J. DIFFERENTIAL GEOMETRY
29 (1989) 615-621

ON THE TOPOLOGY OF POSITIVELY CURVED 4-MANIFOLDS WITH SYMMETRY

WU-YI HSIANG & BRUCE KLEINER

1. Introduction

A positively curved manifold is, by definition, a *complete* Riemannian manifold M with everywhere positive sectional curvature. The work of Gromoll and Meyer [6] gives a thorough understanding of noncompact positively curved manifolds, so we consider only compact positively curved manifolds, henceforth denoted CPCM's. Synge's theorem [10] asserts that an even dimensional, orientable CPCM is simply connected. This theorem together with the topological classification of compact surfaces implies that a 2-dimensional, orientable CPCM is homeomorphic to S^2. Three dimensional CPCM's have been determined by Hamilton [7]; they are diffeomorphic to space forms. However, very little is known about the topology of 4-dimensional CPCM's. The known examples are homeomorphic to S^4, $\mathbf{R}P^4$, and $\mathbf{C}P^2$, while the well-known problem of Hopf remains unsolved:

$$\text{Does } S^2 \times S^2 \text{ admit a positively curved Riemannian metric?}$$

The three known examples of compact 4-manifolds which admit positively curved metrics all admit *homogeneous* positively curved metrics, i.e. metrics with a lot of symmetry. Therefore it is natural to ask the following question: Which compact 4-manifolds admit positively curved Riemannian metrics with at least one infinitesimal isometry, in other words, a nontrivial Killing field? The main result of this paper answers this question.

Theorem 1. *Let M be a 4-dimensional orientable CPCM. If M has a nontrivial Killing vector field, then M is homeomorphic to S^4 or $\mathbf{C}P^2$.*

Corollary 1. *Let M be a 4-dimensional nonorientable CPCM. If M has a nontrivial Killing vector field, then M is two-fold covered by S^4.*

Corollary 2. *$S^2 \times S^2$ does* **not** *admit a positively curved Riemannian metric with a nontrivial Killing field.*

Technically speaking, the existence of a nontrivial Killing vector field on a compact Riemannian manifold M is equivalent to the existence of a nontrivial S^1-action on M. Let $F(S^1, M)$ be the fixed point set of such an S^1-action on

Received January 4, 1988.

M. Then it is easy to prove that the Euler characteristic of $F(S^1, M)$ is equal to that of M, i.e. $\chi(F(S^1, M)) = \chi(M)$, and each connected component of $F(S^1, M)$ is automatically a totally geodesic submanifold. In the special case where M is a 4-dimensional orientable CPCM, we will prove in Lemma 2 that

$$F(S^1, M) = \begin{cases} \chi(M) \text{ isolated points,} \\ \text{or } S^2 \cup (\chi(M) - 2 \text{ isolated points}). \end{cases}$$

The major task in the proof of Theorem 1 is proving that $\chi(F(S^1, M))$ can be at most 3.

Actually, most of the techniques of this paper are equally applicable to the nonnegatively curved case. We believe that the following results are within reach:

Conjecture 1. *A 4-dimensional CPCM with a nontrival Killing vector field should be diffeomorphic to* S^4, $\mathbf{R}P^4$, *or* $\mathbf{C}P^2$.

Conjecture 2. *A compact, simply connected, nonnegatively curved 4-manifold with a nontrivial Killing vector field should be diffeomorphic to either* S^4, $\mathbf{C}P^2$, $\mathbf{C}P^2 \# \pm \mathbf{C}P^2$, *or* $S^2 \times S^2$.

Of course, it is possible that these theorems would remain true without the assumption on infinitesimal symmetry, but then their proofs would require completely new ideas and techniques.

2. The orbital geometry of S^1-Riemannian manifolds

An S^1-Riemannian manifold is, by definition, a Riemannian manifold with a given isometric S^1-action. In this section we will establish some properties of the orbital geometry of a given S^1-Riemannian manifold (S^1, M), especially in the case that M is a 4-dimensional orientable CPCM.

Lemma 1. *Let* (S^1, M) *be a compact* S^1*-Riemannian manifold and let* F *be its fixed point set. Then:*

(i) *The Euler characteristic of* F *is equal to the Euler characteristic of* M.
(ii) *Each connected component of* F *is a totally geodesic submanifold of even codimension.*

Sketch of proof. (For more details, see [8, Theorems 5.3 and 5.6].) (i) Let $\mathbf{Z}_p$ be the unique cyclic subgroup of S^1 of prime order p and let $F(\mathbf{Z}_p, M)$ be the set of fixed points of $\mathbf{Z}_p$ in M. It follows from the long exact sequence of the pair $(M, F(\mathbf{Z}_p, M))$ and the additivity of the Euler characteristic that

$$\begin{aligned} \chi &= \chi(F(\mathbf{Z}_p, M)) + \chi(M, F(\mathbf{Z}_p, M)) \\ &\equiv \chi(F(\mathbf{Z}_p, M)) \pmod{p}. \end{aligned}$$

It is easy to see that $F(\mathbf{Z}_p, M) = F$ for all sufficiently large primes. Hence $\chi(F) \equiv \chi(M) \pmod{p}$ for all sufficiently large primes p, so $\chi(F) = \chi(M)$.

(ii) Let Y be a connected component of F and let $v \in T_y Y$ be an arbitrary tangent vector of Y at $y \in Y$. Then v is fixed under the induced S^1-action on TM. Hence from the existence of a unique geodesic with initial velocity v it follows that such a geodesic is pointwise fixed under the S^1-action, and hence belongs to Y. This proves that Y is a totally geodesic submanifold in M. Since all nontrivial irreducible orthogonal representations of S^1 are two-dimensional, the codimension of Y is necessarily even. q.e.d.

From now on we will always assume, without further specification, that (S^1, M^4, g) is a 4-dimensional, orientable CPCM with a given effective S^1-action and metric tensor g.

Lemma 2. *Let (S^1, M, g) be as above and let F be its fixed point set. Then F is nonempty and*

$$F = \begin{cases} \chi(M) \text{ isolated points,} \\ \text{or } S^2 \cup (\chi(M) - 2 \text{ isolated points}). \end{cases}$$

Proof. Synge's theorem [10] asserts that such an even dimensional manifold is always simply connected. Therefore,

$$H_1(M) = 0 \text{ and by duality } H_3(M) = 0,$$
$$\chi(M) = 2 + \dim H_2(M) \geq 2.$$

Hence by Lemma 1, $\chi(F) \geq 2$ so F is nonempty. Moreover, Frankel's theorem [4] implies that F can have at most one 2-dimensional connected component.

Suppose F contains a 2-dimensional component Y. The normal bundle of Y is oriented by the S^1-action, so Y is orientable. Being totally geodesic as well, Y is positively curved and must therefore be homeomorphic to S^2. q.e.d.

Next let us consider the geometry of the *orbit space* $\overline{M} = M/S^1$. We will equip $\overline{M}$ with the orbital distance metric: the distance between two elements of $\overline{M}$ is the distance between the corresponding orbits in M. Let M_0 be the union of all the principal S^1-orbits in M and let $\overline{M}_0 = \pi(M_0)$ where $\pi\colon M \to \overline{M}$ is the canonical surjection. We give $\overline{M}_0$ the unique smooth structure which makes $\pi\colon M_0 \to \overline{M}_0$ a submersion, and the unique smooth Riemannian metric $\overline{g}$ for which $\pi\colon (M_0, g) \to (\overline{M}_0, \overline{g})$ is a Riemannian submersion.

Lemma 3. *Suppose $F = S^2 \cup \{\text{isolated points}\}$. Let $\overline{S^2} = \pi(S^2) \subset \overline{M}$. Then the Riemannian structure $(\overline{M}_0, \overline{g})$ extends to a Riemannian structure on $N = \overline{M}_0 \cup \overline{S^2}$ with totally geodesic boundary $\overline{S^2}$. The distance function on N induced by this Riemannian structure coincides with the restriction of the orbital distance metric on $\overline{M}$ to $N \subseteq \overline{M}$.*

Proof. The local geometry of $\overline{M}$ near a point $\pi(y) \in \overline{S^2}$ is determined by the geometry of the local representation at $y \in S^2$. This representation is equivalent to

$$\phi\colon S^1 \times \mathbf{C}^2 \to \mathbf{C}^2; \qquad e^{i\theta}(z_1, z_2) = (z_1, e^{i\theta} z_2),$$

where $z_1, z_2 \in \mathbf{C}$, so the local structure of $\overline{M}$ at $\pi(y)$ is of the type

$$\mathbf{C}^2/S^1 \approx \mathbf{C} \times (\mathbf{C}/S^1) \simeq \mathbf{R}^2 \times \mathbf{R}_+ = \text{ a half space,}$$

i.e., $N = \overline{M}_0 \cup \overline{S^2}$ has a boundary structure near $\overline{S}^2$.

Geodesics in $N = \overline{M}_0 \cup \overline{S^2}$ are the projections of geodesics in M which are perpendicular to the S^1 orbits, so it follows that $\overline{S}_2$ is totally geodesic in $\overline{M}$.

The distance function induced on N by the Riemannian structure coincides with the orbital distance metric on the dense subset $\overline{M}_0$, so it coincides with the orbital distance metric on all of N. q.e.d.

Let $y \in M$ be an isolated fixed point. The slice representation at y is orthogonally equivalent to

$$\phi_{k,l}\colon S^1 \times \mathbf{C}^2 \to \mathbf{C}^2; \qquad e^{i\theta}(z_1, z_2) = (e^{ik\theta} z_1, e^{il\theta} z_2),$$

where $z_1, z_2 \in \mathbf{C}$ and $k, l \in \mathbf{Z}$ with g.d.c$(k, l) = 1$. Let $S^3(1) \subseteq \mathbf{C}^2$ be the unit sphere and let $d\colon S^3(1) \times S^3(1) \to \mathbf{R}$ be given by $d(v, w) = \angle(v, w) =$ the angle between v and w. Let (X_{kl}, d_{kl}) be the orbit space of $(\phi_{k,l}, S^3(1), d)$ with orbital distance metric $d_{k,l}$.

Lemma 4. *If x_1, x_2, x_3 are arbitrary points in $X_{k,l}$, then*

$$d_{k,l}(x_1, x_2) + d_{k,l}(x_2, x_3) + d_{k,l}(x_3, x_1) \leq \pi.$$

Proof. The two great circles in $S^3(1)$ given by $z_1 = 0$ and $z_2 = 0$ are orbits of $\phi_{k,l}$ for all k, l with g.c.d.$(k, l) = 1$. Let $\tilde{X}_{k,l} = K_{k,l} \backslash \{\text{these two orbits}\}$. $\tilde{X}_{k,l}$ consists of principal orbits, so we give it the Riemannian submersion metric coming from the canonical Riemannian metric on $S^3(1)$. We will be using the fact that this Riemannian submersion metric induces the distance function $d_{k,l}$ on $\tilde{X}_{k,l}$.

In the special case where $k = l = 1$, the projection $\pi\colon S^3(1) \to X_{1,1}$ is the Hopf fibration and it is easily checked that $X_{1,1}$ is isometric to a $\mathbf{C}P^1$ with diameter $\pi/2$, i.e., $X_{1,1}$ is isometric to $S^2(1/2) \subseteq \mathbf{E}^3$. Hence the inequality $d_{1,1}(x_1, x_2) + d_{1,1}(x_2, x_3) + d_{1,1}(x_3, x_1) \leq \pi$ is obvious.

We now fix $(k, l) \neq (1, 1)$. The isometric T^2-action

$$T^2 \times S^3(1) \to S^3(1); \qquad (e^{i\theta_1}, e^{i\theta_2}) \cdot (z_1, z_2) = (e^{i\theta_1} z_1, e^{i\theta_2} z_2)$$

induces an isometric S^1-action on the Riemannian manifold $\tilde{X}_{k,l}$. $\tilde{X}_{k,l}$ is a connected noncomplete surface of revolution with diameter $\pi/2$, so it admits a coordinate system $(r, \theta)\colon \tilde{X}_{k,l} \to (0, \pi/2) \times S^1$ such that the metric in these coordinates is $ds^2 = dr^2 + (f(r))^2\, d\theta^2$ where $d\theta$ is the standard 1-form on S^1. By replacing r with $\pi/2 - r$ if necessary, we can arrange that the latitude circle $r = c$ corresponds to the orbit space of the torus $T^2(c) = T^2(\cos c, \sin c)) \subseteq S^3(1)$. All the $\phi_{k,l}$ orbits in $T^2(c)$ have the same length and the function $f(r)$ is determined by

$$2\pi f(c)(\text{the length of a } \phi_{k,l} \text{ orbit in } T^2(c)) = 4\pi^2 \cos c \sin c.$$

The orbits of $\phi_{k,l}$ all have length $\geq 2\pi$, so $f(c) \leq \cos c \sin c = \frac{1}{2}\sin 2c$. Hence there is a *length nonincreasing* bijection of $\tilde{X}_{1,1}$ onto $\tilde{X}_{k,l}$ which assigns points in $\tilde{X}_{1,1}$ to points in $\tilde{X}_{k,l}$ with the same coordinates in $(0, \pi/2) \times S^1$. The inequality

$$d_{k,l}(x_1, x_2) + d_{k,l}(x_2, x_3) + d_{k,l}(x_3, x_1) \leq \pi$$

for $x_1, x_2, x_3 \in \tilde{X}_{k,l}$ now follows from the corresponding inequality already proved for $(k, l) = (1, 1)$. Since $\tilde{X}_{k,l}$ is dense in $X_{k,l}$, Lemma 4 follows.

Lemma 5. *If* $\dim F = 2$, *then the local representation of* S^1 *at every isolated fixed point must be equivalent to* $\phi_{1,1}$.

Proof. Let Y be the 2-dimensional component of F. Then from the local representation of S^1 on T_yM, $y \in Y$, it follows that there exists a tubular neighborhood of Y, say U, such that the isotropy group is trivial for all $x \in U \backslash Y$.

Suppose there exists an isolated fixed point $p \in F$ such that the local representation of S^1 on T_pM is equivalent to $\phi_{k,l}$, g.c.d. $(k, l) = 1$ and $k > 1$. Then $F(\mathbf{Z}_k, M)$ contains at least two connected components of dimension 2. This contradicts the theorem of Frankel [4] which asserts that two such totally geodesic surfaces in M cannot be disjoint.

3. The proof of Theorem 1

Let M be a 4-dimensional orientable CPCM. Then by Synge's theorem [10] M is simply connected. We will exploit the orbital geometry of the given S^1-action to prove that $\chi(M)$ is at most 3. It then follows directly from the work of Freedman [5] that M is homeomorphic to either S^4 or $\mathbf{C}P^2$. By Lemmas 1 and 2, $\chi(M) = \chi(F)$ and

$$F(M) = \begin{cases} \chi(M) \text{ isolated points,} \\ \text{or } S^2 + (\chi(M) - 2) \text{ isolated points.} \end{cases}$$

Therefore the proof of the theorem reduces to proving that F consists of at most three isolated points or S^2 plus at most one more isolated point. We will divide the proof into two cases according to $\dim F = 0$ or 2 and we will prove each case by contradiction.

Case 1, $\dim F = 2$. Suppose $F = S^2$ plus at least two isolated fixed points. Let p, q be two isolated fixed points and let γ be a minimizing geodesic segment in M joining p to q. Let η be a minimizing geodesic segment from S^2 to $S^1(\gamma) =$ the S^1 orbit of γ; hence $\text{length}(\eta) = \text{dist}(S^2, S^1(\gamma))$, and η has endpoints $A \in S^2$ and $B \in S^1(\gamma)$. The isotropy group of the S^1-action does not vary along the interior of the minimizing segments γ and η, since otherwise they could be replaced with broken geodesic segments of the same length. Hence it follows from Lemma 5 that the interiors of γ and η lie in $M_0 =$ union of principal orbits in M.

Suppose $B = p$. By Lemma 5 the local representation of S^1 at p is equivalent to $\phi_{1,1}$. Hence $e^{i\theta} \cdot \gamma$ is perpendicular to η at p for all $e^{i\theta} \in S^1$. The second variation formula can now be applied to the geodesic segment η as in the proof of Frankel's theorem [4] to show that $\text{length}(\eta) > \text{dist}(S^2, S^1(\gamma))$. This contradicts the assumption that $\text{length}(\eta) = \text{dist}(S^2, S^1(\gamma))$. The same argument rules out $B = q$.

Now suppose B lies in the interior of γ. Then the isotropy group of B is trivial, forcing $\eta \subseteq M_0 \cup S^2$. Let $\overline{\gamma} = \pi(\gamma \backslash \{p, q\}) \subseteq \overline{M}_0$, and $\overline{\eta} = \pi(\eta) \subseteq \overline{M}_0 \cup \overline{S^2} = N$. By Lemma 3, N is a smooth Riemannian manifold with totally geodesic boundary, and since Riemannian submersions are always curvature nondecreasing (see [4]), N has sectional curvature everywhere $\geq \delta$ for some $\delta > 0$. An application of the second variation formula to the geodesic segment $\overline{\eta} \subset N$ shows once again that $\text{length}(\eta) > \text{dist}(S^2, S^1(\gamma))$, contradicting $\text{length}(\eta) = \text{dist}(S^2, S^1(\gamma))$. Hence F can contain at most one isolated fixed point in addition to the S^2.

Case 2, $\dim F = 0$. Suppose F contains at least four isolated points, p_i, $1 \leq i \leq 4$. Let $l_{ij} = \text{dist}(p_i, p_j)$ and let $C_{ij} = \{\gamma\colon [0, l_{ij}] \to M | \gamma$ is a minimizing geodesic segment from p_i to $p_j\}$, $1 \leq i, j \leq 4$. For each triple $1 \leq i, j, k \leq 4$ set

$$\alpha_{ijk} = \min\{\angle(\gamma_j'(0), \gamma_k'(0)) | \gamma_j \in C_{ij}, \gamma_k \in C_{ik}\}.$$

Note that the minimum exists because M is compact.

Lemma 6. *For each triple of distinct integers* $1 \leq i, j, k \leq 4$,

$$\alpha_{ijk} + \alpha_{kij} + \alpha_{jki} > \pi.$$

Proof. Let us assume, for notational simplicity, that $(i, j, k) = (1, 2, 3)$. Set $1/R^2 = \delta =$ minimum of sectional curvature of M. Choose x_1, x_2, x_3 on $S^2(R)$ such that the spherical triangle $\Delta(x_1, x_2, x_3)$ has l_{12}, l_{23}, l_{31} as its three lengths. Applying Toponogov's theorem [11] to an arbitrary triangle with $\gamma_{12} \in C_{12}$, $\gamma_{23} \in C_{23}$, $\gamma_{13} \in C_{13}$ as its three sides, one gets

$$\angle(\gamma_{12}'(0), \gamma_{13}'(0)) \geq \angle(\overline{x_1x_2}, \overline{x_1x_3}),$$

and hence, by the definition of α_{123}, that $\alpha_{123} \geq \angle(\overline{x_1x_3}, \overline{x_1x_3})$. Therefore $\alpha_{123} + \alpha_{312} + \alpha_{231} \geq$ the sum of interior angles of $\Delta(x_1, x_2, x_3) > \pi$. q.e.d.

From the above lemma it follows easily that

$$\sum_{1 \leq i \leq 4} \sum_{\substack{1 \leq j < k \leq 4 \\ j,k \neq i}} \alpha_{ijk} > 4\pi.$$

But, on the other hand, from Lemma 4 it is easily seen that

$$\sum_{\substack{1 \leq j < k \leq 4 \\ j,k \neq i}} \alpha_{ijk} \leq \pi \quad \text{for each } 1 \leq i \leq 4,$$

which gives a contradiction. Therefore F can have at most three isolated points when $\dim F = 0$. This completes the proof of the theorem.

References

[1] J. Cheeger, *Some examples of manifolds of nonnegative curvature*, J. Differential Geometry **8** (1973), 623–628.

[2] J. Cheeger & D. Ebin, *Comparison theorems in Riemannian geometry*, North-Holland, Amsterdam, 1975.

[3] S. S. Chern, *Differential geometry, its past and future*, International Congress of Mathematicians, Nice, 1970.

[4] T. Frankel, *Manifolds with positive curvature*, Pacific J. Math. **11** (1961), 165–174.

[5] M. H. Freedman, *The topology of four-dimensional manifolds*, J. Differential Geometry **17** (1982) no. 3, 357–453.

[6] D. Gromoll & W. Meyer, *On complete open manifolds of positive curvature*, Ann. of Math. (2) **90** (1969) 45–90.

[7] R. Hamilton, *Three-manifolds with positive Ricci curvature*, J. Differential Geometry **17** (1982) no. 2, 255–306.

[8] S. Kobayashi, *Transformation groups in differential geometry*, Springer, New York, 1972.

[9] B. O'Neill, *The fundamental equations of a submersion*, Michigan J. Math. **13** (1966), 459–469.

[10] J. L. Synge, *On the connectivity of spaces of positive curvature*, Quart. J. Math. Oxford Ser. 7 (1936), 316–320.

[11] V. A. Toponogov, *Riemannian spaces with curvature bounded below*, Uspehi Mat. Nauk. **14** (1959) no. 1, pp. 87–130.

UNIVERSITY OF CALIFORNIA, BERKELEY