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DEFORMATIONS OF SUBALGEBRAS OF LIE ALGEBRAS

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Introduction

In this paper we shall be concerned with "deformations" of subalgebras of Lie algebras. More generally, we are interested in deformations of subalgebras in which the ambient Lie algebra is also allowed to vary. Precisely, we consider the following situation. Let $\mathscr{M}$ be the algebraic set of all Lie algebra multiplications on a finite-dimensional vector space V (taken over $\boldsymbol{R}$ or $\boldsymbol{C}$ for simplicity), and $\Gamma_n(V)$ the Grassmann manifold of all n-dimensional subspaces of V. Let $\mathscr{S}$ be the algebraic subset of $\mathscr{M} \times \Gamma_n(V)$ consisting of all pairs (η, W) such that W is a subalgebra of the Lie algebra (V, η). Let $\mathfrak{g} = (V, \mu)$ be a Lie algebra and $\mathfrak{h}$ a subalgebra of $\mathfrak{g}$. We are interested in geometric properties of $\mathscr{S}$ in a neighborhood of $(\mu, \mathfrak{h})$. Our main result, Theorem 2.5, states that if the Lie algebra cohomology space $H^2(\mathfrak{h}, \mathfrak{g}/\mathfrak{h})$ vanishes, then a neighborhood of $(\mu, \mathfrak{h})$ in $\mathscr{S}$ is isomorphic (as an analytic space) to the product of a neighborhood of μ in $\mathscr{M}$ and an open ball in $\boldsymbol{R}^k$ (or $\boldsymbol{C}^k$), where $k = \dim Z^1(\mathfrak{h}, \mathfrak{g}/\mathfrak{h})$.

As easy consequences of Theorem 2.5, we obtain the results discussed in (a)–(c) below. These results complement and extend the results of two earlier papers [11], [12].

(a) Let $\mathfrak{h} \subset \mathfrak{g}$ be as above. Then $\mathfrak{h}$ is a *weakly stable* subalgebra of $\mathfrak{g}$ if, roughly speaking, for every one-parameter family $(\mathfrak{g}_t) = (V, \mu_t)$ of Lie algebra structures on V with $\mathfrak{g}_0 = \mathfrak{g}$, there exists a one-parameter family $(\mathfrak{h}_t)$ of subspaces of V with $\mathfrak{h}_0 = \mathfrak{h}$ such that $\mathfrak{h}_t$ is a subalgebra of $\mathfrak{g}_t$ for t sufficiently small. It follows from Theorem 2.5 that if $H^2(\mathfrak{h}, \mathfrak{g}/\mathfrak{h}) = 0$, then $\mathfrak{h}$ is weakly stable.

(b) Let $\mathfrak{h} = (U, \eta)$ and $\mathfrak{g} = (V, \mu)$ be Lie algebras, and $\mathscr{N}$ (resp. $\mathscr{M}$) the set of all Lie multiplications on U (resp. V). A homomorphism $\rho: \mathfrak{h} \to \mathfrak{g}$ is *stable* if, for every $\eta' \in \mathscr{N}$ near η and every $\mu' \in \mathscr{M}$ near μ, there exists a homomorphism $\rho': (U, \eta') \to (V, \mu')$ which is near ρ. We show that ρ is stable if $H^2(\mathfrak{h}, \mathfrak{g}) = 0$. If $\mathfrak{h}$ is a subalgebra of $\mathfrak{g}$ and ρ the inclusion map, we obtain a strengthened form of Theorem 6.2 of [11] on stable subalgebras.

(c) Let $\mathfrak{h} \subset \mathfrak{g}$ be Lie algebras. If $(\mathfrak{h}_t)$ is a one-parameter family of subalgebras of $\mathfrak{g}$ with $\mathfrak{h}_0 = \mathfrak{h}$, then it was shown in [12] that the "initial tangent vector" of the family $(\mathfrak{h}_t)$ is an element of $Z^1(\mathfrak{h}, \mathfrak{g}/\mathfrak{h})$. We show that if $H^2(\mathfrak{h}, \mathfrak{g}/\mathfrak{h}) = 0$, then every $\alpha \in Z^1(\mathfrak{h}, \mathfrak{g}/\mathfrak{h})$ occurs as the initial tangent vector of a one-parameter family of subalgebras. In a sense, the elements of $H^2(\mathfrak{h}, \mathfrak{g}/\mathfrak{h})$

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occur as "obstructions" to finding one-parameter families of subalgebras with a given initial tangent vector. (This result was also obtained by A. Nijenhuis [8].)

In a slightly different setting we also obtain a result on "relatively stable" subalgebras of Lie algebras. This result has applications to the study of the variation of isotropy subalgebras for differentiable transformation groups which are discussed in § 10.

Our proofs use only elementary methods, primarily the implicit function theorem. The proofs carry over with no essential changes to the case of subalgebras of associative algebras. Here, Lie algebra cohomology is replaced by the cohomology of associative algebras. Our results are also valid with only minor modifications for Lie and associative algebras over algebraically closed fields.

In conclusion, we would like to point out the remarkable formal analogy between our results and the (much deeper) results of Kodaira on stability and deformations of compact submanifolds of complex-analytic manifolds [4], [5]. Except for the fact that Lie algebra cohomology is replaced by the appropriate sheaf-theoretic cohomology, the statements of many of the main results are almost exactly the same.

1. Preliminaries

Throughout this paper F will denote either the field $\boldsymbol{R}$ of real numbers or the field $\boldsymbol{C}$ of complex numbers. An analytic manifold will be either a real analytic manifold or a complex manifold, depending on whether $F = \boldsymbol{R}$ or $F = \boldsymbol{C}$. Similarly, Lie groups will be either real or complex depending on F. If X is an analytic manifold and $x \in X$, then $T(X, x)$ denotes the tangent space of X at x. If $f\colon X \to Y$ is an analytic map of analytic manifolds, then $df_x\colon T(X, x) \to T(Y, f(x))$ denotes the differential of f at $x \in X$. If X is an analytic submanifold of an analytic manifold Y, and $x \in X$, we shall usually identify $T(X, x)$ with a subspace of $T(Y, x)$. If V is a finite-dimensional vector space over F, and $v \in V$, then $T(V, v)$ is identifield with V in the usual manner.

Analytic spaces and analytic maps of analytic spaces are as in [16]. (The definitions of [16] are given for complex analytic spaces, but they can be carried over with no changes to the real analytic case.)

An algebraic set (over F or an algebraically closed field) is a Zariski-closed subset of an affine space, a projective space, or of a product of such spaces. An algebraic set over F is, in particular, an analytic space.

If V and W are vector spaces, then we denote by $A^m(V, W)$ the vector space of all alternating m-linear maps of $V \times \cdots \times V$ (m times) into W. We set $A(V, W) = \bigoplus_{m \geq 0} A^m(V, W)$.

A representation ρ of a Lie algebra $\mathfrak{g}$ on a vector space W defines on W the structure of a $\mathfrak{g}$-module. If $x \in \mathfrak{g}$ and $w \in W$, we shall often denote $\rho(x) \cdot w$

simply by $x \cdot w$. If W is a $\mathfrak{g}$-module, then we let $C(\mathfrak{g}, W) = \bigoplus_{m \geq 0} C^m(\mathfrak{g}, W)$ denote the usual cochain complex used to compute the cohomology of $\mathfrak{g}$ with coefficients in W; see, e.g., [1, p. 282]. (If V is the underlying vector space of $\mathfrak{g}$, then $C(\mathfrak{g}, W)$ is identical as a graded vector space with $A(V, W)$. However, since we shall frequently consider several different Lie algebra structures on the same vector space V, it is sometimes convenient to distinguish between $C(\mathfrak{g}, W)$ and $A(V, W)$.) As usual, $Z^m(\mathfrak{g}, W)$ (resp. $B^m(\mathfrak{g}, W)$) denotes the space of m-cocycles (resp. coboundaries), and $H^m(\mathfrak{g}, W)$ denotes the m-th cohomology space of $\mathfrak{g}$ with coefficients in W.

Let A be an associative algebra with identity element e over the field K. A representation of the algebra (with identity) $A^e = A \otimes_K A^{op}$ (where A^{op} denotes the opposite algebra of A) on a vector space W determines on W the structure of an A-bimodule. In computing the cohomology of the associative algebra A with coefficients in the A-bimodule W, it is convenient to use the "normalized standard complex" of Cartan and Eilenberg [1, p. 176]. Precisely, we define $C^m(A, W)$ to be the vector space of all m-linear maps φ of $A \times \cdots \times A$ (m times) into W which satisfy the following condition: $\varphi(a_1, \cdots, a_m) = 0$ if there exists an index j such that $a_j = e$. The coboundary operator on $C(A, W) = \bigoplus_{m \geq 0} C^m(A, W)$ is just the usual Hochschild coboundary operator. We denote by $Z^m(A, W)$ (resp. $B^m(A, W)$) the corresponding space of m-cocycles (resp. m-coboundaries), and $H^m(A, W)$ denotes the m-th cohomology space. $H^m(A, W)$ is canonically isomorphic to the usual m-th cohomology space of A with coefficients in W as originally defined by Hochschild.

2. Deformations of Lie algebras and their subalgebras

Let V be a finite-dimensional vector space over F, and $\mathscr{M}$ the set of all Lie algebra multiplications on V; then $\mathscr{M}$ is an algebraic set in $A^2(V, V)$. Let $\mu \in M$, $\mathfrak{g} = (V, \mu)$ be the corresponding Lie algebra, and $\mathfrak{h}$ be an n-dimensional subalgebra of $\mathfrak{g}$ with underlying subspace U. Let $\Gamma_n(V)$ be the Grassmann manifold of n-dimensional subspaces of V. Then $\Gamma_n(V)$ is a projective algebraic variety. Denote by $\mathscr{S}$ the algebraic set in $A^2(V, V) \times \Gamma_n(V)$ consisting of all pairs (η, U') such that $\eta \in \mathscr{M}$ and U' is an n-dimensional subalgebra of the Lie algebra (V, η). We shall be interested in geometric properties of $\mathscr{S}$ in a neighborhood of (μ, U).

Let W be a vector subspace of V such that V is the direct sum of U and W, and $P\colon V \to W$ and $Q\colon V \to U$ the projections corresponding to the direct sum decomposition $V = U \oplus W$. Define a surjective linear map $r\colon A(V, V) \to A(U, W)$ as follows: if $\varphi \in A^m(V, V)$, then $r\varphi \in A^m(U, W)$ is defined by $r\varphi(u_1, \cdots, u_m) = P(\varphi(u_1, \cdots, u_m))$. Similarly, if $\phi \in A^m(U, W)$, we define $s\phi \in A^m(V, V)$ by $s\phi(v_1, \cdots, v_m) = \phi(Qv_1, \cdots, Qv_m)$. Then $s\colon A(U, W) \to A(V, V)$ is a monomorphism, and $r \circ s$ is the identity map on $A(U, W)$. To simplify notation, we shall frequently identity $A^m(U, W)$ with a subspace of

$A^m(V, V)$ by means of the monomorphism s. In particular, $\mathrm{Hom}_F(U, W) = A^1(U, W)$ is identified with a subspace of $\mathrm{Hom}_F(V, V) = A^1(V, V)$. We also consider P and Q as elements of $\mathrm{Hom}_F(V, V)$ in the obvious way.

Let Γ_W be the (Zariski) open subset of $\Gamma_n(V)$ consisting of all n-dimensional subspaces U' such that $U' \cap W = \{0\}$. (Γ_W is an open Schubert cell in $\Gamma_n(V)$.) We denote by 1_V the identity map on V. If $T \in \mathrm{Hom}_F(U, W) \subset \mathrm{Hom}_F(V, V)$, we denote by $\Phi(T)$ the n-dimensional subspace $(1_V + T)(U)$ of V. Then $\Phi(T) \in \Gamma_W$, and $\Phi\colon \mathrm{Hom}_F(U, W) \to \Gamma_W$ is an analytic manifold isomorphism; more precisely, Φ is an isomorphism of algebraic varieties. Let $\Psi\colon \Gamma_W \to \mathrm{Hom}_F(U, W)$ denote the inverse of Φ. Then Ψ is a chart for the analytic manifold $\Gamma_n(V)$.

There is a natural representation of $GL(V)$, the group of vector space automorphisms of V, on $A^2(V, V)$ defined as follows: if $g \in GL(V)$ and $\varphi \in A^2(V, V)$, then $(g\cdot\varphi)(v_1, v_2) = g(\varphi(g^{-1}v_1, g^{-1}v_2))$. The set $\mathscr{M}$ of Lie multiplications on V is stable under the corresponding action of $GL(V)$ on $A^2(V, V)$, and the orbits of $GL(V)$ on $\mathscr{M}$ are just the isomorphism classes of Lie algebra structures on V. Let $g \in GL(V)$ and $\eta \in \mathscr{M}$. Then one checks easily from the definitions that the following conditions are equivalent: (a) $r(g^{-1}\cdot\eta) = 0$; (b) U is a subalgebra of the Lie algebra $(V, g^{-1}\cdot\eta)$; (c) $g(U)$ is a subalgebra of the Lie algebra (V, η).

Now let $T \in \mathrm{Hom}_F(U, W) \subset \mathrm{Hom}_F(V, V)$, and $\eta \in \mathscr{M}$. Since $T^2 = 0$, we have $(1_V + T)(1_V - T) = 1_V$. Thus $(1_V + T) \in GL(V)$ and $(1_V + T)^{-1} = 1_V - T$. It follows from the equivalence of (a), (b) and (c) above that $(\eta, \Phi(T)) \in \mathscr{S}$ (i.e., $\Phi(T)$ is a subalgebra of (V, η)) if and only if $r((1_V - T)\cdot\eta) = 0$. Define $f\colon A^2(V, V) \times A^1(U, W)$ by $f(\varphi, T) = r((1_V - T)\cdot\varphi)$, and set $\mathscr{S}_0 = \{(\varphi, T) \in A^2(V, V) \times A^1(U, W) \mid \varphi \in \mathscr{M} \text{ and } f(\varphi, T) = 0\}$. Then $\mathscr{S}_0$ is an algebraic set in $A^2(V, V) \times A^1(U, W)$. Define $\Theta\colon A^2(V, V) \times A^1(U, W) \to A^2(V, V) \times \Gamma_W$ by $\Theta(\varphi, T) = (\varphi, \Phi(T))$. Then Θ is an isomorphism of analytic manifolds, and maps $\mathscr{S}_0$ onto the open subset $\mathscr{S} \cap (A^2(V, V) \times \Gamma_W)$ of $\mathscr{S}$. Thus the properties of $\mathscr{S}$ in a neighborhood of (μ, U) are the same as those of $\mathscr{S}_0$ in a neighborhood of $(\mu, 0)$.

The adjoint representation of $\mathfrak{h}$ on $\mathfrak{g}$ defines an $\mathfrak{h}$-module structure on $\mathfrak{g}$; $\mathfrak{h}$ is an $\mathfrak{h}$-submodule of $\mathfrak{g}$ and thus there is an induced $\mathfrak{h}$-module structure on the quotient space $\mathfrak{g}/\mathfrak{h}$ (more precisely, on V/U). Let $\pi\colon \mathfrak{g} \to \mathfrak{g}/\mathfrak{h}$ be the canonical projection, and $\beta\colon W \to \mathfrak{g}/\mathfrak{h}$ denote the restriction of π. We use the vector space isomorphism β to transport to W the $\mathfrak{h}$-module structure of $\mathfrak{g}/\mathfrak{h}$. Precisely, if $x \in \mathfrak{h}$ and $w \in W$, then $x\cdot w = \beta^{-1}(\pi([x, w]))$. We have $C^m(\mathfrak{h}, W) = A^m(U, W)$ and $C^m(\mathfrak{g}, \mathfrak{g}) = A^m(V, V)$. Let $\delta_1\colon C(\mathfrak{g}, \mathfrak{g}) \to C(\mathfrak{g}, \mathfrak{g})$, and $\delta_2\colon C(\mathfrak{h}, W) \to C(\mathfrak{h}, W)$ denote the coboundary operators on the complexes $C(\mathfrak{g}, \mathfrak{g})$ and $C(\mathfrak{h}, W)$. One checks easily that $r\colon C(\mathfrak{g}, \mathfrak{g}) \to C(\mathfrak{h}, W)$ is a chain mapping, i.e., $r \circ \delta_1 = \delta_2 \circ r$. We also note that $H^m(\mathfrak{h}, \mathfrak{g}/\mathfrak{h})$ is isomorphic to $H^m(\mathfrak{h}, W)$.

If $\varphi, \psi \in A^2(V, V)$, we define $\varphi \barwedge \psi \in A^3(V, V)$, the "hook product" of φ

and ψ, by

$$\varphi \barwedge \psi(v_1, v_2, v_3) = \varphi(\psi(v_1, v_2, v_3) + \varphi(\psi(v_2, v_3), v_1) + \varphi(\psi(v_3, v_1), v_2) .$$

Then $(\varphi, \psi) \mapsto \varphi \barwedge \psi$ is a bilinear map of $A^2(V, V) \times A^2(V, V)$ into $A^3(V, V)$. Furthermore it follows immediately that $\varphi \in A^2(V, V)$ satisfies the Jacobi identity if and only if $\varphi \barwedge \varphi = 0$. If $\varphi \in C^2(g, g) = A^2(V, V)$, then one checks easily that $\delta_1\varphi = -\mu \barwedge \varphi - \varphi \barwedge \mu$.

Let E be a vector subspace of $C^2(\mathfrak{h}, W)$ which is supplementary to $Z^2(\mathfrak{h}, W)$, and let $\pi_Z\colon C^2(\mathfrak{h}, W) \to Z^2(\mathfrak{h}, W)$ and $\pi_E\colon C^2(\mathfrak{h}, W) \to E$ be the projections corresponding to the direct sum decomposition $C^2(\mathfrak{h}, W) = Z^2(\mathfrak{h}, W) \oplus E$. Similarly, let $C = \boldsymbol{r}^{-1}(Z^2(\mathfrak{h}, W))$, D be a subspace of $C^2(\mathfrak{g}, \mathfrak{g})$ such that $\boldsymbol{r}$ maps D isomorphically onto E, and $\pi_C\colon C^2(\mathfrak{g}, \mathfrak{g}) \to C$ and $\pi_D\colon C^2(\mathfrak{g}, \mathfrak{g}) \to D$ be the projections corresponding to the direct sum decomposition $C^2(\mathfrak{g}, \mathfrak{g}) = C \oplus D$. It follows easily that $\boldsymbol{r} \circ \pi_C = \pi_Z \circ \boldsymbol{r}$ and $\boldsymbol{r} \circ \pi_D = \pi_E \circ \boldsymbol{r}$. We note also that the restriction of $\delta_2 \circ \boldsymbol{r}$ to D is a monomorphism.

If $(\varphi, T) \in A^2(V, V) \times A^1(U, W)$, the equation "$f(\varphi, T) = 0$" is equivalent to the pair of equations "$\pi_Z(f(\varphi, T)) = 0$" and "$\pi_E(f(\varphi, T)) = 0$". The following lemma shows that, for $\varphi \in \mathscr{M}$ and (φ, T) sufficiently $(\mu, 0)$, if (φ, T) satisfies the first of these equations, it automatically satisfies the second.

Lemma 2.1. *There exists an open neighborhood $N(\mu, 0)$ of $(\mu, 0)$ in $A^2(V, V) \times A^1(U, W)$ such that if $(\varphi, T) \in N(\mu, 0)$, $\varphi \in \mathscr{M}$, and $\pi_Z(f(\varphi, T)) = 0$, then $f(\varphi, T) = 0$.*

Proof. Let $(\varphi, T) \in A^2(V, V) \times A^1(U, W)$ be such that $\varphi \in \mathscr{M}$ and $\pi_Z(f(\varphi, T)) = 0$, and set $\alpha = (1_V - T)\cdot\varphi$. Then $f(\varphi, T) = \boldsymbol{r}\alpha$. Since $\varphi \in \mathscr{M}$ and $\mathscr{M}$ is stable under the action of $GL(V)$, it follows that $\alpha \barwedge \alpha = 0$. Hence we have

$$\begin{aligned} 0 = \boldsymbol{r}(\alpha \barwedge \alpha) &= \boldsymbol{r}(\alpha \barwedge (\pi_D\alpha + \pi_C\alpha)) \\ &= \boldsymbol{r}(\alpha \barwedge \pi_D\alpha) + \boldsymbol{r}((\pi_D\alpha + \pi_C\alpha) \barwedge \pi_C\alpha) . \end{aligned} \tag{2.1}$$

One checks easily that if $\gamma \in A^2(V, V)$ is such that $\boldsymbol{r}\gamma = 0$, then $\boldsymbol{r}(\gamma \barwedge \gamma) = 0$. We have

$$0 = \pi_Z(f(\varphi, T)) = \pi_Z(\boldsymbol{r}\alpha) = \boldsymbol{r}(\pi_C\alpha) .$$

Thus $\boldsymbol{r}(\pi_C\alpha \barwedge \pi_C\alpha) = 0$, and consequently (2.1) becomes

$$0 = \boldsymbol{r}(\alpha \barwedge \pi_D\alpha) + \boldsymbol{r}(\pi_D\alpha \barwedge \pi_C\alpha) . \tag{2.2}$$

For every $\gamma \in A^2(V, V)$ we define a linear map $\lambda_\gamma\colon A^2(V, V) \to A^3(U, W)$ by

$$\lambda_\gamma(\psi) = \boldsymbol{r}(\gamma \barwedge \psi + \psi \barwedge \pi_C\gamma) .$$

With this notation, (2.2) becomes

$$0 = \lambda_\alpha(\pi_D\alpha) . \tag{2.3}$$

Since $\mu \in C$, we have $\pi_C \mu = \mu$, and hence

$$\begin{aligned}\lambda_\mu(\phi) &= r(\mu \barwedge \phi + \phi \barwedge \mu) \\ &= -r(\delta_1 \phi) = -\delta_2(r\phi) .\end{aligned}$$

Thus $\lambda_\mu = -\delta_2 \circ r$. Since $\gamma \to \lambda_\gamma$ is a polynomial mapping of $A^2(V, V)$ into $\mathrm{Hom}_F(A^2(V, V), A^3(U, W))$, and the restriction of $\lambda_\mu = -\delta_2 \circ r$ to D is a monomorphism, it follows that there exists a (Zariski) open neighborhood $N(\mu)$ of μ in $A^2(V, V)$ such that, if $\gamma \in N(\mu)$, then the restriction of λ_γ to D is a monomorphism. We choose an open neighborhood $N(\mu, 0)$ of $(\mu, 0)$ in $A^2(V, V) \times A^1(U, W)$ such that if $(\varphi', T') \in N(\mu, 0)$, then $(1_V - T')\cdot\varphi' \in N(\mu)$. Assume now that $(\varphi, T) \in N(\mu, 0)$; thus $\alpha \in N(\mu)$. By (2.3), $\lambda_\alpha(\pi_D\alpha) = 0$. Since the restriction of λ_α to D is a monomorphism, this implies that $\pi_D\alpha = 0$. Thus

$$f(\varphi, T) = r\alpha = r(\pi_D\alpha + \pi_C\alpha) = r(\pi_C\alpha) = \pi_Z(r\alpha) = \pi_Z(f(\varphi, T)) = 0,$$

which proves Lemma 2.1.

Remark. Lemma 2.1 is the key result in the proof of Theorem 2.5. The idea behind the proof is as follows: The relation "$\alpha \barwedge \alpha = 0$" allows us to replace the equation "$f(\varphi, T) = 0$" by the simpler equation "$\pi_Z(f(\varphi, T)) = 0$". For a more detailed discussion of the underlying circle of ideas, see [10].

Lemma 2.2. *The differential*

$$df_{(\mu,0)}: A^2(V, V) \times A^1(U, W) \to A^2(U, W)$$

is given by $df_{(\mu,0)}(\varphi, T) = r\varphi + \delta_2 T$.

Proof. Let $a: GL(V) \to A^2(V, V)$ be defined by $a(g) = g\cdot\mu$. Then it follows easily from the definitions that $da_{1_V}: A^1(V, V) \to A^2(V, V)$ is equal to $-\delta_1$ (see [12, § 7]); it follows easily that the differential at 0 of the map $T \mapsto f(\varphi, T) = r((1_V - T)\cdot\varphi)$ is just the coboundary operator δ_2. The above formula for $df_{(\mu,0)}$ follows easily from this remark.

We now wish to use the implicit function theorem to study $\mathscr{S}_0$ locally. The trouble is that μ may be a singular point of $\mathscr{M}$. Thus it is necessary to replace $\mathscr{M}$ by a larger algebraic set which has μ as a simple point (i.e., is a submanifold in a neighborhood of μ) and which has the "correct" tangent space at μ. This is accomplished by the following (elementary) result, which is proved in [11, § 6.2]:

2.3. There exist an algebraic set $\mathscr{M}_1 \supset \mathscr{M}$ in $A^2(V, V)$ and an open neighborhood $N(\mu)$ of μ in $A^2(V, V)$ such that $\mathscr{M}' = \mathscr{M}_1 \cap N(\mu)$ is a closed analytic submanifold of $N(\mu)$ and $T(\mathscr{M}', \mu) = C$.

Now let $h: \mathscr{M}' \times A^1(U, W)$ denote the map $(\varphi, T) \mapsto \pi_Z(f(\varphi, T))$. Then it follows from Lemma 2.2 that $dh_{(\mu,0)}: C \times A^1(U, W)$ is the map $(\varphi, T) \mapsto \pi_Z(r\varphi) + \delta_2 T$. Since $\pi_Z \circ r$ maps C onto $Z^2(\mathfrak{h}, W)$, $dh_{(\mu,0)}$ is surjective. Thus the implicit function theorem implies that there exists an open neighborhood

$N_1(\mu, 0)$ of $(\mu, 0)$ in $\mathscr{M}' \times A^1(U, W)$ such that $\boldsymbol{h}^{-1}(0) \cap N_1(\mu, 0) = \mathscr{N}$ is a closed analytic submanifold of $N_1(\mu, 0)$ and $T(\mathscr{N}, (\mu, 0))$ is equal to the kernel of $d\boldsymbol{h}_{(\mu,0)}$.

Let $\boldsymbol{q}: \mathscr{N} \to \mathscr{M}'$ denote the restriction to $\mathscr{N}$ of the projection $\mathscr{M}' \times A^1(U, W) \to \mathscr{M}'$.

Lemma 2.4. *If* $H^2(\mathfrak{h}, \mathfrak{g}/\mathfrak{h}) = 0$, *then* $d\boldsymbol{q}_{(\mu,0)}$ *is surjective.*

Proof. Let $\varphi \in C = T(\mathscr{M}', \mu)$. Then $\boldsymbol{r}\varphi \in Z^2(\mathfrak{h}, W)$; in particular $\boldsymbol{r}\varphi = \pi_Z(\boldsymbol{r}\varphi)$. Since $H^2(\mathfrak{h}, W) = 0$, we have $Z^2(\mathfrak{h}, W) = B^2(\mathfrak{h}, W)$. Thus there exists $T \in A^1(U, W)$ such that $\pi_Z(\boldsymbol{r}\varphi) + \delta_2 T = 0$. Hence $(\varphi, T) \in \text{kernel}\,(d\boldsymbol{h}_{(\mu,0)}) = T(\mathscr{N}, (\mu, 0))$. Since $d\boldsymbol{q}_{(\mu,0)}(\varphi, T) = \varphi$, the proof is complete.

We assume for the rest of § 2 that $H^2(\mathfrak{h}, \mathfrak{g}/\mathfrak{h}) = 0$. Let $k = \dim Z^1(\mathfrak{h}, W)$. An easy counting argument, using $H^2(\mathfrak{h}, W) = 0$, shows that the dimension of the kernel of $d\boldsymbol{q}_{(\mu,0)}$ is k. Since $d\boldsymbol{q}_{(\mu,0)}$ is surjective we may apply the implicit function theorem. Thus we see that there exist an open ball $\mathscr{B}$ about 0 in F^k, an open neighborhood $N_2(\mu)$ of μ in $\mathscr{M}'$ and an analytic map $\boldsymbol{u}_0: N_2(\mu) \times \mathscr{B} \to A^1(U, W)$ with $\boldsymbol{u}_0(\mu, 0) = 0$ such that the map $\Omega_0: (\varphi, x) \to (\varphi, \boldsymbol{u}_0(\varphi, x))$ is an analytic manifold isomorphism of $N_2(\mu) \times \mathscr{B}$ onto an open neighborhood $N(\mu, 0)$ of $(\mu, 0)$ in $\mathscr{N}$. We may further assume that $N(\mu, 0)$ satisfies the conditions of Lemma 2.1. If $\varphi \in \mathscr{M} \cap N_2(\mu)$ and $x \in \mathscr{B}$, then it follows from Lemma 2.1 that $(\varphi, \boldsymbol{u}_0(\varphi, x)) \in \mathscr{S}_0$. If we set $N_3(\mu) = N_2(\mu) \cap \mathscr{M}$, then the restriction Ω_1 of Ω_0 to $N_3(\mu) \times \mathscr{B}$ is an analytic space isomorphism of $N_3(\mu) \times \mathscr{B}$ onto neighborhood of $(\mu, 0)$ in $\mathscr{S}_0$. Let $\Omega: N_3(\mu) \times \mathscr{B} \to S$ denote the map $\Theta \circ \Omega_1$. Then Ω is an analytic space isomorphism of $N_3(\mu) \times \mathscr{B}$ onto an open neighborhood of (μ, U) in $\mathscr{S}$.

We summarize the results of this section as follows:

Theorem 2.5. *Let* $\mathfrak{g}$, $\mathfrak{h}$, $\mathscr{M}$ *and* $\mathscr{S}$ *be as above, and* $k = \dim Z^1(\mathfrak{h}, \mathfrak{g}/\mathfrak{h})$. *Assume that* $H^2(\mathfrak{h}, \mathfrak{g}/\mathfrak{h}) = 0$. *Then there exist an open neighborhood* $N_3(\mu)$ *of* μ *in* M, *an open ball* $\mathscr{B}$ *about* 0 *in* F^k *and an analytic map* $\boldsymbol{u}: N_3(\mu) \times \mathscr{B} \to \Gamma_n(V)$ *with* $\boldsymbol{u}(\mu, 0) = U$ *such that the map* $\Omega: (\varphi, x) \to (\varphi, \boldsymbol{u}(\varphi, x))$ *is an isomorphism (of analytic spaces) of* $N_3(\mu) \times \mathscr{B}$ *onto an open neighborhood of* (μ, U) *in* $\mathscr{S}$.

Remark. All of the results of this section are essentially results in (real or complex) algebraic geometry. We have phrased our results in terms of analytic manifolds to make them accessible to readers unfamiliar with algebraic geometry. The case of Lie algebras over algebraically closed fields is discussed in § 4.

3. Preliminary algebraic geometry

With the exception of § 3.1 below, we shall need only elementary results from algebraic geometry. In particular, we shall consider only Zariski-closed subsets of affine spaces, projective spaces, or of products of such spaces. However, since no completely satisfactory reference is available, a few comments on terminology are in order. Our basic references for algebraic geometry are [2] and [15]; the transition from the terminology of [2] to that of [15] is easy. We use the word *algebraic space* (resp. *morphism*) for what is called an *algebraic variety* (resp. *regular map*) in [15]; an irreducible algebraic space is an algebraic variety. In particular, we work over a fixed algebraically closed field and all algebraic spaces are considered as topological spaces, supplied with the Zariski topology. See [2, Chap. 6] for a discussion of tangent spaces and differentials of morphisms (of algebraic varieties); we follow the notation of § 1 for tangent spaces and differentials of morphisms. For an algebraic space X (not necessarily irreducible), $x \in X$ is a *simple* point of X, if x belongs to exactly one irreducible component X_1 of X and is a simple point of X_1. (This is equivalent to the condition that the local ring of X at x be a regular local ring.) In this case we write $T(X, x) = T(X_1, x)$. In particular, if $f\colon X \to Y$ is a morphism of algebraic spaces, and x (resp. $f(x)$) is a simple point of X (resp. Y), then the differential $df_x\colon T(X, x) \to T(Y, f(x))$ is well defined. As in the case of analytic manifolds, if $Y \subset X$ are algebraic varieties and $x \in Y$, then $T(Y, x)$ is often identified with a subspace of $T(X, x)$. If V is a vector space and $x \in V$, then $T(V, x)$ is identified with V.

(As a matter of fact, the definition of the tangent space $T(X, x)$ of an algebraic variety X at a point x given in [2, Chap. 6] carries over with no major changes to the case in which X is not necessarily irreducible. This eliminates the awkward distinction made above between the irreducible case and the general case. See [7, Chap. 3, 4] for a brief discussion of the "correct" definition-given in a much more general setting.)

The following result, due to Chevalley, can be considered as an algebro-geometric analogue of the implicit function theorem.

3.1. Let $f\colon X \to Y$ be a morphism of algebraic spaces, $x \in X$, and $y = f(x)$. Suppose that x (resp. y) is a simple point of X (resp. Y) and that the differential $df_x\colon T(X, x) \to T(Y, y)$ is a surjective map. Then f maps every neighborhood of x in X onto a neighborhood of y in Y. Moreover, if $Z = f^{-1}(y)$, then x is a simple point of Z and $T(Z, x) = \text{kernel } (df_x)$.

For a proof, see [11, Prop. 8.1].

4. The case of algebraically closed fields

Except for the statement concerning the local product structure of a neighborhood of (μ, U) in $\mathcal{S}$, all of the results of § 2 carry over to the case of Lie algebras over algebraically closed fields. We sketch the arguments.

Let V be a finite-dimensional vector space over an algebraically closed field K. Otherwise let the notation be as in § 2. All topological concepts refer to the Zariski topology. The proofs of Lemmas 2.1 and 2.2 carry over with no changes to the case at hand. Using § 3.1 one can show that there exists a closed set $\mathcal{M}_1$ in $A^2(V, V)$ such that $\mathcal{M} \subset \mathcal{M}_1$, μ is a simple point of $\mathcal{M}_1$ and

$T(\mathscr{M}_1, \mu) = C$ (here $C = r^{-1}(Z^2(\mathfrak{h}, W))$). Let h: $\mathscr{M}_1 \times A^1(U, W) \to Z^2(\mathfrak{h}, W)$ denote the morphism $(\varphi, T) \to \pi_Z(f(\varphi, T))$. Then the calculation of $dh_{(\mu,0)}$ in § 2 is also valid here. It follows from § 3.1 that $(\mu, 0)$ is a simple point of $\mathscr{N} = h^{-1}(0)$ and that $T(\mathscr{N}, (\mu, 0))$ is the kernel of $dh_{(\mu,0)}$. Let q: $\mathscr{N} \to \mathscr{M}_1$ denote the restriction to $\mathscr{N}$ of the projection of $\mathscr{M}_1 \times A^1(U, W)$ onto $\mathscr{M}_1$. Then the argument given in Lemma 2.4 shows that the differential $dq_{(\mu,0)}$ is surjective if $H^2(\mathfrak{h}, \mathfrak{g}/\mathfrak{h}) = 0$. In particular, it follows from 3.1 that q maps every neighborhood of $(\mu, 0)$ in $\mathscr{S}_0$ onto a neighborhood of μ in $\mathscr{M}$. Several consequences of this will be discussed in § 6–§ 8.

5. Subalgebras of associative algebras

All of the results of § 2 and § 4 carry over with no essential changes to the case of subalgebras of associative algebras. We shall sketch the details in this section. Associative algebras A are always assumed to have an identity and subalgebras of A are required to contain the identity of A.

If X and Y are vector spaces, we denote by $L^m(X, Y)$ the vector space of all m-linear maps of $X \times \cdots \times X$ (m times) into Y. Now let V be a finite-dimensional vector space over F, and e a non-zero element of V. We are interested in the algebraic set $\mathscr{M}$ of all associative multiplications on V for which e is an identity element. Let X be a vector space over F, U be an n-dimensional subspace of V containing e, and $\mathscr{L}^m(V, X)$ (resp. $\mathscr{L}^m(U, X)$) denote the subspace of $L^m(V, X)$ (resp. $L^m(U, X)$) consisting of all $\varphi \in L^m(V, X)$ (resp. all $\psi \in L^m(U, X)$) which satisfy the following condition: $\varphi(v_1, \cdots, v_m) = 0$ (resp. $\psi(u_1, \cdots, u_m) = 0$) if there exists an index j such that $v_j = e$ (resp. $u_j = e$). We set $L(V, X) = \oplus_{m \geq 0} L^m(V, X)$ and $L(U, X) = \oplus_{m \geq 0} L^m(U, X)$.

Let W be a vector subspace of V such that V is the direct sum of U and W, and let P: $V \to W$ and Q: $V \to U$ be the projections corresponding to the direct sum decomposition $V = U \oplus W$. We define linear maps r: $L^m(V, V) \to L^m(U, W)$ and s: $L^m(U, W) \to L^m(V, V)$ as follows: if $\varphi \in L^m(V, V)$, then $r\varphi(u_1, \cdots, u_m) = P(\varphi(u_1, \cdots, u_m))$; if $\psi \in L^m(U, W)$, then $s\psi(v_1, \cdots, v_m) = \psi(Qv_1, \cdots, Qv_m)$. Then $r \circ s$ is the identity map on $L^m(U, W)$. We note that $r(\mathscr{L}^m(V, V)) \subset \mathscr{L}^m(U, W)$ and $s(\mathscr{L}^m(U, W)) \subset \mathscr{L}^m(V, V)$. We shall often identify $L^m(U, W)$ with the vector subspace $s(L^m(U, W))$ of $L^m(V, V)$ by means of s.

Let $\Gamma_n(V)'$ denote the closed analytic submanifold of $\Gamma_n(V)$ consisting of all n-dimensional subspaces of V which contain e. Let Γ_W be as in § 2, and $\Gamma'_W = \Gamma_W \cap \Gamma_n(V)'$; Γ'_W is an open submanifold of $\Gamma_n(V)'$. Let Φ: $L^1(U, W) \to \Gamma_W$ be defined as in § 2. Then the restriction of Φ to $\mathscr{L}^1(U, W)$ defines an analytic manifold isomorphism Φ': $\mathscr{L}^1(U, W) \to \Gamma'_W$.

Now let

$$\mathscr{L}^2 = \{\varphi \in L^2(V, V) \mid \varphi(e, v) = \varphi(v, e) = v \text{ for every } v \in V\} .$$

Then $\mathscr{L}^2$ is an affine subspace of $L^2(V, V)$. We note that $\boldsymbol{r}$ maps $\mathscr{L}^2$ onto $\mathscr{L}^2(U, W)$. The algebraic set $\mathscr{M}$ of associative multiplications on V with $\boldsymbol{e}$ as identity is included in $\mathscr{L}^2$. Let $\mu \in \mathscr{M}$, and $A = (V, \mu)$ be the corresponding associative algebra. Then $\mathscr{L}^2 = \mu + \mathscr{L}^2(V, V)$; thus $\mathscr{L}^2$ is parallel to $\mathscr{L}^2(V, V)$ and the tangent space $T(\mathscr{L}^2, \mu)$ is (canonically isomorphic to) $\mathscr{L}^2(V, V)$. Let B be an n-dimensional subalgebra of A with underlying subspace U. Then A has a canonical structure of A-bimodule and A/B has a canonical structure of B-bimodule. As in § 2, we transport to W the canonical B-module structure on A/B by means of the canonical vector space isomorphism between W and A/B. It follows from the definitions that $C(A, A) = \mathscr{L}(V, V)$ and $C(B, W) = \mathscr{L}(U, W)$. We denote by $\delta_1\colon C(A, A) \to C(A, A)$ and $\delta_2\colon C(B, W) \to C(B, W)$ the coboundary operators on the complexes $C(A, A)$ and $C(B, W)$.

As in § 2 we denote by $\mathscr{S}$ the algebraic subset of $\mathscr{M} \times \Gamma_n(V)'$ consisting of all pairs (η, U') such that U' is a subalgebra of the associative algebra (V, η).

If $T \in \mathscr{L}^1(U, W)$ and $\varphi \in \mathscr{L}^2$, then one checks easily that $(1_V - T)\cdot\varphi \in \mathscr{L}^2$. Let $f\colon \mathscr{L}^2 \times \mathscr{L}^1(U, W) \to \mathscr{L}^2(U, W)$ be defined by $f(\varphi, T) = \boldsymbol{r}((1_V - T)\cdot\varphi)$. For $\varphi \in \mathscr{M}$, it is easy to verify that $\Phi'(T)$ is a subalgebra of (V, φ) if and only if $f(\varphi, T) = 0$.

If $\varphi, \psi \in L^3(V, V)$, then, following Gerstenhaber [3], we define $\varphi \circ \psi \in L^3(V, V)$, the "composition product" of φ and ψ, by

$$\varphi \circ \psi(v_1, v_2, v_3) = \varphi(\psi(v_1, v_2), v_3) - \varphi(v_1\, \psi(v_2, v_3)) .$$

It follows immediately that $\varphi \in \mathscr{L}^2$ is an associative multiplication on V if and only if $\varphi \circ \varphi = 0$. If $\varphi \in \mathscr{L}^2(V, V) = C^2(A, A)$, then $\delta_1\varphi = -\mu \circ \varphi - \varphi \circ \mu$.

Following § 2, we set $C = \{\varphi \in \mathscr{L}^2(V, V) \mid \boldsymbol{r}\varphi \in Z^2(B, W)\}$, and choose direct sum decompositions $\mathscr{L}^2(V, V) = C \oplus D$ and $\mathscr{L}^2(U, W) = Z^2(B, W) \oplus E$ such that $\boldsymbol{r}$ maps D isomorphically onto E. We denote by π_C, π_D, π_Z and π_E the corresponding projections.

Lemma 5.1. *There exists an open neighborhood $N(\mu, 0)$ of $(\mu, 0)$ in $\mathscr{L}^2 \times \mathscr{L}^1(U, W)$ such that if $\varphi \in \mathscr{M}$, $(\varphi, T) \in N(\mu, 0)$, and $\pi_Z(f(\varphi, T)) = 0$, then $f(\varphi, T) = 0$.*

If one replaces the hook product $\varphi \barwedge \psi$ by the composition product $\varphi \circ \psi$, the proof of Lemma 5.1 is the same as that of Lemma 2.1.

Lemma 5.2. *The differential*

$$df_{(\mu,0)}\colon \mathscr{L}^2(V, V) \times \mathscr{L}^1(U, W) \to \mathscr{L}^2(U, W)$$

is given by $df_{(\mu,0)}(\varphi, T) = \boldsymbol{r}\varphi + \delta_2 T$.

The proof is exactly the same as that of Lemma 2.2.

Lemma 5.3. *There exist an algebraic set $\mathscr{M}_1 \supset \mathscr{M}$ in $\mathscr{L}^2$ and an open neighborhood $N(\mu)$ of μ in $\mathscr{L}^2$ such that $\mathscr{M}' = \mathscr{M}_1 \cap N(\mu)$ is a closed analytic submanifold of $N(\mu)$ and $T(\mathscr{M}', \mu) = C$.*

The proof is essentially the same as that of [11, § 6.2].

Now let $\boldsymbol{h}\colon \mathscr{M}' \times \mathscr{L}^1(U, W) \to Z^2(B, W)$ be the map $(\varphi, T) \mapsto \pi_Z(f(\varphi, T))$. Then, using the implicit function theorem and Lemma 5.2, we see that there exists an open neighborhood $N_1(\mu, 0)$ of $(\mu, 0)$ in $\mathscr{M}' \times \mathscr{L}^1(U, W)$ such that $\mathscr{N} = \boldsymbol{h}^{-1}(0) \cap N_1(\mu, 0)$ is a closed analytic submanifold of $N_1(\mu, 0)$ and $T(\mathscr{N}, (\mu, 0))$ is equal to the kernel of $d\boldsymbol{h}_{(\mu,0)}$. Let $\boldsymbol{q}\colon \mathscr{N} \to \mathscr{M}'$ denote the restriction to $\mathscr{N}$ of the projection $\mathscr{M}' \times \mathscr{L}^1(U. W) \to \mathscr{M}'$. Then the same argument used in the proof of Lemma 2.4 shows that if $H^2(B, A/B) = 0$, then the differential $d\boldsymbol{q}_{(\mu,0)}$ is surjective. Following the reasoning in § 2 we finally obtain:

Theorem 5.4. *Let the notation be as above. Assume that* $H^2(B, A/B) = 0$ *and let* $k = \dim Z^1(B, A/B)$. *Then there exist an open neighborhood* $N(\mu)$ *of* μ *in* $\mathscr{M}$, *an open ball* $\mathscr{B}$ *about* 0 *in* F^k, *and an analytic map* $\boldsymbol{u}\colon N(\mu) \times \mathscr{B} \to \Gamma_n(V)'$ *with* $\boldsymbol{u}(\mu, 0) = U$ *such that the map* $\Omega\colon (\varphi, x) \mapsto (\varphi, \boldsymbol{u}(\varphi, x))$ *is an analytic space isomorphism of* $N(\mu) \times \mathscr{B}$ *onto an open neighborhood of* (μ, U) *in* $\mathscr{S}$.

We remark that the results of this section can be carried over to the case of associative algebras over algebraically closed fields in exactly the same manner as was done in § 4 for the case of Lie algebras. We omit the details.

6. Weakly stable subalgebras

Let V be a finite-dimensional vector space over F. In the case of Lie algebras we let $\mathscr{M}$ be the algebraic set of all Lie multiplications on V. In the case of associative algebras we assume that V has a distinguished element $\boldsymbol{e} \neq 0$ and let $\mathscr{M}$ denote the algebraic set of all associative multiplications on V for which $\boldsymbol{e}$ is an identity element.

Definition 6.1. Let $\mu \in \mathscr{M}$, and $\mathfrak{g}$ (resp. A) be the corresponding Lie (resp, associative) algebra. Let $\mathfrak{h}$ (resp. B) be an n-dimensional subalgebra of $\mathfrak{g}$ (resp. A) with underlying subspace U. Then $\mathfrak{h}$ (resp. B) is a *weakly stable* subalgebra of $\mathfrak{g}$ (resp. A) if there exist an open neighborhood $N(\mu)$ of μ in $\mathscr{M}$ and an analytic map $\boldsymbol{u}$ of $N(\mu)$ into $\Gamma_n(V)$ (resp. $\Gamma_n(V)'$) with $\boldsymbol{u}(\mu) = U$ such that, for every $\varphi \in N(\mu)$, $\boldsymbol{u}(\varphi)$ is subalgebra of the Lie (resp. associative) algebra (V, φ).

Theorem 6.2. *Let* $\mathfrak{g}$ *(resp. A) be a finite-dimensional Lie (resp. associative) algebra over F, and* $\mathfrak{h}$ *(resp. B) a subalgebra of* $\mathfrak{g}$ *(resp. A).*

(a) *If* $H^2(\mathfrak{h}, \mathfrak{g}/\mathfrak{h}) = 0$, *then* $\mathfrak{h}$ *is a weakly stable subalgebra of* $\mathfrak{g}$.

(b) *If* $H^2(B, A/B) = 0$, *then B is a weakly stable subalgebra of A.*

The proof of (a) follows immediately from Theorem 2.5, and that of (b) from Theorem 5.4.

For the case of Lie and associative algebras, over algebraically closed fields, we need to modify our definition of weakly stable subalgebras. Let V be a finite-dimensional vector space over an algebraically closed field K, and $\mathscr{M}$ be as above.

Definition 6.3. Let $\mu \in \mathscr{M}$, $\mathfrak{g}$ (resp. A) be the corresponding Lie (resp. associative) algebra, $\mathfrak{h}$ (resp. B) be a subalgebra of $\mathfrak{g}$ (resp. A) with underlying subspace U. Then $\mathfrak{h}$ (resp. B) is a *weakly stable* subalgebra of $\mathfrak{g}$ (resp. B) if, for every neighborhood $N(U)$ of U in $\Gamma_n(V)$ (resp. $\Gamma_n(V)'$), there exists a neighborhood $N(\mu)$ of μ in $\mathcal{M}$ such that if $\eta \in N(\mu)$, there exists $U_\eta \in N(U)$ such that U_η is a subalgebra of the Lie (resp. associative) algebra (V, η).

With this definition one can again show that if $H^2(h, g/h) = 0$ (resp. $H^2(B, A/B) = 0$), then $\mathfrak{h}$ (resp. B) is a weakly stable subalgebra of $\mathfrak{g}$ (resp. A). The proof for the case of Lie algebras follows from § 3.1 and the results of § 4. The proof for the case of associative algebras follows from § 3.1 and the results indicated at the end of § 5.

Example. Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over F, and $\mathfrak{c}$ a Cartan subalgebra of $\mathfrak{g}$. Then an argument similar to that given in [12, § 12(b)] shows that $H^2(\mathfrak{c}, \mathfrak{g}/\mathfrak{c}) = 0$. Thus $\mathfrak{c}$ is a weakly stable subalgebra of $\mathfrak{g}$.

Remark. Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over F. In [11] we defined a subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ to be *stable*, if (roughly) $\mathfrak{h}$ is a weakly stable subalgebra of $\mathfrak{g}$, and the map $\boldsymbol{u}$: $N(\mu) \to \Gamma_n(V)$ of Definition 6.1 can be chosen so that, for every $\varphi \in N(\mu)$, the subalgebra $\boldsymbol{u}(\varphi)$ of (V, φ) is isomorphic to $\mathfrak{h}$. Theorem 6.1 of [11] states that $\mathfrak{h}$ is a stable subalgebra of $\mathfrak{g}$ if $H^2(\mathfrak{h}, \mathfrak{g}) = 0$. We shall show in § 7 that a stronger form of this theorem is an easy consequence of Theorem 6.2 above.

7. Stable homomorphisms

Let U and W be finite-dimensinoal vector spaces over F, and $\mathcal{M}$ (resp. $\mathcal{N}$) denote the algebraic set of all Lie multiplications on U (resp. W). Let $\mu \in \mathcal{M}$, $\eta \in \mathcal{N}$, $\mathfrak{h} = (U, \mu)$ and $\mathfrak{g} = (W, \eta)$ be the corresponding Lie algebras, and ρ: $\mathfrak{h} \to \mathfrak{g}$ be a homomorphism of Lie algebras.

Definition 7.1. ρ is a *stable homomorphism* if there exist open neighborhoods $N(\mu)$ and $N(\eta)$ of (respectively) μ and η in $\mathcal{M}$ and $\mathcal{N}$ and an analytic map u: $N(\mu) \times N(\eta) \to \operatorname{Hom}_F(U, W)$ with $u(\mu, \eta) = \rho$ such that, for every $(\mu', \eta') \in N(\mu) \times N(\eta)$, $u(\mu', \eta')$: $(U, \mu') \to (W, \eta')$ is a homomorphism of Lie algebras.

Let V be the product vector space $U \times W$. We identify U and W with vector subspaces of V in the usual manner. Let $n = \dim U$, and Γ_W be the open submanifold of $\Gamma_n(V)$ consisting of all n-dimensional subspaces X of V such that $X \cap W = \{0\}$. Let Φ: $\operatorname{Hom}_F(U, W) \to \Gamma_W$ be the analytic manifold isomorphism defined in § 2; we recall that $\Phi(T)$ is the graph of T. Thus every subspace $X \in \Gamma_W$ is the graph of a linear map T: $U \to W$. One checks easily that $X = \Phi(T)$ is a subalgebra of the product Lie algebra $\mathfrak{h} \times \mathfrak{g}$ if and only if T: $\mathfrak{h} \to \mathfrak{g}$ is a homomorphism of Lie algebras.

Let $\mathfrak{p}$ denote the graph of the homomorphism ρ: $\mathfrak{h} \to \mathfrak{g}$. The adjoint representation of $\mathfrak{p}$ on $\mathfrak{h} \times \mathfrak{g}$ determines a $\mathfrak{p}$-module structure on the quotient space $(\mathfrak{h} \times \mathfrak{g})/\mathfrak{p}$. We consider $\mathfrak{g}$ as an $\mathfrak{h}$-module by means of the representation $ad_{\mathfrak{g}} \circ \rho$

(here $ad_{\mathfrak{g}}$ denotes the adjoint representation of $\mathfrak{g}$). Let $\gamma: \mathfrak{h} \to \mathfrak{g}$ denotes the map $x \to (x, \rho(x))$; then γ is an isomorphism of Lie algebras. Let $\pi: \mathfrak{h} \times \mathfrak{g} \to (\mathfrak{h} \times \mathfrak{g})/\mathfrak{p}$ be the canonical projection, and define $f: \mathfrak{g} \to (\mathfrak{h} \times \mathfrak{g})/\mathfrak{p}$ by $f(x) = \pi(0, x)$. Then f is a vector space isomorphism. Furthermore it is easy to see that, for $x \in \mathfrak{h}$ and $y \in \mathfrak{g}$, $f(x \cdot y) = \gamma(x) \cdot f(y)$, from which is follows easily that $H^j(\mathfrak{h}, \mathfrak{g})$ is canonically isomorphic to $H^j(\mathfrak{p}, (\mathfrak{h} \times \mathfrak{g})/\mathfrak{p})$.

Theorem 7.2. *Let $\mathfrak{g}$ and $\mathfrak{h}$ be finite-dimensional Lie algebras over F, and $\rho: \mathfrak{h} \to \mathfrak{g}$ a homomorphism. Consider $\mathfrak{g}$ as an h-module via the representation $ad_{\mathfrak{g}} \circ \rho$. If $H^2(\mathfrak{h}, \mathfrak{g}) = 0$, then ρ is a stable homomorphism.*

Proof. Since $H^2(\mathfrak{h}, \mathfrak{g}) = 0$, we have $H^2(\mathfrak{p}, (\mathfrak{h} \times \mathfrak{g})/\mathfrak{p}) = 0$ by the remarks above. Using the (local) correspondence between subalgebras of a product Lie algebra $\mathfrak{h}' \times \mathfrak{g}'$ and homomorphisms of $\mathfrak{h}'$ into $\mathfrak{b}'$ described above, we see that Theorem 7.2 is an immediate consequence of Theorem 6.2.

If we apply Theorem 7.2 to the case, in which $\mathfrak{h}$ is a subalgebra of $\mathfrak{g}$ and ρ is the inclusion map, we obtain a strengthened form of [11, Theorem 6.1].

Now for the case of algebraically closed fields, let $\mathfrak{h} = (U, \mu)$ and $\mathfrak{g} = (W, \eta)$ be finite-dimensional Lie algebras over an algebraically closed field K, $\mathscr{M}$ and $\mathscr{N}$ denote (respectively) the algebraic sets of all Lie algebra multiplications on U and W, and $\rho: \mathfrak{h} \to \mathfrak{g}$ be a Lie algebra homomorphism. As usual, $\mathscr{M}$, $\mathscr{N}$, and $\mathrm{Hom}_K(U, W)$ are given the Zariski topology.

Definition 7.3. ρ is a *stable* homomorphism if, for every neighborhood $N(\rho)$ of ρ in $\mathrm{Hom}_K(U, W)$, there exists a neighborhood $N(\mu, \eta)$ of (μ, η) in $\mathscr{M} \times \mathscr{N}$ such that, if $(\mu', \eta') \in N(\mu, \eta)$, then there exists $\rho' \in N(\rho)$ such that $\rho': (U, \mu') \to (W, \eta')$ is a homomorphism of Lie algebras.

Theorem 7.4. *If $H^2(\mathfrak{h}, \mathfrak{g}) = 0$, then ρ is a stable homomorphism.*

The proof of Theorem 7.4 follows from the results of § 6 and the local correspondence between subalgebras of product algebras and homomorphisms of their factors.

All of the theorems, proofs, definitions, etc., of this carry over easily to the case of homomorphisms of associative algebras. We leave their formulation to the reader.

7.5. Remark. The neighborhood $N(\mu, \eta)$ in Definition 7.3 cannot necessarily be taken in the form $N(\mu) \times N(\eta)$, where $N(\mu)$ and $N(\eta)$ are neighborhoods of μ and η in M and N.

8. Deformations of subalgebras

Let $\mathfrak{g} = (V, \mu)$ be a finite-dimensional Lie algebra over F, and $\mathscr{A}$ the set of all n-dimensional subalgebras of $\mathfrak{g}$. Then $\mathscr{A}$ is a Zariski closed subset of $\Gamma_n(V)$. Let $\mathfrak{h}$ be an n-dimensional subalgebra of $\mathfrak{g}$ with underlying subspace U, W be a subspace of V of codimension n which is transversal to U, $\Phi: \mathrm{Hom}_F(U, W) \to \Gamma_W$ be the analytic manifold isomorphism defined in § 2, and $\Psi: \Gamma_W \to \mathrm{Hom}_F(U, W)$ denote the inverse of Φ, and let $\mathscr{A}' = \Psi(\mathscr{A} \cap \Gamma_W)$. It follows from the results of [12, § 6] that if $t \to \mathfrak{h}_t$ is an analytic curve in $\mathscr{A}$ with $\mathfrak{h}_0 = \mathfrak{h}$, then the tangent vector at 0 of the curve $t \mapsto \Psi(\mathfrak{h}_t)$ is an element of $Z^1(\mathfrak{h}, W)$ (recall that $Z^1(\mathfrak{h}, W)$ is canonically isomorphic to $Z^1(\mathfrak{h}, \mathfrak{g}/\mathfrak{h})$). The following theorem shows that if $H^2(\mathfrak{h}, \mathfrak{g}/\mathfrak{h}) = 0$, then every $\alpha \in Z^1(h, W)$ occurs as the tangent vector of a one-parameter family of subalgebras in the sense described above.

Theorem 8.1. *Let the notation be as above and assume that $H^2(\mathfrak{h}, \mathfrak{g}/\mathfrak{h}) = 0$. Let u denote the point of $\Gamma_n(V)$ corresponding to the subspace U of V. Then there exists an open neighborhood $N(u)$ of u in $\Gamma_n(V)$ such that $\mathscr{A} \cap N(u) = \mathscr{N}$ is a closed analytic submanifold of $N(u)$. Furthermore $d\Psi_u$ maps $T(N, u)$ isomorphically onto $Z^1(\mathfrak{h}, U)$.*

The proof follows from the results of § 2.

Let G denote the adjoint group of $\mathfrak{g}$. Then G acts in an obvious way as a transformation group on $\mathscr{A}$. A subset $\mathscr{F}$ of $\mathscr{A}$ containing u is said to be a *locally complete* family of subalgebras of $\mathfrak{g}$ at $\mathfrak{h}$ if the orbit $G(\mathscr{F})$ of $\mathscr{F}$ under G is a neighborhood of u in $\mathscr{A}$. Intuitively, this says that every subalgebra near $\mathfrak{h}$ is conjugate to a subalgebra in the family $\mathscr{F}$. By a straightforward argument using the implicit function theorem one can show that there exist an open neighborhood $N(0)$ of 0 in $H^1(\mathfrak{h}, \mathfrak{g}/\mathfrak{h})$, an analytic map $u\colon N(0) \to H^2(\mathfrak{h}, \mathfrak{g}/\mathfrak{h})$ with $u(0) = 0$, and an injective analytic map τ of $\mathscr{C} = u^{-1}(0)$ into $\mathscr{A}$ with $\tau(0) = u$ such that $\mathscr{K} = \tau(\mathscr{C})$ is a locally complete family of subalgebras of $\mathfrak{g}$ at $\mathfrak{h}$. A more detailed formulation and a proof are given in [8], so we omit them here. See also [10, Theorem 20.3] where a similar proof is given in a slightly different setting.

8.2. Remarks. (a) The locally complete family $\mathscr{K}$ described above is an analogue of the locally complete family of complex-analytic structures on a compact manifold constructed by Kuranishi in [6],

(b) The locally complete family $\mathscr{K}$ above is analytically parametrized by the set of zeros of an analytic map of a neighborhood of 0 in $H^1(\mathfrak{h}, \mathfrak{g}/\mathfrak{h})$ into $H^2(\mathfrak{h}, \mathfrak{g}/\mathfrak{h})$. Thus, in a sense, the elements of $H^2(\mathfrak{h}, \mathfrak{g}/\mathfrak{h})$ can be interpreted as "obstructions" to finding one-parameter families of subalgebras with a given initial tangent vector. From another point of view which we shall not go into here, the elements of $H^2(\mathfrak{h}, W)$ occur as "obstructions" to finding a formal power series solution $p(t) = \sum_{n=1}^{\infty} a_n t^n$ ($a_n \in \mathrm{Hom}_F(U, W)$) to the equations defining $\mathscr{A}'$ with a given initial term $a_1 \in Z^1(\mathfrak{h}, W)$.

Theorem 8.1 carries over to the case of Lie algebras over algebraically closed fields. In this case the conclusion is that u is a simple point of A and that $d\Psi_u$ maps $T(A, u)$ isomorphically onto $Z^1(\mathfrak{h}, W)$. An analogue of the result on locally complete families is also valid in this case but will be omitted. See [10, Theorem 23.4] for a precise formulation in a similar case.

All of the results and proofs of this section are also valid for the case of associative algebras. We omit the details.

8.3. Remark. The results of this section were obtained independently and at approximately the same time by A. Nijenhuis and by the author. Nijenhuis's results are given in [8].

9. Relatively stable subalgebras

In this section we shall be dealing with subalgebras of a fixed Lie algebra. Thus there is no point in distinguishing between a Lie algebra (resp. subalgebra) and its underlying vector space and we shall not do so.

Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over F, and G a connected Lie group with Lie algebra $\mathfrak{g}$. (Our proofs are independent of the choice of G.) Let $\mathscr{A}$ be the set of all n-dimensional subalgebras of $\mathfrak{g}$; $\mathscr{A}$ is a Zariski-closed subset of the Grassmann variety $\Gamma_n(\mathfrak{g})$. We denote by Ad the adjoint representation of G on $\mathfrak{g}$. The map $(g, U) \mapsto (\mathrm{Ad}\, g)(U)$ of $G \times \Gamma_n(\mathfrak{g})$ onto $\Gamma_n(\mathfrak{g})$ determines an analytic action (actually an algebraic action) of G on $\Gamma_n(\mathfrak{g})$, and $\mathscr{A}$ is stable under this action of G. Two n-dimensional subalgebras $\mathfrak{a}$ and $\mathfrak{b}$ of $\mathfrak{g}$ are *conjugate* if they lie on the same orbit under the action of G on A.

To avoid confusing notation it is often convenient to distinguish between points of the algebraic set $\mathscr{A}$ and n-dimensional subalgebras of $\mathfrak{g}$. If x is a point of $\mathscr{A}$, we shall often denote the corresponding subalgebra of $\mathfrak{g}$ by $\mathfrak{a}_x$.

Definition 9.1. Let $\mathfrak{k} \subset \mathfrak{h}$ be subalgebras of $\mathfrak{g}$ with $n = \dim \mathfrak{h}$, and h the point of $\mathscr{A}$ corresponding to $\mathfrak{h}$. Then $\mathfrak{k}$ is a *stable subalgebra of* $\mathfrak{g}$ *relative to* $\mathfrak{h}$ if there exist a neighborhood $N(h)$ of h in $\mathscr{A}$ and an analytic map $\boldsymbol{u}: N(h) \to G$ with $\boldsymbol{u}(h) = e$ such that $(\mathrm{Ad}(\boldsymbol{u}(x)))(\mathfrak{k}) \subset \mathfrak{a}_x$ for every $x \in N(h)$.

Roughly speaking, then, $\mathfrak{k}$ is stable relative to $\mathfrak{h}$ if every n-dimensional subalgebra of $\mathfrak{g}$ near $\mathfrak{h}$ contains a subalgebra which is conjugate to $\mathfrak{k}$.

As in § 2, we define an $\mathfrak{h}$-module structure on $\mathfrak{g}/\mathfrak{h}$ by means of the adjoint representation of $\mathfrak{h}$ on $\mathfrak{g}$. By restriction we obtain a $\mathfrak{k}$-module structure on $\mathfrak{g}/\mathfrak{h}$.

Theorem 9.2. *Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over F, and $\mathfrak{k} \subset \mathfrak{h}$ be subalgebras of $\mathfrak{g}$. If $H^1(\mathfrak{k}, \mathfrak{g}/\mathfrak{h}) = 0$, then $\mathfrak{k}$ is a stable subalgebra of $\mathfrak{g}$ relative to $\mathfrak{h}$.*

Remark. If $\mathfrak{k} = \mathfrak{h}$, we obtain the rigidity theorem of [12].

The proof of Theorem 9.2 will be given in a series of lemmas. The general plan of the proof is the same as that of Theorem 2.5.

Let $n = \dim \mathfrak{h}$, W be a subspace of $\mathfrak{g}$ of codimension n such that $W \cap \mathfrak{h} = \{0\}$, V (resp. U) be the underlying subspace of $\mathfrak{g}$ (resp. $\mathfrak{h}$), and $\boldsymbol{r}$, $\boldsymbol{s}$, P, Q, Γ_W, Φ and Ψ be as in § 2. As in § 2 we use the canonical vector space isomorphism between W and $\mathfrak{g}/\mathfrak{h}$ to transport to W the $\mathfrak{h}$-module structure on $\mathfrak{g}/\mathfrak{h}$; by restriction this induces a $\mathfrak{k}$-module structure on W. As in § 2, we shall identify $C^m(\mathfrak{h}, W)$ with a vector subspace of $C^m(\mathfrak{g}, \mathfrak{g})$ by means of the monomorphism $\boldsymbol{s}$. Let $\boldsymbol{p}: C(\mathfrak{h}, W) \to C(\mathfrak{k}, W)$ be the obvious restriction map. Then $\boldsymbol{p}$ is surjective and a chain mapping. Set $C = \{T \in C^1(\mathfrak{h}, W) \,|\, \boldsymbol{p}(T) \in Z^1(\mathfrak{k}, W)\}$. Since $\boldsymbol{p}$ is a chain map, $Z^1(\mathfrak{h}, W) \subset C$. Choose direct sum decompositions $C^1(\mathfrak{h}, W) = C \oplus D$

and $C^1(\mathfrak{k}, W) = Z^1(\mathfrak{k}, W) \oplus E$ such that $\boldsymbol{p}$ maps D isomorphically onto E, and let $\pi_C\colon C^1(\mathfrak{h}, W) \to C$, $\pi_D\colon C^1(\mathfrak{h}, W) \to D$, $\pi_Z\colon C^1(\mathfrak{k}, W) \to Z^1(\mathfrak{k}, W)$ and $\pi_E\colon C^1(\mathfrak{k}, W) \to E$ be the corresponding projections.

If $S, T \in \mathrm{Hom}_F(\mathfrak{g}, \mathfrak{g})$ $(= C^1(\mathfrak{g}, \mathfrak{g}))$, we define $[S, T] \in C^2(\mathfrak{g}, \mathfrak{g})$ to be the map $(x, y) \mapsto [Sx, Ty] + [Tx, Sy]$ (here $[Sx, Ty]$ and $[Tx, Sy]$ denote products in the Lie algebra $\mathfrak{g}$). Then $(S, T) \mapsto [S, T]$ is a bilinear map and $[S, T] = [T, S]$.

Let $\mathscr{C} = \Psi(\mathscr{A} \cap \Gamma_W)$; $\mathscr{C}$ is the algebraic set in $C^1(\mathfrak{h}, W)$ which corresponds to $\mathscr{A} \cap \Gamma_W$ via the chart Ψ. We note that $\Psi(\mathfrak{h}) = 0$. Now we define a polynomial map $\boldsymbol{a}\colon C^1(\mathfrak{h}, W) \to C^2(\mathfrak{h}, W)$ by $\boldsymbol{a}(T) = (P - T) \circ [Q + T, Q + T]$. (As defined, $\boldsymbol{a}$ is a map of $C^1(\mathfrak{h}, W)$ into $C^2(\mathfrak{g}, \mathfrak{g})$; however, one checks easily that the image of $\boldsymbol{a}$ lies in $C^2(\mathfrak{h}, W)$, which is considered as a subspace of $C^2(\mathfrak{g}, \mathfrak{g})$ via $\boldsymbol{s}$.) It is shown in [12, Lemma 3.1] that $T \in \mathscr{C}$ if and only if $\boldsymbol{a}(T) = 0$. (Let $\boldsymbol{f}$ be as in § 2, and μ the multiplication on $\mathfrak{g}$. Then it is easy to check that $\boldsymbol{f}(\mu, T) = \boldsymbol{a}(T)$, for $T \in C^1(\mathfrak{h}, W)$. However the notation we have just introduced is more convenient for the computations we shall make in this section.)

Lemma 9.3. *In $C^1(\mathfrak{h}, W)$ there exist an algebraic set $\mathscr{M}_1$ containing $\mathscr{C}$ and an open neighborhood $N(0)$ of 0 such that $\mathscr{M}' = N(0) \cap \mathscr{M}_1$ is a closed analytic submanifold of $N(0)$ and $T(\mathscr{M}', 0) = C$.*

Proof. It is shown in [12, Prop. 6.1] that the kernel of the differential $d\boldsymbol{a}_0$ is $Z^1(\mathfrak{h}, W)$. Since $Z^1(\mathfrak{h}, W) \subset C$, it follows by elementary linear algebra that there exist a vector subspace A of $C^2(\mathfrak{h}, W)$ and a surjective linear map $\boldsymbol{b}\colon C^2(\mathfrak{h}, W) \to A$ such that the differential $d(\boldsymbol{b} \circ \boldsymbol{a})_0\colon C^1(\mathfrak{h}, W) \to A$ is surjective and the kernel of $d(\boldsymbol{b} \circ \boldsymbol{a})_0$ is C. We set $\mathscr{M}_1 = (\boldsymbol{b} \circ \boldsymbol{a})^{-1}(0)$. Then $\mathscr{C} \subset \mathscr{M}_1$, and the other conclusions of Lemma 9.3 follow from the implicit function theorem as formulated, e.g., in [11, § 5].

Let $\delta_1\colon C(\mathfrak{h}, W) \to C(\mathfrak{h}, W)$ and $\delta_2\colon C(\mathfrak{k}, W) \to C(\mathfrak{k}, W)$ denote the coboundary operators on the complexes $C(\mathfrak{h}, W)$ and $C(\mathfrak{k}, W)$. We choose neighborhoods $N(e)$ and $N(0)$ of e and 0 in G and $C^1(\mathfrak{h}, W)$ such that, if $g \in N(e)$ and $T \in N(0)$, then $(\mathrm{Ad}\, g)(\Phi(T)) \in \Gamma_W$. We may further assume that $N(0)$ satisfies the conditions of Lemma 9.3, and define $\boldsymbol{m}_0\colon N(e) \times N(0) \to C^1(\mathfrak{h}, W)$ by $\boldsymbol{m}_0(g, T) = \Psi((\mathrm{Ad}\, g)(\Phi(T)))$. Roughly, $\boldsymbol{m}_0$ defines the action of G on $C^1(\mathfrak{h}, W)$ which corresponds via the chart $\Psi\colon \Gamma_W \to C^1(\mathfrak{h}, W)$ to the natural action of G on $\Gamma_n(\mathfrak{g})$. Let $\boldsymbol{m}\colon N(e) \times \mathscr{M}' \to C^1(\mathfrak{h}, W)$ denote the restriction of $\boldsymbol{m}_0$ to $N(e) \times \mathscr{M}'$. It follows from [12, § 7] that if $x \in \mathfrak{g}$, then $d\boldsymbol{m}_{(e,0)}(x, 0) = \delta_1(Px)$. (Recall that $C^0(\mathfrak{h}, W) = W$.) Thus $d\boldsymbol{m}_{(e,0)}(x, T) = \delta_1(Px) + T$. Since $\boldsymbol{p}$ maps C onto $Z^1(\mathfrak{k}, W)$, it follows immediately that the differential at $(e, 0)$ of the map $\pi_Z \circ \boldsymbol{p} \circ \boldsymbol{m}\colon N(e) \times \mathscr{M}' \to Z^1(\mathfrak{k}, W)$ is surjective. Let $\mathscr{N}' = (\pi_Z \circ \boldsymbol{p} \circ \boldsymbol{m})^{-1}(0)$. Then it follows from the implicit function theorem that there exists an open neighborhood $N(e, 0)$ of $(e, 0)$ in $N(e) \times \mathscr{M}'$ such that $\mathscr{N} = \mathscr{N}' \cap N(e, 0)$ is a closed submanifold of $N(e, 0)$ and $T(\mathscr{N}, (e, 0))$ is equal to the kernel of $d(\pi_Z \circ \boldsymbol{p} \circ \boldsymbol{m})_{(e,0)}$. Let $\boldsymbol{q}\colon \mathscr{N} \to \mathscr{M}$ denote the restriction to $\mathscr{N}$ of the Projection $N(e) \times \mathscr{M}' \to \mathscr{M}'$.

Lemma 9.4. *The differential $dq_{(e,0)}$ is surjective.*

Proof. We shall use the fact that $H^1(\mathfrak{k}, W) = H^1(\mathfrak{k}, \mathfrak{g}/\mathfrak{h}) = 0$. Let $T \in T(\mathcal{M}', 0) = C$. Then $p(T) \in Z^1(\mathfrak{k}, W) = B^1(\mathfrak{k}, W)$. Hence there exists $w \in W$ such that $p(T) = \delta_2(w) = p(\delta_1 w)$. (Since $C^0(\mathfrak{h}, W) = W = C^0(\mathfrak{k}, W)$, we have $w = p(w)$.) Using the formula derived above for $dm_{(e,0)}$ we see that

$$d(\pi_Z \circ p \circ m)_{(e,0)}(-w, T) = \pi_Z(p(-\delta_1 w + T)) = 0 .$$

Thus $(-w, T) \in \text{kernel } (d(\pi_Z \circ p \circ m)_{(e,0)}) = T(\mathcal{N}, (e, 0))$. Since $dq_{(e,0)}(-w, T) = T$, we see that $dq_{(e,0)}$ is surjective. This proves Lemma 9.4.

If $T \in \mathcal{C}$, then the subalgebra $\Phi(T)$ contains $\mathfrak{k}$ if and only if the restriction of T to $\mathfrak{k}$ is 0, i.e., if and only if $p(T) = 0$. The following lemma, which is analogous to Lemma 2.1, shows that if $T \in \mathcal{C}$ is sufficiently near 0 and $\pi_Z(p(T)) = 0$, then $p(T) = 0$.

Lemma 9.5. *There exists an open neighborhood $N(0)$ of 0 in $C^1(\mathfrak{h}, W)$ such that, if $T \in N(0) \cap \mathcal{C}$ and $\pi_Z(p(T)) = 0$, then $p(T) = 0$.*

Proof. Let $T \in \mathcal{C}$ be such that $\pi_Z(p(T)) = 0$. We may write $T = X + Y$, with $X \in C$ and $Y \in D$. By definition of D we have $\pi_Z(p(Y) = 0$. Consequently

$$0 = \pi_Z(p(T)) = \pi_Z(p(X + Y)) = \pi_Z(p(X)) = p(X) .$$

Since $T \in \mathcal{C}$, we have

$$0 = (P - X - Y) \circ [Q + X + Y, Q + X + Y] . \tag{9.1}$$

If we expand the right hand side of (9.1) by multilinearity, then each individual term is an alternating bilinear map of $\mathfrak{g} \times \mathfrak{g}$ into W, i.e., an element of $A^2(\mathfrak{g}, W)$. Let t: $A^2(\mathfrak{g}, W) \to C^2(\mathfrak{k}, W)$ denote the restriction map, and note the following facts:

(a) $P \circ [Q, Q] = 0$,
(b) $t(X \circ [Q, Q]) = 0$,
(c) $t((P - X - Y) \circ [X, Q + X + Y]) = 0$,
(d) $t(2P \circ [Q, Y] - Y \circ [Q, Q]) = 2\delta_2(p(Y))$.

In fact, (a) is equivalent to the statement that $\mathfrak{h}$ is a subalgebra, (b) follows from the facts that $\mathfrak{k}$ is a sub-algebra and that $p(X)$, the restriction of X to $\mathfrak{k}$, vanishes, (c) is an easy consequence of the vanishing of $p(X)$, and the proof of (d) is by an easy direct computation.

If we expand the right hand side of (9.1) by multilinearity, applying t, and using (a), (b), (c) and (d) above we obtain

$$0 = 2\delta_2(p(Y)) + t(P \circ [Y, Y]) - t((X + Y) \circ ([Y, Y] + 2[Q, Y])) .$$

Finally, using the fact that $p(X) = 0$, one checks that this is equivalent to

$$\begin{aligned} 0 = 2\delta_2(p(Y)) &+ t(P \circ [X + Y, Y]) \\ &- t((X + Y) \circ ([X + Y, Y] + 2[Q, Y])) . \end{aligned} \tag{9.2}$$

For every $T' \in C^1(\mathfrak{h}, W)$, we define a linear map $\lambda_{T'}\colon D \to C^2(\mathfrak{k}, W)$ by

$$\lambda_{T'}(S) = 2\delta_2(p((S)) + t(P \circ [T', S]) - t(T' \circ ([T', S] + 2[Q, S])) .$$

With this notation, (9.2) becomes

$$\lambda_{X+Y}(Y) = 0 . \tag{9.3}$$

We observe that $T' \to \lambda_{T'}$ is a polynomial mapping (and hence continuous), and further that λ_0 is the restriction of $2\delta_2 \circ p$ to D. Thus λ_0 is a monomorphism, and hence there exists a neighborhood $N(0)$ of 0 in $C^1(\mathfrak{h}, W)$ such that if $T' \in N(0)$, then $\lambda_{T'}$ is a monomorphism. Assume now that $T = X + Y \in N(0)$. Since λ_T is a monomorphism, it follows from (9.3) that $Y = 0$. Thus $p(T) = p(X) = 0$, which proves Lemma 9.5.

The proof of Theorem 9.2 is now essentially complete; it is just a matter of putting together the pieces. It follows from Lemma 9.4 and the implicit function theorem that there exist an open neighborhood $N_1(0)$ of 0 in $\mathscr{M}'$ and an analytic map $u_1\colon N_1(0) \to G$ with $u_1(0) = e$ such that $(T, u_1(T)) \in \mathscr{N}$ for every $T \in N_1(0)$. We may assume further that $m(T, u_1(T)) \in N(0)$ for every $T \in N_1(0)$, where $N(0)$ is as in Lemma 9.5. Now let $N_2(0) = N_1(0) \cap \mathscr{C}$, and $T \in N_2(0)$. Then, since $(T, u_1(T)) \in \mathscr{N}$, we have $\pi_Z(p(m(T, u_1(T)))) = 0$ and consequently, by Lemma 9.5, $p(m(T, u_1(T))) = 0$. But this implies that the subalgebra $(\mathrm{Ad}\, u_1(T))(\Phi(T))$ contains $\mathfrak{k}$, or, equivalently, that $(\mathrm{Ad}\,(u_1(T)^{-1})(\mathfrak{k})$ is included in the subalgebra $\Phi(T)$. Let $N(h) = \Phi(N_2(0))$; $N(h)$ is a neighborhood of h in $\mathscr{A}$. We define the analytic map $u\colon N(h) \to G$ by $u(x) = u_1(\Psi(x))^{-1}$. Then it follows from the remarks above that if $x \in N(h)$, then $(\mathrm{Ad}\, u(x))(\mathfrak{k}) \subset \mathfrak{a}_x$. Thus $\mathfrak{k}$ is a stable subalgebra of $\mathfrak{g}$ relative to $\mathfrak{h}$. This completes the proof of Theorem 9.2.

9.6. Examples. (a) Let $\mathfrak{k} \subset \mathfrak{h}$ be subalgebras of $\mathfrak{g}$, and assume $\mathfrak{k}$ is a semi-simple Lie algebra. Then it is known that $H^1(\mathfrak{k}, M) = 0$ for every finite-dimensional $\mathfrak{k}$-module M. In particular $H^1(\mathfrak{k}, \mathfrak{g}/\mathfrak{h}) = 0$, and thus $\mathfrak{k}$ is a stable subalgebra of $\mathfrak{g}$ relative to $\mathfrak{h}$.

(b) Let $\mathfrak{k} \subset \mathfrak{h}$ be subalgebras of $\mathfrak{g}$, and assume that $\mathfrak{k}$ is a Cartan subalgebra of $\mathfrak{g}$. Then $H^1(\mathfrak{k}, \mathfrak{g}/\mathfrak{h}) = 0$. (The proof is similar to that of [12, Prop. 12.5].) Thus $\mathfrak{k}$ is stable in $\mathfrak{g}$ relative to $\mathfrak{h}$, and hence every subalgebra of $\mathfrak{g}$ near $\mathfrak{h}$ contains a Cartan subalgebra. If, in addition, $\mathfrak{g}$ is a semi-simple Lie algebra, then it follows from [12, Cor. 12.9] that every subalgebra of $\mathfrak{g}$ near $\mathfrak{h}$ is actually conjugate to $\mathfrak{h}$.

9.7. A generalization. Theorem 9.2 can be slightly generalized as follows. Let $\mathrm{Aut}\,(\mathfrak{g})$ be the group of all automorphisms of the Lie algebra $\mathfrak{g}$, and $\mathrm{Der}\,(\mathfrak{g})$ the Lie algebra of all derivations of $\mathfrak{g}$. Then $\mathrm{Der}\,(\mathfrak{g})$ is the Lie algebra of the Lie group $\mathrm{Aut}\,(\mathfrak{g})$. Let $\mathfrak{G}$ be a subalgebra of the Lie algebra $\mathrm{Der}\,(\mathfrak{g})$, and G the corresponding connected Lie subgroup of $\mathrm{Aut}\,(\mathfrak{g})$. Define D^1 to be the vector subspace of $\mathrm{Hom}_F\,(\mathfrak{k}, \mathfrak{g}/\mathfrak{h}) = C^1(\mathfrak{k}, \mathfrak{g}/\mathfrak{h})$ consisting of all linear maps of the form $x \to \pi(Dx)$ for some derivation $D \in \mathfrak{G}$, where $\pi\colon \mathfrak{g} \to \mathfrak{g}/\mathfrak{h}$ denotes the canonical map. One checks easily that $D^1 \subset Z^1(k, \mathfrak{g}/\mathfrak{h})$. We have the following theorem:

Theorem 9.8. *Let the notation be as above and assume that $Z^1(\mathfrak{k}, \mathfrak{g}/\mathfrak{h}) = D^1$. Then there exist an open neighborhood $N(h)$ of h in $\mathscr{A}$ and an analytic map $\boldsymbol{u}: N(h) \to G$ such that $(\boldsymbol{u}(x))(\mathfrak{k}) \subset \mathfrak{a}_x$ for every $x \in N(h)$.*

Theorem 9.8 implies that if $Z^1(k, g/h) = D^1$, then every subalgebra of $\mathfrak{g}$ near $\mathfrak{h}$ contains a subalgebra which is conjugate to $\mathfrak{k}$ under G. Theorem 9.5 corresponds to the special case of Theorem 9.8 in which $\mathfrak{G} = \operatorname{ad}(\mathfrak{g})$. The proof of Theorem 9.8 is essentially the same as that of Theorem 9.5 if one replaces $\operatorname{ad}(\mathfrak{g})$ and $\operatorname{Ad}(G)$ by $\mathfrak{G}$ and G respectively. We omit the details.

9.9. The case of algebraically closed fields. Let G be a linear algebraic group over an algebraically closed field K, $\mathfrak{g}$ the Lie algebra of G, $\mathfrak{k} \subset \mathfrak{h}$ subalgebras of $\mathfrak{g}$ with $n = \dim \mathfrak{h}$, and $\mathscr{A}$ the (Zariski) closed subset of $\Gamma_n(g)$ consisting of all n-dimensional subalgebras of $\mathfrak{g}$. As usual, all spaces involved will be considered as topological spaces, supplied with the Zariski topology.

Definition 9.10. $\mathfrak{k}$ is a *stable subalgebra of* $\mathfrak{g}$ *relative to* $\mathfrak{h}$ if, for every open neighborhood $N(e)$ of e in G, there exists an open neighborhood $N(\mathfrak{h})$ of $\mathfrak{h}$ in $\mathscr{A}$ such that the following property holds: if $x \in N(\mathfrak{h})$, there exists $g \in N(e)$ such that $(\operatorname{Ad} g)(\mathfrak{k}) \subset \mathfrak{a}_x$.

Theorem 9.11. *If $H^1(\mathfrak{k}, \mathfrak{g}/\mathfrak{h}) = 0$, then $\mathfrak{k}$ is a stable subalgebra of $\mathfrak{g}$ relative to $\mathfrak{h}$.*

The proof of Theorem 9.11 is essentially the same as that of Theorem 9.2. The modifications necessary in the proof of Theorem 9.2 are similar to those made in § 4 in the proof of Theorem 2.5. We omit the details.

An obvious analogue of Theorem 9.8 is also valid in the case at hand, provided that $\mathfrak{G}$ is the Lie algebra of an *algebraic* sub-group G of $\operatorname{Aut}(\mathfrak{g})$.

9.12. Relatively stable subalgebras of associative algebras. Let A be a finite-dimensional associative algebra (with identity) over either F or an algebraically closed field K, and G the algebraic group of all invertible elements of A. We define a Lie algebra structure on the underlying vector space of A as follows: if $x, y \in A$, then $[x, y] = xy - yx$. Let $\mathfrak{g}$ denote this Lie algebra. Then $\mathfrak{g}$ is the Lie algebra of G, and the adjoint action of G on $A(=\mathfrak{g})$ is simply the action of G on A by inner automorphisms.

With these preliminaries, all of the definitions, theorems, and proofs of § 9 carry over to the case of associative algebras with only trivial modifications. Details are left to the reader.

10. Application to transformation groups

Let G be a real (resp. complex) Lie group with Lie algebra $\mathfrak{g}$, and assume that G acts differentiably (resp. holomorphically) on the differentiable (resp. complex) manifold M. For each $x \in M$, let G_x be the isotropy group of G at x, and $\mathfrak{g}_x$ the Lie algebra of G_x. Then $\mathfrak{g}_x$ is the isotropy subalgebra at x. For each non-negative integer n let M_n denote the set of points $x \in M$ such that the orbit $G(x)$ is n-dimensional. If $q = \dim G$ and $x \in M_n$, then $\dim \mathfrak{g}_x = q - n$. (In the complex case, we refer, of course, to the complex dimension.) For each $x \in M_n$, the isotropy subalgebra $\mathfrak{g}_x$ can be considered as a point of the Grassmann manifold $\Gamma_{q-n}(g)$. It is shown in [13] that the map $x \to \mathfrak{g}_x$ is a differentiable (resp. holomorphic) map of M_n into $\Gamma_{q-n}(\mathfrak{g})$. Furthermore it is shown that, for each integer k, $\cup_{n \geq k} M_n$ is an open subset of M. As an easy consequence of Theorem 9.2, we obtain

Theorem 10.1. *Let G and M be as above, $x \in M_n$, and $\mathfrak{a}$ be a subalgebra of the isotropy subalgebra $\mathfrak{g}_x$ such that the Lie algebra cohomology space $H^1(\mathfrak{a}, \mathfrak{g}/\mathfrak{g}_x)$ vanishes. Then there exist an open neighborhood $N(x)$ of x in M_n and a differentiable (resp. analytic) map $y \mapsto g_y$ of $N(x)$ into G such that $(\mathrm{Ad}\, g_y)(\mathfrak{a}) \subset \mathfrak{g}_y$ for every $y \in N(x)$.*

As a particular example, let $n = \max_{x \in M} \dim G(x)$ and assume that there exists $x \in M_n$ such that $\mathfrak{g}_x$ contains a Cartan subalgebra of $\mathfrak{g}$. Then there exists an open neighborhood $N(x)$ of x in M such that $\mathfrak{g}_y$ contains a Cartan subalgebra for every $y \in N(x)$.

References

[1] H. Cartan & S. Eilenberg, *Homological algebra*, Princeton University Press, Princeton, 1956.

[2] C. Chevalley, *Fondements de la géometrie algébrique*, Course notes at the Institut Henri Poincaré, Paris, 1958.

[3] M. Gerstenhaber, *The cohomology structure of an associative ring*, Ann. of Math. (2) **78** (1963) 267–289.

[4] K. Kodaira, *A theorem of completeness of characteristic systems for analytic families of compact submanifolds of complex manifolds*, Ann. of Math. (2) **75** (1962) 146–162.

[5] ——, *On stability of compact submanifolds of complex manifolds*, Amer. J. Math. **85** (1963) 79–94.

[6] M. Kuranishi, *On the locally complete families of complex analytic structures*, Ann. of Math. (2) **75** (1962) 536–577.

[7] D. Mumford, *Introduction to algebraic geometry*, Lecture notes at Harvard University.

[8] A. Nijenhuis, *Composition systems and deformations of subalgebras*, Indag. Math. **30** (1968) 119–136.

[9] A. Nijenhuis & R. W. Richardson, Jr., *Commutative algebra cohomology and deformations of Lie and associative algebras*, J. Algebra, **9** (1968) 42–53.

[10] ——, *Cohomology and deformations in graded Lie algebras*, Bull. Amer. Math. Soc. **72** (1966) 1–29.

[11] S. Page & R. W. Richardson, Jr., *Stable subalgebras of Lie algebras and associative algebras*, Trans. Amer. Math. Soc. **127** (1967) 302–312.

[12] R. W. Richardson, Jr., *A rigidity theorem for subalgebras of Lie and associative algebras*, Illinois J. Math. **11** (1967) 92–110.

[13] ——, *On the variation of isotropy subalgebras*, Proc. Conf. Transformation Groups, Springer, Berlin, 1968, 429–440.

[14] ——, *Deformations of Lie subgroups* to appear.

[15] J. P. Serre, *Faisceaux algébriques cohérents*, Ann. of Math. (2) **61** (1965) 197–278.

[16] ——, *Géométrie algébrique et géométrie analytique*, Ann. Inst. Fourier (Grenoble) **6** (1955–56) 1–42.

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