\times[0,T]} |R_{ijkl}(x,t)|^2 \leq c(n, k_0), \\ \sup_{B(x_0,\gamma_0/2)\times[0,T]} |\nabla_i V_j|^2 \leq c(n, k_0). \end{aligned}$$

If (132) is not true at x_0, then let

$$(134)\qquad \exp_{x_0} : \widehat{B}(0, \pi(1/k_0)^{1/4}) \to M$$

be the nonsingular map defined in (9) of §5; thus we can pull everything back from M to $\widehat{B}(0, \pi(1/k_0)^{1/4})$ and do the analysis on $\widehat{B}(0, \pi(1/k_0)^{1/4})$.

For any integer $m \geq 1$, similar to Lemma 6.5 we have

$$\int_0^T \int_{\widehat{B}(x_0,\gamma_0+\gamma_0/m)} [|R_{ijkl}(x,t)|^2 + |\nabla_i V_j|^2]^{m-1} |\nabla \widetilde{\nabla} g|^2 \, dw_t \, dt \leq c(n, m, k_0),$$

$$\max_{0\leq t\leq T}\int_{\widehat{B}(x_0,\gamma_0+\gamma_0/m)}[|R_{ijkl}(x,t)|^2+|\nabla_i V_j|^2]^m\,dw_t\leq c(n,m,k_0),$$

$$\int_0^T\int_{\widehat{B}(x_0,\gamma_0+\gamma_0/m)}[|R_{ijkl}(x,t)|^2+|\nabla_i V_j|^2]^{m-1}\cdot(|\nabla\mathrm{Rm}|^2+|\nabla\nabla V|^2)\,dw_t\,dt\leq c(n,m,k_0),$$

where $0<c(n,m,k_0)<+\infty$ depends only on n, m, and k_0. Thus (130) and (131) are also true on $\widehat{B}(x_0,\gamma_0)$. By the same reason as that for (133) we get

$$\sup_{\widehat{B}(x_0,\frac{1}{2}\gamma_0)\times[0,T]}|R_{ijkl}|^2\leq C(n,k_0),\tag{135}$$

$$\sup_{\widehat{B}(x_0,\frac{1}{2}\gamma_0)\times[0,T]}|\nabla_i V_j|^2\leq C(n,k_0).\tag{136}$$

Pushing forward to M from (135) and (136) we know that (133) is also true in the case when (132) does not hold.

But $x_0\in M$ is arbitrary, so from (133) we get

$$|R_{ijkl}(x,t)|^2\leq C(n,k_0),\quad|\nabla_i V_j|^2\leq C(n,k_0)\quad\text{on } M\times[0,T];\tag{137}$$

thus the theorem is true.

Theorem 6.7. *For the constant $T=T(n,k_0)>0$ in* (2) *the unmodified evolution equation*

$$\begin{aligned}&\frac{\partial}{\partial t}\hat{g}_{ij}(x,t)=-2\widehat{R}_{ij}(x,t),\\&\hat{g}_{ij}(x,0)=\tilde{g}_{ij}(x)\quad\forall x\in M\end{aligned}\tag{138}$$

has a smooth solution $\hat{g}_{ij}(x,t)>0$ on $M\times[0,T]$ and satisfies the following estimate:

$$\begin{aligned}&\frac{1}{C_1}\tilde{g}_{ij}(x)\leq\hat{g}_{ij}(x,t)\leq C_1\tilde{g}_{ij}(x),\\&\|\widehat{R}_{ijkl}(x,t)\|^2\leq C_2\quad\text{on } M\times[0,T],\end{aligned}\tag{139}$$

where $0<c_1$, $c_2<+\infty$ are constants depending only on n and k_0, and $\|\ \|^2$ denotes the norm with respect to the metric $\hat{g}_{ij}(x,t)$.

Proof. Suppose $\hat{g}_{\alpha\beta}(y,t)$ is the metric defined on (4). Then

$$\hat{g}_{\alpha\beta}(y,t)=\frac{\partial x^i}{\partial y^\alpha}\frac{\partial x^j}{\partial y^\beta}\,g_{ij}(x,t).\tag{140}$$

From (3) we have

$$\frac{\partial x^k}{\partial y^\alpha}\frac{\partial y^\alpha}{\partial t}=g^{\beta\gamma}[\Gamma^l_{\beta\gamma}-\widetilde{\Gamma}^k_{\beta\gamma}],\qquad y^\alpha(x,0)=x^\alpha.\tag{141}$$

Since by definition

$$g^{\beta\gamma}[\Gamma^k_{\beta\gamma} - \widetilde{\Gamma}^k_{\beta\gamma}] = g^{kl} \cdot V_l,$$

from (141) it follows that

$$\frac{\partial x^k}{\partial t} = g^{kl}(x,t) \cdot V_l(x,t), \qquad y^\alpha(x,0) = x^\alpha. \tag{142}$$

If we replace x by y and y by x, then from (7), (141), and (142) we know that if we define

$$\widehat{g}_{\alpha\beta}(x,t) = \frac{\partial y^i}{\partial x^\alpha}\frac{\partial y^j}{\partial x^\beta} g_{ij}(y,t), \tag{143}$$

where $y^k = y^k(x,t)$ satisfies the quasilinear ordinary differential equation

$$\frac{\partial y^k}{\partial t} = g^{kl}(y,t) \cdot V_l(y,t), \qquad y^k(x,0) = x^k, \tag{144}$$

then the metric $d\hat{s}^2 = \hat{g}_{\alpha\beta}(x,t)\,dx^\alpha dx^\beta > 0$ satisfies the evolution equation

$$\begin{aligned} &\frac{\partial}{\partial t}\hat{g}_{ij}(x,t) = -2\widehat{R}_{ij}(x,t) \quad \text{on } M \times [0,T], \\ &\hat{g}_{ij}(x,0) = \tilde{g}_{ij}(x) \quad \forall x \in M. \end{aligned} \tag{145}$$

Since $w^k(x,t) = g^{kl}(x,t) \cdot V_l(x,t)$ is a smooth vector field on $M \times [0,T]$, from (79) and Theorem 6.6 it follows that

$$|w^k(x,t)|^2 \le c_0(n,k_0), \quad |\nabla_i w^k|^2 \le c_0(n,k_0) \quad \text{on } M \times [0,T]. \tag{146}$$

Thus using the standard theory of ordinary differential equations we know that the system of ordinary differential equations (144) has a unique smooth solution $y^k = y^k(x,t)$ on $M \times [0,T]$. Therefore by (143), $\hat{g}_{\alpha\beta}(x,t) \in c^\infty(M \times [0,T])$ is well defined and satisfies the evolution equation (145).

From (143) we get

$$\widehat{R}_{\alpha\beta\gamma\delta}(x,t) = \frac{\partial y^i}{\partial x^\alpha}\frac{\partial y^j}{\partial x^\beta}\frac{\partial y^k}{\partial x^\gamma}\frac{\partial y^l}{\partial x^\delta} R_{ijkl}(y,t), \tag{147}$$

which together with (143) implies

$$\begin{aligned} \|\widehat{R}_{ijkl}(x,t)\|^2 &= \hat{g}^{i\alpha}\hat{g}^{j\beta}\hat{g}^{k\gamma}\hat{g}^{l\delta}\widehat{R}_{ijkl}(x,t) \cdot \widehat{R}_{\alpha\beta\gamma\delta}(x,t) \\ &= g^{i\alpha}g^{j\beta}g^{k\gamma}g^{l\delta}R_{ijkl}(y,t) \cdot R_{\alpha\beta\gamma\delta}(y,t) \\ &= |R_{ijkl}(y,t)|^2. \end{aligned} \tag{148}$$

A use of Theorem 6.6 gives

$$\|\widehat{R}_{ijkl}(x,t)\|^2 \le c(n,k_0) \quad \text{on } M \times [0,T], \tag{149}$$

or

$$\|\widehat{R}_{ij}(x,t)\|^2 \le n^2 c(n,k_0) \quad \text{on } M \times [0,T],$$

which together with (145) implies

$$\left\|\frac{\partial}{\partial t}\hat{g}_{ij}(x,t)\right\|^2 \le 4n^2 \cdot c(n,k_0). \tag{150}$$

Thus we have

$$-2n\sqrt{c}\hat{g}_{ij} \le \frac{\partial}{\partial t}\hat{g}_{ij} \le 2n\sqrt{c}\hat{g}_{ij} \quad \text{on } M \times [0,T],$$

and therefore

$$e^{-2n\sqrt{c}T}\hat{g}_{ij}(x,0) \le \hat{g}_{ij}(x,t) \le e^{2n\sqrt{c}T}\hat{g}_{ij}(x,0), \qquad 0 \le t \le T.$$

Let $c_1 = e^{2n\sqrt{c}T}$. Then

$$\frac{1}{c_1}\tilde{g}_{ij}(x) \le \hat{g}_{ij}(x,t) \le c_1\tilde{g}_{ij}(x) \quad \text{on } M \times [0,T]. \tag{151}$$

From (149) and (151) we know that the theorem is true.

7. Higher derivatives estimate

In this section we use $g_{ij}(x,t)$ to denote $\hat{g}_{ij}(x,t)$, and $|\ |^2$ to denote $\|\ \|^2$, i.e., $g_{ij}(x,t) > 0$ is a smooth metric on $M \times [0,T]$ and satisfies the following:

$$\begin{aligned} &\frac{\partial}{\partial t}g_{ij}(x,t) = -2R_{ij}(x,t) \quad \text{on } M \times [0,T], \\ &g_{ij}(x,0) = \tilde{g}_{ij}(x) \quad \forall x \in M, \end{aligned} \tag{1}$$

$$\begin{aligned} &\frac{1}{c}\tilde{g}_{ij}(x) \le g_{ij}(x,t) \le c\tilde{g}_{ij}(x), \\ &|R_{ijkl}(x,t)|^2 \le c_0 \quad \text{on } M \times [0,T], \end{aligned} \tag{2}$$

where $T = T(n,k_0) > 0$ and $0 < c,\ c_0 < +\infty$ are constants depending only on n and k_0, and $|\ |^2$ is the norm with respect to $g_{ij}(x,t)$.

From Theorem 6.7 we know that such a solution $g_{ij}(x,t)$ exists; thus to prove Theorem 1.1 we only need to prove the following lemma.

Lemma 7.1. *For any integer $m \ge 1$, there exist constants $c_m > 0$ depending only on n, m, and k_0 such that*

$$|\nabla^m R_{ijkl}(x,t)|^2 \le c_m/t^m \quad \textit{on } M \times [0,T]. \tag{3}$$

Proof. From Theorem 7.1 of [3] we have

$$\begin{aligned}\frac{\partial}{\partial t}R_{ijkl} = \Delta R_{ijkl} &+ 2(B_{ijkl} - B_{ijlk} - B_{iljk} + B_{ikjl}) \\ &- g^{pq}(R_{pjkl}R_{qi} + R_{ipkl}R_{qj} + R_{ijpl}R_{qk} + R_{ijkp}R_{ql}),\end{aligned}$$

where

$$B_{ijkl} = g^{p\gamma}g^{qs}R_{piqj}R_{\gamma ksl};$$

thus

$$\frac{\partial}{\partial t}R_{ijkl} = \Delta R_{ijkl} + g^{-2} * \mathrm{Rm} * \mathrm{Rm}. \tag{4}$$

From (4) it follows that

$$\begin{aligned}\frac{\partial}{\partial t}|R_{ijkl}|^2 &= 2R_{ijkl}\frac{\partial}{\partial t}R_{ijkl} + g^{-3} * \frac{\partial g^{-1}}{\partial t} * \mathrm{Rm} * \mathrm{Rm} \\ &= 2R_{ijkl}\cdot\Delta R_{ijkl} + g^{-6} * \mathrm{Rm} * \mathrm{Rm} * \mathrm{Rm}, \\ \frac{\partial}{\partial t}|R_{ijkl}|^2 &= \Delta|R_{ijkl}|^2 - 2|\nabla R_{ijkl}|^2 + g^{-6} * \mathrm{Rm} * \mathrm{Rm} * \mathrm{Rm}.\end{aligned} \tag{5}$$

Using (2) we get

$$\frac{\partial}{\partial t}|R_{ijkl}|^2 \le \Delta|R_{ijkl}|^2 - 2|\nabla R_{ijkl}|^2 + \tilde{c}_0, \tag{6}$$

where $0 < \tilde{c}_0 < +\infty$ means some constants depending only on n and k_0; they may not be the same as each other.

Again by (4) we have

$$\frac{\partial}{\partial t}\nabla R_{ijkl} = \Delta(\nabla R_{ijkl}) + g^{-2} * \mathrm{Rm} * \nabla\mathrm{Rm}, \tag{7}$$

$$\begin{aligned}\frac{\partial}{\partial t}|\nabla R_{ijkl}|^2 &= \Delta|\nabla R_{ijkl}|^2 - 2|\nabla^2 R_{ijkl}|^2 \\ &\quad + g^{-7} * \mathrm{Rm} * \nabla\mathrm{Rm} * \nabla\mathrm{Rm}, \\ \frac{\partial}{\partial t}|\nabla R_{ijkl}|^2 &\le \Delta|\nabla R_{ijkl}|^2 - 2|\nabla^2 R_{ijkl}|^2 + \tilde{c}_0|\mathrm{Rm}|\cdot|\nabla\mathrm{Rm}|^2,\end{aligned} \tag{8}$$

which becomes, in consequence of (2),

$$\frac{\partial}{\partial t}|\nabla R_{ijkl}|^2 \le \Delta|\nabla R_{ijkl}|^2 - 2|\nabla^2 R_{ijkl}|^2 + \tilde{c}_0|\nabla\mathrm{Rm}|^2. \tag{9}$$

Suppose $a > 0$ is a constant to be determined later. From (6) it follows that

$$\frac{\partial}{\partial t}(a + |R_{ijkl}|^2) \le \Delta(a + |R_{ijkl}|^2) - 2|\nabla R_{ijkl}|^2 + \tilde{c}_0, \tag{10}$$

which together with (9) yields

$$\begin{aligned}(11)\qquad &\frac{\partial}{\partial t}[(a+|R_{ijkl}|^2)|\nabla R_{ijkl}|^2] \\ &\quad \le \Delta[(a+|R_{ijkl}|^2)|\nabla R_{ijkl}|^2] - 2\nabla_p|R_{ijkl}|^2\cdot\nabla_p|\nabla R_{ijkl}|^2 \\ &\qquad - 2|\nabla R_{ijkl}|^4 + \tilde{c}_0|\nabla R_{ijkl}|^2 - 2(a+|R_{ijkl}|^2)|\nabla^2 R_{ijkl}|^2 \\ &\qquad + \tilde{c}_0(a+|R_{ijkl}|^2)|\nabla \mathrm{Rm}|^2.\end{aligned}$$

Since

$$\begin{aligned}-2\nabla_p|R_{ijkl}|^2\cdot\nabla_p|\nabla R_{ijkl}|^2 &= \mathrm{Rm} * \nabla\mathrm{Rm} * \nabla\mathrm{Rm} * \nabla^2\mathrm{Rm} \\ &\le \tilde{c}_0|\nabla \mathrm{Rm}|^2\cdot|\nabla^2\mathrm{Rm}| \\ &\le 2a|\nabla^2 R_{ijkl}|^2 + \frac{\tilde{c}_0^2}{8a}|\nabla R_{ijkl}|^4,\end{aligned}$$

substituting this into (11) we get

$$\begin{aligned}(12)\qquad \frac{\partial}{\partial t}[(a+|R_{ijkl}|^2)|\nabla R_{ijkl}|^2] &\le \Delta[(a+|R_{ijkl}|^2)|\nabla R_{ijkl}|^2] \\ &\quad - \left(2-\frac{\tilde{c}_0^2}{8a}\right)|\nabla R_{ijkl}|^4 + \tilde{c}_0|\nabla\mathrm{Rm}|^2 \\ &\quad + \tilde{c}_0(a+|R_{ijkl}|^2)|\nabla \mathrm{Rm}|^2.\end{aligned}$$

If we choose

$$(13)\qquad a = \frac{\tilde{c}_0^2}{8} + c_0,$$

where c_0 is the constant in (2) and $\tilde{c}_0$ is the constant in (12), then we have

$$(14)\qquad 2-\frac{\tilde{c}_0^2}{8a}\ge 1, \qquad a \le a+|R_{ijkl}|^2 \le a + c_0 \le 2a.$$

Substituting (14) into (12) gives

$$\begin{aligned}(15)\qquad &\frac{\partial}{\partial t}[(a+|R_{ijkl}|^2)|\nabla R_{ijkl}|^2] \\ &\quad \le \Delta[(a+|R_{ijkl}|^2)|\nabla R_{ijkl}|^2] \\ &\qquad - |\nabla R_{ijkl}|^4 + \tilde{c}_0(1+1/a)(a+|R_{ijkl}|^2)|\nabla R_{ijkl}|^2 \\ &\quad \le \Delta[(a+|R_{ijkl}|^2)|\nabla R_{ijkl}|^2] - \frac{1}{4a^2}(a+|R_{ijkl}|^2)^2|\nabla R_{ijkl}|^4 \\ &\qquad + \tilde{c}_0(1+1/a)(a+|R_{ijkl}|)|\nabla R_{ijkl}|^2 \\ &\quad \le \Delta[(a+|R_{ijkl}|^2)|\nabla R_{ijkl}|^2] - \frac{1}{8a^2}(a+|R_{ijkl}|^2)^2|\nabla R_{ijkl}|^4 \\ &\qquad + \tilde{c}_0(n,k_0,a).\end{aligned}$$

If we let

$$\varphi(x,t) = (a + |R_{ijkl}|^2) \cdot |\nabla R_{ijkl}|^2 \cdot t \quad \text{on } M \times [0,T], \tag{16}$$

then

$$\varphi(x,0) \equiv 0 \quad \forall x \in M. \tag{17}$$

From (15) we know that

$$\begin{aligned} \frac{\partial \varphi}{\partial t} &\le \Delta\varphi - \frac{1}{8a^2 t}\varphi^2 + \tilde{c}_0(n,k_0,a) + \frac{\varphi}{t} \quad \text{on } M \times [0,T], \\ \frac{\partial \varphi}{\partial t} &\le \Delta\varphi + \frac{\varphi}{t}\left[1 + \tilde{c}_0(n,k_0,a)T - \frac{\varphi}{8a^2}\right], \\ \frac{\partial \varphi}{\partial t} &\le \Delta\varphi + \frac{\varphi}{t}(\tilde{c}_1 - \tilde{c}_2\varphi) \quad \text{on } M \times [0,T], \end{aligned} \tag{18}$$

where $0 < \tilde{c}_1, \tilde{c}_2 < +\infty$ depend only on n, k_0, and a.

For any point $x_0 \in M$, by (39), (40), and (45) of §4 we can find a function $\xi(x)$ such that

$$\begin{aligned} &\xi(x) \equiv 1, && x \in B(x_0,1), \\ &\xi(x) \equiv 0, && x \in M \backslash B(x_0,2), \\ &0 \le \xi(x) \le 1, && x \in M; \end{aligned} \tag{19}$$

$$\begin{aligned} &|\tilde{\nabla}\xi(x)|_0^2 \le 4^2\xi(x) \quad \forall x \in M, \\ &\tilde{\nabla}_\alpha\tilde{\nabla}_\beta\xi(x) \ge -\tilde{c}_3(k_0)\tilde{g}_{\alpha\beta}(x), \qquad x \in M, \end{aligned} \tag{20}$$

where $|\ |_0^2$ denotes the norm with respect to the metric $\tilde{g}_{\alpha\beta}$. Consider the function

$$F(x,t) = \xi(x)\varphi(x,t), \qquad (x,t) \in M \times [0,T]. \tag{21}$$

Then from (16), (17), and (19) it follows that

$$\begin{aligned} &F(x,0) \equiv 0, x \in M, \\ &F(x,t) \equiv 0, (x,t) \in (M \backslash B(x_0,2)) \times [0,T], \\ &F(x,t) \ge 0, (x,t) \in M \times [0,T]. \end{aligned} \tag{22}$$

If $F(x,t) \not\equiv 0$, by (22) we know that there exists a point $(x_1,t_1) \in B(x_0,2) \times [0,T]$ such that

$$F(x_1,t_1) = \max_{M \times [0,T]} F(x,t) > 0, \tag{23}$$

which together with (22) implies that

$$t_1 > 0, \tag{24}$$

and therefore that

$$\frac{\partial F}{\partial t}(x_1,t_1) \geq 0, \quad \nabla F(x_1,t_1) = 0, \quad \Delta F(x_1,t_1) \leq 0. \tag{25}$$

Thus

$$\xi(x_1)\frac{\partial \varphi}{\partial t}(x_1,t_1) \geq 0. \tag{26}$$

Since $\xi \geq 0$, from (18) and (26) we get

$$\xi\Delta\varphi + \frac{\xi\varphi}{t_1}(\tilde{c}_1 - \tilde{c}_2\varphi) \geq 0 \quad \text{at } (x_1,t_1). \tag{27}$$

On the other hand, by (25) we have

$$\xi\Delta\varphi + 2g^{\alpha\beta}\nabla_\alpha\xi \cdot \nabla_\beta\varphi + \varphi\Delta\xi \leq 0 \quad \text{at } (x_1,t_1), \tag{28}$$

which together with (27) gives

$$\frac{\xi\varphi}{t_1}(\tilde{c}_2\varphi - \tilde{c}_1) \leq -2g^{\alpha\beta}\nabla_\alpha\xi \cdot \nabla_\beta\varphi - \varphi\Delta\xi. \tag{29}$$

Since $\nabla F(x_1,t_1) = 0$, we have

$$\begin{gathered} \xi \cdot \nabla_\beta\varphi + \varphi \cdot \nabla_\beta\xi = 0, \\ -2g^{\alpha\beta}\nabla_\alpha\xi \cdot \nabla_\beta\varphi = \frac{2\varphi}{\xi} \cdot g^{\alpha\beta}\nabla_\alpha\xi \cdot \nabla_\beta\xi \quad \text{at } (x_1,t_1). \end{gathered} \tag{30}$$

Using (29) and (30) we get

$$\frac{\xi\varphi}{t_1}(\tilde{c}_2\varphi - \tilde{c}_1) \leq \frac{2\varphi}{\xi} \cdot g^{\alpha\beta}\nabla_\alpha\xi \cdot \nabla_\beta\xi - \varphi\Delta\xi. \tag{31}$$

Since $F(x_1,t_1) = \xi(x_1) \cdot \varphi(x_1,t_1) > 0$, from (19) it follows that

$$\xi(x_1) > 0, \qquad \varphi(x_1,t_1) > 0, \tag{32}$$

and since $\xi(x)$ is a function, we have

$$\begin{gathered} \nabla_\alpha\xi = \widetilde{\nabla}_\alpha\xi, \\ g^{\alpha\beta}\nabla_\alpha\xi \cdot \nabla_\beta\xi = g^{\alpha\beta} \cdot \widetilde{\nabla}_\alpha\xi \cdot \widetilde{\nabla}_\beta\xi. \end{gathered} \tag{33}$$

Using (2) and (20) we get

$$g^{\alpha\beta}\nabla_\alpha\xi \cdot \nabla_\beta\xi \leq 16c \cdot \xi(x). \tag{34}$$

Substituting (32) and (34) into (31) we find

$$\frac{\xi\varphi}{t_1}(\tilde{c}_2\varphi - \tilde{c}_1) \leq 32c \cdot \varphi - \varphi\Delta\xi \quad \text{at } (x_1,t_1). \tag{35}$$

We also have

$$\begin{aligned}\Delta\xi &= g^{\alpha\beta}\nabla_\alpha\nabla_\beta\xi = g^{\alpha\beta}\nabla_\alpha\widetilde{\nabla}_\beta\xi\\ &= g^{\alpha\beta}[\widetilde{\nabla}_\alpha\widetilde{\nabla}_\beta\xi - (\Gamma^\gamma_{\alpha\beta} - \widetilde{\Gamma}^\gamma_{\alpha\beta})\widetilde{\nabla}_\gamma\xi],\\ -\Delta\xi &= -g^{\alpha\beta}\widetilde{\nabla}_\alpha\widetilde{\nabla}_\beta\xi + g^{\alpha\beta}(\Gamma^\gamma_{\alpha\beta} - \widetilde{\Gamma}^\gamma_{\alpha\beta})\cdot\widetilde{\nabla}_\gamma\xi.\end{aligned}\tag{36}$$

From (20) and (2) it follows respectively that

$$-g^{\alpha\beta}\widetilde{\nabla}_\alpha\widetilde{\nabla}_\beta\xi \le \tilde{c}_3 g^{\alpha\beta}\cdot\tilde{g}_{\alpha\beta},$$

$$-g^{\alpha\beta}\widetilde{\nabla}_\alpha\widetilde{\nabla}_\beta\xi \le n\tilde{c}_3\cdot c.\tag{37}$$

Substituting (37) into (36) we get

$$-\Delta\xi \le n\tilde{c}_3\cdot c + g^{\alpha\beta}(\Gamma^\gamma_{\alpha\beta} - \widetilde{\Gamma}^\gamma_{\alpha\beta})\cdot\widetilde{\nabla}_\gamma\xi.\tag{38}$$

Since

$$\Gamma^\gamma_{\alpha\beta} = \frac{1}{2}g^{\gamma\delta}\left(\frac{\partial g_{\alpha\delta}}{\partial x^\beta} + \frac{\partial g_{\beta\delta}}{\partial x^\alpha} - \frac{\partial g_{\alpha\beta}}{\partial x^\delta}\right),$$

$$\frac{\partial}{\partial t}\Gamma^\gamma_{\alpha\beta} = \frac{1}{2}g^{\gamma\delta}\left[\nabla_\beta\left(\frac{\partial g_{\alpha\delta}}{\partial t}\right) + \nabla_\alpha\left(\frac{\partial g_{\beta\delta}}{\partial t}\right) - \nabla_\delta\left(\frac{\partial g_{\alpha\beta}}{\partial t}\right)\right].$$

Using (1) we have

$$\frac{\partial}{\partial t}\Gamma^\gamma_{\alpha\beta} = g^{\gamma\delta}(\nabla_\delta R_{\alpha\beta} - \nabla_\alpha R_{\beta\delta} - \nabla_\beta R_{\alpha\delta})\tag{39}$$

We still have

$$|\nabla R_{ij}|^2 \le n^2|\nabla R_{ijkl}|^2.\tag{40}$$

From (14) and (16) it follows that

$$|\nabla R_{ijkl}|^2 \le \varphi/(at),$$

which together with (40) gives

$$|\nabla R_{ij}|^2 \le \frac{n^2}{at}\varphi.\tag{41}$$

Using (39) and (41) we know that

$$\left|\frac{\partial}{\partial t}\Gamma^\gamma_{\alpha\beta}\right| \le 3|\nabla R_{ij}| \le \frac{3n}{\sqrt{at}}\varphi^{1/2}.\tag{42}$$

From (23) we have

$$\begin{aligned}\xi(x_1)\varphi(x_1,t) &= F(x_1,t) \le F(x_1,t_1), \qquad t\in[0,T],\\ \varphi(x_1,t) &\le \frac{F(x_1,t_1)}{\xi(x_1)}, \qquad t\in[0,T],\end{aligned}\tag{43}$$

which together with (42) yields

$$\left|\frac{\partial}{\partial t}\Gamma^{\gamma}_{\alpha\beta}(x_1,t)\right| \le 3n\left(\frac{F(x_1,t_1)}{a\xi(x_1)}\right)^{1/2}\cdot\frac{1}{\sqrt{t}}, \qquad t\in[0,T]. \tag{44}$$

Thus

$$\begin{aligned}\Gamma^{\gamma}_{\alpha\beta}(x_1,t_1)-\widetilde{\Gamma}^{\gamma}_{\alpha\beta}(x_1) &= \Gamma^{\gamma}_{\alpha\beta}(x_1,t_1)-\Gamma^{\gamma}_{\alpha\beta}(x_1,0)\\ &= \int_0^{t_1}\frac{\partial}{\partial t}\Gamma^{\gamma}_{\alpha\beta}(x_1,t)\,dt,\end{aligned}$$

$$\begin{aligned}|\Gamma^{\gamma}_{\alpha\beta}(x_1,t_1)-\widetilde{\Gamma}^{\gamma}_{\alpha\beta}(x_1)| &\le \int_0^{t_1}\left|\frac{\partial}{\partial t}\Gamma^{\gamma}_{\alpha\beta}(x_1,t)\right|\cdot dt\\ &\le \left|\int_0^{T}\left|\frac{\partial}{\partial t}\Gamma^{\gamma}_{\alpha\beta}(x_1,t)\right|\cdot dt\right.\\ &\le 3n\left(\frac{F(x_1,t_1)}{a\xi(x_1)}\right)^{1/2}\int_0^T\frac{dt}{\sqrt{t}}\\ &\le 6n\left(\frac{T}{a}\right)^{1/2}\cdot\left(\frac{F(x_1,t_1)}{\xi(x_1)}\right)^{1/2}.\end{aligned} \tag{45}$$

From (20) we know that

$$|\widetilde{\nabla}\xi(x_1)|_0 \le 4\xi(x_1)^{1/2}. \tag{46}$$

Using (45), (46) and (2) we get

$$g^{\alpha\beta}(\Gamma^{\gamma}_{\alpha\beta}-\widetilde{\Gamma}^{\gamma}_{\alpha\beta})\cdot\widetilde{\nabla}_{\gamma}\xi \le 24n^2\left(\frac{Tc}{a}\right)^{1/2}\cdot F(x_1,t_1)^{1/2}. \tag{47}$$

By means of (38) and (47) we find

$$-\Delta\xi \le n\tilde{c}_3\cdot c+\tilde{c}_4F(x_1,t_1)^{1/2} \quad \text{at } (x_1,t_1), \tag{48}$$

where $\tilde{c}_4 = 24n^2(Tc/a)^{1/2}$.

Using (32), (35) and (48) we get

$$\begin{aligned}\xi\varphi(\tilde{c}_2\varphi-\tilde{c}_1) &\le (32c+n\tilde{c}_3c)t_1\varphi+\tilde{c}_4t_1\varphi F(x_1,t_1)^{1/2} \quad \text{at } (x_1,t_1)\\ &\le (32c+n\tilde{c}_3c)\cdot T\cdot\varphi+\tilde{c}_4\cdot T\cdot\varphi F(x_1,t_1)^{1/2},\end{aligned}$$

$$\xi\varphi(\tilde{c}_2\varphi-\tilde{c}_1) \le \tilde{c}_5\varphi+\tilde{c}_6\varphi F(x_1,t_1)^{1/2} \quad \text{at } (x_1,t_1).$$

By (32) we have

$$\tilde{c}_2(\xi\varphi)^2-\tilde{c}_1\xi^2\varphi \le \tilde{c}_5\xi\varphi+\tilde{c}_6\xi\varphi\cdot F(x_1,t_1)^{1/2} \quad \text{at } (x_1,t_1).$$

But $\xi\varphi = F(x_1,t_1)$, so

$$\tilde{c}_2F(x_1,t_1)^2 \le (\tilde{c}_1\xi(x_1)+\tilde{c}_5)F(x_1,t_1)+\tilde{c}_6F(x_1,t_1)^{3/2}.$$

Since $0 \leq \xi(x_1) \leq 1$,

$$\tilde{c}_2 F(x_1, t_1)^2 \leq \tilde{c}_7 F(x_1, t_1) + \tilde{c}_6 F(x_1, t_1)^{3/2}, \tag{49}$$

where $0 < \tilde{c}_3$, $\tilde{c}_4$, $\tilde{c}_5$, $\tilde{c}_6$, $\tilde{c}_7 < +\infty$ are constants depending only on n, k_0, and a.

Since $\tilde{c}_2 > 0$, from (49) it follows that

$$F(x_1, t_1) \leq \tilde{c}_8(n, k_0, a), \tag{50}$$

which together with (23) gives

$$F(x,t) \leq \tilde{c}_8(n, k_0, a) \quad \text{on } M \times [0,T],$$
$$\xi(x)\varphi(x,t) \leq \tilde{c}_8(n, k_0, a) \quad \text{on } M \times [0,T].$$

Using (19) we get

$$\varphi(x,t) \leq \tilde{c}_8(n, k_0, a) \quad \text{on } B(x_0, 1) \times [0,T].$$

Since $x_0 \in M$ is arbitrary,

$$\varphi(x,t) \leq \tilde{c}_8(n, k_0, a) \quad \text{on } M \times [0,T]. \tag{51}$$

From (16) and (51) it follows that

$$\begin{aligned} a|\nabla R_{ijkl}|^2 \cdot t &\leq \tilde{c}_8 \quad \text{on } M \times [0,T], \\ |\nabla R_{ijkl}|^2 &\leq \tilde{c}_9 / t \quad \text{on } M \times [0,T], \end{aligned} \tag{52}$$

where $0 < \tilde{c}_9 < +\infty$ depends only on n, k_0, and a. By (13) we know that a depends only on n and k_0, and therefore $\tilde{c}_9$ depends only on n and k_0. Hence the lemma is true in the case $m = 1$.

By induction, suppose for $s = 1, 2, \ldots, m-1$ we have

$$|\nabla^s R_{ijkl}|^2 \leq c_s(n, k_0)/t^s \quad \text{on } M \times [0,T]. \tag{53}$$

In the case $s = m \geq 2$, we define a function

$$\psi(x,t) = (a + t^{m-1}|\nabla^{m-1} R_{ijkl}|^2) \cdot |\nabla^m R_{ijkl}|^2 t^m \tag{54}$$

and choose a large enough. Then similarly to (18) we have

$$\frac{\partial \psi}{\partial t} \leq \Delta \psi + \frac{\psi}{t}(\tilde{c}_{10} - \tilde{c}_{11}\psi) \quad \text{on } M \times [0,T], \tag{55}$$

where $0 < \tilde{c}_{10}$, $\tilde{c}_{11} < +\infty$ depend only on n, m, and k_0.

Let Δ_t denote the Laplacian operator of the metric $g_{ij}(x,t)$. Then using (53) and reasoning similar to (48) we can show that

$$-\Delta_t \xi(x) \leq \tilde{c}_{12}(n, k_0) \quad \forall (x,t) \in M \times [0,T], \tag{56}$$

where $\xi(x)$ is the function defined by (19) and (20). Thus similar to (51) from (55) and (56) it follows that

$$\psi(x,t) \le \tilde{c}_{13}(n,m,k_0) \quad \text{on } M \times [0,T], \tag{57}$$

which together with (54) implies

$$|\nabla^m R_{ijkl}|^2 \le c_m(n,k_0)/t^m \quad \text{on } M \times [0,T]. \tag{58}$$

This completes the proof of Lemma 7.1, and hence Theorem 1.1 is true.

8. Remark

In this section we want to generalize Theorem 1.2. In Theorem 1.2 we proved that if (M, ds^2) is a complete noncompact Riemannian manifold with bounded curvature tensor, then one can find a metric $d\tilde{s}^2$ on M, which is equivalent to ds^2 and has bounded curvature tensor and all of the covariant derivatives. Now we want to prove that if the curvature of ds^2 is not bounded but satisfies some growth condition, we can still get some kind of estimate for the covariant derivative of the curvature of $d\tilde{s}^2$.

Suppose (M, ds^2) is an n-dimensional complete noncompact Riemannian manifold with metric

$$ds^2 = g_{ij}(x)dx^i dx^j > 0, \tag{1}$$

and satisfies the curvature growth condition

$$|R_{ijkl}(x)| \le \beta_0[1 + \gamma(x,x_0)]^\alpha \quad \forall x \in M, \tag{2}$$

where $x_0 \in M$ is a fixed point, $\gamma(x,x_0)$ denotes the distance between x_0 and x, and $\beta_0 > 0$, $\alpha \ge 1$ are some constants.

Define a function φ on M as follows:

$$\varphi(x) = [1 + \gamma(x,x_0)^2]^{\alpha/2}, \qquad x \in M. \tag{3}$$

Using curvature condition (2) and the comparison theorem in Riemannian geometry we know that at the smooth points of $\gamma(x,x_0)$ one has

$$\begin{aligned} &|\nabla\gamma(x,x_0)| \le 1, \\ &|\nabla_i\nabla_j\gamma(x,x_0)| \le \frac{\beta_1}{\gamma(x,x_0)} + \beta_1[1+\gamma(x,x_0)]^{\alpha/2}; \end{aligned} \tag{4}$$

thus at the smooth points of $\varphi(x)$ we have

$$\begin{aligned} &|\nabla_i\varphi(x)| \le \beta_2[1+\gamma(x,x_0)]^{\alpha-1}, \\ &|\nabla_i\nabla_j\varphi(x)| \le \beta_3[1+\gamma(x,x_0)]^{3\alpha/2-1} \quad \forall x \in M. \end{aligned} \tag{5}$$

Since $\varphi(x)$ may not be smooth on the whole manifold M, we are going to use the mollifier technique to smooth φ on M. Suppose $\{\theta_k(x)\}$ for $k = 1, 2, 3, \dots$ is a partition of unity on M:

$$\begin{aligned}
&\theta_k \in c^\infty(M), \\
&0 \le \theta_k(x) \le 1 \quad \forall x \in M, \\
&\theta_k(x) \equiv 0 \quad \text{if } \gamma(x, x_0) \ge 2^k + \tfrac{3}{2} \\
&\qquad\qquad\quad \text{or } \gamma(x, x_0) \le 2^{k-1} - \tfrac{3}{2}, \\
&\sum_{k=1}^{\infty} \theta_k(x) \equiv 1 \quad \forall x \in M.
\end{aligned} \tag{6}$$

Then we have

$$\begin{aligned}
&\varphi(x) \equiv \sum_{k=1}^{\infty} \theta_k(x)\varphi(x), \\
&\operatorname{supp}(\theta_k\varphi) \subseteq B\left(x_0, 2^k + \frac{3}{2}\right) \backslash B\left(x_0, 2^{k-1} - \frac{3}{2}\right).
\end{aligned} \tag{7}$$

Since the support of function $\theta_k\varphi$ is contained in a compact subset of M and the injectivity radius of M in that compact subset is bounded away from zero, using the mollifier technique we can find a function $\psi_k \in c^\infty(M)$ such that

$$\operatorname{supp}\psi_k \subseteq B(x_0, 2^k + 2)\backslash B(x_0, 2^{k-1} - 2), \qquad k = 1, 2, 3, \dots,$$

$$\begin{aligned}
&|\psi_k - \theta_k\varphi| \le \left(\tfrac{1}{4}\right)^k, \\
&|\nabla_i(\psi_k - \theta_k\varphi)| \le \left(\tfrac{1}{4}\right)^k, \qquad k = 1, 2, 3, \dots, \\
&|\nabla_i\nabla_j(\psi_k - \theta_k\varphi)| \le \left(\tfrac{1}{4}\right)^k.
\end{aligned} \tag{8}$$

Define

$$\psi(x) = \sum_{k=1}^{\infty} \psi_k(x), \qquad x \in M. \tag{9}$$

Then we have

$$\begin{aligned}
&\psi(x) \in c^\infty(M), \\
&\tfrac{1}{4}[1 + \gamma(x, x_0)]^\alpha \le \psi(x) \le 4[1 + \gamma(x, x_0)]^\alpha, \\
&|\nabla_i\psi(x)| \le \beta_4[1 + \gamma(x, x_0)]^{\alpha - 1}, \\
&|\nabla_i\nabla_j\psi(x)| \le \beta_5[1 + \gamma(x, x_0)]^{3\alpha/2 - 1} \quad \forall x \in M.
\end{aligned} \tag{10}$$

Now define a new metric $d\tilde{s}^2$ on M:

$$d\tilde{s}^2 = \psi(x) ds^2 = \tilde{g}_{ij}(x) dx^i dx^j, \tag{11}$$

where

$$\tilde{g}_{ij}(x) = \psi(x) g_{ij}(x). \tag{12}$$

Then the curvature tensor of $d\tilde{s}^2$ is

(13)

$$\begin{aligned}\widetilde{R}_{ijkl} = \psi R_{ijkl} &+ \frac{1}{2}(g_{jk}\nabla_i\nabla_l\psi - g_{jl}\nabla_i\nabla_k\psi - g_{ik}\nabla_j\nabla_l\psi + g_{il}\nabla_j\nabla_k\psi) \\ &+ \frac{3}{4\psi}(g_{ik}\nabla_j\psi\cdot\nabla_l\psi - g_{jk}\nabla_i\psi\cdot\nabla_l\psi \\ &\qquad + g_{jl}\nabla_i\psi\cdot\nabla_k\psi - g_{il}\nabla_j\psi\cdot\nabla_k\psi) \\ &+ \frac{1}{4\psi}(g_{jk}g_{il} - g_{ik}g_{jl})g^{pq}\nabla_p\psi\cdot\nabla_q\psi.\end{aligned}$$

Using (12) and (13) one gets

$$|\widetilde{R}_{ijkl}| \le \frac{\beta_6}{\psi}|R_{ijkl}| + \frac{\beta_6}{\psi^3}|\nabla_i\psi|^2 + \frac{\beta_6}{\psi^2}|\nabla_i\nabla_j\psi|, \tag{14}$$

where ∇ denotes the covariant derivative with respect to ds^2. From (2) and (10) it follows that

$$\sup_{x\in M}|\widetilde{R}_{ijkl}(x)| \le \beta_7 < +\infty. \tag{15}$$

Thus by Theorem 1.2 we know that there exist a constant $\beta_8 > 0$ and a metric $d\hat{s}^2$ on M,

$$d\hat{s}^2 = \hat{g}_{ij}(x)dx^i dx^j > 0, \tag{16}$$

such that

$$\begin{aligned}&\frac{1}{\beta_8}\tilde{g}_{ij}(x) \le \hat{g}_{ij}(x) \le \beta_8\tilde{g}_{ij}(x), \qquad x\in M, \\ &\sup_{x\in M}|\widehat{\nabla}^k\widehat{\mathrm{Rm}}(x)| \le c_k < +\infty, \qquad k = 0,1,2,3,\dots,\end{aligned} \tag{17}$$

where $\widehat{\mathrm{Rm}}$ denotes the curvature tensor of $\hat{g}_{ij}$, $\widehat{\nabla}$ the covariant derivatives with respect to $\hat{g}_{ij}$, and $\widehat{\nabla}^k$ the kth order covariant derivatives.

Now we define the metric

$$ds_*^2 = g^*_{ij}(x)dx^i dx^j, \qquad g^*_{ij}(x) = \frac{1}{\psi(x)}\hat{g}_{ij}(x), \quad x\in M. \tag{18}$$

From (12), (17), and (18) one has

$$\frac{1}{\beta_8}g_{ij}(x) \le g^*_{ij}(x) \le \beta_8 g_{ij}(x), \tag{19}$$

where Rm* denotes the curvature tensor of ds_*^2. Using the same reasoning as in (14) we get

$$|\mathrm{Rm}^*| \le \beta_9 \psi |\widehat{\mathrm{Rm}}| + \frac{\beta_9}{\psi}|\widehat{\nabla}\psi|^2 + \beta_9 |\widehat{\nabla}_i \widehat{\nabla}_l \psi|. \tag{20}$$

If we differentiate both sides of (13), similar to (14) we get the estimate for the covariant derivatives:

$$\begin{aligned} |\nabla^* \mathrm{Rm}^*| \le \beta_{10} \Big(& \psi^{3/2} |\widehat{\nabla}\widehat{\mathrm{Rm}}| + \psi^{1/2} |\widehat{\nabla}\psi|^2 |\widehat{\mathrm{Rm}}| + \frac{1}{\psi^{3/2}} |\widehat{\nabla}\psi|^3 \\ & + \frac{1}{\psi^{1/2}} |\widehat{\nabla}\psi| \cdot |\widehat{\nabla}_i \widehat{\nabla}_j \psi| + \psi^{1/2} |\widehat{\nabla}_i \widehat{\nabla}_j \widehat{\nabla}_k \psi| \Big). \end{aligned} \tag{21}$$

From (10) and (12) it follows that

$$|\widetilde{\nabla}_i \psi(x)| = \frac{1}{\sqrt{\psi}} |\nabla_i \psi(x)| \le \beta_{11} [1 + \gamma(x, x_0)]^{\alpha/2 - 1}, \tag{22}$$

where $\widetilde{\nabla}$ is the covariant derivative with respect to $d\tilde{s}^2$. Using (17) and (22) we get

$$|\widehat{\nabla}_i \psi(x)| \le \beta_{12} [1 + \gamma(x, x_0)]^{\alpha/2 - 1}. \tag{23}$$

If the second and the third order covariant derivatives of ψ with respect to $d\hat{s}^2$ are not well controlled, we can use the heat equation

$$\frac{\partial}{\partial t}\psi(x,t) = \widehat{\Delta}\psi(x,t), \qquad \psi(x,0) = \psi(x), \tag{24}$$

to deform $\psi(x)$ for a small time interval $[0, \delta]$. Using the estimate arguments derived in the previous sections we can control the second and the third order covariant derivatives of $\psi(x,t)$, and $\psi(x,t)$ still has growth order $[1 + \gamma(x, x_0)]^\alpha$. Thus without loss of generality we can assume that

$$\begin{aligned} & \tfrac{1}{8}[1 + \gamma(x, x_0)]^\alpha \le \psi(x) \le 8[1 + \gamma(x, x_0)]^\alpha, \\ & |\widehat{\nabla}_i \psi(x)| \le \beta_{13} [1 + \gamma(x, x_0)]^\alpha, \\ & |\widehat{\nabla}_i \widehat{\nabla}_j \psi(x)| \le \beta_{13} [1 + \gamma(x, x_0)]^\alpha, \\ & |\widehat{\nabla}_i \widehat{\nabla}_j \widehat{\nabla}_k \psi(x)| \le \beta_{13} [1 + \gamma(x, x_0)]^\alpha. \end{aligned} \tag{25}$$

Of course here we have to use (17), and the fact that all of the covariant derivatives for the curvature of $d\hat{s}^2$ are bounded on M. Substituting (25) into (20) and (21) and using (17) again we find

$$\begin{aligned} & |\mathrm{Rm}^*(x)| \le \beta_{14} [1 + \gamma(x, x_0)]^\alpha, \\ & |\nabla^* \mathrm{Rm}^*(x)| \le \beta_{15} [1 + \gamma(x, x_0)]^{3\alpha/2}, \qquad x \in M. \end{aligned} \tag{26}$$

From (19) and (26) we get the following theorem.

Theorem 8.1. *Suppose M is an n-dimensional complete noncompact Riemannian manifold with metric*

$$ds^2 = g_{ij}(x)dx^i dx^j > 0,$$

and satisfies the curvature growth condition

$$|\mathrm{Rm}(x)| \le \beta_0[1 + \gamma(x, x_0)]^\alpha, \qquad x \in M.$$

Then there exists another metric

$$ds_*^2 = g_{ij}^*(x)dx^i dx^j > 0$$

on M such that

$$\begin{aligned} &\frac{1}{\beta_8} g_{ij}(x) \le g_{ij}^*(x) \le \beta_8 g_{ij}(x), \\ &|\mathrm{Rm}^*(x)| \le \beta_{14}[1 + \gamma(x, x_0)]^\alpha, \qquad x \in M, \\ &|\nabla^* \mathrm{Rm}^*(x)| \le \beta_{15}[1 + \gamma(x, x_0)]^{3\alpha/2}, \end{aligned} \tag{27}$$

where $0 < \beta_8, \beta_{14}, \beta_{15} < +\infty$ *are some constants depending only on n,* α, *and* β_0.

Similarly one can get a control for the higher order covariant derivatives of $\mathrm{Rm}^*(x)$. Furthermore, if the growth of $\mathrm{Rm}(x)$ is larger than $[1 + \gamma(x, x_0)]^\alpha$, then one can still get similar results.

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