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A FIBRE BUNDLE DESCRIPTION OF TEICHMÜLLER THEORY

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1. Introduction

(A) In this paper we prove the theorems which we announced in [14] concerning the diffeomorphism groups of a closed surface, and, in addition, the corresponding theorems for the diffeomorphism groups of the closed non-orientable surfaces. Our method is to construct a certain principal fibre bundle, whose total space is the space of smooth conformal structures of a closed surface, whose base is a Teichmüller space, and whose structural group is a subgroup of the diffeomorphism group of the surface. Our bundle has the further property that its tangent bundle sequence embodies the infinitesimal deformation of structure theory (for surfaces) of Kodaira-Spencer [22].

Set theoretically, the construction of our bundle is a modification of the Ahlfors-Bers development of Teichmüller theory. To show that we have produced a topological fibre bundle, we need a new theorem about the continuity of solutions to Beltrami equations with smooth coefficients (see § 3). We have provided a fairly detailed account of our construction, because even where it closely follows the Ahlfors-Bers developments, certain adjustments are needed. Consequently we believe that the reader will find the paper relatively self-contained. For expositions of Teichmüller theory, and for guides to the literature, we refer to Ahlfors [2], Bers [6], Rauch [26], and Teichmüller [30].

(B) We now formulate precisely our main results. Let X be an oriented smooth (= class C^∞) 2-dimensional manifold which is compact and without boundary. We denote by $\mathbf{D}(X)$ the topological group of all orientation preserving diffeomorphisms of X, endowed with the C^∞-topology of uniform convergence of differentials of all orders; $\mathbf{D}_0(X)$ is the subgroup consisting of the diffeomorphisms which are homotopic to the identity. (We shall find later that $\mathbf{D}_0(X)$ is the arc component in $\mathbf{D}(X)$ of the neutral element.)

We denote by $\mathbf{M}(X)$ the space of smooth complex structures on X compatible with its given orientation, and give $\mathbf{M}(X)$ the C^∞-topology. Then (viewing the elements of $\mathbf{M}(X)$ as smooth tensor fields on X) we have a natural action

$$\mathbf{M}(X) \times \mathbf{D}(X) \to \mathbf{M}(X) .$$

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The following results are established in §§ 5, 6, 8.

Theorem. *Assume that X has genus $g > 1$.*

1. $\mathbf{M}(X)$ *is a contractible complex analytic manifold modeled on a Fréchet space.*
2. $\mathbf{D}(X)$ *acts continuously, effectively, and properly on* $\mathbf{M}(X)$.
3. *If*

$$\Phi\colon \mathbf{M}(X) \to \mathbf{T}(X) = \mathbf{M}(X)/\mathbf{D}_0(X) \tag{1.1}$$

denotes the indicated quotient map (where $\mathbf{T}(X)$ *is given the quotient topology), then* (1.1) *is a universal principal* $\mathbf{D}_0(X)$*-fibre bundle.*

4. *Let G be the Lie group of automorphisms of the upper half plane. Then* $\mathbf{T}(X)$ *can be embedded as a real analytic submanifold of* G^{2g}. *The complex structure of* $\mathbf{M}(X)$ *induces a complex structure on* $\mathbf{T}(X)$, *with Φ holomorphic.*

$\mathbf{T}(X)$ is the *Teichmüller space* of the oriented surface X; its complex structure is the standard one. The quotient group $\mathbf{D}(X)/\mathbf{D}_0(X)$ acts properly discontinuously on $\mathbf{T}(X)$, and its quotient space $\mathbf{R}(X)$ is the *Riemann space of moduli of X*.

Part 4 of our theorem is known [1], [7], since $\mathbf{T}(X)$ can be identified with the classical Teichmüller space of closed surfaces of genus g.

There are an analogous result for the case $g = 1$ (Theorem 10F) and a suitable statement for the case $g = 0$ (Theorem 9B). We also have a formulation, in the context of conformal structures, for non-orientable surfaces (§ 11).

In broad terms, our proof proceeds by transfering our activities from X to its universal cover, and studying Beltrami's equation there. A technical fact (Theorem 3B) of importance throughout is the continuous dependence of the solution of Beltrami's equation on its coefficients.

(C) Teichmüller's theorem [6] asserts that $\mathbf{T}(X)$ is a cell. Together with the covering homotopy theorem this implies that the fibration (1.1) is topologically trivial. We outline in § 8E an alternative proof of that triviality by constructing a continuous section of Φ, based on the existence theorem for harmonic maps [16]. There is a holomorphic section if $g = 1$; but none for $g > 1$ [12].

(D) The next results are interpretations of the development in § 7, in the spirit of Kodaira-Spencer [22] and Weil [32], [33]. We appeal to § 7 for an explanation of the terminology.

Theorem. *Assume that X has genus $g > 1$. Fix any complex structure* $J \in \mathbf{M}(X)$.

1. *The tangent space of* $\mathbf{M}(X)$ *at J consists of the space of $\bar\partial$-closed* 1*-forms on X with values in the vector bundle* $T^{1,0}(X)$. *The kernel of the differential* $d\Phi(J)$ *is identified with the space of such $\bar\partial$-derived* 1*-forms.*
2. *The tangent space of* $\mathbf{T}(X)$ *at* $\Phi(J)$ *is given by the cohomology space*

$H^1(X, \Theta)$, where Θ is the sheaf of germs of smooth sections of $T^{1,0}(X) \otimes T^{*0,1}(X)$. $H^1(X, \Theta)$ is conjugate to the space of J-holomorphic quadratic differentials on X.

3. *Suppose we represent X (using J) as the quotient of the upper half plane U by a Fuchsian group Γ, acting freely on U. Then the differential of Φ induces an isomorphism of $H^1(X, \Theta)$ onto $H^1(\Gamma, \mathfrak{g})$.*

Here $H^1(\Gamma, \mathfrak{g})$ denotes the cohomology space of the discrete group Γ relative to its adjoint representation on the Lie algebra of G. It measures the infinitesimal deformations of Γ in G.

(E) The following is a purely topological conclusion; it assembles results from §§ 8–11.

Let X be a closed surface. We extend the notation $\mathbf{D}(X)$ to non-orientable X, defining it for that case as the topological group of all diffeomorphisms.

Corollary.

1. *If X is the sphere or projective plane, then $\mathbf{D}(X) = \mathbf{D}_0(X)$ has $SO(3)$ as strong deformation retract.*
2. *If X is the torus, then $\mathbf{D}_0(X)$ has X as strong deformation retract.*
3. *If X is the Klein bottle, then $\mathbf{D}_0(X)$ has $SO(2)$ as strong deformation retract.*
4. *In all other cases $\mathbf{D}_0(X)$ is contractible.*

The case of the sphere was first established by Smale [29], using different methods.

Remark. In case 4, it follows that all fibre bundles with structural group $\mathbf{D}_0(X)$ are topologically trivial. In particular, that is true of the bundle over $\mathbf{T}(X)$ with fibre model X, associated with the principal bundle $\Phi: \mathbf{M}(X) \to \mathbf{T}(X)$ using the natural action of $\mathbf{D}_0(X)$ on X. The total space of that bundle has a natural complex structure, making it a holomorphic family of compact Riemann surfaces [3], [22].

Remark. The spaces $\mathbf{D}(X)$, $\mathbf{D}_0(X)$, $\mathbf{M}(X)$, and $\mathbf{T}(X)$ are absolute neighborhood retracts, being metrizable manifolds modeled on Fréchet spaces. In particular, they are absolute retracts if they are contractible.

(F) **Remark.** Theorems 1C and 1D suggest the form of a global deformation theory for structures on closed manifolds X: Start with a smooth bundle $\gamma: V \to X$ associated with the principal bundle of X. Then the space $\mathscr{C}(\gamma)$ of C^r-sections $(0 \leq r \leq \infty)$ of γ is an infinite dimensional manifold. Specify a subgroup $\mathscr{G}$ of $\mathbf{D}(X)$; then $\mathscr{G}$ acts continuously on $\mathscr{C}(\gamma)$, and we can form the quotient space $\mathbf{T}(\gamma; \mathscr{G})$. In a large variety of cases the differential of the quotient map $\Phi: \mathscr{C}(\gamma) \to \mathbf{T}(\gamma; \mathscr{G})$ determines the infinitesimal deformation theory of Kodaira-Spencer.

2. Complex structures

(A) A complex structure on the oriented vector space $\boldsymbol{R}^2$ is an endomorphism J of square $-I$ such that det $(v, Jv) > 0$ for $v \in \boldsymbol{R}^2$. The space M

of all such structures is the homogeneous space $GL^+(\boldsymbol{R}^2)/GL(\boldsymbol{C}^1)$. Here $GL^+(\boldsymbol{R}^2)$ is the group of real 2×2 matrices with positive determinant, and $GL(\boldsymbol{C}^1)$ is the multiplicative group of non-zero complex numbers, embedded in $GL^+(\boldsymbol{R}^2)$ by

$$a + ib \mapsto \begin{pmatrix} a & -b \\ b & a \end{pmatrix}.$$

On the other hand, if we write $a + ib$ in the form $r \exp i\theta$, $r > 0$, we can identify $GL(\boldsymbol{C}^1)$ with $GL^+(\boldsymbol{R}^1) \times SO(\boldsymbol{R}^2)$, where $SO(\boldsymbol{R}^2)$ is the rotation subgroup of $GL^+(\boldsymbol{R}^2)$. The corresponding homogeneous space is the space of conformal structures on $\boldsymbol{R}^2$, and we have the canonical identification

$$GL^+(\boldsymbol{R}^2)/GL(\boldsymbol{C}^1) = M = GL^+(\boldsymbol{R}^2)/GL(\boldsymbol{R}^1) \times SO(\boldsymbol{R}^2) \tag{2.1}$$

of the complex and conformal structures on $\boldsymbol{R}^2$. (We recall that a conformal structure on $\boldsymbol{R}^2$ is an equivalence class of positive definite quadratic forms on $\boldsymbol{R}^2$, where two such forms are equivalent if they are proportional.)

As is well known, the homogeneous space M can be represented as the open unit disk $\Delta = \{z \in \boldsymbol{C}\colon |z| < 1\}$ in $\boldsymbol{R}^2$. We do so by associating with each $\mu \in \Delta$ the equivalence class of the quadratic form

$$Q(x, y) = |z + \mu\bar{z}|^2\,, \qquad z = x + iy\,. \tag{2.2}$$

(B) Let X be an oriented connected smooth ($=C^\infty$) 2-manifold. From its principal $GL^+(\boldsymbol{R}^2)$-bundle we construct the associated homogeneous bundle with fibre M. We denote by $\mathbf{M}(X)$ the space of smooth sections of this bundle, endowed with the C^∞-topology, i.e., the topology of uniform convergence of all differentials on compact subsets of X. The elements of $\mathbf{M}(X)$ are well known to be the almost complex structures on X which are compatible with its orientation. Since X is 2-dimensional, every almost complex structure is integrable, and so $\mathbf{M}(X)$ is the space of complex structures on X [31, Ch. II N°3]. Of course the identification (2.1) means that $\mathbf{M}(X)$ can equally well be considered as the space of conformal structures on X.

3. Beltrami's equation

(A) Let D be a subregion of $\boldsymbol{R}^2$. The Fréchet space $C^\infty(D, \boldsymbol{C})$ is the vector space of smooth complex-valued functions on D with the C^∞-topology. The space $\mathbf{M}(D)$ of complex structures on D may be identified with the subset $C^\infty(D, \Delta)$ of $C^\infty(D, \boldsymbol{C})$ through our identification (2.2) of Δ with M. Explicitly, each $\mu\colon D \longrightarrow \Delta$ induces the conformal ($=$complex) structure on D represented by

$$ds = |dz + \mu(z)d\bar{z}|\,. \tag{3.1}$$

We note that the zero function induces the usual complex structure on D.

Suppose that D has the structure (3.1) and C its usual complex structure. Then the map $w: D \to C$ is holomorphic if and only if it satisfies Beltrami's equation

$$w_{\bar{z}} = \mu w_z , \tag{3.2}$$

where

$$w_z = \frac{1}{2}\left(\frac{\partial w}{\partial x} - i\frac{\partial w}{\partial y}\right), \qquad w_{\bar{z}} = \frac{1}{2}\left(\frac{\partial w}{\partial x} + i\frac{\partial w}{\partial y}\right).$$

(B) Since $|\mu(z)| < 1$ for all $z \in D$, the Beltrami equation (3.2) is elliptic. (3.2) is uniformly elliptic in D if and only if there is a number k such that

$$|\mu(z)| \leq k < 1 , \qquad z \in D .$$

The theory of uniformly elliptic Beltrami equations is thoroughly developed [3], [4], [10], [24].

Every such equation has a solution which is a diffeomorphism of D onto a region in the plane. If D is the plane C, there is a unique solution of (3.2), denoted by w_μ, which is a diffeomorphism of C onto itself and leaves the points $0, 1, \infty$ fixed. If D is the upper half plane $U = \{z \in C: \operatorname{Im} z > 0\}$, there is a unique solution of (3.2), again denoted by w_μ, which is a homeomorphism of the closure of U onto itself and leaves $0, 1, \infty$ fixed. In both cases w_μ will be called the normalized solution of (3.2).

We shall need the following theorem about the dependence of w_μ on μ. For its proof we refer to the companion paper [15]. (The theorems of our announcement [14] were based on a more primitive version, proved by us somewhat differently, following [10]). In the statement of the theorem, D is either U or C.

Theorem. *For each positive number $k < 1$, the map $\mu \mapsto w_\mu$ is a homeomorphism of the set of $\mu \in \mathbf{M}(D)$ with* $\sup\{|\mu(z)| : z \in D\} \leq k$ *onto its image in $C^\infty(D, C)$.*

Remark. The construction of homeomorphisms and diffeomorphisms as global solutions of elliptic systems provides a promising tool in topology. For instance,

1) the above theorem implies almost immediately Smale's theorem that the identity component of the diffeomorphism group of the 2-sphere has the rotation group as strong deformation retract-as we shall find in § 9;

2) the homotopy types of the groups of diffeomorphisms of closed surfaces of higher genera can be determined by constructing harmonic maps [16] (diffeomorphic solutions of a second order elliptic system, namely the Euler-Lagrange equation of the energy integral of § 8E below), utilizing the results of [20] and [28]. Further discussion will be given in § 8E.

4. Fuchsian groups

(A) The uniformization theorem says that every simply connected Riemann surface (= surface with complex structure) is conformally equivalent to the Riemann sphere, to $\boldsymbol{C}$, or to the upper half plane U (each with its usual complex structure). A complex structure on the surface X induces a complex structure on its universal covering surface $\tilde{X}$, which is therefore (equivalent to) one of the above.

With four exceptions (X the plane, punctured plane, torus, or sphere), $\tilde{X} = U$, and the cover group Γ is a properly discontinuous group of holomorphic automorphisms of U, acting freely on U. Such a group is called a *Fuchsian group*. (By requiring a Fuchsian group to act freely we are violating standard usage; for our purposes it is convenient to do so).

(B) The group G of all holomorphic automorphisms of U consists of the Möbius transformations

$$Az = (az + b)(cz + d)^{-1}; \quad a, b, c, d \in \boldsymbol{R}; \quad ad - bc = 1 .$$

G is therefore a 3-dimensional Lie group, isomorphic to $SL(\boldsymbol{R}^2)$ modulo its center. Its Lie algebra $\mathfrak{g}$ is $\mathrm{sl}(\boldsymbol{R}^2)$, the algebra of 2×2 real matrices of trace zero. The *adjoint representation* $u \mapsto u^A$ of G on $\mathfrak{g}$ is defined by $u^A = (\mathrm{Ad}\, A)u$, where $\mathrm{Ad}\, A \colon \mathfrak{g} \to \mathfrak{g}$ is the differential at the identity in G of the map $B \mapsto A^{-1}BA$.

The elements of G are conveniently classified by the positions of their fixed points. An element $A \in G$, not the identity, is called *hyperbolic*, *parabolic*, or *elliptic* according as A has two fixed points in $\boldsymbol{R} \cup \{\infty\}$, one fixed point in $\boldsymbol{R} \cup \{\infty\}$ (and no others), or two conjugate non-real fixed points. For us, the hyperbolic and parabolic transformations are of special importance because Γ acts freely and therefore cannot have elliptic elements.

If $A \in G$ is hyperbolic, one of its fixed points is *attractive*, the other *repulsive*. The attractive fixed point z_1 is described by the condition $A^n z \to z_1$ as $n \to \infty$ for any $z \in U$. The attractive fixed point of A is the repulsive fixed point of A^{-1}. These assertions are readily verified by noting that every hyperbolic transformation is conjugate in G to a homothetic expansion $z \mapsto kz$ $(k > 1)$.

Lemma 1. *If Γ is not cyclic, the centralizer of Γ in G is trivial.*

This classical fact is proved in two steps, both easy. First one proves that two non-trivial elements of G commute if and only if their fixed points coincide. Next one verifies that a discrete subgroup of G whose elements all have the same fixed points is cyclic.

Lemma 2. *If X is compact, Γ consists of hyperbolic transformations. If two elements of Γ have a common fixed point, they commute.*

This lemma is also classical. The first assertion is proved in [6, p. 97]. The second assertion follows from the first, because if two non-commuting hyperbolic transformations have a (unique) common fixed point, then their commutator is parabolic.

(C) Let X be a compact Riemann surface of genus $g > 1$. As we have seen, there exists a holomorphic covering map $\pi: U \to X$. π is of course not unique; it may be composed with any element of G. To specify one such π, we *mark* the surface X by choosing a basepoint $x_0 \in X$ and a canonical system of loops $a_1, \cdots, a_g, b_1, \cdots, b_g$ generating the fundamental group $\pi_1(X, x_0)$.

Lemma. *For each complex structure $J \in \mathbf{M}(X)$ there is a unique J-holomorphic covering map $\pi: U \to X$ with Fuchsian cover group Γ such that, for some $z_0 \in \pi^{-1}(x_0)$,*

1) *the element $A_1 \in \Gamma$ determined by a_1 has its fixed points at 0 and ∞,*
2) *the element $B_1 \in \Gamma$ determined by b_1 has its attractive fixed point at 1.*

Proof. Given J, choose any holomorphic covering map $\pi_1: U \to X$ and any $z_1 \in \pi_1^{-1}(x_0)$. Denote the cover group by Γ_1. Then the elements A_1 and B_1 of Γ_1 determined by a_1 and b_1 do not commute. Thus, by Lemma 2 of § 4B, the fixed points of A_1 and the attractive fixed point of B_1 are distinct. Hence there is a unique $A \in G$ which moves the fixed points of A_1 to 0 and ∞, and the attractive fixed point of B_1 to 1. $\pi = \pi_1 \circ A^{-1}$ is the required covering map.

5. The action of $\mathbf{D}(X)$ on $\mathbf{M}(X)$, $g > 1$

From now until § 9, X will be a compact oriented surface of genus $g > 1$, marked as in § 4C. In this section we study the action of $\mathbf{D}(X)$ on $\mathbf{M}(X)$. It is convenient to avoid the use of charts on X, employing the uniformization theorem to lift $\mathbf{M}(X)$ and $\mathbf{D}(X)$ to U. We carry out the lifting in §§ 5A and B.

Many results of this section are true under less stringent assumptions on X. We use the compactness of X only in Propositions 5A and 5D.

(A) Since X is marked, by Lemma 4C each complex structure J in $\mathbf{M}(X)$ determines a smooth covering map $\pi: U \to X$ whose cover group Γ is Fuchsian. The map π induces a map $\pi^*: \mathbf{M}(X) \to \mathbf{M}(U)$ whose image we denote by $\mathbf{M}(\Gamma)$; its elements are the Γ-invariant complex structures on U. Recall from § 3 that $\mathbf{M}(U)$ is $C^\infty(U, \Delta)$. The uniformization theorem assures that for each $\mu \in \mathbf{M}(U)$ there is a diffeomorphism $w: U \to w(U) \subset C$ which satisfies Beltrami's equation (3.2). Moreover, μ is Γ-invariant if and only if $w \circ \gamma$ satisfies (3.2), which happens when and only when

$$(\mu \circ \gamma)\bar{\gamma}'/\gamma' = \mu \qquad \text{for all } \gamma \in \Gamma \, . \tag{5.1}$$

For reasons which will become evident in § 7A we denote by $A^1(\Gamma)$ the Fréchet space of all $\mu \in C^\infty(U, C)$ which satisfy (5.1).

Proposition. *$\mathbf{M}(\Gamma)$ is the convex open set in $A^1(\Gamma)$ consisting of those $\mu \in A^1(\Gamma)$ such that* $\sup \{|\mu(z)|: z \in U\} < 1$; *and $\pi^*: \mathbf{M}(X) \to \mathbf{M}(\Gamma)$ is a homeomorphism.*

Proof. Since X is compact, Γ has a compact fundamental domain ω. Equation (5.1) shows that $\sup\{|\mu(z)|: z \in U\} = \max\{|\mu(z)|: z \in \omega\}$ for all $\mu \in A^1(\Gamma)$. Thus μ maps U into Δ if and only if that maximum is less than one. The assertion concerning π^* requires no proof.

As an open set in the complex Fréchet space $A^1(\Gamma)$, $\mathbf{M}(\Gamma)$ has a natural complex structure. The map π^* therefore induces a complex structure on $\mathbf{M}(X)$. Any choice of $J \in \mathbf{M}(X)$ leads to the same complex structure on $\mathbf{M}(X)$ because a diffeomorphism $w: U \to U$ induces a holomorphic automorphism $w^*: \mathbf{M}(\Gamma) \to \mathbf{M}(w\Gamma w^{-1})$. Thus we obtain the

Corollary. *$\mathbf{M}(X)$ is a contractible complex analytic manifold modeled on a Fréchet space.*

(B) Let $\mathbf{D}(U)$ be the group of orientation preserving diffeomorphisms of U. As a subset of $C^\infty(U, C)$, $\mathbf{D}(U)$ is metrizable. Furthermore, it is a topological group, by an easy application of Arens' theorem [5]. Let $\mathbf{D}(\Gamma)$ be the normalizer of Γ in $\mathbf{D}(U)$. Then the covering map π induces a continuous epimorphism $\pi_*: \mathbf{D}(\Gamma) \to \mathbf{D}(X)$ with kernel Γ, given by $\pi_*(f) \circ \pi = \pi \circ f$.

Lemma. *π_* is an open map.*

Proof. The hyperbolic metric $ds = |z - \bar{z}|^{-1}|dz|$ defines on U a complete Γ-invariant Riemannian structure of constant curvature -4. Any two points z_1, z_2 in U can be joined by a unique geodesic segment whose length is the hyperbolic distance $\rho(z_1, z_2)$.

Let (g_n) be a sequence in $\mathbf{D}(X)$ converging to the identity 1. Choose z_0 in U and a sequence (f_n) in $\mathbf{D}(\Gamma)$ so that $\pi_*(f_n) = g_n$ and $f_n(z_0) \to z_0$. The hypothesis on (g_n) means that for each small open set 0 in U there is a sequence (γ_n) in Γ such that $\gamma_n \circ f_n \to 1$ in $C^\infty(0, C)$. Hence on each compact subset of 0

$$\rho(f_n(z_1), f_n(z_2)) = \rho(\gamma_n(f_n(z_1)), \gamma_n(f_n(z_2))) \leq K\rho(z_1, z_2)$$

for some number K. It follows that the same inequality (with different K) holds on compact subsets of U. Because $f_n(z_0) \to z_0$, a subsequence (still called (f_n)) converges, uniformly on compact subsets of U, to a map $f: U \to U$. But $\pi(f(z)) = \lim g_n(\pi(z)) = \pi(z)$. Thus $f \in \Gamma$; in fact $f = 1$ because $f(z_0) = z_0$ and Γ acts freely. We conclude that $f_n \to 1$ in $\mathbf{D}(\Gamma)$, for in the above convergence $\gamma_n \circ f_n \to 1$ in $C^\infty(0, C)$, γ_n must be the identity for large n. The lemma is proved.

Corollary. *π_* induces an isomorphism between the topological groups $\mathbf{D}(\Gamma)/\Gamma$ and $\mathbf{D}(X)$.*

Let $\mathbf{D}_0(\Gamma) = \{f \in \mathbf{D}(\Gamma): f \circ \gamma = \gamma \circ f$ for all $\gamma \in \Gamma\}$, the centralizer of Γ in $\mathbf{D}(\Gamma)$. Recall that $\mathbf{D}_0(X) = \{g \in \mathbf{D}(X): g$ is homotopic to the identity$\}$.

Proposition. *$\pi_*: \mathbf{D}_0(\Gamma) \to \mathbf{D}_0(X)$ is an isomorphism of topological groups.*

Proof. It is well known that $\pi_*(\mathbf{D}_0(\Gamma)) = \mathbf{D}_0(X)$; see for instance [6, pp. 98–100]. We have already noted that the kernel of $\pi_*: \mathbf{D}(\Gamma) \to \mathbf{D}(X)$ is Γ.

Since $\mathbf{D}_0(\Gamma)\cap\Gamma$, the center of Γ, is trivial by Lemma 1 of § 4B, $\pi_*: \mathbf{D}_0(\Gamma) \to \mathbf{D}_0(X)$ is bijective.

It remains to show that $\pi_*^{-1}: \mathbf{D}_0(X) \to \mathbf{D}_0(\Gamma)$ is continuous. Given f in $\mathbf{D}_0(\Gamma)$, let (g_n) be a sequence in $\mathbf{D}_0(X)$ converging to $g = \pi_*(f)$. We must prove that $w_n = \pi_*^{-1}(g_n) \to f$. By the lemma, there is a sequence (f_n) in $\mathbf{D}(\Gamma)$ such that $f_n \to f$ and $\pi_*(f_n) = g_n$. Now $h_n = f_n \circ w_n^{-1} \in$ kernel $\pi_* = \Gamma$, and

$$h_n \circ \gamma \circ h_n^{-1} = f_n \circ \gamma \circ f_n^{-1} \to f \circ \gamma \circ f^{-1} = \gamma$$

for all $\gamma \in \Gamma$. Choose non-commuting elements γ_1 and γ_2 of Γ. For sufficiently large n, h_n commutes with both γ_1 and γ_2, whence h_n is the identity. (Otherwise the fixed points of h_n would coincide with those of both γ_1 and γ_2, which is impossible).

(C) The covering map π transfers the natural action (pulling back the complex structure) of $\mathbf{D}(X)$ on $\mathbf{M}(X)$ to an action of $\mathbf{D}(\Gamma)$ on $\mathbf{M}(\Gamma)$, given by

$$(\pi^*J)\cdot g = \pi^*(J\cdot\pi_*g) \qquad \text{for } g \in \mathbf{D}(\Gamma),\ J \in \mathbf{M}(X)\ . \tag{5.2}$$

Of course (5.2) is the restriction of the natural action of $\mathbf{D}(U)$ on $\mathbf{M}(U)$. That action has a convenient expression when $\mu \in \mathbf{M}(U)$ is of the form $\mu_f = f_{\bar z}/f_z$, $f \in \mathbf{D}(U)$. Indeed, $\mu_f = 0\cdot f$, the pullback by f of the usual complex structure on U. Thus

$$\mu_f\cdot g = (0\cdot f)\cdot g = 0\cdot(\mathrm{f}\circ \mathrm{g}) = \mu_{\mathrm{f}\circ\mathrm{g}}\ . \tag{5.3}$$

Each μ in $\mathbf{M}(\Gamma)$ has the form μ_f; for we may take $f = w_\mu$, the solution of (3.2) introduced in § 3B, since Proposition 5A insures that μ is bounded by some $k < 1$.

Proposition.

1. *The action* $\mathbf{M}(\Gamma)\times\mathbf{D}(\Gamma)\to\mathbf{M}(\Gamma)$ *defined by* (5.2) *is continuous.*
2. *The isotropy group of* $0 \in \mathbf{M}(\Gamma)$ *is* $\mathbf{D}(\Gamma)\cap G = N(\Gamma)$, *the normalizer of* Γ *in* G.
3. $\Gamma = \{g \in \mathbf{D}(\Gamma)$: *g acts trivially on* $\mathbf{M}(\Gamma)\}$.
4. $\mathbf{D}_0(\Gamma)$ *acts freely on* $\mathbf{M}(\Gamma)$.

Proof. 1. The continuity of (5.2) follows from general principles. For an alternative proof using (5.3) and Theorem 3B, we observe that each of the following maps is continuous:

$$(\mu, g) \mapsto (w_\mu, g) \mapsto w_\mu \circ g \mapsto \mu\cdot g\ .$$

2. The isotropy group of $0 \in \mathbf{M}(\Gamma)$ consists of all $g \in \mathbf{D}(\Gamma)$ which are holomorphic automorphisms of U with its usual complex structure; that group is $\mathbf{D}(\Gamma)\cap G$.

3. Since $\mathbf{M}(\Gamma)$ consists of the Γ-invariant complex structures on U, it is evident that Γ acts trivially on $\mathbf{M}(\Gamma)$. Thus Γ is a subgroup of the group Γ_0

of all g which act trivially; by part 2, Γ_0 in turn is a subgroup of G. If $\Gamma_0 \neq \Gamma$, there would exist a fundamental domain ω for Γ and a pair of Γ_0-equivalent points $z_1, z_2 \in \omega$ with $z_1 \in \operatorname{Int} \omega$. Let μ be a smooth function on $\operatorname{Int} \omega$ which has compact support containing z_1 but not z_2. Extending the definition of μ to U by (5.1) we obtain an element of $\mathbf{M}(\Gamma)$ which is not Γ_0-invariant. We conclude that $\Gamma_0 = \Gamma$.

Part 4 is equivalent to the assertion that $\mathbf{D}_0(X)$ acts freely on $\mathbf{M}(X)$, because π_* is an isomorphism on $\mathbf{D}_0(\Gamma)$. Since the complex structure $J \in \mathbf{M}(X)$ corresponding to $0 \in \mathbf{M}(\Gamma)$ was chosen arbitrarily, we need only consider the isotropy group of $0 \in \mathbf{M}(\Gamma)$ relative to $\mathbf{D}_0(\Gamma)$. That group is $\mathbf{D}_0(\Gamma) \cap N(\Gamma)$, the centralizer of Γ in G, which we know to be trivial.

Corollary. *The natural action of* $\mathbf{D}(X)$ *on* $\mathbf{M}(X)$ *is continuous and effective.* $\mathbf{D}_0(X)$ *acts freely.*

(D) **Proposition.** $\mathbf{D}(X)$ *acts properly on* $\mathbf{M}(X)$.

Proof. The condition of proper action means that the map $\Theta: \mathbf{M}(X) \times \mathbf{D}(X) \to \mathbf{M}(X) \times \mathbf{M}(X)$ defined by $\Theta(J, f) = (J, J \cdot f)$ is proper. We shall prove the corresponding assertion in U.

First, let $K \subset \mathbf{M}(\Gamma) \times \mathbf{D}(\Gamma)/\Gamma$ be a closed set, and $((\mu_n, \nu_n))$ a sequence in $\Theta(K)$, converging to (μ, ν). Fix z_0 in U and a compact fundamental domain ω for Γ, and choose a sequence (f_n) in $\mathbf{D}(\Gamma)$ so that $\nu_n = \mu_n \cdot f_n$, $(\mu_n, f_n\Gamma) \in K$, and $z_n = f_n(z_0) \in \omega$.

Let $w_n = w_{\mu_n}$, $w = w_\mu$, and $h = w_\nu$. By Theorem 3B, $w_n \to w$. Determine a sequence (g_n) in G so that $g_n \circ w_n \circ f_n$ fixes the points 0, 1, ∞. Then (5.3) and Theorem 3B imply that $g_n \circ w_n \circ f_n \to h$; in particular, $g_n(w_n(z_n)) \to h(z_0) \in U$. Since the points $w_n(z_n)$ lie in a compact subset of U, we can pass to a subsequence so that $g_n \to g \in G$. Then $f_n \to w^{-1} \circ g \circ h = f \in \mathbf{D}(\Gamma)$. Obviously $(\mu_n, f_n) \to (\mu, f)$, and $(\mu, \nu) = (\mu, \mu \cdot f)$ is in the image of K. Thus, Θ is a closed map.

It remains to prove that $\Theta^{-1}(J_1, J_2)$ is compact for any $(J_1, J_2) \in \mathbf{M}(X) \times \mathbf{M}(X)$. We may use $J = J_1$ to determine $\pi: U \to X$; then (J_1, J_2) corresponds to $(0, \nu) \in \mathbf{M}(\Gamma) \times \mathbf{M}(\Gamma)$. If $\Theta(\mu_1, f_1\Gamma) = \Theta(\mu_2, f_2\Gamma) = (0, \nu)$, then $0 = \mu_1 = \mu_2 = 0 \cdot f_1 \circ f_2^{-1}$, and $f_1 \circ f_2^{-1} \in N(\Gamma)$ by Proposition 5C. We conclude that $\Theta^{-1}(0, \nu)$ either is empty or can be mapped bijectively onto $N(\Gamma)/\Gamma$. But $N(\Gamma)/\Gamma$ is a finite group [34, Ch. II].

Corollary 1. $\mathbf{D}_0(X)$ *acts properly on* $\mathbf{M}(X)$.

In fact, every closed subgroup acts properly.

Corollary 2. *The natural action of* $\mathbf{D}(X)/\mathbf{D}_0(X)$ *on* $\mathbf{M}(X)/\mathbf{D}_0(X)$ *is properly discontinuous.*

The proposition implies that the action is proper. But $\mathbf{D}(X)/\mathbf{D}_0(X)$ is discrete because $\mathbf{D}_0(X)$, for compact X, is open in $\mathbf{D}(X)$. Hence the corollary.

The group $\mathbf{D}(X)/\mathbf{D}_0(X)$ is the modular group of genus g. The first proof that its action is properly discontinuous was given by Kravetz [23].

6. The map P

To complete the proof that the action of $\mathbf{D}_0(X)$ on $\mathbf{M}(X)$ defines a principal fibre bundle, we need local cross-sections, which are provided by the Bers coordinates on Teichmüller space [1], [7]. To obtain those coordinates we follow the classical path [1], [6], [7], imbedding Teichmüller space as a smooth manifold of dimension $6g-6$ in G^{2g}, where again G is the real Möbius group. The imbedding is accomplished by a smooth map $P: \mathbf{M}(X) \to G^{2g}$ which factors through $\mathbf{M}(X)/\mathbf{D}_0(X)$. In § 7 we shall prove that the differential of P establishes an isomorphism between the theories of infinitesimal deformations of complex structures and of Fuchsian groups.

(A) The assumption introduced in § 5, that X is a marked surface of genus $g>1$, is still in force. We define $P: \mathbf{M}(X) \to G^{2g}$ by $P(J)=(A_1, B_1, \cdots, A_g, B_g)$. Here A_i and B_i are the elements of Γ determined by the loops a_i, b_i, and Γ is the group determined by J as in Lemma 4C. Of course the set $\{A_1, \cdots, B_g\}$ generates Γ. In the spirit of [1], [6], we denote by $\mathcal{S}$ the set of points $(A_1, \cdots, B_g) \in G^{2g}$ such that

(6.1) the product of commutators $\Pi_{1\le i\le g}[A_i, B_i] = 1$,

(6.2) the fixed points of A_g and B_g are real and distinct,

(6.3) $A_1(0) = 0, \quad A_1(\infty) = \infty, \quad B_1(1) = 1.$

It is clear that P maps $\mathbf{M}(X)$ into $\mathcal{S}$.

Proposition. *$\mathcal{S}$ is a real analytic submanifold of G^{2g} of dimension* $6g-6$.

Proof. Let N be the set of $(A_1, \cdots, B_g) \in G^{2g}$ which satisfy (6.2) and (6.3). It is clear that N is a real analytic $(6g-3)$-dimensional submanifold of G^{2g}. The map $\phi: N \to G$ given by

$$\phi(A_1, \cdots, B_g) = \Pi_{1\le i\le g}[A_i, B_i]$$

is real analytic, and $\mathcal{S} = \phi^{-1}(1) \subset N$. The proposition will therefore follow from the implicit function theorem as soon as we prove that the differential of ϕ at every $s \in \mathcal{S}$ is surjective.

Choose $s = (A_1, \cdots, B_g) \in \mathcal{S}$ and $u, v \in \mathfrak{g}$. Let

$$C(t) = \phi(A_1, \cdots, B_{g-1}, A_g \exp tu, B_g \exp tv), \qquad t \in \boldsymbol{R} .$$

An easy calculation gives

$$C(t) = \exp\{t(u^B - u + v - v^A)^{A^{-1}B^{-1}} + o(t)\} ,$$

where $A = A_g$ and $B = B_g$. Thus

$$(u^B - u + v - v^A)^{A^{-1}B^{-1}}$$

is in the image of the differential $d\phi(s)$, and all we need to prove is the following lemma, which the reader can easily verify.

Lemma. *If A, $B \in G$ have distinct real fixed points, the map*

$$(u, v) \mapsto u^B - u + v - v^A \tag{6.4}$$

from $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ is surjective.

Remark. $u^B - u + v - v^A = w$ is the infinitesimal form of the equation $[A, B] = C$ studied by Ahlfors [1, Lemma 3] in a similar context.

(B) Take any $J_0 \in \mathbf{M}(X)$, and let $\pi : U \to X$ be the covering map determined by J_0 and the marking of X. Then the cover group Γ is generated by $s = P(J_0) \in G^{2g}$. Composing P with the inverse of the map $\pi^* : \mathbf{M}(X) \to \mathbf{M}(\Gamma)$ produces a map, still called $P : \mathbf{M}(\Gamma) \to \mathscr{S}$.

Lemma. $P(\mu) = w_\mu \circ s \circ w_\mu^{-1}$ *for all* $\mu \in \mathbf{M}(\Gamma)$.

Proof. For any μ in $\mathbf{M}(\Gamma)$, $\pi_\mu = \pi \circ w_\mu^{-1} : U \to X$ is a covering map, holomorphic from U with its usual complex structure to X with the complex structure $(\pi^*)^{-1}\mu$. The cover group $\Gamma_\mu = w_\mu \circ \Gamma \circ w_\mu^{-1}$ is Fuchsian, and the loops a_1 and b_1 on X determine the transformations $w_\mu A_1 w_\mu^{-1}$ and $w_\mu B_1 w_\mu^{-1}$ in Γ_μ. Because w_μ fixes the points 0, 1, and ∞, π_μ is the cover map determined by Lemma 4C from the marking of X and the complex structure $(\pi^*)^{-1}\mu$, and hence the lemma is proved.

Proposition. *$P : \mathbf{M}(\Gamma) \to \mathscr{S}$ is continuous. The restriction of P to any finite dimensional affine subspace is real analytic. Moreover, the kernel* Ker $dP(0)$ *of the differential at* 0 *consists of all $\nu \in A^1(\Gamma)$ such that*

$$\dot{\gamma}(\nu)(z) = \lim_{t \to 0} \frac{\gamma_{t\nu}(z) - \gamma(z)}{t} \tag{6.5}$$

vanishes for all $z \in U$ and $\gamma \in \Gamma$.

Proof. The continuity of P follows at once from the last lemma and Theorem 3B. For any $\gamma \in \Gamma$ consider the map $\mu \mapsto \gamma_\mu = w_\mu \gamma w_\mu^{-1} \in G$, which is real analytic on finite dimensional subspaces by [4], and whose directional derivative at 0 in the direction ν vanishes if and only if $\dot{\gamma}(\nu)(z)$ vanishes for all $z \in U$. The required real analyticity of P is now obvious, for each component map of P has the form $\mu \mapsto \gamma_\mu$. Furthermore, $\nu \in \text{Ker}\, dP(0)$ if and only if (6.5) vanishes for all γ in a set of generators of Γ, hence for all γ.

(C) **Lemma.** *$P(J_0) = P(J_1)$ if and only if J_0 and J_1 are $\mathbf{D}_0(X)$-equivalent.*

Proof. We shall prove that $P(0) = P(\mu)$, $\mu \in \mathbf{M}(\Gamma)$, if and only if 0 and μ are $\mathbf{D}_0(\Gamma)$-equivalent. By Lemma 6B, $P(0) = P(\mu)$ if and only if $w_\mu \in \mathbf{D}_0(\Gamma)$. But 0 and μ are $\mathbf{D}_0(\Gamma)$-equivalent if and only if $\mu = \mu_f$ for some $f \in \mathbf{D}_0(\Gamma)$. That f can only be w_μ. In fact $A_1 = f \circ A_1 \circ f^{-1}$ and $w_\mu \circ A_1 \circ w_\mu^{-1}$ both fix 0 and ∞, while $f \circ B_1 \circ f^{-1} (= B_1)$ and $w_\mu \circ B_1 \circ w_\mu^{-1}$ both have the attractive fixed point 1. Thus, $g = f \circ w_\mu^{-1}$ leaves 1 fixed and maps the set $\{0, \infty\}$ on itself; this implies g is the identity and $f = w_\mu$, because $g \in G$.

7. The infinitesimal theory

Here we investigate the connection between the global space of complex structures on X, described by $\mathbf{M}(X)/\mathbf{D}_0(X)$, and the theory of infinitesimal variations of complex structures, measured by appropriate cohomology spaces. There is also a connection with the theory of infinitesimal deformations of Fuchsian groups. In fact, the cohomology spaces associated with those two theories are isomorphic, the isomorphism being given by the differential of P. In a sense, $P: \mathbf{M}(X) \to \mathscr{S}$ is the envelope of the cohomology isomorphisms. Our point of view in this section has been influenced by Weil's paper [32].

(A) A complex structure J_0 on X defines on each tangent vector space $T_x(X)$ an endomorphism $J_0(x)$ of square $= -I$. This extends to a complex endomorphism $J_0(x)$ of $CT_x(X) = C \otimes_R T_x(X)$; that space has the direct sum decomposition $T_x^{1,0} \oplus T_x^{0,1}$, where $T_x^{1,0}$ (resp. $T_x^{0,1}$) is the image of the projection operator $\frac{1}{2}(I - iJ_0(x))$ (resp. $\frac{1}{2}(I + iJ_0(x))$). This induces a similar decomposition on all tensor products of $CT_x(X)$ and its dual space $CT_x(X)^*$.

Let A^p be the vector space of smooth differential forms on X of type (o, p) with values in the vector bundle $T^{1,0}(X)$. The (0, 1)-component $\bar{\partial}$ of the exterior differential maps A^p into A^{p+1}. Following Kodaira-Spencer [22], let Θ denote the sheaf of germs of smooth sections of $T^{1,0} \otimes T^{*0,1}$. The $\bar{\partial}$-cohomology group $H^1(X, \Theta)$ measures the infinitesimal variations of J_0; because $\bar{\partial}$ is zero on A^1, $H^1(X, \Theta) = A^1/\bar{\partial}A^0$.

Remark. The complex structure J_0 identifies the vector space of smooth real vector fields on X with A^0. Indeed, suppose $v \in C^\infty(CT(X))$ is expressed as $v = v^{1,0} + v^{0,1}$; then v is real if and only if $(v^{1,0})^- = v^{0,1}$.

(B) Once more we pass to the universal covering space U by the holomorphic covering map π. In the notation of § 5A, the space A^0 lifts to

$$A^0(\Gamma) = \{f \in C^\infty(U, C): (f \circ \gamma)/\gamma' = f \text{ for all } \gamma \in \Gamma\} ;$$

the space A^1 lifts to $A^1(\Gamma)$. Of course, with this interpretation $\bar{\partial} f = f_{\bar{z}}$.

Let $Q(\Gamma)$ be the lift of the vector space $H^0(X, T^{*1,0} \odot T^{*1,0})$ of holomorphic quadratic differentials; then $Q(\Gamma)$ consists of the holomorphic functions φ on U which satisfy

$$(\varphi \circ \gamma)(\gamma')^2 = \varphi \qquad \text{for all } \gamma \in \Gamma .$$

The vector spaces $H^0(X, T^{*1,0} \odot T^{*1,0})$ and $H^1(X, \Theta)$ are conjugate. This special case of Serre's duality theorem [27] - also known as Teichmüller's Lemma - is a consequence of the next

Proposition. Ker $dP(0) = \bar{\partial}A^0(\Gamma) = Q(\Gamma)^\perp$, *where*

$$Q(\Gamma)^\perp = \left\{\nu \in A^1(\Gamma): \int_X \nu\varphi d\bar{z} \wedge dz = 0 \quad \textit{for all } \varphi \in Q(\Gamma)\right\}.$$

Proof. Let $\nu \in \operatorname{Ker} dP(0)$. By [3, p. 138], [1],

$$\dot{\gamma}(\nu) = f \circ \gamma - \gamma' f , \quad \text{where} \quad f_{\bar{z}} = \nu .$$

But $\dot{\gamma}(\nu)$ vanishes for all $\gamma \in \Gamma$ by Proposition 6B. Therefore $f \in A^0(\Gamma)$, and we have proved

$$\operatorname{Ker} dP(0) \subset \bar{\partial} A^0(\Gamma) . \tag{7.1}$$

Next, take any $f \in A^0(\Gamma)$ and set $\nu = f_{\bar{z}}$. Then for each $\varphi \in Q(\Gamma)$, $\omega = f\varphi dz$ is a 1-form on X. By Stokes' theorem

$$\int_X \nu\varphi d\bar{z} \wedge dz = \int_X d\omega = 0 .$$

Thus

$$\bar{\partial} A^0(\Gamma) \subset Q(\Gamma)^{\perp} . \tag{7.2}$$

From (7.1) and (7.2), codim $\operatorname{Ker} dP(0) \geq \dim Q(\Gamma)$, which is $6g - 6$ by the Riemann-Roch theorem. Since the kernel of $dP(0)$ has codimension no greater than $6g - 6$, the dimension of $\mathscr{S}$, we conclude that $Q(\Gamma)^{\perp} = \operatorname{Ker} dP(0)$.

(C) We now define $H^1(\Gamma, \mathfrak{g})$, the 1-dimensional cohomology space of Γ relative to the adjoint representation, as follows. A 1-cocycle is a map $f: \Gamma \to \mathfrak{g}$ satisfying

$$f(\gamma_1 \circ \gamma_2) = f(\gamma_1)^{\gamma_2} + f(\gamma_2) , \tag{7.3}$$

and the coboundary δu of $u \in \mathfrak{g}$ is the 1-cocycle

$$\delta u(\gamma) = u^{\gamma} - u .$$

Thus $H^1(\Gamma, \mathfrak{g})$ is the quotient vector space of cocycles modulo coboundaries, which measures the infinitesimal deformations of Γ in G [32], [33].

Proposition. *The tangent space* $T_s(\mathscr{S}) = H^1(\Gamma, \mathfrak{g})$, *where* $s = P(0) \in \mathscr{S}$.

Proof. We construct a linear map $L: T_s(\mathscr{S}) \to H^1(\Gamma, \mathfrak{g})$ as follows: From each smooth curve $c: (-1, 1) \to \mathscr{S}$ with $c(0) = s$, construct a curve of homomorphisms $\varphi_t: \Gamma \to G$ by setting $c(t) = (\varphi_t(A_1), \cdots, \varphi_t(B_g))$. The curve φ_t gives rise to a 1-cocycle $f: \Gamma \to \mathfrak{g}$ in the usual way:

$$\gamma^{-1}\varphi_t(\gamma) = \exp(tf(\gamma) + \circ(t)) \qquad \text{for all } \gamma \in \Gamma .$$

Clearly f depends only on the tangent vector $c'(0) \in T_s(\mathscr{S})$. We define $L(c'(0))$ to be the image of f in $H^1(\Gamma, \mathfrak{g})$.

We show next that L is injective. Suppose that the cocycle f determined by φ_t is a coboundary: $f(\gamma) = u^{\gamma} - u$. Then the curves $\varphi_t(\gamma)$ and $\exp(tu)_{\gamma} \exp(-tu)$

are tangent at $t = 0$ for all $\gamma \in \Gamma$. Because A_1 and $\varphi_t(A_1)$ fix 0 and ∞, u is a diagonal matrix. Because B_1 and $\varphi_t(B_1)$ leave 1 fixed and B_1 has distinct fixed points, u is the zero matrix.

It remains to show that L is surjective, a consequence of the

Lemma. $\dim H^1(\Gamma, \mathfrak{g}) = 6g - 6 = \dim \mathscr{S}$.

Proof. We already know that $\dim H^1(\Gamma, \mathfrak{g}) \geq \dim \mathscr{S} = 6g - 6$ because $L\colon T_s(\mathscr{S}) \to H^1(\Gamma, \mathfrak{g})$ is injective. On the other hand, it is easy to verify, using (7.3), (6.1) and Lemma 6A, that the space of 1-cocycles has dimension not exceeding $6g - 3$. Finally, $u \mapsto \delta u$ maps $\mathfrak{g}$ injectively to the space of 1-coboundaries: if $u^{A_1} = u$, then u is diagonal; if also $u^{B_1} = u$, then u is the zero matrix.

(D) **Theorem.** *The differential*

$$dP(0)\colon A^1(\Gamma) \to H^1(\Gamma, \mathfrak{g}) = T_s(\mathscr{S})$$

of $P\colon M(\Gamma) \to \mathscr{S}$ at 0 induces an isomorphism $H^1(X, \Theta) \to H^1(\Gamma, \mathfrak{g})$.

Proof. Proposition 7B says that $\operatorname{Ker} dP(0) = \bar{\partial} A^0(\Gamma)$, and that the real dimension of $H^1(X, \Theta) = \dim Q(\Gamma) = 6g - 6$.

8. The Teichmüller space $T(X)$, $g > 1$

(A) **Proposition.** *$P\colon \mathbf{M}(X) \to \mathscr{S}$ is an open map with local sections; i.e., for each $s \in P(\mathbf{M}(X))$ there exist a neighborhood N of s in $\mathscr{S}$ and a real analytic map $f\colon N \to \mathbf{M}(X)$ with $P \circ f$ as the identity on N.*

Proof. Because J_0 was chosen arbitrarily in § 6B, we need only show that the map $P\colon \mathbf{M}(\Gamma) \to \mathscr{S}$ is open at the origin and has a local section $f\colon N \to \mathbf{M}(\Gamma)$ defined in a neighborhood N of $s = P(0)$. This is immediate from Propositions 7B, 7C, and the implicit function theorem.

Recall that *Teichmüller's space* $\mathbf{T}(X)$ is the quotient $\mathbf{M}(X)/\mathbf{D}_0(X)$ with quotient topology. In view of Lemma 6C, Proposition 8A has the immediate

Corollary. *$P\colon \mathbf{M}(X) \to \mathscr{S}$ has the form $P = h \circ \Phi$, where $\Phi\colon \mathbf{M}(X) \to \mathbf{T}(X)$ is the quotient map, and $h\colon \mathbf{T}(X) \to P(\mathbf{M}(X))$ is a homeomorphism.*

Remark. Map $Q(\Gamma)$ into $A^1(\Gamma)$ by $\varphi \mapsto \bar{\varphi}\lambda^{-2}$, where $\lambda(z)|dz|$ is the hyperbolic metric on U, and denote the image by $\mathscr{H}^1(\Gamma)$. Proposition 7B implies that $A^1(\Gamma)$ is the direct sum of $\operatorname{Ker} dP(0)$ and $\mathscr{H}^1(\Gamma)$; this can be viewed as a case of Hodge's theorem. Hence $\mu \mapsto P_\mu$ is a diffeomorphism from a neighborhood of the origin in $\mathscr{H}^1(\Gamma)$ to an open set in $\mathscr{S}$, and the set of all such diffeomorphisms provides complex local coordinate charts, the *Bers coordinates*, on $P(\mathbf{M}(X))$. These charts define a complex analytic structure [7] which is the quotient by P of the complex analytic structure of $\mathbf{M}(X)$ defined in § 5A. Each Bers coordinate chart can be extended (uniquely) to a global coordinate chart f, which is a holomorphic homeomorphism of $P(\mathbf{M}(X))$ onto an open subset of $\mathscr{H}^1(\Gamma)$. The restriction of f to $f^{-1}(\mathbf{M}(\Gamma))$ is a local section of P, the Ahlfors-Weill section [14], and the image of f is a bounded domain of holomorphy in $\mathscr{H}^1(\Gamma)$ [7], [9].

(B) A principal fibre bundle is determined by a continuous action of a topological group on a space, which is free, proper, and locally trivial [21]; the local triviality amounts to the existence of local sections of the quotient map.

Theorem. *The quotient map $\Phi: \mathbf{M}(X) \to \mathbf{T}(X)$ defines a universal principal fibre bundle with structure group $\mathbf{D}_0(X)$.*

Proof. The theorem consolidates the results of §§ 5C, 5D, and 8A. The bundle is universal because $\mathbf{M}(X)$ is contractible by Corollary 5A.

(C) **Teichmüller's Theorem.** *$T(X)$ is homeomorphic to $\boldsymbol{R}^{6g-6}$.*

We refer to [6] for a particularly simple proof.

Corollary 1. *The bundle $\Phi: \mathbf{M}(X) \to \mathbf{T}(X)$ is topologically trivial.*

Proof. By Teichmüller's theorem there is a map $g: \mathbf{T}(X) \times [0, 1] \to \mathbf{T}(X)$ with $g(\tau, 0) = \tau_0$ and $g(\tau, 1) = \tau$. By the covering homotopy theorem there is a map $f: \mathbf{T}(X) \times [0, 1] \to \mathbf{M}(X)$, which covers g. $\sigma(\tau) = f(\tau, 1)$ defines a section of the map Φ.

Corollary 2. *$\mathbf{M}(X)$ is homeomorphic to $\mathbf{T}(X) \times \mathbf{D}_0(X)$. In particular, $\mathbf{D}_0(X)$ is contractible.*

Proof. Let $\sigma: \mathbf{T}(X) \to \mathbf{M}(X)$ be any section of Φ. Then $(\tau, g) \mapsto \sigma(\tau)\cdot g$ is a homeomorphism from $\mathbf{T}(X) \times \mathbf{D}_0(X)$ to $\mathbf{M}(X)$.

(D) **Remarks.**

1. Recently M. E. Hamstrom [19] has computed the homotopy groups of the homeomorphism group $\mathscr{H}(X)$ (a topological group with compact-open topology) of any compact surface X with or without boundary. Comparison of her results with ours shows that in every case $\mathscr{H}_0(X)$ and $\mathbf{D}_0(X)$ have the same homotopy groups. It is reasonable to guess that the identity map $i: \mathbf{D}_0(X) \to \mathscr{H}_0(X)$ is a homotopy equivalence, which could be established if it were true that $\mathscr{H}_0(X)$ is an absolute neighborhood retract. It is not known whether $\mathscr{H}_0(X)$ enjoys the last property, although it is a locally contractible metrizable group.

2. Recall that $\mathbf{D}_0(X)$ consists of all $f \in \mathbf{D}(X)$ homotopic to the identity. In the topological category, R. Baer's theorem [17] states that homotopic homeomorphisms of X are isotopic. The fact that $D_0(X)$ is connected (by Corollary 2 above) gives Baer's theorem in the smooth category.

3. Corollary 1, the contractibility of $\mathbf{T}(X)$, and the contractibility of $\mathbf{D}_0(X)$ are equivalent properties. A. Grothendieck conjectured such a relationship [18], emphasizing the importance of a topological proof that $\mathbf{D}_0(X)$ is contractible. We sketch an analytical proof (therefore violating the spirit of Grothendieck's conjecture) in § 8E, and construct an explicit section of Φ.

4. By Remark 8A, $\mathbf{T}(X)$ has a complex structure such that $\Phi: \mathbf{M}(X)$

$\to \mathbf{T}(X)$ is holomorphic. Moreover, $\mathbf{T}(X)$ is a Stein manifold [9]. Since the bundle $\Phi\colon \mathbf{M}(X) \to \mathbf{T}(X)$ is topologically trivial, one might ask whether it is holomorphically trivial. The answer is no; there are no holomorphic cross-sections of Φ [12]. By contrast, in § 10 we shall define a holomorphic section of Φ when $g = 1$.

5. In the work of Ahlfors and Bers [2], [7], X is endowed with a fixed conformal structure, and one considers the space $M(X)$ of all conformal structures whose Teichmüller distance [6], [13] from the given one is finite. Let $Q_0(X)$ be the group of homeomorphisms of X, which are quasiconformal (relative to the given conformal structure) [6] and homotopic to the identity. $Q_0(X)$ operates on $M(X)$, and the quotient is $\mathbf{T}(X)$. Let $\Psi\colon M(X) \to \mathbf{T}(X)$ be the quotient map. Then Ψ does not define a fibre bundle with group $Q_0(X)$, for $Q_0(X)$ is not a topological group relative to the topology of $M(X)$. Still, Ψ is a locally trivial map [13], globally trivial if and only if $T(X)$ is contractible. The Ahlfors-Bers theory applies to non-compact surfaces; it is not known in general whether $\mathbf{T}(X)$ is contractible.

6. We should verify that for compact X the Teichmüller space of Ahlfors and Bers coincides with ours. It is clear that there are a continuous injection $j\colon \mathbf{M}(X) \to M(X)$, and an open map $Q\colon M(X) \to \mathcal{S}$ satisfying $P = Q \circ j$, whose image is the classical Teichmüller space [3], [6]. That P and Q have the same image follows, for instance, from [11, Theorem 3]; the point is simply that every homeomorphism of X is homotopic to a diffeomorphism.

(E) We shall now outline an alternative proof that the action of $\mathbf{D}_0(X)$ on $\mathbf{M}(X)$ produces a trivial fibre bundle. Our proof makes essential use of an unpublished theorem of J. Sampson.

Each complex structure on X gives rise to a holomorphic covering map $\pi\colon U \to X$, and the hyperbolic metric on U thereby induces a metric on X of constant curvature -4. Therefore we may interpret $\mathbf{M}(X)$ as the space of Riemannian metric structures of curvature -4 on X.

Given the metrics μ, ν in $\mathbf{M}(X)$ and a smooth map $f\colon X \to X$ we form its Dirichlet integral (energy)

$$E(f) = \frac{1}{2} \int_X \rho^2(f(z))(|f_z|^2 + |f_{\bar z}|^2)dxdy .$$

Here $z = x + iy$ is an isothermal parameter relative to μ, and ν is given in isothermal parameters by $ds = \rho(w)\,|dw|$.

It was proved by Sampson and Eells [16] and by Shibata [28] that there is a smooth map $f\colon X \to X$ which has minimal energy among maps homotopic to the identity. (Such an f is called a *harmonic map*, relative to μ and ν.) The strictly negative curvature of ν and the formula for the second variation of E imply that the harmonic f is unique; we denote it by $f(\mu, \nu)$. Shibata [28] proved that $f(\mu, \nu)$ is a homeomorphism. Theorems of Lewy [25] and Heinz

[20] imply that f is a diffeomorphism. Thus, for any fixed μ in $\mathbf{M}(X)$, we obtain a map $\nu \mapsto f(\mu, \nu)$ from $\mathbf{M}(X)$ into $\mathbf{D}_0(X)$. Sampson has proved that this map is continuous (oral communication).

Let (X, ν) denote the manifold X endowed with the Riemannian metric ν. Since the composite of a harmonic map and an isometry is harmonic, we obtain the commutative diagram, where $g \in \mathbf{D}_0(X)$:

$$\begin{array}{ccc} (X, \mu) & \xrightarrow{f(\mu, \nu)} & (X, \nu) \\ & {\scriptstyle f(\mu, \nu\cdot g)} \searrow & \uparrow {\scriptstyle g} \\ & & (X, \nu\cdot g) \end{array}$$

Thus

$$g \circ f(\mu, \nu\cdot g) = f(\mu, \nu) \qquad \text{for all } g \in \mathbf{D}_0(X) \ . \tag{8.1}$$

We now define a map $F\colon \mathbf{M}(X) \to \mathbf{T}(X) \times \mathbf{D}_0(X)$ by

$$F(\nu) = (\Phi(\nu), f(\mu, \nu)^{-1}) \ ,$$

where of course $\Phi\colon \mathbf{M}(X) \to \mathbf{T}(X)$ is the quotient map. Sampson's theorem implies that F is continuous. Moreover, (8.1) yields

$$F(\nu\cdot g) = F(\nu)\cdot g \qquad \text{for all } g \in \mathbf{D}_0(X) \ , \tag{8.2}$$

where $\mathbf{D}_0(X)$ acts on $\mathbf{T}(X) \times \mathbf{D}_0(X)$ in the obvious way: $(\tau, f)\cdot g = (\tau, f \circ g)$. It follows that F is injective, for if $F(\nu) = F(\nu')$, then $\Phi(\nu) = \Phi(\nu')$, so $\nu' = \nu\cdot g$, $g \in \mathbf{D}_0(X)$. Thus, by (8.2),

$$F(\nu) = F(\nu') = F(\nu\cdot g) = F(\nu)\cdot g \ ,$$

so $g = 1$ and $\nu = \nu'$. That F is surjective and a homeomorphism now follows from the identity

$$(\Phi(\nu), g) = F(\nu)\cdot f(\mu, \nu) \circ g = F(\nu\cdot f(\mu, \nu) \circ g) \ ,$$

valid for all $g \in \mathbf{D}_0(X)$. We conclude at once, *without appealing to either Teichmüller's theorem or* §§ 5–7, that $\mathbf{D}_0(X)$ and $\mathbf{T}(X)$ are contractible Hausdorff spaces, *and* that $\Phi\colon \mathbf{M}(X) \to \mathbf{T}(X)$ a trivial fibre bundle with structure group $\mathbf{D}_0(X)$. In fact, (8.2) means that F defines a bundle equivalence between $\Phi\colon \mathbf{M}(X) \to \mathbf{T}(X)$ and the trivial bundle $\pi_1\colon \mathbf{T}(X) \times \mathbf{D}_0(X) \to \mathbf{T}(X)$. An explicit section $\sigma\colon \mathbf{T}(X) \to \mathbf{M}(X)$ of Φ is given by

$$\sigma(\Phi(\nu)) = F^{-1}(\Phi(\nu), 1) = \nu\cdot f(\mu, \nu) \ .$$

9. The sphere

In this section X will be the Riemann sphere. Hence $\mathbf{D}(X)$ and $\mathbf{D}_0(X)$ coincide.

(A) **Proposition.** *$\mathbf{D}_0(X)$ is homeomorphic to $G_C \times \mathbf{D}_0(X; 0, 1, \infty)$, where G_C is the group of holomorphic automorphisms of the sphere, and $\mathbf{D}_0(X; 0, 1, \infty)$ denotes the subgroup of $\mathbf{D}_0(X)$ of elements holding* 0, 1, ∞ *fixed.*

Proof. The map $(A, f) \mapsto A \circ f$ from $G_C \times \mathbf{D}_0(X; 0, 1, \infty)$ to $\mathbf{D}_0(X)$ is continuous, because $\mathbf{D}_0(X)$ is a topological group. Moreover, it is bijective, the inverse map being $f \mapsto (A_f, A_f^{-1} \circ f)$, where A_f is the unique member of G_C taking 0, 1, ∞ to $f(0), f(1), f(\infty)$. Finally, $f \mapsto A_f$ is continuous by compactness properties of holomorphic functions.

Remark. G_C has the rotation group SO(3) as maximal compact subgroup, and hence as strong deformation retract.

(B) Define the charts h_1 and h_2 on X by stereographic projection from 0 and ∞ respectively. Each $J \in \mathbf{M}(X)$ gives rise to a pair of functions $\mu_1, \mu_2 \in C^\infty(C, \Delta)$ related (compare (5.1)) by

$$\mu_2(f(z))\overline{f'(z)}/f'(z) = \mu_1(z), \quad z \in C - \{0\}, \tag{9.1}$$

where $f = h_2 \circ h_1^{-1}: C - \{0\} \to C - \{0\}$ is the map $z \mapsto 1/z$.

Let $w_i: C \to C$ be the normalized solution of Beltrami's equation $w_{\bar z} = \mu_i w_z (i = 1, 2)$. Then $f^{-1} \circ w_2 \circ f = w_1$ because of (9.1). In other words,

$$w_J = h_1^{-1} \circ w_1 \circ h_1 = h_2^{-1} \circ w_2 \circ h_2 \in \mathbf{D}_0(X; 0, 1, \infty).$$

Of course w_J is the unique element of $\mathbf{D}_0(X; 0, 1, \infty)$ which is a holomorphic map from X with complex structure J to X with its usual complex structure.

Theorem. *The map $J \mapsto w_J$ is a homeomorphism from $\mathbf{M}(X)$ onto $\mathbf{D}_0(X; 0, 1, \infty)$.*

Proof. The map is clearly bijective, and is a homeomorphism by applying Theorem 3B to both w_1 and w_2.

Corollary (*Smale* [29]). SO(3) *is a strong deformation retract of* $\mathbf{D}(X)$.

10. The torus

In this section X is a torus, and x_0 is a point of X. Since our arguments are quite similar to those we have already given for $g > 1$, we shall omit many details.

(A) Fix a point x_0 in X, and mark X by choosing a pair of simple loops a and b, which generate $\pi_1(X; x_0)$, so that a crosses b from left to right at x_0 and there are no other intersections. Analogous to Lemma 4C we have the

Lemma. *For each J in $\mathbf{M}(X)$ there is a unique (J-)holomorphic covering map $\pi: C \to X$ with cover group Γ such that*

1) *The loop a determines the translation*

$$Az = z + 1 \qquad \text{in } \Gamma .$$

2) *The loop b determines the translation*

$$Bz = z + \tau \quad \text{in } \Gamma, \quad \operatorname{Im} \tau > 0 .$$

3) $\pi(x_0) = 0$.

Now choose $J_0 \in \mathbf{M}(X)$, and let $\pi: \boldsymbol{C} \to X$ and Γ be determined by the lemma. As in § 5 A, there is an induced map $\pi^*: \mathbf{M}(X) \to \mathbf{M}(\boldsymbol{C})$ whose image is $\mathbf{M}(\Gamma)$, the space of Γ-invariant complex structures on $\boldsymbol{C}$. Because of the simple form of Γ, the equation for Γ-invariance of $\mu \in \mathbf{M}(\boldsymbol{C})$ becomes

$$\mu \circ \gamma = \mu \qquad \text{for all } \gamma \in \Gamma . \tag{10.1}$$

As before, we denote by $A^1(\Gamma)$ the Fréchet space of all $\mu \in C^\infty(\boldsymbol{C}, \boldsymbol{C})$ which satisfy (10.1). The following assertions are proved in the same way as the corresponding ones in § 5A.

Proposition. *$\mathbf{M}(\Gamma)$ is the convex open set in $A^1(\Gamma)$ consisting of the $\mu \in A^1(\Gamma)$ such that* $\sup\{|\mu(z)|: z \in \boldsymbol{C}\} < 1$, *and $\pi^*: \mathbf{M}(X) \to \mathbf{M}(\Gamma)$ is a diffeomorphism.*

Corollary. *$\mathbf{M}(X)$ is a contractible complex analytic manifold modeled on a Fréchet space.*

(B) Let $\mathbf{D}_0(\Gamma)$ be the centralizer of Γ in $\mathbf{D}(\boldsymbol{C})$, and $\mathbf{D}_0(\Gamma; 0)$ the subgroup fixing 0. As in § 5B, define $\pi_*: \mathbf{D}_0(\Gamma; 0) \to \mathbf{D}(X)$ by $\pi_*(f) \circ \pi = \pi \circ f$.

Proposition. *$\pi_*: \mathbf{D}_0(\Gamma; 0) \to \mathbf{D}_0(X; x_0)$ is an isomorphism of topological groups.*

We follow the reasoning of § 5B, with the Euclidean metric in place of the hyperbolic metric.

(C) Once again, the natural action of $\mathbf{D}_0(X; x_0)$ on $\mathbf{M}(X)$ is transferred by π to the action

$$\mu_f \cdot g = \mu_{f \circ g} \tag{10.2}$$

of $\mathbf{D}_0(\Gamma; 0)$ on $\mathbf{M}(\Gamma)$. Analogous to Propositions 5C and 5D we have the

Proposition. *The action $\mathbf{M}(\Gamma) \times \mathbf{D}_0(\Gamma; 0) \to \mathbf{M}(\Gamma)$ given by* (10.2) *is free, continuous, and proper.*

Corollary. *The natural action $\mathbf{M}(X) \times \mathbf{D}_0(X; x_0) \to \mathbf{M}(X)$ is free, continuous, and proper.*

(D) Define $P: \mathbf{M}(X) \to U$ by $P(J) = \tau$, where $Bz = z + \tau$ is determined by Lemma 10A. Composing P with the inverse of $\pi^*: \mathbf{M}(X) \to \mathbf{M}(\Gamma)$ produces a map, still called $P: \mathbf{M}(\Gamma) \to U$. Analogous to §§ 6B and C we have

Lemma 1. *Let $\tau_0 = P(0) \in U$. Then*

$$P(\mu) = w_\mu(\tau_0) \qquad \textit{for all } \mu \in \mathbf{M}(\Gamma) .$$

Proof. For any $\mu \in \mathbf{M}(\Gamma)$, $\pi_\mu = \pi \circ w_\mu^{-1}: C \to X$ is the covering map determined by Lemma 10A, and $\Gamma_\mu = w_\mu \Gamma w_\mu^{-1}$. In particular, $B_0 z = z + P(0)$ and $B_\mu(z) = z + P(\mu)$ are related by $B_\mu = w_\mu \circ B_0 \circ w_\mu^{-1}$.

Lemma 2. *$P(\mu) = P(\nu)$ if and only if μ and ν are $\mathbf{D}_0(\Gamma; 0)$-equivalent.*

Proof. Because J_0 was arbitrary we may assume $\nu = 0$. By Lemma 1, $P(\mu) = P(0)$ if and only if w_μ commutes with $z \mapsto z + \tau_0$ and hence with Γ (for w_μ always commutes with $z \mapsto z + 1$). But 0 and μ are $\mathbf{D}_0(\Gamma; 0)$-equivalent if and only if $\mu = \mu_f$ for some $f \in \mathbf{D}_0(\Gamma; 0)$, which, being normalized, can only be w_μ.

(E) **Proposition.** *$P: \mathbf{M}(\Gamma) \to U$ is continuous and surjective. Further, $\sigma: U \to \mathbf{M}(\Gamma)$ defined by*

$$\sigma(z) = \frac{\tau_0 - z}{z - \bar{\tau}_0}$$

is a holomorphic section of P.

Proof. The continuity of P is immediate from Theorem 3B. By (10.1), all constant maps $\lambda: U \to \Delta$ are Γ-invariant complex srtuctures; these form the image $\sigma(U)$. To verify that $P \circ \sigma: U \to U$ is the identity map, note that

$$w_\lambda(z) = (1 + \lambda)^{-1}(z + \lambda \bar{z}) .$$

Corollary. *$P: \mathbf{M}(X) \to U$ is an open map.*

In fact, $P: \mathbf{M}(\Gamma) \to U$ maps each neighborhood of $0 \in \mathbf{M}(\Gamma)$ to a neighborhood of $\tau_0 \in U$. But $0 \in \mathbf{M}(\Gamma)$ corresponds to an arbitrary $J_0 \in \mathbf{M}(X)$.

Remark. The holomorphic section σ was discovered by Teichmüller. Teichmüller's theorem [6] gives a section $\sigma: \mathbf{T}(X) \to M(X)$ for any compact X, taking its values in the space $M(X)$ of bounded measurable complex structures. But if the genus of X is greater than one, Teichmüller's section is not continuous.

(F) **Theorem.** *The quotient map $\Phi: \mathbf{M}(X) \to \mathbf{T}(X) = \mathbf{M}(X)/\mathbf{D}_0(X; x_0)$ defines a trivial principal fibre bundle, and $\mathbf{T}(X)$ is homeomorphic to U.*

Corollary. *$D_0(X; x_0)$ is contractible. Thus every fibre bundle with structure group $\mathbf{D}_0(X; x_0)$ is topologically trivial.*

Those assertions merely consolidate the results of §§ 10C, D, E.

(G) **Proposition.** *The map $X \times \mathbf{D}_0(X; x_0) \to \mathbf{D}_0(X)$ defined by $(\tau, f) \mapsto \tau \circ f$ is a homeomorphism.*

Proof. Write any $f \in \mathbf{D}(X)$ in the form $\tau_f \circ f_0$ where $f_0(x_0) = x_0$. f is homotopic to f_0.

Corollary. *$\mathbf{D}_0(X)$ has X as strong deformation retract. In particular, it is the identity component of $\mathbf{D}(X)$.*

11. Non-orientable surfaces

A closed non-orientable surface X cannot have a complex structure, but one can still consider the space $\mathbf{M}(X)$ of *conformal* structures on X. Moreover, for any conformal structure there always exists a universal covering map $\pi: \tilde{X} \to X$ such that the cover transformations are conformal maps. Here $\tilde{X}$ is the sphere, Euclidean plane, or hyperbolic plane (with its usual conformal structure). The methods of the previous sections can thus be applied to the study of non-orientable surfaces, with only minor changes of details. We shall outline here the principal results. In all cases we find that the diffeomorphism group $\mathbf{D}(X)$ has the same homotopy groups as the homeomorphism group (Hamstrom [19]).

(A) If X is not the real projective plane or the Klein bottle, there is a covering map $\pi: U \to X$, whose cover group Γ consists of conformal automorphisms of U. There is of course an induced map $\pi^*: \mathbf{M}(X) \to \mathbf{M}(\Gamma)$, where $\mathbf{M}(\Gamma)$ is the space of Γ-invariant conformal structures. The equation for Γ-invariance takes a new form for orientation-reversing elements of Γ; $\mu \in \mathbf{M}(U)$ is Γ-invariant if and only if

(11.1) $$(\mu \circ \gamma)\overline{\gamma'}/\gamma' = \mu \qquad \text{if } \gamma \in \Gamma \text{ is holomorphic,}$$

(11.2) $$(\mu \circ \gamma)\overline{\gamma_{\bar{z}}}/\gamma_{\bar{z}} = \bar{\mu} \qquad \text{if } \gamma \in \Gamma \text{ reverses orientation.}$$

Let $A^1(\Gamma)$ be the Fréchet space of $\mu \in C^\infty(U, C)$ which satisfy (11.1) and (11.2). Because of (11.2), $A^1(\Gamma)$ is a real but not a complex linear space. In fact, let $\Gamma_0 \subset \Gamma$ be the normal subgroup of orientation preserving (holomorphic) maps. Then $A^1(\Gamma_0)$ is the direct sum of $A^1(\Gamma)$ and $iA^1(\Gamma)$. Still we have

Proposition. *$\mathbf{M}(\Gamma)$ is an open convex set in $A^1(\Gamma)$, and $\pi^*: \mathbf{M}(X) \to \mathbf{M}(\Gamma)$ is a diffeomorphism. In particular, $\mathbf{M}(X)$ is contractible.*

(B) Mimicing the reasoning of § 5 we obtain the

Proposition. *The natural action of $\mathbf{D}_0(X)$ on $\mathbf{M}(X)$ is free, proper, and continuous.*

Here $\mathbf{D}_0(X)$ is the group of diffeomorphisms homotopic to the identity, and X is not the projective plane nor the Klein bottle. To complete the story for such X, we note that our construction in § 8 of a harmonic section $\sigma: \mathbf{M}(X)/\mathbf{D}_0(X) \to \mathbf{M}(X)$ did not require X to be oriented. Defining the Teichmüller space $\mathbf{T}(X) = \mathbf{M}(X)/\mathbf{D}_0(X)$ we have

Theorem. *The quotient map $\Phi: \mathbf{M}(X) \to \mathbf{T}(X)$ defines a trivial principal fibre bundle.*

Corollary. *$\mathbf{T}(X)$ and $\mathbf{D}_0(X)$ are contractible. In particular, $\mathbf{D}_0(X)$ is connected.*

(C) The real projective plane X is the quotient of the Riemann sphere $C \cup \{\infty\}$ by the group Γ of order two generated by the antipodal map $\gamma(z) = -1/\bar{z}$. The space $\mathbf{M}(X)$ of conformal structures on X is diffeomorphic to $\mathbf{M}(\Gamma)$, the space of $\mu \in \mathbf{M}(C)$ satisfying (11.2). (Comparison with (9.1) reveals that each $\mu \in \mathbf{M}(C)$ which satisfies (11.2) is also smooth at ∞.)

Similarly, the group $\mathbf{D}(X)$ of all diffeomorphisms of X is diffeomorphic to $\mathbf{D}(\Gamma)$, the centralizer ($=$ normalizer) of Γ in $\mathbf{D}(C)$. As in § 9, let G_C be the group of all conformal automorphisms of $C \cup \{\infty\}$. The intersection of G_C and $\mathbf{D}(\Gamma)$ is SO(3), the group of rotations of the sphere. Let N_0 be the set (*not* a group) of f in $\mathbf{D}(\Gamma)$ with $f(0) = 0$ and $f_z(0)$ real and positive. Since for any f in $\mathbf{D}(\Gamma)$, $|f_z(0)| \geq |J_f(0)| > 0$ (where J_f is the Jacobian of f), we have the

Lemma. *$\mathbf{D}(\Gamma)$ is homeomorphic to* $\mathrm{SO}(3) \times N_0$.

Proposition. *The map $\mu \mapsto \mu_f = f_{\bar{z}}/f_z$ is a homeomorphism from N_0 onto $\mathbf{M}(\Gamma)$.*

Proof. $\mu \mapsto \mu_f$ is clearly a continuous map into $\mathbf{M}(C)$. It takes its values in $\mathbf{M}(\Gamma)$ because each $f \in N_0$ commutes with γ. It is injective because if $\mu_f = \mu_g$, then $f \circ g^{-1} \in G_C \cap \mathbf{D}(\Gamma) = \mathrm{SO}(3)$; the normalization at 0 makes $f = g$. Finally, we must exhibit a continuous inverse map from $\mathbf{M}(\Gamma)$ to N_0. Given $\mu \in \mathbf{M}(\Gamma)$, let $w = w_\mu$ be the normalized solution of (3.2). $w \circ \gamma \circ w^{-1} = h$ is an orientation-reversing conformal involution of the sphere. Since w is normalized, h interchanges 0 and ∞. Further, h has no fixed points. It follows that $h(z) = r/\bar{z}$, where $r = h(1) = w(-1) < 0$. Put

$$f_\mu = (-r)^{-1/2} |w_z(0)| w_z(0)^{-1} w .$$

Clearly, $f_\mu \in N_0$ and satisfies (3.2). Theorem 3B implies that the map $\mu \mapsto f_\mu$ is continuous.

Corollary. *The group of diffeomorphisms of the real projective plane has* SO(3) *as strong deformation retract.*

(D) It remains to consider the Klein bottle. We take $X = C/\Gamma$, where Γ is generated by $Az = \bar{z} + 1/2$ and $Bz = z + i$; as usual, $\pi\colon C \to X$ is the natural map. The space of Γ-invariant conformal structures is

$$\mathbf{M}(\Gamma) = \{\mu \in \mathbf{M}(C)\colon \mu \circ A = \bar{\mu}, \quad \mu \circ B = \mu\} .$$

Let $\mathbf{D}_0(\Gamma)$ be the centralizer of Γ in $\mathbf{D}(C)$, and $\pi^*\colon \mathbf{D}_0(\Gamma) \to \mathbf{D}_0(X)$ the natural map. The kernel of π^* is the group of all *real* translations $z \mapsto z + t$, $t \in R$. Let N_0 be the set (*not* a group) of f in $\mathbf{D}_0(\Gamma)$ such that the real part of $f(0)$ vanishes.

Proposition.

(a) *$\mathbf{D}_0(X)$ is homeomorphic to* $\mathrm{SO}(2) \times N_0$.

(b) *N_0 is homeomorphic to*

$$\mathbf{M}_0(\Gamma) = \{\mu \in \mathbf{M}(\Gamma)\colon w_\mu \circ B = B \circ w_\mu\} .$$

(c) *Define $\sigma\colon R^+ \to \mathbf{M}(\Gamma)$ by $\sigma(r) = (1 - r)(1 + r)^{-1}$. For $\mu \in \mathbf{M}(\Gamma)$, $w_{\sigma(r)}^{-1} \circ w_\mu$ commutes with B if and only if $w_\mu(i) = ri$.*

(d) *The map $(r, \lambda) \mapsto \mu$, where $w_\mu = w_{\sigma(r)} \circ w_\lambda$, is a homeomorphism from $R^+ \times \mathbf{M}_0(\Gamma)$ onto $\mathbf{M}(\Gamma)$.*

The proofs, which we omit, are analogous to several others in §§ 10 and 11.

Corollary. *Let X be the Klein bottle. Then $\mathbf{D}_0(X)$ has* SO(2) *as strong deformation retract.*

Remark. For every X except the projective plane and Klein bottle, we have found a subgroup G_0 of $\mathbf{D}(X)$ acting freely on $\mathbf{M}(X)$ such that the natural map from $\mathbf{M}(X)/G_0$, the Teichmüller space, onto $\mathbf{M}(X)/\mathbf{D}(X)$ is a ramified covering map. For the projective plane and Klein bottle, however, our luck ran out. We were compelled to use subsets N_0 of $\mathbf{D}(X)$ which were not subgroups. Alternatively, we could have chosen subgroups G_0 contained in N_0, at the cost of accepting quotient space $\mathbf{M}(X)/G_0$ of higher dimension. For more general manifolds X and spaces of structures, of course, the unlucky cases are the rule. It seems very unusual to have a subgroup of $\mathbf{D}(X)$ which acts freely and produces a finite dimensional quotient.

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